3305 Mathematics for General Relativity Notes
Based on the autumn 2012 lectures by
Dr J Burnett.

3/10/11

S. Carroll
Space time and Geometry → http://arxiv.org/abs/gr-qc/9712019
Addison Wesley
Lewis Ryder
Introduction to GR.
Cambridge University Press.

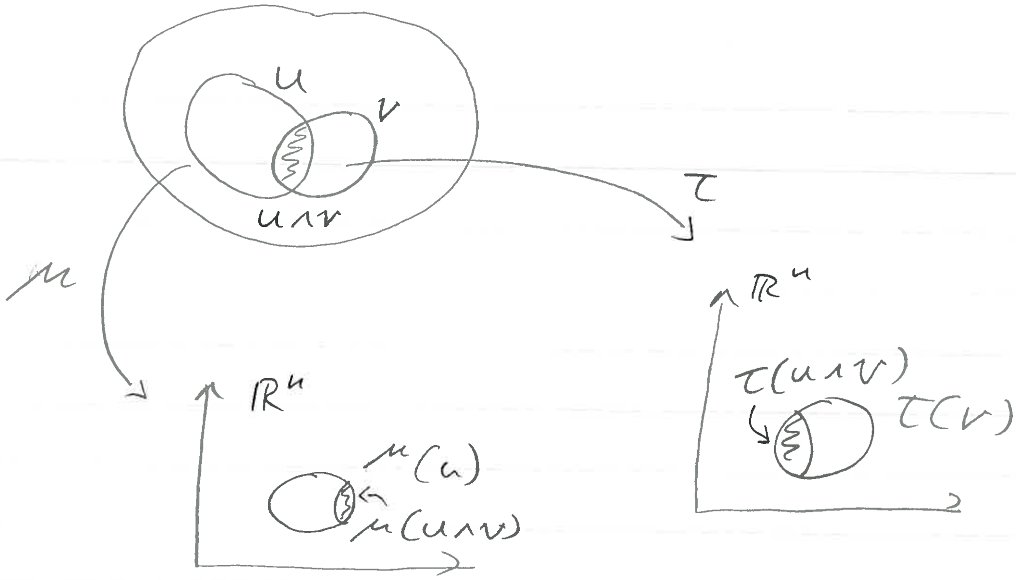
712 - Room, James B.

Moddle: MATH3305, Key: 3305EGR

—/—

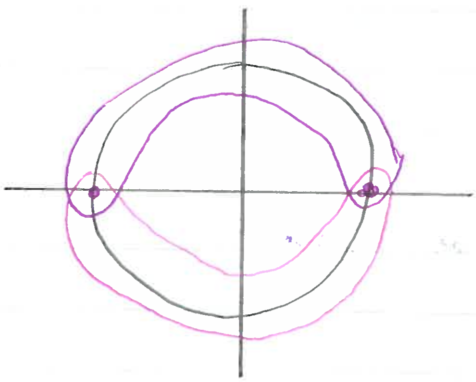
1. Manifolds:
A Manifold generlies what we called a surface or a space.

Def 1.11: A manifold M:
i) M is a set of points which can be mapped int $\mathbb{R}^n$, $n \in \mathbb{N}$, where n is called the dimension of the manifold.
ii) this mapping must be one-to-one.
iii) If two mappings overlap, one must be a differen-tiable function of the other.



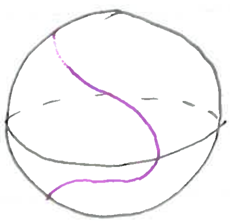
$\mu : u \longrightarrow \mathbb{R}^n$
$\tau : v \longrightarrow \mathbb{R}^n$

<u>Ex 1.11</u>: S^1 = Circle.

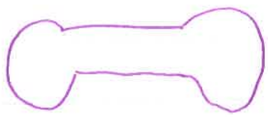


We can choose the angle as our co-ord which we can't choose to go from $0 \to 2\pi$ as $0 = 2\pi$ and then you break one-to-one so choose two.

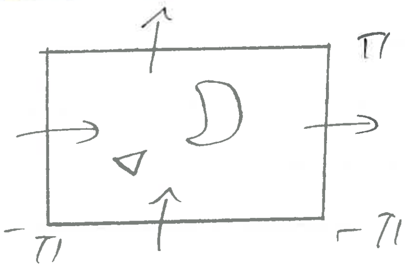
Ex 1.2: Sphere S^2

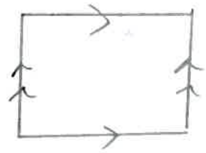


Sphere has a similar problem as the circle with the circle with the periodic co-ordinates.

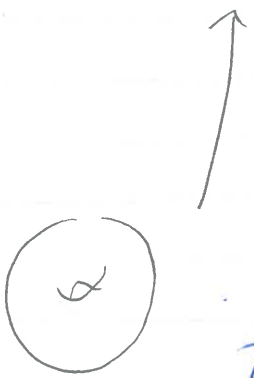


Ex 1.4 $M = \mathbb{T}^2$



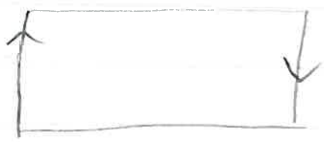


this is that



$$T^2 = S^1 \times S^1$$

Ex 1.5 : Möbius Strip.



Only one sided and one boundary 4π to come back to the start. Like an electron

1.2 Coordinate transformation :

Let U be an open set of points and let a point $p \in U \subset M$ (dim $= n$)

$$X = (X^1, \dots, X^n) \quad \text{e.g } X = (x, y)$$

$$Y = (Y^1, \dots, Y^n) \quad \text{e.g } Y = (r, \theta)$$

These have differentiable functions at each other if they overlap

$$Y^1 = Y^1(X^1, \dots X^n)$$

$$\vdots$$

$$Y^n = Y^n(X^1, \dots, X^n)$$

And vice-versa.

Recall, the Jacobi matrix from vector calculus.

$$\frac{\partial Y^a}{\partial X^b} = J^a{}_b$$

row

columns

a labels the rows
and b labels the columns.

E.g: $X = (x, y)$, $X' = (r, \theta)$.

$$X^1 = X^1(r, \theta) = x = r\cos\theta.$$

$$X^2 = X^2(r, \theta) = y = r\sin\theta.$$

$$J^1{}_1 = \frac{\partial X^1}{\partial X'^1} = \frac{\partial x}{\partial r} = \cos\theta.$$

1.3 Notations and conventions:

Def 1.2: Einstein summation convention.

Given two objects one indexed with superscripts $A = A(A^1, \dots, A^n)$ and the other with subscripts $B = B(B_1, \dots, B_n)$ one defines:

$$A^a B_a = \sum_{a=1}^{n} A^a B_a.$$

Eg: $A^a B_a C^b D_b$.

Def 1.3: Partial Derivative: We abbreviate the notation in the following names:

$$\frac{\partial f}{\partial x^a} = \partial_a f = f_{,a}.$$

f is a function $f : M \to \mathbb{R}$.

Def : 1.4 : Coordinate systems are denoted by capital letters. We can prime them and unprime them if needed.

2. Vectors, tensors and matrices.

What we need are tools and objects, to describe our universe

2.1 Definitions

Def 2.1 : Scalar fields.
It assigns numbers to points on the manifold. More precisely it is a function f which maps M to the real line (parts of the real line)

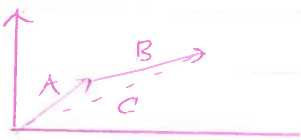
$$f : M \longrightarrow \mathbb{R} .$$

Def 2.2 : Vector or Contravariant vector.

A vector is an object with one superscript that transforms under coordinate transformation as follows:

prime

$$V'^{a} = \frac{\partial x'^{a}}{\partial x^{b}} V^{b}$$

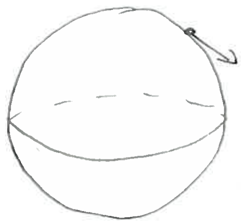
Old bad way : 

$A + B = C$ NO MORE!

Def 2.3 : 1-form or covariant vector. This is an object with one subscript that transform under a co-ordinate transformation a follows

$$W'_b = \frac{\partial X'^a}{\partial X^b} w_b$$

5/10/12



Chapter 2. cont:

We have something with no indices this was the Scalar field. Then there are two objects with 1 index; vector and 1-form (one was superscript and the other subscript).

Def^n 2.4: Tensor. A type $\binom{p}{q}$ tensor is an object with p superscript and q subscript. It is said to be of rank p+q.

Under co-ordinate transformations a $\binom{p}{q}$ tensor transform according to

$$T'^{a_1 \dots a_p}{}_{b_1 \dots b_q} = \frac{\partial x'^{a_1}}{\partial x^{c_1}} \cdots \frac{\partial x'^{a_p}}{\partial x^{c_p}} \frac{\partial x^{d_1}}{\partial x'^{b_1}} \cdots \frac{\partial x^{d_q}}{\partial x'^{b_q}} T^{c_1 \dots c_p}{}_{d_1 \dots}$$

Scalars are rank 0 tensors, vectors and 1-form of type $\binom{1}{0}$ and $\binom{0}{1}$ respectively. A rank 2 tensor can be visualised by a n×n matrix.

Lemma 2.1 The transformation (2.2) and (2.3) are inverse.

Proof: Consider the product.

$$A^a{}_c = \frac{\partial x'^a}{\partial x^b}\frac{\partial x^b}{\partial x'^c} = \frac{\partial x'^a}{\partial x'^c} = \delta^a{}_c$$

Chain rule

—/—

Eg: $x = (x, y)$, $x^1 = x$
$x^2 = y$.

$$\frac{\partial x^2}{\partial x^1} = \frac{\partial y}{\partial x} = 0, \quad \frac{\partial x^1}{\partial x^1} = \frac{\partial x}{\partial x} = 1$$

$$\frac{\partial x^2}{\partial x^2} = \frac{\partial y}{\partial y} = 1, \quad \frac{\partial x^1}{\partial x^2} = \frac{\partial x}{\partial y} = 0$$

—/—

$$\delta^a{}_c = \begin{cases} 1 & a = c \\ 0 & a \neq c \end{cases}$$

—/—

2.2 Tensor Algebra. ← Will appear in exam

Def. 2.5: Addition: Two tensor can be added together if they are of the same type

$$R^a{}_b{}^c + S^a{}_b{}^c = T^a{}_b{}^c$$

but not:

$$A^a + B_a$$

Def 2.6 : Composition : Given a type $\binom{p}{q}$ and another type $\binom{r}{s}$ tensor, these can be combined to give $\binom{p+r}{q+s}$ tensor

Ex 2.2 :

$$V^a = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{matrix} = V^1 \\ = V^2 \end{matrix} \qquad W_b = (\overset{= W_1}{2} \quad \overset{= W_2}{3})$$

$$M^a{}_b = V^a W_b = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}$$

Def^n 2.7 : Contraction, Given a type $\binom{p}{q}$ tensor we can form a $\binom{p-1}{q-1}$ tensor. i.e. we can sum over one upper and one lower index:

$$T^{a_1 \ldots d \ldots a_p}{}_{b_1 \ldots d \ldots b_q} = U^{a_1 \ldots a_p}{}_{b_1 \ldots b_q} \quad \begin{matrix} (p-1) \text{ terms} \\ (q-1) \text{ terms} \end{matrix}$$

Ex :

$$M^a{}_a = \sum_{a=1}^{2} M^a{}_a = M^1{}_1 + M^2{}_2 = 2+6=8$$

10/10/12

Trace :

$$\operatorname{tr} M = M^a{}_a$$

M is a rank 2 tensor of type $\binom{1}{1}$

$$M^a{}_b = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}$$

$$M^a{}_a = M^1{}_1 + M^2{}_2 = 2+6 = 8$$

Defn: 2.9 : Let T^{ab} be a rank 2 tensor of type $\binom{2}{0}$

We define symmetrisation as follows

$$T^{(ab)} = \frac{1}{2}\left(T^{ab} + T^{ba}\right)$$

They are different and important.

We define anti-symmetrisation as follows :

$$T^{[ab]} = \frac{1}{2}\left(T^{ab} - T^{ba}\right)$$

Def 2.10 : T^{ab} is symmetric if

$$T^{ab} = T^{(ab)}$$

Eg :

$$T^{ab} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

$$T^{12} = T^{21}$$

$$T^{(ab)} = \frac{1}{2}\left(T^{ab} + T^{ba}\right) = T^{ab}$$

T^{ab} is antisymmetric if

$$T^{ab} = T^{[ab]}$$

Eg: $A_{ab} \Rightarrow A_{ab} = -A_{ba}$

$S^{ab} \Rightarrow S^{ab} = S^{ba}$

If you contract A_{ab} and S^{ab}

$$A_{ab} S^{ab}$$

relabel $a \longleftrightarrow b$

$$A_{ba} S^{ba}$$

$$A_{ba} S^{ab}$$

$$-A_{ab} S^{ab}$$

Any tensor can be written in terms of its antisymmetric part and symmetric part as follows.

Defn 2.11: Levi-Civita tensor

The Levi-Civita tensor is a totally anti-symmetric tensor of rank n in n dimension.

$$\varepsilon^{ab\dots n} = \begin{cases} 0 & \text{if any two of the indices are equal} \\ +1 & \text{if } (a,b,\dots,n) \text{ is an even permutation} \\ -1 & \text{if } (a,b,\dots,n) \text{ is an odd permutation} \end{cases}$$

e.g: 2 dimensions

$$\varepsilon^{ab} = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}$$

$$\varepsilon^{12} = +1$$

3 dimensions

$$\varepsilon^{123} = +1$$

$$\varepsilon^{132} = -1$$

$$\varepsilon^{321} = +1$$

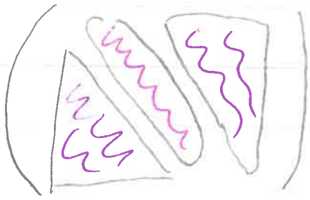
$$\varepsilon^{111} = 0.$$

In 2 dimensions take a rank 2 tensor of type $\binom{2}{0}$
$\rightarrow T^{ab}$

T^{ab} has 4 independent components in 2 dimensions.

In n dimensions T^{ab} has n^2 independent components.

$T^{(ab)}$ has $\frac{n^2}{2} + \frac{n}{2} = \frac{n(n+1)}{2}$ independent comp.



$T^{[ab]}$ has $\frac{n(n-1)}{2}$ independent comp.

Eg $n = 3$, (dim) Let $M = \mathbb{E}^3$, Euclidean 3-space with cartesian co.ord. (x, y, z)

$$(\vec{\nabla} \times \vec{A})^i = \varepsilon^{ijk} \partial_j A_k$$

Let $i = y$

$$(\vec{\nabla} \times \vec{A})^y = \varepsilon^{yjk} \partial_j A_k$$

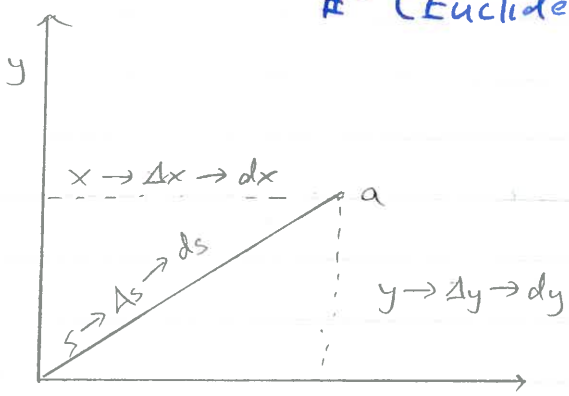
$$(\vec{\nabla} \times \vec{A})^y = \varepsilon^{yxz} \partial_x A_z + \varepsilon^{yzx} \partial_z A_x + 0$$

Def^n $\varepsilon^{xyz} = +1$

$$= -\partial_x A_z + \partial_z A_x$$

<u>Metrics and Geodesics. I</u>

$\mathbb{E}^2$ (Euclidean 2-space)



$$s^2 = x^2 + y^2$$

$$\Delta s^2 = \Delta x^2 + \Delta y^2$$

$$ds^2 = dx^2 + dy^2$$

Defn 2.12 : Metric

Let $(X^1, \dots, X^n)$ and $(X^1 + dX^1, \dots, X^n + dX^n)$ be two nearby points. We can define the distance by introducing g_{ab} the metric tensor. The distance satisfies

$$ds^2 = g_{ab}\, dX^a dX^b$$

g_{ab} in general is some arbitrary function of the co-ords.

It is non-degenerate (non-zero determinant) and therefore its inverse exists. $g_{ab} g^{bc} = \delta_a^c$.

Eg. in Euclidean 2-space

$$g_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$g_{11} = g_{22} = 1$$
$$g_{12} = g_{21} = 0$$

ds^2 is called the line element

g_{ab} is defined to the symmetric therefore it has $\frac{n(n+1)}{2}$ independant components

We can use g_{ab} to raise and lower indices.

Eg $V^a g_{ab} = V_b$

Defn 2.13: Total Differential.

Let f be a function of several variables $f = f(x^1, \dots, x^n)$ the total differential is defined as

$$df = f_{,a} \, dx^a$$

$$= \underbrace{\partial_a}_{\text{1 form}} \overbrace{dx^a}^{\text{basis}}$$

$$= \frac{\partial f}{\partial x^a} dx^a$$

Ex 2.6: $M = \mathbb{E}^2$ (Euclidean 2-space)

In cartesian co-ordinates $(x^1, x^2) = (x, y)$

$$ds^2 = dx^2 + dy^2 \,. \quad g_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$x = r\cos\theta \quad , \quad dx = \cos\theta \, dr + r(-\sin\theta)\, d\theta \,.$$

$$y = r\sin\theta \quad , \quad dy = \sin\theta \, dr + r\cos\theta \, d\theta \,.$$

$$dx^2 = \cos^2\theta \, dr^2 + r^2\sin^2\theta \, d\theta^2 - 2r\sin\theta\cos\theta \, dr\, d\theta$$

$$dy^2 = \sin^2\theta \, dr^2 + r^2\cos^2\theta \, d\theta^2 + 2r\sin\theta\cos\theta \, dr\, d\theta$$

$$ds^2 = dx^2 + dy^2 = dr^2 + r^2 d\theta^2$$

$$g_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

Def 2.14: Signature of a metric

Using the results from linear algebra one can show at any point $p \in M$ the metric can be diagonlised.

In general, one can choose the diagonal elements to be ± 1 in a suitable basis. The number of $+$ signs and number of $-$ signs is independent of that choice.

We call the number of $+$ and $-$ signs the **signature**. Their sum can be used.

e.g: $M = \mathbb{E}^3$ $\quad dr^2 = dx^2 + dy^2 + dz^2$ has signature $(+,+,+)$ or signature 3.

e.g: $M =$ Rindler spacetime:

$$ds^2 = x\,dt^2 + dx^2$$

with co-ords $-\infty < t < \infty$ and $0 < x < \infty$. This metric has signature $(-,+)$ or 0.

Defn 2.15: Riemannian metric
Let $V^a \neq 0$ be a contravariant non-vanishing vector. The metric g_{ab} is called Riemannian if

$$g_{ab} V^a V^b > 0 \quad \forall V^a \neq 0.$$

A manifold equipped with a Riemannian is called a Riemannian manifold.

Lemma 2.2: Let M be a n-dim manifold of signature n. Then M is a Riemannian manifold.

Proof: Let $V^a \neq 0$.
$$g_{ab} V^a V^b = (V^1)^2 + (V^2)^2 + \ldots + (V^n)^2 > 0$$

$\square$

Def 2.16: Pseudo-Riemannian metric which is not Riemannian

Eg: Rindler Space time has a Psuedo-Riemannian metric.

Eg: $M = \mathbb{M}^4$ Minkowski spacetime is given by the metric

$$ds^2 = dt^2 + dx^2 + dy^2 + dz^2 .$$

12/10/12

① $ds^2 = dt^2 - dx^2 - dy^2 - dz^2$ $(+,-,-,-)$ or -2

② $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$ $(-,+,+,+)$ or 2

Def^n: 2.17: Lorentzian metrics. Metrics with either $(-+++\ldots+)$ or $(+,-,\ldots,-)$ are Lorentzian metrics (only one sign is different). A manifold with a Lorentzian metric is called a Lorentian manifold.

We have already defined what a function is:

$$f : M \to \mathbb{R}.$$

(a map from the manifold to the real line)
Next, we are going define a curve which is the opposite of a function.

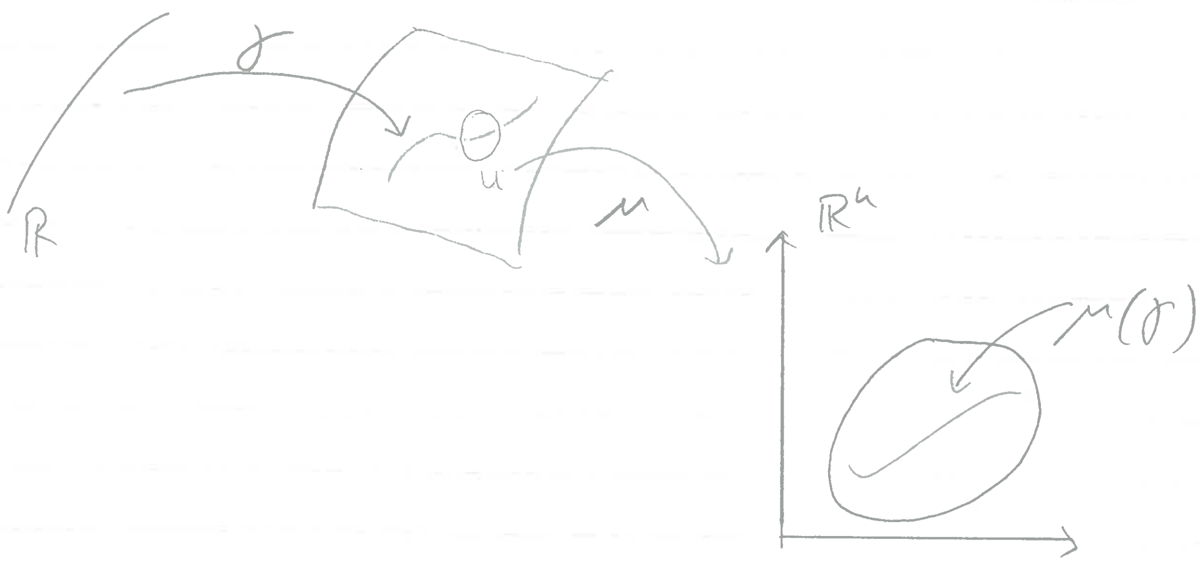
Def^n 2.18 : Curve. A curve is a mapping from the real line into the manifold

$$\gamma : \mathbb{R} \to M.$$

A smooth curve γ on M is a C^∞ mapping from $\mathbb{R}$ to the manifold.

Let us assume that γ lies in an open region $u \subset M$. By definition there exist a set of local coordinates u that map u into $\mathbb{R}^n$. Then the curve provides a set of n coordinate functions of the parameter λ.

$$\gamma(\lambda) = \begin{pmatrix} x^1(\lambda) \\ \vdots \\ x^n(\lambda) \end{pmatrix} = x^a(\lambda).$$



2012 1e) <u>Defⁿ 2.19</u>: Tangentvector to the curve: Let γ be a smooth curve on M. The tangent to the curve γ in any co-ord basis:

$$T^a = \frac{dx^a}{d\lambda}$$

<u>Defⁿ 2.20</u>: Let V^a be a vector on a Riemannian manifold with metric g_{ab}, the norm of V^a is defined as

$$|V| = \sqrt{g_{ab} V^a V^b}$$

Lets assume that a curve $x^a(\lambda)$ connects two points on a manifold M. We can compute the length of the curve:

$$\int ds = \int \frac{ds(\lambda)}{d\lambda} \, d\lambda$$

$$\int ds = \int \sqrt{g_{ab} \dot{x}^a \dot{x}^b} \, d\lambda .$$

$$ds^2 = g_{ab} \, dx^a dx^b$$

$$\dot{x}^a = \frac{dx^a}{d\lambda}$$

We want to follow the principle of least action and find the shorest length.

Def^n: 2.21: Geodesics: Let $x^a(\lambda)$ be a curve. We define a geodesic to be curve whose path extremeises the functional

$$\int ds = \int \sqrt{g_{ab} \dot{x}^a \dot{x}^b} \, d\lambda .$$

"Without loss of generality (length is a parameterisation independent) we may assume that

$$g_{ab} \dot{x}^a \dot{x}^b = 1$$

which is called the affine parameterisation.

We will call the Lagrangian of our system $L = g_{ab} \dot{x}^a \dot{x}^b$

Lemma 2.3 Geodesic equation. A geodesic satisfies the following equations of motion:

$$\frac{d^2 x^a}{d\lambda^2} + \Gamma^a_{bc} \frac{dx^b}{d\lambda} \frac{dx^c}{d\lambda} = 0 .$$

where:

$$\Gamma_{bc}{}^{a} = \frac{1}{2} g^{ad} \left(\partial_b g_{dc} + \partial_c g_{db} - \partial_d g_{bc} \right)$$

<u>Proof</u>:

$$L = g_{ab} x^a x^b$$

The Euler-Lagrange eq^{n}s are

$$\frac{d}{dx} \frac{\partial L}{\partial \dot{x}^c} = \frac{\partial L}{\partial x^c}$$

N.B: we will write ei away that is eaiser to read for now but is slightly bad

$$L = g_{ab}(x^c)\, x^a x^b$$

2.4 A glance forward.

The geodesic eq^n looks like this:

$$\frac{d^2x^a}{d\tau^2} = -\Gamma^a_{bc}\,\dot{x}^b\,\dot{x}^c$$

From Newtonian gravity

$$m\ddot{\underline{r}} = -m\nabla\Phi(\underline{r})$$

$$\frac{d^2x^i}{dt^2} = -\frac{\partial\Phi}{\partial x^i}$$

The Christoffel contains the first derivative of the metric tensor. The metric must contain the gravitational potential.

Newtonian's theory of gravity also gives:

$$\Delta\Phi(\underline{r}) = 4\pi G\rho(\underline{r}).$$

3. A little Special Relativity.

Minkowski metric is what we use for Special Relativity

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2$$

Its spacetime co-ordinates are:

$$(x^0, x^1, x^2, x^3) = (t, x, y, z).$$

$$\eta_{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Conventially η_{ab} and g_{ab}

Reelly what we have is either $x^0 = ct$ or $x^1 = x/c$ $x^2 = y/c$, $x^3 = z/c$, c being the speed of light (m/s).
We set $c = 1$.

Don't forget there is another convention where

$$\eta_{ab} = \text{dig}(-1, +1, +1, +1)$$

The purely spatial part is still Euclidean 3-space. This part is invariant rotations and translation.

But we are considering now space and time we now have an extra translation and more rotations into time.

<u>Defn 3.1.</u> Boost.
A boost is a transformation to a coordinate system moving at a constant velocity with respect to the original one:

<u>Defn 3.2</u> Inertial reference frame.
A Inertial refrence frame is a coord system with Cartesian co-ord, where there are no interial (fictitous) forces.

17/10/12

$$\int ds = \int \sqrt{g_{ab}\dot{x}^a\dot{x}^b}\, d\lambda.$$

— / —

$$L = g_{ab}(x^c)\dot{x}^c\dot{x}^b$$

Euler-Lagrange equations are:

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^c}\right) = \frac{\partial L}{\partial x^c}$$

$$\frac{\partial L}{\partial x^c} = \frac{\partial g_{ab}}{\partial x^c}\dot{x}^a\dot{x}^b = g_{ab,c}\dot{x}^a\dot{x}^b$$

$$\frac{\partial L}{\partial \dot{x}^c} = g_{ab}\frac{\partial \dot{x}^a}{\partial \dot{x}^c}\dot{x}^b + g_{ab}\dot{x}^a\frac{\partial \dot{x}^b}{\partial \dot{x}^c}$$

$$= g_{ab}\delta^a{}_c\dot{x}^b + g_{ab}\dot{x}^a\delta^b{}_c.$$

$$= g_{cb}\dot{x}^b + g_{ac}\dot{x}^a.$$

2012

$$= 2g_{ac}\dot{x}^a \qquad (g_{ab} = g_{ba})$$

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^c}\right) = 2\frac{dg_{ac}}{d\lambda}\dot{x}^a + 2g_{ab}\ddot{x}^a$$

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^c}\right) = 2g_{ac,b}\dot{x}^b\dot{x}^a + 2g_{ac}\ddot{x}^a$$

Hence, we find:

$$g_{ab,c}\dot{x}^a\dot{x}^b = 2g_{ca}\ddot{x}^a + g_{ca,b}\dot{x}^a\dot{x}^b + g_{cb,a}\dot{x}^a\dot{x}^b$$

$$0 = 2g_{ca}\ddot{x}^a + (g_{ca,b} + g_{cb,a} - g_{ab,c})\dot{x}^a\dot{x}^b$$

Next we apply g^{cd}

$$0 = 2\delta^d_c\,\ddot{x}^a + g^{cd}(g_{ca,b} + g_{cb,a} - g_{ab,c})\dot{x}^a\dot{x}^b$$

Divide by 2

$$0 = \ddot{x}^d + \frac{1}{2}g^{cd}(g_{ca,b} + g_{cb,a} - g_{ab,c})\dot{x}^a\dot{x}^b$$

Finally we will relabel:

$a \to b$, $b \to c$, $c \to d$, $d \to a$.

$$0 = \ddot{x}^a + \frac{1}{2}g^{da}(g_{db,c} + g_{dc,b} - g_{bc,d})\dot{x}^b\dot{x}^c$$

Finally, finally we can write:

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0$$

very important

where

$$\Gamma^a_{bc} = \frac{1}{2}g^{da}(g_{db,c} + g_{dc,b} - g_{bc,d})$$

and is called the Chrisstoffel symbol. (IS NOT A TENSOR).

Defn 3.3 Einstein's axiom of SR.

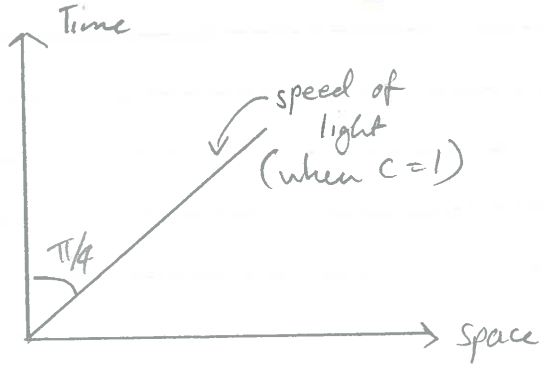
i) The laws of physics are invariant under translations, rotations and boosts.

ii) The speed of light is the same in all interial reference frame.

3.2 Spacetime Diagrams.

Defn 3.4: World-line.
The world-line of an object is the path (curve) it traces in space-time.

Remember $dx/dt = v$ then the gradient of the world-line is equal to the velocity



Defn 3.5 Proper Time.

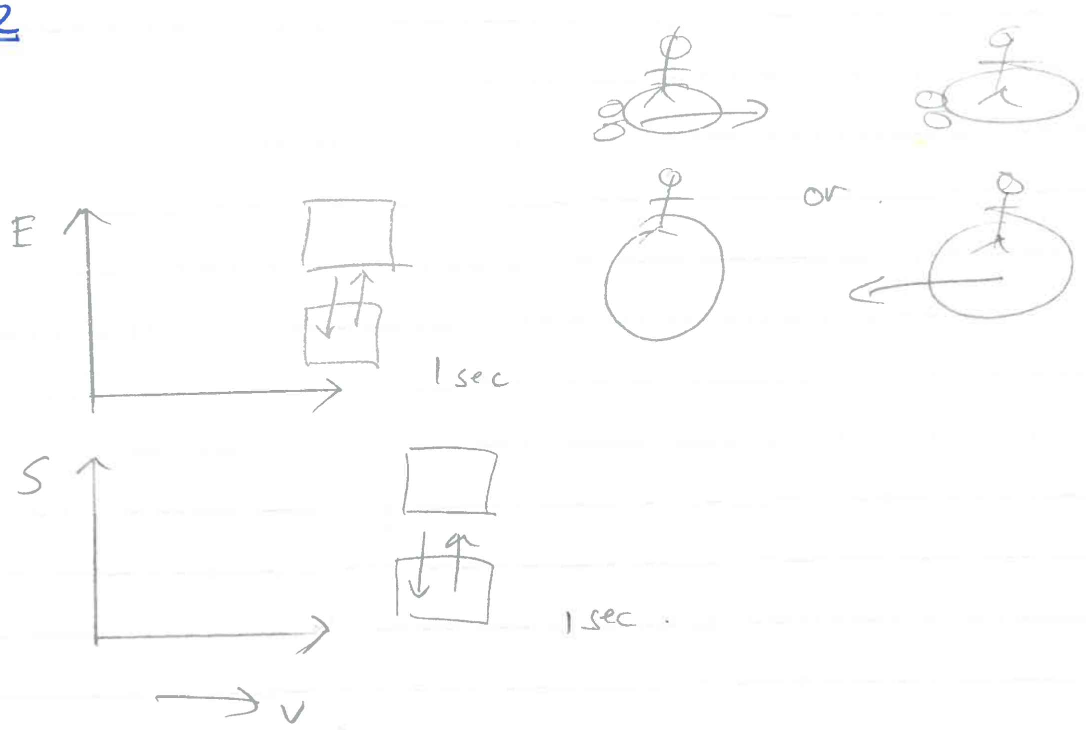
Choose a point p on an object's world line to be at τ. Let τ be the arc length away from p.

$$\tau = \int \sqrt{g_{ab} \dot{x}^a \dot{x}^b} \, d\lambda .$$

The arc-length τ along the world line is called proper time.

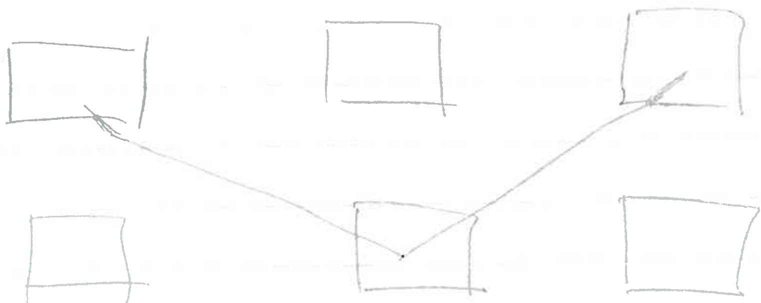
19/10/12

Time ?:



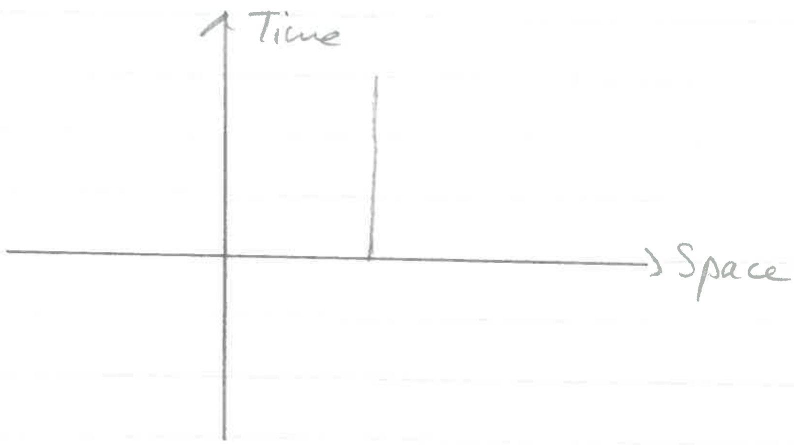
If we are on the Earth and the ship is moving fast enough close to c (speed of light)

What would see is this:



Proper time

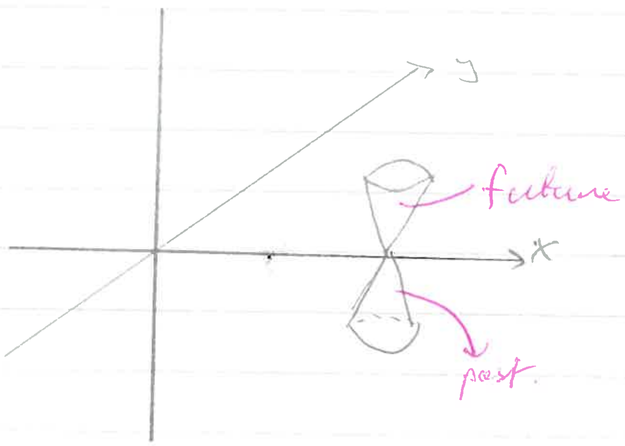
$$\tau = \int \sqrt{g_{ab}\, dx^a dx^b} = \int \sqrt{dt^2 - dx^2 - dy^2 - dz^2}$$



In your own rest frame your $dx = dy = dz = 0$ then $\tau = t$. Proper time equals co-ordinate time.

Def 3.6: Event. A event p is a point in spacetime

Suppose at an event p a light signal is sent out. This will look like a spherical wave front leaving some point, which looks as follows time.



Def 3.7: Let p and q be two event. The interval pq is called time-like if $\Delta\tau^2 > 0$, space-like [$(+,-,-,-)$] if $\Delta\tau^2 < 0$ and light-like if $\Delta\tau^2 = 0$.

Since all massive objects moves slower than c, they must travel within the future light cone. They can only reach time-like parts.

A particle 3-velocity $\underline{v} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$ is

$$|\underline{v}|^2 dt^2 = dx^2 + dy^2 + dz^2 .$$

For the photon $|\underline{v}| = 1$, hence

$$dt^2 = dx^2 + dy^2 + dz^2 \Rightarrow dt^2 - dx^2 - dy^2 - dz^2 = 0.$$

Since $d\tau^2$ is our interval between points we have

$$d\tau^2 = dt^2 - dx^2 - dy^2 - dz^2 = 0 \quad \text{(for a photon)}$$

Let $x^a(\tau)$ be a world-line and τ proper time.
Let $T^a = \frac{dx^a}{d\tau}$ be the tangent vector to the world line

Then we find

$$\eta_{ab} T^a T^b = \left(\frac{dt}{d\tau}\right)^2 - \left(\frac{dx}{d\tau}\right)^2 - \left(\frac{dy}{d\tau}\right)^2 - \left(\frac{dz}{d\tau}\right)^2$$

$$= \frac{dt^2 - dx^2 - dy^2 - dz^2}{d\tau^2} = 1$$

— / —

$$\eta_{ab} = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

3.3 Lorentz Transformations

Consider Minkowski spacetime $\mathbb{M}^4$

$$ds^2 = \eta_{ab}\, dx^a dx^b$$

There exists co-ord transformation that leave line element invarient:

$$x'^a = \Lambda^a{}_b\, x^b + A^a.$$

These are called Poincaré transformations:

The vector A^a changes the orgin. (translations)

$\Lambda^a{}_b$ leaves the orgin unchanged and is responsible for boost and rotations. The set of all transformations that leave η_{ab} unchanged is called the Poincaré group. Likewise the group of transformations that leave η_{ab} unchanged but also do not move the orgin is called the Lorentz group.

If we consider:

$$\eta_{ab} = \Lambda^c{}_a\, \Lambda^d{}_b\, \eta_{cd} = \Lambda^T \eta \Lambda$$

Take the determinant of both sides, we get

$$(\det \Lambda)^2 = 1$$

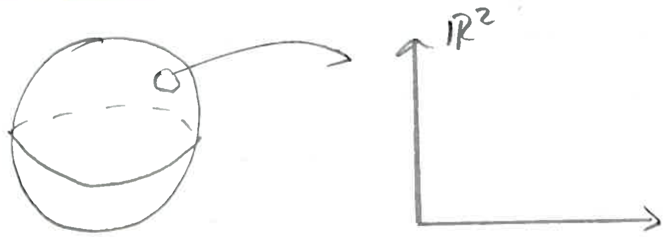
the transformatcoin split into those with $\det \Lambda = 1$ and $\det \Lambda = -1$.

29/10/12

NFE: (Not for exam):

Tangent Spaces

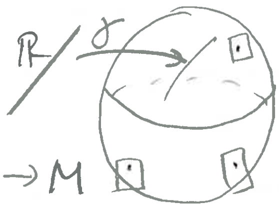
S^2



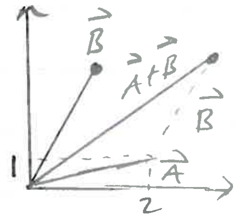
Vector V^a was something with apply index and transforms such

$$V'^a = \frac{\partial x'^a}{\partial x^b} V^b .$$

S^2



$\gamma : \mathbb{R} \to M$



$$\boxed{V = V^a \hat{e}_a}$$

Vector in proper geometric sense.

$$V = V^a \frac{\partial}{\partial x^a} \qquad V : f \to \mathbb{R} .$$

End of NFE.

-/-

$$V' = V$$

$$V' = V'^a \frac{\partial}{\partial x'^a} , \qquad V = V^a \frac{\partial}{\partial x^a}$$

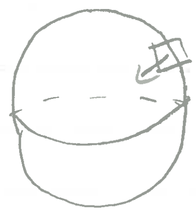
$$v'^a \frac{\partial}{\partial x'^a} = v^a \frac{\partial}{\partial x^a}$$

$$v'^a \frac{\partial x^a}{\partial x'^a} \frac{\partial}{\partial x^b} = v^a \frac{\partial}{\partial x^a}$$

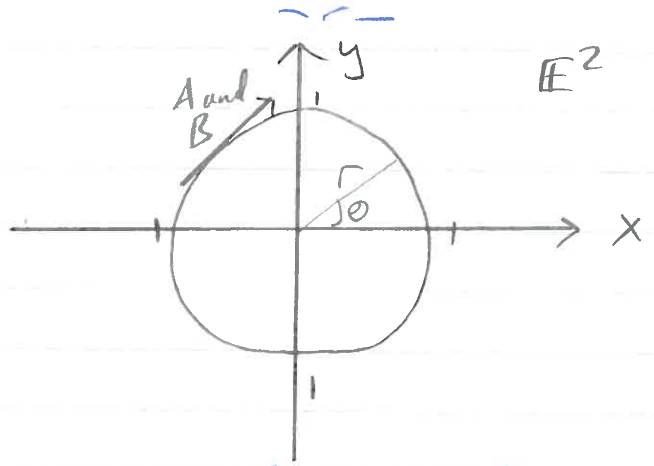
$$v'^a \frac{\partial x^b}{\partial x'^a} \frac{\partial}{\partial x^b} = v^b \frac{\partial}{\partial x^b}$$

$$v'^a \frac{\partial x^b}{\partial x'^a} = v^b$$ (eq^n coefficients of the basis vectors)

$$V : f \to \mathbb{R}$$



$$V = v^a \hat{e}_a = v^a \frac{\partial}{\partial x^a}$$



<u>The co-ords</u>

$$X = (X'^1, X'^2) = (x, y)$$

$$Y = (Y^1, Y^2) = (r, \theta).$$

$\gamma : \mathbb{R} \to M$.

<u>The curve</u>.

N.B: don't be confused that we are reusing the x^1, x^2

In cartesian:

$$X^1_\gamma = x = \cos\lambda$$

$$X^2_\gamma = y = \sin\lambda.$$

In polar

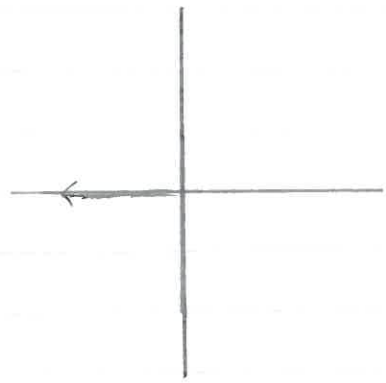
$$Y^1_\gamma = r = 1$$

$$Y^2_\gamma = \theta = \lambda.$$

If we pick a $\lambda = \pi$

$$X^1_\gamma = x = -1$$

$$X^2_\gamma = y = 0$$



This is called a position "vector" but do not be fooled for it is not a vector

$$X^a_\gamma = \frac{\partial X^a}{\partial Y^b} Y^b_\gamma$$

$$X^1 = x = r\cos\theta$$
$$X^2 = y = r\sin\theta.$$

$$\frac{\partial x^1}{\partial y^1} = \cos\theta \quad , \quad \frac{\partial x^1}{\partial y^2} = -r\sin\theta .$$

$$\frac{\partial x^2}{\partial y^1} = \sin\theta \quad , \quad \frac{\partial x^2}{\partial y^2} = r\cos\theta .$$

$$\dot{x}^1_\gamma = \frac{\partial x^1}{\partial y^b} \dot{y}^b_\gamma$$

$$\dot{x}^2_\gamma = \frac{\partial x^2}{\partial y^b} \dot{y}^b_\gamma$$

So $$\dot{x}^1_\gamma = \frac{\partial x^1}{\partial y^1} \dot{y}^1_\gamma + \frac{\partial x^1}{\partial y^2} \dot{y}^2_\gamma$$

$$\dot{x}^2_\gamma = \frac{\partial x^2}{\partial y^1} \dot{y}^1_\gamma + \frac{\partial x^1}{\partial y^2} \dot{y}^2_\gamma$$

(on the curve)

So $$\dot{x}^1_\gamma = \cos\theta - r\sin\theta\,\lambda = \cos\lambda - (\sin\lambda)\lambda$$

(on the curve)

$$\dot{x}^2_\gamma = \sin\theta + r\cos\theta\,\lambda = \sin\lambda + (\cos\lambda)\lambda$$

$$A^1 = \frac{d\dot{x}^1_\gamma}{\partial\lambda} = -\sin\lambda \quad , \quad A^2 = \frac{d\dot{x}^2_\gamma}{d\lambda} = \cos\lambda$$

$$B^1 = \frac{d\dot{y}^1_\gamma}{d\lambda} = 0 \qquad B^2 = \frac{d\dot{y}^2_\gamma}{d\lambda} = 1 .$$

So $$A^a = \frac{\partial x^a}{\partial y^b} B^b .$$

$$A^1 = \frac{\partial X^1}{\partial Y^b} B^b$$

$$A^2 = \frac{\partial X^2}{\partial Y^b} B^b$$

$$A^1 = \frac{\partial X^1}{\partial Y^1} B^1 + \frac{\partial X^1}{\partial Y^2} B^2$$

$$A^2 = \frac{\partial X^2}{\partial Y^1} B^1 + \frac{\partial X^2}{\partial Y^2} B^2$$

$$A^1 = \cos\lambda \times 0 - \sin\lambda \times 1 = -\sin\lambda$$
$$A^2 = \sin\lambda \times 0 + \cos\lambda \times 1 = \cos\lambda$$

3.4 Lorentz Boosts.

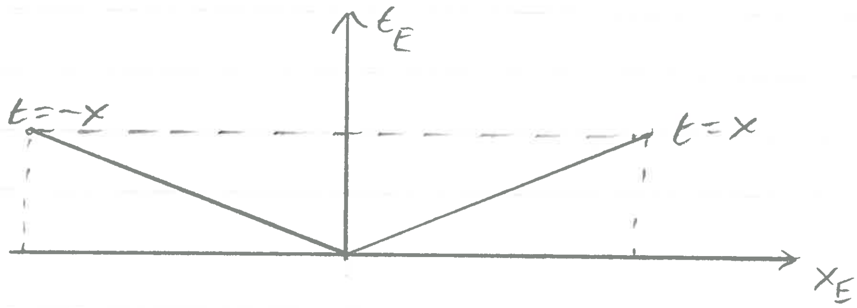
$$X'^a = \underbrace{\Lambda^a{}_b}_{\text{Lorentz part.}} X^b + A^a .$$

2011 ci) Suppose a space ship moves with velocity v in the x-direction with respect to the Earth. We have the Ship's rest frame S and Earth's rest frame E. We will assume their origins coincide at some event p. What is the form of the transformation.

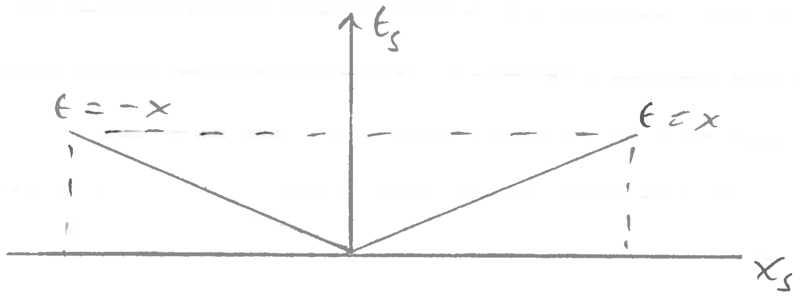
$$\begin{pmatrix} t \\ x \end{pmatrix}_S = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}_E .$$

At the event p two beams of light are emitted in +ve and -ve x direction.

In the Earth's frame.



In the Ship's frame.



The right moving photon passes through $(t, x) = (t, t)$ and the left moving has $(t, x) = (t, -t)$. Suppose in the Earth frame a photon passes through some event.

$$\begin{pmatrix} t \\ x_0 \end{pmatrix} = \begin{pmatrix} t_0 \\ t_0 \end{pmatrix}_E$$

In the co-ords of the ship

$$\begin{pmatrix} t \\ x \end{pmatrix}_S = \begin{pmatrix} t \\ t \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} t_0 \\ t_0 \end{pmatrix}_E .$$

$$\begin{pmatrix} t \\ x \end{pmatrix}_S = \begin{pmatrix} t \\ t \end{pmatrix}_S = \begin{pmatrix} \alpha t_0 + \beta t_0 \\ \gamma t_0 + \delta t_0 \end{pmatrix}.$$

$$\Rightarrow \alpha + \beta = \gamma + \delta .$$

The left moving photon goes through an event:

$$\begin{pmatrix} t \\ x \end{pmatrix}_E = \begin{pmatrix} t_0 \\ -t_0 \end{pmatrix}_E .$$

$$\begin{pmatrix} t \\ x \end{pmatrix}_S = \begin{pmatrix} t \\ -t \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} t_0 \\ -t_0 \end{pmatrix}$$

$$\begin{pmatrix} t \\ x \end{pmatrix}_S = \begin{pmatrix} t \\ -t \end{pmatrix} = \begin{pmatrix} \alpha t_0 - \beta t_0 \\ \gamma t_0 - \delta t_0 \end{pmatrix}$$

$$\Rightarrow \alpha - \beta = \delta - \gamma$$

$$\alpha = \delta , \; \beta = \gamma .$$

ii) Follow spatial origin in the ship's co-ordinates.

$$\begin{pmatrix} t \\ x \end{pmatrix}_S = \begin{pmatrix} t \\ 0 \end{pmatrix}_S$$

However, we see it move with velocity v from the Earth's frame so $(t, vt)_E$

$$\begin{pmatrix} t \\ 0 \end{pmatrix}_S = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} t \\ vt \end{pmatrix}_E$$

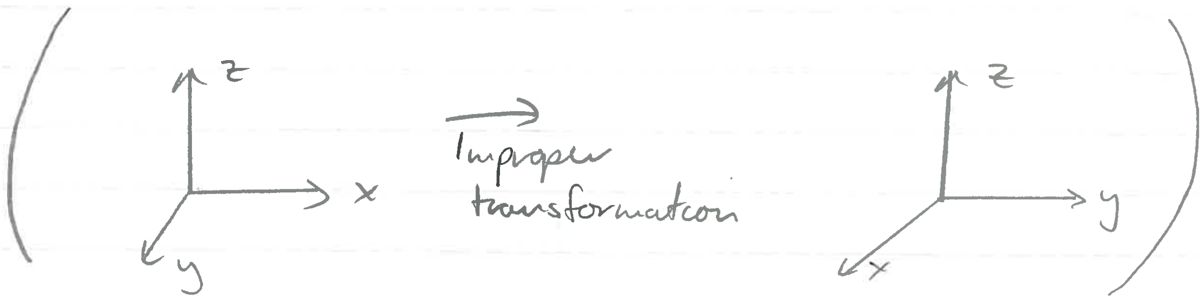
Same thing

$$\begin{pmatrix} t \\ 0 \end{pmatrix}_S = \begin{pmatrix} \alpha t + \beta vt \\ \beta t + \alpha vt \end{pmatrix}$$

$$\Rightarrow \beta = -\alpha v .$$

iii) Finally we assume proper transformations

$$\det \begin{vmatrix} \alpha & -\alpha v \\ -\alpha v & \alpha \end{vmatrix} = \alpha^2 - \alpha^2 v^2 = 1.$$



Rewrite α as γ, this then gives

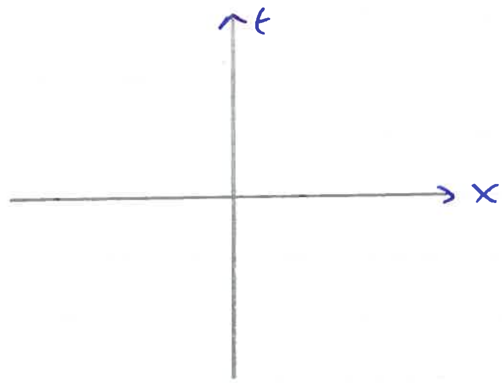
$$\gamma = \frac{1}{\sqrt{1-v^2}}$$

The famous γ factor $(c=1)$.

26/10/12

$$X'^a = \Lambda^a{}_b X^b$$

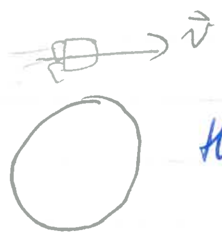
$$\begin{pmatrix} t \\ x \end{pmatrix}_S = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}_E .$$



— / —

End of section 3.4

$$\Lambda^a{}_b = \begin{pmatrix} \gamma & -\gamma v \\ -\gamma v & \gamma \end{pmatrix}$$ for a boost from one reference frame to another with is moving with velocity $\vec{v}$ in the x-direction.



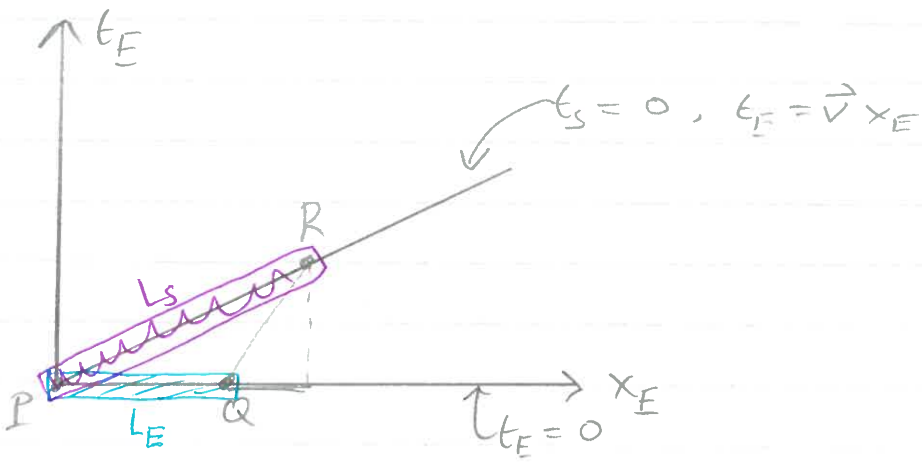
The Earth thinks the ship is moving to the right with $\vec{v}$ and the ship thinks the Earth to the left with velocity $-\vec{v}$.

3.5 Simultaneity.

A "surface of simultaneity" (SOS) is a set of points where $t = c$ is some reference frame. Let's look at the ship's SOS in the Earth's co-ordinate frame. We will look at $t_S = 0$.

$$\begin{pmatrix} t_E \\ x_E \end{pmatrix} = \begin{pmatrix} \gamma & \gamma \vec{v} \\ \gamma \vec{v} & \gamma \end{pmatrix} \begin{pmatrix} 0 \\ x_S \end{pmatrix}$$ which gives

$$t_E = \gamma \vec{v} x_S \;, \; x_E = \gamma x_S \;, \;\Rightarrow t_E = \vec{v} x_E .$$



Events P and Q are simultaneous for the Earth (in the Earth's frame) and P and R are simultaneous for the Ship. The Earth and ship don't agree!

3.6 Length Contraction

Def 3.9: Length: The Length of an L is the spatial distance between the ends, measured simultaneously in some reference frame.

Consider a metre stick at rest on the Millennium Falcon. Hans and Chewbacca would measure the ends of the stick simultaneously at P, R. Luke on Tatooine sees P, Q as simultaneous events corresponding to the end of the sticks at $t_E = 0$.

We need to find Q in the ship frame and R in the Earth's frame.

$$Q: \begin{pmatrix} t_Q \\ x_Q \end{pmatrix}_S = \begin{pmatrix} \gamma & -\gamma\vec{v} \\ -\gamma\vec{v} & \gamma \end{pmatrix} \begin{pmatrix} 0 \\ L_E \end{pmatrix} = \gamma \begin{pmatrix} -\vec{v} L_E \\ L_E \end{pmatrix}$$

$$R: \begin{pmatrix} t_R \\ x_R \end{pmatrix}_E = \begin{pmatrix} \gamma & \gamma\vec{v} \\ \gamma\vec{v} & \gamma \end{pmatrix} \begin{pmatrix} 0 \\ L_s \end{pmatrix} = \gamma \begin{pmatrix} \vec{v} L_s \\ L_s \end{pmatrix}$$

The right end of the stick moves $Q \to R$ with speed v w.r.t Earth.

$$\Rightarrow x_{RE} = x_{QE} + v(t_{RE} - t_{QE})$$

$$= x_{QE} + v t_{RE}.$$

$$\gamma L_s = L_E + v^2 \gamma L_s.$$

$$L_E = \gamma L_s (1 - v^2)$$

$$= L_s \gamma \gamma^{-2}$$

$$= L_s \gamma^{-1}$$

$$\gamma = \frac{1}{\sqrt{1-v^2}}$$

since $\gamma > 1$

$$L_s > L_E.$$

31/10/12

3.7 Relativistic Dynamics.

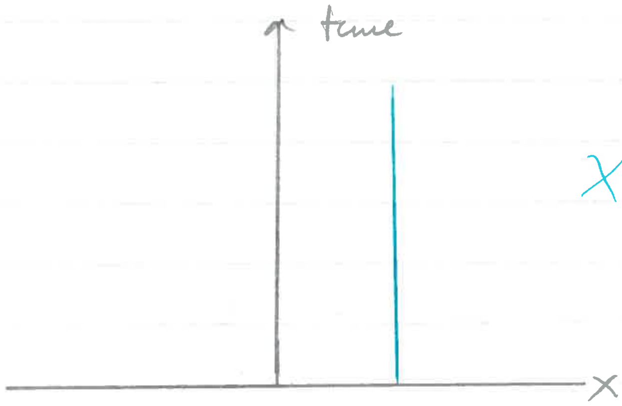
3-momentum is defined as $\vec{p} = m\vec{v}$. For an object travelling with a velocity $\vec{v}$ a Lorentz-transformation from its rest frame gives a

$$u^a = \begin{pmatrix} \gamma \\ \gamma\vec{v} \end{pmatrix}$$

(In its own rest frame $u^a_{object} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$)

$$u^a = \Lambda^a{}_b \, u^b{}_{object} = \begin{pmatrix} \gamma & \gamma\vec{v} \\ \gamma\vec{v} & \gamma \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad u^a = \begin{pmatrix} \gamma \\ \gamma v \end{pmatrix}$$



$$x^a = \begin{pmatrix} \tau \\ 0 \end{pmatrix}$$

u^a is the 4-velocity in relativistic dynamics.

$$u^a = \frac{dx^a}{d\tau}$$

You can verify that $u^a u_a = 1$.

$$u^a u_a = u^a u^b \eta_{ab} = \frac{dx^a}{d\tau}\frac{dx^b}{d\tau}\eta_{ab}$$

Defn 3.10 : 4-momentum.
The 4-momentum is defined as $p_a = u_a m$.

$$p_0 = E$$
$$p_i = -\vec{p}$$
$$i = 1, 2, 3.$$

(The convariant form).

We can write $p^a = m u^a$.

$$\Rightarrow p^0 = \gamma m \quad , \quad p^i = \gamma m \vec{v} . \quad i = 1, 2, 3.$$

$$E = p^0 = \gamma m = m \frac{1}{\sqrt{1-v^2}} \quad \text{for } v \ll 1.$$

$$= m + \frac{1}{2} m v^2 + O(v^2)$$

—/—

$$p_a \eta^{ab} = p^b$$

—/—

for massive particles:

$$E^2 = |\vec{p}|^2 + m^2.$$

Def 3.11: Newton's force law:
In special relativity; Newton's Force becomes

$$f^b = m a^b$$

where a^b is the 4-acceleration defined by

$$a^b = \frac{du^b}{d\tau} = \frac{d^2 X^b}{d\tau^2}$$

where u^b is 4-velocity and X^b is the curve.

Lemma 3.1.

2011

$$a_b u^b = 0.$$

Proof

$$a_b u^b = a^c u^b \eta_{cb}. = \frac{du^c}{d\tau} u^b \eta_{cb}$$

$$= \eta_{cb} \frac{1}{2} \frac{d}{d\tau} (u^c u^b)$$

$$= \frac{1}{2} \frac{d}{d\tau} \left(\eta_{cb} u^c u^b \right) = \frac{1}{2} \frac{d}{d\tau} (1)$$

$$= 0$$

In the limit of $v \to 1$ the γ factor diverges. Photons are massless so we can still keep $E = \gamma m$ constant.

For massless particles:

$$|\vec{p}|^2 = E^2.$$

3-vector $\vec{v}$ is just a unit vector telling you the direction of travel

Important: For a photon $d\tau = 0$ so we can't use τ for a photon.

We can use other parameters (like our own co-ordinate time!).

4 Curvature.

4.1 Covariant derivative and parallel transport.

Definition 4.1: Covariant Derivative.
A covariant derivative ∇_a on a manifold M is a map which takes a type $\binom{p}{q}$ tensor to a type $\binom{p}{q+1}$ tensor.

It satisfies the following properties.

i) Linearity: $\forall \alpha, \beta \in \mathbb{R}$.

$$\nabla_c (\alpha A^{a_1 \dots a_n}{}_{b_1 \dots b_n} + \beta B^{a_1 \dots a_m}{}_{b_1 \dots b_m})$$

$$= (\alpha \nabla_c A^{a_1 \dots a_n}{}_{b_1 \dots b_n} + \beta \nabla_c B^{a_1 \dots a_m}{}_{b_1 \dots b_m})$$

ii) Leibnitz Rule.

$$\nabla_c (A^{\dots}{}_{\dots} B^{\dots}{}_{\dots}) = \nabla_c A^{\dots}{}_{\dots} B^{\dots}{}_{\dots} + A^{\dots}{}_{\dots} \nabla_c B^{\dots}{}_{\dots}$$

iii) Commutivity with contraction:

$$\nabla_c \left(\alpha A^{a_1 \dots k \dots a_m}{}_{b_1 \dots k \dots b_n}\right)$$

$$= \alpha \nabla_c A^{a_1 \dots k \dots a_m}{}_{b_1 \dots k \dots b_n}.$$

iv) Torsion free: for all $f \in C^\infty(M)$.

$$\nabla_a \nabla_b f = \nabla_b \nabla_a f.$$

In GR we are in a torsion free setting, Einstein-Cartan theory has torsion.

In Euclidean 3-space with Cartesian co-ords the covarient derivative should just look like the partial derivative. Moreover for any function in any space time the following holds.

$$\nabla_a f = \partial_a f = f_{,a}.$$

Given a covariant derivative ∇_a, its action on a vector A^a should depend only on the value of quantities defined at the same point.

$\nabla_a A^b$ What does this look like?

We know it can't just be $\partial_a A^b$ because it doesn't transform like a tensor. We could combine this with the linearity assumption and look at the difference between $\partial_a A^b$ and $\nabla_a A^b$

$$\nabla_a A^b - \partial_a A^b = C^b{}_{ac} A^c.$$

Property (iv) means we have:

$$C'^{b}_{ac} = C'^{b}_{ca}.$$

Since $\partial_a A^b$ is not a tensor but $\nabla_a A^b$ is then we can conclude that $C^b_{ac} A^c$ is not a tensor either.

$$\boxed{\nabla_a A^b = \partial_a A^b + C^b_{ac} A^c}$$

the second order non-tensor like transformations terms cancel on the RHS.

How does $\nabla_a A_b$ look?

$$\nabla_a (A_b A^b) = (\nabla_a A_b) A^b + A_b (\nabla_a A^b)$$

$$\partial (A_b A^b) = (\nabla_a A_b) A^a + A_b (\cancel{\partial_a A^b} + C^b_{ac} A^c)$$

$$(\partial_a A_b) A^b + A_b (\cancel{\partial_a A^b}) =$$

$$(\partial_a A_b) A^b - A_b C^b_{ac} A^c = (\nabla_a A_b) A^b$$

$$(\partial_a A_c) A^c - A_b C^b_{ac} A^c = (\nabla_a A_c) A^c$$

$$\nabla_a A_c = \partial_a A_c - C^b_{ac} A_b.$$

Now we can write it for an tensor:

$$\nabla_c A^{a_1 \ldots a_p}{}_{b_1 \ldots b_q} = \partial_c A^{a_1 \ldots a_p}{}_{b_1 \ldots b_q}$$
$$+ C^{a_1}{}_{ck} A^{k \ldots a_q}{}_{b_1 \ldots b_q} + \ldots$$
$$+ C^{a_p}{}_{ck} A^{a_1 \ldots k}{}_{b_1 \ldots b_q} - C^{k}{}_{cb_1} A^{a_1 \ldots a_p}{}_{k \ldots b_q}$$
$$- \ldots - C^{k}{}_{cb_q} A^{a_1 \ldots a_p}{}_{b_1 \ldots k}.$$

Theorem 4.1: Let M be a manifold and g_{ab} be a metric. Then there exist a unique covariant derivative operator satisfying

$$\nabla_a g_{bc} = 0.$$

which is called the metricity condition

Proof: Apply the ∇_a to g_{bc} and try and slove for $C^a{}_{bc}$.

$$\nabla_a g_{bc} = \partial_a g_{bc} - C^d{}_{ab} g_{dc} - C^d{}_{ac} g_{bd} = 0 \quad (1)$$
$$\nabla_c g_{ab} = \partial_c g_{ab} - C^d{}_{ca} g_{db} - C^d{}_{cb} g_{ad}. = 0 \quad (2)$$
$$\nabla_b g_{ca} = \partial_b g_{ca} - C^d{}_{bc} g_{da} - C^d{}_{ba} g_{cd} = 0. \quad (3).$$

(1) + (2) − (3)

$$\partial_a g_{bc} + \partial_c g_{ab} - \partial_b g_{ca} - 2 C^d{}_{ca} g_{db} = 0.$$

apply g^{bm} to the above expression.

$$\overbrace{g^{bm} g_{bd}}^{\delta^m{}_d} C^d{}_{ca} = \frac{1}{2} g^{bm} \left(\partial_a g_{bc} + \partial_c g_{ab} - \partial_b g_{ca} \right)$$

$$C^m{}_{ca} = \Gamma^m_{ca} = \frac{1}{2} g^{bm} \left(\partial_a g_{bc} + \partial_c g_{ab} - \partial_b g_{ca} \right).$$

16/11/12

Riemann Curvature tensor

1. We examined $(\nabla_a \nabla_b - \nabla_b \nabla_a) A^c = ? = R_{bad}{}^c A^d$

$$R_{bad}{}^c = \partial_a \Gamma_{bd}^c - \partial_b \Gamma_{ad}^c + \Gamma_{bd}^e \Gamma_{ea}^c - \Gamma_{ad}^e \Gamma_{eb}^c$$

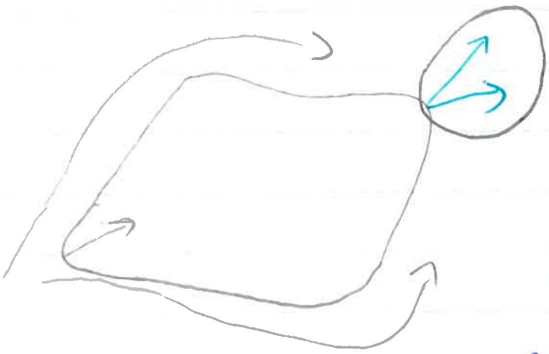
Riemann Curvature tensor.

Remember. Newton's equation:

$$\frac{\partial^2 x^i}{\partial t^2} = -\frac{\partial \Phi}{\partial x^i} \quad \text{AND} \quad \Delta \Phi(r) = 4\pi G \rho(s)$$

related to the metric, g.

2.



Path dependence of the parallel transport on a general manifold. $\propto R_{bad}{}^c$

$R_{bad}{}^c$ is basically curvature and this gravity!

—/—

There are a few symmetries which we need to examine. In n dim the Riemann curvature tensor has

2011 3a)

$$\frac{1}{12} n^2(n^2-1)$$

Lemma 4.2: The Riemann curvature tensor has following properties:

i. $R_{abcd} = -R_{bacd}$.

2012

ii. $R_{abcd} + R_{cabd} + R_{bcad} = 0$

iii. $R_{abcd} = -R_{abdc}$

iv. $\nabla_e R_{abcd} + \nabla_d R_{abec} + \nabla_c R_{abde} = 0$ (The famous Bianchi identity)

Proof:

i. Trivial. Follows from definition.

ii: We consider the permutation of $\nabla_a \nabla_b W_c$,

$$\nabla_a \nabla_b W_c - \nabla_b \nabla_a W_c = R_{abc}{}^d W_d.$$
$$\nabla_c \nabla_a W_b - \nabla_a \nabla_c W_b = R_{cab}{}^d W_d.$$
$$\nabla_b \nabla_c W_a - \nabla_c \nabla_b W_a = R_{bca}{}^d W_d.$$

$$\nabla_a \nabla_b W_c - \nabla_a \nabla_c W_c = \nabla_a (\partial_b W_c - \cancel{\Gamma_{bc}^e W_e} - \partial_c W_b + \cancel{\Gamma_{cb}^e W_e})$$

$$= \nabla_a (\partial_b W_c - \partial_c W_b)$$

$$T_{bc} = \partial_b W_c - \partial_c W_b.$$

$$\Rightarrow \nabla_a T_{bc} + \nabla_b T_{ca} + \nabla_c T_{ab} = R_{abc}{}^d W_d + R_{cab}{}^d W_d + R_{bca}{}^d W_d.$$

Because $T_{bc} = -T_{cb}$ then

$$\nabla_a T_{bc} = \partial_a T_{bc} - \Gamma_{ab}^e T_{ec} - \Gamma_{ac}^e T_{be}.$$

$$\Rightarrow \underbrace{\partial_a T_{ac} + \partial_b T_{ca} + \partial_c T_{ab}}_{0} = R_{abc}{}^{d} W_d + R_{cab}{}^{d} W_d$$
$$+ R_{bca}{}^{d} W_d .$$

iii) We will use $\nabla_a g_{bc} = 0$ (the metricity condition)

$$(\nabla_a \nabla_b - \cancel{\nabla_b \nabla_a})\, g_{cd} = R_{abc}{}^{e} g_{ed} + R_{abd}{}^{e} g_{ce}$$
$$0 = R_{abcd} + R_{abdc}$$
$$\Rightarrow R_{abcd} = -R_{abdc} .$$

i) - iii) imply another symmetry:

Ricci, Weyl and Einstein Tensors

The Ricci tensor: (Def 4.3)

$$R_{ab} := R_{acb}{}^{c} \Rightarrow R_{ab} = R_{ba} .$$

The Ricci Scalar: (Def 4.4).

$$R = R^{a}{}_{a} .$$

Weyl Tensor (Def 4.5)

The "trace free part" of the Riemann tensor is the Weyl tensor.

$$C_{abcd} = R_{abcd} - \frac{2}{n-2}\left(g_{a[c}R_{d]b} - g_{b[c}R_{d]a}\right) + \frac{2}{(n-2)(n-1)} \cdot R\, g_{a[c}\, g_{d]b}$$

21/11/12

Cont. Ricci, Weyl and Einstein tensor

We will contract the Bianchi identity. (apply the metric once)

$$g^{eb}\left(\nabla_e R_{abcd} + \nabla_d R_{abec} + \nabla_c R_{abde}\right)$$

$$\nabla_e\left(g^{ed} R_{abcd}\right) + \nabla_d\left(g^{ed} R_{abec}\right) + \nabla_c\left(g^{ed} R_{abde}\right)$$

$$\nabla_e\left(g^{eb} R_{cdab}\right) - \nabla_d\left(g^{eb} R_{abce}\right) + \nabla_c R_{abd}{}^{b} = 0.$$

$$\nabla_e R_{cda}{}^{e} - \nabla_d R_{ac} + \nabla_c R_{ad} = 0.$$

Relabel $e \to a$, $c \to b$, $d \to c$, $a \to d$.

$$\nabla_a R_{bcd}{}^{a} + \nabla_b R_{cd} - \nabla_c R_{bd}$$

Apply g^{bd} (twice contracted)

$$\Rightarrow \nabla_a R_c{}^{a} + \nabla_c R_c{}^{b} - \nabla_c R = 0$$

$$\Rightarrow 2\nabla_a R_c{}^{a} - \nabla_c R = 0 \qquad g^{bd} R_{bd} = R$$

$g^{cf} R_{bcdf} = R_{bd}$

or

$g^{bd} R_{bcdf} = R_{cf}$.

$$\Rightarrow 2\nabla^a R_{ca} - \nabla^b g_{bc} R = 0$$

$$\Rightarrow \nabla^a\left(R_{ca} - \tfrac{1}{2} g_{ac} R\right) = 0.$$

Def 4.6 Einstein tensor.

Finally we have arrived at the Einstein tensor:

$$G_{ab} := R_{ab} - \frac{1}{2} g_{ab} R$$

Geodesics II

Use the motion of parallel transport to derive the geodesic equations. The first method used the principle of shortest distance, this method, based on parallel transport, uses the principle of straightest lines.

Lemma 4.3: Let ∇_a covariant derivative. A geodesic is a curve whose tangent vector is parallelly transported along itself, this means that vector T^a satisfies

$$T^a \nabla_a (T^b) = 0.$$

Proof:

Let γ be a curve with affine parametrisation $X^a(\lambda)$. the tangent is defined as $T^a = \frac{dX^a}{d\lambda}$

$$\nabla_a T^b = \partial_a T^b + \Gamma_{ac}^{b} T^c$$

$$T^a \nabla_a T^b = T^a \partial_a T^b + T^a \Gamma_{ac}^{b} T^c$$

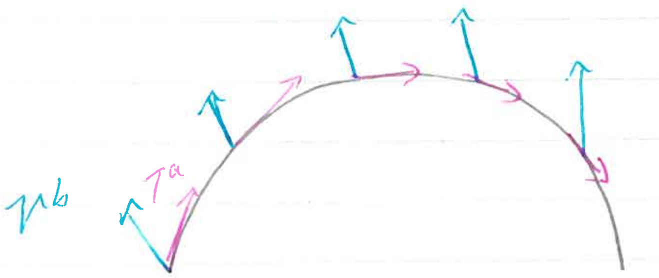
$$= \frac{dX^a}{d\lambda} \frac{\partial}{\partial X^a}\left(\frac{dX^b}{d\lambda}\right) + \Gamma_{ac}^{b} \frac{dX^a}{d\lambda} \frac{dX}{d\lambda}$$

$$= \frac{d^2X^b}{d\lambda^2} + \Gamma_{ac}^{b} \frac{dX^a}{d\lambda} \frac{dX^c}{d\lambda} = 0.$$

Because we have Geodesic eq^n.

$$\frac{d}{d\lambda}(T^a) + \Gamma^a_{bc} T^b T^c = 0.$$

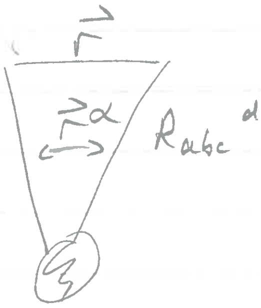
If we were in Euclidean this turns into $\frac{dT^a}{d\lambda} = 0$.



$$T^a \nabla_a (V^b) = 0.$$

Next, we are going to look at one more interpretation of the Riemann curvature tensor, where it measures the relative acceleration between two bodies falling in a gravitational field.

—/—

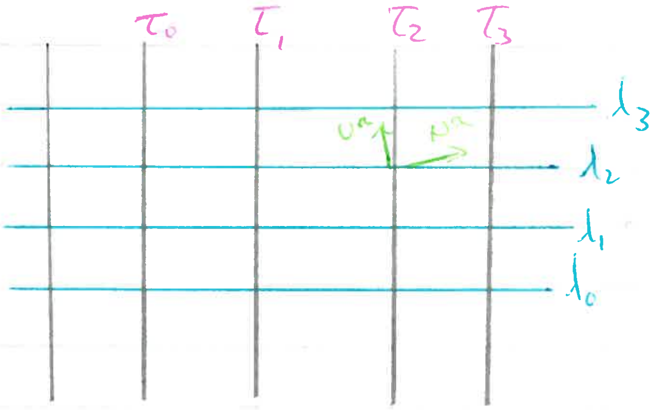


—/—

Another interpretation of $R_{abc}{}^{d}$

We will need families of geodesics:

$$\ddot{x} = 0 \Rightarrow X(\lambda) = a\lambda + b.$$



X^a is now a function of two parameters $X^a(\tau, \lambda)$

$$U^a = \frac{dX^a}{d\lambda}, \qquad N^a = \frac{dX^a}{d\tau}.$$

U^a is the tangent vector of the curves parameterised by λ and N^a is the tangent vector of the curves parameterised by τ.

We choose $U^a U_a = 1$ then also $U^a N^b g_{ab} = 0$.

$$\frac{\partial^2 X^a}{\partial\lambda\partial\tau} = \frac{\partial^2 X^a}{\partial\tau\partial\lambda}.$$

$$U^a N^b{}_{,a} = N^a U^b{}_{,a}$$

$$U^a \frac{\partial N^b}{\partial X^a} = N^a \frac{\partial U^b}{\partial X^a}$$

LHS:

$$\frac{dX^a}{d\lambda}\frac{\partial}{\partial X^a}\left(\frac{dX^b}{d\tau}\right)$$

Since the connection is symmetric we can replace the partial derivative with a covariant derivative.

$$\underbrace{U^a \nabla_a N^b} = N^a \nabla_a U^b$$

This gives the rate of change of the distance to the closest geodesic N^a as you move along a curve parametrised by t:

$$V^b = U^a \nabla_a N^b$$

We will define the acceleration N^a as:

$$a^b := U^a \nabla_a V^b .$$

$$\begin{aligned} a^b = U^a \nabla_a V^b &= U^a \nabla_a \left(U^c \nabla_c N^b \right) \\ &= U^a \nabla_a \left(N^c \nabla_c U^b \right) \end{aligned}$$

$$= U^a \nabla_a N^c \nabla_c U^b + U^a N^c \nabla_a \nabla_c U^b$$

$$= N^a \nabla_a U^c \nabla_c U^b + U^a N^c \nabla_c \nabla_a U^b$$

$$- U^a N^c R_{acd}{}^b U^d$$

N.B: $(\nabla_a \nabla_c - \nabla_c \nabla_a) U^b = R_{cad}{}^b U^d$

$$= N^c \nabla_c (U^a) \nabla_a U^b + U^a N^c \nabla_c (\nabla_a U^b) - U^a N^c R_{acd}{}^b U^d$$

$$= N^c \nabla_c \cancelto{0}{\left(U^a \nabla_a U^b \right)} - U^a N^c R_{acd}{}^b U^d$$

$$a^b = - U^a N^c R_{acd}{}^b U^d .$$

So if we are in a flat geometry $R_{acb}{}^{d} = 0$, then $a^b = 0$ and the geodesics. DO NOT converge.

Einstein's Field Equations

In Newtonian gravity, the gravitational field is only described by potential Φ, The total accceleration between two nearby objects.

$$-(\vec{x} . \vec{\nabla}) \vec{\nabla} \Phi$$ where $\vec{x}$ is the seperation vector.

$$R_{cbd}{}^{a} U^c U^d \leftrightarrow \partial_b \partial^a \Phi .$$

However, Poisson's eq^n read

$$\Delta \Phi = 4\pi \rho .$$

From Tensor analysis you can show that the energy-momentum tensor T_{ab}.

$$T_{ab} U^a U^b = \rho .$$

$$\Rightarrow R_{cad}{}^{a} U^c U^d = 4\pi T_{cd} U^c U^d$$

$$\Rightarrow R_{cd} = 4\pi T_{cd} \ ?$$

LHS is not divergent free, but RHS is: $\nabla^a T_{ab} = 0$.

So Einstein used his own tensor G_{ab}

$$\boxed{G_{ab} := R_{ab} - \frac{1}{2} R g_{ab} = 8\pi T_{ab}.}$$

Matter tells spacetime how to curve and spacetime tells matter how to move.

23/11/12

$$G_{ab} := \boxed{\underbrace{R_{ab} - \frac{1}{2} g_{ab} R}_{\text{Geometry}} = \underbrace{8\pi T_{ab}}_{\text{Matter}}.}$$

$\nabla^a T_{ab} = 0$, conservation of energy and momentum.

Contract with g^{ab} : $\Rightarrow R - \frac{1}{2}(4)R = 8\pi T_{ab} g^{ab}$

$$-R = 8\pi T \qquad (T_{ab} g^{ab}) = T.$$

$$R_{ab} = 8\pi \left(T_{ab} - \frac{1}{2} T g_{ab} \right).$$

$$g^{ab} g_{bc} = \delta^a{}_c \Rightarrow g^{ab} g_{ba} = \delta^a{}_a = n \text{ (dim)}$$

If there were no matter at point where you were then $T = 0$ and $T_{ab} = 0$.

$$R_{ab} = 0 \quad \text{(vacuum solution)}.$$

(Ricci flat)

$$\frac{d^2 X^a}{d\lambda^2} + \Gamma^a_{bc} \frac{dX^a}{d\lambda} \frac{dX^b}{d\lambda} = 0$$

$$L = g_{ab} \frac{dX^a}{d\lambda} \frac{dX^b}{d\lambda} = \begin{cases} \pm 1 & \text{massive particles} \\ 0 & \text{mass less particles.} \end{cases}$$

<u>Affine parameterisation</u>
If you have a parameter λ then $\lambda' = a\lambda + b$ for some real numbers a, b.

$+1$ for $\{+,-,-,-\}$ (signature)

-1 for $\{-,+,+,+\}$ (signature).

5. Schwarzschild Solution.

$$R_{ab} - \frac{1}{2} g_{ab} R = 8\pi T_{ab}.$$

$$\propto \frac{\partial^2 g_{ab}}{\partial x^{a2}} \cdots + \frac{\partial g_{ab}}{\partial x} \cdots + c g_{ab} = 8\pi T_{ab}.$$

At first people guessed spherically and static solution.

5.1 Metric Ansatz and Christoffel symbols.

The most general static spherically symmetric metric has the form

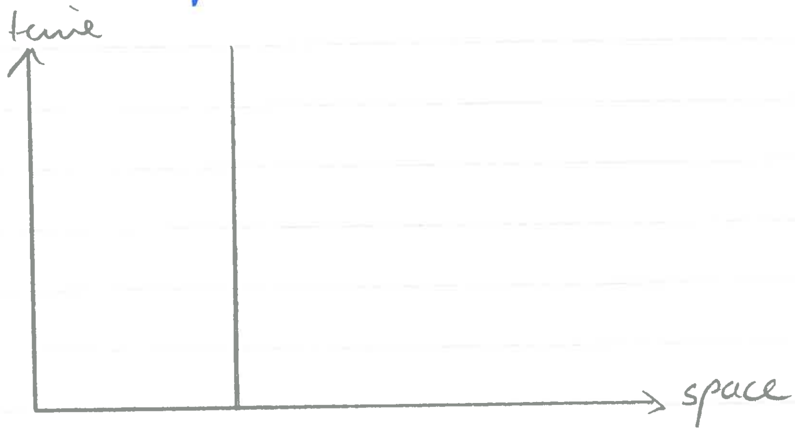
$$d^2 s = -e^{\nu(r)} dt^2 + e^{\lambda(r)} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

Turn this into the E-L format.

$$L = -e^{\nu(r)} \dot{t}^2 + e^{\lambda(r)} \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\phi}^2$$

28/11/12

Why Affine parameterisation?



Someone in a rest frame the curve they draw out has the following form:

$$X^a(\lambda) = \begin{pmatrix} \lambda \\ 0 \end{pmatrix}$$

$$\frac{dX^a}{d\lambda} = T^a = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad (+, -)$$

$$L = \sqrt{g_{ab} T^a T^b} = 1 \qquad \lambda' = a\lambda + b$$

$$\tau = \int \sqrt{g_{ab}\, dX^a dX^b} \qquad \text{(proper time)}$$

$$S' = \int \sqrt{g_{ab}\, dX^a dX^b} \qquad \text{(arc length)}$$

—/—

$$L = -e^{\nu}\dot{t}^2 + e^{\lambda}\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}$$

Note that $g_{ab} \to (g^{ab})^{-1}$

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^a}\right) = \frac{\partial L}{\partial x^a} \quad , \quad \nu' = \frac{\partial \nu}{\partial r}$$

$x^0 = t$:

$$\frac{\partial L}{\partial t} = 0 \qquad \frac{\partial L}{\partial \dot{t}} = -2e^{\nu}\dot{t}$$

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{t}}\right) = -2\nu' e \dot{t}\dot{r} - 2e^{\nu}\ddot{t}$$

$x^2 = \theta$

$$\frac{\partial L}{\partial \theta} = 2r^2 \operatorname{sen}\theta \cos\theta\, \dot{\phi}^2$$

$$\frac{\partial L}{\partial \dot{\theta}} = 2r^2\dot{\theta} \qquad \frac{d}{d\tau}(2r^2\dot{\theta}) = 4r\dot{r}\dot{\theta} + 2r^2\ddot{\theta}$$

$x^3 = \phi$

$$\frac{\partial L}{\partial \phi} = 0 \qquad \frac{\partial L}{\partial \dot{\phi}} = 2r^2 \operatorname{sen}^2\theta\, \dot{\phi}$$

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = 4r\dot{r}\operatorname{sen}^2\theta\,\dot{\phi} + 4r^2\operatorname{sen}\theta\cos\theta\,\dot{\theta}\dot{\phi} + 2r^2\operatorname{sen}^2\theta\,\ddot{\phi}$$

$$x^1 = r \,, \quad \frac{\partial L}{\partial r} = -\nu' e^{\nu}\dot{t}^2 + \lambda' e^{\lambda}\dot{r}^2 + 2r\dot{\theta}^2 + 2r\operatorname{sen}^2\theta\,\dot{\phi}^2$$

$$\frac{\partial L}{\partial \dot r} = 2e^{\lambda}\dot r$$

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot r}\right) = 2\lambda' e^{\lambda}\dot r^2 + 2e^{\lambda}\ddot r$$

$\ddot t + \nu' \dot t \dot r = 0 \quad t\text{-equ} \Rightarrow \Gamma^t_{rt} = \Gamma^t_{tr} = \frac{\nu'}{2}$.

$\ddot t + \Gamma^t_{tr}\dot t \dot r + \Gamma^t_{rt}\dot r \dot t = 0$.

$$2r^2\ddot\theta + 4r\dot r\dot\theta = 2r^2 \sin\theta\cos\theta\,\dot\phi^2$$

$\Gamma^\theta_{r\theta} = \frac{1}{r}$, $\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$.

$\Gamma^t_{rt} = \frac{\nu^2}{2}$, $\Gamma^\theta_{\phi\phi} = -\sin\theta\cdot\cos\theta$, $\Gamma^\theta_{r\theta} = \frac{1}{r}$.

$2\ddot\phi r^2\sin^2\theta + 4r\dot r\sin^2\theta\,\dot\phi + 4r^2\sin\theta\cos\theta\,\dot\theta\dot\phi = 0$.

$\ddot\phi + \frac{2}{r}\dot r\dot\phi + 2\cot\theta\,\dot\theta\dot\phi = 0 \Rightarrow \Gamma^\phi_{r\phi} = \frac{1}{r}$, $\Gamma^\phi_{\theta\phi} = \cot\theta$

$$2\ddot r e^{\lambda} + 2\lambda' e^{\lambda}\dot r^2 = -\nu' e^{\nu}\dot t^2 + \lambda' e^{\lambda}\dot r^2 + 2r\dot\theta^2 + 2r\sin^2\theta\,\dot\phi^2$$

$$\ddot r + \frac{\lambda'}{2}\dot r^2 + \frac{\nu'}{2}e^{\nu-\lambda}\dot t^2 - re^{-\lambda}\dot\theta^2 - r\sin\theta\, e^{-\lambda}\dot\phi^2 = 0$$

$\Gamma^r_{rr} = \frac{\lambda'}{2}$, $\Gamma^r_{tt} = \frac{\nu'}{2}e^{\nu-\lambda}$, $\Gamma^r_{\theta\theta} = -re^{-\lambda}$,

$\Gamma^r_{\phi\phi} = -r\sin^2\theta e^{-\lambda}$

Ricci tensor components

$$R_{ab} = \Gamma^n_{ba,n} - \Gamma^n_{bn,a} + \Gamma^m_{ba}\Gamma^n_{nm} - \Gamma^m_{bn}\Gamma^n_{am}.$$

$$R_{tt} = \Gamma^n_{tt,n} - \Gamma^n_{tn,t} + \Gamma^n_{tt}\Gamma^n_{nm} - \Gamma^m_{tn}\Gamma^n_{tm}$$

$$\Gamma^n_{tt,n} = \left(\frac{\nu' e^{\nu-\lambda}}{2}\right)_{,r} = \frac{\nu''}{2}e^{\nu-\lambda} + \frac{\nu'(\nu'-\lambda')e^{\nu-\lambda}}{2}$$

$$-\Gamma^n_{tn,t} = 0.$$

$$\Gamma^m_{tn}\Gamma^n_{mt} = \Gamma^t_{tn}\Gamma^n_{tt} + \Gamma^r_{tn}\Gamma^n_{tr}$$

$$= 2\Gamma^t_{tr}\Gamma^r_{tt} = \frac{\nu'^2}{2}e^{\nu-\lambda}.$$

$$R_{tt} = \frac{\nu''}{2}e^{\nu-\lambda} + \frac{\nu'(\nu'-\lambda')}{2}e^{\nu-\lambda}$$

$$- \frac{\nu'^2}{2}e^{\nu-\lambda} + \frac{1}{2}\nu' e^{\nu-\lambda}$$

$$\left(\frac{\nu'}{2} + \frac{\lambda'}{2} + \frac{2}{r}\right)$$

5.3 Schwarschild Solution

The vacuum field equations where $R_{ab} = 0$

$$(tt): \cancel{\frac{1}{2}\nu''} + \cancel{\frac{1}{4}(\nu')^2} + \frac{1}{r}\nu' - \cancel{\frac{1}{4}\nu'\lambda'} = 0$$

$$(rr): -\cancel{\frac{1}{2}\nu''} - \cancel{\frac{1}{4}(\nu')^2} + \cancel{\frac{1}{4}\nu'\lambda'} + \frac{1}{r}\lambda' = 0$$

$$\frac{1}{r}(\lambda' + \nu') = 0$$

Integrate $\Rightarrow \lambda + \nu = \tilde{c}$

$$\Rightarrow e^{\nu} = C' e^{-\lambda} \quad \text{where } C = e^{\tilde{c}}$$

The constant of integration to one by rescaling the time component $t \to \sqrt{C}t$

$$\Rightarrow e^{\nu} = e^{-\lambda}$$

If we substitute this into the $\theta\theta$ component of the Ricci tensor.

$$1 - e^{\nu} - \frac{1}{2} r \nu' e^{\nu} - \frac{1}{2} r \nu' e^{\nu} = 1 - e^{\nu} - r\nu' e^{\nu} = 0$$

first note that $\frac{d}{dr}(re^{\nu}) = e^{\nu} + \nu' r e^{\nu}$.

$$\frac{d}{dr}\left(r - re^{\nu}\right) = 0 .$$

$$\Rightarrow r - re^{\nu} = \hat{C} \quad \text{which gives us}$$

$$e^{\nu} = 1 - \frac{\hat{C}}{r} .$$

We have shown that the metric which is spherically symmeteric and static that solves the vaccum field equations ($R_{ab} = 0$) is

$$ds^2 = -\left(1 - \frac{\hat{C}}{r}\right)dt^2 + \left(1 - \frac{\hat{C}}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta \, d\phi^2$

$r \to \infty$ gives you the flat metric (Minkowski) in polar coords

$r \to \hat{C}$ we see that the r-component explodes.

$r \to 0$ we also see that the t-comp explodes

If we were to reintroduce the physical constant which we set to 1 we would have in the time component say $-\left(1 - \frac{G\hat{C}}{c^2 r}\right)dt^2$ (G - Newton gravitational constant, c^2 - speed of light squared)

Since $1 - \frac{G\hat{C}}{c^2 r}$ is dimensionless the $\hat{C}$ has dimension mass.

$$ds^2 = -\left(1 - \frac{G\hat{C}}{c^2 r}\right)dt^2 + \left(1 - \frac{G\hat{C}}{c^2 r}\right)^{-1} dr^2 + \dots$$

If we take $c^2 \to \infty$

$$\underset{c^2 \to \infty}{\Gamma^r_{tt}} = \frac{G\hat{C}}{2r^2}$$ (The only Christoffel symbol which survives)

Compare this to Newton's law per unit mass.

$$\nabla \Phi = \frac{GM}{r^2}$$

These equate to give $\Rightarrow \hat{c} = 2M$.

$$\Rightarrow ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2$$

$$+ r^2 d\Omega^2$$

$$\ddot{r} + \frac{G\hat{c}}{2r^2}\dot{t}^2 = 0$$

2/11/12

$$\partial'_a W'_b = \frac{\partial x^c}{\partial x'^a} \frac{\partial}{\partial x^c}\left(\frac{\partial x^d}{\partial x'^b} W_d\right)$$

$$= \frac{\partial x^c}{\partial x'^a} \frac{\partial x^d}{\partial x'^b} \partial_c W_d + \frac{\partial x^c}{\partial x'^a} \frac{\partial^2 x^d}{\partial x^c \partial x'^b} W_d.$$

$$\neq \dots + \frac{\partial^2 x^d}{\partial x'^b \partial x^c} W_d.$$

—/—

$$\nabla_a A^b = \partial_a A^b + C^b_{ac} A^c$$

because we set $\nabla_a g_{ac} = 0 \Rightarrow C^b_{ac} = \Gamma^b_{ac}$.

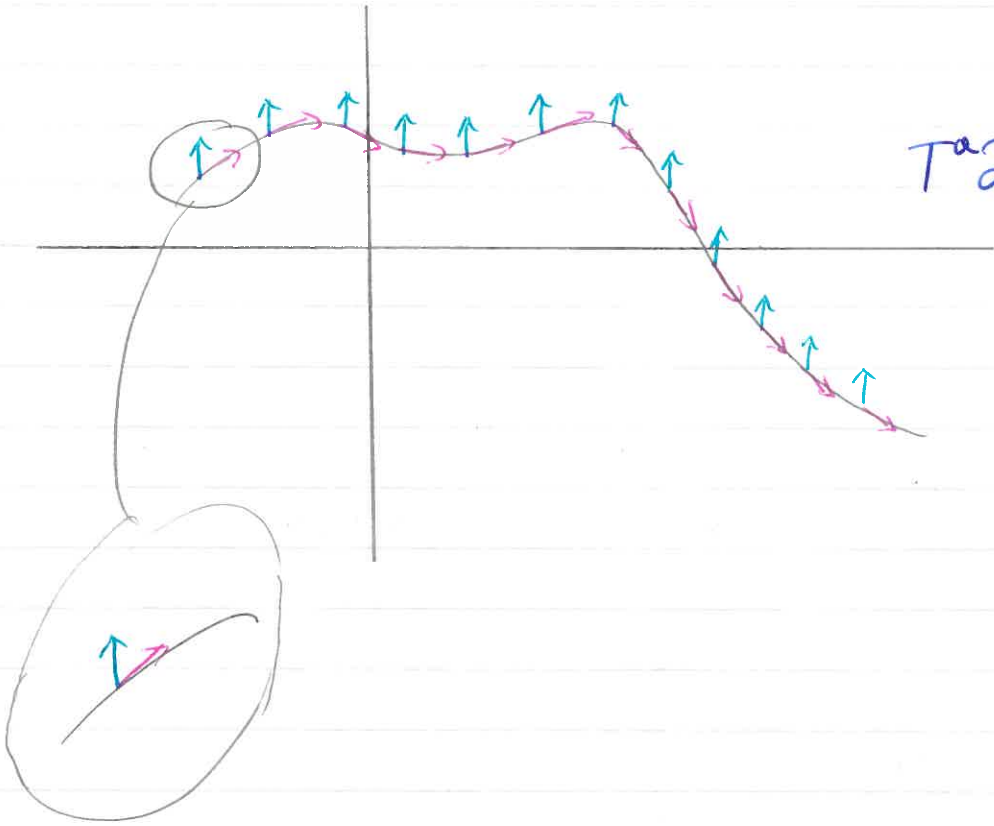
Defn 4.2 Parallel transport.

Let ∇_a be covariant derivative and γ be a curve with tangent vector T^a. A vector V^a at each point on the curve is said parallely transported along γ if

$$T^a \nabla_a V^b = 0.$$

is satisfied along the curve.

Let's say we are in Euclidean 2-space with Cartesian co-ords.



$T^a \partial_a V^b = 0$

Parallel transport of arbitrary type $\binom{p}{q}$ tensor is defined by

$$T^a \nabla_a A^{a_1 \ldots a_p}{}_{b_1 \ldots b_q} = 0 .$$

Use the definition of the covarient derivate

$$T^a \partial_a V^b + T^a \Gamma^b_{ac} V^c = 0 .$$

If the curve is parameterised by τ then $X^a = X^a(\tau)$ then the tangent vector is just $T^a = \frac{dX^a}{d\tau}$.

$$T^a \partial_a V^b = \frac{dX^a}{d\tau} \frac{dV^b}{\partial X^a} = \frac{dV^b}{d\tau}$$ (looks like acceleration)

$$\Rightarrow \frac{dV^b}{d\tau} + T^a \Gamma^b_{ac} V^c = 0$$

In Euclidean 2-space with cartesian $\frac{dV^b}{d\tau} = 0$

This is a first order D.E, and with an initial V^b can be uniquely solved.

<u>Lemma 4.1</u>. Let V^a and W^a be two vectors parallely transport them along a curve γ. Then the scalar $V^a W_a$ remains unchanged if also parallely transported along γ.

<u>Proof</u>:

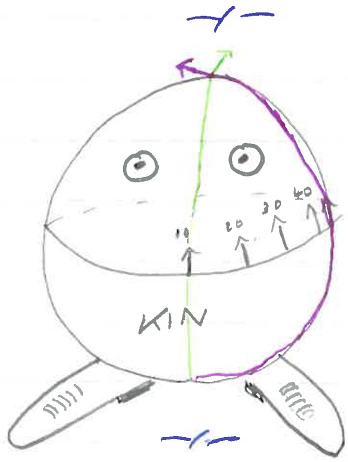
$$T^a \nabla_a (V^b W_b)$$

$$T^a \nabla_a (g_{bc} V^b W^c)$$

$$T^a \cancelto{0}{(\nabla_a g_{bc})} V^b W^c + T^a g_{bc} \cancelto{0}{(\nabla_a V^b)} W^c + T^a g_{ac} V^b \cancelto{0}{\nabla_a W^c}$$

Metricity cond.

Parallel transport of V^b and W^c.



4.2 Riemann Curvature Tensor.

From the definition of the covarient derivative we know how it acts on a scalar.

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) f = 0. \quad (\text{It commutes}).$$

However, it does not for a vector. Let us work.

14/11/12

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) A^c$$

—/—

4.2 Riemann Curvature Tensor

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) f = 0.$$

In conjunction with $\nabla_a g_{bc} = 0$ this gives you $\Gamma^a_{bc} = \Gamma^a_{cb}$, the symmetry on the two lowers indices. (torsion free).

It does not commute for a vector (or any higher rank tensors). Let us work out what it is equal to.

$$\nabla_a \nabla_b A^c = \nabla_a (\partial_b A^c + \Gamma^c_{bd} A^d)$$

$$= \partial_a (\partial_b A^c) - \Gamma^d_{ab} \partial_d A^c + \Gamma^c_{ad} \partial_b A^d + \partial_a (\Gamma^c_{bd} A^d)$$
$$- \cancel{\Gamma^e_{ad} \Gamma^c_{ed} A^d} + \Gamma^c_{ae} \Gamma^e_{bd} A^d$$

$$\nabla_b \nabla_a A^c = \nabla_b (\partial_a A^c + \Gamma^c_{ad} A^d)$$

$$= \partial_b (\partial_a A^c) - \Gamma^d_{ba} \partial_d A^c + \Gamma^c_{bd} \partial_a A^d + \partial_b (\Gamma^c_{ad} A^d)$$
$$- \cancel{\Gamma^e_{ba} \Gamma^c_{ed} A^d} + \Gamma^c_{be} \Gamma^e_{ad} A^d$$

Since we are going to write $(\nabla_a \nabla_b - \nabla_b \nabla_a) A^c$.

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)A^c = \cancel{\Gamma^c_{ad}\partial_b A^d} + (\partial_a \Gamma^c_{bd})A^d + \cancel{\Gamma^c_{bd}(\partial_a A^d)} - \cancel{\Gamma^c_{bd}\partial_a A^d} - \partial_b \Gamma^c_{ad} A^d - \cancel{\Gamma^c_{ad}\partial_b A^d}$$
$$+ \Gamma^c_{ae}\Gamma^e_{bd}A^d - \Gamma^c_{be}\Gamma^e_{ad}A^d$$

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)A^c = \left[\partial_a \Gamma^c_{bd} - \partial_b \Gamma^c_{ad} + \Gamma^c_{ac}\Gamma^e_{bd} - \Gamma^c_{be}\Gamma^e_{ad}\right]A^d.$$

We write
$$(\nabla_a\nabla_b - \nabla_b\nabla_a)A^c = R_{bad}{}^c A^d$$

where

very important

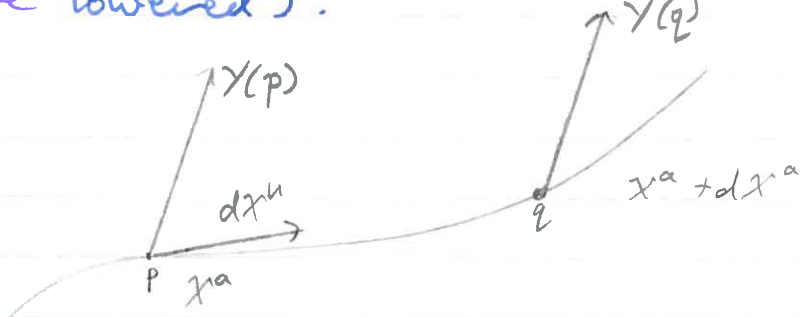
$$\boxed{R_{bad}{}^c := \partial_a \Gamma^c_{bd} - \partial_b \Gamma^c_{ad} + \Gamma^e_{bd}\Gamma^c_{ea} - \Gamma^e_{ad}\Gamma^c_{eb}}$$

which is called the Riemann curvature tensor. People also define it with the first index up.

It is antisymmetric in b and a. If you lower the last index.

$$R_{badc} = -R_{bacd}.$$

It is antisymmetric in the last two (only when the last two are lowered).



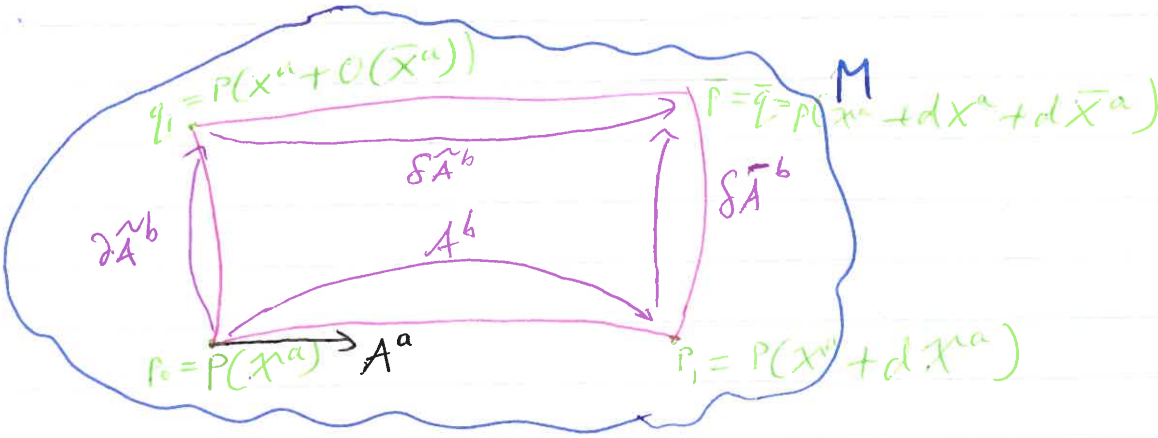
$Y^c(x^a)$ is the vector at point p. At the point q we have:

$$Y^c(x^a + dx^a) = Y^c(x^a) + dx^a \frac{\partial Y^c}{\partial x^a}(x^a) + O(dx^a)^2$$

$$= Y^c(x^a) + dx^a \cancelto{0}{\nabla_a Y^c} - dx^a \Gamma^c_{ab} Y^b + O(dx^a)^2.$$

Since we assumed that the vector was parallel transported then the second term is zero.

$$\delta Y^c = Y^c_{(q)} - Y^c_{(p)} = -dx^a \Gamma^c_{ab} Y^b.$$



Let us transport A^b from p_0 to p_1

$$A^b + \delta A^b = A^b_{p_0} - dx^a \Gamma^b_{(p_0)ac} A^c_{(p_0)}$$

Next we are going to move it from p to $\bar{p}$

$$\delta \bar{A}^b = -d\bar{x}^a \Gamma^b_{ac(p_1)} A^c_{(p_1)}$$

$$= -d\bar{x}^a \Gamma^b_{ac(p_1)} (A^c_{(p_0)} + \delta A^c)$$

$$= -d\bar{x}^a (\Gamma^b_{ac(p_0)} + \delta\Gamma^b_{ac})(A^c_{(p_0)} + \delta A^c)$$

(Taylor expan.

$$\simeq -d\bar{x}^a(\Gamma^b_{ac(p_0)} + dx^d \partial_d \Gamma^b_{ac})(A^c - dx^e \Gamma^c_{ef(p_0)} A^f_{(p_0)})$$

$$\delta \bar{A}^b \simeq -d\bar{x}^a \Gamma^b_{ac} A^c + d\bar{x}^a \Gamma^b_{ac} dx^e \Gamma^c_{ef} A^f.$$
$$- d\bar{x}^a dx^d (\partial_d \Gamma^b_{ac}) A^c$$

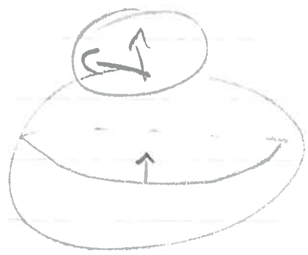
If we consider the other way

$$\delta \tilde{A}^b \simeq -dx^a \Gamma^b_{ac} A^c - dx^a d\bar{x}^d (\partial_d \Gamma^b_{ac}) A^c$$
$$+ dx^a d\bar{x}^d \Gamma^b_{ac} \Gamma^c_{de} A^e$$

$$\Delta A^b = (A^b + \delta A^b + \delta \bar{A}^b) - (A^b + \delta \hat{A}^b + \delta \tilde{A}^b)$$
$$= -dx^a \Gamma^b_{ac} A^c \cancel{- d\bar{x}^a \Gamma^b_{ac} A}$$
$$+ d\bar{x}^a \Gamma^b_{ac} dx^e \Gamma^c_{ef} A^f - d\bar{x}^a dx^a (\partial_d \Gamma^b_{ac}) A_c$$
$$-\big(\cancel{-d\bar{x}^a \Gamma^b_{ac} A^c} \cancel{- dx^a \Gamma^b_{ac}} A^c$$
$$- dx^a d\bar{x}^d (\partial_d \Gamma^b_{ac}) A^c + dx^a d\bar{x}^d \Gamma^b_{ac} \Gamma^c_{de} A^e \big)$$

$$\Delta A^b = -d\bar{x}^a dx^d A^e \left[\partial_d \Gamma^b_{ae} - \partial_a \Gamma^b_{de} + \Gamma^b_{ac} \Gamma^c_{de} - \Gamma^b_{dc} \Gamma^c_{ae}\right]$$

$$= -d\bar{x}^a dx^d R_{dae}{}^b A^e$$



30/11/12

Correction

$$ds^2 = -\left(1 - \frac{G\hat{C}}{c^2 r}\right)c^2 dt^2 + \left(1 - \frac{G\hat{C}}{c^2 r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

If we take the limit of $c^2 \longrightarrow \infty$, then you can show that the only Christoffel symbol with a $\hat{C}$ in it that survives

$$\Gamma^r_{tt} = \frac{1}{2} g^{rr}\left(-\partial_r g_{tt}\right) = \frac{G\hat{C}}{2r^2}\left(1 - \frac{G\hat{C}}{c^2 r}\right)$$

$$\overset{c^2 \to \infty}{=} \frac{G\hat{C}}{2r^2}$$

$\ddot{r} + \Gamma^r_{tt} = 0 \implies \ddot{r} = -\Gamma^r_{tt}$ This looks a Force eq^n, with unit mass. $\Phi = \pm GM/r$, where Φ is the gravitational potential. (unit mass)

$$\nabla \Phi = \pm \frac{GM}{r^2} = \mp F.$$

we can see that $\Gamma^r_{tt} = G\hat{C}/2r^2$ and $-F = GM/r^2$ so $\hat{C} = 2M$.

Weak field limit or Newtonian limit.

$$m\underline{\ddot{r}} = -m\nabla\underline{\Phi}.$$

in index notation $\quad \dfrac{\partial^2 x^i}{\partial t^2} = -\dfrac{\partial \Phi}{\partial x^i}$ (slight abuse of notation)

Let us consider the metric given by:

$$g_{ab} = \eta_{ab} - h_{ab}$$

where η_{ab} is the Minkowski (flat metric) and h_{ab} is some weak perturbation away from a flat space-time such that

$$|h_{ab}| << 1 .$$

So the $g^{ab} = \eta^{ab} + h^{ab}$ which can be seen by

$$g_{ab}\, g^{bc} = (\eta_{ab} - h_{ab})(\eta^{ab} - h^{ab}) = \delta_a^{\ c} - \cancel{h_{ab}\eta^{bc}} + \cancel{\eta_{ab} h^{bc}}$$

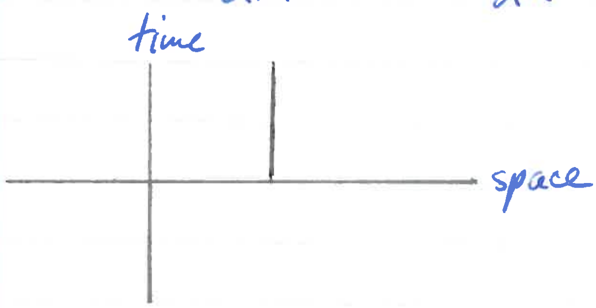
$$= \delta_a^{\ c} .$$

remember:

$$\frac{d^2 x^a}{d\lambda^2} + \Gamma^a_{\ bc}\, \dot{x}^b \dot{x}^c = 0 .$$

If we have velocities which are much slower than than the speed of light.

$$U^a = \frac{dx^a}{d\lambda} \Rightarrow \frac{dt}{d\lambda} >> \frac{dx^i}{d\lambda}$$



$$\Gamma^a_{\ bc} \frac{dx^b}{d\lambda} \frac{dx^c}{d\lambda} \simeq \Gamma^a_{\ tt} \frac{dt}{d\lambda} \frac{dt}{d\lambda}$$

$$\frac{d^2x^a}{d\lambda^2} + \Gamma^a_{bc}\frac{dx^b}{d\lambda}\frac{dx^c}{d\lambda} \simeq \frac{d^2x^a}{d\lambda^2} + \Gamma^a_{tt}\left(\frac{dt}{d\lambda}\right)^2$$

$$\Gamma^a_{tt} = \frac{1}{2}g^{ad}\left(g_{td,t} + g_{td,t} - g_{tt,d}\right)$$

$$\Gamma^a_{tt} = -\frac{1}{2}g^{ad}\left(g_{tt,d}\right)$$

$$\Gamma^t_{tt} = -\frac{1}{2}\left(\eta^{tt} + h^{tt}\right)\left(\cancel{\eta_{tt,t}}^{\,0} - \cancel{h_{tt,t}}^{\,0}\right) = 0$$

$$i=1,2,3,\quad \Gamma^i_{tt} = -\frac{1}{2}\left(\eta^{ii} + h^{ii}\right)\left(\cancel{\eta_{tt,i}}^{\,0} - h_{tt,i}\right) = \frac{1}{2}\eta^{ii}h_{tt} = \frac{1}{2}h_{tt,i}$$

where $\eta^{ii} = +1$

$$\frac{d^2x^i}{d\lambda^2} + \Gamma^i_{tt}\left(\frac{dt}{d\lambda}\right)^2 \simeq \frac{d^2x^i}{d\lambda^2} + \frac{1}{2}h_{tt,i}\left(\frac{dt}{d\lambda}\right)^2$$

Now reintroduce c (the speed of light)

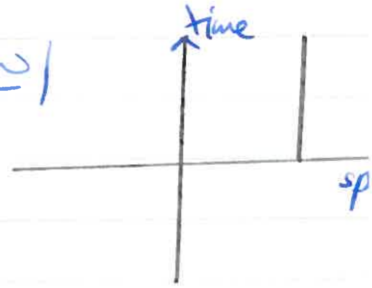
$$\frac{d^2x^i}{d\lambda^2} + \frac{1}{2}\frac{\partial h_{tt}}{\partial x^i}\, c\frac{dt}{d\lambda}\, c\frac{dt}{d\lambda} = 0$$

$$\frac{d^2x^i}{d\lambda^2} = -\frac{c^2}{2}\frac{\partial h_{tt}}{\partial x^i}$$

$$U^a U_a = 1$$

$$\frac{dt}{d\lambda} \simeq \frac{dx^i}{d\lambda}$$

$$\left(\frac{dt}{d\lambda}\right)^2 \simeq 1$$



Comparison with Newton's law.

$$-c^2 \frac{1}{2} \frac{\partial h_{tt}}{\partial x^i} = \frac{\partial \Phi}{\partial x^i} \Rightarrow h_{tt} = \frac{2\Phi}{c^2}$$

$$g_{tt} = \eta_{tt} - h_{tt} = -1 - \frac{2\Phi}{c^2} = -\left(1 + \frac{2\Phi}{c^2}\right)$$

$$\Phi = -\frac{GM}{r} \Rightarrow g_{tt} = -\left(1 - \frac{2GM}{c^2 r}\right).$$

5/12/12

6. Geodesic of the Schwarschild solution.

$$V_{eff} = \frac{L^2}{2r^2} - \frac{ML^2}{r^3} + \frac{LM}{r} - \frac{L}{2}.$$

$\frac{L^2}{2r^2}$ centrifrigual barrier.

$-\frac{ML^2}{r^3}$ new dominates over $\frac{1}{r}$ for small r.

$\frac{LM}{r}$ normal Newtonian term $-\frac{M}{r}$

We have the co-ords, t, r, θ, ϕ.

We will set $\theta = \pi/2$.

— / —

Aside: $L = g_{ab} \dot{X}^a \dot{X}^b = \sqrt{d\tau}$.

If you are only moving in time, at rest then

$$X^t = a\lambda + b \qquad a, b \in \mathbb{R}.$$

$$X^i = 0. \qquad i = 1, 2, 3 \quad (x, y, z).$$

$$d\tau = \int \sqrt{g_{ab}\, dx^a dx^b} = ds.$$

$$\lambda = \tau.$$

—/—

<u>cont</u>: So we want have some notion of ϕ in terms of r and vice vérsa:

$$\frac{d\phi}{dr} = \frac{d\phi}{d\lambda}\frac{d\lambda}{dr} = \frac{\dot\phi}{\dot r} = \frac{L}{r^2 \dot r}$$

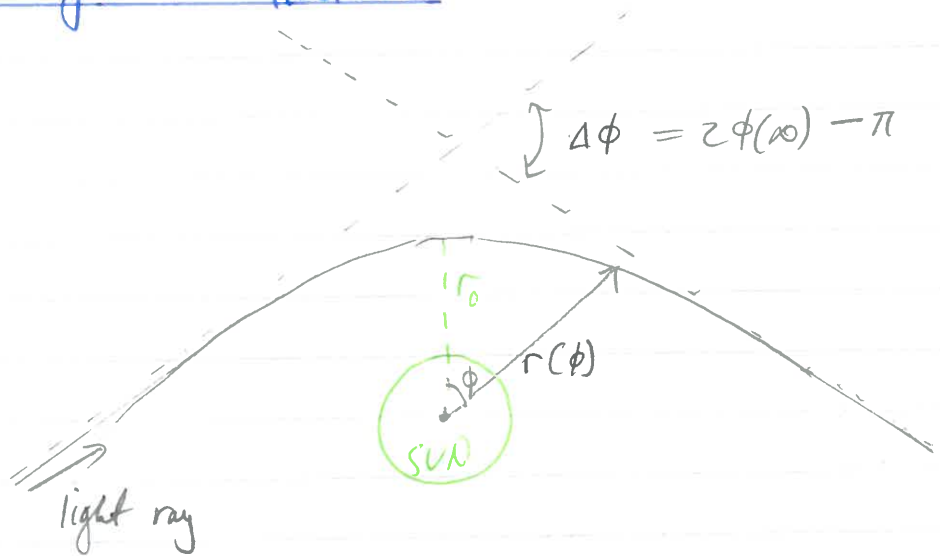
$$\dot r^2 = E^2 - \left(1 - \frac{2M}{r}\right)\left(\frac{L^2}{r^2} - L\right)$$

$$\frac{1}{\dot r} = \left(E^2 - \left(1 - \frac{2M}{r}\right)\left(\frac{L^2}{r^2} - L\right)\right)^{-\frac{1}{2}}.$$

$$\frac{d\phi}{dr}\frac{r^2}{L} = \left(E^2 - \left(1 - \frac{2M}{r}\right)\left(\frac{L^2}{r^2} - L\right)\right)^{-\frac{1}{2}}$$

$$\frac{d\phi}{dr} = \frac{L}{r^2}\left(E^2 - \left(1 - \frac{2M}{r}\right)\left(\frac{L^2}{r^2} - L\right)\right)^{-\frac{1}{2}}$$

6.2 Light Deflection



$$\phi(r_0) = 0$$

$$\Delta\phi = \phi(\infty) - \phi(-\infty)$$

$$= 2\phi(\infty) - \pi .$$

For light rays, $L = 0$. This gives us

$$\frac{d\phi}{dr} = \frac{\ell}{r^2}\left[E^2 - \left(1 - \frac{2M}{r}\right)\frac{\ell^2}{r^2}\right]^{-\frac{1}{2}}$$

since $r(\phi)$ is minimal at r_0.

$$\left.\frac{dr}{d\phi}\right|_{r=r_0} = 0 .$$

$$0 = \frac{r_0^2}{\ell}\left[E^2 - \left(1 - \frac{2M}{r_0}\right)\frac{\ell^2}{r_0^2}\right]^{-\frac{1}{2}} .$$

$$E^2 = \left(1 - \frac{2M}{r_0}\right)\frac{\ell^2}{r_0^2}.$$

Therefore:

$$\phi(r) = \int_{r_0}^{r} \frac{\ell}{r'^2}\left[\left(1-\frac{2M}{r_0}\right)\frac{\ell^2}{r_0^2} - \left(1-\frac{2M}{r'}\right)\frac{\ell^2}{r'^2}\right]^{-\frac{1}{2}} dr'$$

$$\phi(\infty) = \int_0^{\infty} \frac{dr'}{\sqrt{\left(1-\frac{2M}{r_0}\right)\frac{r'^4}{r_0^2} - \left(r'^2 - 2Mr'\right)}}$$

$u = \frac{1}{r}$ we will change variables.

$$\phi(0) = \int_0^{1/r} \frac{du}{\sqrt{\left(1-\frac{2M}{r_0}\right)\frac{1}{r_0^2} - u^2 + 2Mu^3}}$$

To first in the mass M one can use the following trick:

$$\phi(0) = \phi(0)[M=0] + \frac{\partial\phi}{\partial M}[M=0]M + O(M^2)$$

$$\phi(0)[M=0] = \int_0^{1/r_0} \frac{du}{\sqrt{1/r_0^2 - u^2}} = \frac{\pi}{2}.$$

$$\frac{\partial \phi}{\partial M}[M=0] = \int_0^{1/r_0} \frac{1/r_0^3 - u^3}{(1/r_0^2 - u^2)^{3/2}} \, du .$$

$$= -\frac{2 + r_0 u}{1 + r_0 u} \sqrt{1/r_0^2 - u^2} \Bigg|_0^{1/r_0} = \frac{2}{r_0} .$$

which leads to :

$$\Delta\phi = 2\phi(\infty) - \pi = \pi + \frac{4}{r_0} M - \pi$$

$$\simeq \frac{4M}{r_0}$$

$r_0 \simeq R_0 \simeq 7 \times 10^5$ km .

$M = \frac{G M_0}{c^2} \simeq 1.5$ km .

Using that $\pi = 180.3600''$

$\Delta\phi = 1.75''$ This $\frac{1.75}{3600}$ of a degree

Eddington and Einstein, English and German. The measurement made by Eddington had a greater error than 1.75" and so didn't really prove.

7/12/12

6.3 Orbits

$$\dot{r} = \frac{dr}{d\phi}\dot{\phi} = \frac{dr}{d\phi}\frac{\ell}{r^2}$$

$$1 = \frac{E^2}{\left(1-\frac{2M}{r}\right)} - \frac{\ell}{\left(1-\frac{2M}{r}\right)r^4}\left(\frac{dr}{d\ell}\right)^2 - \frac{\ell^2}{r^2}$$

$$\frac{1}{r^4}\left(\frac{dr}{d\phi}\right)^2 = \left[\frac{d}{d\phi}\left(\frac{1}{r}\right)\right]^2$$

Multiply by $\left(1-\frac{2M}{r}\right)/\ell^2 \Rightarrow \frac{\left(1-\frac{2M}{r}\right)}{\ell^2} = \frac{E^2}{\ell} - \left[\frac{d}{d\phi}\left(\frac{1}{r}\right)\right]^2 - \frac{\left(1-\frac{2M}{r}\right.}{r^2}$

$$\left[\frac{d}{d\phi}\left(\frac{1}{r}\right)\right]^2 + \frac{1}{r^2} = \frac{E^2}{\ell^2} + \frac{2M}{r^3} + \frac{2M}{r\ell^2} - \frac{1}{\ell^2}$$

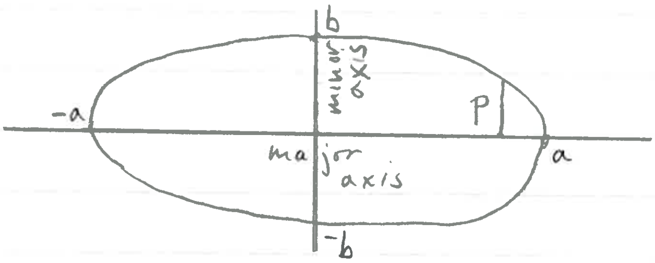
$$2\left[\frac{d}{d\phi}\frac{1}{r}\right]\left[\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right)\right] + \frac{2}{r}\left[\frac{d}{d\phi}\left(\frac{1}{r}\right)\right] = 3\left[\frac{d}{d\phi}\left(\frac{1}{r}\right)\right]\frac{2M}{r^2} + \frac{2M}{\ell^2}\left[\frac{d}{d\phi}\left(\frac{1}{r}\right)\right]$$

We will cancel away $\frac{d}{d\phi}\left(\frac{1}{r}\right)$ since we are not worried for this example about $\frac{d}{d\phi}\left(\frac{1}{r}\right) = 0$ since this the equation of a circular orbit.

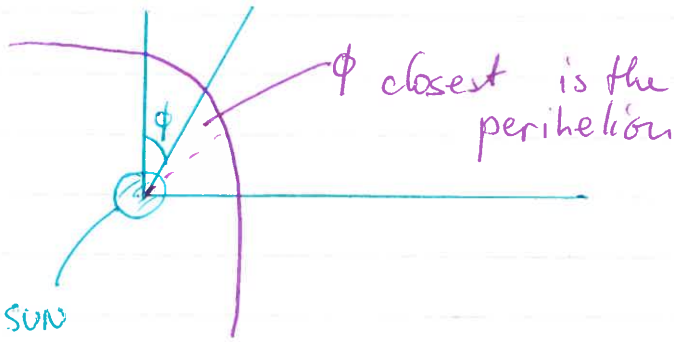
$$\frac{d^2u}{d\phi^2} + u = 3mu^2 + \frac{M}{\ell^2} \quad \text{where } u = \frac{1}{r}.$$

This is looking a lot like:

$$\frac{d^2u}{d\phi^2} + u = \frac{1}{p}, \quad p \text{ is a constant and is called semi-latus rectum.}$$



a is the semi-major axis (i.e. half of the major axis)
b " " " -minor axis.



$$V_{eff} = \frac{Mc^2}{2}\left[-\frac{r_s}{r} + \frac{\ell}{r^2} - \frac{r_s \ell^2}{r^3}\right], \quad r_s = 2M.$$

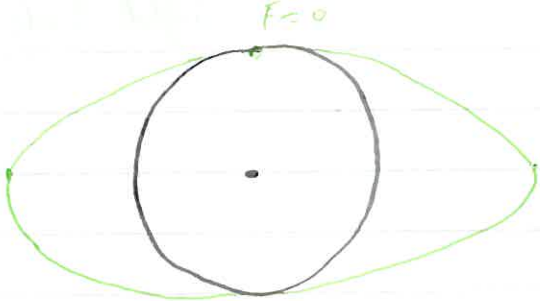
Newtonian form

Centrifigal force

new term from GR

Look at circular orbits which is when the

$$F_{eff} = -\frac{dV_{eff}}{dr} = 0.$$



$$(*) = -\frac{Mc^2}{2r^4}\left[r_s r^2 - 2\ell^2 r + 3r_s \ell^2\right] = 0.$$

$$r_{outer} = \frac{\ell^2}{r_s}\left(1 + \sqrt{1 - \frac{3r_s^2}{\ell^2}}\right),$$

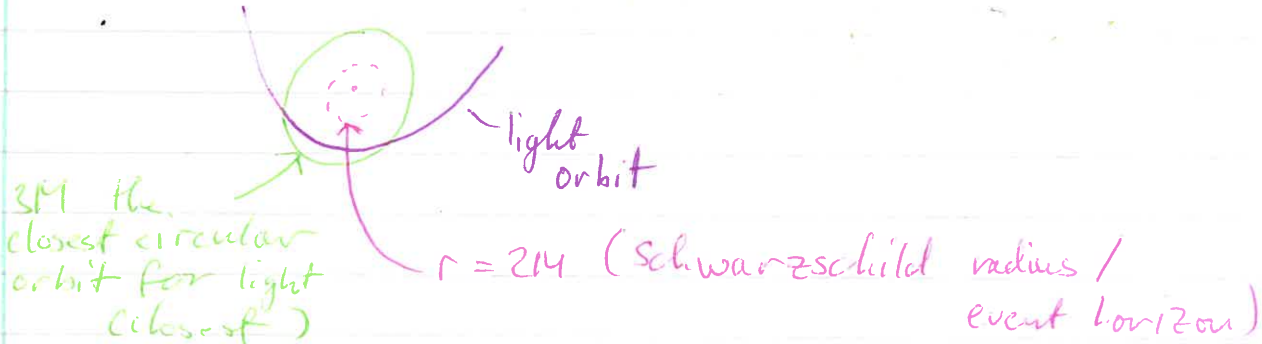
$$r_{inner} = \frac{\ell^2}{r_s}\left(1 - \sqrt{1 - \frac{3r_s^2}{\ell^2}}\right) = \frac{3\ell^2}{r_{outer}}$$

r_{outer}, r_{inner} are the two solutions to $(*)$ $(F_{eff} = 0)$
r_{inner} is unstable with respect to small perturbations.
$r_s = 2M$ is called the schwarzchild radius. Remember the metric

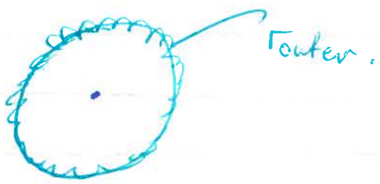
$$ds^2 = \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + \ldots$$

$r_s = 2M$ is where g_{11} term is ill defined. The point at which light has a chance of escaping.

So r_{inner} for light is $= 3M$. This is called the light shell.



We want to take the r_{outer} which is stable and let it have a frequency which is really close to the frequency orbit.



<u>12/12/12</u>

Precession of Eliptical orbit

r_{inner} and r_{outer}.

We can take a formula from Newtonian physics

$$r_{outer}^3 \simeq \frac{GM}{\omega_\varphi^2}$$

ω_φ is the orbital angular speed for a body orbiting M at a radius r_{outer}.

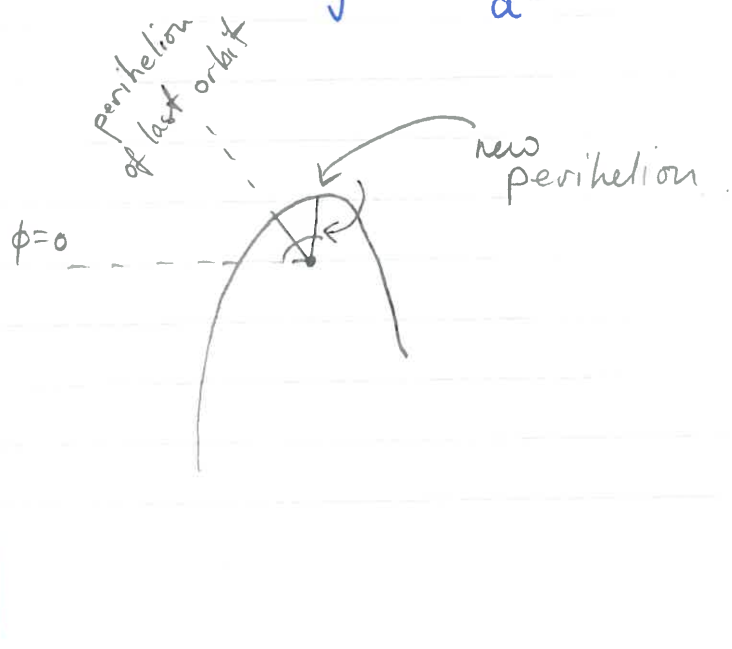
$$\omega_r^2 = \left[\frac{d^2V}{dr^2}\right]_{r=r_{outer}}$$

A small radial deviation will oscillate according to this

V is the potential from our $\dot{r}^2 + V_{eff} = C'$.

$$\delta\varphi = T(\omega_\varphi - \omega_r)$$

$$\omega_r^2 = \omega_\varphi^2 \sqrt{1 - \frac{3r_s^2}{a^2}}$$



$$\phi_{new} - \phi_{old} = \delta\phi.$$

$$\delta\phi = \frac{6\pi MG}{c^2 a_0(1-e^2)} \qquad \text{where } \left[\frac{a^2}{GM} = a_0(1-e^2)\right]$$

$$\simeq \text{"}0.1\text{"}$$

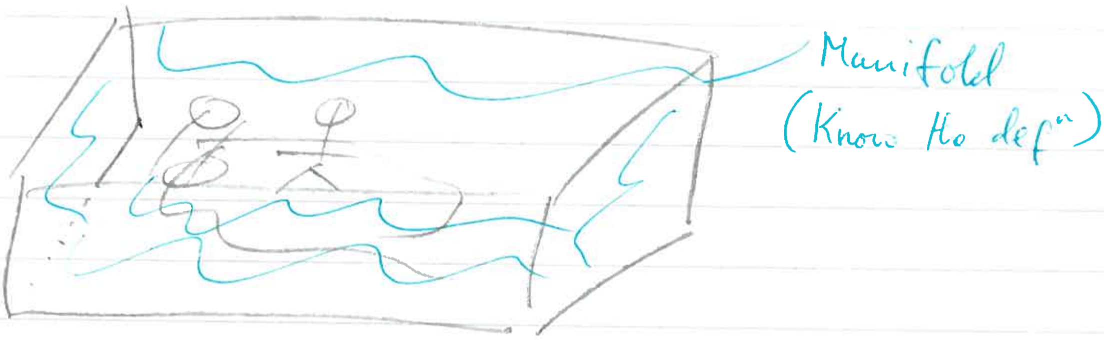
$$\delta\phi_{100} = 43.03''$$

NFE

END OF COURSE

—/—

In the start.



—/—

- Path dependence of the parallel transport linked to the Riemann curvature tensor.
- May include special relativity, Lorentz boost and length contraction.
- Look at the past two years of papers.

Basic definition

- Definition of manifold
- Coordinate transformation.
- Vectors, covectors, tensors., scalars.
- Tensor algebra.

Metrics + Geodesics.

- Definition of both
- Recall the different types
  - Riemannian
  - Pseudo-Riemannian.
- Curves
- Tangent vector (all vectors in GR are tangent vectors).

SR.

This is flat metric.

- Lorentz - boost
- Rotations
- Light cones.
- World lines.
- Length contraction.

Curvature.

- Covariant derivative

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) V^d = R_{bac}{}^{d} V^c$$

- torsion free.
- Geodesic equation.
- Metricity condition $\nabla_a g_{bc} = 0$.

- The famous R's: $R_{abc}{}^{d}$, R_{ab}, R.
- Einstein Tensor.
- Einstein's field Equations.

<u>Schwarschild solution</u>.

- Know its origin and how to derive it from the fully spherically symmetrized and static metric.
- $R_{ab} = 0$ vacuum field eq^{n}s.
- Solutions to the geodesic equations.
- $\dot{r}$ equation.

You get
- Christoffel symbol in terms of metric.
- Geodesic equation
- Riemann curvature tensor
  - a couple of identities.
- Ricci tensor, scalar $R_{mr} = R_{mnr}{}^{n}$
- Einstein tensor
- Schwarzchild metric.

5 questions, best 4, no calculator.

Q4. 2012.
4a) Show that the Riemann curvature tensor satisfies

$$R_{abcd} + R_{bcad} + R_{cabd} = 0. \quad (*)$$

b) Simply $R_{ab}{}^{ab}$, $R_{abc}{}^{c}$.

c) Use eq (*) show that G_{ij} is symmetric.

d) For a constant $\lambda > 0$ let h_{ij} be the metric tensor $h_{ij} = \lambda g_{ij}$. If R, R_{ab}, G_{ij}, R_{abcd} and $R_{abc}{}^{d}$ are the curvature tensors for g_{ij} express the corresponding one for h_{ij}

$$h^{ij} = \frac{1}{\lambda} g^{ij}$$ Introduce a symbol for h.

$$\hat{\Gamma}^{i}_{jk} = \Gamma^{i}_{jk}$$

$$\hat{R}_{abc}{}^{d} = R_{abc}{}^{d}.$$

$$\hat{R}_{ab} = R_{ab}$$

$$\hat{R} = \frac{1}{\lambda} R. \quad , \quad \hat{G}_{ab} = G_{ab}.$$

(6 points)

Q2. 2011

a) Write down the equation for a hypersphere with radius R in n-dimension (Hint: think circle).

$$x^2 + y^2 + z^2 = r^2.$$

$$(x^1)^2 + (x^2)^2 + \dots + (x^n)^2 = R^2.$$

Show that the line element on the hypersphere with radius R in 3-dim is given by:

$$ds^2 = R^2(d\theta^2 + \sin^2\theta \, d\phi^2)$$

$$ds^2 = dx^2 + dy^2 + dz^2$$
$$x = R\sin\theta\cos\phi$$
$$y = R\sin\theta\sin\phi$$
$$z = R\cos\theta .$$

$$dx = \ldots$$
$$dy = \ldots$$
$$dz = \ldots$$

c) Write down the geodesic equations and find all Christoffel symbols.

$$L = R^2(\dot\theta^2 + \sin^2\theta\,\dot\phi^2)$$

d) Compute all non-vanishing components of the Riemann tensor $R_{abc}{}^{d}$.

$$R_{\theta\phi\theta\phi} = (1.-\cos^2\theta)$$

e) find the Ricci Scalar and tensor.

$$R_{abcd}\, g^{de} = R_{abc}{}^{d}$$

$$R_{abc1}\, g^{1e} + R_{abc2}\, g^{2e} = R_{abc}{}^{e} .$$

$$e=1 \quad R_{abc1}\, g^{11} + \cancel{R_{abc2}\, g^{21}} = R_{abc}{}^{1}$$