3308 Maxwell's Theory of Electrodynamics Notes

Based on the 2011 spring lectures by Prof Y Kurylev

OBSOLETE

3308 MAXWELL'S THEORY OF ELECTROMAGNETISM

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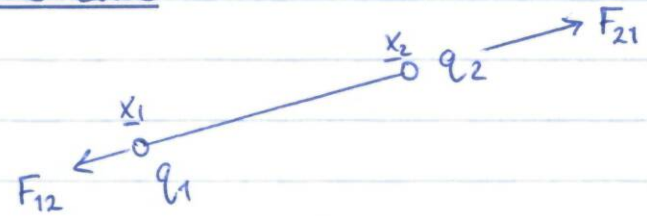
"It's Pheesics. The name of the course is pheesics".

Short-ish book: http://www.phys.ufl.edu/%7Edorsey/phy6346-00/lectures

ELECTROSTATICS

Coulomb's Law



"it will heal you or keal you"

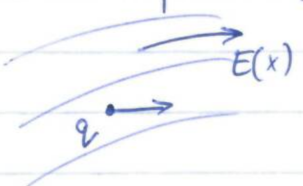
If we have two charged particles of charges q_1, q_2 at points $\underline{x}_1, \underline{x}_2 \in \mathbb{R}^3$; then

$$F_{12} = F_{1\leftarrow 2} = kq_1q_2 \frac{\underline{x}_{12}}{|\underline{x}_{12}|^3}$$

k is a constant to do with units

How do they exert a force on each other? They used to think there was ether which gave a transfer material to go through. This was later rubbished, of course. Turns out when you place a charged particle, it propogates a magnetic field. Now we're looking for Higgs bosons!

If we have an electric field $\underline{E}(\underline{x})$ $[E^1, E^2, E^3]$



$$F_q(x) = qE(x)$$

So by looking at Coulomb's law, we can see the field created by q_2 if we place q_1:

$$F_q(x) = kqq_2 \frac{x-x_2}{|x-x_2|^3}$$

$$\Rightarrow E(x) = kq_2 \frac{x-x_2}{|x-x_2|^3}$$

sometimes seen as $k = \frac{1}{2\pi\varepsilon_0}$

sometimes seen as $\frac{\widehat{x-x_2}}{|x-x_2|^2}$

Electric fields add linearly.

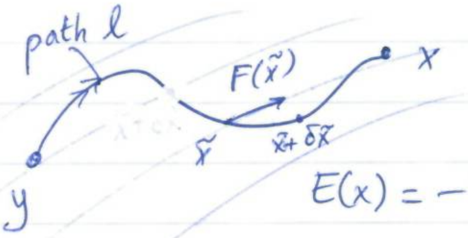
if you have electric field E
and potential Φ, (scalar electric potential)
physicists use

$$E(x) = -\nabla \Phi(x)$$

$$\left(\begin{array}{l} \text{Note that } \operatorname{curl} E = 0 \\ \text{and domain simply connected} \end{array} \right\} \Rightarrow \text{it has a potential } \Phi \text{ by } 1402 \Bigg)$$

Obviously Φ is defined up to a constant. So Φ has no real physical meaning, but the <u>difference</u> of potentials does, of course.

Say we have $\Phi(x) - \Phi(y)$



$$F(\tilde{x}) = q\,E(\tilde{x})$$

$$E(x) = -\nabla \Phi(x)$$

Work done getting from $\tilde{x}$ to $\tilde{x}+\delta\tilde{x}$: $\quad dA = F(\tilde{x}) \cdot dx$

$$\text{Work} = A = \int_\ell F(\tilde{x}) \cdot d\tilde{x} = q \int_\ell E(\tilde{x}) \cdot d\tilde{x}$$

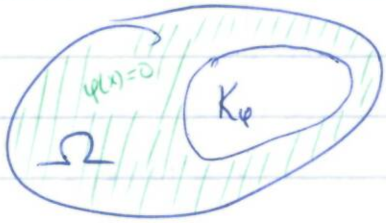
$$= q \int_\ell \nabla \Phi(\tilde{x})\, d\tilde{x} = -q\left[\Phi(x) - \Phi(y)\right]$$

$$= q\left[\Phi(y) - \Phi(x)\right]$$

The convention is that $\Phi = 0$ at infinity.

THEORY OF DISTRIBUTIONS

__DuBois-Raymond lemma:__ Say we have a domain Ω and a f^n $f \in \underset{\text{cts}}{C}(\Omega)$



Take a f^n $\varphi \in C_0^\infty(\Omega)$, ie φ can be differentiated arbitrarily many times.

There is a compact region (closed, bounded set) $K_\varphi \subset \Omega$ s.t. $\varphi(x) = 0$, $x \notin K_\varphi$.

__Lemma 1.1:__ Let $f \in C(\Omega)$ and for all $\varphi \in C_0^\infty(\Omega)$,

$$\int_\Omega f(x)\varphi(x)\,dx = 0. \qquad \text{Then } f \equiv 0.$$

__Corollary:__ If $f_1, f_2 \in C(\Omega)$, $\forall \varphi \in C_0^\infty(\Omega)$,

$$\int_\Omega f_1(x)\varphi(x)\,dx = \int_\Omega f_2(x)\varphi(x)\,dx. \qquad \text{Then } f_1 = f_2$$

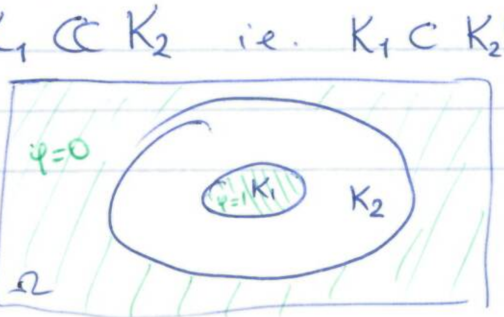
__Proof of corollary:__ Subtract, then use lemma.

This is useful. φ is called 'test f^n' because we can see if $f_1 = f_2$ by using their integrals. Cool.

$\mathcal{D}(\Omega) = C_0^\infty(\Omega) =$ space of test f^ns

C_0^∞ is called the space of test f^ns, and is often denoted by $\mathcal{D}(\Omega)$.

__Prop^n 1.2__ $K_1 \subset\subset K_2$ ie. $K_1 \subset K_2^{int}$



For any two compacts,

there exists $\varphi \geq 0$, $\varphi \in C_0^\infty(K_2)$, i.e. $\varphi = 0$ outside K_2.
s.t. $\varphi = 1$ in K_1.

<u>Proof of Lemma 1.1</u> Say we have $x_0 \in \Omega$ s.t. $f(x_0) > 0$.

$f \in C(\Omega) \Rightarrow \exists$ a bowl of radius δ $B_\delta(x_0)$ s.t. $f(x) > \frac{1}{2} f(x_0)$, $x \in B_\delta(x_0)$

$\forall \varepsilon > 0 \; \exists \delta$ s.t. $|f(x) - f(x_0)| < \varepsilon \Rightarrow |x - x_0| < \delta$

Let $\varepsilon = \frac{1}{2} f(x_0)$.

Choose φ from C_0^∞ s.t. (a) $\varphi \geq 0$

(b) $\varphi = 1$, $x \in B_{\delta/2}(x_0)$ $(= K_1)$

(c) $\varphi = 0$ $x \notin B_\delta(x_0)$ $(= K_2)$

Look at $$\underbrace{\int_\Omega f(x)\varphi(x)\,dx}_{=0} = \int_{B_\delta(x_0)} \underbrace{f(x)\varphi(x)}_{\text{all these terms are positive}}\,dx$$ $\because$ φ is zero outside this bowl.

$$\geq \int_{B_{\delta/2}(x_0)} f(x) \underbrace{\varphi(x)}_{=1}\,dx \geq \frac{1}{2} f(x_0) \cdot \mathrm{Vol}(B_{\delta/2}(x_0)) > 0$$

\#

$\Rightarrow$ f is zero everywhere. $\square$

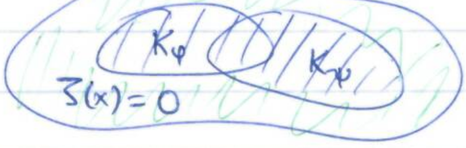
$\mathcal{D}(\Omega)$ is a <u>topological vector space</u>. What does that mean?

(1) it's a vector space — you can $\begin{cases}\text{add vectors} \\ \text{mult. by scalar}\end{cases}$

if we have $\varphi, \psi \in C_0^\infty(\Omega)$, $\lambda, \mu \in \mathbb{R}$

then $\zeta(x) = \lambda\varphi(x) + \mu\psi(x) \in C_0^\infty(\Omega)$.

DBR lemma $\rightarrow$ $\zeta(x) = 0$ for $x \notin K_\varphi \cup K_\psi$



(2) topological $\Leftrightarrow$ there is a notion of convergence,

ie. there is a meaning to $\varphi_k \xrightarrow[k\to\infty]{} \varphi$.

<u>Two conditions to convergence:</u>

(a) For any n-dimensional vector $\underline{\alpha} = (\alpha_1, \ldots, \alpha_n)$, $\Omega \subset \mathbb{R}^n$,

$\alpha_1 \in \mathbb{Z}^+ = \mathbb{N}_0$

all derivatives cvg

$$\partial^\alpha \varphi_k(x) \longrightarrow \partial^\alpha \varphi(x), \quad x \in \Omega$$

$$\partial^\alpha \varphi_k(x) = \frac{\partial^{\alpha_1+\alpha_2+\cdots+\alpha_n}\varphi_k}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}(x)$$

whatever derivative at whatever point you choose, it will tend to the derivative at that pt

<u>Notation:</u>

$|\alpha| = \alpha_1 + \cdots + \alpha_n$

$\alpha! = \alpha_1! \cdots \alpha_n!$

(b) $\exists K \subset \Omega$ s.t. $\left.\begin{aligned}\varphi_k(x) &= 0 \\ \varphi(x) &= 0\end{aligned}\right\} x \notin K$

region where both satisfy DBR.

<u>Def^n 1.4</u> $\varphi_k \to \varphi$ in $\mathcal{D}(\Omega)$ if (a) and (b) are satisfied.

<u>Examples of vector space</u>: $\mathbb{R}^2 = (x, y) = \underline{x}$ with norm $\sqrt{x^2+y^2}$

$\|x\|_\infty = \max(|x|, |y|)$

$\|x\|_p = (|x|^p + |y|^p)^{1/p}$, $p \geq 1$

Def^n: a f^n $f: V \to \mathbb{R}$ mapping onto the reals is a <u>functional</u>

Def^n: a functional f is <u>linear</u> if $f(\lambda \underline{v} + \mu \underline{w}) = \lambda f(\underline{v}) + \mu f(\underline{w})$
for $\underline{v}, \underline{w} \in V$

Def^n: The space of all <u>linear</u> <u>continuous</u> f^n's form a linear vector space called a <u>dual</u> vector space.

$f, g \in V'$ ← prime means dual space

$(\lambda f + \mu g)(\underline{v}) = \lambda f(\underline{v}) + \mu g(\underline{w})$

A functional is continuous if when $x_n \to x$, $f(x_n) \to f(x)$.

What are the f^{nl}'s in $\mathbb{R}^2$? $f = (f_1, f_2)$
$f(x) = f_1 x + f_2 y$.
They all have this form; it gives you nothing new.
The dual space of $\mathbb{R}^2$ is $\mathbb{R}^2$ itself.

L2

Recall $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$.

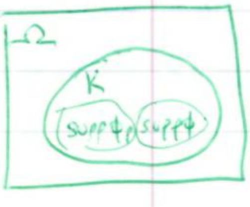
Def^n: $\mathcal{N}(f) \in C(\Omega)$ is the largest <u>open</u> set where $f=0$. (nullset)

Def^n: The <u>support</u> $\text{supp}(f) = \Omega \setminus \mathcal{N}(f)$ and it is closed.
$\text{supp}(f) = \text{cl}\{x : f(x) \neq 0\}$
↑ closure

Def^n of convergence (again)

Def^n 1.4 $\varphi_p \to \varphi$ in $\mathcal{D}(\Omega)$ iff

(a) $\partial^\alpha \varphi_p(x) \to \partial^\alpha \varphi(x)$, $x \in \Omega$, where α is any mult. index
$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$



(b) there is a compact set $K \Subset \Omega$ (cl. bounded)
s.t. $\text{supp}(\varphi_p)$, $\text{supp}(\varphi) \subset K$.
↑ and

Problem: if we have sequences $\varphi_1^p, \varphi_2^p \in \mathcal{D}(\Omega)$,
$\lambda_1^p, \lambda_2^p \in \mathbb{R}$

and $\varphi_i^p \underset{p\to\infty}{\rightrightarrows} \varphi_i$, $\lambda_i^p \to \lambda_i$ $i=1,2$

$$\lambda_1^p \varphi_1^p + \lambda_2^p \varphi_2^p \longrightarrow \lambda_1 \varphi_1 + \lambda_2 \varphi_2 \text{ in } \mathcal{D}(\Omega) \quad \cdots (1)$$

recall topological means notion of convergence

Since $\mathcal{D}(\Omega)$ is a vector space which has the notion of convergence which satisfies (1), $\mathcal{D}(\Omega)$ is a topological vector space

If V is a topological vector space
V' is a dual space, i.e. the space of all linear continuous functionals on V

↳ the corresponding

Recall a functional is a map $F: V \longrightarrow \mathbb{R}$

Defn 1.7

it is linear: $F(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 F(v_1) + \lambda_2 F(v_2)$
it has continuity: $F(v_p) \to F(v)$ if $v_p \to v$.

These F form a topological vector space called dual to V.

DUAL = LINEAR CTS FNLS

If $F_1, F_2 \in V'$, $\lambda_1, \lambda_2 \in \mathbb{R}$:
$$(\lambda_1 F_1 + \lambda_2 F_2)(\underline{v}) = \lambda_1 F_1(\underline{v}) + \lambda_2 F_2(\underline{v})$$

Need to check that this object is linear and continuous.

(i) Linearity: $(\lambda_1 F_1 + \lambda_2 F_2)(\mu_1 v_1 + \mu_2 v_2)$
$= \lambda_1 F_1(\mu_1 v_1 + \mu_2 v_2)$
$+ \lambda_2 F_2(\mu_1 v_1 + \mu_2 v_2)$

each F_1, F_2 are linear fn!s $\Rightarrow$

$$= \lambda_1 \left(\mu_1 F_1(v_1) + \mu_2 F_1(v_2) \right) + \lambda_2 \left(\mu_1 F_2(v_1) + \mu_2 F_2(v_2) \right)$$

$$= \mu_1 \left[\lambda_1 F_1(v_1) + \lambda_2 F_2(v_1) \right] + \mu_2 \left[\lambda_1 F_1(v_2) + \lambda_2 F_2(v_2) \right]$$

$$= \mu_1 \left(\lambda_1 F_1 + \lambda_2 F_2 \right)(v_1) + \mu_2 \left(\lambda_1 F_1 + \lambda_2 F_2 \right)(v_2)$$

(2) Cont. : $\overset{\text{want}}{(\lambda_1 F_1 + \lambda_2 F_2)(v_p)} \longrightarrow (\lambda_1 F_1 + \lambda_2 F_2)(v)$ if $v_p \to v$

$\quad \| \text{ by defn}$

$$\lambda_1 F_1(v_p) + \lambda_2 F_2(v_p)$$

$$\downarrow \qquad\qquad \downarrow$$

$$\lambda_1 F_1(v) + \lambda_2 F_2(v) = (\lambda_1 F_1 + \lambda_2 F_2)(v). \qquad \square$$

Say

(1) $F_1, F_2 \in V'$, $\lambda_1, \lambda_2 \in \mathbb{R}$, $(\lambda_1 F_1 + \lambda_2 F_2)(v) = \lambda_1 F_1(v) + \lambda_2 F_2(v)$

(2) $F_p \to F$ in V' if $\forall\, v \in V$, $F_p(v) \to F(v)$.

H/W prove this

$$\text{If } \left.\begin{array}{l} \lambda_1^p \to \lambda_1 \\ \lambda_2^p \to \lambda_2 \\ F_1^p \to F_1 \\ F_2^p \to F_2 \end{array}\right\} \text{ then } \begin{array}{c} \lambda_1^p F_1^p + \lambda_2^p F_2^p \longrightarrow \lambda_1 F_1 + \lambda_2 F_2 \\ \Updownarrow \\ \forall\, v \in V \ (\lambda_1^p F_1^p + \lambda_2^p F_2^p)(v) \to (\lambda_1 F_1 + \lambda_2 F_2)(v) \end{array}$$

Def^{n} 1.11 : The space of distributions is the topological vector space dual to $\mathcal{D}(\Omega)$, ie. the space of linear continuous f^{n}'s on $\mathcal{D}(\Omega)$.

Prob 1.5 $\forall \varphi \in C_0^\infty(\Omega), \forall \beta$ multiindex then $\psi = \partial^\beta \varphi \in C_0^\infty(\Omega)$.

Proof: more zero areas after diff^n $\rightarrow \underset{\mathcal{N}(\partial^\beta\varphi)}{\underset{\shortparallel}{\mathcal{N}(\psi)}} \supset \mathcal{N}(\varphi) \Rightarrow \operatorname{supp}(\partial^\beta\varphi) \subset \operatorname{supp}(\varphi) \subset\subset \Omega$.

If $\zeta \in C^\infty(\Omega)$, $\forall \varphi \in \mathcal{D}(\Omega)$ $\quad \psi = \zeta\varphi \in \mathcal{D}(\Omega)$

$\downarrow$

$\operatorname{supp}(\psi) \subset \operatorname{supp}(\varphi) \subset\subset \Omega$

end of def^{n}s. Now coming onto essential stuff.

If $f(x) \in C(\Omega)$, we can look at integrals

$$\int_\Omega f(x)\varphi(x)\,dx \qquad \varphi \in \mathcal{D}(\Omega)$$ ← space of test f^{n}s, remember!

Prop^n: For any $f \in C(\Omega)$ (and even $f \in L^1_{loc}(\Omega)$), the integrals $\int_\Omega f(x)\varphi(x)\,dx$ form a linear cont. f^{n} in $D(\Omega)$.

Proof: linearity : $\int_\Omega f(x)\left[\lambda_1\varphi_1(x) + \lambda_2\varphi_2(x)\right] dx$

$$= \lambda_1 \int_\Omega f(x)\varphi_1(x)\,dx + \lambda_2 \int_\Omega f(x)\varphi_2(x)\,dx$$

cont : let $\varphi_\ell \to \varphi$ in $\mathcal{D}(\Omega)$.

Then for <u>any</u> multiindex $\underline{\alpha}$, $\partial^{\alpha}\varphi_p(x) \to \partial^{\alpha}\varphi(x)$, $x \in \Omega$.
Take $\underline{\alpha} = (0,0,\dots,0)$.

Then $\partial^0\varphi = \varphi$ by def^n so, $\varphi_p(x) \to \varphi(x) \quad \forall x \in \Omega$.

Then $f(x)\varphi_p(x) \to f(x)\varphi(x) \quad \forall x \in \Omega$.

But we can't say $\int \cdots \to \int \cdots$ because
this is not true!! Why?

counter-example

Take interval $(0 \quad 1)$ $\quad \varphi_p(x) = \begin{cases} p & \text{on } (0, \frac{1}{p}) \\ 0 & \text{on } [\frac{1}{p}, 1) \end{cases}$

$\forall x \in (0,1)$, $\varphi_p(x) \longrightarrow 0$.

$\int_0^1 \varphi_p(x)\, dx = 1$

What <u>extra condition</u> do we need to make the integrals converge?

$$\text{If } \int_\Omega X_p(x)\,dx \to \int_\Omega X(x)\,dx$$

then $X_p \to X$ <u>uniformly</u>

$$\forall \varepsilon > 0 \ \exists\, p(\varepsilon) \text{ s.t. } p > p(\varepsilon) \quad |X_p(x) - X(x)| < \varepsilon \quad \forall x \in \Omega.$$

There was a part of def^n of convergence:
that $\forall \varphi_p$, $\operatorname{supp}(\varphi_p)$ lie in the same compact.

Reminder: If $X_p \in C(\Omega)$ and $X_p(x) \to X(x)$ at any $x \in \Omega$. Then for any compact $K \subset\subset \Omega$, $X_p \to X$ uniformly in K.

By (b) there is compact $K \subset\subset \Omega$ s.t.
$\mathrm{supp}(\varphi_p), \mathrm{supp}(\varphi) \subset K$,
i.e. $\varphi_p(x) = \varphi(x) = 0, \qquad x \notin K$.

$$\text{So} \quad \int_\Omega f(x)\varphi_p(x)\,dx = \int_K f(x)\varphi_p(x)\,dx \longrightarrow \int_K f(x)\varphi(x)\,dx \qquad \square$$

by the reminder, the integrands converge uniformly, so the integrals converge.

Summarise: every $f \in C(\Omega)$ defines a distribution in $\mathcal{D}'(\Omega)$ which we denote $F_f \in \mathcal{D}'(\Omega)$.
dash means dual

$$F_f(\varphi) = \int_\Omega f(x)\varphi(x)\,dx$$

$$C(\Omega) \subset \mathcal{D}(\Omega)$$
embedding

This embedding is continuous
la la la la la

Prob 1.12. We say that $f_p \to f$ pointwise if $f_p(x) \to f(x) \ \forall x \in \Omega$.
Then if $f_p, f \in C(\Omega)$ and $f_p \to f$ ptwise then
$$F_{f_p} \to F_f \text{ in } \mathcal{D}'(\Omega).$$

Proof. $F_{f_p} \to F_f$ in $\mathcal{D}'(\Omega)$ means $\forall \varphi \in \mathcal{D}'(\Omega)$, $F_{f_p}(\varphi) \longrightarrow F_f(\varphi)$

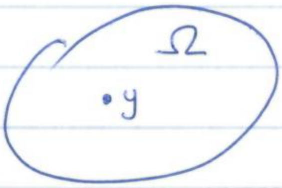
$$\iff \int_\Omega f_p(x)\varphi(x)\,dx \longrightarrow \int_\Omega f(x)\varphi(x)\,dx.$$

$K = \mathrm{supp}(\varphi)$ $\qquad \int_K f_p(x)\varphi(x)\,dx \qquad \int_K f(x)\varphi(x)\,dx$

Since $f_p\varphi \to f\varphi$ pointwise and $f_p\varphi$ and $f\varphi$ are continuous
then $f_p\varphi \to f\varphi$ uniformly in K. D

For every f^n there is one distribution.

<u>Example 1.13</u>:



dualspace = linear cts f^{n}'s on Ω

Let $\delta_y \in \mathcal{D}'(\Omega)$ be the following distribution:

$$\delta_y(\varphi) := \varphi(y)$$

<u>linearity</u>: trivial (in print out)

<u>cont^y</u>: if $\varphi_p \to \varphi$ in $\mathcal{D}(\Omega)$, then for any $\underline{\alpha}$, in particular $\underline{\alpha} = (0,\dots,0)$, and any $x \in \Omega$, in particular $x = y$, we have $\quad \partial^\alpha \varphi_p(x) \longrightarrow \partial^\alpha \varphi(x)$.

For $\underline{\alpha} = (0,\dots,0)$, $x = y$, we get $\varphi_p(y) \to \varphi(y)$.

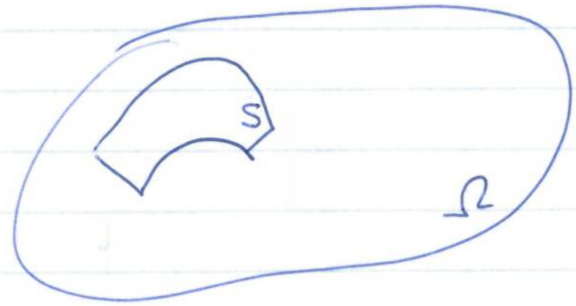
$$\underset{\delta_y(\varphi_p)}{\underset{\parallel}{\varphi_p(y)}} \to \underset{\delta_y(\varphi)}{\underset{\parallel}{\varphi(y)}}$$

<u>Prob 1.14</u>: There are no $f \in C(\Omega)$ or $f \in L^1_{loc}(\Omega)$ s.t. $F_f = \delta_y$.

<u>Ex. 1.15</u>: $\Omega = \mathbb{R}^3$. Consider the plane, say $x_3 = a$.
Let $h(x_1, x_2)$ be a cts f^n on $\mathbb{R}^2$, with complex
$\in C(\mathbb{R}^2)$

$$F_{a,h}(\varphi) = \int_{\mathbb{R}^2} h(x_1,x_2)\,\varphi(x_1,x_2,a)\,dx_1\,dx_2$$

$$= \int_{\text{plane } x_3=a} h(x)\, \varphi|_{\overset{\in \text{ plane } \subset \mathbb{R}^3}{\mathbb{R}}}\, dA$$



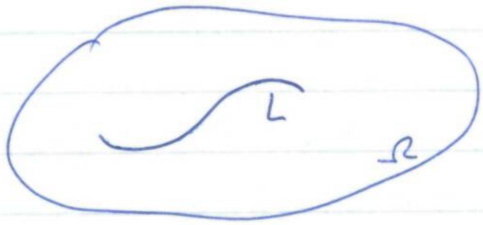
surface

$S \subset \Omega$

h cts on S.

$$F_{h,S}(\varphi) = \int_S h\varphi|_S \, dA$$

surface integral from 1402.



L line in $\Omega \subset \mathbb{R}^2$ or $\mathbb{R}^3$

h cts on L

$$F_{h,L}(\varphi) = \int_L h\varphi|_L \, dl$$

length element.

1.2 SUPPORT OF A DISTRIBUTION

If $f \in C(\Omega)$ then $\mathcal{N}(f)$ is the largest open set where $f \equiv 0$. (by defn)

Say we have open $U \subset \Omega$. Is it possible to check that $f \equiv 0$ in U without evaluating $f(x)$ pointwise?

$$\int_{\Omega} f(x)\varphi(x)\,dx = 0 \text{ if } \operatorname{supp}(\varphi) \subset U$$

By DuBois-Raymond, also $f|_U = 0$ if $\int_U f(x)\varphi(x)\,dx = 0$ for $\forall \varphi \in C_0^\infty(U)$.

$$C_0^\infty(U) = \{\varphi \in C_0^\infty(\Omega) : \operatorname{supp}(\varphi) \subset U\}$$

Thus $f|_U = 0$ iff $\int_{\Omega} f(x)\varphi(x)\,dx = 0 \quad \forall \varphi \in \mathcal{D}(\Omega)$, $\operatorname{supp}(\varphi) \subset U$.

$$f|_U = 0 \iff F_f(\varphi) = 0 \text{ if } \operatorname{supp}(\varphi) \subset U.$$

Def$^{\underline{n}}$ We say $F|_U = 0$, where $F \in \mathcal{D}'(\Omega)$ if $F(\varphi) = 0$ for $\varphi \in \mathcal{D}(\Omega)$, $\operatorname{supp}(\varphi) \subset U$.

Def$^{\underline{n}}$: If $F \in \mathcal{D}'(\Omega)$ then $\mathcal{N}(F)$ is the largest open set in Ω where $F = 0$.

$F = 0 \iff F(\varphi) = 0 \ \forall \varphi \in \mathcal{D}(\Omega)$, $\operatorname{supp}(\varphi) \subset U$

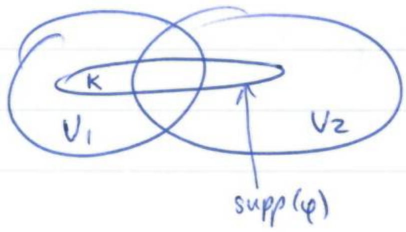
If we have $F(\varphi) = 0$ if $\operatorname{supp}(\varphi) \subset U_1$

$F(\varphi) = 0$ if $\operatorname{supp}(\varphi) \subset U_2$

$\Downarrow$

$F(\varphi) = 0$ if $\operatorname{supp}(\varphi) \subset U_1 \cup U_2$ (not trivial but TRUE).

Why? Suppose we have $X \in C^\infty$ s.t. $0 \le X \le 1$, $X = 1$ in K_1, $X = 0$ in K_2.



we can write $\varphi = \varphi_1 + \varphi_2$ $\quad supp(\varphi_1) \subset U_1$

$\quad supp(\varphi_2) \subset U_2$

$$F(\varphi) = \underset{0}{\underset{\shortparallel}{F(\varphi_1)}} + \underset{0}{\underset{\shortparallel}{F(\varphi_2)}}.$$

Not a proof but a <u>feeling</u>.

def^n continued...

$$supp(F) = \Omega \setminus \mathcal{N}(F)$$

<u>Example 1.18</u> $\quad supp(\delta_y) = \{y\}$. $\quad$ Recall $\delta_y(\varphi) = \varphi(y)$

$\Omega \setminus \{y\} \subset \mathcal{N}(\delta_y)$.

We need to show that if $supp(\varphi) \subset \Omega \setminus \{y\} \Rightarrow$

$\underset{\varphi(y)}{\underset{\shortparallel}{\delta_y(\varphi)}} = 0$ and $supp(\varphi) \cap \{y\} = \emptyset \Rightarrow \varphi(y) = 0$.

Assume that $y \in \mathcal{N}(\delta_y)$. Then $\mathcal{N}(\delta_y) = \Omega$

$\Updownarrow$

$\delta_y(\varphi) = 0 \quad \forall \varphi \in \mathcal{D}(\Omega)$

However, by Prop 1.2 there exist $\varphi \in C_0^\infty(\Omega)$ s.t. $\varphi(y) = 1$.

But then $\delta_y(\varphi) = \varphi(y) = 1$ $\quad \#$

$\Rightarrow y \notin \mathcal{N}(\delta_y)$. $\square$

<u>Def^n</u>: $F \in \mathcal{E}'(\Omega)$ [distributions with compact support]
if $\text{supp}(F) = \Omega \setminus \mathcal{N}(F)$ is compact in Ω.

Then we can define $F(\psi)$, $\psi \in C^\infty(\Omega) = \mathcal{E}(\Omega)$

$\psi_n \to \psi$ in $C^\infty(\Omega)$ if $\forall \alpha$, $\forall x \in \Omega$, $\partial^\alpha \psi_n(x) \to \partial^\alpha \psi(x)$

By Prop 1.2, $\exists\, \chi \in C_0^\infty(\Omega) = \mathcal{D}(\Omega)$ s.t. $\chi = 1$ on K

By def^n, $F(\psi) := F(\underbrace{\chi\psi}_{\in \mathcal{D}(\Omega)})$. [Pick some χ with this property]

Note that if $\hat{\chi} \in C_0^\infty$ and $\hat{\chi} = 1$ on K, then $F(\hat{\chi}\psi) = F(\chi\psi)$

Showing true for any χ

(indeed $F(\chi\psi) - F(\hat{\chi}\psi) = F((\chi - \hat{\chi})\psi)$.

but $\chi(x) = \hat{\chi}(x) = 1$ on $x \in \text{supp}(F)$

$\Rightarrow \text{supp}(\chi - \hat{\chi}) \cap \text{supp} F = \emptyset$ by def^n of supp F

$\Rightarrow F((\chi - \hat{\chi})\psi) = 0$

<u>Differentiation of Distributions</u>

Take $f \in C^\infty(\Omega)$, then $\forall \beta$, $\partial^\beta f = \dfrac{\partial^\beta f}{\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}} := g$.

Obviously $g \in C^\infty(\Omega)$ as well.

So $F_g(\varphi) = \int_\Omega g(x)\varphi(x)\, dx$.

$= \int_\Omega \partial^\beta f(x) \varphi(x)\, dx =$ ($f, \varphi = 0$ at boundaries which is where first term goes)

int. by parts $= (-1)^{|\beta|} \int_\Omega f \underbrace{\partial^\beta \varphi}_{\in \mathcal{D}(\Omega)}\, dx$

$$= (-1)^{|\beta|} F_f(\partial^\beta \varphi)$$

$$\Rightarrow \quad F_{\partial^\beta f}(\varphi) = (-1)^{|\beta|} F_f(\partial^\beta \varphi)$$

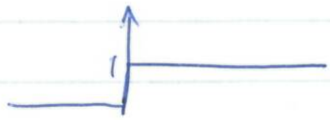
<u>Defn 1.21</u> Let $F \in \mathcal{D}'$, β multiindex.

Then $\partial^\beta F \in \mathcal{D}'$ is the distribution of the form

$$\partial^\beta F(\varphi) = (-1)^{|\beta|} F(\partial^\beta \varphi).$$

<u>Example</u>: $H(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \\ \text{gazillion} & x = 0 \end{cases}$ ← val. at 0 plays no role when integrate!

Heaviside!!



What is H'?

$$H'(\varphi) = (-1) H(\varphi') = -\int_0^\infty \varphi'(x)\,dx = \varphi(0).$$

$H(x) = 1$ for $x > 0$

($\varphi(\infty) = 0$ because of compact!!)

$= \delta(\varphi)$

$$\Rightarrow H' = \delta.$$

Check that $\partial^\beta F(\varphi) \in \mathcal{D}(\Omega)$, i.e. it's cts and linear.

(i) Linearity: $\partial^\beta F(\lambda\varphi + \mu\psi) = (-1)^{|\beta|} F[\underbrace{\partial^\beta(\lambda\varphi + \mu\psi)}_{= \lambda\partial^\beta\varphi + \mu\partial^\beta\psi}]$

$$= (-1)^{|\beta|}\left[\lambda F(\partial^\beta\varphi) + \mu F(\partial^\beta\psi)\right] \quad \because F \in \mathcal{D}'$$

$$= \lambda\,\partial^\beta F(\varphi) + \mu\,\partial^\beta F(\psi).$$

(ii) Cont: Let $\varphi_n \to \varphi$ in $\mathcal{D}(\Omega)$.
We need $\partial^\beta F(\varphi_n) \to \partial^\beta F(\varphi)$.

i.e. $(-1)^{|\beta|} F(\partial^\beta \varphi_n) \to (-1)^{|\beta|} F(\partial^\beta \varphi)$.

By Prob 1.5(i), we know that $\partial^\beta \varphi_n \to \partial^\beta \varphi$ in $\mathcal{D}(\Omega)$

$\Rightarrow F(\partial^\beta \varphi_n) \to F(\partial^\beta \varphi)$ $\because$ F is a distribution and distr's have cty.

as required. $\square$

If f is a distribution, $f \in \mathcal{D}'(\Omega)$ and $\psi \in C^\infty(\Omega)$
then $(\psi F)(\varphi) = \underset{\underset{\mathcal{D}(\Omega)}{\cap}}{F(\psi\varphi)}$, $\quad \varphi \in \mathcal{D}(\Omega)$.

Notation: $\underset{\text{coeffs depend on } x}{P}(x, \underset{\text{differential}}{\partial}) u = \sum_{|\beta| \le M} \psi_\beta(x) \partial^\beta \underset{\underset{\mathcal{D}(\Omega)}{\cap}}{u} \in \mathcal{D}'(\Omega)$.

If $u \in \mathcal{D}'(\Omega)$ solves A then u is called the Green f^n for PDE $P(x, \partial)$ and denoted $G(x, \partial)$

Convolution

Let $\Omega = \mathbb{R}^n$. $\quad f \in C_0(\mathbb{R}^n)$ — i.e. cont. with compact support.
$g \in C(\mathbb{R}^n)$.

$$h(x) = (f * g)(x) = \int_{\mathbb{R}^n} f(x-y) g(y)\, dy$$

$$= \int_{\mathbb{R}^n} f(y) g(x-y)\, dy = (g * f)(x).$$

f has compact support so we aren't integrating over whole $\mathbb{R}^n$ just the f is nonzero bit.

$$F_h(\varphi) = \int_{\mathbb{R}^n} h(x)\varphi(x)\,dx$$

$$= \int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n} f(x-y)g(y)\,dy\right]\varphi(x)\,dx$$

$$= \int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n} f(x-y)\varphi(x)\,dx\right]g(y)\,dy \qquad \text{Let } z=x-y.$$

$$= \int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n} f(z)\varphi(z+y)\,dz\right]g(y)\,dy \qquad \text{Let } x=z$$

$$= \int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n} f(x)\underbrace{\varphi(x+y)}_{\varphi^y(x)}\,dx\right]g(y)\,dy$$

$$= \int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n} f(x)\varphi^y(x)\,dx\right]g(y)\,dy$$

$$= \int_{\mathbb{R}^n} \underbrace{F_f(\varphi^y)}_{\Phi(y)}\,g(y)\,dy$$

$$= F_g(\Phi) = F_g(F_f(\varphi^y))$$

<u>Def^n</u>: Let $F \in \mathcal{E}'(\mathbb{R}^n)$, $G \in \mathcal{D}'(\Omega)$.

Then the convolution $H = F * G \in \mathcal{D}'(\Omega)$ is

$$(F*G)(\varphi) = G(F(\varphi^y))$$

Questions: Is it well-defined?
Is it linear?
Is it cts?

Question: Does $F(\varphi^y) \in \mathcal{D}(\mathbb{R}^n)$

(1) Support: since $F \in \mathcal{E}'(\mathbb{R}^n)$

$\Rightarrow \operatorname{supp}(F) \Subset \mathbb{R}^n$, ie. $\exists\, b>0$ s.t.

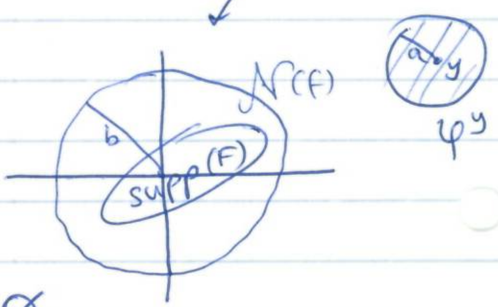
$\mathbb{R}^n \setminus$ bowl of radius b centred at 0 $\subset \mathcal{N}(F)$.

$\varphi \in \mathcal{D}(\Omega)$

$\Downarrow \exists\ a>0$ s.t. $\operatorname{supp}\varphi \in B_a$

$\Downarrow$

$\operatorname{supp}\varphi^y \in B_a(y)$



Thus if $|y| > b+a$,

$\operatorname{supp} F \cap \operatorname{supp}\varphi^y = \emptyset$.

$\Downarrow$

$F(\varphi^y) = 0$

Denote $\Phi(y) = F(\varphi^y)$. Then $\operatorname{supp}\Phi \in B_{b+a}(0)$.

(2) Does $\Phi(y) \in C^\infty(\mathbb{R}^n)$?

Use induction.

Assume that $\Phi \in C^m(\mathbb{R}^n)$, ie. $\exists\ \partial^\alpha \Phi$, $|\alpha| \le m$.

ignore

Want $\Phi \in C^{m+1}(\mathbb{R}^n)$

Look for $\partial^\beta \Phi$, $|\beta| = m+1$

where $\underline{\beta} = \underline{\alpha} + \underline{e}_i$ where $\underline{e}_i = (0, 0, \dots, 1, \dots, 0)$ i^{th} place

$\Rightarrow \partial^\beta \Phi = \partial_i \partial^\alpha \Phi$.

$$= \lim_{s\to 0} \frac{\partial^\alpha \Phi(x+se_i) - \partial^\alpha \Phi(x)}{s}$$

We shall prove that $\underline{\partial^\alpha \Phi = F(\partial^\alpha \varphi^y)}$.

We use induction. Assume $\partial^\alpha \Phi = F(\partial^\alpha \varphi^y)$, $|\alpha| \leq m$.

Let us show that $\partial^\beta \Phi = F(\partial^\beta \varphi^y)$ if $|\beta| = m+1$

Let $\beta = \underline{\alpha} + \underline{e_i}$,

$\underline{e_i} = (0, \ldots, \overset{i^{th}\text{ place}}{1}, \ldots, 0)$

$$\partial_i \partial^\alpha \Phi \ \underset{\text{hypothesis}}{\overset{\text{inductive}}{=}} \ \partial_i F(\partial^\alpha \varphi^y)$$

need to prove this exists.

$$\partial_i F(\partial^\alpha \varphi^y) = \lim_{s \to 0} \frac{F(\partial^\alpha \varphi^{y+se_i}) - F(\partial^\alpha \varphi^y)}{s}$$

$$= \lim F\left(\frac{\partial^\alpha \varphi^{y+se_i} - \partial^\alpha \varphi^y}{s} \right) \quad \text{by linearity.}$$

Claim $\dfrac{\partial^\alpha \varphi^{y+se_i} - \partial^\alpha \varphi^y}{s} \xrightarrow{s \to 0} \partial^\beta \varphi^y$ in $\mathcal{D}(\mathbb{R}^n)$

$$\frac{\partial^\alpha \varphi(x+y+se_i) - \partial^\alpha \varphi(x+y)}{s} \xrightarrow{s \to 0} \partial^{\beta}_{\alpha+e_i} \varphi(x+y)$$

Let $\underline{\gamma}$ be an arbitrary multiindex

Want to show $\partial^\gamma \left[\dfrac{\partial^\alpha \varphi(x+y+se_i) - \partial^\alpha \varphi(x+y)}{s} \right] \xrightarrow{s \to 0} \partial^{\gamma+\beta} \varphi(x+y)$.

Let, for ease of notation, $f(x) = \partial^{\gamma+\alpha} \varphi(x+y) \in C_0^\infty(\mathbb{R}^n)$.

Then

$$\frac{f(x+se_i) - f(x)}{s} = \partial_i f(x)$$

$$= \partial_i \partial^{\gamma+\alpha} \varphi(x+y)$$

$$= \partial^{\gamma+\alpha+e_i} \varphi(x+y)$$

$$= \partial^{\gamma+\beta}\varphi(x+y) \qquad \square.$$

Due to claim,

$$F\left(\frac{\partial^\alpha\varphi^{y+se_i} - \partial^\alpha\varphi^y}{s}\right) \xrightarrow{s\to 0} F(\partial^{\alpha+e_i}\varphi^y) \quad \text{for all } y$$

$$\|$$

$$\frac{1}{s}\Big[F(\partial^\alpha\varphi^{y+se_i}) - \underbrace{F(\partial^\alpha\varphi^y)}_{\Phi(y)}\Big]$$

$$\|$$

$$\frac{\Phi(y+se_i) - \Phi(y)}{s}$$

Thus $\Phi(y)$ is differentiable and $\partial_i\Phi(y) = F(\partial^{\alpha+e_i}\varphi^y)$.

$$\partial_i\partial^\alpha F(\varphi^y) \underset{\substack{\text{induct.}\\ \text{hypoth.}}}{=} \partial_i F(\partial^\alpha\varphi^y) = F(\partial^{\alpha+e_i}\varphi^y)$$

$$\|$$

$$\partial^{\alpha+e_i}F(\varphi^y) \qquad \text{and so} \quad \partial^\alpha F(\varphi^y) = F(\partial^\alpha\varphi^y) \qquad |\alpha| \leq m$$

$\Rightarrow F(\varphi^y) \in \mathcal{D}(\Omega)$. Thus $G(F(\varphi^y))$ is well defined..

Is it linear? Trivial.

continuous? Also a pain, but same type of pain.

Let $\varphi_k \to \varphi$ in $\mathcal{D}$. We need

$(F*G)(\varphi_n) \to (F*G)\varphi_k \to (F*G)(\varphi)$

i.e. $G(F(\varphi_k^y)) \xrightarrow[k\to\infty]{} G(F(\varphi^y))$.

It is enough to show that $\Phi_k(y) \left(= F(\varphi_k^y)\right) \longrightarrow \Phi(y)^{\left(= F(\varphi^y)\right)}$ in $\mathcal{D}$

<u>Check support</u>: Let us show that $\mathrm{supp}(\Phi_k)$, $\mathrm{supp}(\Phi)$ lie in the same bowl.

As $\varphi_k \to \varphi$ in $\mathcal{D}$, $\exists\, a > 0$ s.t. $\left.\begin{matrix}\mathrm{supp}(\varphi_k), \\ \mathrm{supp}(\varphi)\end{matrix}\right\} \subset B_a$

Then $\mathrm{supp}(\Phi_k), \mathrm{supp}(\Phi) \subset B_{b+a}$

$(\mathrm{supp}\, F \subset B_b)$.

$$\underline{\partial^\alpha \Phi_k(y) \to \partial^\alpha \Phi(y)}$$
$$\begin{matrix} \| & & \| \\ F(\partial^\alpha \varphi_k^y) & & F(\partial^\alpha \varphi^y) \end{matrix}.$$

~~By Prop 1.2, since~~

Since $\varphi_k \to \varphi$ in $\mathcal{D}(\mathbb{R}^n) \Rightarrow \varphi_k^y \to \varphi^y$ in $\mathcal{D}(\mathbb{R}^n)$

$\overset{\text{Prop 1.2}}{\Longrightarrow} \partial^\alpha \varphi_k^y \to \partial^\alpha \varphi^y$ in $\mathcal{D}(\Omega)$ $\forall\, \underline{\alpha}$.

$\Rightarrow F(\partial^\alpha \varphi_k^y) \xrightarrow[k\to\infty]{} F(\partial^\alpha \varphi^y)$

<u>Propn. 1.25</u> Suppose $F_p \to F$ in $\mathcal{E}'(\mathbb{R}^n)$
$G_p \to G$ in $\mathcal{D}'(\mathbb{R}^n)$.
Then $F_p * G_p \to F * G$ in $\mathcal{D}'(\mathbb{R}^n)$

Not proved ~~it~~ $\because$ difficult.

Problem Extra* If $F_p \to F$ in $\mathcal{E}'(\mathbb{R}^n)$,
show $F_p * G \longrightarrow F * G$

$$(F*G)(\varphi) = F(\overset{\in C^\infty}{G(\varphi^y)})$$

has compact support

Lemma: $G * F = F * G$

(Observe: $F = F_f$ $G = F_g$. Obvious for from the f^ns:

$$G*F = F_h \qquad h(x) = \int g(x-y) f(y)\, dy$$

$$F*G = F_{\hat{h}} \qquad \hat{h}(x) = \int f(x-y)\, g(y)\, dy$$

)) equal)

Problem 1.27 Show that $(\partial^\alpha \delta) * G = \partial^\alpha G$

Note you can split derivatives.

Super flows to (†)

$$\partial^\alpha (G*F) = \partial^\beta G * \partial^\gamma F \qquad \forall \beta, \gamma \quad \beta + \gamma = \alpha$$

$G = F_g \in C^\infty$
$F = F_f \in C^\infty$

$G*F = F_h$
$h(x) = \int g(x-y) f(y)\, dy$

$\cancel{\partial^\alpha (G*F) = F_{\partial^\alpha h}}$

$$\partial^\alpha (G*F) = F_{\partial^\alpha h}$$

$$\partial^\alpha h(x) = \int \partial_x^\alpha g(x-y) f(y)\, dy \qquad [\alpha = \beta + \gamma]$$

$$= \int \partial_x^\beta (-1)^{|\gamma|} \partial_y^\gamma g(x-y)\, dy \qquad [\text{int. by parts}]$$

$$= \int \partial_x^\beta g(x-y) \partial^\gamma f(y)\, dy$$

$$\Rightarrow F_{\partial^\alpha h} = F_{\partial^\beta g} * F_{\partial^\gamma f}$$

<u>Prop^n 1.28</u> $\partial^\alpha(F * G) = F * \partial^\alpha G = \partial^\alpha F * G$.

$$\partial^\alpha(F * G)(\varphi) = (-1)^{|\alpha|}(F * G)(\partial^\alpha \varphi) \quad \text{by differentiation}$$

$$= (-1)^{|\alpha|} G(F(\partial^\alpha \varphi^y)) \quad \text{by def}^n \text{ of } *$$

$$\begin{aligned} F(\partial^\alpha \varphi^y) &= F(\partial_x^\alpha \varphi(x+y)) \\ &= F(\partial_y^\alpha \varphi(x+y)) \\ &= \partial_y^\alpha(F(\varphi^y)) \end{aligned}$$

like last lecture
when we proved $F(\varphi^y) \in C^\infty$
$\partial^\alpha F(\varphi^y) = F(\partial^\alpha \varphi^y)$

$$\Rightarrow \partial^\alpha(F * G)(\varphi) = (-1)^{|\alpha|} G(\partial^\alpha(F(\varphi^y)))$$

$$= (-1)^{|\alpha|} \partial^\alpha G(F(\varphi^y))$$

$$= (F * \partial^\alpha G) \quad \square$$

And since $F * G = G * F$, swapping F and G holds.

(f) <u>Also</u> $\partial^{\beta+\gamma}(F * G) = \partial^\beta F * \partial^\gamma G$. as a corollary.

<u>Recall Prop 1.25</u>: $F_p \to F$, $G_p \to G \Rightarrow F_p * G_p \to F * G$

We know smaller version, $F_f * F_g = F_{f*g} = F_{g*f} = F_g * F_f$.
if the distribs come from functions

The proposition is v. difficult to prove.

<u>Thm 1.29 (Density)</u> If we have $\Omega \subset \mathbb{R}^n$
$F \in \mathcal{D}'(\Omega)$,
$\exists$ a sequence $f_p \in C_0^\infty(\Omega)$
s.t.
$$F_{f_p} \xrightarrow[p\to\infty]{} F \text{ in } \mathcal{D}'(\Omega)$$

$$F * G = \lim_{p\to\infty} F_{f_p} * F_{g_p}$$ by approximating G by F_{g_p} (a thm tells us we can always find a seq. $F_{g_p} \to G \ \forall G$...)

$$= \lim \big(\underset{\substack{\downarrow \\ G}}{F_{g_p}} * \underset{\substack{\downarrow \\ F}}{F_{f_p}} \big) = G * F.$$

<u>Corollary</u>: If $F \in \mathcal{E}'(\Omega)$, then $F_{f_p} \to F$ in $\mathcal{E}'(\Omega)$.

<u>Sketch of proof</u>: (1) From $\mathcal{D}'(\Omega)$ to $\mathcal{E}'(\Omega)$

Let $K_n = \{ x \in \Omega : |x| \le n, \ d(x, \delta\Omega) \ge \frac{1}{n} \}$

$$\Omega = \bigcup_{n=1}^{\infty} K_n$$

Also $K_n \subset K_{n+1}^{int}$

By Prop^n 1.2, $X_n \in C_0^\infty(\Omega)$ $0 \le X_n$ s.t.
$X_n = 1$ in K_n
$X_n = 0$ outside K_{n+1}

Let $F \in \mathcal{D}'(\Omega)$, $F_n = X_n F$.

$\mathrm{Supp}(F_n) \subset K_{n+1}$.

Indeed, if $\varphi \in \mathcal{D}(\Omega)$ and $\mathrm{supp}(\varphi) \cap K_{n+1} = \emptyset$ then $F_n(\varphi) = (X_n F)(\varphi) = F(\underset{\overset{\shortparallel}{0}}{X_n\varphi}) = 0$.

Also $F_p \to F$. To prove this: $\forall \varphi \in \mathcal{D}(\Omega)$,

$$\underset{\overset{\shortparallel}{F(X_p\varphi)}}{F_p(\varphi)} \to F(\varphi).$$

Indeed, for large p, $K \subset K_p$.

Proof: $d(K, \delta\Omega) = d > 0$ as $K \subset\subset \Omega$. (is compact in)

As K-compact $K \subset B_R$, for some $R > 0$. Thus, for $p > \max(R, \frac{1}{d})$, $K \subset K_p$.

Any distribution can be approximated by distributions with compact support.

(2) $\forall F \in \underset{\text{dists. with compact support}}{\mathcal{E}'(\Omega)}$, there are $f_p \in C_0^\infty(\Omega)$ s.t.

$f_p \rightsquigarrow F$ in $\mathcal{E}'(\Omega)$

$F_{f_p} \to F$

$X(x) \in C_0^\infty(\mathbb{R}^n)$, $\quad X = X(|x|)$ radial $\quad X \geq 0$

$X = 0$ for $|x| > 1$

$\int X(x)\,dx = 1$

$X^\varepsilon(x) = \varepsilon^{-n} X\left(\frac{x}{\varepsilon}\right)$ ← has support in $B_\varepsilon(0)$.

$\int X^\varepsilon(x)\,dx = 1.$

[<u>Problem*</u>: Prove $\chi^\varepsilon \xrightarrow[\varepsilon\to 0]{} \delta$.]

$F^{\varepsilon} = \chi^\varepsilon * F \in \mathcal{E}'(\Omega)$ $\qquad \varphi \in \mathcal{D}(\Omega)$.

$(\chi^\varepsilon * F)(\varphi) = F(\chi^\varepsilon(\varphi^y))$ $\qquad$ by defⁿ of $*$.

$$F_{\chi^\varepsilon}(\varphi^y) := \int_{\Omega = \mathbb{R}^n} \chi^\varepsilon(x)\varphi(x+y)\,dx$$

$$\in C_0^\infty(\mathbb{R}^n)$$

$F^\varepsilon(\varphi) = (\chi^\varepsilon * F)(\varphi) = F(\chi^\varepsilon\varphi^y) = \int \chi^\varepsilon(x)\varphi(x+y)\,dx$

$\cdot\ \chi^\varepsilon(\varphi^y) \to \varphi(y)$.

Moreover, we can prove that $(\chi^\varepsilon\varphi^y) \to \varphi$ in $C^\infty(\Omega)$.

Call $\chi^\varepsilon(\varphi^y) = \psi^\varepsilon(y)$. Then

$$\psi^\varepsilon(y) = \int_{|x|\le\varepsilon} \chi^\varepsilon(x)\varphi(x+y)\,dx = \int_{|x|\le\varepsilon} \chi^\varepsilon(x)\varphi(y)\,dx = \varphi(y).$$

$$= \varphi(y) + \int_{|x|\le\varepsilon} \chi^\varepsilon(x)\left[\varphi(x+y)-\varphi(y)\right]dx$$

Observe $\sup_{|x|\le\varepsilon} |\varphi(x+y)-\varphi(y)| \xrightarrow[\varepsilon\to 0]{} 0$

$\Rightarrow [\varphi(x+y)-\varphi(y)]$ in integral $\longrightarrow 0$
uniformly for $y \in K \subset \Omega$.

$\chi^\varepsilon(\varphi^y)$

Therefore $\varphi^\varepsilon(y) \to \varphi(y)$ $\forall y \in \Omega$ (and uniformly on any compact).

Same for derivative, if we replace $\varphi^\varepsilon(y)$ by $\partial^\alpha \varphi^\varepsilon(y)$, we do the same trick,

so $\partial^\alpha \varphi^\varepsilon(y) \longrightarrow \partial^\alpha \varphi(y)$ $\forall y \in \Omega$ and α.

This means $\varphi^\varepsilon \to \varphi$ in $\mathcal{E}(\Omega) = C^\infty(\Omega)$

$F(\varphi^\varepsilon) \xrightarrow[\varepsilon\to 0]{} F(\varphi)$ since $F \in \mathcal{E}'(\Omega)$.

$F(\chi^\varepsilon(\varphi^y)) = (\chi^\varepsilon * F)(\varphi) \Rightarrow \chi^\varepsilon * F \longrightarrow F$ in $\mathcal{D}'(\Omega)$.

Note $\operatorname{supp}\left(\int \chi^\varepsilon(x)\, \varphi(x+y)\, dx\right) \subset \varepsilon$-vicinity of $\operatorname{supp}(\varphi)$

$F^\varepsilon(\varphi) = (\chi^\varepsilon * F)(\varphi) = F(\chi^\varepsilon(\varphi^y))$

Claim $\operatorname{supp}(\chi^\varepsilon * F) \subset \varepsilon$-vicinity of $\operatorname{supp}(F)$

$= \{x : d(x, \operatorname{supp} F) \leq \varepsilon\} = \operatorname{supp}(F)^\varepsilon$

Let $\operatorname{supp}(\varphi) \cap (\operatorname{supp} F)^\varepsilon = \emptyset$

$\iff d(\operatorname{supp}(\varphi), \operatorname{supp}(F)) > \varepsilon$

$\Rightarrow [\varepsilon \text{ vicinity of } \operatorname{supp}(\varphi)] \cap \operatorname{supp} F = \emptyset$.

Thus $\operatorname{supp}(\varphi^\varepsilon) \cap \operatorname{supp}(F) = \emptyset$.

$\Rightarrow F(\varphi^\varepsilon) = (\chi^\varepsilon * F)(\varphi) = 0$. $\square$

This proves that $\operatorname{supp}(\chi^\varepsilon(\varphi^y))$ is compact

i.e. $\chi^\varepsilon(\varphi^y) \in \mathcal{E}' \Rightarrow \chi^\varepsilon * F \to F$ in $\mathcal{D}'(\Omega)$

Therefore, $\operatorname{supp}(X^{\varepsilon} * F) \subset$ ε_0-vicinity of $\operatorname{supp}(F)$
if $\varepsilon < \varepsilon_0$.

Thus $X^{\varepsilon} * F \to F$ in $\mathcal{E}'(\Omega)$ if $\varepsilon \le \varepsilon_0$
$= \frac{1}{2} d[\operatorname{supp}(F), \sigma\Omega]$

<u>Claim</u>: $X^{\varepsilon} * F \in C_0^{\infty}(\Omega)$ $\quad$ [when $\varepsilon < \frac{1}{2} d[\operatorname{supp}(F), \sigma\Omega]$]

ie. $\exists$ a f^{n} $h^{\varepsilon} \in C_0^{\infty}(\Omega)$ s.t. $X^{\varepsilon} * F = F_{h^{\varepsilon}}$

Sketch of proof: Let $F = F_f$. Then $(X^{\varepsilon} * F_f)(\varphi)$

$$= F(X^{\varepsilon}(\varphi^{y}))$$

$$= \int f(y) \left[\int X^{\varepsilon}(x) \varphi(x+y) dx \right] dy$$

let $z = x+y$
$\Rightarrow x = z-y$

$$= \int \varphi(z) \left[\int f(y) X^{\varepsilon}(z-y) dy \right] dz$$

$= F_{h^{\varepsilon}}(\varphi)$, where

$$h^{\varepsilon}(z) = \int f(y) X^{\varepsilon}(z-y) dy.$$

because we take abs. value $\quad = \int f(y) X^{\varepsilon}(y-z) dy$

$$= F_f(X^{\varepsilon - z})$$

$\varphi^{y} = \varphi(x+y)$
$\varphi^{-z} = \varphi(x-z)$.

$$(X^\varepsilon * F)(\varphi) = F_{h^\varepsilon}(\varphi)$$

$$\overset{\shortparallel}{R^\varepsilon}(z) = F(X^{\varepsilon - z}).$$

$$F(\varphi^y) \in C^\infty(\mathbb{R}^n) \Rightarrow F(\varphi^{-y}) \in C^\infty(\mathbb{R}^n)$$

Thus $F(X^{\varepsilon-z}) \in C_0^\infty(\mathbb{R}^n)$.

$$\psi^\varepsilon(y) = \int X^\varepsilon(x)\varphi(x+y)\,dx = \int_K X^\varepsilon(z-y)\varphi(z)\,\underset{\overset{\cap}{C^\infty(\Omega)}}{dz}.$$

$$= \lim_{\delta\to 0} \sum_i \underbrace{X^\varepsilon(z^i - y)\varphi(z^i)\delta^n}_{= R^\varepsilon_\delta(y)} \qquad \text{in } C^\infty(\Omega).$$

$$\partial^\alpha \psi^\varepsilon(y) = \int_K (-1)^{|\alpha|}\partial^\alpha X(z-y)\varphi(z)\,dz$$

$$= (-1)^{|\alpha|} \lim_{\delta\to 0} \sum \partial^\alpha X^\varepsilon(z^i - y)\varphi(z^i)\delta^n$$

Thus $\psi^\varepsilon = \lim_{\delta\to 0} R^\varepsilon_\delta$ in $C^\infty(\Omega)$

$$\Rightarrow F(\psi^\varepsilon) = \lim F(R^\varepsilon_\delta)$$

'We can take all these guys out'

$$= \lim_{\delta\to 0} F\left(\sum X^\varepsilon(z^i - y)\varphi(z^i)\delta^n\right)$$

$$= \lim_{\delta\to 0} \sum \left(F(X^{\varepsilon - z^i})\right)\varphi(z^i)\delta^n$$

$$= \int F(X^{\varepsilon - z})\varphi(z)\,dz$$

$$(X^\varepsilon * F)(\varphi) = F_{h^\varepsilon}(\varphi), \qquad \underset{\overset{\cap}{C_0^\infty(\Omega)}}{h^\varepsilon(z)} = F(X^{\varepsilon - z}).$$

Stuff missing!!

Integration of distrbs wrt τ

Let F_τ, $\tau \in A \subset \mathbb{R}^d$ is a distribution continuously depending on parameter τ, $F_\tau \in \mathcal{D}(\Omega)$ and $\tau = (\tau_1, \dots, \tau_d)$

If $\tau \to \tau_0$, then $F_\tau \xrightarrow[\mathcal{D}']{} F_{\tau_0}$.

Let $\rho \in C_0(A)$ [recall means cont.s f.n with compact support]

Then let $G_\rho = \int_A \rho(\tau) F_\tau \, d\tau \in \mathcal{D}'(\Omega)$

then $G_\rho(\varphi) = \int_A \rho(\tau) F_\tau(\varphi) \, d\tau$

We can check that if $F_\tau = F_{f(\tau)}$ where $f(\tau) = f(x,\tau) \in C(\Omega \times A)$

then $G_\rho = F_h$, $h(x) = \int_A \rho(\tau) f(x,\tau) \, d\tau$

Let's consider $\Omega = \mathbb{R}^n$ and $A = \mathbb{R}^n$.

Def.n: For $F \in \mathcal{D}'(\mathbb{R}^n)$, $y \in \mathbb{R}^n$, we define the translation of F by y (notated $T_y F$)

If $F = F_f$, then $T_y F = F_{T_y F}$ $(T_y F)(x) = f(x-y)$

Observe that $F_{T_y F}(\varphi) = \int \varphi(x) f(x-y) \, dx$

$$= \int f(x) \varphi(x+y) \, dx$$

$$= F_f(\varphi^y)$$

$\underline{(T_y F)(\varphi) = F_f(\varphi^y)}$ ← Definition of translation

Example: $T_y \delta = \delta_y$:

$$(T_y\delta)(\varphi) = \delta(\varphi^y) = \varphi^y(0) = \varphi(y) = \delta_y(\varphi)$$

Notation: $T_y F = F(\cdot - y)$

Observe that $T_y F := F_y$ is a distribution in $\mathcal{D}'(\mathbb{R}^n)$, which depends continuously on y.

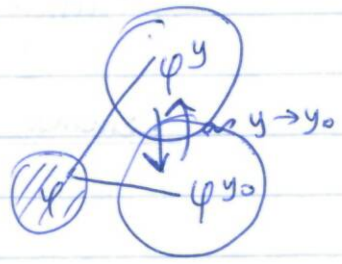
$$T_y F \to T_{y_0} F \text{ as } y \to y_0$$
$$(T_y F)(\varphi) = F(\varphi^y) \xrightarrow[?]{} (T_{y_0}F)(\varphi) = F(\varphi^{y_0})$$

This is true $\because$ $\varphi^y \to \varphi^{y_0}$ in $\mathcal{D}(\mathbb{R}^n)$ if $y \to y_0$

$$\varphi(x+y) \xrightarrow[\mathcal{D}(\mathbb{R}^n)]{} \varphi(x+y_0)$$

need to do this to show it goes in $\mathcal{D}(\mathbb{R}^n)$ Clearly $\partial^\alpha \varphi(x+y) \to \partial^\alpha\varphi(x+y_0)$ $\forall \alpha$ $\forall x \in \mathbb{R}^n$ when $y \to y_0$

The supports lie in the same compact



Let $g \in C_0(\mathbb{R}^n)$. Then we can

$$G_g = \int_{\mathbb{R}^n} g(y)(T_y F)\, dy$$

Problem: $G_g = F_g * F$

Laplace operator and Green's function

$$DF = \sum_{|\alpha| \leq m} \psi_\alpha(x)\, \partial^\alpha F \qquad \psi_\alpha \in C^\infty(\Omega)$$
$$F \in \mathcal{D}'(\Omega)$$

Can we solve the eq^n $DF = \delta_y$?

If we can we have a distribution $G_y \in \mathcal{D}'(\Omega)$ called Green's function

Typically G_y depends continuously on y

Recall the Laplacian in $\mathbb{R}^n$ (or $\Omega \subset \mathbb{R}^n$)

$$\Delta = -\sum_{i=1}^{n} \partial_i^{\,2}$$

Green's f^n for the Laplacian is also called the fundamental solution and denoted $\Phi_y(x) = G_y$

Lemma 1.38 $\quad \Phi_y(x) = \dfrac{1}{4\pi|x-y|}$ is the fundamental sol^n for $\Omega = \mathbb{R}^3$.

i.e. need to prove $\Delta\Phi = \delta_y$.

Proof: $(\Delta\Phi_y)(\varphi) = \Phi_y(\Delta\varphi)$

$$= \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\Delta\varphi(x)}{|x-y|}\,dx$$

$$= \frac{1}{4\pi}\lim_{\varepsilon\to 0}\int_{\mathbb{R}^3\setminus B^\varepsilon(y)} \frac{\Delta\varphi(x)}{|x-y|}\,dx$$

Since $\frac{1}{4\pi|x-y|}$ is smooth in $\mathbb{R}^3 \setminus B_\varepsilon(y)$ we can integrate by parts in the above integral.

Recall Green's second identity from 1402:

$$\int_V \Delta f g = \int_V f \Delta g + \int_S \left(\frac{\partial f}{\partial n} g - f \frac{\partial g}{\partial n} \right) dS$$

Using this we get

$f = \varphi$
$g = \frac{1}{|x-y|}$

$$\lim \left[\frac{1}{4\pi} \int_{\mathbb{R}^3 \setminus B_\varepsilon} \varphi(x) \underbrace{\Delta_x \left(\frac{1}{|x-y|} \right)}_{0} dx \right.$$

why

$$\left. - \frac{1}{4\pi} \int_{S_\varepsilon} \left(\frac{\partial \varphi}{\partial n} \cdot \frac{1}{|x-y|} - \varphi \frac{\partial}{\partial n} \left(\frac{1}{|x-y|} \right) \right) dS_x \right]$$

S_ε: sphere of rad. ε

dS_x: y is fixed

Let $x \to z = x - y$
$z \to (\rho, \theta, \varphi)$ spherical coords

Observe $\frac{\partial}{\partial n} = -\frac{\partial}{\partial \rho}$.

$$\Rightarrow = \frac{1}{4\pi} \lim_{\varepsilon \to 0} \int_{\rho = \varepsilon} \left[+ \frac{\partial \varphi}{\partial \rho} (y + \rho\omega) \frac{1}{\varepsilon} \cancel{+} \varphi \underbrace{\frac{\partial}{\partial \rho} \left(\frac{1}{\rho} \right) \Big|_{\rho = \varepsilon}}_{-\frac{1}{\varepsilon^2}} \right] dS_\varepsilon$$

where ω is a unit vector made of θ, φ
$\omega = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$

$$\text{So } (\Delta \Phi_y)(\varphi) = \frac{1}{4\pi} \lim_{\varepsilon \to 0} \underbrace{\int_{\rho = \varepsilon}}_{\iint_{\substack{0 \le \theta \le \pi \\ 0 \le \varphi \le 2\pi}}} \left(+ \frac{\partial \varphi}{\partial \rho} \cdot \frac{1}{\varepsilon} + \frac{1}{\varepsilon^2} \varphi(y + \rho\omega) \right) \cdot \underbrace{\varepsilon^2 \sin\theta \, d\theta \, d\varphi}_{dS_\varepsilon}$$

Now note (1) $\iint \underset{\uparrow \text{bounded}}{\frac{\partial \varphi}{\partial \rho}} \frac{1}{\varepsilon} \varepsilon^2 \sin\theta \, d\theta \, d\varphi \xrightarrow[\varepsilon \to 0]{} 0$

and (2) $\frac{1}{4\pi} \iint_{\theta,\varphi} \varphi(y + \varepsilon\omega) \sin\theta \, d\theta \, d\varphi$ ($\rho = \varepsilon$ on the sphere)

$$\downarrow$$

$$\varphi(y)$$

$$\xrightarrow[\varepsilon \to 0]{} \frac{1}{4\pi} \varphi(y) \underbrace{\iint_{\theta,\varphi} \sin\theta \, d\theta \, d\varphi}_{\substack{= \text{vol. of sphere of} \\ \text{radius } 1 \; = 4\pi}}$$

$\square$

So we have shown $-\Delta\left(\frac{1}{4\pi|x-y|}\right) = \delta_y$

Want to let $y=0$: $-\Delta\left(\frac{1}{4\pi|x|}\right) = \delta$?

$$-\Delta\left(T_y\left(\frac{1}{4\pi|x|}\right)\right) = T_y \delta$$

Noting $T_y \circ \partial^\alpha = \partial^\alpha \circ T_y$

$$T_y\left(-\Delta \frac{1}{4\pi|x|}\right) = T_y \delta = \delta_y$$

So it's sufficient to prove $-\Delta\left(\frac{1}{4\pi|x|}\right) = \delta$ and use commutativity to solve $-\Delta\left(\frac{1}{4\pi|x-y|}\right) = \delta_y$.

Why is knowledge of Green's f^n such a useful thing?

Say we want to solve ~~ΔF =~~
an operator with constant coeffs

$$D = \sum_{|\alpha| \leq m} c_\alpha \partial^\alpha$$

Assume we know G s.t. $DG = \delta$ (Green's f^n)

$DF = H \in \mathcal{E}'(\mathbb{R}^n)$ sol^n $F = G * H$

Recall if $DF = \sum c^\alpha \partial^\alpha (G * H)$

$$= \underbrace{\left(\sum c^\alpha \partial^\alpha G\right)}_{\delta} * H = \delta * H = H$$

So if you ever want to solve $-\Delta \cdot F = \rho$ in $\mathbb{R}^3$ or $\Omega \subset \mathbb{R}^3$

Then $F = \dfrac{1}{4\pi|x|} * \rho$

Observe that if $\rho \in C_0(\mathbb{R}^3)$

$$F(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\rho(y)}{|x-y|}\, dy$$

If we have an electrostatic potential

$(q_i, y_i) \quad i=1,\dots,p$

$$\Phi^t(x) = \sum_{i=1}^{p} \frac{q_i}{4\pi\varepsilon_0(x-y_i)}$$

$$E^t(x) = -\nabla \Phi^t$$

$$\rho = \sum_{i=1}^{p} q_i \delta_{y_i}$$

$$\Phi * \frac{\rho}{\varepsilon_0} = \frac{1}{4\pi|x|} * \frac{1}{\varepsilon_0}\sum_{i=1}^{p} q_i\delta_{y_i} = \frac{1}{\varepsilon_0}\sum_{i=1}^{p} \underbrace{\frac{1}{4\pi|x|} * q_i\delta_{y_i}}_{\frac{q_i}{4\pi|x-y_i|}}$$

Recall $\cancel{\delta_y * H = H}$

$$(\delta_z * H)(\varphi)$$

since $\varphi^y(z) = \varphi^z(y)$

$$\underset{\varphi^y(z)}{\overset{}{H(\delta_z\varphi^y)}} = H(\varphi^y(z)) = H(\varphi^z) = (T_z H)(\varphi)$$

So we get $\delta_z * H = T_z H$

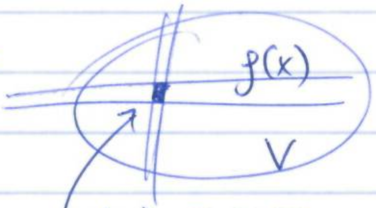
So $\Phi^t(x) = \Phi * \frac{1}{\varepsilon_0}\left(\sum_{i=1}^{p} q_i\delta_{y_i}\right)$

What is the electrostatic potential due to a distribution $\rho \in \mathcal{E}'(\mathbb{R}^3)$ of charges

$$\Phi^\rho = \Phi * \frac{\rho}{\varepsilon_0}$$

Remember triple integrals

cubes:



$\rho(x_i)\,\delta x \delta y \delta z$ ← inside this point there is approx. this charge

$$\frac{\rho(x_i)\,\delta x \delta y \delta z}{4\pi\varepsilon_0 |x - x_i|}$$

Take $\displaystyle \lim_{\substack{\delta x\\ \delta y \to 0\\ \delta z}} \sum_i \frac{\rho(x_i)\,\delta x \delta y \delta z}{4\pi\varepsilon_0 |x - x_i|}$

$$= \frac{1}{4\pi\varepsilon_0} \int_{\substack{V\\ \| \\ \mathbb{R}^3}} \frac{\rho(y)}{|x-y|}\,dy = \frac{1}{4\pi|x|} * \frac{\rho}{\varepsilon_0}$$

$$= \Phi * \frac{\rho}{\varepsilon_0} = \Phi^\rho$$

$$-\nabla^2 \Phi^\rho = \nabla^2 \Phi * \frac{\rho}{\varepsilon} = \delta * \frac{\rho}{\varepsilon} = \frac{\rho}{\varepsilon}$$

And $-\nabla^2 \Phi^\rho = \dfrac{\rho}{\varepsilon_0}$ Poisson eqn.

$E^\rho = -\nabla \Phi^\rho$

electric field due to particle distribution ρ

Gauss' law

$$\left.\begin{aligned} \nabla \cdot E^\rho &= \frac{\rho}{\varepsilon_0} \\ \nabla \times E^\rho &= 0 \end{aligned}\right\} \text{Fundamental eqns of electrostatics!!}$$

NFE

$\rho \in \mathcal{E}'$, $\quad \rho_p \in C_0(\mathbb{R}^3)$

$\rho_p \to \rho$ in $\mathcal{E}'(\Omega)$

$$\Phi^{\rho_p} = \Phi * \frac{\rho_p}{\varepsilon_0} \longrightarrow \Phi * \frac{\rho}{\varepsilon_0}$$

Green's

$$-\nabla^2 \Phi = \delta_1 \quad \Phi = \frac{1}{4\pi|x|}, \quad \mathbb{R}^3$$

If $\rho \in C_0(\mathbb{R}^3)$ – cont. charge densities with compact support.

$\Phi * \frac{\rho}{\varepsilon_0} = \Phi^\rho$ – electrostatic potential of the charge density ρ.

$$\Phi^\rho(x) = \frac{1}{4\pi\varepsilon_0} \int_{\mathbb{R}^3} \frac{\rho(y)}{|x-y|}\, dy \qquad -\nabla^2 \Phi^\rho = \frac{\rho}{\varepsilon_0}$$

If $\rho \in \mathcal{E}'(\mathbb{R}^3)$, $\quad \Phi^\rho = \frac{1}{\varepsilon_0} \Phi * \rho$

$$\Phi^q_y = \frac{q}{4\pi\varepsilon_0 |x-y|}$$

$$E^\rho = -\nabla \Phi^\rho = -\nabla \Phi * \frac{\rho}{\varepsilon_0}$$

$$-\nabla \Phi = \frac{x}{4\pi|x|^3}$$

$$E^\rho(x) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(y)(x-y)}{|x-y|^3}\, dy$$

3D vector

If ρ is cts this integral is cvgt.

Problem: $F \in \mathcal{D}'(\Omega) \qquad \Omega \subset \mathbb{R}^3$

$\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right) = \nabla F \in \mathcal{D}'(\Omega)^3 \qquad$ 3D vector distribution

$$\nabla \times \nabla F \stackrel{?}{=} 0$$

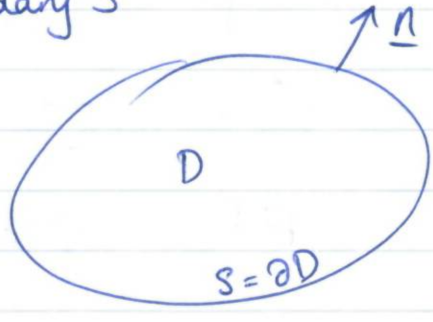
$$\nabla \cdot (\nabla F) = \nabla^2 F$$

$$
\begin{aligned}
\nabla \cdot E^\rho &= -(\nabla \cdot \nabla \Phi^\rho) && \text{by defn of } E^\rho \\
&= -\nabla^2 \Phi^\rho \\
&= \frac{\rho}{\varepsilon_0} && \text{Gauss' law in differential form}
\end{aligned}
$$

$$
\begin{aligned}
\nabla \times E^\rho &= -\nabla \times (\nabla \Phi^\rho) \\
&= -\nabla \times \left(\nabla \Phi * \frac{\rho}{\varepsilon}\right) \\
&= \underbrace{(-\nabla \times \nabla \Phi)}_{0} * \frac{\rho}{\varepsilon} \\
&= 0. && \text{ie. each component is zero.}
\end{aligned}
$$

basic laws of electro-statics

Now, if we have a domain D with boundary S



Write down divergence thm:

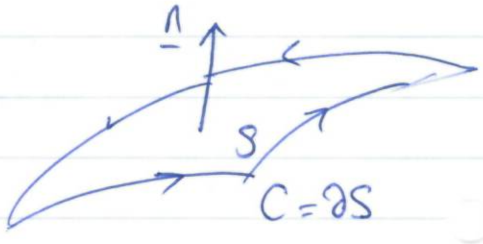
$$\int_S \underline{E}^\rho \cdot \underline{n} \, dS = \int_D \underbrace{\nabla \cdot E^\rho}_{\parallel \atop \frac{\rho}{\varepsilon_0}} dV$$

$$= \frac{1}{\varepsilon_0} \int_D \rho \, dV$$

$$\parallel$$

total charge in D $\longrightarrow$ $\dfrac{Q(D)}{\varepsilon_0}$

(Gauss' law in integral form)



P.C.

Problem Set $\rho = \rho(r)$. Find E^s then Φ^s.

Because of spherical symmetry

$$E = E(r) = \mathcal{E}(r)\underline{e}_r$$ unit vec. in radial dir?.

Kav! where you going!

By Gauss law in integral form, we have

$$\iint_{S_r} E\cdot\underline{n}\,dS = \frac{1}{\varepsilon_0} Q(B_r)$$

$$= \frac{1}{\varepsilon_0}\int_{B_r} \rho\, dV$$

$$= \frac{1}{\varepsilon_0}\underbrace{\int_0^r \rho(r')(r')^2\,dr'}_{R(r)}\underbrace{\int_{S^2} d\omega}_{4\pi} = \frac{4\pi}{\varepsilon_0}R(r)$$

$\sin\theta\, d\theta\, d\phi$

25/3

But $\iint_{S_r} E\cdot\underline{n}\,dS = \iint E\cdot\underline{r}\,dS = \mathcal{E}_r\iint_{S_r} dS = 4\pi r^2\mathcal{E}(r)$

$$\rightarrow \mathcal{E}(r) = \frac{R(r)}{\varepsilon_0 r^2} \xrightarrow[r\to 0]{} 0 \quad \text{as } R(r)\sim r^3$$

Now to find Φ^s.

$$-\nabla\Phi(r) = E.$$

$$\rightarrow \Phi(r) = \int_r^\infty \mathcal{E}(r)\,dr \qquad \text{because normally } -\int_\infty^r \mathcal{E}(r)\,dr$$

(set potential to be 0 at infinity

$$\Phi(r) \xrightarrow[r\to\infty]{} 0 \qquad \| \quad \frac{1}{\varepsilon_0}\int_r^\infty \frac{R(r')}{(r')^2}\,dr'$$

We can use integration by parts on $\int_r^\infty \frac{R(r')}{r'^2} \, dr'$

$\because$ R is itself an integral.

sec ~~First~~ $= \frac{1}{r^2}$

~~sec~~ first $= R(r)$

$$= \left[R(r) \cdot \left(-\frac{1}{r} \right) \right]_r^\infty + \int_r^\infty R'(r) \frac{1}{r} \, dr$$

$R' = \cancel{\rho r^2}$ ρr^2

$$= \frac{R(r)}{r} + \int_r^\infty \rho(r') \, r' \, dr'$$

P.C.

1.71 Q^n: $f \in C(\mathbb{R}^n)$ satisfies

$$\int_{\mathbb{R}^n} f(x)\,dx = 1 \qquad \int_{\mathbb{R}^n} |f(x)|\,dx < \infty .$$

Let $f_p(x) = p^n f(px) \quad p = 1, 2, \ldots$

Show $f_p \to \delta$ as $p \to \infty$ in $\mathcal{D}'(\mathbb{R}^n)$.

Ans: $F_{f_p}(\varphi) = p^n \int_{\mathbb{R}^n} f(px)\varphi(x)\,dx$

$p^n\,d\underline{x} = d\underline{y}$

$\underline{y} = p\underline{x} \Rightarrow \quad = \int_{\mathbb{R}^n} f(y)\varphi\left(\frac{y}{p}\right)\,dy \qquad \because \underline{x}$ is n-dim vector

$$= \underbrace{\int_{\mathbb{R}^n} f(y)\varphi(0)\,dy}_{=\varphi(0) \text{ since } \int f(y)dy = 1} + \int_{\mathbb{R}^n} f(y)\Big[\underbrace{\varphi\left(\tfrac{y}{p}\right) - \varphi(0)}_{\to 0 \text{ pointwise}}\Big]\,dy$$

Pointwise is OK but we really need uniformly.

Want to show $\forall \varepsilon\ \exists p(\varepsilon)$ s.t. $p > p(\varepsilon)$, $\left|\int_{\mathbb{R}^n} f(y)\left[\varphi\left(\frac{y}{p}\right) - \varphi(0)\right]dy\right| < \varepsilon$

$$= \int_{B_A} + \int_{\mathbb{R}^n \setminus B_A} .$$

But $\int_{\mathbb{R}^n} |f(x)|\,dx = C < \infty \Rightarrow \forall \delta\ \exists A(\delta)$ s.t. $\int_{\mathbb{R}^n \setminus B_A} |f(x)|\,dx < \delta$

Let $M = \max|\varphi(y)|$

let $\delta = \frac{\varepsilon}{4\cancel{2}M}$

$$\cancel{A(\delta)} = \int_{\mathbb{R}^n \setminus B_A} \left| f(y)\left[\varphi\left(\tfrac{y}{\rho}\right) - \varphi(0)\right]\right| dy$$

$$\leq 2M \int_{\mathbb{R}^n \setminus B_A} |f(y)|\, dy \leq \varepsilon/2$$

Look for

$$\underbrace{\int_{B_A} |f(y)|\left|\varphi\left(\tfrac{y}{\rho}\right) - \varphi(0)\right| dy}$$

$$\left|\frac{y}{\rho}\right| \leq \frac{A}{\rho} \xrightarrow[\rho\to\infty]{} 0$$

So $\left|\varphi\left(\frac{y}{\rho}\right) - \varphi(0)\right| \leq \frac{\varepsilon}{2C}$ if $\rho \geq \rho(\varepsilon)$

So

$$\leq \frac{\varepsilon}{2C}\int_{\mathbb{R}^n} |f(y)|\, dy = \frac{\varepsilon}{2}.$$

$$0 = \int_S (\nabla \times E^\rho)\cdot \underline{n}\, dS = \oint_C E^\rho \cdot dr$$

[Stoke's law]

recall only valid for simply connected domains

Problem: $\rho = \rho(r)$ $r^2 = x^2+y^2+z^2 = |\underline{x}|^2$

i.e. assume charge distribution is f^n of r alone

Find $\Phi^\rho(\underline{x})$ in terms of $\rho(r)$. and find E^ρ to get it

So $\Phi^\rho = \Phi^\rho(r)$ by symmetry since $\rho = \rho(r)$.

(PET) Problem*: $\rho = \rho(d)$ in cylindrical coords (d, φ, z)
$d = x^2+y^2$

Find E^ρ (use Gauss' law)

<u>Dipoles</u>

If we have charge distribution $\underline{p}\cdot\nabla\delta_y$ ($\underline{p} = p\underline{u}$ ← unit vector)

$$\underline{p}\cdot\nabla\delta_y = p\left(\frac{\partial\delta_y}{\partial x}u_1 + \frac{\partial\delta_y}{\partial y}u_2 + \frac{\partial\delta_y}{\partial z}u_3\right)$$

$$= p\frac{\partial\delta_y}{\partial u} \quad \text{directional derivative}$$

$\underline{p}\cdot\nabla\delta_y$ is called the dipole of size p in the direction $\underline{u}$ at the point y.

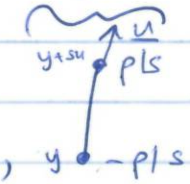
(What is the directional derivative of an arbitrary f^n f?

$$\frac{\partial}{\partial \underline{u}} f(\underline{x}) = \lim_{s\to 0}\frac{f(\underline{x}+s\underline{u}) - f(\underline{x})}{s}$$)

$$\text{So } p\frac{\partial\delta_y}{\partial \underline{u}} = \lim_{s\to 0}\frac{p\,\delta_{y+s\underline{u}} - p\,\delta_y}{s}$$

$$= \lim_{s\to 0}\left(\frac{p}{s}\delta_{y+su} - \frac{p}{s}\delta_y\right)$$

overall charge 0



If at the point $y+su$ we put charge p/s and at the point y we put charge $-p/s$, this is dipole.

then Φ^s ← electrostatic potential due to $\left(\frac{p}{s}\delta_{y+su} - \frac{p}{s}\delta_y\right)$ ← the two pt charges is

$$\Phi^s_{p,y} \;//\; \Phi^s = \frac{p}{\varepsilon_0 s}\left[\frac{1}{4\pi|x-(y+su)|} - \frac{1}{4\pi|x-y|}\right]$$

$$\lim_{s\to 0}\Phi^s(x) = \frac{p}{\varepsilon_0}\frac{\partial}{\partial \underline{u}_y}\Phi_y(x)$$

$$= -\frac{p}{\varepsilon_0}\frac{\partial}{\partial \underline{u}_x}\Phi_y(x)$$

Electrostatic potential

But we know that $\Phi^{d}_{\underline{p},y}$ ← dipole $= \frac{1}{\varepsilon_0}\Phi * (\underline{p}\cdot\nabla\delta_y)$

dipole, size p dir^n $\underline{u}$ sitting on the point y.

$$= \frac{1}{\varepsilon_0}\underline{p}\cdot\nabla(\Phi * \delta_y)$$

$$= \frac{1}{\varepsilon_0}\underbrace{(\underline{p}\cdot\nabla\Phi)}_{p\frac{\partial\Phi}{\partial \underline{u}}} * \delta_y$$

← convolution with δ f^n mean take at a point y

<u>Problem</u>: Evaluate $\Phi^d_{\underline{p},y}$ and $E^d_{\underline{p},y}$

OK. Recall $F_{h,S}(\varphi) = \int_S h(x)\,\varphi(x)\,dS_x$ $\qquad h \in C_0(S)$

h-surface charge distribution

$\Phi_{h,S}$ $\qquad E_{h,S}$

Want to prove $\Phi_{h,S}(x) = \frac{1}{\varepsilon_0}\,\Phi * F_{h,S} = \frac{1}{4\pi\varepsilon_0}\int_S \frac{h(y)}{|x-y|}\,dS_y$. $\qquad \dots (*)$

$\frac{1}{4\pi\varepsilon_0|x|}$

$\Rightarrow$ … $\varphi^y))$

not right

and … $)\,dS_x$ ← y is fixed $\quad \in C_0^\infty(\mathbb{R}^3)$

What are we showing here?

$\Rightarrow$ … $\left[\int_S h(x)\,\varphi(x+y)\,dS_x\right] dy$

… $\int_S$ … $\left[\int_{\mathbb{R}^3} \frac{\varphi(x+y)}{|y|}\,dy\right] dS_x$

$$= \frac{1}{4\pi\varepsilon_0}\int_S h(x)\left[\int_{\mathbb{R}^3}\frac{\varphi(z)}{|x-z|}\,dz\right] dS_x$$

$$= \int_{\mathbb{R}^3} \varphi(z)\underbrace{\left[\frac{1}{4\pi\varepsilon_0}\int \frac{h(x)}{|x-z|}\,dS_x\right]}_{f} dz$$

$= F_f(\varphi)$. $\qquad f = \Phi_{h,S}(x)$ in (*) above !!

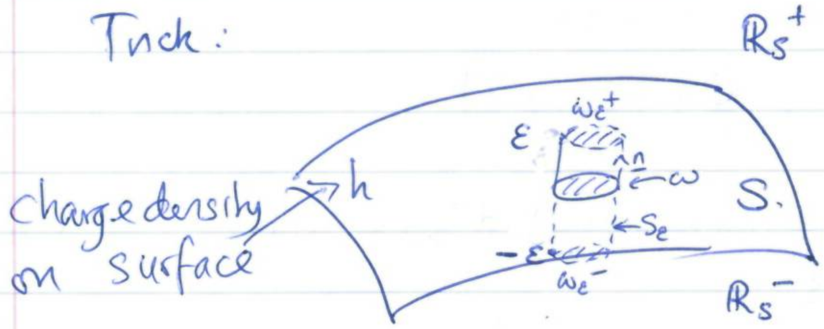
$\square$

how??

Note that $\Phi_{h,S}$ is continuous across S.
However, $E_{h,S}$ has a jump discontinuity across S.

To find E we need to take ∇ of Φ, and $\frac{h(y)}{|x-y|}$ becomes like $\sim \frac{1}{r^2} r\, dr$ and we have problems. Hopefully explained now:

Trick:



Consider $C_\omega^\varepsilon = \omega \times (-\varepsilon, \varepsilon)\underline{n}$

cylinder

Write Gauss law in integral form for this cylinder.

$$\int_{\partial C_\omega^\varepsilon} [E_{h,S} \cdot \underline{\nu}]\, dS = \frac{Q(C_\omega^\varepsilon)}{\varepsilon_0}$$

normal to $\partial C_\omega^\varepsilon$

$$= \frac{1}{\varepsilon_0} \int_\omega h\, dS$$

because only charge is in the disc

Now let's take a look at LHS:

$\omega_\varepsilon^\pm = \{\omega \pm \varepsilon \underline{n}\}$, the top and bottom of cylinder

$S_\varepsilon = \{\partial\omega \times (-\varepsilon, \varepsilon)\underline{n}\}$, the surface of the cylinder

$$\rightarrow \int_{\partial C_\omega^\varepsilon} = \int_{\omega_\varepsilon^+} + \int_{\omega_\varepsilon^-} + \int_{S_\varepsilon}$$

Take $\varepsilon \to 0$.

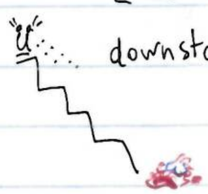
$$\int_{S_\varepsilon} E \cdot \underline{\nu}\, dS \longrightarrow 0 \quad \text{since } \operatorname{Area}(S_\varepsilon) \to 0.$$

$$\int_{\omega_\varepsilon^+} E \cdot \underline{\nu}\, dS = \int_\omega \left[E(x+\varepsilon \underline{n}) \cdot \underline{n} \right] dS_x$$

$$\xrightarrow[\varepsilon \to 0^+]{} \int_\omega \lim_{\varepsilon \to 0} E(x+\varepsilon \underline{n}) \cdot \underline{n}\, dS_x$$

$$= \int_\omega E^+(x) \cdot \underline{n}\, dS_x$$

$\underline{u}$ looks downstairs!



$$\int_{\omega_\varepsilon^-} E \cdot \underline{\nu}\, dS = \int_\omega \left[E(x-\varepsilon \underline{n}) \cdot (-\underline{n}) \right] dS_x$$

$$\longrightarrow -\int_\omega E^-(x) \cdot \underline{n}\, dS_x$$

$$\Rightarrow \lim_{\varepsilon \to 0} \int_{\partial C_\omega^\varepsilon} E_{h,S} \cdot \underline{\nu}\, dS = \int_\omega \left[E^+(x) - E^-(x) \right] \cdot \underline{n}\, dS$$

$$\overset{\text{told}}{=} \frac{1}{\varepsilon_0} \int h(x)\, dS \quad (\text{the RHS of the eq}^{\underline{n}})$$

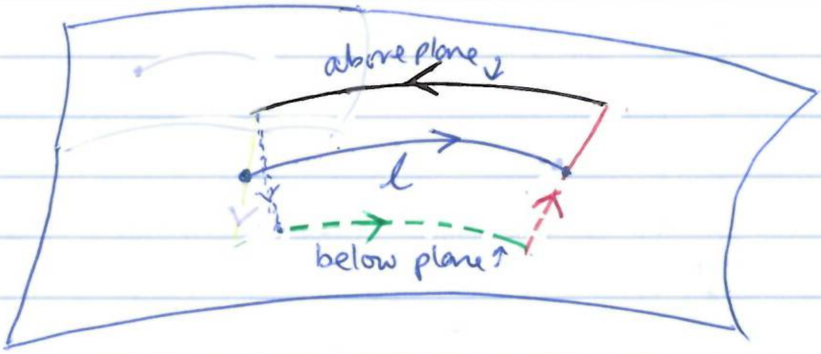
$$\Longrightarrow \underbrace{\left[E^+(x) - E^-(x) \right] \cdot \underline{n}}_{\text{jump of normal component across } S} = \frac{h(x)}{\varepsilon_0}$$

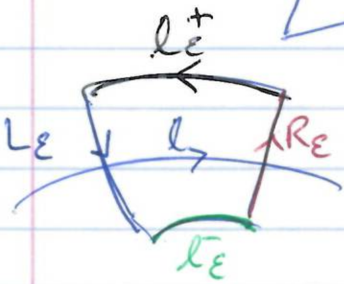
$\Rightarrow$ the normal component of $E_{h,S}$ has a <u>jump</u> across S of size $\dfrac{h(x)}{\varepsilon_0}$.

Thu 1pm

Btw, check DOESN'T bend!

This time take a line instead of a section





$x_0 \times (\varepsilon, -\varepsilon)\,\underline{n}$	L_ε
$x_1 \times (-\varepsilon, \varepsilon)\,\underline{n}$	R_ε
$l \times -\varepsilon \underline{n}$	l_ε^-
$-l \times \varepsilon n$	l_ε^+

$$\oint_{C_\varepsilon^l} \underline{E} \cdot d\underline{r} = 0$$

$$\oint_{C_\varepsilon^l} = \int_{l_\varepsilon^-} + \int_{l_\varepsilon^+} + \underbrace{\int_{L_\varepsilon} + \int_{R_\varepsilon}}_{\to 0 \ \because \text{ length is } 2\varepsilon \to 0}$$

$$\lim_{\varepsilon \to 0} \left[\int_{l_\varepsilon^-} + \int_{l_\varepsilon^+} \right] = \int_l \left[E^-(x) - E^+(x) \right] \cdot \underset{\dot{\underline{r}}\,dt}{\underset{\shortparallel}{d\underline{r}}} \overset{\text{by}}{=} 0 \quad (\dagger)$$

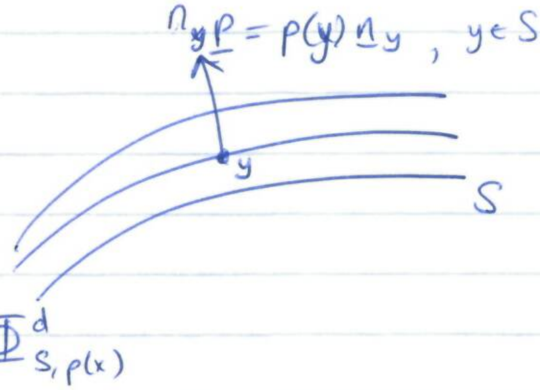
$\dot{\underline{r}}$ - tangential vector to S $:= \underline{t}$

Change l through any $x \in S$ (i.e. rotate l) to have all possible tangential dir^{n}s

Then $(\dagger)$ means that $[E^+(x) - E^-(x)] \cdot \underline{t} = 0$ for any tangential $\underline{t}$ $\Rightarrow$ tgtal part of Elect. Field is cts !!

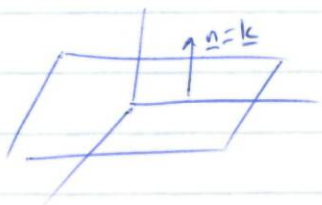
$\underline{p}\,\delta_y$ – dipole at y

$$\Phi_{\underline{p},y}(x) = \frac{(\underline{p},\, x-y)}{|x-y|^3}$$



$$\Phi^d_{S,p(x)}(x) = \int_S \frac{p(y)\cdot(x-y)}{|x-y|^3}\, dS_y$$

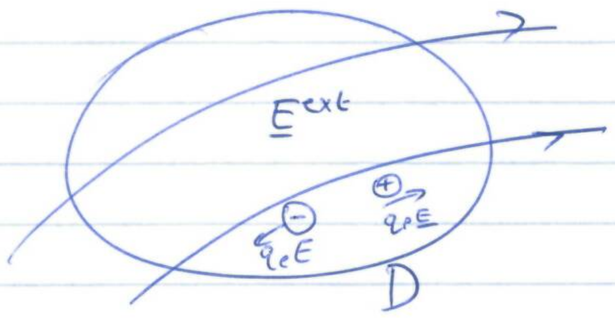
<u>Problem *</u>: $S = \{z=0\}$, the xy plane.
Assume the magnitude of the dipole is constant
$p = p_0$.



- Find $\Phi(x)$, the corresponding electrostatic potential of this surface
- Find the jump of $\Phi(x)$ across $z=0$.

CONDUCTORS

A conductor is a very good ~~metal~~ man.
Assume we have a conductor in a domain D, put into an electrostatic field $\underline{E}^{ext}$

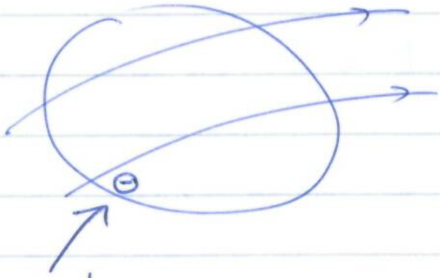


hand waving

too heavy don't move.

+vely charged particles move along $\underline{E}^{ext}$
−vely - - - - - - - - $-\underline{E}^{ext}$,
under the action of the electric field.

Eventually inside this conductor, the electric field is 0 for statics.



−vely charged particles at the boundary, somehow competing with $\underline{E}^{ext}$ creating $\underline{E}=0$.

So electrons move to the boundary on one side

In conductors, elementary charges can move freely. As a result, there will be a redistribution of these elementary charges inside D and the total electric field becomes 0 in D.

$$\overset{E^{ext} + E}{\overset{\shortparallel}{}}$$

As $\underline{E}^t = \underline{E}^{total} = 0$ in $D \Rightarrow \Phi = \text{const}$ inside D.

By Poisson eq^n, $-\nabla^2 \Phi = \dfrac{\rho}{\varepsilon_0}$.

$$\Rightarrow \underline{\rho = 0} \text{ inside } D.$$

← you made the kitten sad!

Therefore the charges are concentrated on $S = \partial D$.

i.e. we have surface charge $F_{s,\sigma}$ where σ is the charge distribution over the surface

$\left(\underline{E}^t = \underline{E}\right)_{\text{notation}}$

Recall for jumps of electric field ~~as~~ across charged surface,

$$[E^+(y) - E^-(y)] \cdot \underline{n}_y = \frac{\sigma(y)}{\varepsilon_0}$$

$$[E^+(y) - E^-(y)] \cdot \underline{t} = 0 \quad \text{for any tangential vector } \underline{t}$$

But $E^- = 0$ since $E \equiv 0$ inside D

$$\Rightarrow E^+(y) = \frac{1}{\varepsilon_0} \sigma(y)\, \underline{n}_y$$

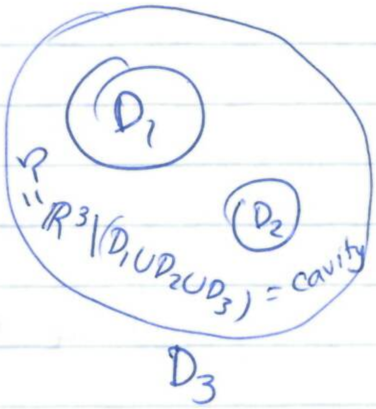
$\Phi^+(y) = \Phi^-(y)$ because ~~E~~ Φ is continuous across the boundary

~~on~~ for y on the surface

and since Φ is const on the surface

$$\Phi^+(y) = \Phi^-(y)\Big|_{y \in S} = \text{const}.$$

Suppose we have 2 conductors that are insulated from each other



~~val of Φ on boundary of D_1 : $\Phi|_{D_1} = C_1$~~

~~$\Phi|_{D_2} = D_2$~~

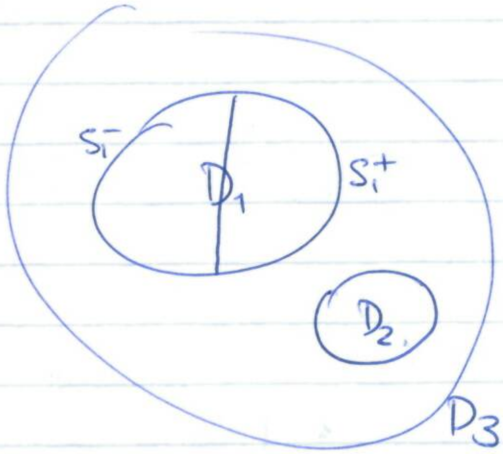
~~$\Phi|_{D_3} = C_3$~~

for $x \in \overset{\text{cavity}}{\Omega}$, $\quad \Phi|_{S_1 = \partial D_1} = C_1$

$$\Phi|_{S_2 = \partial D_2} = C_2$$

$$\Phi|_{S_3 = \partial D_3} = C_3$$

Now divide D_1 by an insulator



Electrostatic potential in two halves can be different.

$$\Phi|_{S_1^+} = C_1^+$$

$$\Phi|_{S_1^-} = C_1^-$$

Keep on dividing D_1 into smaller and smaller portions, each giving a different value of Φ.

" By dividing conductors by insulating layers into infinitezimally small pieces, we can obtain

$$\Phi|_{\partial\Omega} = \varphi \text{ arbitrary continuous function}$$ "

where φ tells you the charge at a point.

Therefore, when considering $\Phi(\underline{x})$, $\underline{x} \in \Omega$, we can often assume that when we know

$$\Phi|_{\partial\Omega} = \varphi \quad \text{given f}^{\underline{n}}. \quad \left(\begin{matrix}\text{Dirichlet} \\ \text{bdy condition}\end{matrix}\right) \quad (1)$$

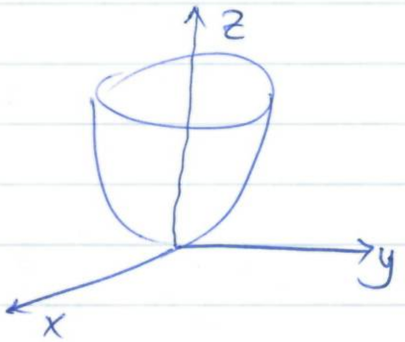
(e.g. when conductor is grounded, $\Phi_{\text{conductor}} = 0$.)

but we know also that Φ in Ω satisfies Poisson's eq$^{\underline{n}}$:

$$-\nabla^2 \Phi = \frac{\rho}{\varepsilon_0} \quad (2)$$

(1) + (2) is called in mathematics 'the Dirichlet boundary value problem for Poisson's eq$^{\underline{n}}$.'

Problem 16: $\Phi(\underline{x}) = \begin{cases} \sin x & z \geq x^2+y^2 \\ \sin x + e^z(z-x^2-y^2) & z < x^2+y^2 \end{cases}$

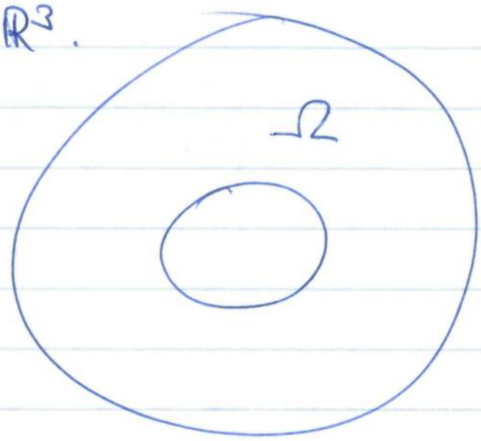


- Find corresponding ρ, the charge density outside the paraboloid
- Find σ, the surface charge density on ~~the paraboloid~~ the paraboloid $z-(x^2+y^2)=0$.

Recall that is your surface is given by the eq$^{\underline{n}}$ $S = \{f(x,y,z)=0\}$, then $\underline{n} = \dfrac{\nabla f}{|\nabla f|}$

BOUNDARY VALUE PROBLEMS OF ELECTROSTATICS

Say we have domain (cavity) $\Omega \subset \mathbb{R}^3$.



We have the following conditions

$$(0)\begin{cases} \bullet \quad -\nabla^2 \Phi = \dfrac{\rho}{\varepsilon_0} \text{ in } \Omega \\ \bullet \quad \Phi|_{\partial\Omega} = \varphi \text{, given} \\ \bullet \quad \Phi(x) \to 0 \text{ if } |x| \to \infty \end{cases}$$

(not necessary in ↑ this case)
(have it with a bit of salt)

Our problem is linear in the following case:
Φ_1, Φ_2 solve problems (A) and (B):

$$(A): \begin{cases} -\nabla^2 \Phi_1 = \rho/\varepsilon_0 \\ \Phi_1|_{\partial\Omega} = 0 \\ \Phi_1 \to 0 \text{ at } \infty \end{cases}$$

$$(B): \begin{cases} -\nabla^2 \Phi_2 = 0 & (\rho \equiv 0) \quad \text{(Laplace's eq}^{n}\text{)} \\ \Phi_2|_{\partial\Omega} = \varphi \\ \Phi_2 \to 0 \text{ at } \infty. \end{cases}$$

Then $\Phi = \Phi_1 + \Phi_2$ solves (0). (Nice, huh!)

Note that (B).1 is Laplace's eq^n $\Rightarrow$ Φ_2 is a harmonic f^n.

When ρ is delta f^n

Now let's look for Green's f^n corresponding to the problem (A).
It is a distribution (actually a discontinuous f^n) in $\mathcal{D}'(\Omega)$:

$G(x,y)$, $y \in \Omega$ is a fixed parameter

s.t.

$$(A_\rho)\begin{cases} -\nabla_x^2 G(x,y) = \delta(x-y) = \delta_y(x) \ (\delta \text{ f}^{\underline{n}} \text{ at a pt } y) \\ G(x,y)\big|_{x\in\partial\Omega} = 0 \\ G(x,y) \underset{x\to\infty}{\longrightarrow} 0 \end{cases}$$

we can speak about its value on the bdy $\because$ the distribution is from a f$^{\underline{n}}$.

If $\Omega = \mathbb{R}^3$ then $G_0 = \dfrac{1}{4\pi|\underline{x}-\underline{y}|}$ $G_0 \leftarrow$ 0 means free space, i.e. $\mathbb{R}^3$

Let $\Psi_y(x) = G(x,y) - G_0(x,y)$.

Then $-\nabla^2\Psi_y = -\nabla_x^2 G - (-\nabla_x^2 G_0)$ (y is const.)

$$= \delta_y(x) - \delta_y(x) = 0.$$

So $-\nabla^2\Psi_y = 0$ in Ω.

$$\Psi_y\big|_{x\in\partial\Omega} = \underbrace{G\big|_{x\in\partial\Omega}}_{0} - G_0\big|_{x\in\partial\Omega}$$

but G on the bdy is 0.

$$= -\frac{1}{4\pi|\underline{x}-\underline{y}|}\bigg|_{x\in\partial\Omega}$$

$\Rightarrow \Psi_y(x) \to 0$

i.e. this is

$$(B_\rho)\begin{cases} -\nabla^2\Psi_y = 0 \text{ in } \Omega \\ \Psi_y(x)\big|_{x\in\partial\Omega} = -\dfrac{1}{4\pi|x-y|}\bigg|_{x\in\partial\Omega} \\ \Psi_y(x) \to 0 \quad x\to\infty \end{cases}$$

Tue 4pm 206
come for fun

So we have reduced the problem (A_ρ) to type (B_ρ)

<u>Now</u> what about solving Φ for any $\rho \in C_0(\Omega)$.

$$\Phi = \frac{1}{\varepsilon_0}\int G(x,y)\,\rho(y)\,dy$$

$$-\nabla_x^2 \Phi = \frac{1}{\varepsilon_0}\int \underbrace{-\nabla_x^2 G(x,y)}_{\delta(x-y)}\,\rho(y)\,dy$$

$$= \frac{1}{\varepsilon_0}\int \delta(x-y)\,\rho(y)\,dy = \frac{1}{\varepsilon_0}\rho(x)$$

~~but~~ $\cancel{\frac{1}{\varepsilon_0}\int G(x,y)\,\rho(y) = \frac{1}{\varepsilon_0} G(\rho)}$

<u>Lemma</u>: let $G(x,y)$ be known, $x, y \in \Omega$.
Then for any $\rho \in C_0(\Omega)$,

$$\Phi^\rho(x) = \frac{1}{\varepsilon_0}\int_\Omega G(x,y)\,\rho(y)\,dy$$

solves
$$\begin{cases} -\nabla^2 \Phi^\rho = \frac{1}{\varepsilon_0}\rho \\ \Phi^\rho\big|_{\partial\Omega} = 0 \end{cases}$$

Proof precedes the lemma.

Let's summarise:

We had
$$\begin{cases} -\nabla^2 \Phi = \dfrac{\rho}{\varepsilon_0} & \text{Poisson's eq}^{n} \\ \Phi\big|_{\partial\Omega} = 0 \\ \Phi \to 0 \text{ at } \infty \end{cases}$$

Reduced to finding the Green's f^n
$$\begin{cases} -\nabla_x^2 G = \delta_y & y \in \Omega \\ G\big|_{\partial\Omega} = 0 \\ G(x,y) \longrightarrow 0 \text{ as } x \to \infty \end{cases}$$

and then since $G(x,y) = G_0(x,y) + \Psi_y(x)$, reduces to find Ψ s.t.

Special Dirichlet boundary value problems for Laplace's eq^n
$$\begin{cases} -\nabla_x^2 \Psi_y(x) = 0 & y \in \Omega \\ \Psi_y(x)\big|_{x \in \partial\Omega} = -\dfrac{1}{4\pi|x-y|} \\ \Psi_y(x) \longrightarrow 0 \text{ as } x \to \infty. \end{cases}$$

Dirichlet boundary value problem for Laplace's eq^n

$$\begin{cases} \nabla^2 u = 0 & \text{in } \Omega \\ u|_{\partial\Omega} = \varphi \in C(\partial\Omega) \\ u(x) \to 0 & \text{as } x \to \infty \end{cases}$$

u instead of Φ so as not to imply $\mathbb{R}^3$: this result holds true for any $\mathbb{R}^n$.

$\nabla^2 u = 0$ in $\Omega \iff u(\underline{x})$ is a harmonic $f^{\underline{n}}$ in Ω.

Problem*: Assume $u \in C^2(\Omega)$ is harmonic.

Show That $\displaystyle\int_{\partial\Omega} \underbrace{\frac{\partial u}{\partial \underline{n}}}_{\text{normal distribution}} dS = 0$.

(hint: use Green's formula)

Thm: (Mean Value Theorem)

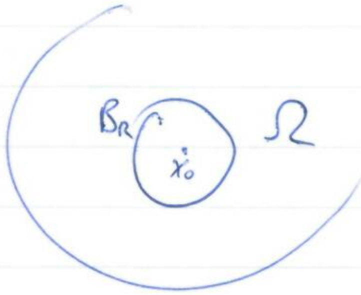
Let u be harmonic in $\Omega \subset \mathbb{R}^3$ (i.e. $\nabla^2 u = 0$) and $B_R(x_0) \subset \Omega$.

Then $\displaystyle u(x_0) = \frac{1}{4\pi R^2} \int_{\substack{\partial B_R(x_0) \\ \parallel \\ S_R(x_0)}} u(x)\, dS_x$ "mean value"

s. area of sphere

in 2D case $u(x_0) = \frac{1}{2\pi R}\int_{C_R(x_0)} u(x)\, dl_x$

$G(x) = \ln|x| \frac{1}{2\pi}$ ← can't remember!



Proof of 3D case:

$$0 = \int_{B_R(x_0)} \overbrace{\nabla^2 u}^{=0} \left(\frac{-1}{4\pi|x-x_0|}\right) dx$$

by Green's formula $= \underbrace{\int_{B_R(x_0)} u \overbrace{\nabla^2\left(\frac{-1}{4\pi|x-x_0|}\right)}^{\delta(x-x_0)} dx}_{u(x_0)} + \int_{S_R(x_0)} \frac{\partial u}{\partial n}\left(\frac{-1}{4\pi\underbrace{|x-x_0|}_{R}}\right) dS$

$$+ \int_{S_R(x_0)} u \frac{\partial}{\partial n}\left(\frac{1}{4\pi|x-x_0|}\right) dS$$

$$= u(x_0) - \frac{1}{4\pi R}\overbrace{\int_{S_R(x_0)} \frac{\partial u}{\partial n} dS}^{=0 \text{ by problem}} + \int_{S_R(x_0)} u \frac{\partial}{\partial n}\left(\frac{1}{4\pi|x-x_0|}\right) dS$$

$$0 = u(x_0) + \frac{1}{4\pi}\int_{S_R(x_0)} u(x) \underbrace{\underbrace{\frac{\partial}{\partial n}\left(\frac{1}{|x-x_0|}\right)}_{\left.\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\right|_{r=|x-x_0|}}}_{-\frac{1}{r^2}} dS$$

$$\Rightarrow u(x_0) = \frac{1}{4\pi}\int_{S_R(x_0)} u(x) \frac{1}{\underbrace{|x-x_0|}_{R}^2} dS$$

$$= \frac{1}{4\pi R^2}\int u(x)\, dS \qquad \square$$

Problem* Prove for 2D case.

$$\int_{B_r(x_0)} u(x)\,dx = \int_{r'=0}^{r} \left(\int_{S_{r'}(x_0)} u(x)\,dS \right) dr'$$

$$= \int_{r'=0}^{r} 4\pi r'^2 u(x_0)\,dr'$$

$$= 4\pi\, u(x_0)\, \frac{r^3}{3}$$

$$u(x_0) = \frac{1}{\frac{4\pi}{3} r^3} \int_{B_r(x_0)} u(x)\,dx$$

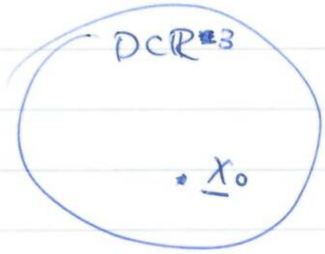
Lecture 8.

'Chalk'
Problem: Show MVT in this case:

$$u(\underline{x}_0) = \frac{1}{\frac{4\pi}{3}r^3} \int_{B_R(x_0)} u(\underline{x})\, dV$$

(Corollary apparently.

Thm: Maximum Principle



Bounded domain $D \subset \mathbb{R}^3$,
Let u be s.t. $\nabla^2 u = 0$
$\hookrightarrow u \in C^2(D^{int}) \cap C(D)$
$\hookrightarrow$ ie. u harmonic and D-bounded.

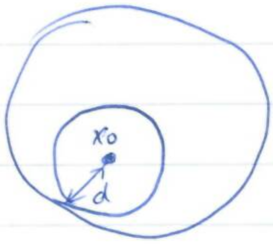
Let $M = \max_{\underline{x} \in \partial D} u(\underline{x})$

Then $u(\underline{x}) \le M$, $\forall \underline{x} \in D$.

~~Mor~~ if D is connected

Moreover, ~~if~~ & $\exists\, \underline{x}_0 \in D^{int}$ s.t. $u(\underline{x}_0) = M$,
then $u(\underline{x}) \equiv M$, $x \in D$

Proof: Assume that $\exists\, \underline{x}_0 \in D^{int}$
s.t. $u(\underline{x}_0) = A = \max_{\underline{x} \in D} u(\underline{x})$.



Let $d = \text{dist}(\underline{x}_0, \partial D) > 0$.

Consider $B_d(\underline{x}_0) \subset D$.

By Problem 'Chalk',

$$A = u(\underline{x}_0) = \frac{1}{\frac{4\pi}{3} d^3} \int_{B_d} u(\underline{x})\, dV.$$

We have $u(\underline{x}) \leq A$ in B_d.

Therefore $$\frac{1}{\frac{4\pi}{3} d^3} \int_{B_d} u(\underline{x})\, d\underline{x} \leq A \cdot \underbrace{\frac{1}{\frac{4\pi}{3} d^3} \int dV}_{1 \ \left(\frac{\text{vol}}{\text{vol}}\right)!!} = A$$

If $u(\underline{x}) < A$ in some $y \in B_d$ (and $\therefore$ nearby),

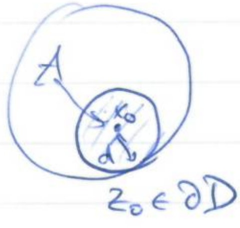
then $$\frac{1}{\frac{4\pi}{3} d^3} \int_{B_d} u(\underline{x})\, d\underline{x} < A$$

Therefore, assuming that $\exists\, \underline{y} \in B_d$ s.t. $u(\underline{y}) < A$, we get

$$A = u(\underline{x}_0) = \frac{1}{\frac{4\pi}{3} d^3} \int_{B_d} u(\underline{x})\, d\underline{x} < A. \quad \#$$

$\Rightarrow u(\underline{x}) \equiv A$ inside B_d.

As $d = \text{dist}(\underline{x}_0, \partial B) \Rightarrow S_d \cap \partial D \neq \emptyset$



$\Rightarrow \exists\, \underline{z}_0 \in \partial D$ s.t. $u(\underline{z}_0) = A = \max\limits_{\underline{x} \in D} u(\underline{x})$

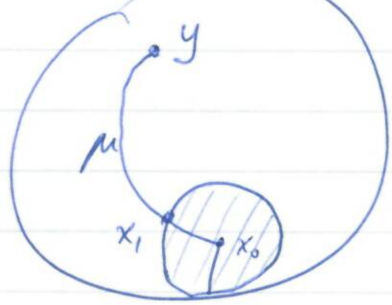
Thus $$\underbrace{\max_{\underline{z} \in \partial D} u(\underline{z})}_{M} = \underbrace{\max_{\underline{x} \in D} u(\underline{x})}_{A}$$

$$M = A.$$

1st part $\square$.

Now, $u(\underline{x}_0) = A = \max\limits_{x \in D} u(x)$.

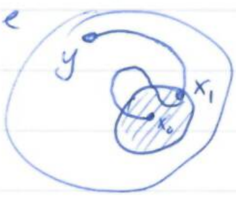
Take any $y \in D$ and connect $\underline{x}_0$ and $\underline{y}$ by a curve μ lying in D^{int}.



Then $\mathrm{dist}(\mu, \partial D) = \rho > 0$

Assume $\underline{y}$ is not in the ball, since if $\underline{y} \in B_d \Rightarrow u(y) = A$.

Take $\underline{x}_1$ = the last point on μ in $\cancel{B_d}$ S_d. $\rightarrow$ could be



Then $u(\underline{x}_1) = A = \max\limits_{x \in D} u(\underline{x})$.

So then take a ball of radius ρ ($\because$ it doesn't touch the bdy).

Consider $B_\rho(\underline{x}_1)$. The same arguments show that $u(\underline{x}) = A$ in $B_\rho(\underline{x}_1)$.

We continue the process until ~~wherever~~ $y \in B_\rho(x_n)$. We need only a finite n° of steps as μ has finite length. $\square$

'Coke'

<u>Problem</u>: Prove, as a corollary, the minimum principle thm,
$$\min_{z \in \partial D}(u(z)) = \min_{x \in D} u(x).$$
[Can rewrite or sget $\tilde{u}(x) = -u(x)$].

Def^n: $\underline{x}_0$ is a local maximum ~~minimum~~ of $u(\underline{x})$ if $\exists\ B_\varepsilon(\underline{x}_0)$ s.t.

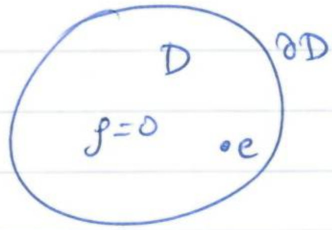
$$u(\underline{x}_0) = \sup_{x \in B_\varepsilon(x_0)} u(\underline{x})$$

Strong version of the theorem: If D is connected and $u(x) \not\equiv$ const and $\nabla^2 u = 0$ then u has no local maxima and minima inside D^{int}. Not proved.

Now, gentlemen, we move to physics.

Let's have a domain D with no charges inside

$$\nabla^2 \Phi = 0 \text{ in } D.$$



Introduce a point charge e.

Then $e\Phi$ is (electrostatic potential) energy of the charge e.

Now recall if we have Φ, we have $\underline{E}$ s.t. $\underline{E} = -\nabla\Phi$.

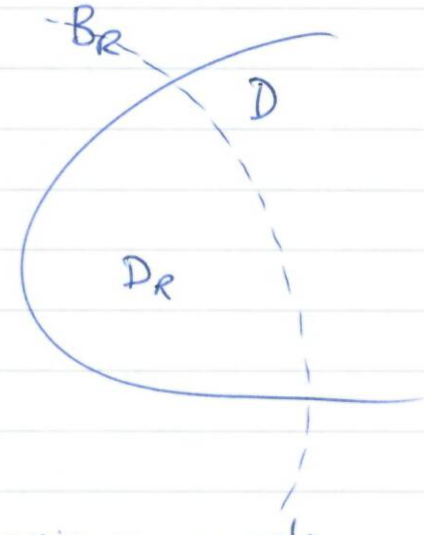
Then there is a force $\underline{F} = e\underline{E}$.

this electrostatic force tries to decrease the electrostatic energy of e.

If you get out of this window you will fall down

Since min $e\Phi$ is achieved on ∂D (by cor. Coke), the charge ends up on the boundary.

Unbounded domain



Let $\begin{cases} \nabla^2 u = 0 & \text{in } D \\ u|_{\partial D} = \varphi \\ u \to 0 & \text{as } x \to \infty \end{cases}$.

Let $D_R = D \cap B_R$

$$u(x)\Big|_{x\in D_R} \leq \max_{z\in\partial D_R} u(z) \qquad \text{by maximum principle}$$

$$\partial D_R = (\partial D \cap B_R) \cup (S_R \cap D)$$

$$\leq \max\left[\max_{z\in\partial D\cap B_R} u(z),\ \max_{x\in S_R\cap D} u(x)\right]$$

smaller set

should be larger set

$$\leq \max\left[\max_{z\in\partial D} u(z),\ \max_{x\in S_R\cap D} u(x)\right]$$

does not depend on R.

$\downarrow$ as $R\to\infty$

0 since $u\to 0$ as $x\to\infty$.

$$\Rightarrow \quad u(x)\Big|_{x\in D} \leq \max\left[\max_{z\in\partial D} u(z),\ 0\right]$$

Also use the minimum principle

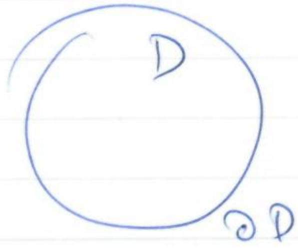
$$u(x)\Big|_{x\in D} \geq \min\left[\min_{z\in\partial D}(u(z)),\ 0\right]$$

So if $D = \mathbb{R}^3$, $\partial D = \emptyset \Rightarrow u(x)\Big|_{x\in\mathbb{R}^3} = 0$.

Now, let us have a bounded domain

Dirichlet b.c. ??(check)??

$$\begin{cases} \nabla^2 u = 0 & \text{in } D \\ u|_{\partial D} = \varphi \in C(\partial D) \end{cases}$$



<u>Claim</u>: For any bounded D and any $\varphi \in C(\partial D)$, $\exists$ a unique u^φ which solves the Dirichlet b.c.s.

<u>Proof</u>: not given, but proven using potential theory for Dirichlet problems for Laplace's eqn.

<u>Note</u>: $u^\varphi \in C^2(D^{int}) \cap C(D)$.

But we <u>can</u> prove uniqueness.

<u>Proof of uniqueness</u>: Assume we have u_1^φ, u_2^φ which solve the Dirichlet b.c.s.

Let $u = u_1^\varphi - u_2^\varphi \Rightarrow \begin{cases} \nabla^2 u = 0 \\ u|_{\partial D} = 0 \end{cases}$

By maximum principle, $u(x) \leq 0 \quad x \in D$
By minimum principle, $u(x) \geq 0 \quad x \in D$

$\Rightarrow u \equiv 0$ and $u_1^\varphi = u_2^\varphi$ $\square$.

What happened to distributions?

Let's look at $\mathcal{D}'_\varphi(D) \subset \mathcal{D}'(D)$, ie. those which have a 'meaningful' restriction on ∂D which is equal to φ. ~~If~~

If $u \in \mathcal{D}'_\varphi(D)$ is harmonic, $\nabla^2 u = 0$

$$\Rightarrow \quad u \in C^\infty(D^{int}).$$

MAGNETOSTATICS

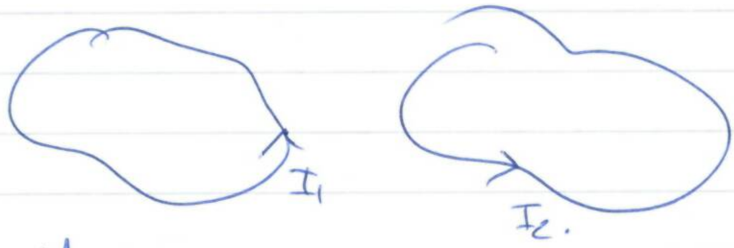
1980, German guy Orsted found that if we have electric ~~field~~ current, it effects magnetic dipole, ie $\exists$ a force which acts on magnets.

~~Biot-Savart Law~~

1820 - Biot-Savart.

~~1820 Biot Savart~~

18



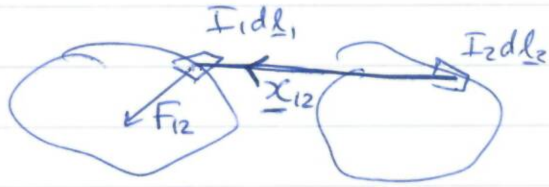
two electric circuits.

Turn on I_1

Turn on I_2. Then $\exists$ a force applied to the first circuit.

Take small part of each circuit

$$I_1 d\underline{l}_1 \qquad I_2 d\underline{l}_2$$



They found the relation

$$F_{12} = \frac{\mu_0}{4\pi} \frac{(I_1 d\underline{l}_1) \times (I_2 d\underline{l}_2 \times \underline{x}_{12})}{|x_{12}|^3} = \text{force by 2 on 1}$$

Biot-Savart Law

$$\underline{B}(x) = \frac{\mu_0}{4\pi} \frac{I_2 d\underline{l}_2 \times (x - x_2)}{|x - x_2|^3}$$

"magnetic field"

in Russian "magnetic flux density" and H is the magnetic field

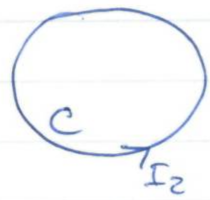
Suppose we have a flow of charges $\underline{J}(y)$

Then $$\underline{B}(\underline{x}) = \frac{\mu_0}{4\pi} \int_{\mathbb{R}^3} \frac{\underline{J}(y) \times (\underline{x} - \underline{y})}{|\underline{x} - \underline{y}|^3} \, dV_y \quad \text{------} (*)$$

$\underline{J}(\underline{y}) = \underline{\dot{y}}(s) \, \delta_{I_2, C}$ ← like $\delta_{h,i}$ ← Prob. 146

← I_2 is constant

tangent to curve



$C = \{ y = y(s) \}$

$\underline{\dot{y}}(s) \, ds = d\underline{\ell}$

convolution of cross products

then $\underline{B}(\underline{x}) = \cancel{\frac{\mu_0}{4\pi} \int_{\mathbb{R}^3}} \quad \underline{J} \underset{\times}{*} \nabla \Phi$

(vector valid convolution

valid for each component of J_i)

Let us have an electric field $\underline{E}(\underline{x})$. If we have a charge q, in electrostatics

$$\underline{F}(\underline{x}) = q \underline{E}(\underline{x}).$$

If $\rho(x)$ is an electric charge distribution (i.e. we don't have pt charges but continuous instead).

Then $$F^{total} = \int_D \rho(\underline{x}) E(\underline{x}) \, dV$$

not $J(x)$ which caused $\underline{B}$ initially.

Now suppose we have a body with some $\underline{\tilde{J}}(\underline{x})$ inside.

"absorbed?"

Then $$\underline{F}^{total}_{mag} = \int_D \underline{\tilde{J}}(\underline{x}) \times \underline{B}(\underline{x}) \, dV$$ ← Ampère's law of magnetic forces

Ampère was a big boy; he had lots of laws.

<u>Ampère's law</u>, analagous to Gauss' law

~~Let $\underline{J}(y) \in \mathcal{D}(\mathbb{R}^3)$~~ Look at $(*)$.

<u>Prob</u>: Show that

$$\nabla \times (\underline{a}\, f(\underline{x})) = -\underline{a} \times \nabla f$$

Note $$-\frac{(x-y)}{|x-y|^3} = \underline{\nabla} \frac{1}{(x-y)}$$

$\underline{J}(\underline{y})$ is constant w.r.t. $\underline{x}$. Let $\underline{J}(\underline{y}) \in \mathcal{D}(\mathbb{R}^3)$

~~Then (*) becomes So look at~~ Then $(*)$ becomes

$$\underline{B}(\underline{x}) = \cancel{\underline{B}(\underline{x})} = \frac{\mu_0}{4\pi} \nabla \times \int_{\mathbb{R}^3} \frac{\underline{J}(\underline{y})}{|\underline{x}-\underline{y}|}\, dV_{\underline{y}}$$

So $\underline{B}$ is the curl of some magnetic field
and remember $\underline{E}$ was the gradient of $-\Phi \Rightarrow \nabla \times E = 0$.

$$\underline{B} = \nabla \times \left(\frac{\mu_0}{4\pi} \int_{\mathbb{R}^3} \frac{\underline{J}(\underline{y})}{|\underline{x}-\underline{y}|}\, dy \right)$$

$$\Rightarrow \nabla \cdot \underline{B} = 0 \qquad \text{since } \operatorname{div}(\operatorname{curl}()) = 0.$$

What is $\nabla \times \underline{B}$?

<u>Problem</u>: show that $\nabla \times \nabla \times \underline{F} = \nabla(\nabla \cdot \underline{F}) - \nabla^2 \underline{F}$

vector field

$$\text{So } \nabla \times \underline{B} = \frac{\mu_0}{4\pi}\int \left[\nabla_x \times \nabla_x \left(\frac{\underline{J}(\underline{y})}{|\underline{x}-\underline{y}|}\right)\right] dV_y$$

$$= \overbrace{\frac{\mu_0}{4\pi}\int \left[\nabla_x \left(\nabla_x \cdot \frac{\underline{J}(\underline{y})}{|\underline{x}-\underline{y}|}\right)\right] dV_y}^{B}$$

$$\underbrace{- \frac{\mu_0}{4\pi}\int \left[\nabla_x^2 \left(\frac{\underline{J}(\underline{y})}{|\underline{x}-\underline{y}|}\right)\right] dV_y}_{A}$$

$$A = -\frac{\mu_0}{4\pi}\int_{\mathbb{R}^3} \left[\underline{J}(\underline{y})\, \nabla_x^2 \left(\frac{1}{|\underline{x}-\underline{y}|}\right)\right] dV_y$$

$$= \mu_0 \int_{\mathbb{R}^3} \underline{J}(\underline{y})\, \delta(\underline{x}-\underline{y})\, dV_y$$

$$= \mu_0\, \underline{J}(\underline{x})$$

$$B = \int \left[\nabla_x \left(\nabla_x \cdot \frac{\underline{J}(\underline{y})}{|\underline{x}-\underline{y}|}\right)\right] dV_y$$

$$= \nabla_x \left[\int \nabla_x \cdot \frac{\underline{J}(\underline{y})}{|\underline{x}-\underline{y}|}\, dV_y\right]$$

[Problem: show $\nabla\cdot(\underline{a}\, f(\underline{x})) = \underline{a}\cdot\nabla f$]

$$= \nabla_x \int \underline{J}(\underline{y}) \cdot \nabla_x \left(\frac{1}{|\underline{x}-\underline{y}|}\right) dV_y$$

$$= -\nabla_x \int \underline{J}(\underline{y}) \cdot \nabla_y \left(\frac{1}{|\underline{x}-\underline{y}|}\right) dV_y$$

[Problem: show $\int \underline{F}\cdot\nabla f\, dV = -\int (\nabla\cdot F) f\, dV$]

hint: use div. thm. $\to \nabla\cdot(Ff) = F\cdot\nabla f + (\nabla\cdot F) f$

$$= + \nabla_x \times \left[\int (\nabla \cdot \underline{J})(y) \cdot \frac{1}{|\underline{x}-\underline{y}|} \, dV_y \right] \quad \cdots\cdots (*)$$

[Claim: in magnetostatics (electro), $\nabla \cdot \underline{J} = 0$.]

because of philosophy of life. Nothing comes from nothing.

$$= 0 \qquad \Rightarrow \qquad \cancel{\nabla \times \underline{B} = 0} \qquad \square$$

"Ampère's law of electro magneto statics" $\longrightarrow$ $\nabla \times \underline{B} = \mu_0 \underline{J}(\underline{x})$

Not deus ex machina. Else we should shut our doors and go to church.

<u>Proving the claim</u>

Take a region V and look for total charge in the region

$$\underset{Q(t)}{\overset{}{Q^{total}}}(t) = \int_V \rho(x,t) \, dV_x$$

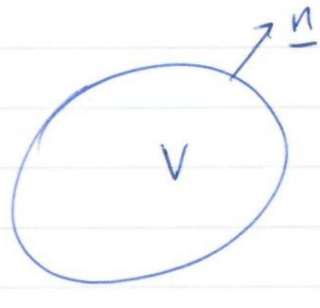
We believe charge doesn't appear from nothing. If it changes, charge comes in and out.

If Q as a f^n of time changes, there should be an in/outflux of charge in the region.

If $\frac{\partial Q}{\partial t} \neq 0$, there should be <u>current</u> flux through $S = \partial V$

But this current flux is

$$\int_S \underline{J}(\underline{x},t) \cdot \underline{n} \, dS$$



Thus the continuity law is

$$\int_V \frac{\partial \rho}{\partial t}(\underline{x},t)\, dV = -\int_S \underline{J}(t) \cdot \underline{n}\, dS$$

minus since if ρ increases, $\underline{J}$ points inwards

by div. thm, $$= -\int_V \nabla \cdot \underline{J}(t)\, dV$$

Since V is arbitrary,

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \underline{J}(\underline{x},t)$$

i.e. $$\frac{\partial \rho}{\partial t} + \nabla \cdot \underline{J} = 0$$ continuity eqn.

We are talking about statics, so $\frac{\partial \rho}{\partial t} = 0$

time!!

$$\nabla \cdot \underline{J} = 0.$$ Ampère's law

So we have the 4 laws of electromagnetostatics:

diff. form

$$\nabla \cdot \underline{E} = \frac{\rho}{\varepsilon_0} \quad \text{(Gauss)} \qquad \nabla \times \underline{B} = \mu_0 \underline{J} \quad \text{(Ampère)}$$

$$\nabla \times \underline{E} = 0 \qquad \nabla \cdot \underline{B} = 0$$

int. form

$$\int_{S=\partial V} \underline{E} \cdot \underline{n}\, dS = \frac{1}{\varepsilon_0} \int_V \rho\, dV \qquad \oint_{C=\partial S} \underline{B} \cdot d\underline{r} = \mu_0 \int_S \underline{J} \cdot \underline{n}\, dS$$

$$\oint_C \underline{E} \cdot d\underline{r} = 0 \qquad \int_{S\,(\text{closed})} \underline{B} \cdot \underline{n}\, dS = 0$$

<u>Dynamical case</u>

$\nabla \cdot \underline{J} \neq 0$, instead $\nabla \cdot \underline{J} = -\frac{\partial \rho}{\partial t}$.

$$\nabla \times \underline{B} = -\frac{\mu_0}{4\pi} \nabla_x \frac{\partial}{\partial t} \underbrace{\int \frac{\rho(\underline{y},t)}{|\underline{x}-\underline{y}|}\, dV_y}_{\text{depends on } \underline{x},\, t.} + \mu_0 \underline{J}(\underline{x}) \qquad \text{from } (*) \quad \text{- - - - } (\dagger)$$

But what is $\frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\underline{y},t)}{|\underline{x}-\underline{y}|}\, dV_y$? what is that what is that?

$$= \Phi^\rho(\underline{x}, t) \quad !!$$

$$(\dagger) = \varepsilon_0 \mu_0 \frac{\partial}{\partial t}\left(-\nabla_x \Phi^\rho\right) + \mu_0 \underline{J}(\underline{x})$$

$$= \varepsilon_0 \mu_0 \frac{\partial}{\partial t} \underline{E}(\underline{x},t) + \mu_0 \underline{J}(\underline{x})$$

$$\Rightarrow \nabla \times (\mu_0^{-1} \underline{B}) = \frac{\partial}{\partial t}(\varepsilon_0 \underline{E}) + \underline{J}$$

Maxwell Ampère eq^n. in vacuum

What are μ_0, ε_0? In vacuum they are just some constants (but specific to vacuum)

If we go to the brick, ε, μ_0 will change.

ε_0, μ_0 are <u>matrices</u>, actually different for each material. They depend on $\underline{x}, t$.

Sometimes it goes to 10,000.

Sometimes $\underline{H} := \mu_0^{-1} \underline{B}$

"magnetic field" or "mag. ~~flux density~~ field strength"

$\underline{D} := \varepsilon_0 \underline{E}$

"electric displacement in vacuum"

and the equation is even easier.

Why call $\underline{H}$ magnetic field instead of $\underline{B}$?

Recall $\nabla \cdot \underline{E} = \frac{\rho}{\varepsilon_0}$ but this is bad since ε is a matrix

$\nabla \cdot \underline{D} = \rho$ better.

and recall $\nabla \cdot \underline{B} = 0$.

* Recall $\nabla \times \underline{E} = 0$ for statics. We need to change this for dynamics.

Max-Amp says $$\nabla \times \underline{H} = \frac{\partial}{\partial t}\underline{D} + \underline{J}$$ Max-Amp. eq^n

(not nec. in vacuum!!)

where $\left.\begin{array}{l}\underline{D} = \varepsilon \underline{E} \\ \underline{B} = \mu \underline{H}\end{array}\right\}$ constitutive relation

How does $\nabla \times \underline{E} = 0$ change in dynamics, ~~giving~~ we call the new version

In 1830s, Faraday ~~made the~~ carried out his famous expt. He takes a closed piece of wire

and he produced mag. flux through it by placing another wire with current next to it. (by Bio-Sav. law)

If you change $\underline{B}$, there appears a current in the wire
We change flux through $\underline{B}$ which creates a force through it $\propto$ to the force. What produces forces? E. fields!

So he concluded when you change through time an electric field you change a mag. field.

After all these fantastic experiments, he found

$$\nabla \times \underline{E} = -\frac{\partial B}{\partial t}. \longleftarrow \text{Max. Faraday eq}^n$$

So we have

→ " Adam [#] Where are horses?"

– "..."

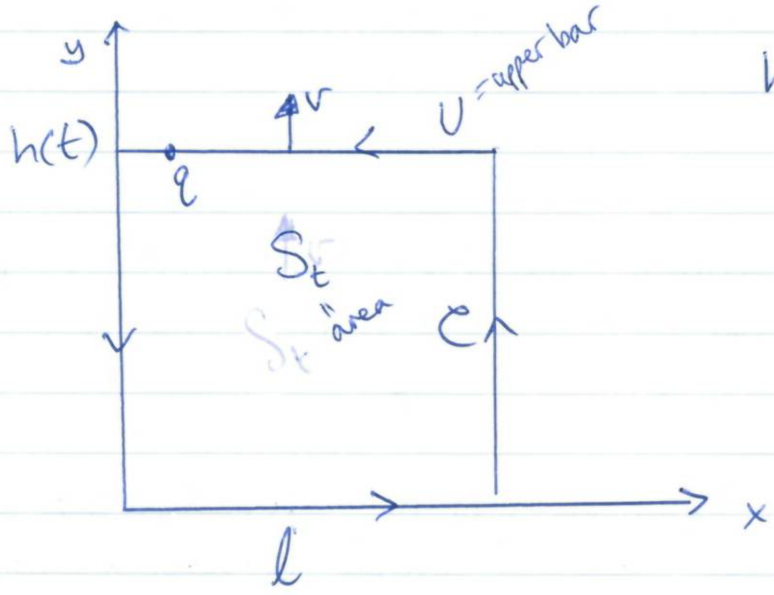
– "Stables!"

$$\nabla \cdot \underline{D} = \rho \qquad \nabla \cdot \underline{B} = 0$$

$$\nabla \times \underline{E} = -\frac{\partial B}{\partial t} \qquad \nabla \times \underline{H} = \frac{\partial}{\partial t}\underline{D} + \underline{J}$$

And now, gentlemen and ladies, we ~~can~~ will derive from Ampère, Max-Far. law.

Let us have a circuit



$h'(t) = v$

Let $\underline{B} = B_0 \hat{\underline{k}}$, B_0 is magnetic field of the Earth.

Take a charge q along the bar at height $h(t)$.

$$\underline{J} = q\underline{v} = qv\hat{\underline{j}}$$

Then by Ampère's law of forces,

$$\underline{F} = q\underline{v} \times \underline{B}$$

(usually integrate but this is delta fn : one point)

$$= qvB_0\hat{\underline{i}}$$

but we know $\underline{F} = q\underline{E}$

$$\Rightarrow \quad \underline{E} = v B_0 \hat{\underline{i}}$$

$$= \underline{v} \times \underline{B}$$

Let us look at $\oint_{C(t)}$.

$$\oint_{C(t)} \underline{E} \cdot d\underline{r} = \int_{U} v B_0 \hat{\underline{i}} \cdot d\underline{r}$$

$$= -v B_0 l \qquad \text{- - - - - (1)}$$

$$\int_{S_t} \underline{B} \cdot \underline{n} \, dS = \int_{S_t} \underline{B} \cdot \hat{\underline{k}} \, dS$$

$$= \int_{S_t} B_0 \hat{\underline{k}} \cdot \hat{\underline{k}} \, dS$$

$$= B_0 \int_{S_t} dS = B_0 \, \text{Area}(S_t)$$

$$= B_0 l h(t)$$

$$\rightarrow \frac{\partial}{\partial t} \int_{S_t} \underline{B} \cdot \underline{n} \, dS = \frac{d}{dt} B_0 l h(t)$$

$$= B_0 l v \qquad \text{- - - - (2)}$$

$$\Rightarrow -\oint_{C(t)} \underline{E} \cdot d\underline{r} = \frac{\partial}{\partial t} \int_{S_t} \underline{B} \cdot \underline{n} \, dS \qquad \text{by } -(1) = (2)$$

or $$-\oint_{C(t)} \underline{E}(t) \cdot d\underline{r} = \frac{\partial}{\partial t} \int_{S_t} \underline{B}(t) \cdot \underline{n} \, dS$$

Now we say this change is not $\because$ of changing area, it's $\because$ of changing flux

Thus,

$$-\oint_{C} \underline{E}(\underline{x},t)\cdot d\underline{r} = \frac{\partial}{\partial t}\int_S \underline{B}(t)\cdot\underline{n}\, dS$$

$\Vert$ stokes $\qquad\qquad$ $\Vert$ $\because$ S not dpt on t

$$\int_S (-\nabla\times\underline{E})\cdot\underline{n}\, dS = \int_S \frac{\partial \underline{B}}{\partial t}\cdot\underline{n}\, dS$$

S arbitrary $\Rightarrow$ $\nabla\times\underline{E} = -\dfrac{\partial B}{\partial t}$

why is this step valid? It's a version of DuBois-Reymond:

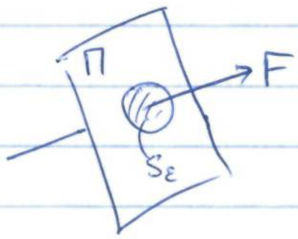
$$\int\left(\nabla\times\underline{E} + \frac{\partial \underline{B}}{\partial t}\right)\cdot\underline{n}\, dS = 0$$

$$\Leftrightarrow \underbrace{\nabla\times\underline{E} + \frac{\partial B}{\partial t}}_{F} = \underline{0}\,.$$

Why? Assume it is not zero at some point.

ie $\exists\ \underline{x}_0$ s.t. $\underline{F}(\underline{x}_0)\neq 0$.

Take a plane $\perp^r$ to F through $\underline{x}_0$.



S_ε disc of radius ε in Π

Then I need $\displaystyle\int_{S_\varepsilon} \underline{F}\cdot\underline{n}\, dS = 0$

but $\underline{n} = \dfrac{F(\underline{x}_0)}{|F(\underline{x}_0)|}$ by clever choice of plane

$\Rightarrow \underline{F}\cdot\underline{n}\big|_{\underline{x}=\underline{x}_0} = |F(\underline{x}_0)| > 0$.

but $\underline{F}$ is cts on Π.

$\Rightarrow \underline{F}\cdot\underline{n} > \frac{1}{2}|\underline{F}(\underline{x}_0)|$ if ε is sufficiently small.

Thus $\int_{S_\varepsilon} \underline{F}\cdot\underline{n}\, dS > \frac{1}{2}|\underline{F}(\underline{x}_0)|\, \text{Area}(S_\varepsilon) > 0$ ✗

$\Rightarrow \underline{F}(\underline{x}) \equiv \underline{0}\ \forall \underline{x}$

So we get Maxwell-Faraday.

So we have Maxwell's eq^{n}s: (in any material)

$$\begin{cases} \nabla\cdot\underline{D} = \rho & \nabla\cdot\underline{B} = 0 \\ \nabla\times\underline{E} = -\dfrac{\partial B}{\partial t} & \nabla\times\underline{H} = \dfrac{\partial D}{\partial t} + \underline{J} \end{cases}$$

where $\begin{cases} \underline{D} = \varepsilon\underline{E} \\ \underline{B} = \mu\underline{H} \end{cases}$

In integral form:

$$\begin{cases} \int_S \underline{D}\cdot\underline{n}\, dS = \int_V \rho\, dV & \int_S \underline{B}\cdot\underline{n}\, dS = 0 \\ \int_C \underline{E}\cdot d\underline{r} = -\int_S \dfrac{\partial B}{\partial t}\, dS & \int_C \underline{H}\cdot d\underline{r} = \int_S \dfrac{\partial D}{\partial t} + \underline{J}\, dS \end{cases}$$

Étude on differential forms

$$\int_C \underline{E}\cdot d\underline{r} = \int_C E_1\, dx^1 + E_2\, dx^2 + E_3\, dx^3 \quad \cdots\cdots (L)$$

Let $x^1 = x^1(t)$, $x^2 = x^2(t)$, $x^3 = x^3(t)$

$$= \int_a^b \Big[E_1(\underline{x}(t))\,\dot{x}^1(t) + \dots + E_3\big(\underline{x}(t)\big)\,\dot{x}^3(t) \Big]\, dt$$

Let $x^i = x^i(y^k)$ $\quad k,i = 1,2,3.$

i.e. our x coords are in terms of y coords

e.g. polars in cartesians

Then $dx^1 = \frac{\partial x^1}{\partial y^1} dy^1 + \frac{\partial x^1}{\partial y^2} dy^2 + \frac{\partial x^1}{\partial y^3} dy^3$

$$(L) = \int \Big[E_1 \frac{\partial x^1}{\partial y^1} + E_2 \frac{\partial x^2}{\partial y^1} + E_3 \frac{\partial x^3}{\partial y^1} \Big] dy^1$$
$$+ \Big[E_1 \frac{\partial x^1}{\partial y^2} + E_2 \frac{\partial x^2}{\partial y^2} + E_3 \frac{\partial x^3}{\partial y^2} \Big] dy^2$$
$$+ \Big[E_1 \frac{\partial x^1}{\partial y^3} + E_2 \frac{\partial x^2}{\partial y^3} + E_3 \frac{\partial x^3}{\partial y^3} \Big] dy^3$$

$$= \int_C \hat{\underline{E}} \cdot d\underline{y}$$

$\hat{\underline{E}} = \underline{E}$ in y-coordinates

$$\hat{E}_1 = \frac{\partial \underline{x}}{\partial y^1} \cdot \underline{E} = \sum_{i=1}^{3} \frac{\partial x^i}{\partial y^1} E_i$$

$j = 1,2,3$ $\quad \hat{E}_j = \sum_{i=1}^{3} \frac{\partial x^i}{\partial y^j} E_i$

So $\displaystyle\sum_{i=1}^{3} E_i(\underline{x})\, dx^i \longleftrightarrow \sum_{j=1}^{3} \hat{E}_j(\underline{y})\, dy^j$

differential form of first order

Vectors and covectors: velocity is a real vector

~~v~~

$$\cancel{v^i = \frac{dx^i}{dt} = \frac{d(x^i(y))}{dt}}$$

$$\hat{v}^j = \frac{dy^j}{dt}$$

$$v^i = \frac{dx^i}{dt} = \frac{d(x^i(y))}{dt} = \sum_{j=1}^{3} \frac{\partial x^i}{\partial y^j} \frac{\partial y^j}{\partial t} = \sum_{j=1}^{3} \frac{\partial x^i}{\partial y^j} \hat{v}^j$$

sum over j, in contrast to $\hat{E}$, where we sum over i.

$$\rightarrow \hat{v} = \left(\frac{\partial x}{\partial y}\right)^{-1} v$$

$$\hat{E} = \left(\frac{\partial x}{\partial y}\right)^{T} E$$

v is a vector, E is a covector } when you look at a point

At any point in E you have a covector.

:)

Ohm's law: $\underline{J} = \underline{J}^{ext} + \sigma \underline{E}$

σ conductivity
σ matrix
f^n of x, t

$\sigma \geqslant 0$ (positive definite)

Electromagnetic Energy

(ε, μ are time indpt ; $\sigma = 0$) $\Rightarrow$ the system has losses

$\therefore$ From now on, $\varepsilon = \varepsilon(x)$, $\mu = \mu(x)$, $\sigma = 0$.

So what is the energy of an EM field.

EM wave is a solution to Maxwell's eq^{n}s and Ohm's when $\rho = 0$, $J^{ext} = 0$.

Energy density $\mathcal{E}(x,t)$ of an EM is

$$\mathcal{E}(x,t) = \frac{1}{2}\left[\underline{D} \cdot \underline{E} + \underline{B} \cdot \underline{H}\right]$$

$$= \frac{1}{2}\left[(\varepsilon \underline{E}) \cdot \underline{E} + (\mu \underline{H}) \cdot \underline{H}\right]$$

Now we look on how $\mathcal{E}$ changes with time in some volume V.

$$\mathcal{E}_V(t) = \int_V \mathcal{E}(x,t)\, dV_x$$

$$\Rightarrow \frac{\partial \mathcal{E}_V(t)}{\partial t} = \int_V \frac{\partial}{\partial t} \mathcal{E}(x,t)\, dV_x$$

Do you like Medieval paintings?

$$= \int_V \left[\left(\frac{\partial}{\partial t}(\varepsilon \underline{E}) \cdot \underline{E}\right) + \left(\frac{\partial}{\partial t}(\mu \underline{H}) \cdot \underline{H}\right)\right] dV$$

since $\frac{\partial}{\partial t}\left[(\varepsilon \underline{E}) \cdot \underline{E}\right] = 2\left[\frac{\partial}{\partial t}(\varepsilon E) \cdot \underline{E}\right]$

$$= \int_V \left[\frac{\partial D}{\partial t} \cdot \underline{E} + \frac{\partial B}{\partial t} \cdot \underline{H} \right] dV$$

Recall my Maxwell's eq^ns that $\frac{\partial D}{\partial t} = \nabla \times \underline{H}$ $(J = 0)$

$$-\frac{\partial B}{\partial t} = \nabla \times \underline{E}$$

$$= \int_V \left[(\underline{\nabla} \times \underline{H}) \cdot \underline{E} - \underline{H} \cdot (\nabla \times \underline{E}) \right] dV$$

[Prob. A1*: Show $\nabla \cdot (\underline{a} \times \underline{b}) = (\nabla \times \underline{a}) \cdot \underline{b}) - \underline{a} \cdot (\nabla \times \underline{b})$]

$$= -\int_V \nabla \cdot (\underline{E} \times \underline{H}) \, dV \quad \text{.......(1)}$$

Call $\underline{S} = \underline{E} \times \underline{H}$ pointing vector

[Prob A2*: Show $\frac{\partial \mathcal{E}}{\partial t}(x,t) + \nabla \cdot \underline{S}(\underline{x},t) = 0$

and $\dfrac{\partial \mathcal{E}_V(t)}{\partial t} = -\int_{\partial V} \underline{S} \cdot \underline{n} \, dA$]

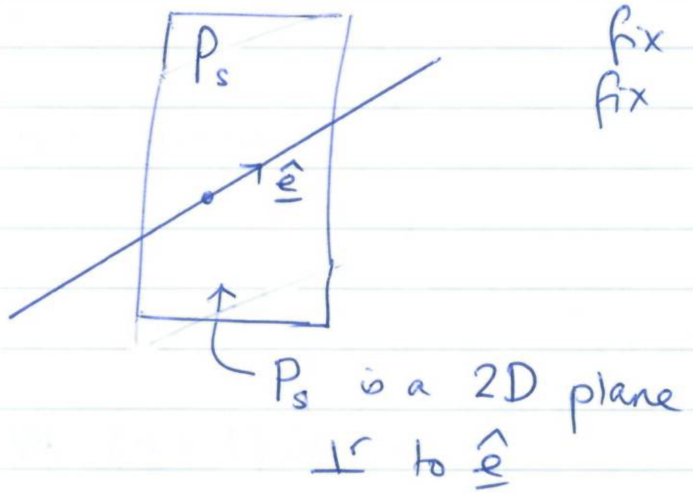
Now, let ε, μ be scalars and constants > 0.

$$\left.\begin{matrix} \rho = 0 \\ \underline{J} = 0 \end{matrix}\right\} \text{still}.$$

(Isotropic homogeneous medium w/o dissipation)

EM wave is called a "plane wave" if ~~it depends~~ there is a direction $\hat{\underline{e}}$ (unit vector) s.t.

$$\underline{E} = \underline{E}(\underline{x}\cdot\hat{\underline{e}}, t)$$
$$\underline{H} = \underline{H}(\underline{x}\cdot\hat{\underline{e}}, t)$$



fix t
fix $s = \underline{x}\cdot\hat{\underline{e}}$

Then $\underline{E}, \underline{H}$ do not vary along P_s.

Since we are in isotropic material,
take $\hat{\underline{e}} = \hat{\underline{\imath}}$.

$$\Rightarrow \quad \underline{E} = \underline{E}(x^1, t)$$
$$\underline{H} = \underline{H}(x^1, t).$$

[Prob 3.4 "B1*" is now star]

[Prob B2*: Show that $\nabla\cdot\underline{E} = \nabla\cdot\underline{H} = 0$ for isotrophic, homogeneous medium]

Let us differentiate one of Maxwell's eq^{n}s.

$$\frac{\partial}{\partial t}(\nabla\times\underline{H}) = \frac{\partial}{\partial t}\left(\frac{\partial D}{\partial t}\right) \qquad \text{(Maxwell-Ampère)}$$

$$\nabla\times\frac{\partial \underline{H}}{\partial t} = \frac{\partial^2}{\partial t^2}(\varepsilon\underline{E})$$

// Max-Far.

$$-\frac{1}{\mu}(\nabla\times\nabla\times\underline{E})$$

|| Prob 3.4 and B2* (ie. $\nabla\cdot\underline{E}=0$)

$$\frac{1}{\mu}\nabla^2\underline{E}$$

ie. $\frac{\partial^2 E_j}{\partial t^2} = \frac{1}{\varepsilon\mu}\nabla^2 E_j$ componentwise

$$\Rightarrow \quad \frac{\partial^2 \underline{E}}{\partial t^2} = \frac{1}{\varepsilon\mu}\nabla^2\underline{E}$$

[Prob B3*] Derive $\dfrac{\partial^2\underline{H}}{\partial t^2} = \dfrac{1}{\varepsilon\mu}\nabla^2 H$]

These are the wave eq^n with speed $c = \frac{1}{\sqrt{\varepsilon\mu}}$

$\because \frac{\partial^2 f}{\partial t^2} = c^2\nabla^2 f$ is usual wave eq^n.

Note: when deriving wave eq^n we don't require that $\underline{E}, \underline{H}$ are plane waves.

[Prob B4*: Show that if $\underline{E} = \underline{E}(x^1, t)$

$\underline{H} = \underline{H}(x^1, t)$,

then

$$E_n(\underline{x},t) = E_n^+(x^1+ct) + E_n^-(x^1-ct)$$

$$H_n(\underline{x},t) = \cancel{E}H_n^+(x^1+ct) + H_n^-(x^1-ct) \qquad n=1,2,3.$$

for some f^{n}s $E_n^\pm(s)$, $H_n^\pm(s)$. $\qquad (s = x^1)$

Note: just look at wave eq^n $\frac{\partial^2 f}{\partial t^2} = c^2\nabla^2 f$ assuming that $f = f(x^1,t)$. Use substitution $\zeta = x^1 - ct$, $\eta = x^1+ct$ to show that $\frac{\partial^2 f}{\partial\zeta\partial\eta} = 0$.

Adam likes Cheese. But doesn't like Geese

Then for $F = \frac{\partial f}{\partial\zeta}$ we get $\frac{\partial F}{\partial\eta} = 0$

ie $F(\zeta,\eta)$ is an arbitrary f^n $F(\zeta)$

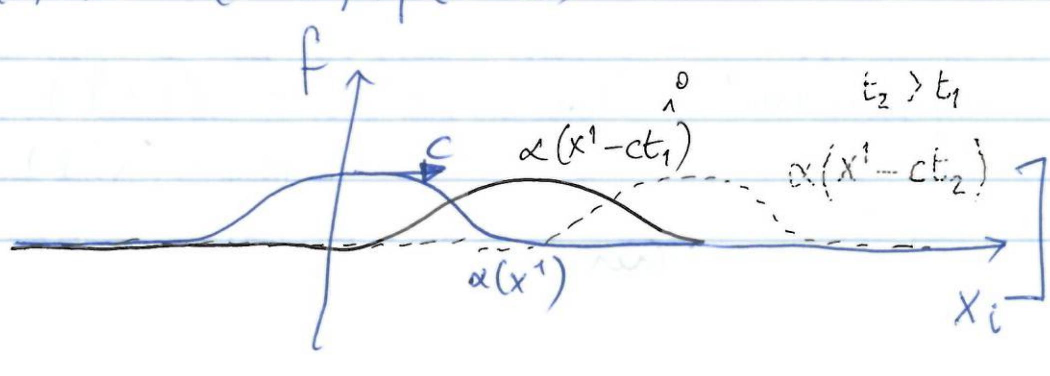
ie $\frac{\partial f(\zeta,\eta)}{\partial\zeta}$ is an arbitrary f^n of ζ

$$\frac{\partial f}{\partial\zeta}(\zeta,\eta) = F(\zeta)$$

Show that these imply $f(\zeta,\eta)$
$= \alpha(\zeta) + \beta(\eta)$

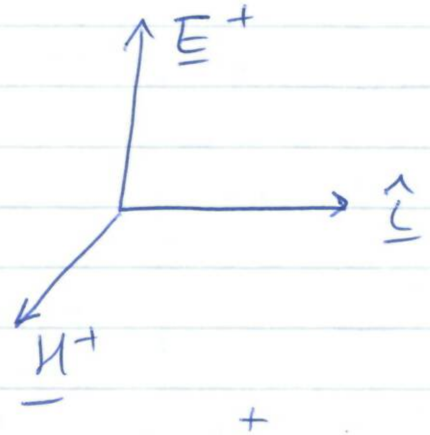
where α, β arbitrary f^{n}s

$$f(x^1,t) = \alpha(x^1-ct) + \beta(x^1+ct)$$



So we get $\underline{E}, \underline{H}$ are superpositions of two waves, one going to right and one going to left.

Suppose $E_1, H_1 = 0$, ie. $\underline{E}, \underline{H} \perp^r \underline{\hat{l}}$
and $\underline{E}^+ \perp^r \underline{H}^+$



Look at poinling vector $\underline{S}^+$ (pointing to right)

$$\underline{S}^+ = \underline{E}^+ \times \underline{H}^+ = |\underline{S}^+| \, \underline{\hat{l}}$$

EM wave is harmonic if it's periodic in t and its dependence on t is of the form $e^{\pm i\omega t}$ (ω = freq.)

Harmonic plane wave is then of the form

$$\underline{E} = \underline{E}^+ e^{i(kx_e - \omega t)} + \underline{E}^- e^{i(kx_e + \omega t)} + \text{comp. conj.}$$

$\underline{E}^+, \underline{E}^-$ constant complex vectors

$$x_e = \underline{x} \cdot \underline{e}$$

Plane waves $\Rightarrow$ $e^{ik(x_e - \frac{\omega}{k} t)}$ $\qquad k = \frac{\omega}{c}$

↳ depend on $x - ct$

So $\underline{E} = \sum$ four terms. All should satisfy Max Eq$^{\underline{n}}$s $\because$ they're indpt.

We need that each term $E^{\pm} e^{i(kx^1 \mp ct)}$ satisfies Max Eq$^{\underline{n}}$s.

$$H(x,t) = H^+ e^{ik(x^1-ct)} + H^- e^{ik(x^1+ct)} + c.c.$$

We are studying EM waves of form

$$\underline{E}_0 e^{ik(x^1-ct)}, \ \underline{H}_0 e^{ik(x^1-ct)}$$

$\left.\begin{matrix}\underline{E}_0 \\ \underline{H}_0\end{matrix}\right\}$ constant complex vectors

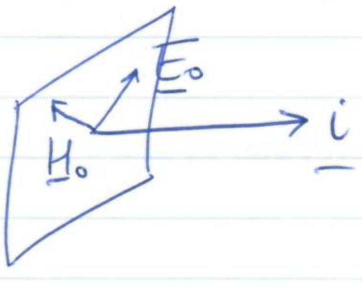
Let's apply divergence to this formula.
(recall $\nabla\cdot\underline{E} = \nabla\cdot\underline{B} = 0$)

$$\nabla\cdot\left(\underline{E}_0 e^{ik(x^1-ct)}\right) = E_0^1 ik e^{ik(x^1-ct)} = 0$$

$$\Rightarrow E_0^1 = 0$$

$$\Rightarrow (\underline{E}_0 \cdot \underline{i}) = 0$$

and similarly $(\underline{H}_0 \cdot \underline{i}) = 0$.



Now recall $\nabla \times \underline{E} = -\frac{\partial B}{\partial t} = -\mu \frac{\partial H}{\partial t}$

Καγβ

Ι ♥ Καβ

ΚVΕΣΣΙΟVΣ
ΚβΕΣΣΙΩΩ

Greek sigma is ς at end of word

$$\Rightarrow -\mu \frac{\partial \underline{H}}{\partial t} = ick\mu \underline{H}$$

$$\underline{\nabla} \times \underline{E} = ik(\hat{\underline{\imath}} \times \underline{E}) \quad \text{— check.}$$

$$\Rightarrow \underline{H} = \frac{1}{c\mu}(\hat{\underline{\imath}} \times \underline{E})$$

$$= \sqrt{\frac{\varepsilon}{\mu}}(\hat{\underline{\imath}} \times \underline{E}) = \frac{1}{Z}(\hat{\underline{\imath}} \times \underline{E})$$

$$Z = \sqrt{\frac{\mu}{\varepsilon}} \quad \text{electromagnetic impedance}$$

$\hat{\underline{\imath}}, \underline{E}_0, \underline{H}_0$ form orthogonal basis

~~By shifting~~

$$\underset{\text{not squared}}{E_0^{2}} e^{ik(x-ct)} = |E_0^2| e^{i\beta} \quad \text{for some } \beta.$$

$$E_0 = K_2 \hat{\underline{j}} + K_3 e^{i\alpha} \hat{\underline{k}} \qquad K_1, K_2 \geq 0 \qquad \alpha \text{ phase}$$

$$H_0 = \frac{K_2}{Z}\hat{\underline{k}} - \frac{K_3}{Z} e^{i\alpha} \hat{\underline{j}}$$

Polarisation.

Where $\alpha = 0, \pm\pi$, we have linearly polarised waves
ie $e^{i\alpha} \in \mathbb{R}$ (it's ± 1)

$$\underline{E}_0 = (K_2 \hat{\underline{j}} \pm K_3 \hat{\underline{k}}) \quad \text{real vector}$$

$$\underline{H}_0 = \frac{1}{Z}(K_2 \hat{\underline{k}} \mp K_3 \hat{\underline{j}}) \quad \text{real vector}$$

$$\mathrm{Re}\left[E_0 e^{ik(x^1-ct)}\right] = E_0 \cos\left[k(x^1-ct)\right]$$

and same with H_0.

Take $\alpha = \frac{\pi}{2}$. (~~Recall E_0~~

Recall $\underline{E}_0 = \cancel{k_3}\ k_2 \underline{\hat{j}} + i k_3 \underline{\hat{k}}$

$$\mathrm{Re}\left(E_0 e^{ik(x^1-ct)}\right) = k_2 \cos(k(x^1-ct))\underline{\hat{j}} - k_3 \sin(k(x^1-ct))\underline{\hat{k}}$$

gives you an ellipse

[Problem C: Let $\alpha \neq \frac{\pi}{2}$, $k_3 \neq 0$; ~~but~~ let $e^{i\alpha}$ is not real.
find the formula of $\underline{E}_0$] → show we get