3401 Mathematical Methods 5 Notes

Based on the 2011 autumn lectures by Dr R I Bowles

Chapter 1: ODEs

Consider $y'' + Py' + Qy = R$ for $y(x)$ with P, Q, R being functions of x. This is a linear eq^n, so the sol^n is of the form $y = CF + PI$ where:

CF: sol^n of $y'' + Py' + Qy = 0$ including constant of integration

PI: sol^n of $y'' + Py' + Qy = R$

The CF has the form $y = Ay_1(x) + By_2(x)$ (2nd order) where y_1 and y_2 satisfy the eq^n $y'' + Py' + Qy = 0$ and are linearly independent, i.e. there does not exist C_1 and C_2 s.t $c_1 y_1 + c_2 y_2 = 0$ $\forall x$.

Reduction of Order

Consider $y'' + a_1 y' + a_0 y = 0$ with $a_0(x)$ and $a_1(x)$. If we know $y = u$ is a sol^n (where $y(x) = u(x)$) then it is possible to find another by looking for one of the form $y = uv$ and solving for $V(x)$.

$$y = uv \Rightarrow y' = uv' + vu'$$
$$\Rightarrow y'' = uv'' + u'v' + u'v' + u''v$$
$$= uv'' + 2u'v' + u''v$$

Substitution will give:

$$uv'' + 2u'v' + u''v + a_1(uv' + u'v) + a_0(uv) = 0$$

and $v\underbrace{(u'' + a_1 u' + a_0 u)}_{=0 \text{ as } u \text{ is a sol}^n} = 0$

$$\Rightarrow uv'' + v'(2u' + a_1 u) = 0$$

Let $z = v' \Rightarrow z' = v''$ and divide by u

$$\Rightarrow z' + z(2u'/u + a_1) = 0$$ this is a first order DE.

This has sol^n using the integrating factor

$$e^{\int \frac{2u'}{u} + a_1 dx} = e^{2\ln u + \int^x a_1 dx'}$$
$$= u^2 e^{\int^x a_1 dx'}$$

So,

$$z'\left(u^2 e^{\int^x a_1 dx'}\right) + z\left(2uu' e^{\int^x a_1 dx} + u^2 a_1 e^{\int^x a_1 dx}\right) = 0$$

$$\Rightarrow \frac{d}{dx}\left[z\left(u^2 e^{\int^x a_1 dx'}\right)\right] = 0$$

$$\Rightarrow z\left(u^2 e^{\int^x a_1 dx'}\right) = A$$

$$z = \frac{A}{u^2} e^{-\int^x a_1 dx'} = v'$$

$$\Rightarrow v = A \int^x \frac{1}{u^2(t)} e^{-\int a_1(s) ds} dt + B$$

But, $y = uv$, so sol^n is:

$$y = \underbrace{Au \int^x \frac{1}{u^2(t)} e^{-\int^t a_1(s) ds} dt}_{\text{new } sol^n} + \underbrace{Bu}_{\text{old } sol^n}$$

This follows the form $y = Ay_1(x) + By_2(x)$

Legendre's eq^n

$$(1-x^2)y'' - 2xy' + 2y = 0 \qquad (\text{of order } 1)$$

We can spot the fact that $y = x$ is a sol^n and we can look for a second sol^n of the form $y = xv$.

Use Leibnitz or $\quad y = xv \Rightarrow y' = xv' + v$

$$\Rightarrow y'' = xv'' + 2v'$$

Reduction of order

Example: Legendre's eq^n

same method works if this $\neq 0$, ✓ spot y as if this is zero.

$$(1-x^2)y'' - 2xy' + 2y = 0$$

We spot $y = x$ as a sol^n

Look for second of the form $y = xv$

$$y' = v + xv'$$
$$y'' = x''v + 2x'v' + xv'' \qquad \text{(Leibniz' rule)}$$
$$= 2v' + xv''$$

v terms should always cancel $\because$ y is a sol^n

$$\Rightarrow (1-x^2)(2v' + xv'') - 2x(\cancel{v} + xv') + 2\cancel{xv} = 0$$

$$\Rightarrow 2v' + xv'' - x^2 2v' - x^3 v'' - 2x^2 v' = 0$$

Let $z = v'$

$$\Rightarrow 2z + xz' - 4x^2 z - x^3 z' = 0$$

$$\Rightarrow z'(x - x^3) + z(2 - 4x^2) = 0$$

$$\Rightarrow z' = \frac{z(4x^2-2)}{x(1-x^2)}$$

$$\Rightarrow \frac{1}{z}\frac{dz}{dx} = \frac{4x^2-2}{x(1-x^2)}$$

PARTIAL FRACS

$$\Rightarrow \frac{1}{z}\frac{dz}{dx} = 2\left(\frac{x}{1-x^2} - \frac{1}{x}\right)$$

$$\Rightarrow \ln z = 2\left(-\tfrac{1}{2}\ln(1-x^2) - \ln x\right) + c$$

$$\Rightarrow \ln z = \ln\left(\frac{c}{x^2(1-x^2)}\right)$$

$$\Rightarrow \quad z = \frac{c}{x^2(1-x^2)} = v'$$

$$= c\left[\frac{p}{x^2} + \frac{q}{1-x} + \frac{r}{1+x}\right]$$

$$\Rightarrow \quad p(1-x^2) + qx^2(1+x) + rx^2(1-x) = 1$$

$$\Rightarrow \quad p = 1 \qquad \left.\begin{array}{l} q+r=1 \\ q=r \end{array}\right\} \qquad p=1, \; q=r=\tfrac{1}{2}$$

$$\Rightarrow \quad v' = c\left[\frac{1}{x^2} + \frac{1}{2}\frac{1}{(1-x)} + \frac{1}{2}\frac{1}{(1+x)}\right]$$

$$\Rightarrow \quad v = c\left[-\frac{1}{x} + \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)\right]$$

And second sol^n is $y = xv$. Dropping constants,

$$y = -1 + \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right).$$

Variation of Parameters

$$y'' + Py' + Qy = R \quad \cdots\cdots (*)$$

Suppose y_1 and y_2 are indpt sol^{n}s of $y'' + Py' + Qy = 0$.

We look for a sol^n of the forced eq^n $(R \neq 0)$ of the form
$y(x) = A(x)y_1(x) + B(x)y_2(x)$ $\cdots\cdots (**)$

Note there is redundancy here since we could say

$$A \to A + By_2/y_1$$

but we'll use this later.

$$y' = \underline{A'y_1} + Ay_1' + \underline{B'y_2} + By_2'$$

Use the freedom (redundancy) to insist that

$$A'y_1 + B'y_2 = 0 \qquad \left(\begin{array}{c}\text{to avoid}\\ A'' \text{ \& } B''\end{array}\right) \quad \cdots (\dagger)$$

Then $y'' = A'y_1' + Ay_1'' + B'y_2' + By_2''$

and plugging this into $(*)$:

$$(A'y_1' + \underline{Ay_1''} + B'y_2' + \underline{By_2''}) + \underline{P(Ay_1'} + \underline{By_2')}$$

$$+ \underline{Q(Ay_1} + \underline{By_2)} = R$$

these are A.O.

these are B.O

$\because y_1, y_2$ are sol^{n}s to unforced eq^n

$$\to \quad A'y_1' + B'y_2' = R. \qquad (1)$$

and $A'y_1 + B'y_2 = 0$ from our choice $(\dagger)$. (2)

Just sim. eq^{n}s.

$$y_2(1) - y_2'(2) \;:\; A'(y_1'y_2 - y_1y_2') = y_2R$$

$$-y_1(1) + y_1'(2) \;:\; B'(y_2y_1' - y_1y_2') = -y_1R$$

The expression $y_1 y_2' - y_2 y_1' = \begin{vmatrix} y_1 & y_1' \\ y_2 & y_2' \end{vmatrix} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

The Wronskian $W(x)$ of the two f^{n}s y_1 & y_2.

$$\Rightarrow \quad A' = -\frac{y_2 R}{W}$$

$$B' = \frac{y_1 R}{W}$$

$$\Rightarrow \quad A = -\int^x \frac{y_2(s) R(s)}{W(s)}\, ds + A_0$$

$$B = \int^x \frac{y_1(s) R(s)}{W(s)}\, ds + B_0$$

and the sol^{n} is $A y_1(x) + B y_2(x)$ [from (**)]

$$y = A_0 y_1 + B_0 y_2 + \int^x \frac{y_1(s) y_2(x) - y_1(x) y_2(s)}{W(s)} R(s)\, ds$$

Example: Solve $y'' + y = \sec(x)$

The CF is $y = A\cos(x) + B\sin(x)$ and we look for a PI of the form $y = A(x)\cos(x) + B(x)\sin(x)$.

So $y' = A'c + A(-s) + B'(s) + Bc$

and we choose $A'c + B's = 0$

So $y' = A(-s) + Bc$.

and $y'' + y = A'(-s) + \cancel{A(-c)} + B'(c) + B(-s)$
$+ \cancel{Ac} + Bs = \sec(x)$.

$$A'(-s) + B'(c) = \sec(x) \qquad (1)$$
$$A'(c) + B'(s) = 0 \qquad (2)$$

$c(1) + s(2)$: $B'\underbrace{(c^2 + s^2)}_{1} = \cos x \sec x = 1$

$\Rightarrow$ $B = x + B_0$

$s(1) - c(2)$: $-A'\underbrace{(c^2 + s^2)}_{1} = \sin x \sec x = \tan x$

$\Rightarrow A = \ln(\cos x) + A_0$

and $y(x) = \cos(x)\ln(\cos(x)) + x\sin(x)$
$+ A_0\cos(x) + B_0\sin(x)$

Properties of the Wronskian

The Wronskian of y_1 and y_2 is

$$W = y_1 y_2' - y_1' y_2 = \begin{vmatrix} y_1 & y_1' \\ y_2 & y_2' \end{vmatrix}$$

where y_1 and y_2 satisfy

$$y_1'' + Py_1' + Qy_1 = 0 \qquad (1)$$
$$y_2'' + Py_2' + Qy_2 = 0 \qquad (2)$$

$$y_2(1) - y_1(2): \quad (y_2 y_1'' - y_2'' y_1) + P(\overbrace{y_1' y_2 - y_1 y_2'}^{-w}) = 0$$

and $\frac{dw}{dx} = \cancel{y_1' y_2'} + y_1 y_2'' - \cancel{y_1' y_2'} - y_1'' y_2$

$$\Rightarrow \frac{dw}{dx} - Pw = 0$$

and $w(x) = Ce^{-\int^x P(s)\,ds}$

Observe: this is never zero unless $C = 0$.

Generalised Transforms

We will look for sol^{n}s to

$$(a_1 x + a_0) y'' + (b_1 x + b_0) y' + (c_1 x + c_0) y = 0$$

[note degree of the polynomial coefficients (1) is less than the degree of the diff. eq^n (2). This method works only if this is the case.]

sometimes e^{-xt}

of the form $y(x) = \int_C e^{xt} f(t)\,dt$

If $y(x)$ is defined in this way, then

$$\frac{dy}{dx} = \int_C \frac{\partial}{\partial x}\left[e^{xt} f(t)\right] dt$$

(from Analysis 2)

$$= \int_C e^{xt}\, t f(t)\,dt$$

and $$\frac{d^2y}{dx^2} = \int_{\mathcal{C}} e^{xt}\, t^2 f(t)\, dt$$

Let us choose, for the moment, $a_1 = b_1 = c_1 = 0$ and so

$$a_0 y'' + b_0 y' + c_0 y = 0$$

and $y(x) = \int_{\mathcal{C}} e^{xt} f(t)\, dt$ is a sol^n if

$$\int_{\mathcal{C}} \underbrace{\left[a_0 t^2 + b_0 t + c_0\right] e^{xt} f(t)}_{g}\, dt = 0$$

We must choose f s.t. there are no singularities of g within $\mathcal{C}$ but there are singularities of f within $\mathcal{C}$.

If the roots of the auxiliary eq^n are α and β, $\in \mathbb{C}$ this requires, $\alpha \neq \beta$,

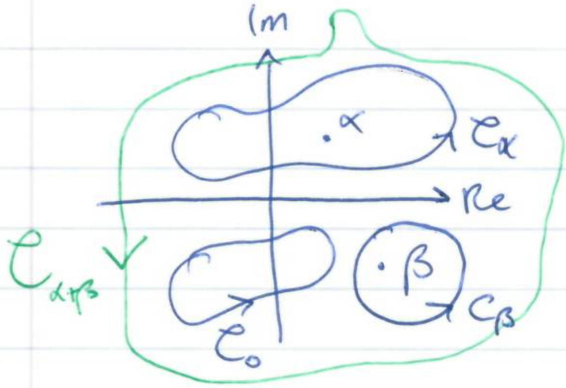
$$a_0 \int_{\mathcal{C}} (t-\alpha)(t-\beta) f(t)\, e^{xt}\, dt = 0$$

Let us choose $f(t) = \frac{A}{t-\alpha} + \frac{B}{t-\beta}$

then we require $\int_{\mathcal{C}} \left[A(t-\beta) + B(t-\alpha)\right] e^{xt}\, dt = 0$

which is true for any closed $\mathcal{C}$.

Now, $$y(x) = \int_{\mathcal{C}} \left(\frac{A}{t-\alpha} + \frac{B}{t-\beta}\right) e^{xt}\, dt$$



C_0 : get zero $\because$ no sings

C_α: nonzero, get α part

Recall Cauchy's integral theorem:

$$\oint_C \frac{g(t)}{t-t_0}\,dt = 2\pi i\, g(t_0)$$

if $g(t)$ is analytic with C and t_0 is in C.

If $C = C_\alpha$ above, then we obtain $\tilde{A}e^{\alpha x}$
If $C = C_\beta$ . . . . . . $\tilde{B}e^{\beta x}$
If $C = C_{\alpha+\beta}$ . . . . . . $\tilde{A}e^{\alpha x} + \tilde{B}e^{\beta x}$

$$a_0 \int_C (t-\alpha)^2 e^{xt} f(t) = 0$$

$$\int_C e^{xt} f(t)\,dt \neq 0$$

If α is a repeated root, we choose $f(t)$ to be

$$\frac{A}{(t-\alpha)^2} + \frac{B}{t-\alpha}$$

and then

$$a_0 \int_C (t-\alpha)^2 f(t) e^{xt}\,dt$$

$$= \int_C \left[A + B(t-\alpha)\right] e^{xt}\,dt = 0$$

but $y(x) = \int_{C_\alpha} e^{xt}\left[\frac{A}{(t-\alpha)^2} + \frac{B}{t-\alpha}\right] dt$

To find the residue, the coefficient of $\frac{1}{t-\alpha}$ in the Laurent expansion about $t=\alpha$, write the integrand as

$$e^{\alpha x} e^{x(t-\alpha)}\left[\frac{A}{(t-\alpha)^2} + \frac{B}{t-\alpha}\right]$$

and expand in $(t-\alpha)$:

$$\approx x e^{\alpha x}\left[1 + x(t-\alpha) + \dots\right]\left[\frac{A}{(t-\alpha)^2} + \frac{B}{t-\alpha}\right]$$

and so the residue is

$$e^{\alpha x}(1 \cdot B + x \cdot A)$$

and redefining our A & B with a factor of $2\pi i$, we obtain the sol^n.

$$e^{\alpha x}(Ax + B)$$

Note: If we had a third repeated root, we'd end up with an x^2 term in the equation above ↑.

$$(a_1 x + a_0) y'' + (b_1 x + b_0) y + (c_1 x + c_0) y = 0$$

$$y(x) = \int_{C} e^{xt} f(t)\, dt.$$

Substitution leads to

$$\int_{C} \left[x(a_1 t^2 + b_1 t + c_1) + (a_0 t^2 + b_0 t + c_0)\right] e^{xt} f(t)\, dt = 0$$

If we can write the integrand as $\frac{d}{dt}(\ )$, say $\frac{d}{dt}\left(e^{xt} g(t)\right)$. then this leads to the requirement

that $\left[e^{xt} g(t)\right]_{\text{start of contour}}^{\text{end of contour}} = 0$.

$$\frac{d}{dt}\left[e^{xt} g(t)\right] = x e^{xt} g + e^{xt} g'$$

So we can identify
$$g(t) = (a_1 t^2 + b_1 t + c_1) f(t)$$
$$g'(t) = (a_0 t^2 + b_0 t + c_0) f(t)$$

and $\dfrac{g'}{g} = \dfrac{a_0 t^2 + b_0 t + c_0}{a_1 t^2 + b_1 t + c_1}$. Integrate to find $g(t)$.

To find f we have

$$f = \frac{g}{a_1 t^2 + b_1 t + c_1} = \frac{g'}{a_0 t^2 + b_0 t + c_0}$$

and we can now choose $\mathcal{C}$ s.t.

$$\int_{\mathcal{C}} f e^{xt}\, dt \neq 0 \quad \text{but} \quad \left[g e^{xt}\right]_{\mathcal{C}} = 0.$$

Example $\quad xy'' + 4y' - xy = 0, \qquad x > 0$

Look for a solⁿ of the form $y(x) = \int_C e^{xt} f(t)\, dt$.

Substitution requires

$$\int_C \left[x(t^2-1) + 4t \right] e^{xt} f(t)\, dt = 0$$

If this is $\int_C \frac{d}{dt}\left[e^{xt} g\right] dt$, then we need

$$\begin{aligned} g &= (t^2-1) f \\ g' &= 4tf \end{aligned} \quad \Rightarrow \quad \frac{g'}{g} = \frac{4t}{t^2-1}$$

$$\Rightarrow \ln g = 2 \ln (t^2-1)$$

$$\Rightarrow \quad g(t) = (t^2-1)^2$$

$$\text{and } f(t) = t^2 - 1$$

So we must choose C s.t.

$$y(x) = \int_C e^{xt}(t^2-1)\, dt \neq 0 \quad \leftarrow$$

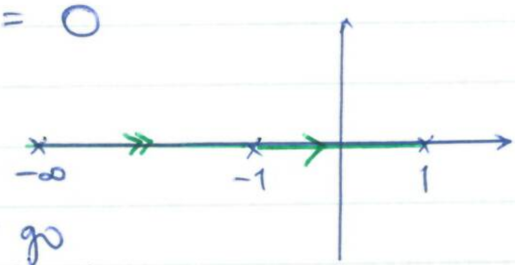
no sings so we can't have a closed contour

$$\text{and } \underbrace{\left[e^{xt}(t^2-1)^2 \right]}_{\text{zeroes at } \pm 1}{}^{\text{end of } C}_{\text{start of } C} = 0$$

$e^{xt} g$

and at $-\infty$ $(\because x>0)$

so make contour go from one zero to the other.



One solⁿ is $\quad y_1 = \int_{-1}^{1} e^{xt}(t^2-1)\, dt$

Second is $\quad y_2 = \int_{-\infty}^{-1} e^{xt}(t^2-1)\, dt$

and so $y(x) = Ay_1(x) + By_2(x)$.

What happens to $y_1(x)$ and $y_2(x)$ as $x \to 0, \infty$?

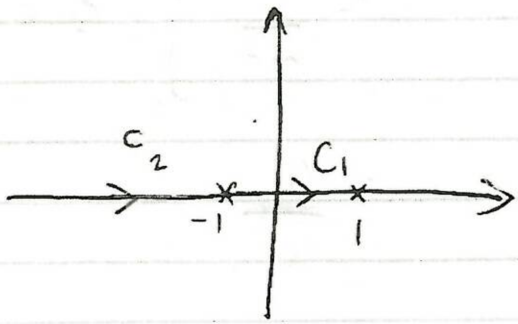
$y_1(0) = \int_{-1}^{1} (t^2-1)\,dt$, finite, as $x \to 0$, $e^{xt} \to 1$ for $t \in [-1,1]$

y_2 is singular at $x = 0$.

same example continued

$$xy'' + 4y' - xy = 0 \qquad x > 0$$

$$y = \int_C e^{xt} f(t)\, dt$$



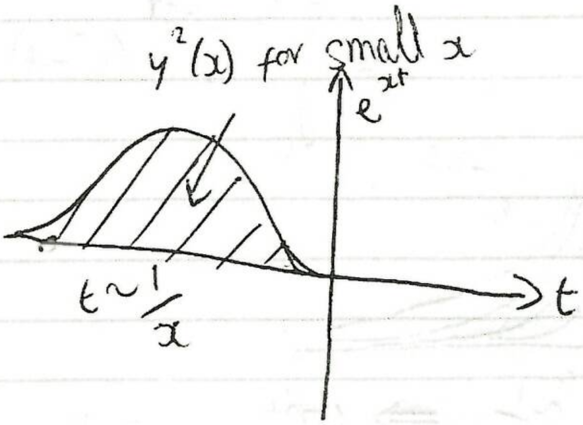
$$f(t) = t^2 - 1$$

$$\left[e^{xt} (t^2-1)^2 \right]_C = 0$$

$$y_1 = \int_{-1}^{1} e^{xt} (t^2-1)\, dt$$

$$y_2 = \int_{-\infty}^{-1} e^{xt} (t^2-1)\, dt$$

<u>$x \to 0$</u> $\quad y_1(0)$ finite
$y_2(0)$ undefined



Examine more carefully

$$y_2(x) = \int_{-\infty}^{-1} e^{xt} (t^2-1)\, dt$$

A more sensible variable for the integration is one which has xt of size one when $x \to 0$ is $s = -xt$

$$dt = \frac{-ds}{x}$$

$$y_2(x) = \int_x^\infty e^{-s}\left(\frac{s^2}{x^2}-1\right)\frac{ds}{x}$$

$$\simeq \frac{1}{x^3}\int_0^\infty s^2 e^{-s}\,ds \qquad \text{as } x \to 0$$

$$= \frac{2}{x^3}$$

$\underline{x \to \infty}$

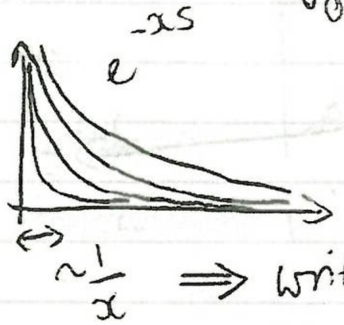
For y_1 in the interval $[0,1]$ over which we integrate e^{xt} is going to get exponentially large.

write $$y_1 = \int_{-1}^{0} e^{xt}(t^2-1)\,dt + \int_0^1 e^{xt}(t^2-1)\,dt$$

dominant contribution (exp large is from $t=1$)

$$= \int_0^1 e^{-sx}(s^2-1)\,ds + \int_0^1 e^{xt}(t^2-1)\,dt$$

$s=-t$



Basically want to integrate w.r.t a variable of size 1 ?

$\Rightarrow$ write $s = \dfrac{u}{x}$

$$= \int_0^x e^{-u}\left(\frac{u^2}{x^2}-1\right)\frac{du}{x} + \int_0^1 e^{xt}(t^2-1)\,dt$$

To deal with second term, let $t = 1 - \frac{v}{x}$

$$= \int_0^x e^{-u}\left(\frac{u^2}{x^2} - 1\right)\frac{du}{x} + e^x \int_0^x e^{-v}\left(\frac{-v}{x}\right)\left(2 - \frac{v}{x}\right)\frac{dv}{x}$$

$$\approx \underbrace{\int_0^\infty e^{-u}\left(\frac{u^2}{x^2} - 1\right)\frac{du}{x}}_{} + \underbrace{e^x \int_0^\infty \underbrace{e^{-v}\left(\frac{-v}{x}\right)\left(2 - \frac{v}{x}\right)\frac{dv}{dx}}_{\text{error size } e^{-x} \text{ (same reason as other integral)}}}_{\text{error size } 1}$$

makes error of size

$$\int_x^\infty \ldots$$

since integrand is exponentially small because of e^{-u}.

So error is of size e^{-x}

The first term is of size 1, so there's no point having it, because it's the same size as the error of the second term.

$$\approx -\frac{2e^x}{x^2}\underbrace{\int_0^\infty ve^{-v}\,dv}_{1} + \frac{e^x}{x^3}\underbrace{\int_0^\infty v^2 e^{-v}\,dv}_{2}$$

1 comes from $\left(\frac{-v}{x}\right)(2)$, 2 comes from $\left(\frac{-v}{x}\right)\left(\frac{-v}{x}\right)$

These integrals can be found explicitly

$$y_1(x) = 2\int_0^1 (t^2-1)\cosh(xt)\,dt$$

$$= \frac{4}{x^3}\left(\sinh x - x\cosh x\right)$$

Use l'hôpital's rule as $x \to 0$ finite result

$$y_2(x) = \int_1^\infty (t^2-1)e^{-xt}\,dt$$

$$= \frac{2e^{-x}}{x^3}(1+x)$$

Back to $\quad x'' + 4y' - xy = 0 \quad$ another method

$$y = \int_C e^{xt} f(t)\,dt$$

$$\int_C \left[x(t^2-1) + 4t\right] e^{xt} f(t)\,dt = 0$$

$$\int_C (t^2-1) f(t) \frac{d}{dt}\left(e^{xt}\right) dt + \int_C 4t\, f(t)\, e^{xt}\,dt = 0$$

$$\Big[(t^2-1)f(t)e^{xt}\Big]_C + \int_C \left[4tf - \frac{d}{dt}\left[(t^2-1)f\right]\right] e^{xt}\,dt = 0$$

these bits are integration ...

which is true if f satisfies

$$4t f = \frac{d}{dt}\left[(t^2-1) f\right] \Rightarrow 2t f = (t^2-1) f'$$

& then f is chosen so

$$\left[(t^2-1) f e^{xt}\right]_c = 0$$

$$\Rightarrow \quad \frac{f'}{f} = \frac{2t}{t^2-1}$$

$$f = (t^2-1)$$

& we need $\left[(t^2-1)^2 e^{xt}\right]_c = 0$

Another example

$$xy'' + (3x-1)y' - 9y = 0 \quad , \quad x > 0$$

$$y = \int_c e^{xt} f(t)\, dt$$

Substitution requires $\displaystyle\int_c \left\{x(t^2+3t) - (t+9)\right\} e^{xt} f(t)\, dt = 0$

Using integration by parts

$$= \left[t(t+3) f(t) e^{xt}\right]_c - \int_c \left[\frac{d}{dt}\left((t^3+3t)f\right) + (t+9) f\right] e^{xt}\, dt = 0$$

Could possibly ask for soln of form $y = \int_C e^{-xt} f(t)\,dt$

be very careful with - signs!

We require $\frac{d}{dt}\left((t^2+3t)f\right) + (t+9)f = 0$

$$(2t+3)f + t(t+3)f' + (t+9)f = 0$$

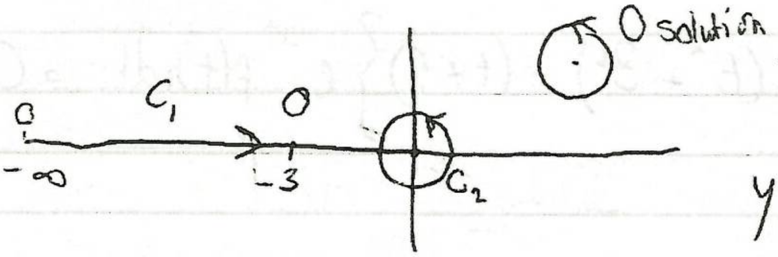
$$\frac{f'}{f} = -\left(\frac{3t+12}{t(t+3)}\right)$$

$$= \frac{1}{t+3} - \frac{4}{t}$$

$$f = \frac{(t+3)}{t^4}$$

& the solution is $y(x) = \int_C \frac{(t+3)}{t^4} e^{xt}$

where c is such that $\left[e^{xt}\frac{(t+3)^2}{t^3}\right]_C = 0$



So one solution is

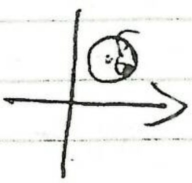
$$y_1(x) = \int_{-\infty}^{-3} \frac{(t+3)e^{xt}}{t^4}\,dt$$

As $e^{xt}\frac{(t+3)^2}{t^3}$ has no branch cuts, $\left[e^{xt}\frac{(t+3)^2}{t^3}\right]_{C_2} = 0$

& $y_2(x) = \int_{C_2} e^{xt} \frac{(t+3)}{t^4}\, dt \neq 0$ due to pole at 0

Cauchy's Integral Theorem for derivatives is

analytic within c



$$\frac{n!}{2\pi i} \int_C \frac{f(t)}{(t-t_0)^{n+1}}\, dt$$

$$= f^{(n)}(t_0)$$

$$y_2 = \frac{1}{2\pi i} \int_{C_2} \frac{e^{xt}(t+3)\,dt}{t^4} = \frac{1}{3!} \frac{d^3}{dt^3} \left(e^{xt}(t+3)\right)\Big|_{t=0}$$

y_2 unique up to multiplicative constant, added this to be nice

$$= \frac{1}{3!}\left(x^3 e^{xt}(t+3) + 3x^2 \cdot 1\right)\Big|_{t=0}$$

$$= \frac{1}{2}(x^3 + x^2)$$

<u>Look at y_1</u>

$$y_1(x) = \int_{-\infty}^{-3} \frac{(t+3)e^{xt}}{t^4}\, dt$$

$$= \int_3^{\infty} \frac{(3-t)}{t^4} e^{-xt}\, dt \rightarrow 0 \text{ as } x \to \infty$$

At $x = 0$, $y_1(0)$ exists as $\int_3^\infty \frac{(3-t)}{t^4}\, dt$ exists as integrand $\sim \frac{1}{t^3}$ as $t \to \infty$

Look for singularities at 0

$$y_1'(0) = -\int_3^\infty \frac{-(3-t)}{t^3}\,dt \quad < \infty$$

$$y_1''(0) = \int_3^\infty \frac{(3-t)}{t^2}\,dt \quad \text{which diverges.}$$

<u>New example</u>

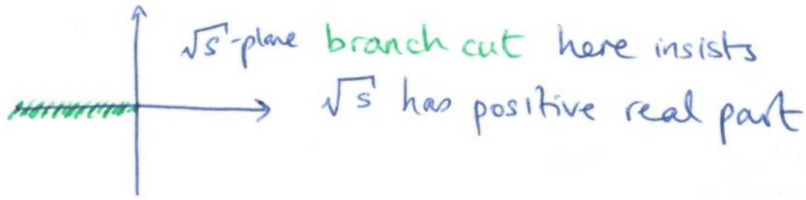
$$xy'' + (1-x)y' + ay = 0$$

has sol$^{\underline{n}}$s $$y = \int_C e^{xt}\frac{t^{a-1}}{(t-1)^a}\,dt$$

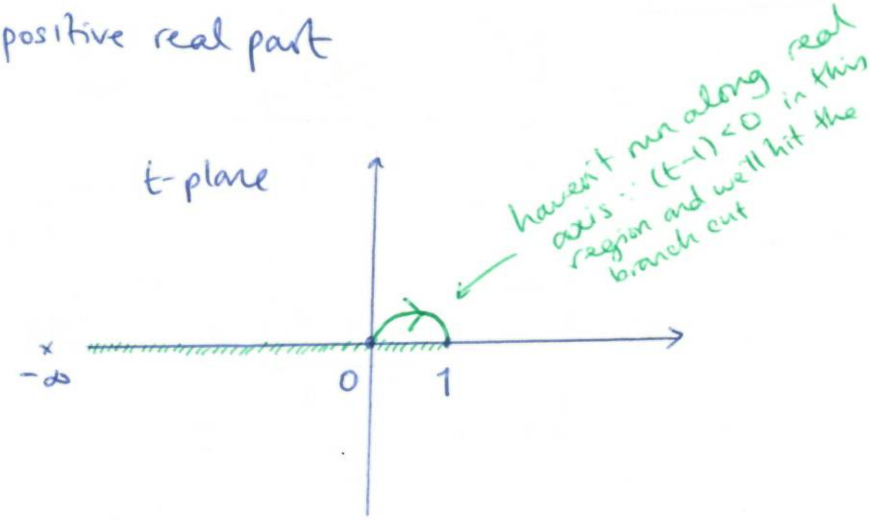
if $$\left[\frac{t^a e^{xt}}{(t-1)^{a-1}}\right]_C = 0$$

If $a = \frac{1}{2}$, $$y = \int_C \frac{e^{xt}}{\sqrt{t}\sqrt{t-1}}\,dt$$

and $$\left[\sqrt{t}\sqrt{t-1}\,e^{xt}\right]_C = 0$$



Where are the zeroes?



Parameterise by running along the interval $[0,1]$ in t just above the branch cut.

likely exam q^n type

$$\int_0^1 \frac{e^{xt}}{\sqrt{t}\sqrt{1-t}\,(+i)}\,dt$$

$\because$ we are above real axis. If below, we would have $(-i)$ here.

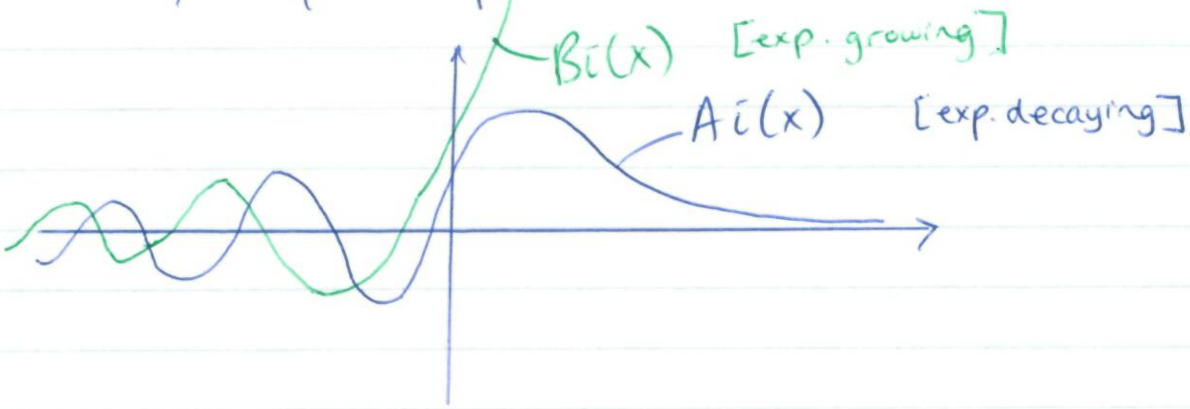
and the corresponding real sol^n is

$$\int_0^1 \frac{e^{xt}}{\sqrt{t}\sqrt{1-t}}\,dt$$

Airy's Equation

$$y'' - xy = 0$$

If $x<0$, expect oscillatory sol^{n}s
$x>0$, expect exponential sol^{n}s



Try $\quad y = \int_c f(t)\, e^{xt}\, dt$

and substitution gives $\displaystyle\int_c (t^2 - x) f(t)\, e^{xt}\, dt = 0$

parts $\Rightarrow \left[-f e^{xt}\right]_c + \displaystyle\int_c (t^2 f + f')\, e^{xt}\, dt = 0$

So choose f s.t. $f' + t^2 f = 0$, ie $f = e^{-t^3/3}$

and our sol^n is $\quad y(x) = \int_C e^{xt - \frac{1}{3}t^3}\, dt$

where $\left[e^{xt - \frac{1}{3}t^3} \right]_C = 0$

need to find zeroes.

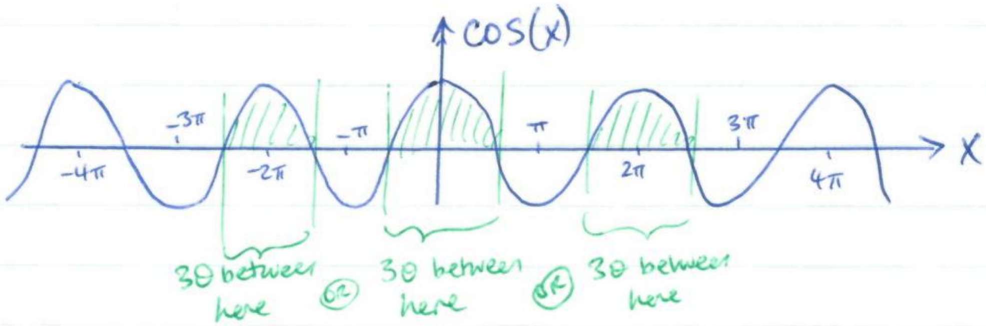
Where are they? At <u>infinities</u>. Investigate!

C must start & end at infinity in a direction in which $e^{xt-\frac{1}{3}t^3}$ is exponentially small.

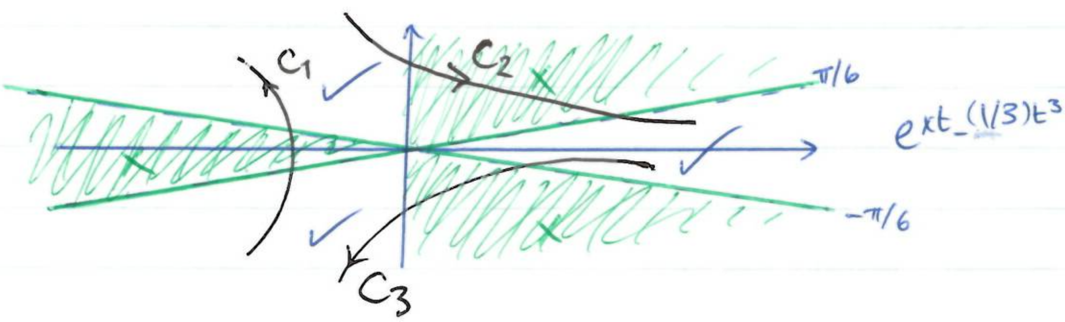
Write $t = Re^{i\theta}$ and the exponential is

$$e^{xRe^{i\theta} - \frac{1}{3}R^3(\cos 3\theta + i\sin 3\theta)}$$

which is exponentially <u>small</u> for $R \to \infty$ where $\cos 3\theta > 0$.



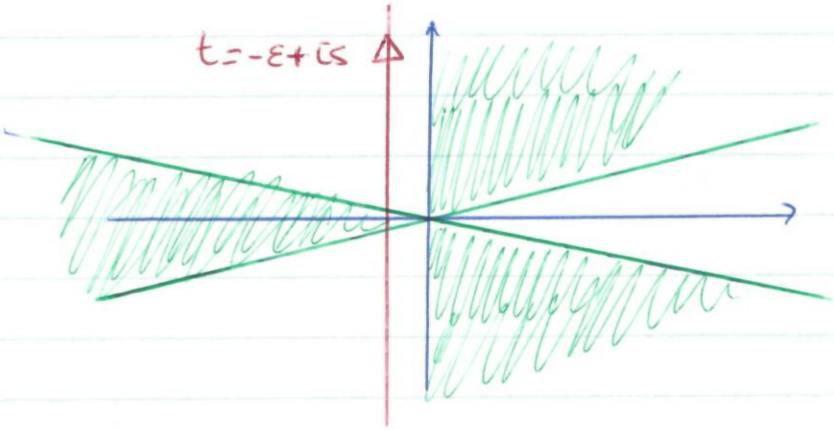
$-\frac{5\pi}{6} < \theta < -\frac{\pi}{2} \qquad -\frac{\pi}{6} < \theta < \frac{\pi}{6} \qquad \frac{\pi}{2} < \theta < \frac{5\pi}{6}$



$\square$ Note $y_1 + y_2 + y_3 = 0$.

$$Ai(x) = \frac{1}{2\pi i} y_1(x)$$

$$Bi(x) = \frac{1}{2\pi} (y_2(x) - y_3(x))$$



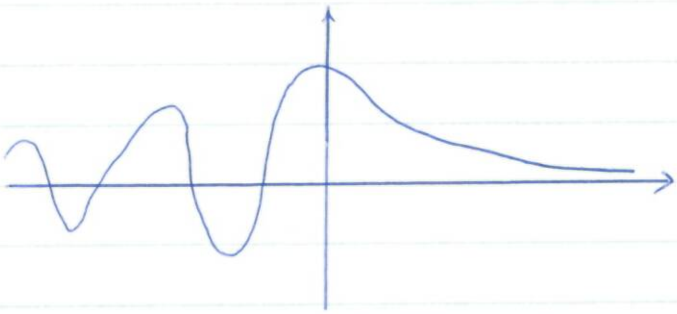
$$Ai(x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{xis + is^3/3} \, i \, ds \qquad \begin{aligned} t &= is \\ dt &= i\,ds \end{aligned}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(xs + s^3/3)} \, ds$$

← this integral doesn't quite exist but if we use $t = -\varepsilon + is$, the $e^{-\varepsilon s^2}$ terms come out and make this OK

$\left(= \int (\cos + i \sin), \text{ which doesn't decay} \right)$

$$= \frac{1}{\pi} \int_0^{\infty} \cos\left(xs + \frac{s^3}{3}\right) ds$$



RIB hates this topic

PHASE PLANE ANALYSIS OF ODEs

It is not possible to find explicit sol^{n}s to all first-order ODEs. Our collection of "standard f^{n}s" is not big enough. Also although for separable eq^{n}s, we might find sol^{n}s in terms of integrals, we don't know how the sol^{n}s behave.

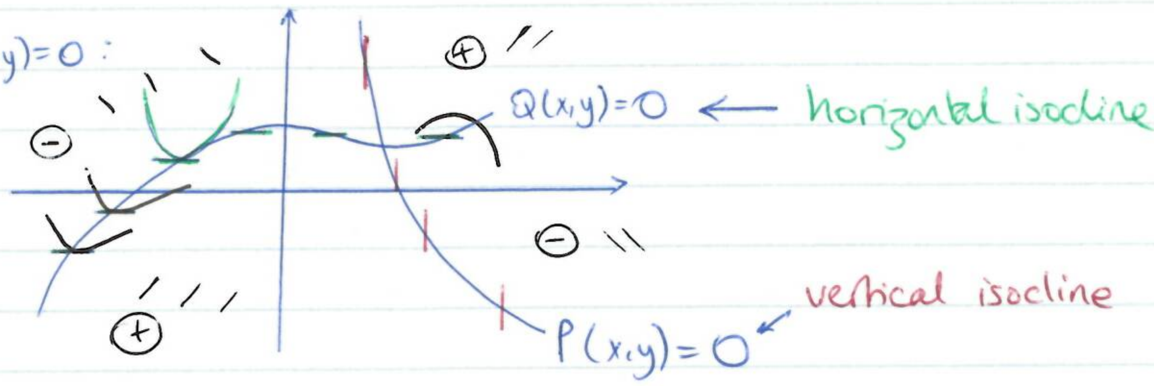
Phase plane analysis allows us to find out a great deal about the qualitative nature of the sol^n to first order ODEs.

An ODE is $\quad \dfrac{dy}{dx} = F(x,y) = \dfrac{Q(x,y)}{P(x,y)}$

The sol^n curves drawn in the x-y plane ("phase plane") are known as <u>integral curves</u>, or <u>trajectories</u>.

If Q and P are single-valued (ie they output a number) then the integral curves cannot cross (unless $Q=P=0$).

If $Q(x,y)=0$:



Points where P & Q are both zero are called "critical points" and at such points integral curves <u>can</u> cross.

Every time we cross an isocline, $\frac{dy}{dx} = \frac{Q}{P}$ changes sign. [beware not always, e.g. $Q=1$, $P=x^2$]

Gin's fault

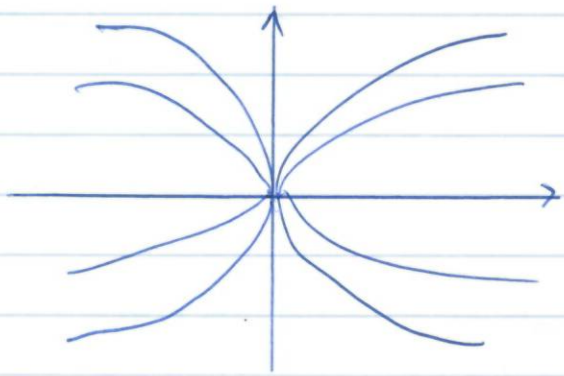
Suppose slopes are negative in top left.

Easy illustrative examples of sol^{n}s at critical points

(1) $\dfrac{dy}{dx} = \dfrac{y}{2x} = \dfrac{Q}{P}$ $\quad Q(x,y) = y$, $P(x,y) = 2x$ $\quad \left\{\begin{matrix} x=0 \\ y=0 \end{matrix}\right\}$ critical pt.

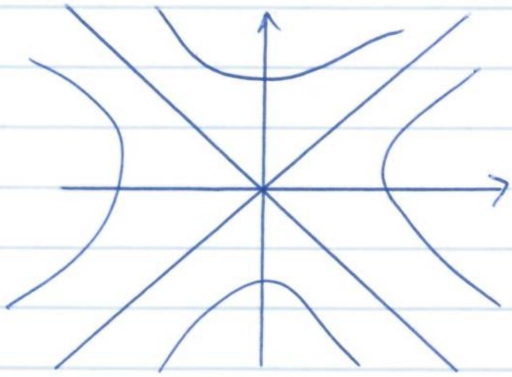
$$\frac{dy}{y} = \frac{1}{2}\frac{dx}{x} \Rightarrow \ln y = \frac{1}{2}\ln x + C$$

$$y^2 = Cx$$



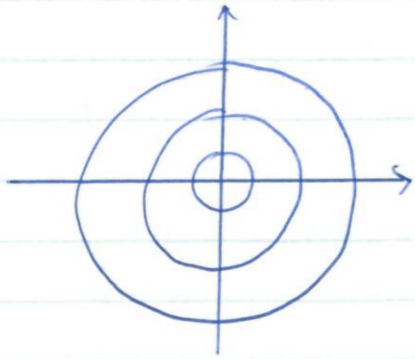
all trajectories cross

(2) $\dfrac{dy}{dx} = \dfrac{x}{y} \Rightarrow y^2 - x^2 = C$



only two special trajectories cross

(3) $\frac{dy}{dx} = -\frac{x}{y} \Rightarrow y^2 + x^2 = C$



here none cross.

Problems which have x and y as f$^{\underline{n}}$s of a third variable, say t, have for example

$$\frac{dx}{dt} = P\left(x(t), y(t)\right) \qquad \frac{dy}{dt} = Q\left(x(t), y(t)\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}$$

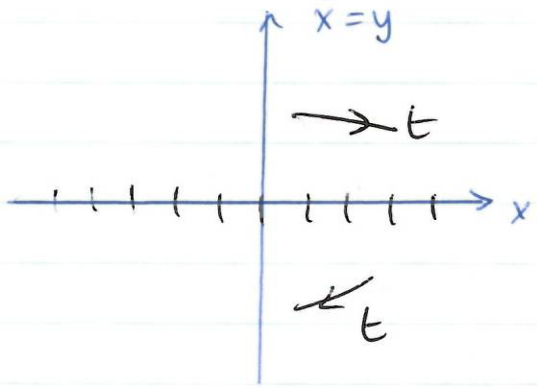
Consider too the second-order equation

$$\frac{d^2x}{dt^2} = Q\left(x, \frac{dx}{dt}, t\right)$$

If Q does not depend on t, so

$$\frac{d^2x}{dt^2} = Q\left(x, \frac{dx}{dt}\right),$$

then this eq$^{\underline{n}}$ is called <u>autonomous</u> and the substitution $y = \frac{dx}{dt}$ allows us to write



$$\frac{dy}{dt} = Q(x, y)$$

$$\frac{dx}{dt} = y = P(x, y)$$

$$\Rightarrow \frac{dy}{dx} = \frac{Q}{P} = \frac{Q}{y}$$

Note all trajectories must cross the x-axis being vertical ($dy/dx = \infty$) unless $Q = 0$.

<u>Examination of the trajectories near critical points</u>

Let (x_0, y_0) be a critical point, ie $P(x_0, y_0) = Q(x_0, y_0) = 0$.

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}.$$

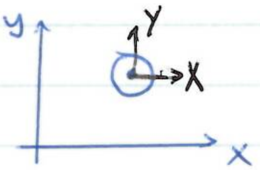
Close to these critical points, we have

$$P(x,y) = P(x_0,y_0) + \left.\frac{\partial P}{\partial x}\right|_{(x_0,y_0)} (x-x_0) + \left.\frac{\partial P}{\partial y}\right|_{(x_0,y_0)} (y-y_0) + \cdots$$

$$Q(x,y) = Q(x_0,y_0) + \left.\frac{\partial Q}{\partial x}\right|_{(x_0,y_0)} (x-x_0) + \left.\frac{\partial Q}{\partial y}\right|_{(x_0,y_0)} (y-y_0) + \cdots$$

Now, let $X = x - x_0$

$Y = y - y_0$



Then $P(x_0,y_0) = Q(x_0,y_0) = 0$.

Then $\left(\frac{dy}{dx}\right) = \frac{dY}{dX} = \frac{Q_x X + Q_y Y}{P_x X + P_y Y}$ $\qquad Q_x = \left.\frac{\partial Q}{\partial x}\right|_{(x_0,y_0)}$

$$= \frac{CX + DY}{AX + BY}$$

where $\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} P_x & P_y \\ Q_x & Q_y \end{pmatrix} \longleftarrow$ Jacobian of the singular point

and $\frac{dX}{dt} = AX + BY$

$\frac{dY}{dt} = CX + DY$

or $\frac{d}{dt}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}$

$$\dot{\underline{x}} = J\underline{x}$$

Brute force silly method

We have
$$\ddot{Y} = C\dot{X} + D\dot{Y}$$
$$= C(AX + BY) + D\dot{Y}$$

But $X = \dfrac{\dot{Y} - DY}{C}$ (rearranging $\frac{dY}{dt} = CX + DY$)

$$\Rightarrow \quad \ddot{Y} = (A+D)\dot{Y} + (CB - AD)Y$$

$$\Rightarrow \quad \ddot{Y} \underbrace{- (A+D)}_{p = -\mathrm{tr}(J)} \dot{Y} + \underbrace{(AD - CB)}_{q = \det(J)} Y = 0$$

Look for roots of the auxiliary eq^n λ_1, λ_2, $\lambda_2 \leq \lambda_1$, which satisfy
$$\lambda^2 + p\lambda + q = 0$$

$$\Rightarrow \quad \lambda = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$$

$$\Rightarrow \quad Y(t) = \alpha e^{\lambda_1 t} + \beta e^{\lambda_2 t}$$

But $\dot{X} = AX + BY$

$$\dot{X} - AX = BY = B\alpha e^{\lambda_1 t} + B\beta e^{\lambda_2 t}$$

$$\frac{d}{dt}\left[e^{-At}X\right] = B\alpha e^{(\lambda_1 - A)t} + B\beta e^{(\lambda_2 - A)t}$$

$$\Rightarrow \quad X = \frac{B\alpha}{\lambda_1 - A} e^{\lambda_1 t} + \frac{B\beta}{\lambda_2 - A} e^{\lambda_2 t} + \gamma e^{At}$$

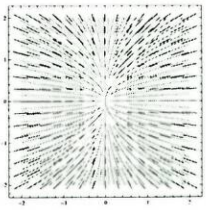
Nonlinear differential equations - phase plane analysis

We consider the general first order differential equation for $y(x)$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y) = \frac{Q(x,y)}{P(x,y)}. \tag{1}$$

1 Revision

Curves in the (x,y)-plane which satisfy this equation are called ***integral curves*** or ***trajectories***. There is a family of such curves, paremterised by the constant of integration associated with solving the equation. The slope of an integral curve that passes through the point (x_0, y_0) is $f(x_0, y_0) = P(x_0, y_0)/Q(x_0, y_0)$ and hence is a unique slope, except perhaps where $f(x_0, y_0)$ is undetermined, i.e. $P(x_0, y_0) = Q(x_0, y_0) = 0$. Hence the only place that the trajectories can intersect is at points where $P = Q = 0$. These are called ***singular points***, or ***equilibrium points***. We will investigate the trajectories in the vicinity of such points below.



Example

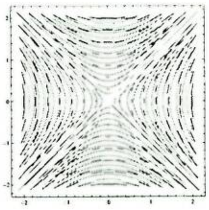
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{2x} \Rightarrow \int \frac{2\mathrm{d}y}{y} = \int \frac{\mathrm{d}x}{x} \Rightarrow \ln y^2 = \ln x + C' \Rightarrow y^2 = Cx.$$

ll trajectories cross at $(0,0)$ where $f(x,y) = y/2x$ is undetermined.

```
VectorPlot[{2x,y},{x,-2,2},{y,-2,2},StreamScale->None,
 StreamPoints->Fine,StreamStyle->Red,VectorStyle->Arrowheads[0]]
```

Mathematica 7+

Example



$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y} \Rightarrow \int y\,\mathrm{d}y = \int x\,\mathrm{d}x \Rightarrow y^2/2 = x^2/2 + C' \Rightarrow y^2 - x^2 = C.$$

Only two trajectories cross at $(0,0)$ where $f(x,y) = x/y$ is undetermined. These are given by $C = 0$.

```
VectorPlot[{y,x},{x,-2,2},{y,-2,2},StreamScale->None,
 StreamPoints->Fine,StreamStyle->Red,VectorStyle->Arrowheads[0]]
```

2 Second-order equations

The most general form is for a second order equation for $x(t)$ is $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = Q(x, \frac{\mathrm{d}x}{\mathrm{d}t}, t)$. However such an equation is called ***utonomous*** if the coefficients do not depend explicitly on t so that

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = Q\left(x, \frac{\mathrm{d}x}{\mathrm{d}t}\right). \tag{2}$$

For these equations we may introduce

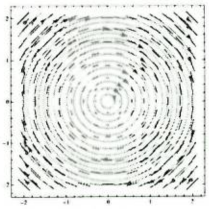
$$y = \frac{\mathrm{d}x}{\mathrm{d}t} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = Q\left(x, \frac{\mathrm{d}x}{\mathrm{d}t}\right) = Q(x,y) \text{ and } \frac{\mathrm{d}x}{\mathrm{d}t} = y = P(x,y) \text{ giving } \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = \frac{Q(x,y)}{P(x,y)} = \frac{Q}{y}.$$

So (2) can be written as a special case of (1). In this case the (x,y)-plane is an $(x,\dot{x})$-plane, known as a ***phase-plane*** and the integral curve/trajectory may also be called a ***phase-trajectory***. The trajectories are solutions of the equations $\dot{x} = y$, $\dot{x} = Q(x,y)$, with t as an effective parameter taking us along a trajectory. The trajectories are therefore traversed in a particular direction as t increases. This direction is easy to identify as it is in the direction of increasing x ($\dot{x} > 0$) in the upper-half plane $y = \dot{x} > 0$. Singular points are more often called equilibrium points in this context since at such a point, $x = x_0$, $y = 0$,say, $P = Q = \dot{x} = \dot{y} = \ddot{x} = 0$ and, if x represents the displacement of a particle, for example, in some physical system, a particle placed exactly at $x = x_0$ so that $y = 0$ will stay there, in equilibrium.

Example

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -x, \quad \text{so} \quad \dot{y} = -x, \quad Q = -x, \quad \dot{x} = y, \quad P = y.$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-x}{y} \Rightarrow \int y \,\mathrm{d}y = -\int x \,\mathrm{d}x \Rightarrow y^2/2 = -x^2/2 + C' \Rightarrow y^2 + x^2 = C.$$



Here no trajectories cross at $(0,0)$ where $f(x,y) = -x/y$ is undetermined.

```
VectorPlot[{y,-x},{x,-2,2},{y,-2,2},StreamScale->{Full, All, 0.03},
StreamPoints->Fine,StreamStyle->Directive[Red],VectorStyle->Arrowheads[0]]
```

We have seen that the time-dependent system (2) can be rewritten as (1). Similarly (1) can be written as a pair of first order equations for $x(t)$ and $y(t)$, with t as a parameter in describing the solution trajectories. If

$$\frac{\mathrm{d}x}{\mathrm{d}t} = P(x(t), y(t)), \quad \frac{\mathrm{d}y}{\mathrm{d}t} = Q(x(t), y(t)), \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\dot{y}}{\dot{x}} = \frac{Q(x,y)}{P(x,y)}. \tag{3}$$

A direction of travel along the trajectories can then be assigned, moving to the right, in the direction of increasing x in regions of the (x,y)-plane where $P > 0$ and up, in the direction of increasing y in regions where $Q > 0$.

3 Solution near singular points

We examine the solutions to (1) in the vicinity of critical points (x_0, y_0) where $P(x_0,y_0) = Q(x_0,y_0) = 0$. We have seen above that there are several different forms for the trajectories. Expanding about these points we find

$$P(x,y) \approx P(x_0,y_0) + \frac{\partial P}{\partial x}\,|_{(x_0,y_0)}\,(x - x_0) + \frac{\partial P}{\partial y}\,|_{(x_0,y_0)}\,(y - y_0) = P_x X + P_y Y$$
$$Q(x,y) \approx Q(x_0,y_0) + \frac{\partial Q}{\partial x}\,|_{(x_0,y_0)}\,(x - x_0) + \frac{\partial Q}{\partial y}\,|_{(x_0,y_0)}\,(y - y_0) = Q_x X + Q_y Y,$$

where $X = (x - x_0)$, $Y = (y - y_0)$, giving

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{CX + DY}{AX + BY}, \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} P_x & P_y \\ Q_x & Q_y \end{pmatrix}\Bigg|_{(x_0,y_0)} = \mathbf{J}, \tag{4}$$

where $\mathbf{J}$ is called the ***Jacobian*** of the equilibrium point.

Equation (4) is straightforward enough to solve in individual cases, by putting $Y(X) = XZ(X)$.
(see `http://www.ucl.ac.uk/Mathematics/geomath/level2/deqn/MHde.html` and
`http://en.wikipedia.org/wiki/Homogeneous_differential_equation`.)
However it is difficult to undertake a general analysis of the solutions this way. Instead we introduce a time t and use (3) to write

$$\frac{\mathrm{d}X}{\mathrm{d}t} = AX + BY, \quad \frac{\mathrm{d}Y}{\mathrm{d}t} = CX + DY, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}, \qquad \dot{\mathbf{u}} = \mathbf{J}\mathbf{u} \tag{5}$$

with $\mathbf{u} = (X, Y)^T$. We will present two analyses of this system.

As a single second order equation, using brute force

Eliminating $X(t)$ from (5) in favour of $Y(t)$ gives

$$\ddot{Y} = C\dot{X} + D\dot{Y} = C(AX + BY) + D\dot{Y} = A(\dot{Y} - DY) + CBY + D\dot{Y}$$
$$\Rightarrow \quad \ddot{Y} - (A + D)\dot{Y} + (AD - BC)Y = 0. \tag{6}$$

The same equation is derived for X upon eliminating Y in a similar fashion. Note that $A + D = \operatorname{tr}\mathbf{J} = -p$, say and $AD - BC = \det\mathbf{J} = q$, the trace and determinant of $\mathbf{J}$. The auxiliary equation for (6) is

$$\lambda^2 + p\lambda + q = 0, \quad p = -(A + D), \quad q = AD - BC \Rightarrow \lambda = \lambda_{1,2} = (-p \pm \sqrt{p^2 - 4q})/2. \tag{7}$$

This gives

$$Y(t) = \alpha e^{\lambda_1 t} + \beta e^{\lambda_2 t}.$$

This contains two arbitrary constants, which is all we would expect as our original system is a pair of first-order equations. The solution for $X(t)$ can be found corresponding to this $Y(t)$. From (5)

$$\dot{X} - AX = BY \quad \Rightarrow \quad X(t) = B\left(\frac{\alpha e^{\lambda_1 t}}{\lambda_1 - A} + \frac{\beta e^{\lambda_2 t}}{\lambda_2 - A}\right) + \gamma e^{At},$$

but this solution must be consistent with

$$\dot{Y} - DY = \alpha(\lambda_1 - D)e^{\lambda_1 t} + \beta(\lambda_2 - D)e^{\lambda_2 t} = CX = CB\left(\frac{\alpha e^{\lambda_1 t}}{\lambda_1 - A} + \frac{\beta e^{\lambda_2 t}}{\lambda_2 - A}\right) + C\gamma e^{At},$$

which requires, firstly,

$$\gamma = 0,$$

and also

$$(\lambda_{1,2} - A)(\lambda_{1,2} - D) = CB \quad \text{i.e.} \quad \lambda_{1,2}^2 - (A+D)\lambda_{1,2} + (AD - CB) = 0,$$

which we know is true. Hence we have expressions for $X(t)$, $Y(t)$ which we can use the arbitrainess in α and β to write as

$$X(t) = r_1 e^{\lambda_1 t} + r_2 e^{\lambda_2 t}, \quad Y(t) = s_1 e^{\lambda_1 t} + s_2 e^{\lambda_2 t}, \quad \frac{s_1}{r_1} = \frac{\lambda_1 - A}{B} = \frac{C}{\lambda_1 - D}, \quad \frac{s_2}{r_2} = \frac{\lambda_2 - A}{B} = \frac{C}{\lambda_2 - D}. \tag{8}$$

There are two arbitrary constants since, for example choosing r_1 and r_2 fixes s_1 and s_2. These constants determine which trajectory the solution (8) describes in the vicinity of the critical point - we can pick a particular point that the trajectory passes through by, for example evaluating (8) at $t = 0$. We also have an expression for $\frac{dY}{dX}$,

$$\frac{dY}{dX} = \frac{\dot{Y}}{\dot{X}} = \frac{\lambda_1 s_1 e^{\lambda_1 t} + \lambda_2 s_2 e^{\lambda_2 t}}{\lambda_1 r_1 e^{\lambda_1 t} + \lambda_2 r_2 e^{\lambda_2 t}}. \tag{9}$$

The behaviour of the solution depends on the values of $\lambda_{1,2}$ and hence on p and q.

1. If $q > 0$, so that, if real, $\sqrt{p^2 - 4q} < p$

   (a) $q > 0$, $p^2 > 4q$. Here λ_1 and λ_2 are both real. Since $\lambda_1 > \lambda_2$, as $t \to \infty$ $e^{\lambda_1 t} >> e^{\lambda_2 t}$, whereas as $t \to -\infty$, $e^{\lambda_1 t} << e^{\lambda_2 t}$.

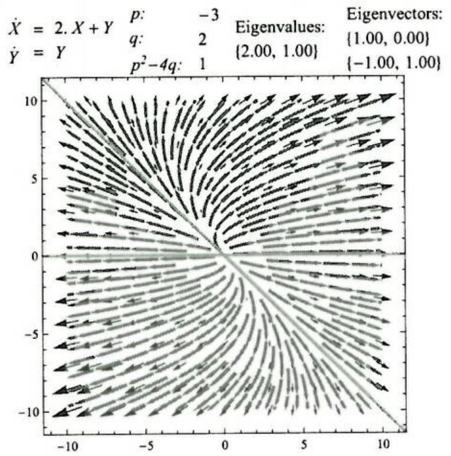
      i. $q > 0$, $p^2 > 4q$, $p > 0$. Here $\lambda_2 < \lambda_1 < 0$

      $$\text{As } t \to \infty, \quad X \to 0, \quad Y \to 0, \quad Y \approx (s_1/r_1)X.$$

      $$\text{As } t \to -\infty, \quad X \to \infty, \quad Y \to \infty, \quad Y \approx (s_2/r_2)X.$$

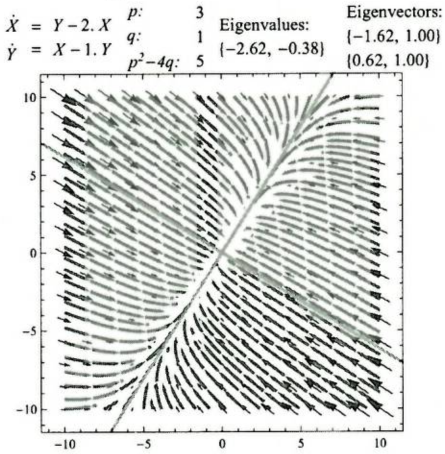
      There are special trajectories that are straight lines in the vicinity of the critical point. These are generated by the choices

      $$r_1 = s_1 = 0, \quad Y = (s_2/r_2)X, \qquad r_2 = s_2 = 0, \quad Y = (s_1/r_1)X$$



All the trajectories pass through $(0,0)$ and such a point is called a ***stable node***. Note that the straight lines (not shown) $Y = -2X$ and $Y = 0$ delineate regions of increasing/decreasing X and increasing/decreasing Y respectively. The straight lines shown are the special trajectories which are exactly straight lines.

ii. $q > 0$, $p^2 > 4q$, $p < 0$. Here $0 < \lambda_2 < \lambda_1$. The qualitative solution is as above, but with the effects of the limits $t \to \infty$ and $t \to -\infty$ interchanged as the values of λ have changed sign.

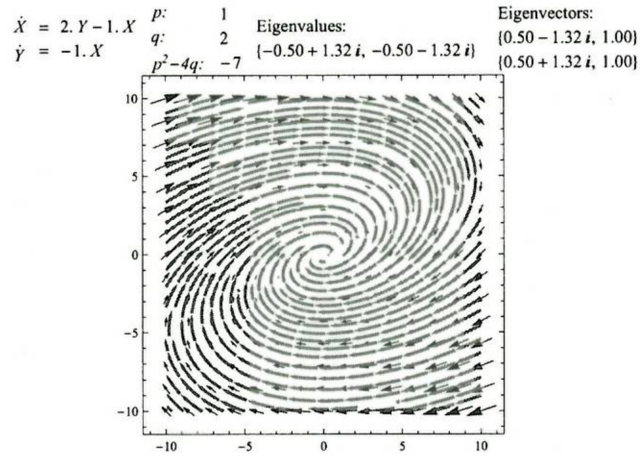


This is known as an ***unstable node***. Again look for the change of direction of the trajectories along $Y = 2X$ and $Y = X$, again not shown.

(b) $q > 0$, $p^2 < 4q$, $p > 0$. In this case the roots are complex, with negative real part. If we write $\lambda_{1,2} = -\mu_1 \pm i\mu_2$, $\mu_{1,2} > 0$. Instead of the exponential solutions given in (8) we have the solutions

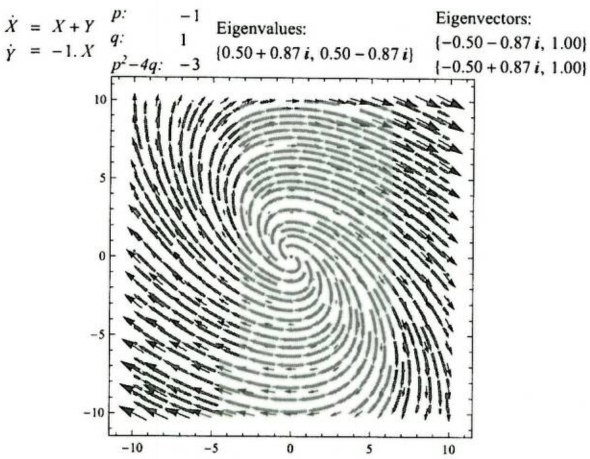
$$X(t) = k_1 \mathrm{e}^{-\mu_1 t} \cos(\mu_2 t + \epsilon_1), \quad Y(t) = k_2 \mathrm{e}^{-\mu_1 t} \cos(\mu_2 t + \epsilon_2).$$

As before, only two of the constants $k_{1,2}$ and $\epsilon_{1,2}$ can be independently chosen. It is clear that the trajectories are spiral, spiraling in towards the origin $(0,0)$ - as t is increased by a value $2\pi/\mu_2$, both X and Y are multiplied by the same factor $\mathrm{e}^{-2\pi\mu_1/\mu_2}$.



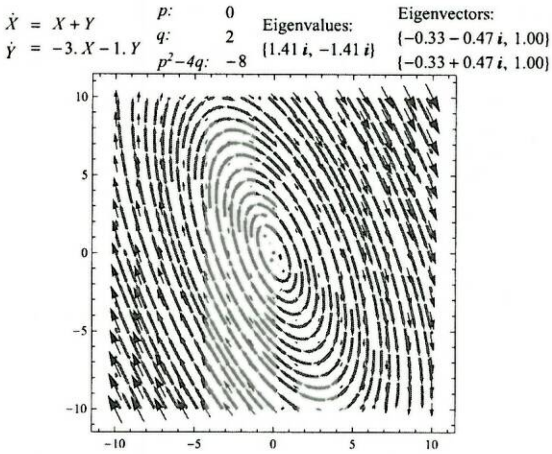
All trajectories approach the origin. The singular point is known as a ***stable spiral point or focus***.

(c) $q > 0$, $p^2 < 4q$, $p < 0$. This case again has imaginary roots, but with a positive real part.



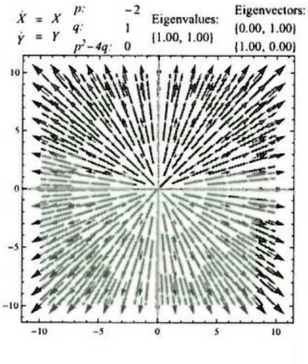
All trajectories depart from the origin. The singular point is known as a ***unstable spiral point or focus***.

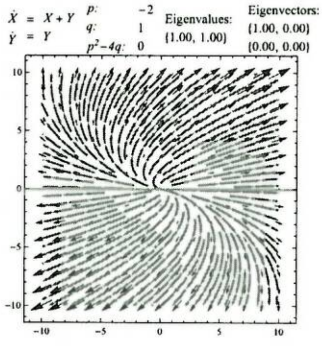
(d) $q > 0$, $p = 0$. This case again has purely imaginary roots, $\mu_1 = 0$ and the trajectories are circles/ellipses. No trajectories pass through $(0, 0)$ except for the trajectory consisting of a single point at $(0, 0)$



The critical point is called a ***centre***. Again it is illustrative to pick out the lines $Y = -3X$ and $Y = -X$ and note that the individual trajectories have turning points on these lines.

(e) $q > 0$, $p^2 = 4q$, $p > 0$. This corresponds to two equal negative roots for λ. The trajectories still form an stable node. However this can be of two types known as a firstly a ***star*** and secondly an ***improper node***. They are indistinguishable simply using the values of p and q



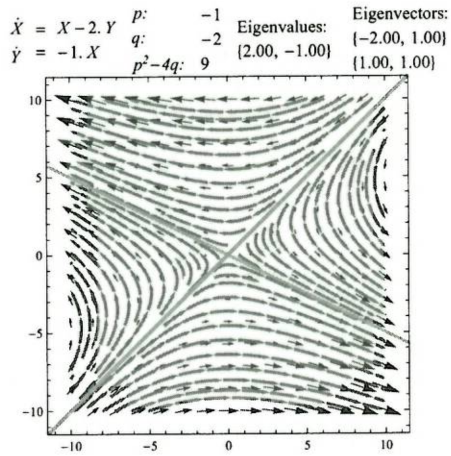


(f) $q > 0$, $p^2 = 4q$, $p < 0$. This corresponds to two equal positive roots for λ. The trajectories form an unstable node, which may be of star type.

2. $q < 0$ so that $\sqrt{p^2 - 4q}$ is real but $\sqrt{p^2 - 4q} > p$ and the roots differ in sign. Here $\lambda_2 < 0 < \lambda_1$

$$\text{As } t \to -\infty, \quad X \approx r_2 e^{\lambda_2 t} \to \infty \text{ (in modulus)}, \quad Y \approx s_2 e^{\lambda_2 t} \to \infty \text{ (in modulus)}, \quad Y \approx (s_2/r_2)X.$$

As $t \to \infty$, $\quad X \approx r_1 e^{\lambda_1 t} \to \infty$ (in modulus), $\quad Y \approx s_1 e^{\lambda_1 t} \to \infty$ (in modulus), $\quad Y \approx (s_1/r_1)X$.

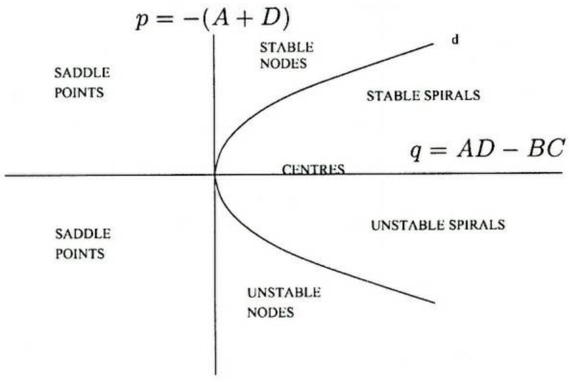


Only the two special straight line trajectories pass through $(0, 0)$. The others approach the critical point, from the direction of one of these straight lines and leave the critical point in the direction of the other. The critical point is known as a ***saddle point***. A change in the sign of p interchanges the roles of λ_1 and λ_2 as before.

The figures above have all been generated with the following *Mathematica* commands, varying the coefficients of the matrix `m`.

```
m = {{1,1},{0,1}};{{a,b},{c,d}}=m;p=-(a+d);q=ad-bc;disc=p^2-4q;
Show[VectorPlot[m.{x,y},{x,-10,10},{y,-10,10},StreamPoints->Fine,StreamStyle->{Red,Thick},
ImageSize->{460,310}],Graphics[{Thick,Orange,Map[Line[{-100 #, 100 #}]&,
Select[Eigenvectors[m],(Im[#[[1]]]==0&&Im[#[[2]]]==0)&]]}],
PlotLabel->Row[{Column[{Row[{Column[{Style["\!\(\*OverscriptBox[\"X\",\".\"]\)",Italic],
Style["\!\(\*OverscriptBox[\"Y\", \".\"]\)", Italic]}],Column[{" = ", " = "}],
TableForm[m.{Style["X", Italic], Style["Y", Italic]}]//N}]}],"  ",
Column[{Style["p:",Italic],Style["q:",Italic],Style["\!\(\*SuperscriptBox[\"p\", \"2\"]\)-4q:", Italic]}], "  ",Column[{p, q, disc}], "  ",
Column[{"Eigenvalues:",NumberForm[Chop@N@Eigenvalues[m],{4, 2}]}],"  ", ,
Column[{"Eigenvectors:",NumberForm[Chop@N@Eigenvectors[m][[1]],{4, 2}], NumberForm[Chop@N@Eigenvectors[m][[2]], {4, 2}] }]}]]
```

We can summarise what we have found with this diagram



As a first order matrix/vector equation

Equation (5) is $\dot{\mathbf{u}} = \mathbf{J}\mathbf{u}$ for $\mathbf{u}(t)$ with $\mathbf{J}$ a constant matrix. Comparison with a differential equation of the form $\dot{x} = ax$, with solution $x(t) = Ae^{at}$, with A and a constant, suggests we try the solution $\mathbf{u} = \mathbf{v}e^{\lambda t}$. Direct substitution leads to $\lambda \mathbf{v} e^{\lambda t} = \mathbf{J}\mathbf{v}e^{\lambda t}$ or $\lambda \mathbf{v} = \mathbf{J}\mathbf{v}$ so that λ is an eigenvalue of $\mathbf{J}$ and $\mathbf{v}$ the corresponding eigenvector. The general solution is a sum over the possible eigenvalue/eigenvector pairs. The matrix $\mathbf{J}$ is 2×2 so there are a maximum of two and, if they are real, distinct and non-zero, $\lambda_{1,2}$ say,

$$\mathbf{u}(t) = A_1 \mathbf{v}_1 e^{\lambda_1 t} + A_2 \mathbf{v}_2 e^{\lambda_2 t}.$$

As above we have two degrees of freedom in this solution and $A_{1,2}$ can be found to specify a particular trajectory uniquely. As the eigenvalues are real, distinct and non-zero, then we know the eigenvectors are independent. If we form the matrix $\mathbf{P} = (\mathbf{v}_1, \mathbf{v}_2)$ with the eigenvectors as columns then the transformation to the new variables $(\bar{X}, \bar{Y})$

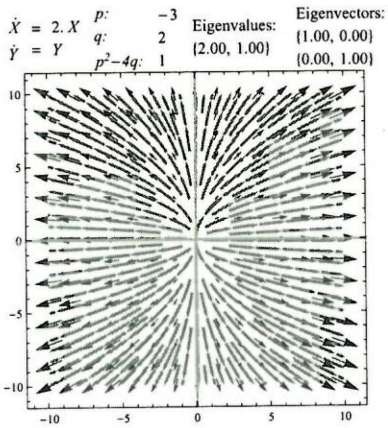
rather than (X, Y) through the definition $\mathbf{u} = \mathbf{P}\bar{\mathbf{u}}$, $\bar{\mathbf{u}} = \mathbf{P}^{-1}\mathbf{u}$, with $\bar{\mathbf{u}} = (\bar{X}, \bar{Y})^T$. Also, as $\mathbf{P}$ has columns made of the eigenvectors of $\mathbf{J}$, $\mathbf{JP} = (\lambda_1\mathbf{v}_1, \lambda_2\mathbf{v}_2) = \mathbf{\Lambda}(\mathbf{v}_1, \mathbf{v}_2) = \mathbf{\Lambda P}$, where $\mathbf{\Lambda}$ is a diagonal matrix $\mathrm{diag}(\lambda_1, \lambda_2)$ with the eigenvalues of $\mathbf{J}$ along its diagonal. We therefore have $\mathbf{J} = \mathbf{P\Lambda P}^{-1}$, or $\mathbf{\Lambda} = \mathbf{P}^{-1}\mathbf{JP}$. (These are standard results on the diagonalisation of matrices.) Therefore

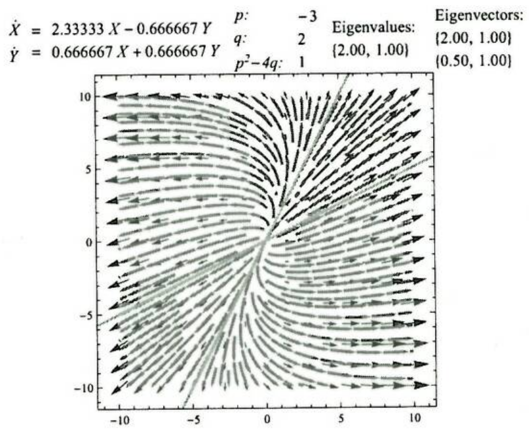
$$\dot{\mathbf{u}} = \mathbf{Ju} = \mathbf{P\Lambda P}^{-1}\mathbf{u}, \quad \Rightarrow \quad \mathbf{P}^{-1}\dot{\mathbf{u}} = \mathbf{\Lambda P}^{-1}\mathbf{u}, \quad \Rightarrow \quad \dot{\bar{\mathbf{u}}} = \mathbf{\Lambda}\bar{\mathbf{u}}, \quad \Rightarrow \quad \dot{\bar{X}} = \lambda_1\bar{X}, \quad \dot{\bar{Y}} = \lambda_2\bar{Y} \quad \Rightarrow$$
$$\bar{X}(t) = \bar{X}_0 e^{\lambda_1 t}, \quad \bar{Y}(t) = \bar{Y}_0 e^{\lambda_2 t} \text{ and, eliminating } t, \quad \bar{Y} = C\bar{X}^a, \quad a = \lambda_2/\lambda_1 \tag{10}$$

1. Real, positive eigenvalues. Here, a in (10) is positive. All trajectories pass through $(\bar{X}, \bar{Y}) = (0, 0)$ (and so the critical point $(X, Y) = (0, 0)$. We have an ***unstable node*** as $\lambda_{1,2}$ are positive so $\bar{X}$ and $\bar{Y}$ (and so (X, Y)) tend to infinity as $t \to \infty$. If $a > 1$, i.e. $\lambda_2 > \lambda_1$, then the trajectories have the character of $\bar{Y} = \pm\bar{X}^2$, but if $a < 1$, $\lambda_2 < \lambda_1$, the roles of $\bar{X}$ and $\bar{Y}$ are interchanged with the trajectories looking more like $\pm\bar{Y} = \sqrt{|\bar{X}|}$. This is in terms of the new coordinates. The trajectories in the original (X, Y) coordinates are similar in character but "skewed" so that the $\bar{X}$ and $\bar{Y}$ axes correspond to lines in the (X, Y) plane that point along the eigenvectors of $\mathbf{J}$.

   Choose $\lambda_{1,2} = 2, 1, \quad a = \dfrac{1}{2}, \quad \mathbf{\Lambda} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. Choose $\mathbf{v}_{1,2} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, giving $\mathbf{P} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \mathbf{P}^{-1} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$

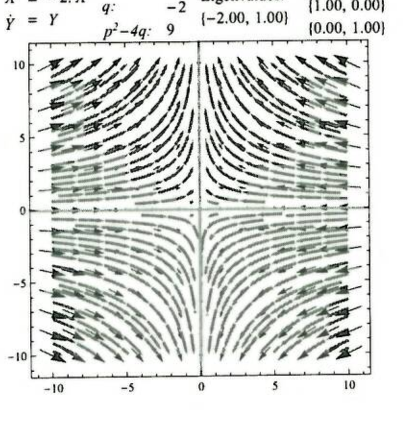
$$\mathbf{J} = \mathbf{P\Lambda P}^{-1} = \begin{pmatrix} \frac{7}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} \end{pmatrix} \tag{11}$$

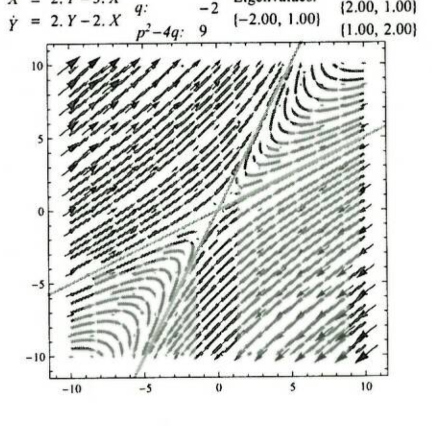




2. Real, negative eigenvalues. This is the same situation as above, but with the direction of t reversed - ***a stable node.***

3. Real eigenvalues, one positive and one negative. Here a is negative and the trajectories generally do not pass through $(X, Y) = (0, 0)$. Also as $t \to \infty$ only one of $\bar{X}$ or $\bar{Y}$ approaches zero. The other approaches ∞. As $t - \to -\infty$ the roles are reversed. We have a ***saddle point***.





$$\dot{Y} - DY = \alpha(\lambda_1 - D)e^{\lambda_1 t} + \beta(\lambda_2 - D)e^{\lambda_2 t} = CX$$

$$= CB\left[\frac{\alpha e^{\lambda_1 t}}{\lambda_1 - A} + \frac{\beta e^{\lambda_2 t}}{\lambda_2 - A}\right] + C\gamma e^{At}$$

$\Rightarrow \gamma = 0$

and $(\lambda - D) = \dfrac{CB}{\lambda - A} \Rightarrow (\lambda - D)(\lambda - A) = CB$

$\Rightarrow \lambda^2 - \lambda(A + D) + AD - BC = 0$

$\Rightarrow \lambda^2 + p\lambda + q = 0$ (which is true!)

Say $X(t) = r_1 e^{\lambda_1 t} + r_2 e^{\lambda_2 t}$

$Y(t) = s_1 e^{\lambda_1 t} + s_2 e^{\lambda_2 t}$

$$\frac{s_1}{r_1} = \frac{\lambda_1 - A}{B} = \frac{C}{\lambda_1 - D}$$

$$\frac{s_2}{r_2} = \frac{\lambda_2 - A}{B} = \frac{C}{\lambda_2 - A}$$

$$\Rightarrow \frac{dY}{dX} = \frac{\lambda_1 s_1 e^{\lambda_1 t} + \lambda_2 s_2 e^{\lambda_2 t}}{\lambda_1 r_1 e^{\lambda_1 t} + \lambda_2 r_2 e^{\lambda_2 t}}$$

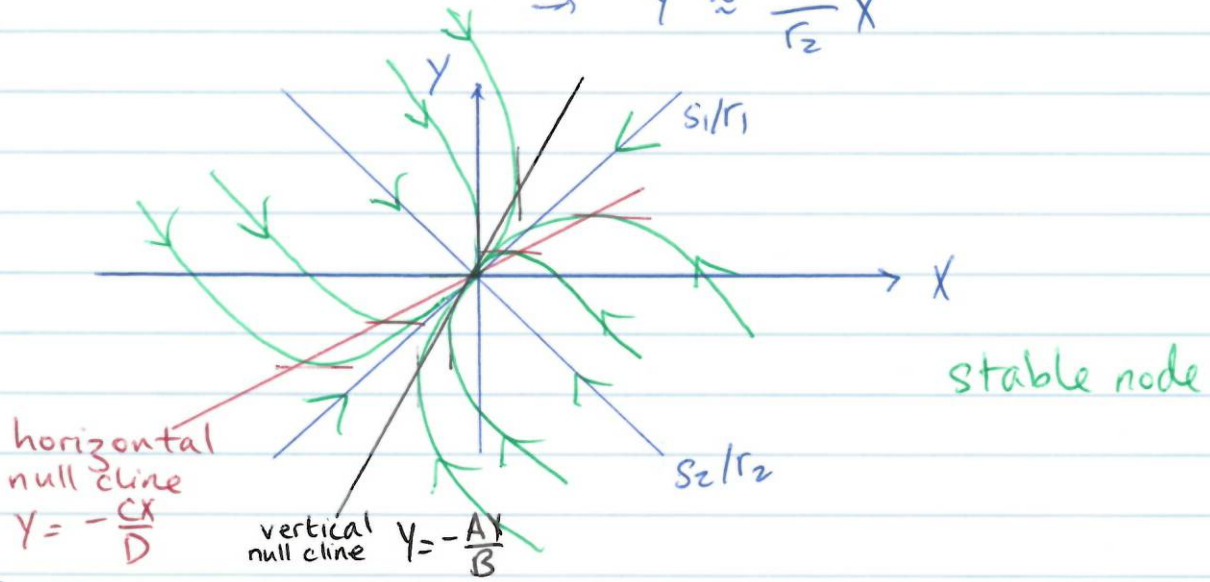
If $q > 0$ and $p^2 > 4q$, and $p > 0$ λ_1 & λ_2 are real, $\lambda_2 < \lambda_1 < 0$.

As $t \to \infty$, $X, Y \to 0$ but $\dfrac{dY}{dX} \to \dfrac{s_1}{r_1}$

$\Rightarrow Y \approx \dfrac{s_1}{r_1} X$

As $t \to -\infty$, $X, Y \to \infty$ but $\frac{dY}{dX} \to \frac{s_2}{r_2}$

$\to Y \approx \frac{s_2}{r_2} X$



The straight lines through the origin are also trajectories given by $s_1 = 0$ and $s_2 = 0$.

FINDING s_1, s_2.

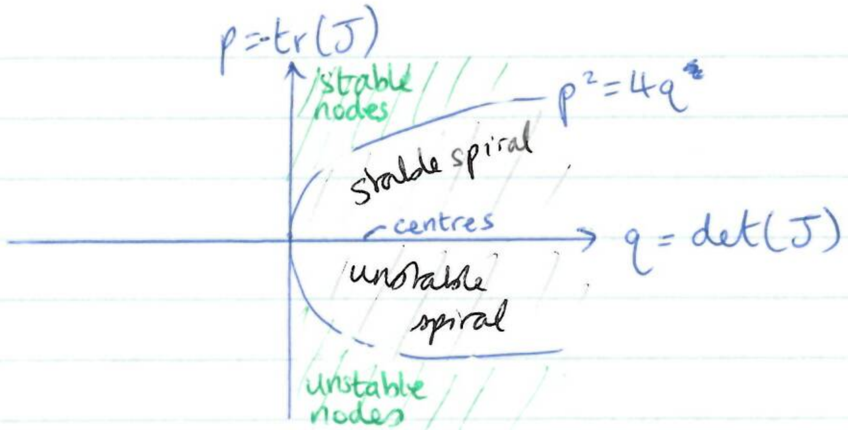
These are straight line sol^{n}s, ie $Y = mX$

$$\frac{dY}{dX} = \frac{CX + DY}{AX + BY} \Rightarrow m = \frac{C + Dm}{A + Bm}$$

a quadratic equation in m.

If $q > 0$, $p^2 > 4q$, $p < 0$, $0 < \lambda_2 < \lambda_1$

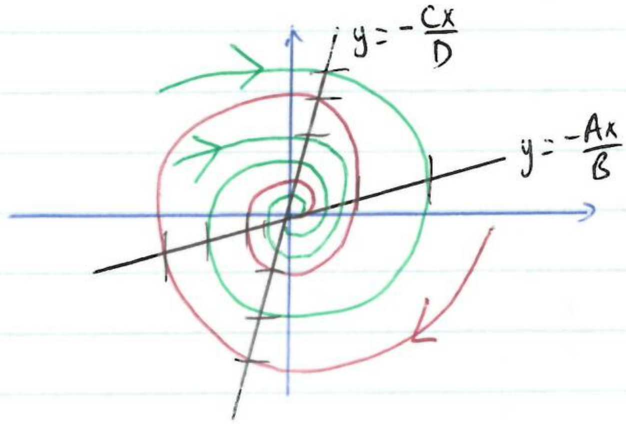
we can reproduce this situation from the previous one by changing $t \to -t$, ie reversing arrows.
This gives us an unstable node.



$\mu_1 > 0$

If $q > 0$, $p^2 > 4q$, $p > 0$, we can write $\lambda_{1,2} = -\mu_1 \pm i\mu_2$

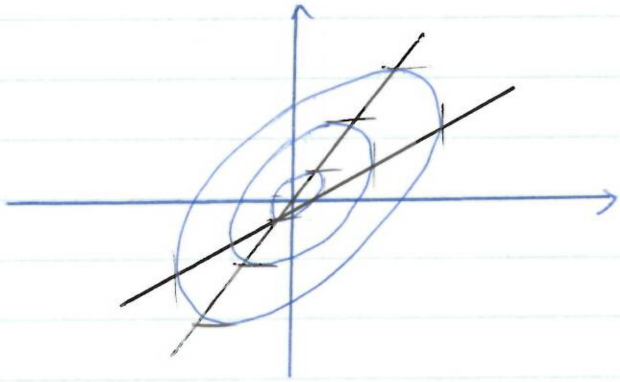
And get $X(t) = k_1 e^{-\mu_1 t} \cos(\mu_2 t + \phi_1)$
$Y(t) = k_2 e^{-\mu_1 t} \cos(\mu_2 t + \phi_2)$



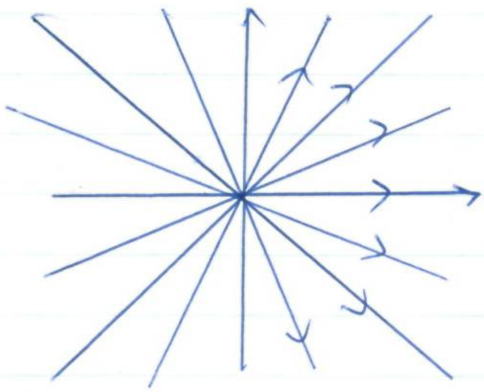
stable spiral point or focus

If $p < 0$, this changes the sign of μ_1 and the spirals spiral outward, ie we have an unstable spiral point

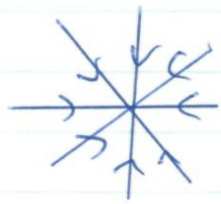
If $q > 0$, $p^2 < 4q$, $p = 0$ we have a centre:



if $p^2 = 4q$, $q > 0$



"star"



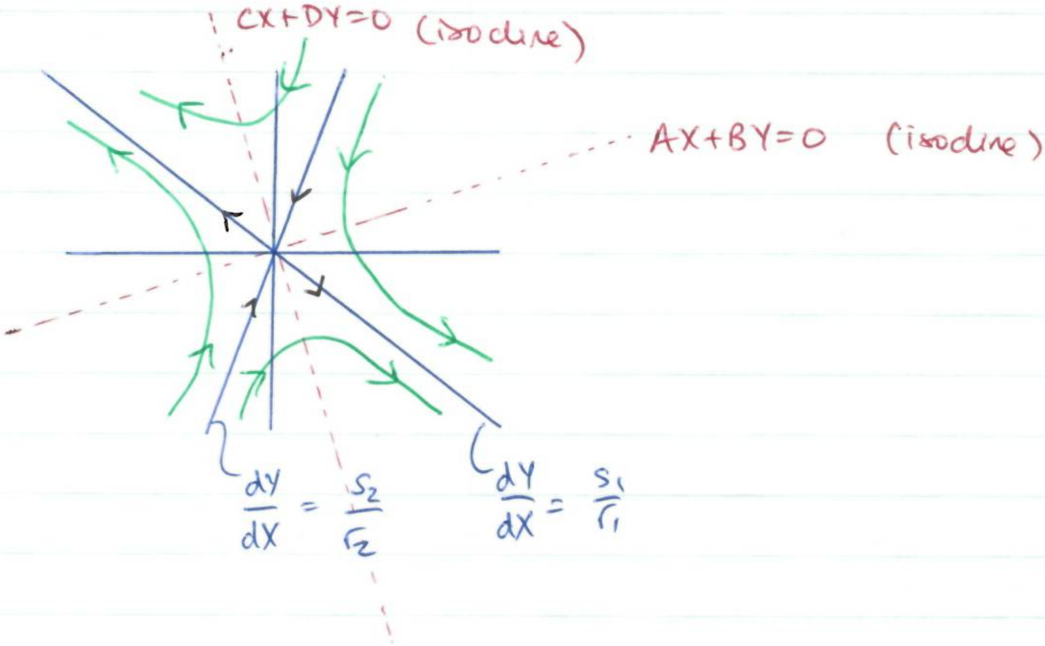
all trajectories are straight lines
any value of m will do.

$\boxed{\text{L}}$ If $q < 0$, the λ are of opposite sign

$$\lambda_2 < 0 < \lambda_1$$

As $t \to \infty$, $\frac{dY}{dX} \to \frac{s_1}{r_1}$ and $X, Y \to \infty$

As $t \to -\infty$ $\frac{dY}{dX} \to \frac{s_2}{r_2}$ and $X, Y \to \infty$



There are two trajectories that pass through $X=Y=0$.
These have slope m where

$$m = \frac{C+Dm}{A+Bm}$$

these are called <u>SADDLE POINTS</u>.

<u>An alternative approach</u>

Our pair of ODEs can be written, $\underline{X}=\begin{pmatrix} X \\ Y \end{pmatrix}$, $J=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$,

$$\dot{\underline{X}} = J\underline{X}$$

If this were an ODE $\dot{X}=jX$, one would look for a sol^n of the form $X=ce^{\lambda t}$ and find λ.

Extending this we look for a sol^n $\underline{X}=\underline{v}e^{\lambda t}$ and substitute to find

$$\lambda\underline{v}e^{\lambda t} = J\underline{v}e^{\lambda t}$$

$$\Rightarrow \quad J\underline{v}=\lambda\underline{v}$$

$$\Downarrow$$

λ is an eigenvalue of J
$\underline{v}$ is the corresponding eigenvector

$$\begin{vmatrix} A-\lambda & B \\ C & D-\lambda \end{vmatrix} = 0$$

If we diagonalise J by introducing new variables ...

$$\begin{pmatrix} \tilde{X} \\ \tilde{Y} \end{pmatrix} = \underline{\tilde{u}}$$

such that

$$\underline{u} = \begin{pmatrix} X \\ Y \end{pmatrix} = P\underline{\tilde{u}}$$

where P has columns made up of eigenvectors $(\underline{v}_1, \underline{v}_2)$

$$\Rightarrow \quad J\underline{u} = JP\underline{\tilde{u}} = (\lambda_1 \underline{v}_1, \lambda_2 \underline{v}_2)\underline{\tilde{u}}$$

$$= P\Lambda\underline{\tilde{u}} \qquad \left(\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\right)$$

$$= P\Lambda P^{-1}\underline{u}$$

$$\Rightarrow \quad J = P\Lambda P^{-1}$$

And recall $\underline{\dot{u}} = J\underline{u} \Rightarrow$

$$\underline{\dot{u}} = P\Lambda P^{-1}\underline{u}$$

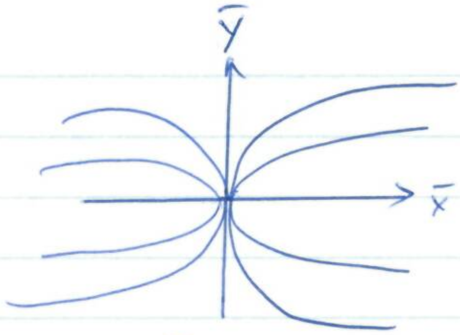
$$\Rightarrow \quad P^{-1}\underline{\dot{u}} = \Lambda P^{-1}\underline{u}$$

$$\Rightarrow \quad \underline{\dot{\tilde{u}}} = \Lambda\underline{\tilde{u}}$$

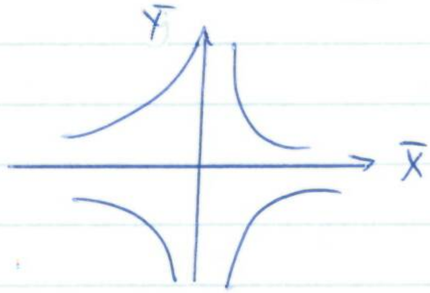
$$\Rightarrow \quad \begin{pmatrix} \dot{\tilde{X}} \\ \dot{\tilde{Y}} \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} \tilde{X} \\ \tilde{Y} \end{pmatrix} \quad \Rightarrow \quad \begin{aligned} \dot{\tilde{X}} &= \lambda_1 \tilde{X} \\ \dot{\tilde{Y}} &= \lambda_2 \tilde{Y} \end{aligned}$$

$$\Rightarrow \quad \left.\begin{aligned} \tilde{X} &= \tilde{x}_0 e^{\lambda_1 t} \\ \tilde{Y} &= \tilde{y}_0 e^{\lambda_2 t} \end{aligned}\right\} \quad \tilde{Y} = c\tilde{X}^a \qquad (a = \lambda_2 / \lambda_1)$$

e.g. $a = \frac{1}{2}$: $\lambda_1, \lambda_2 > 0$:



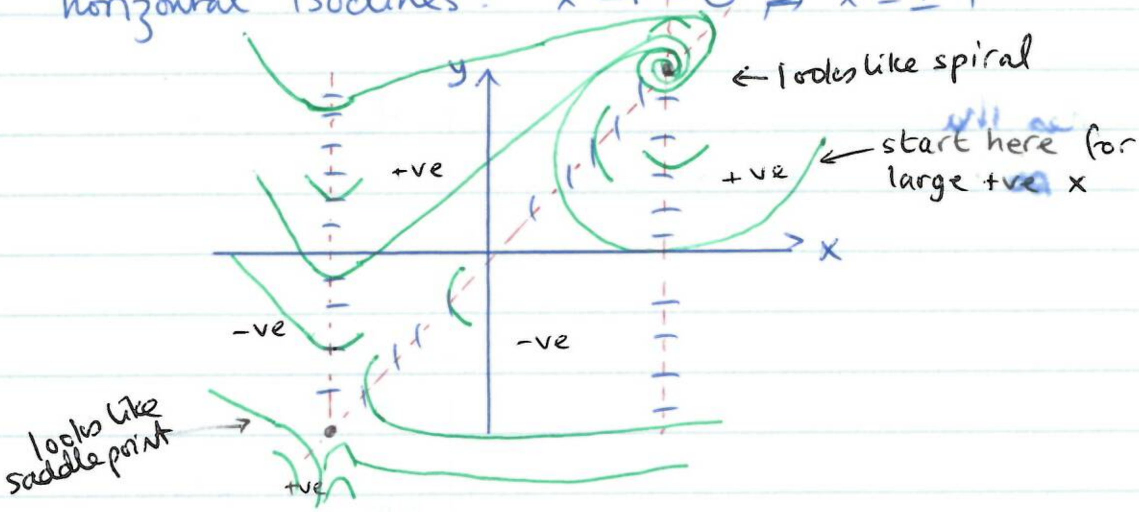
$a < 0$, $\lambda_2 < 0 < \lambda_1$, saddle :



<u>Example</u> Sketch the phase plane for the equation

$$\frac{dy}{dx} = \frac{x^2-1}{x-y}$$

horizontal isoclines: $x^2 - 1 = 0 \Rightarrow x = \pm 1$



vertical isoclines: $x - y = 0 \Rightarrow y = x$

Critical points are where horizontal + vertical isoclines meet (here $(1,1)$, $(-1,-1)$)

Let us examine the point $(-1,-1)$

$(-1,-1)$
(A)

$$x = -1 + X \qquad X, Y \text{ small (perturbations)}$$
$$y = -1 + Y$$

$$\frac{dy}{dx} = \frac{dY}{dX} = \frac{(-1-X)^2 - 1}{(-1+X) - (-1+Y)} = \frac{-2X}{X-Y} \quad \ldots\ldots (*)$$

X^2 small $\Rightarrow$ ignore this term

$$\overset{\text{set}}{=} \frac{CX + DY}{AX + BY} \Rightarrow \begin{array}{l} C = -2,\ D = 0 \\ A = 1,\ B = -1 \end{array}$$

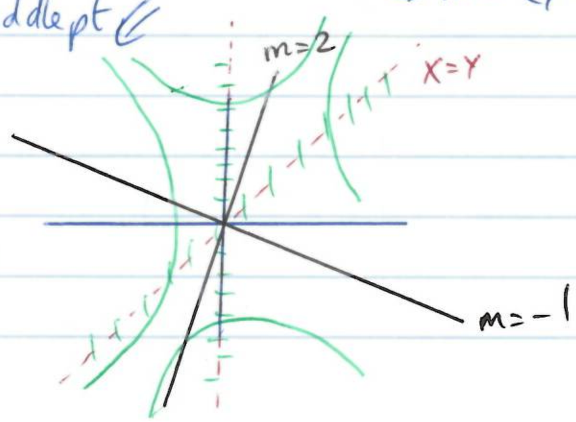
$$\Rightarrow p = -(A+D) = -1$$
$$q = AD - BC = -2 \qquad q < 0 \Rightarrow \text{SADDLE}$$

$$J = \begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix}$$ eigenvalues are different in sign $\Rightarrow$ saddle

Lines that pass exactly through the saddle - the seperatrix do ~~so~~ with slope m where (set $Y = mX$, plug into $(*)$)

$$m = \frac{-2}{1-m} \Rightarrow m - m^2 = -2$$
$$\Rightarrow m^2 - m - 2 = 0$$
$$\Rightarrow (m-2)(m+1) = 0$$
$$\hookrightarrow m = 2, -1$$

at saddlept



(1,1)
(B) Point (1,1): Could say $x = 1+X$
$y = 1+Y$
but won't; will use different method instead.

$$\frac{dy}{dx} = \frac{Q}{P} = \frac{x^2-1}{x-y}$$

$$J = \begin{pmatrix} P_x & P_y \\ Q_x & Q_y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2x & 0 \end{pmatrix}$$

and $J|_{x=1} = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}$

Eigenvalues of J are roots of $\begin{vmatrix} 1-\lambda & -1 \\ 2 & -\lambda \end{vmatrix} = 0$

ie $-\lambda(1-\lambda) + 2 = 0$
$\Rightarrow \lambda^2 - \lambda + 2 = 0 \Rightarrow \lambda = \dfrac{1 \pm \sqrt{1-8}}{2}$

COMPLEX $\Rightarrow$ spiral

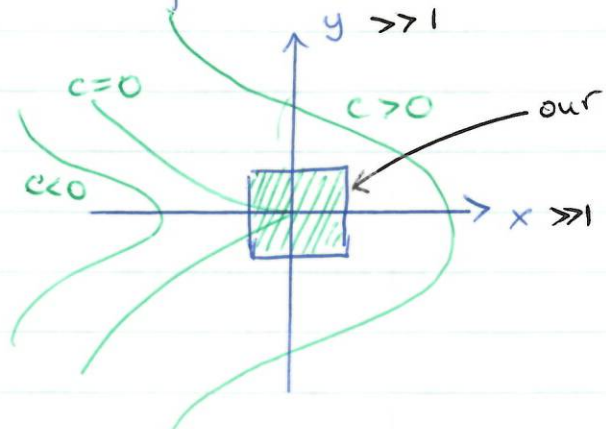
If x and y are large in

$$\frac{dy}{dx} = \frac{x^2-1}{x-y} \approx \frac{x^2}{x-y}$$ and if $|y| \gg |x|$

$$\approx \frac{x^2}{-y} \Rightarrow -y\,dy = x^2\,dx$$

$$\frac{y^2}{2} = \frac{-x^3}{3} + C$$

→ there are trajectories that miss the whole action:



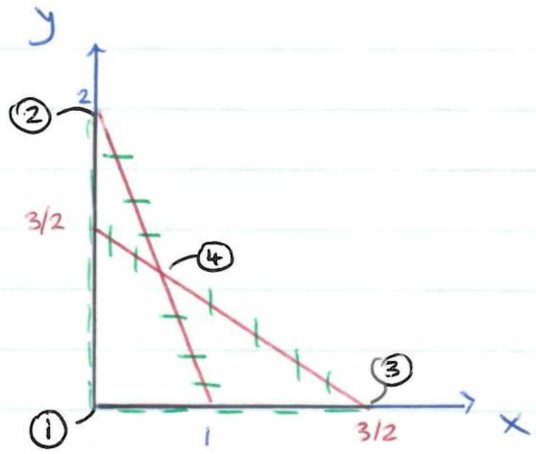
Rabbits and sheep

$$\frac{dx}{dt} = x(3-2x-2y) = P(x,y)$$

$$\frac{dy}{dt} = y(2-2x-y) = Q(x,y)$$

Vertical null clines $\quad \frac{dx}{dt} = 0 \quad \Rightarrow \quad x=0$, $\; y = \frac{3}{2} - x$

Horizontal null clines $\quad \frac{dy}{dt} = 0 \quad \Rightarrow \quad y=0$, $\; y = 2-2x$



Critical points where different null clines cross

$$J = \begin{pmatrix} P_x & P_y \\ Q_x & Q_y \end{pmatrix}, \qquad \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = J \begin{pmatrix} x \\ y \end{pmatrix}$$

$$J = \begin{pmatrix} 3-4x-2y & -2x \\ -2y & 2-2x-2y \end{pmatrix}$$

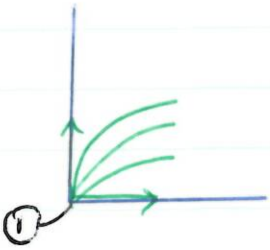
① $x=0$, $y=0$ $\quad J = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$.

Two real eigenvalues $\Rightarrow$ node
positive $\Rightarrow$ unstable

locally, if $x = x_0 + X$
$y = y_0 + Y$,

$$\frac{dX}{dt} = 3X \qquad \frac{dY}{dt} = 2Y$$

$$\Rightarrow \frac{dY}{dX} = \frac{2}{3}\frac{Y}{X} \quad \Rightarrow \quad Y = CX^{2/3}$$



② $x=0$, $y=2$ $\quad J = \begin{pmatrix} -1 & 0 \\ -4 & -2 \end{pmatrix}$

Two real eigenvalues $(-1,-2)$ $\Rightarrow$ node
both negative $\Rightarrow$ stable.

locally, if $x = x_0 + X$
$y = y_0 + Y$,

$$\frac{dX}{dt} = -X \qquad \frac{dY}{dt} = -4X - 2Y$$

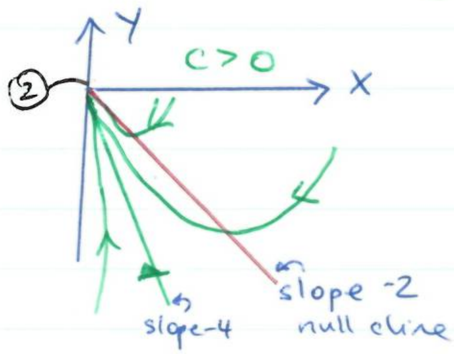
$$\Rightarrow \frac{dY}{dX} = 4 + 2\frac{Y}{X}$$

$$\Rightarrow \frac{dY}{dX} - \frac{2}{X}Y = 4. \qquad \text{IF is } 1/X^2$$

$$\Rightarrow \frac{d}{dX}\left[e^{-\int \frac{2}{X}dX} Y\right] = e^{-\int\frac{2}{X}dX} 4$$

$$\Rightarrow \frac{d}{dX}\left[\frac{Y}{X^2}\right] = \frac{4}{X^2}$$

$$\Rightarrow Y = -4X + CX^2$$



3 $x = 3/2$, $y = 0$ $\qquad J = \begin{pmatrix} -3 & -3 \\ 0 & -1 \end{pmatrix}$

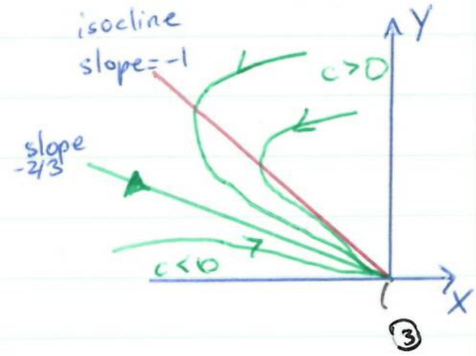
Two real eigenvalues $\Rightarrow$ node
both negative $\Rightarrow$ stable

locally, $\dfrac{dX}{dt} = -3X - 3Y \qquad \dfrac{dY}{dt} = -Y$

$$\frac{dY}{dX} = \frac{Y}{3X+3Y}$$

if $Y = mX$ then $m = \frac{1}{3}\frac{m}{1+m}$

$\Rightarrow 3m^2 + 2m = 0$, $m = 0$ or $-2/3$.



Can solve $\frac{dY}{dX}$ for Y, but easier to solve for X :

$$\frac{dX}{dY} = 3\left(1 + \frac{X}{Y}\right)$$

$$\Rightarrow \frac{dX}{dY} - \frac{3X}{Y} = 3 .$$ I.F. $= \frac{1}{y^3}$

$$\Rightarrow \frac{d}{dY}\left[\frac{X}{Y^3}\right] = \frac{3}{Y^3}$$

$$\Rightarrow X = -\frac{3}{2}Y + CY^3$$

(14) $x = \frac{1}{2}$ $\quad J = \begin{pmatrix} -1 & -1 \\ -2 & -1 \end{pmatrix}$

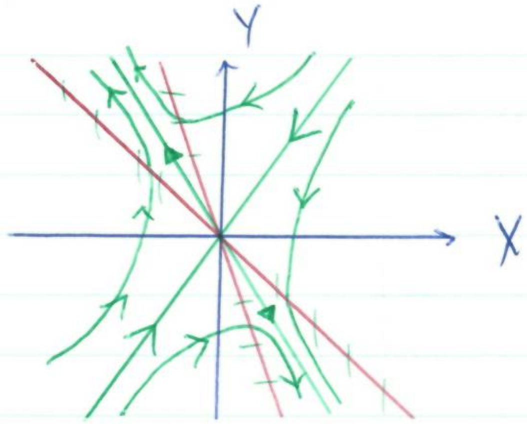
$y = 1$

Eigenvalues are $-1 \pm \sqrt{2}$. Real $\Rightarrow$ different in sign $\Rightarrow$ saddle point.

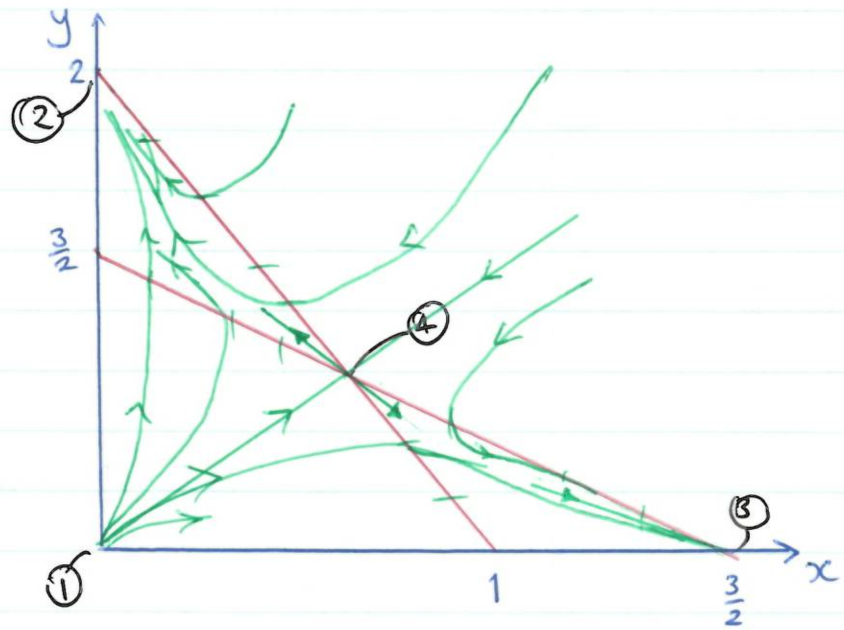
We have $\frac{dX}{dt} = -X - Y$ $\qquad \frac{dY}{dt} = -2X - Y$

$$\Rightarrow \frac{dY}{dX} = \frac{Y + 2X}{Y + X} .$$

If $Y = mX$, $\quad m = \frac{m+2}{m+1} \Rightarrow m = \pm\sqrt{2}$.



So, putting it all together.



$\Rightarrow$ trajectories will end up at 2 or 3.

Periodic solutions

These may arise from a pair of equations

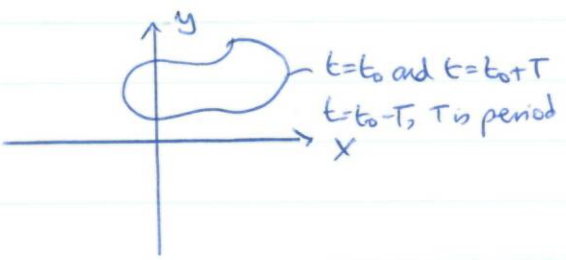
$$\frac{dx}{dt} = P(x,y) \qquad \frac{dy}{dt} = Q(x,y)$$

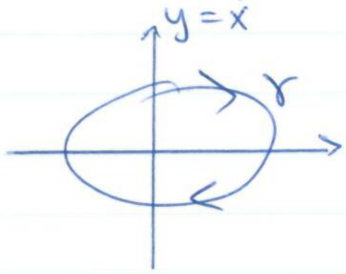
or from the second-order equation

$$\ddot{x} = f(x, \dot{x})$$

and introducing $y = \dot{x}$ so this becomes

$$\frac{dy}{dt} = f(x,y), \qquad \frac{dx}{dt} = y.$$



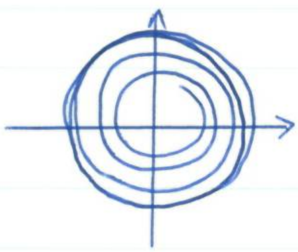


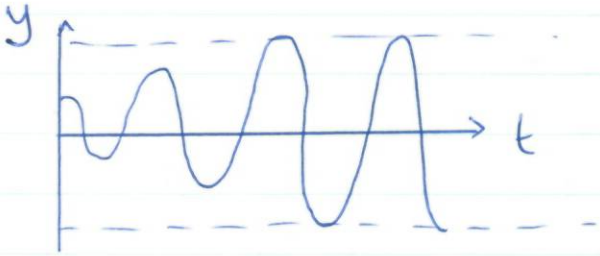
$$x(T+t_0) = x(t_0)$$
$$y(T+t_0) = y(t_0).$$

To find the period we need to evaluate the integral

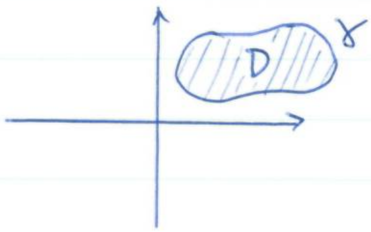
$$T = \int_0^T dt = \int_\gamma \frac{1}{P(x,y)}\,dx = \int_\gamma \frac{1}{Q(x,y)}\,dy.$$

$\boxed{L}$ Sometimes periodic sol^{n}s can be approached in the limit $t \to \infty$, $t \to -\infty$. They are then called limit cycles.





<u>Bendixon's Negative Criteria</u> for a limit cycle, or periodic sol^n is a test that shows where periodic sol^{n}s cannot exist.



Let γ be a closed trajectory and D be its interior.

Consider

$$\iint_D \underbrace{\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right)}_{\operatorname{div}\binom{P}{Q}} dx\,dy$$

$$(\text{Stokes}) \quad = \int_\gamma P\,dy - Q\,dx$$

$$= \int_{t_0}^{t_0+T} P \frac{dy}{dt} dt - Q \frac{dx}{dt} dt$$

$$= \int_{t_0}^{t_0+T} (PQ - QP)\,dt$$

$$= 0.$$

So $\iint_D (P_x + Q_y)\,dx\,dy = 0$

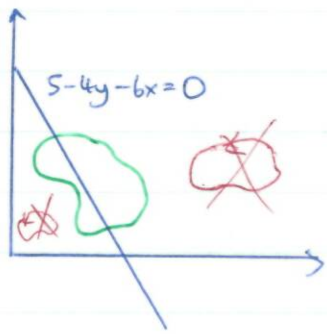
and we deduce that if $P_x + Q_y$ is single-signed in some part of the x-y plane, then we cannot have a closed trajectory, or periodic orbit, within that region.

In our last example:

$$\frac{dx}{dt} = P = x(3-2x-2y)$$

$$\frac{dy}{dt} = Q = y(2-2x-2y)$$

So $P_x + Q_y = (3-4x-2y) + (2-2x-2y)$
$\qquad\qquad\quad = 5-4y-6x$



If there's gonna be a periodic sol^n, it's got to straddle the line $5-4y-6x=0$.

Dulac's Extension of Bendixon's condition

$$\iint_D \operatorname{div}(R\underline{u})\,dx\,dy = \iint_D \left[\frac{\partial}{\partial x}(RP) + \frac{\partial}{\partial y}(RQ)\right] dx\,dy$$

$$\text{(Stokes)} = \int_\gamma RP\,dy - RQ\,dx$$

$$= \int (RPQ - RPQ)\,dt$$

$$= 0$$

$\forall R$, f^n

R would be given in an exam.

If we use this with $R = \frac{1}{xy}$ in this example, then

$$RP = \left(\frac{3}{y} - \frac{2x}{y} - 2\right)$$

$$RQ = \left(\frac{2}{x} - 2 - \frac{y}{x}\right)$$

and $$(RP)_x + (RQ)_y = -\frac{2}{y} - \frac{1}{x}.$$

$$< 0 \quad \text{for } x, y > 0.$$

So there is no closed orbit for +ve x, y. (or zero x, y)
since if we did, $\iint (-ve)\, dx\, dy = 0$.

A limit cycle

Consider
$$\frac{dx}{dt} = x + y - x(x^2 + y^2)$$

$$\frac{dy}{dt} = y - x - y(x^2 + y^2)$$

$$\left(\frac{dy}{dx} = \frac{y - x - y(x^2+y^2)}{y + x - x(x^2+y^2)}\right)$$

Look for critical points, ie

$$x + y - x(x^2+y^2) = 0$$
$$y - x - y(x^2+y^2) = 0.$$

ie $$x+y = x(x^2+y^2)$$
$$y-x = y(x^2+y^2)$$

$\Rightarrow$ $$\frac{x+y}{y-x} = \frac{x}{y}$$

$\Rightarrow$ $$xy+y^2 = xy-x^2$$

$\Rightarrow$ $$x^2+y^2=0$$ ie one critical pt at the origin.

Linearising about the origin gives:

$$\frac{dX}{dt} = X+Y \qquad \frac{dY}{dt} = Y-X \qquad J = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

J has eigenvalues λ where $(1-\lambda)^2+1=0$
$\Rightarrow \lambda = 1 \pm i$

COMPLEX $\rightarrow$ not a node
not a saddle pt
<u>is</u> a spiral pt ($\mathbb{C}$ and $Re() \neq 0$)
<u>is</u> unstable ($Re() > 0$)
not a centre ($Re() \neq 0$ and $Im() = 0$)

<u>Switching to polar coordinates</u>

$$r^2 = x^2+y^2 \qquad \tan\theta = \frac{y}{x}$$

Look to find $\dot{r}, \dot{\theta}$.

- $2r\dot{r} = 2x\dot{x} + 2y\dot{y}$ $\qquad\qquad \dot{x} = P,\ \dot{y} = Q$

$$\Rightarrow \dot{r} = \frac{1}{r}(xP + yQ)$$

- $\theta = \tan^{-1}(y/x)$ $\qquad\qquad \frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$

$$\Rightarrow \dot{\theta} = \frac{1}{1+(\frac{y}{x})^2}\left(\frac{\dot{y}}{x} - \frac{\dot{x}}{x^2}y\right)$$

$$= \frac{1}{r^2}(xQ - yP)$$

$$\Rightarrow \frac{dr}{d\theta} = \frac{\dot{r}}{\dot{\theta}} = \frac{r(xP+yQ)}{xQ-yP}$$

Here we find $\frac{dr}{dt} = r - r^3$

$$\frac{d\theta}{dt} = -1$$

$$\frac{dr}{d\theta} = r^3 - r.$$

An alternative method is to use complex n°s, and write $z = x + iy$, then

$$\frac{dz}{dt} = \frac{dx}{dt} + i\frac{dy}{dt}$$

$$= x + y - x(x^2+y^2) + iy - ix - iy(x^2+y^2)$$

$$= z - iz - z|z|^2$$

$$= (1-i)z - z|z|^2$$

To convert now to polars, write $z = re^{i\theta}$

$$\dot{z} = \dot{r}e^{i\theta} + ir\dot{\theta}e^{i\theta}$$
$$= (1-i)re^{i\theta} - re^{i\theta}r^2$$

Compare real parts: $\dot{r} = r - r^3$

$r\dot{\theta} = -r$

ie $\dot{\theta} = -1$, as before.

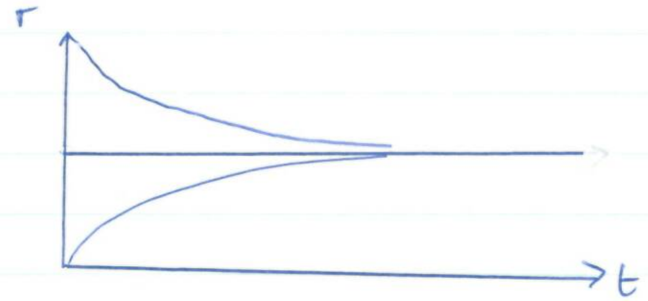
$$\dot{r} = r - r^3 = r(1-r^2).$$

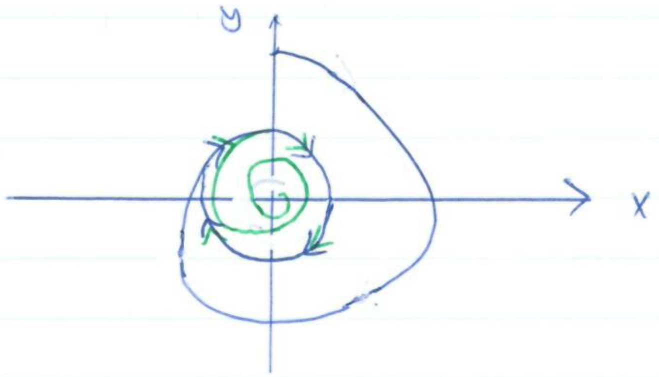
$\dot{\theta} = -1$, $\dot{r} < 0$: $r > 1$

$\dot{\theta} = -1$, $\dot{r} > 0$: $r < 1$

$\dot{\theta} = -1$, $\dot{r} = 0$: $r = 0$

$\dot{\theta} = -1$, $\dot{r} = 0$: $r = 1$





arrows from $\dot{\theta} = -1$

$r = 1$ is a <u>limit cycle</u>

We can find $r(\theta)$ and $r(t)$ explicitly $\quad (\dot{r} = r - r^3)$

$$\left(\frac{dr}{d\theta} = -r + r^3\right)$$

$$\Rightarrow \int \underbrace{\frac{1}{r^3-r}} dr = \int d\theta$$

p. fracs : $\frac{A}{r} + \frac{B}{r-1} + \frac{C}{r+1}$

$\underline{\text{or}}$ $$r\frac{dr}{d\theta} = r^4 - r^2$$

$$\Rightarrow \frac{1}{2}\frac{d}{d\theta}(r^2) = r^4 - r^2 \qquad \text{If } u = r^2,$$

$$\Rightarrow \frac{1}{2}\frac{du}{d\theta} = u^2 - u$$

$$\Rightarrow \int \frac{1}{u^2-u}\,du = 2\int d\theta$$

$$\Rightarrow \int \left(\frac{-1}{u} + \frac{1}{u-1}\right) du = 2\theta + c$$

$$\Rightarrow \ln\left(\frac{u-1}{u}\right) = 2\theta + c$$

$$\Rightarrow \frac{r^2-1}{r^2} = Ae^{2\theta}$$

$$\Rightarrow 1 - \frac{1}{r^2} = Ae^{2\theta}$$

$$\Rightarrow \frac{1}{r^2} = 1 - Ae^{2\theta}$$

$$\Rightarrow r^2 = \frac{1}{1-Ae^{2\theta}}$$

$$= \frac{e^{-2\theta}}{e^{-2\theta}-A}$$

$$\Rightarrow r = \frac{e^{-\theta}}{\sqrt{e^{-2\theta}-A}} \qquad \text{but } \dot{\theta} = -1$$

$$\Rightarrow r = \frac{e^t}{\sqrt{e^{2t} - A}}$$

If at $t=0$, $r>1$, $\Rightarrow$ $1-A<1$ $\Rightarrow A>0$.

and as $t\to\infty$, $r\to 1$.

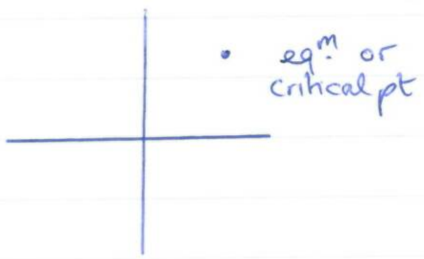
If at $t=0$, $r<1$, so $1-A>1 \Rightarrow A<0$.

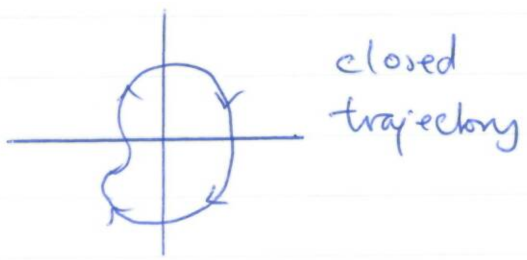
and as $t\to\infty$, $r\to 1$.

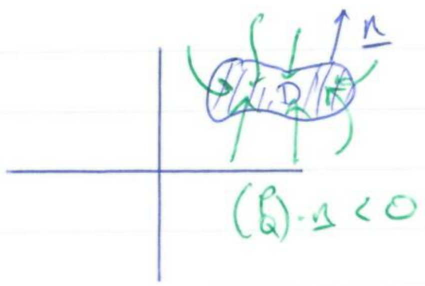
If at $r=0$, $r=1$, so $A=0$ and $r=1$ always.

Poincaré - Bendixon Thm

Def^n: A closed set of points (a single point/region/line) is said to be $\binom{\text{positively}}{\text{negatively}}$ invariant if all the trajectories at time $t=0$ remain in the set for $t\geq 0$, $t\leq 0$.







$$\binom{P}{Q}\cdot\underline{n} \;\begin{matrix}<0\\>0\end{matrix} \quad \text{ie} \quad \binom{\dot{x}}{\dot{y}}\cdot\underline{n} \;\begin{matrix}<0\\>0\end{matrix}$$

The P-B Thm states that if there exists a bounded invariant region of the phase plane with no eq^{m} points, then that region contains at least one limit cycle.

Example: $\frac{dx}{dt} = x - y - 2x(x^2+y^2) = P$

$\frac{dy}{dt} = x + y - y(x^2+y^2) = Q$

In polar coords,

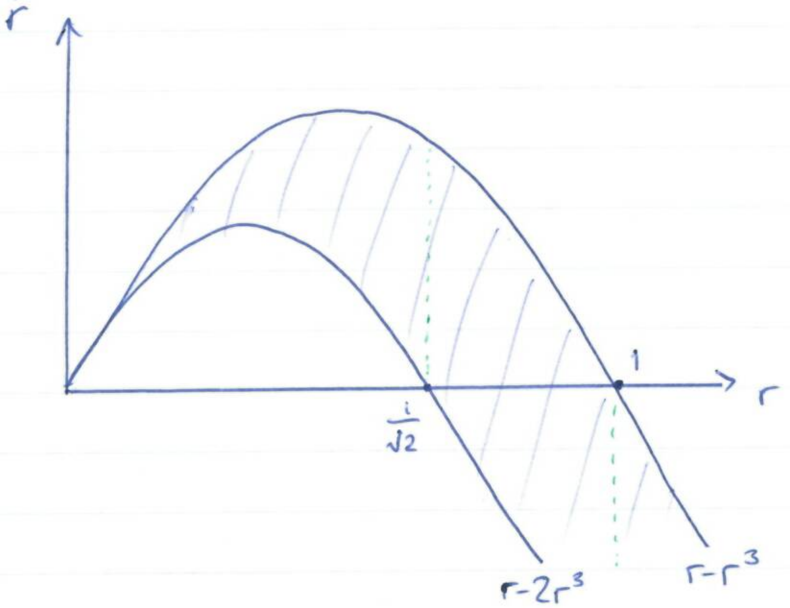
$$r\dot{r} = xP + yQ$$
$$= x^2 - xy - 2x^2(x^2+y^2) + xy + y^2 - y^2(x^2+y^2)$$
$$= r^2(1 - y^2 - 2x^2)$$

$$\Rightarrow \dot{r} = r(1-y^2-2x^2)$$
$$= r(1 - r^2\sin^2\theta - 2r^2\cos^2\theta)$$
$$= r - r^3(1+\cos^2\theta)$$

$$\Rightarrow r^2\dot{\theta} = xQ - yP$$
$$= x^2 + xy - xy(x^2+y^2)$$
$$- xy + y^2 + 2xy(x^2+y^2)$$
$$= r^2 + xyr^2$$

$$\Rightarrow \dot{\theta} = 1 + xy$$
$$= 1 + r^2\sin\theta\cos\theta$$
$$= 1 + \frac{r^2}{2}\sin 2\theta$$

$\boxed{L}$



$\leftarrow$ max/min values of $\dot r$.

if $r < \frac{1}{\sqrt{2}}$ $\quad \dot r > 0$

$r > 1$ $\quad \dot r < 0$

So we see there is a positively invariant "annular" set with inner radius $< \frac{1}{\sqrt{2}}$ and outer > 1.

In polar coordinates a critical pt has $\dot r = 0$ and $\dot\theta = 0$. $\dot\theta$ lies in the range $1 - \frac{1}{2}r^2$ and $1 + \frac{1}{2}r^2$ for fixed r, varying θ.

We can show that the lower bound is > 0 if $r < \sqrt{2}$. This includes the region where we have established the invariant set lies

$\Rightarrow$ we have a limit cycle with $\frac{1}{\sqrt{2}} < r < 1$.

Some special eq^{n}s

1. Consider ODEs of the form

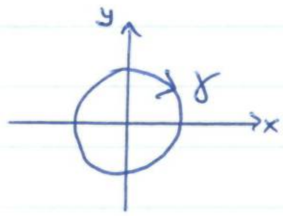
$$\ddot{x} + \phi(\dot{x}) + f(x) = 0$$

Let $y = \dot{x}$, then

$$\dot{y} + \phi(y) + f(x) = 0 .$$

and $\ddot{x} = \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = y\frac{dy}{dx}$ and get

$$y\frac{dy}{dx} + \phi(y) + f(x) = 0$$



Assume a closed orbit γ exists and integrate this eqn round γ.

$$\int_\gamma y\frac{dy}{dx}\,dx + \int_\gamma \phi(y)\,dx + \int_\gamma f(x)\,dx = 0$$

$f = F'$

$$\Rightarrow \quad \underbrace{\left[\tfrac{1}{2}y^2\right]_{start}^{end}}_{=0} + \int_\gamma \phi(y)\,dx + \underbrace{\left[F\right]_{start}^{end}}_{=0} = 0$$

$$\Rightarrow \quad \int_\gamma \phi(y)\,dx = 0 .$$

So if we have a closed orbit, on it $\int_\gamma \phi(y)\,dx = 0$.

ie $\int_0^T \phi(\dot{x})\,\dot{x}\,dt$ (if we use dt to go round the orbit instead of dx)

So we can have no closed orbit if $y\phi(y)$ is single-signed

($\because$ we have to 'sum over' $y\phi(y)$ [$= \dot{x}\phi(\dot{x})$] to get 0) e.g. $\ddot{x} + \dot{x} + x = 0$ SHM (damped)

$$\ddot{x} + \underbrace{\dot{x}}_{\text{damping}} + x = 0$$

Here, $\phi(y) = y$ and $y\phi(y) = y^2$.

This is single-signed $\Rightarrow$ no periodic orbit exists.
(which we know because it damps until it stops)

Lienard's eq^n

2. Consider the equation

$$\ddot{x} + \dot{x}f(x) + g(x) = 0 \qquad [\text{Lienard's eq}^n]$$

Lienard's thm: (1) If $f(x)$ is even in x [e.g. (x^2-1)], and
(2) If $g(x)$ is odd in x and $g(x) > 0$ for $x > 0$, and
(3) If $F(x) = \int_0^x f(t)\,dt$, where $F(x)$ has a single positive zero x_0 and $F(x)$ is positive and monotonic increasing for $x > x_0$ and $F(x) \to \infty$ as $x \to \infty$

then

the eq^n has a unique period sol^n

Example: Van der Pol eq^n

$$\ddot{x} - \underbrace{\varepsilon(1-x^2)\dot{x}}_{\text{damping}} + \overset{\omega^2}{\cancel{1}}x = 0$$

Lienhard's thm tells us that for $\varepsilon > 0$ and $\varepsilon \neq 0$ we have a unique periodic sol^n. However, with $\varepsilon = 0$, we have

$$\ddot{x} + \omega^2 x = 0$$

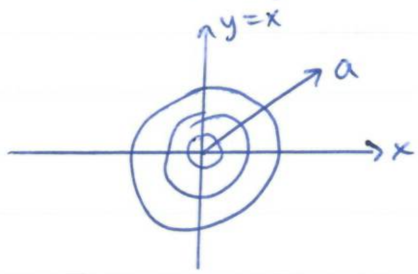
with sol^n:

$$x = a\cos(\omega t + \phi)$$

with a and ϕ arbitrary.

As we are looking for periodic sol^{n}s, we can choose the origin of t conveniently and choose $x = a$, $\dot{x} = 0$ at $t = 0$, ie $\phi = 0$ and

$$x = a\cos\omega t$$



We will investigate the Van der Pol eq^n for small ε.

$$\ddot{x} + x = \varepsilon(1 - x^2)\dot{x}$$

and it seems sensible to solve this for periodic sol^{n}s, writing

$$x = x_0 t + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \cdots$$

However, it doesn't work!! Consider a simpler eq^n to see why not.

Consider

$$\ddot{u} + u + \varepsilon u^3 = 0$$

and look for periodic sol^{n}s of the form

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots$$

Substitution leads to (to ε^2)

$$\ddot{u}_0 + \varepsilon \ddot{u}_1 + \varepsilon^2 \ddot{u}_2 + (u_0 + \varepsilon u_1 + \varepsilon^2 u_2) + \varepsilon(u_0^3 + 3u_0^2 \varepsilon u_1) = 0$$

terms

$\boxed{L}$ ε $[\varepsilon^0]$ $\ddot{u}_0 + u_0 = 0 \Rightarrow u_0 = A\cos t + B\sin t$

Since we are looking for periodic sol^{n}s, we may put $B=0$, ie choose the origin of t to be where $\dot{u}_0 = 0$, at which time $u_0 = a$, $\Rightarrow u_0 = a\cos t$.

$[\varepsilon^1]$ $$\ddot{u}_1 + u_1 = -u_0^3 = -a^3\cos^3 t = -\frac{a^3}{4}(\cos 3t + 3\cos t)$$

We look for a sol^n $u_1 = a_1\cos t + b_1 \sin t$ ← C.F.

$$+ A\cos 3t + B\sin 3t$$
$$+ A t\cos t + D t\sin t$$

a_1 and b_1 are found from relevant initial conditions. Substitution gives:

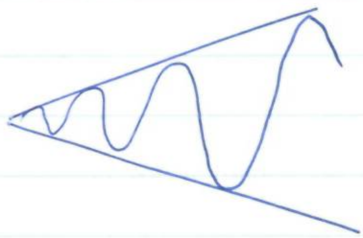
$$A = \frac{a^3}{32} \quad B = 0 \quad C = 0 \quad D = -\frac{3}{8}a^3$$

So $\quad u = u_0 + \varepsilon u_1$

$$= \underline{a\cos t} + \varepsilon\left(a_1\cos t + b_1 \sin t + \frac{a^3}{32}\cos 3t \underline{- \frac{3}{8}a^3 t\sin t}\right)$$

This, however, is not periodic due to the

$$\varepsilon\left(-\frac{3}{8}a^3\right) t\sin t$$



Also if t becomes large enough, εt can become as big as u_0. So $\varepsilon u_1 \not\ll u_0$ and our expansion is invalid.

In fact, our mistake has been to fail to realise that the nonlinear term εu^3 can affect the frequency/period of the motion as well as the amplitude.

You could see this like this: (look at green terms above)

$$a\cos t - \frac{3}{8}\varepsilon a^3 t\sin t \approx a\cos\left[t - \frac{3}{8}\varepsilon a^2 t\right] \qquad \varepsilon \ll 1$$

(Taylor expansion of $\frac{3}{8}\cdots$ $\sin' = \cos$)

$$= a\cos\left[t\left(1 - \frac{3}{8}\varepsilon a^2\right)\right]$$

frequency is $1 - \frac{3}{8}\varepsilon a^2$

Period $\dfrac{2\pi}{1 - \frac{3}{8}\varepsilon a^2}$

We can accommodate this using Lindstedt's method.

We switch to a new variable s, where

$$s = t(1 + \varepsilon c_1 + \varepsilon^2 c_2 + \cdots)$$

and expand ← this is 1 ∵ this is 1.

$$u = u_0(s) + \varepsilon u_1(s) + \cdots$$

where u is 2π-periodic in s

$$\ddot{u} + u + \varepsilon u^3 = 0.$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial s}(1 + \varepsilon c_1 + \varepsilon^2 c_2 + \cdots)$$

$$\frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial s^2}(1 + \varepsilon c_1 + \varepsilon^2 c_2 + \cdots)^2$$

$'' = \frac{\partial^2}{\partial s^2}$

$$\Rightarrow \ddot{u} = (1 + \varepsilon c_1 + \varepsilon^2 c_2 + \cdots)^2 (u_0'' + \varepsilon u_1'' + \cdots)$$

$$\Rightarrow \ddot{u} + u + \varepsilon u^3 = \underbrace{(1 + \varepsilon c_1 + \varepsilon^2 c_2 + \cdots)^2}_{1 + 2\varepsilon c_1 + \cdots}(u_0'' + \varepsilon u_1'' + \cdots)$$

$$+ (u_0 + \varepsilon u_1 + \cdots) + \varepsilon(u_0^3 + \cdots) = 0$$

terms $[\varepsilon^0]$ $\quad u_0'' + u_0 = 0 \quad \Rightarrow \quad \underline{u_0 = a\cos s}$

$[\varepsilon^1]$ $\quad (u_1'' + 2c_1 u_0'') + u_1 + u_0^3 = 0$

$$\Rightarrow u_1'' + u_1 = -a^3\cos^3 s - 2c_1(-a\cos s)$$

$$= -\frac{a^3}{4}(\cos 3s + 3\cos s) + 2c_1 a\cos s$$

We are now able to choose c_1 to kill off the term $-\frac{3}{4}a^3\cos s$ which caused the non-periodicity of our sol^n, previously and so ensure u_1 is 2π periodic in s

$$2c_1 a - \frac{3}{4}a^3 = 0 \quad \Rightarrow \quad c_1 = \frac{3}{8}a^2$$

and $\quad u_1'' + u_1 = -\dfrac{a^3}{4}\cos 3s$

Solution: $\quad u_1 = a_1\cos s + b_1 \sin s + \left(-\dfrac{a^3}{4}\right)\dfrac{\cos 3s}{-9+1}$

Now, $\quad s = t\left(1 + \frac{3}{8}\varepsilon a^2 + \cdots\right)$

$$\Rightarrow \quad u = a\cos s + \varepsilon\left(a_1\cos s + b_1\sin s + \frac{a^3}{32}\cos(3s)\right)$$

Period of motion is $\quad \dfrac{2\pi}{1+\frac{3}{8}\varepsilon a^2} = 2\pi\left(1 - \frac{3}{8}\varepsilon a^2 \cdots\right)$

Rayleigh's equation

$$\ddot{x} - \varepsilon\left[\dot{x} - \frac{1}{3}\dot{x}^3\right] + x = 0$$

We introduce $\theta = nt$, $\qquad n = n_0 + \varepsilon n_1 + \varepsilon^2 n_2 + \cdots$

$$x = x_0(\theta) + \varepsilon x_1(\theta) + \varepsilon^2 x_2(\theta) + \cdots$$

where x_0, x_1, x_2 are 2π-periodic.

$$\frac{d}{dt} \to n\frac{d}{d\theta} \qquad \left(\frac{d}{d\theta} = {}'\right)$$

$$n^2x'' - \varepsilon\left(nx' - \frac{1}{3}n^3x'^3\right) + x = 0$$

$$\Rightarrow \left(n_0^2 + 2\varepsilon n_0 n_1 + \varepsilon^2[2n_2n_0 + n_1^2] + \cdots\right)\cdot$$

$$\cdot\left(x_0'' + \varepsilon x_1'' + \varepsilon^2 x_2''\right)$$

$$- \varepsilon\Big(n_0x_0' + \varepsilon[n_1x_0' + n_0x_1'] + \cdots$$

$$-\frac{1}{3}\left[n_0^3x_0'^3 + \varepsilon\left(n_0^3 3x_0'^2x_1' + 3x_0'^3n_0^2n_1\right) + \cdots\right]\Big)$$

$$+ \left(x_0 + \varepsilon x_1 + \varepsilon^2 x_2\right) = 0$$

$[\varepsilon^0]$: $n_0^2x_0'' + x_0 = 0$

$$x_0 = A_0\cos\left(\frac{\theta}{n_0}\right) + B_0\sin\left(\frac{\theta}{n_0}\right)$$

Choosing origin appropriately gives $B_0 = 0$

2π-periodic in θ $\Rightarrow$ $n_0 = 1$

$[\varepsilon^1]$: $x_1'' + x_1 = -2n_1x_0'' + x_0' - \frac{1}{3}x_0'^3$

$$= 2n_1A_0\cos\theta - A_0\sin\theta + \frac{1}{3}A_0^3\sin^3\theta$$

Now, $\sin^3\theta = \frac{3}{4}\sin\theta - \frac{\sin 3\theta}{4}$

We want to ensure that the RHS has no terms in $\cos\theta$ or $\sin\theta$, so that x_1 remains periodic.

$[\cos\theta]$: $\quad 2n_1 A_0 = 0 \Rightarrow n_1 = 0$

$[\sin\theta]$: $\quad -A_0 + \frac{1}{4}A_0^3 = 0 \Rightarrow A_0 = 2$

$$x_1'' + x_1 = -\frac{1}{12}A_0^3 \sin 3\theta$$

$$= -\frac{2}{3}\sin 3\theta$$

<u>Consider general equations</u>

$$\ddot{x} + \varepsilon f(x, \dot{x}) + \omega^2 x = 0$$

Not using Lindstedt's method leads to an expansion,

$$x = x_0(t) + \varepsilon x_1(t) + \dots$$

$$\ddot{x}_0 + \omega^2 x_0 = 0 \Rightarrow x_0 = A\sin(\overbrace{\omega t + \phi}^{\chi})$$

$$\ddot{x}_1 + \omega^2 x_1 = -f(x_0, \dot{x}_0)$$

$$= -f(A\sin\chi, A\omega\cos\chi) \quad \text{Periodic with period } \omega t$$

We know that any component of the RHS that depends on $\cos\chi$ and $\sin\chi$ will cause x_1 to be non-periodic and grow.

$$\ddot{x}_1 + \omega^2 x_1 = r_0 + \sum_{n=1}^{\infty} r_n \cos(n\omega t) + s_n \sin(n\omega t)$$

where $\int_0^{2\pi/\omega} r_0 \, dt = -\int_0^{2\pi/\omega} f(A\sin\chi, A\omega\cos\chi)\, dt$

$[\chi \equiv \omega t + \phi]$

(or integrate between $-\frac{\pi}{\omega}$ and $\frac{\pi}{\omega}$ if f is even/odd to get some of the integrals explicitly.)

$$\Rightarrow \quad \frac{2\pi}{\omega} r_0 = -\int_0^{2\pi/\omega} f(A\sin\chi, A\omega\cos\chi)\, dt$$

$$\frac{2\pi}{\omega}\frac{1}{2} r_n = -\int_0^{2\pi/\omega} \cos(n\omega t)\, f(A\sin\chi, A\omega\cos\chi)\, dt$$

$$\frac{2\pi}{\omega}\frac{1}{2} s_n = -\int_0^{2\pi/\omega} \sin(n\omega t)\, f(A\sin\chi, A\omega\cos\chi)\, dt$$

We can bring in extra terms that cancel r_1 and s_1 using Linstedt's method

$$\theta = nt, \qquad n = \omega + \varepsilon n_1 + \cdots$$
$$x = x_0(\theta) + \varepsilon x_1(\theta) + \cdots$$

$$\ddot{x} + \omega^2 x = -\varepsilon f(x, \dot{x}) \Rightarrow n^2 x'' + \omega^2 x = -\varepsilon f(x, n x')$$

$$\Rightarrow (\omega^2 + 2\omega n_1 \varepsilon + \cdots)(x_0'' + \varepsilon x_1'' + \cdots) + \omega^2 (x_0 + \varepsilon x_1 + \cdots)$$
$$= -\varepsilon f(x_0, \omega x_0')$$

$\boxed{L}$ $$\ddot{x} + \varepsilon f(x, \dot{x}) + \omega^2 x = 0$$

$\theta = nt$, $\quad n = n_0 + \varepsilon n_1 + \cdots$

$\qquad x = x_0(\theta) + \varepsilon x_1(\theta) + \cdots$

2π-periodic in θ

$[\varepsilon^0]$: $\quad n_0^2 x_0'' + \omega^2 x = 0$

$$x_0 = A\sin(\theta + \phi) = A\sin(\chi)$$

$n_0 = \omega$ gives 2π-periodic.

$[\varepsilon^1:]$ $\quad n_0^2 x_1'' + \omega^2 x_1 + 2n_0 n_1 x_0'' = -f(x_0, n_0 x_0')$

$n_0 = \omega$ and $x_0 = A\sin\chi$

$$x_1'' + x_1 = 2\frac{n_1 A}{\omega}\sin\chi - f(A\sin\chi, \omega A\cos\chi)$$

$$= \frac{2n_1 A\sin\chi}{\omega} + \underbrace{r_0 + \sum_{i=1}^{\infty} r_i\cos(i\chi) + s_i\sin(i\chi)}_{g(\chi)}$$

multiply by $\cos\chi$ and integrate:

$$\int_0^{2\pi} g(\chi)\cos\chi\, d\chi = r_1 \cancel{\cos} 2\pi \cdot \tfrac{1}{2}$$

$$\Rightarrow r_1 = \frac{1}{\pi}\int_0^{2\pi} -\cos\chi\, f(A\sin\chi, \omega A\cos\chi)\, d\chi$$

similarly $s_1 = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} -\sin\chi\, f(A\sin\chi, \omega A\cos\chi)\, d\chi$

Want the RHS to have no sin or cos's $\because$ we don't want a periodic sol^n? so how do we get rid of them

General Application of the Poincaré–Lindstedt Method

$$\ddot{x} + \varepsilon f(x, \dot{x}) + \omega^2 x = 0$$

$\theta = nt$, $n = n_0 + \varepsilon n_1 + \ldots$, $x = x_0 + \varepsilon x_1 + \ldots$ $x_0(\theta), x_1(\theta)$ are 2π-periodic in θ.

$\frac{d}{dt} \to \frac{d}{d\theta}\cdot\frac{d\theta}{dt} = n\frac{d}{d\theta}$, so $n^2 x'' + \omega^2 x = -\varepsilon f(x, nx')$

Substitution $\Rightarrow$ $(n_0 + \varepsilon n_1 \ldots)^2(x_0'' + \varepsilon x_1'') + \omega^2(x_0 + \varepsilon x_1) = -\varepsilon f(x_0, n_0 x_0')$
+ terms in ε^2

ε^0: $n_0^2 x_0'' + \omega^2 x_0 = 0 \Rightarrow x_0 = A\sin\left(\frac{\omega\theta}{n_0} + \varphi\right)$ say

Period $2\pi \Rightarrow$ $\underline{n_0 = \omega}$ & $x_0 = A\sin\chi$
$\chi = \theta + \varphi$

ε^1: $n_0^2 x_1'' + \omega^2 x_1 = -2n_0 n_1 x_0'' - f(x_0, n_0 x_0')$

$$\Rightarrow \quad x_1'' + x_1 = \frac{2n_1}{\omega} A\sin\chi - f(A\sin\chi, \omega A\cos\chi)$$

$$= \frac{2n_1}{\omega} A\sin\chi + r_0 + \sum_{i=1}^{\infty} r_i\cos(i\chi) + s_i\sin(i\chi)$$

with r_i & s_i the Fourier coefficients of $-f(A\sin\chi, \omega A\cos\chi)$ so

$$r_1 = \frac{1}{\pi}\int_0^{2\pi}(-1)\cos\chi\, f(A\sin\chi, \omega A\cos\chi)\,d\chi$$

$$s_1 = \frac{1}{\pi}\int_0^{2\pi}(-1)\sin\chi\, f(A\sin\chi, \omega A\cos\chi)\,d\chi$$

x_1 will be periodic 2π in θ if coefficients of $\sin\chi$ & $\cos\chi$ on r.h.s are zero.

So $\frac{2n_1 A}{\omega} + s_1 = 0 \Rightarrow n_1 = \frac{\omega}{2\pi A}\int_0^{2\pi} \sin\chi\, f(A\sin\chi, \omega A\cos\chi)\,d\chi$

$r_1 = 0$ & $0 = \int_0^{2\pi}\cos\chi\, f(A\sin\chi, \omega A\cos\chi)\,d\chi.$

For Rayleigh's equation $f(x, \dot{x}) = -(\dot{x} - \frac{1}{3}\dot{x}^3)$

So $n_1 = \frac{\omega}{2\pi A}\int_{-\pi}^{\pi}\sin\chi(-1)\left(\omega A\cos\chi - \frac{1}{3}\omega^3 A^3\cos^3\chi\right)d\chi = 0$

$$0 = \int_0^{2\pi}\cos\chi(-1)\left(\omega A\cos\chi - \frac{1}{3}\omega^3 A^3\cos^3\chi\right)d\chi \Rightarrow \underline{A = 2/\omega}.$$

$\underline{\sin\chi}$: $\quad \frac{2n_1 A}{\omega} + s_1 = 0 \qquad$ for coeff of $\sin\chi$ to be 0

$\underline{\cos\chi}$: $\quad r_1 = 0$

$$\int_0^{2\pi} \cos\chi \, f(A\sin\chi, \omega A\cos\chi)\, d\chi = 0$$

$$n_1 = \frac{\omega}{\pi 2A} \int_0^{2\pi} \sin\chi \, f(A\sin\chi, \omega A\cos\chi)\, d\chi$$

Two equations for n_1 and A.

<u>Now apply these to Rayleigh's equation:</u>

$$\ddot{x} \underbrace{- \varepsilon\left(\dot{x} - \tfrac{1}{3}\dot{x}^3\right)}_{f(x,\dot{x})} + \omega^2 x = 0$$

$$f(x, \dot{x}) = -\left(\dot{x} - \tfrac{1}{3}\dot{x}^3\right)$$

$$n_1 = \frac{\omega}{2\pi A} \int_{-\pi}^{\pi} \sin\chi (-1)\left(\omega A\cos\chi - \tfrac{1}{3}\omega^3 A^3 \cos^3\chi\right) d\chi = 0$$

($\because$ sin is odd, $\cos$ & $\cos^3$ even)

$$0 = \int_{-\pi}^{\pi} \cos\chi (-1)\left(\omega A\cos\chi - \tfrac{1}{3}\omega^3 A^3 \cos^3\chi\right) dx$$

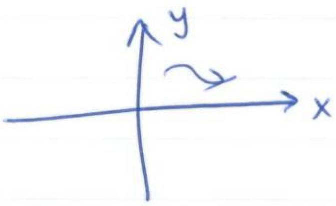
Require $\quad 0 = \omega A \underbrace{\pi}_{\int_0^{2\pi}\cos^2\chi\, d\chi} - \frac{1}{3}\omega^3 A^3 \underbrace{\frac{3\pi}{4}}_{\int_0^{2\pi}\cos^4\chi\, d\chi}$

$$\Rightarrow A = \frac{2}{\omega}$$

(in the example before, $\omega = 1$, so this general method has reproduced the particular case)

Lienhard plane

Instead of plotting $\dot{x}$ vs x, we sometimes plot



where $y = \dot{x} + F(x)$

$$F' = f$$

Note $\dot{y} = \frac{d}{dx}\left(\dot{x} + F(x)\right) = \ddot{x} + F'(x)\dot{x} = \ddot{x} + f(x)\dot{x}$

$= -g(x)$ from ODE

$\Rightarrow$ coupled eq^n:

$$\begin{cases} \dot{x} = y - F(x) \\ \dot{y} = -g \end{cases}$$

"Lienhard transformation"

We will use this transformation to examine the sol^n in the Lienhard plane for the Van der Pol eq^n

$$\ddot{x} - \varepsilon(1 - x^2)\dot{x} + x = 0 \qquad \text{for } \varepsilon \gg 1.$$

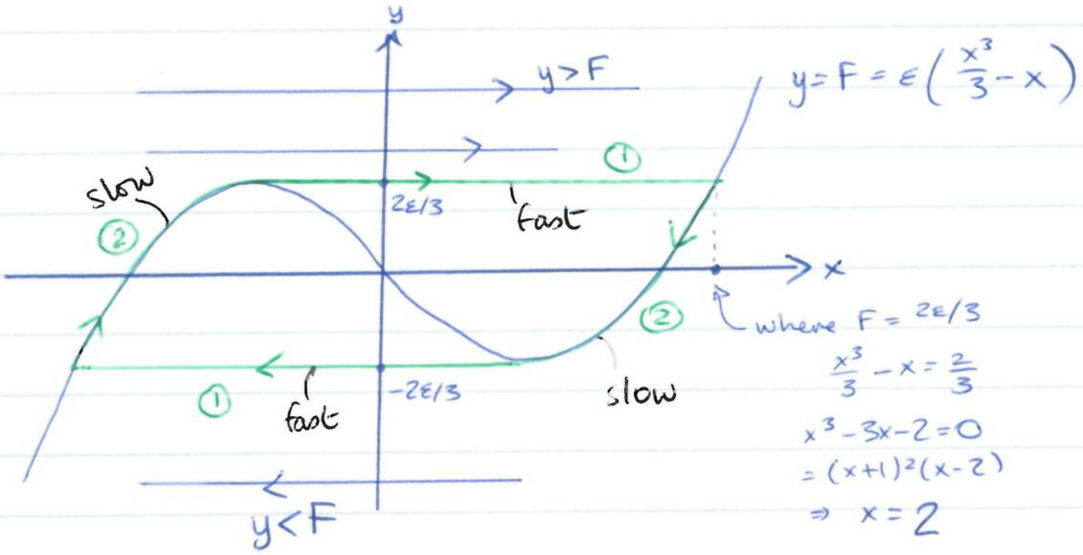
Here, $f(x) = -\varepsilon(1 - x^2)$

$$\Rightarrow \quad F(x) = \varepsilon\left(\frac{x^3}{3} - x\right)$$

Make Lienhard transformation.

$$\begin{cases} \dot{x} = y - \varepsilon\left(\frac{x^3}{3} - x\right) \\ \dot{y} = -x \end{cases}$$

$\varepsilon \gg 1 \Rightarrow \dot{x} \gg 1$ except where $y = \varepsilon\left(\frac{x^3}{3} - x\right)$
$= F(x)$



x can only increase quickly for a short time and in this time, y does not vary as much.

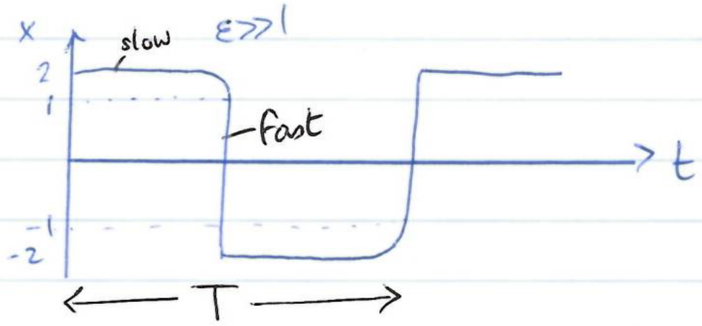
$\therefore$ Trajectories are horizontal lines

If y is close to F, then $\dot{x}$ is not large and x and y vary more slowly.

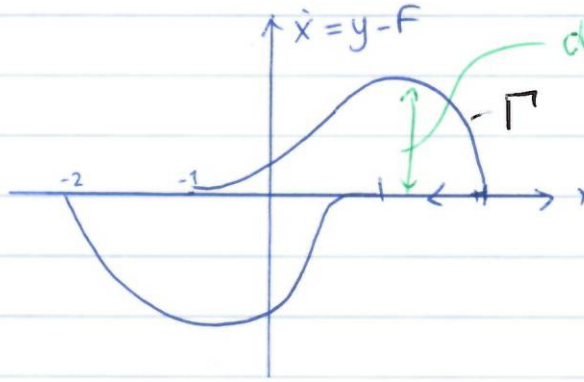
A periodic, finite amplitude orbit is possible, as shown.

① are traversed quickly with x moving from -1 to 2.

② is traversed more slowly with x moving from 2 to 1



Translated back into the phase plane



$$\text{Period} = T = \int_{\Gamma} \frac{dt}{dx}\, dx = \int_{\Gamma} \frac{dt}{dy} \frac{dy}{dx}\, dx$$

small on fast parts
$= f$ on slow parts ($\because y = F$)

$\frac{1}{-g} = -\frac{1}{x}$

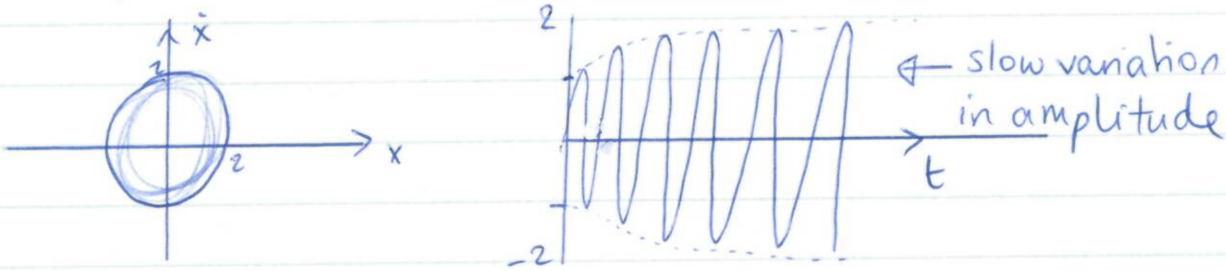
$2 \rightarrow 1$
$= -2 \rightarrow -1$

$$\approx 2 \int_{2}^{1} -\frac{1}{x}(x^2-1)\,\varepsilon\, dx$$

$$= (3 - 2\ln 2)\,\varepsilon \qquad \gg 1 \qquad (\because \varepsilon \gg 1)$$

VdP eq^n: $\ddot{x} - \varepsilon(1-x^2)\dot{x} + x = 0$, $\varepsilon \ll 1$

but sol^n not the limit cycle, amplitude 2, $x = 2\cos t$



Two active timescales – one where time is $O(1)$, over which oscillations occur and one where time is $O(\frac{1}{\varepsilon})$, ie long over which amplitudes vary.

To tackle these sorts of problems we use the method of multiple scales and consider x as a f^n of two independent variables, t and $T = \varepsilon t$, ie $x = x(t, T)$.

Consider $\ddot{x} + \underbrace{\varepsilon f(x, \dot{x})} + \omega^2 x = 0$

This term will have an influence over a time of size $\frac{1}{\varepsilon}$, ie $\varepsilon t = T$ is $O(1)$.

Let $x = x_0(t, T) + \varepsilon x_1(t, T) + \cdots$

and insist solution is periodic in t.

$$\frac{d}{dt} \rightarrow \left(\frac{\partial}{\partial t} + \frac{\partial T}{\partial t} \frac{\partial}{\partial T} \right)$$

$$= \left(\frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial T} \right)$$

$$\frac{d^2}{dt^2} \to \left(\frac{\partial}{\partial t} + \varepsilon\frac{\partial}{\partial T}\right)^2$$

$$= \frac{\partial^2}{\partial t^2} + 2\varepsilon\frac{\partial^2}{\partial T\partial t} + \varepsilon^2\frac{\partial^2}{\partial T^2}$$

Substitution gives

$$\left(\frac{\partial^2}{\partial t^2} + 2\varepsilon\frac{\partial^2}{\partial T\partial t} + \cdots\right)\left(x_0(t,T) + \varepsilon x_1(t,T) + \cdots\right)$$

$$+ \varepsilon f\Big(x_0 + \cdots, \underbrace{\left(\frac{\partial}{\partial t} + \cdots\right)(x_0 + \cdots)}_{\frac{\partial x_0}{\partial t}}\Big) + \omega^2(x_0 + \varepsilon x_1, \cdots)$$

$$= 0.$$

$[\varepsilon^0]$: $\dfrac{\partial^2 x_0}{\partial t^2} + \omega^2 x_0 = 0 \quad \Rightarrow \quad x_0 = A(T)\sin[\omega t + \phi(T)]$

ie amplitude $A(T)$ and phase $\phi(T)$ vary over the longer timescale.

Say $x_0 = A(T)\sin\chi$ where $\chi = \omega t + \phi(T)$

$[\varepsilon^1]$: $\dfrac{\partial^2 x_1}{\partial t^2} + \omega^2 x_1 = -2\dfrac{\partial^2}{\partial T\partial t}x_0 - f\left(x_0, \dfrac{\partial x_0}{\partial t}\right)$

$$= -2\frac{\partial}{\partial T}(\overbrace{\omega A\cos\chi}^{\partial x_0/\partial t}) - f(A\sin\chi, \omega A\cos\chi)$$

$$= -2\omega\frac{\partial A}{\partial T}\cos\chi + 2\omega\frac{\partial\phi}{\partial T}A\sin\chi$$

$$- f(A\sin\chi, \omega A\cos\chi)$$

The sol^{n} for x_1 will be periodic in t if the RHS has no component in $\cos\omega t$ and $\sin\omega t$,

ie $\cos\chi$ and $\sin\chi$.

Or, multiplying the RHS by $\underline{\cos\chi}$ and then $\underline{\sin\chi}$ and integrating over the period (2π in χ, $\frac{2\pi}{\omega}$ in t) gives zero.

$\cos\chi$ $$-2\omega \frac{\partial A}{\partial T} \underbrace{\int_0^{2\pi} \cos^2\chi \, d\chi}_{\pi} + 2\omega \frac{\partial \phi}{\partial T} A \underbrace{\int_0^{2\pi} \cos\chi \sin\chi \, d\chi}_{0}$$

$$= \int_0^{2\pi} \cos\chi \, f(A\sin\chi, \omega A\cos\chi) \, d\chi$$

$$\Rightarrow \quad \frac{\partial A}{\partial T} = -\frac{1}{2\pi\omega} \int_0^{2\pi} \cos\chi \, f(A\sin\chi, \omega A\cos\chi) \, d\chi$$

$\sin\chi$ and $$-2\omega \frac{\partial A}{\partial T} \int_0^{2\pi} \cos\chi \sin\chi \, d\chi + 2\omega \left(\frac{\partial \phi}{\partial T}\right) A \int_0^{2\pi} \sin^2\chi \, d\chi$$

$$= \int_0^{2\pi} \sin\chi \, f(A\sin\chi, \omega A\cos\chi) \, d\chi$$

$$\Rightarrow \quad \frac{\partial \phi}{\partial T} = \frac{1}{2\pi\omega A} \int_0^{2\pi} \sin\chi \, f(A\sin\chi, \omega A\cos\chi) \, d\chi$$

<u>For the Van der Pol eq^{n},</u> $\quad f(x, \dot{x}) = -(1-x^2)\dot{x} \qquad \omega = 1$

$$\Rightarrow \quad \frac{\partial \phi}{\partial T} = \frac{1}{2\pi A} \int_{-\pi}^{\pi} \sin\chi \underbrace{(-1)}_{-1} \underbrace{(1 - A^2\sin^2\chi)}_{1-x^2} \underbrace{A\cos\chi}_{\dot{x}} \, d\chi$$

$$= 0$$

$\Rightarrow$ ϕ does not vary over a timescale of size $\frac{1}{\varepsilon}$.

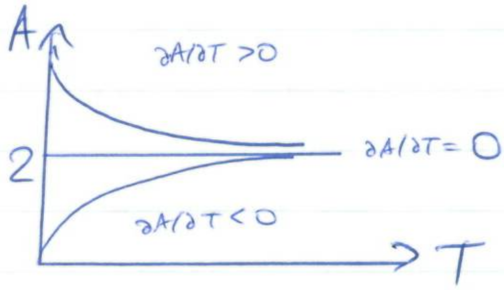
$$\frac{\partial A}{\partial T} = -\frac{1}{2\pi}\int_{-\pi}^{\pi} \cos x\,(-1)(1-A^2\sin^2 x)\,A\cos x\,dx$$

$$\frac{dA}{dT} = A\frac{1}{2\pi}\underbrace{\int_{-\pi}^{\pi}\cos^2 x\,dx}_{2\pi\frac{1}{2}=\pi} - A^3\frac{1}{2\pi}\underbrace{\int_{-\pi}^{\pi}\sin^2 x\cos^2 x\,dx}_{\frac{1}{4}(\sin^2 2x)\,;\ \frac{1}{4}2\pi\frac{1}{2}=\frac{\pi}{4}}$$

$$= A\frac{1}{2} - \frac{A^3}{8}$$

$$= \frac{A}{2}\left(1-\left(\frac{A}{2}\right)^2\right)$$

LOOKS LIKE $\Rightarrow$



But to solve explicitly:

$$Q = A^2 \Rightarrow \frac{dQ}{dT} = 2A\frac{dA}{dT}$$

$$= 2\frac{A^2}{2}\left(1-\left(\frac{A^2}{2}\right)\right)$$

$$= Q - \frac{Q^2}{4}$$

$$= \frac{Q}{4}(4-Q)$$

$$\Rightarrow \int \frac{1}{Q(4-Q)}\,dQ = \int \frac{1}{4}\,dT$$

$$= \frac{1}{4}\int \frac{1}{Q} + \frac{1}{4-Q}\,dQ$$

$$T + T_0 = \ln\left(\frac{Q}{4-Q}\right) \Rightarrow \frac{Q}{4-Q} = \alpha e^{T}$$

and if $A = A_0$ at $T = 0$ then $\alpha = \dfrac{A_0^2}{4-A_0^2}$

$$\Rightarrow \frac{4-Q}{Q} = \frac{4-A_0^2}{A_0^2} e^{-T}$$

$$\Rightarrow \frac{4}{Q} = 1 + \frac{(4-A_0^2)}{A_0^2} e^{-T}$$

$$\Rightarrow Q = A^2 = \frac{4}{\left(1 + \frac{4-A_0^2}{A_0^2} e^{-T}\right)}$$

$$= \frac{4A_0^2}{A_0^2 + (4-A_0^2)e^{-T}}$$

"try using the general method directly onto the VdP equation"

$$\Rightarrow A = \frac{2A_0}{\sqrt{A_0^2 + (4-A_0^2)e^{-T}}}$$

Now have to be patient (possible exam question idea)

Solve $\ddot{y} + \underbrace{\varepsilon\dot{y}}_{} + y + \underbrace{\varepsilon^2 y^3}_{} = 0$ $\quad y(0) = 1$

$\dfrac{dy}{dt}(0) = 0$

$\varepsilon \to 0.$

Small damping implies try multiple scales

Non-linearity implies try the Lindstedt procedure

We combine the two methods.

Introduce $\quad \tau = nt$

infact it is true and you could derive this but takes so damn long...

Lindstedt: $\quad n = \overset{\text{set } n_0 = 1}{n_0} + \underset{\text{set } n_1 = 0}{\varepsilon n_1} + \varepsilon^2 n_2 + \cdots$

$T = \varepsilon t$

Look for sol^ns $\quad y = y_0(\tau, T) + \varepsilon y_1(\tau, T) + \varepsilon^2 y_2(\tau, T) + \cdots$

- $$\frac{d}{dt} = \frac{\partial \tau}{\partial t}\frac{\partial}{\partial \tau} + \frac{\partial T}{\partial t}\frac{\partial}{\partial T}$$

$$= \underbrace{(1 + \varepsilon^2 a + \cdots)}_{n} \frac{\partial}{\partial \tau} + \varepsilon \frac{\partial}{\partial T}$$

- $$\frac{d^2}{dt^2} = \underbrace{(1 + 2\varepsilon^2 a + \cdots)}_{n^2} \frac{\partial^2}{\partial \tau^2} + 2\varepsilon(1 + \cdots)\frac{\partial}{\partial T}\frac{\partial}{\partial \tau} + \varepsilon^2 \frac{\partial^2}{\partial T^2}$$

$$\ddot{y}\left\{\begin{array}{l} (1 + 2\varepsilon^2 a)(y_{0\tau\tau} + \varepsilon y_{1\tau\tau} + \varepsilon^2 y_{2\tau\tau} \cdots) \\ \quad + 2\varepsilon(1 \cdots)(y_{0\tau T} + \varepsilon y_{1\tau T}) \\ \quad + \varepsilon^2 (y_{0TT} + \cdots) \end{array}\right.$$

$$+ \varepsilon\left[(1 + \cdots)(y_{0\tau} + \varepsilon y_{1\tau} + \cdots) + \varepsilon(y_0 + \cdots)\right]$$

$$+ (y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots)$$

$$+ \varepsilon^2 y_0^3 + \cdots$$

$$= 0$$

[L] If we have boundary conditions

$$y(0) = 1 \qquad \dot{y}(0) = 0$$

$$y(0) = 1 \Rightarrow y_0(\overset{\tau}{0}, \overset{T}{0}) + \varepsilon y_1(0,0) + \varepsilon^2 y_2(0,0) + \cdots = 1$$

$$\Rightarrow \underline{y_0(0,0) = 1}, \quad \underline{y_1(0,0) = 0} \quad \ldots$$

$$\dot{y}(0) = 0 \Rightarrow \left[(1+\varepsilon^2 a)\frac{\partial}{\partial \tau} + \varepsilon \frac{\partial}{\partial T}\right](y_0 + \varepsilon y_1 + \cdots) = 0$$

$$\Rightarrow (1+\varepsilon^2 a)\left(\frac{\partial y_0}{\partial \tau} + \varepsilon \frac{\partial y_1}{\partial \tau} + \ldots\right) + \varepsilon \frac{\partial y_0}{\partial T} + \varepsilon^2 \frac{\partial y_1}{\partial T} + \cdots = 0$$

$$\Rightarrow \underline{y_{0\tau}(0,0) = 0}, \quad \underline{y_{1\tau}(0,0) + y_{0T}(0,0) = 0}.$$

$[\varepsilon^0]$: $$y_{0\tau\tau} + y_0 = 0$$

with $y_{0\tau}(0,0) = 0, \quad y_0(0,0) = 1$

$$\Rightarrow y_0(\tau, T) = A_0(T)\cos\tau + B_0(T)\sin\tau$$

b.c. $y_0(0,0) = 1 \Rightarrow A_0(0)\cdot 1 + B_0(0)\cdot 0 = 1$
$\Rightarrow A_0(0) = 1$

b.c. $y_{0\tau}(0,0) = 0 \Rightarrow A_0(0)\cdot 0 + B_0(0)\cdot 1 = 0$
$\Rightarrow B_0(0) = 0$

$[\varepsilon^1]$: $$y_{1\tau\tau} + y_1 = -2y_{0\tau T} - y_{0\tau}$$

$$= -2\frac{\partial}{\partial T}\left(-A_0(T)\sin\tau + B_0(T)\cos\tau\right)$$

$$= -(-A_0(T)\sin\tau + B_0(T)\cos\tau)$$

$= 0$

To avoid growth in y_1 (and so maintain periodicly) we set the coefficients of $\sin\tau$ and $\cos\tau$ on the RHS to zero

$$2\frac{\partial A_0}{\partial T} + A_0 = 0, \qquad A_0(0) = 1$$

$$2\frac{\partial B_0}{\partial T} + B_0 = 0, \qquad B_0(0) = 0$$

$$\Rightarrow B_0(T) = 0, \quad A_0(T) = e^{-T/2}$$

$$\Rightarrow y_0(\tau, T) = e^{-T/2}\cos\tau$$

So $y_{1\tau\tau} + y_1 = 0$

$$\Rightarrow y_1(\tau, T) = A_1(T)\cos\tau + B_1(T)\sin\tau$$

$$y_1(0,0) = 0$$

$$y_{1\tau}(0,0) = -y_{0T}(0,0)$$

$$= -\frac{\partial}{\partial T} e^{-T/2}\cos\tau \Big|_{(0,0)}$$

$$= \frac{1}{2}$$

$$\Rightarrow A_1(0) = 0$$

$$B_1(0) = \frac{1}{2}$$

$[\varepsilon^2]$: $\quad y_{2\tau\tau} + y_2 = -2a y_{0\tau\tau} - 2y_{1\tau T} - y_{0TT}$
$\qquad\qquad\qquad\qquad - y_{1\tau} - y_{0T} - y_0^3$

$$\left[\begin{array}{l} y_0 = e^{-T/2} \cos\tau \\ y_1 = A_1(T)\cos\tau + B_1(T)\sin\tau \end{array}\right]$$

$$= 2ae^{-T/2}\cos\tau - 2\frac{\partial}{\partial T}\left(-A_1 \sin\tau + B_1\cos\tau\right)$$
$$- \frac{1}{4} e^{-T/2} \cos\tau - \left(-A_1\sin\tau + B_1\cos\tau\right)$$
$$+ \frac{1}{2} e^{-T/2} \cos\tau - e^{-3T/2} \underbrace{\cos^3\tau}_{\frac{3}{4}\cos\tau + \frac{1}{4}\cos 3\tau}$$

To keep y_2 periodic in τ, we need to set coefficients of $\cos\tau$ and $\sin\tau = 0$.

$$[\cos\tau]: \quad 2ae^{-T/2} - 2\frac{\partial B_1}{\partial T}$$
$$- \frac{1}{4} e^{-T/2} - B_1$$
$$+ \frac{1}{2} e^{-T/2} - \frac{3}{4} e^{-3T/2} = 0$$

Fortunately, don't need $\sin\tau$ eq^n $\because$ we don't have a coupled eq^n in A_1 and B_1, instead we can just say

$$\frac{\partial B_1}{\partial T} + \frac{1}{2} B_1 = -\frac{3}{8} e^{-3T/2} + \left(a + \frac{1}{8}\right) e^{-T/2}$$

Integrating factor is $e^{T/2}$:

$$\frac{\partial}{\partial T}\left(e^{T/2}B_1\right) = -\frac{3}{8}e^{-T} + \left(a+\frac{1}{8}\right)$$

$$\Rightarrow \quad e^{T/2}B_1 = \frac{3}{8}e^{-T} + \left(a+\frac{1}{8}\right)T + B_{10}$$

$$\Rightarrow \quad B_1 = \frac{3}{8}e^{-3T/2} + \left(a+\frac{1}{8}\right)Te^{-T/2} + B_{10}e^{-T/2}$$

Recall $y = y_0 + \varepsilon y_1 + \cdots$

We've found $y = e^{-T/2}\cos\tau + \varepsilon\left(A_1(T)\cos\tau + B_1(T)\sin\tau\right)$

$$= \underbrace{e^{-T/2}\cos\tau}_{①} + \underbrace{\varepsilon T e^{-T/2}\sin\tau}_{②} + \text{other bits}$$

If $T = O\left(\frac{1}{\varepsilon}\right)$ [very big!] ie $t = O(\frac{1}{\varepsilon^2})$, $O(①) = O(②)$ and we don't want this, so we set $\left(a+\frac{1}{8}\right) = 0$

$$B_1(0) = \frac{1}{2} \Rightarrow B_{10} = \frac{1}{8}.$$

$[\sin\tau]$: $\quad 2\frac{\partial A_1}{\partial T} + A_1 = 0$

$$A_1(0) = 0 \Rightarrow A_1(T) = 0$$

$$\Rightarrow y = y_0 + \varepsilon y_1 + \cdots$$

$$= e^{-T/2}\cos\tau + \varepsilon\left(\frac{3}{8}e^{-3T/2} + \frac{1}{8}e^{-T/2}\right)\sin\tau + O(\varepsilon^2)$$

where $T = \varepsilon t$, $\tau = \left(1 - \frac{\varepsilon^2}{8}\right)t$.

$$\Rightarrow y = e^{-\varepsilon t/2}\cos\left[t\left(1-\frac{\varepsilon^2}{8}\cdots\right)\right] + \frac{\varepsilon}{8}\left[3e^{-3\varepsilon t/2} + e^{-\varepsilon t/2}\right]\sin\left[t\left(1-\frac{\varepsilon^2}{8}\right)\right].$$

ASYMPTOTIC EXPANSION OF INTEGRALS

Q5 of the exam

We will develop techniques that allow us to find approximate expressions for integrals of the type

$$I(x) = \int_0^T e^{-xg(t)} f(t)\,dt$$

where x is large. The approximations we find will be known as <u>asymptotic approximations</u>. To help explain what this means, we need some notation:

(a) We say $f(x) = O(g(x))$

"$f(x)$ is of the order of $g(x)$"

as $x \to \infty$ or $x \to 0$ or $x \to 5$ etc

if we can find K, X s.t.

$$|f| < K|g| \quad \forall\ x > X$$
$$\text{or } x < X$$
$$\text{or } x-5 < X \quad \text{etc.}$$

<u>Note</u> K & X can be of any size.

<u>e.g.</u> if $f(x) = x^2 + x$, $g(x) = x^2$, $x \to \infty$.

Then $f(x) = O(g(x))$:

$$\frac{|f|}{|g|} = \left|\frac{f}{g}\right| = \left|1 + \frac{1}{x}\right| < \underset{K}{\frac{3}{2}} \quad \text{if } x > \underset{X}{2}$$

and $f(x) = O(10\,g(x))$

ie f is well approximated by a constant multiplied by g.

in that the relative error gets small as $x \to \infty$.

If $f(x) = x^2 + x$, $\quad g(x) = x \quad x \to 0$.

then $f(x) = O(g(x))$ as $x \to 0$.

as $\left|\frac{f}{g}\right| = \left|\frac{x^2+x}{x}\right| = |1+x| < \frac{3}{2}$ if $x < \frac{1}{2}$

$K \qquad\qquad X$

<u>Note</u>: $10^{1000} + x^{0.0001} = O(x^{0.0001})$ as $x \to \infty$.

$$x^2 + e^{-x} = O(x^2) \quad x \to \infty$$
$$= O(1) \quad x \to 0.$$

(b) $f(x) = o(g(x))$ as $x \to \infty$, $x \to 0$, $x \to 5$ means

$\left|\frac{f}{g}\right| \to 0$ as $x \to \infty$, $x \to 0$, $x \to 5$

<u>e.g.</u> $x = o(x^2)$ as $x \to \infty$ $\because$ $\left|\frac{x}{x^2}\right| \to 0$ as $x \to \infty$

$x^2 = o(x)$ as $x \to 0$ $\because$ $\left|\frac{x^2}{x}\right| \to 0$ as $x \to 0$.

e.g. $x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + O(\varepsilon^3)$ as $\varepsilon \to 0$

$= x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + o(\varepsilon^2)$ as $\varepsilon \to 0$

(c) $f(x) \sim g(x)$ as $x \begin{smallmatrix}\to\infty\\ \to 0\\ \to 5\end{smallmatrix}$ if $\left|\frac{f}{g}\right| \to 1$

"f is asymptotic to g"

e.g. $x^2 + x \sim x^2$ as $x \to \infty$.

$x^2 + x \sim x^2 + \sin x$

$x^2 + x + 3 \sim x^2 + \sin x$ ← vague.

Example Consider the exponential integral

$$Ei(x) = \int_x^\infty \frac{e^{-t}}{t}\,dt, \qquad x > 0.$$

Put $t = xu \Rightarrow$

$$Ei(x) = \int_1^\infty \frac{e^{-xu}}{xu}\,x\,du$$

$$= \int_1^\infty \frac{e^{-xu}}{u}\,du$$

$$= \int_1^\infty \frac{e^{-xt}}{t}\,dt$$

Integrate by parts

$$\int_1^\infty \frac{e^{-xt}}{t}\,dt = \left[-\frac{1}{x}e^{-xt}\frac{1}{t}\right]_1^\infty - \int_1^\infty -\frac{e^{-xt}}{x}\left(\frac{-1}{t^2}\right)dt$$

$$= e^{-x}\frac{1}{x} - \int_1^\infty \frac{e^{-xt}}{xt^2}\,dt$$

$$= e^{-x}\frac{1}{x} - \frac{1}{x}\int_1^\infty \frac{e^{-xt}}{t^2}\,dt$$

Integrate by parts again

$$= \frac{e^{-x}}{x} - \frac{1}{x}\left[\left[-\frac{1}{x}e^{-xt}\frac{1}{t^2}\right]_1^\infty - \int_1^\infty -\frac{e^{-xt}}{x}\frac{(-2)}{t^3}\,dt\right]$$

$$= e^{-x}\left(\frac{1}{x} - \frac{1}{x^2}\right) + \frac{2}{x^2}\int_1^\infty \frac{e^{-xt}}{t^3}\,dt$$

and so on n times, and get

$$= e^{-x}\left[\sum_{r=1}^{n} \frac{(-1)^{r-1}(r-1)!}{x^r}\right] + \overbrace{\frac{(-1)^n n!}{x^n}\int_1^\infty \frac{e^{-xt}}{t^{n+1}}\,dt}^{R_n}$$

doesn't converge as $n \to \infty$
(D'Alembert's ratio test gives a ROC of 0)

$u = xt \Rightarrow$

$$R_n = \frac{(-1)^n n!}{x^n}\int_x^\infty \frac{e^{-u}}{x\left(\frac{u}{x}\right)^{n+1}}\,\frac{du}{x}$$

$$= (-1)^n n! \int_x^\infty \frac{e^{-u}}{u^{n+1}}\,du$$

$$|R_n| < n!\int_x^\infty \frac{e^{-u}}{x^{n+1}}\,du$$

$$= \frac{n!}{x^{n+1}}\left[-e^{-u}\right]_x^\infty$$

$$= e^{-x}\frac{n!}{x^{n+1}}$$

$$\Rightarrow Ei(x) = e^{-x}\left[\sum_{r=1}^{n} \frac{(-1)^{r-1}}{x^r}(r-1)! + \tilde{R}_n\right]$$

where $|\tilde{R}_n| < \frac{n!}{x^{n+1}}$

For fixed n, $R_n \to 0$ as $x \to \infty$
For fixed x, $R_n \to \infty$ as $x \to \infty$ and the series which emerges,

$$\sum_{r=1}^{\infty} \frac{(-1)^r (r-1)!}{x^r},$$

does not converge.

There is an optimum value of n for a given x for which

$$Ei(x) \approx e^{-x} \sum_{r=1}^{n} \frac{(-1)^{r-1}(r-1)!}{x^r}$$

performs best.

We can write

we are not going to ∞, only some finite value as large as we need.

$$Ei(x) \sim e^{-x}\left[\frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \cdots\right] \quad x \to \infty$$

Factorial function

$$I(n) = \int_0^\infty e^{-u} u^n \, du \qquad \text{exists for } n > -1$$

$$= \left[-e^{-u} u^n\right]_0^\infty + \int_0^\infty e^{-u} n u^{n-1} \, du$$

$$= n\, I(n-1) \quad \cdots\cdots\cdots (*)$$

$$= n(n-1)\, I(n-2)$$

$$\vdots$$

$$= n!\, I(0).$$

$$I(0) = \int_0^\infty e^{-u} \, du = 1 \qquad \Rightarrow \quad I(n) = n!$$

For $n > -1$ and not necessarily an integer, we use the integral definition

$$n! = \int_0^\infty e^{-u} u^n \, du$$

So $\left(-\frac{1}{2}\right)! = \int_0^\infty e^{-u} u^{-1/2} \, du$

$\left.\begin{array}{l} u = v^2 \\ du = 2v\,dv \end{array}\right\} \Rightarrow \quad = \int_0^\infty \frac{e^{-v^2}}{v} 2v \, dv$

$$= 2\int_0^\infty e^{-v^2} \, dv$$

$$= 2 \, \frac{\sqrt{\pi}}{2}$$

$$= \sqrt{\pi}$$

And $\left(\frac{1}{2}\right)! = \frac{1}{2}\left(-\frac{1}{2}\right)!$ using (*)

$$= \frac{\sqrt{\pi}}{2}$$

The Gamma Function

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} \, dt, \qquad x > 0$$

and $\Gamma(x+1) = x!$

$$\Rightarrow \Gamma\left(\frac{1}{2}\right) = \left(-\frac{1}{2}\right)! = \sqrt{\pi}$$

Now, $(n+1)! = (n+1)\,n!$

$$\Rightarrow \quad n! = \frac{(n+1)!}{n+1}$$

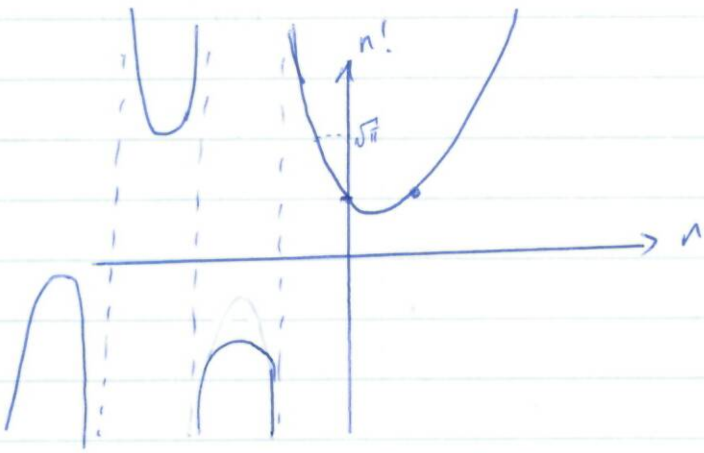
$n \to -1$: $\quad n = -1+\varepsilon, \qquad (-1+\varepsilon)! = \dfrac{\varepsilon!}{\varepsilon} \to \dfrac{1}{\varepsilon}$ as $\varepsilon \to 0$.

So $n!$ has a pole of residue 1 at $n=-1$.

For $n<-1$, say $n = -\frac{3}{2}$,

$$\left(-\tfrac{3}{2}\right)! = \frac{\left(-\frac{3}{2}+1\right)!}{-\frac{3}{2}+1} = \frac{\left(-\frac{1}{2}\right)!}{-\frac{1}{2}} = -2\sqrt{\pi}$$

And the recursion relation allows us to extend the definition to negative n EXCEPT negative integers. because this will involve the calculation of $(-1)!$. The factorial function has poles at all negative integers.



Watson's Lemma

positive

minus

x is here

Consider the integral $I(x) = \displaystyle\int_0^T e^{-xt} f(t)\,dt$

zero

integral must be of this form

Taylor exp. about 0

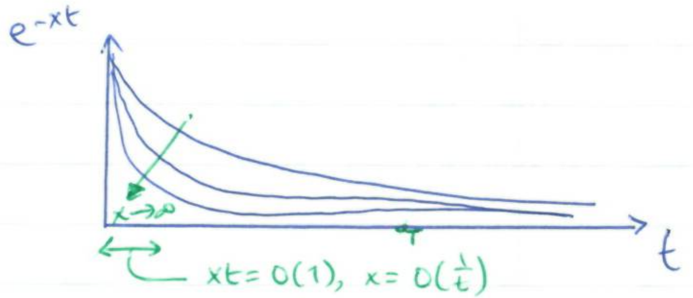
Assume that $f(t) = \sum_{n=0}^{\infty} \frac{f^n(0)\, t^n}{n!}$ with a nonzero radius of convergence, r.

and $f(t)$ does not grow faster than an exponential as $t \to \infty$.

e^{100t} is fine

e^{t^2} is not.

then as $x \to \infty$, $I(x) \sim \sum_{n=0}^{\infty} \frac{f^n(0)}{x^{n+1}}$ may not converge but if x is large enough, it can be asymptotically accurate



because x is huge e^{-xt} is asymptotically small away from regions of t where $xt = O(1)$ ie when t is tiny

Make a change of variable more suitable to the region where $xt = O(1)$, ie $u = xt$, $t = u/x$ So u is $O(1)$

$$\to \quad I(x) = \int_0^{Tx} e^{-u} f\left(\frac{u}{x}\right) \frac{du}{x}$$

As $x \to \infty$, e^{-xT} is exponentially small whatever the value of T and we might as well replace T by ∞ since the induced error is exponentially small.

If the radius of convergence of the series for f is r, if x is sufficiently large, we will have $t = \frac{1}{x} \ll r$ and we may replace $f\left(\frac{u}{x}\right)$ by $\sum_0^{\infty} \frac{f^n(0)}{n!} \left(\frac{u}{x}\right)^n$.

$$\Rightarrow \quad I(x) = \int_0^\infty e^{-u} \underbrace{\sum_{n=0}^{\infty} \frac{f^n(0)}{n!}\left(\frac{u}{x}\right)^n}_{\substack{\text{might have a small ROC} \\ \text{but if } x\to\infty \text{ it is of } o(1/x)}} \cdot \frac{du}{x}$$

$$= \frac{1}{x}\sum_{n=0}^{\infty} \underbrace{\int_0^\infty e^{-u} u^n \, du}_{n!} \, \frac{f^n(0)}{n!} \, \frac{1}{x^n}$$

$$\sim \sum_{n=0}^{\infty} \frac{f^n(0)}{x^{n+1}}$$

← Watson's lemma

"easy to remember"

!!! It turns out this formula is valid if we only have an asymptotic approximation for $f(t)$ as $t \to 0$.

If $\quad f(t) \sim t^\lambda \sum_{n=0}^{\infty} a_n t^{\lambda_n}$, $\lambda > -1$, $\lambda_0 = 0$, then

$$I(x) \sim \sum_{n=0}^{\infty} \frac{a_n (\lambda + \lambda_n)!}{x^{\lambda + \lambda_n + 1}}$$

Example

"not really easy" (∴ examinable q^n?)

$$Ei(x) = \int_x^\infty \frac{e^{-t}}{t} \, dt$$

← try to get this into a form in Watson's lemma

$$[t = ux] \quad = \int_1^\infty \frac{e^{-ux}}{ux} \, du \cdot x$$

$$= \int_1^\infty \frac{e^{-ux}}{u} \, du$$

$$[u = 1+s] \quad = \int_0^\infty \frac{e^{-(1+s)x}}{1+s} \, ds$$

$$= e^{-x}\underbrace{\int_0^\infty \frac{e^{-sx}}{1+s}\,ds}$$

use Watson's lemma with

$$f(s) = \frac{1}{1+s} = 1 - s + s^2 - s^3 + s^4 - \dots$$

$$= e^{-x}\left(\frac{1.0!}{x^{0+1}} - \frac{1.1!}{x^{1+1}} + \frac{1.2!}{x^{2+1}} - \frac{1.3!}{x^{3+1}} + \dots\right)$$

$$= e^{-x}\left(\frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \dots\right)_{x\to\infty}$$

<u>Example</u> $I(x) = \int_0^\infty e^{-xt}\ln(1+t^2)\,dt$

$$\ln(1+t^2) = t^2 - \frac{1}{2}t^4 + \frac{1}{3}t^6 - \dots \qquad ROC = 1$$

Quoting Watson's lemma:

$$I(x) \sim \frac{1.2!}{x^{2+1}} - \frac{1}{2}\frac{4!}{x^{4+1}} + \frac{1}{3}\frac{6!}{x^{6+1}} - \dots \qquad x\to\infty.$$

<u>Example</u> Same as above but from first principles

$$I(x) = \int_0^1 e^{-xt}\ln(1+t^2)\,dt$$

$$u = xt$$

$$= \int_0^x e^{-u}\ln\left(1+\frac{u^2}{x^2}\right)\frac{du}{x}$$

$$\sim \int_0^\infty e^{-u}\underbrace{\left(\frac{u^2}{x^2} - \frac{1}{2}\frac{u^4}{x^4} + \frac{1}{3}\frac{u^6}{x^6} - \dots\right)}_{\text{converges for } u/x<1 \text{ but } x\to\infty \text{ so we're OK.}}\frac{du}{x}$$

$$= \frac{1}{x^3}\int_0^\infty u^2 e^{-u}\,du - \frac{1}{2}\frac{1}{x^5}\int_0^\infty u^4 e^{-u}\,du + \cdots$$

$$= \frac{2!}{x^3} - \frac{1}{2}\frac{4!}{x^5} + \cdots$$

Example $\quad I(x) = \int_0^{\pi/2} e^{\overbrace{-x\cos\theta}^{-ve \text{ for } \theta \in (0,\frac{\pi}{2})}}\,d\theta$

$e^{-x\cos\theta}$ is not exponentially small where θ is close to $\pi/2$.

$$\left.\begin{aligned}\cos\theta &= t \\ -\sin\theta\,d\theta &= dt\end{aligned}\right\} \quad d\theta = \frac{-dt}{\sqrt{1-t^2}}$$

$$\Downarrow$$

$$I(x) = -\int_1^0 \frac{e^{-xt}}{\sqrt{1-t^2}}\,dt$$

$$= \int_0^1 \frac{e^{-xt}}{\sqrt{1-t^2}}\,dt$$

$$(1-t^2)^{-1/2} = 1 + \frac{1}{2}t^2 + \frac{3}{8}t^4 + \cdots$$

$$\rightarrow \quad I(x) \sim \frac{1\,0!}{x^{1+0}} + \frac{1}{2}\frac{2!}{x^{1+2}} + \frac{3}{8}\frac{4!}{x^{1+4}} + \cdots$$

$$= \frac{1}{x} + \frac{1}{x^3} + \frac{9}{x^5} + \cdots$$

Once again, Watson's lemma says

if $\quad f \sim a_0 t^{\lambda_0} + a_1 t^{\lambda_1} + \cdots$

$$\int_0^T e^{-xt} f(t)\,dt \sim \frac{a_0 \lambda_0!}{x^{\lambda_0+1}} + \frac{a_1 \lambda_1!}{x^{\lambda_1+1}} + \cdots$$

Example

$$I(x) = \int_0^1 (1-t^2)^x \, dt$$
$$= \int_0^1 e^{x \ln(1-t^2)} \, dt$$

Want $-u = \ln(1-t^2)$. limits $\int_0^{\infty}$ $\leftarrow -\infty \cdot (-1)$

$$\Rightarrow \quad du = \frac{2t}{1-t^2} dt \qquad e^{-u} = 1-t^2$$
$$\Rightarrow t = \sqrt{1-e^{-u}}$$

$$= \frac{2\sqrt{1-e^{-u}}}{e^{-2u}}$$

$$\Rightarrow I(x) = \int_0^{\infty} e^{-ux} \underbrace{\frac{e^{-u}}{2\sqrt{1-e^{-u}}}}_{f(u)} du$$

as $u \to 0$, $f(u) \sim \dfrac{1-u+}{2(1-1+u-u^2/2)^{1/2}}$

$$= \frac{1}{2\sqrt{u}} \frac{(1-u \ldots)}{(1-u/2 \ldots)^{1/2}}$$

$(1+\varepsilon)^n \approx 1+n\varepsilon$

$$= \frac{1}{2\sqrt{u}} (1-u \cdots)(1+u/4 \cdots)$$

$$= \frac{1}{2\sqrt{u}} - \frac{3}{8}\sqrt{u} \ldots$$

$\Rightarrow$ by Watson's lemma,

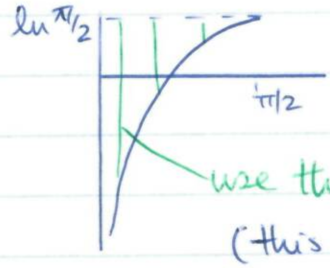
$$I(x) \sim \frac{1}{2} \frac{(-\frac{1}{2})!}{x^{1-1/2}} - \frac{3}{8} \frac{(1/2)!}{x^{1+1/2}}$$

Recall $\left(-\frac{1}{2}\right)! = \sqrt{\pi} \Rightarrow \left(\frac{1}{2}\right)! = \frac{1}{2}\sqrt{\pi}$

$$I(x) \sim \frac{1}{2}\sqrt{\frac{\pi}{x}}\left(1 - \frac{3}{8}x \ldots\right)$$

__Example__ $I(x) = \int_0^{\pi/2} t^x \sin t \, dt$

$$= \int_0^{\pi/2} e^{x \ln t} \sin t \, dt$$



use this distance as a new variable

(this works $\because$ the fn is monotonic)

$$u = \ln\frac{\pi}{2} - \ln t$$

$t=0, u \to \infty$
$t = \pi/2, u = 0$
} this makes it a good substitution choice!

$$\ln t = \ln\left(\frac{\pi}{2}\right) - u$$

$$\Rightarrow e^{x\ln t} = e^{x\ln(\pi/2) - xu}$$

$$= \left(\frac{\pi}{2}\right)^x e^{-xu}$$

$\Rightarrow t = e^{\ln \pi/2} = \frac{\pi}{2}e^{-u}$ $\Rightarrow dt = -\frac{\pi}{2}e^{-u}\,du$

$\Rightarrow \sin t = \sin\left(\frac{\pi}{2}e^{-u}\right)$

$$\Rightarrow I(x) = \left(\frac{\pi}{2}\right)^{x+1}\int_0^\infty e^{-ux} \overbrace{\sin\left(\frac{\pi}{2}e^{-u}\right)e^{-u}}^{f(u)} \, du$$

$$f(0) = \sin\frac{\pi}{2}e^{-0} e^{-0} = 1$$

$$\Rightarrow I(x) \sim \left(\frac{\pi}{2}\right)^{x+1}\frac{1}{x}$$

Alternatively, could've said $ux = s$, then

$$\int_0^\infty e^{-s} \underbrace{\sin\left(\tfrac{\pi}{2} e^{-s/x}\right) e^{-s/x}}_{\substack{\to 1 \\ \text{as } x\to\infty}} \frac{ds}{x}$$

and derived...

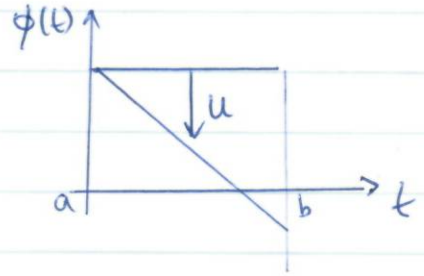
Laplace integrals

The previous example is an example of a Laplace integral

$$I(x) = \int_a^b e^{x\phi(t)} f(t)\, dt \qquad \text{as} \quad x \to \infty.$$

(a) ϕ monotonic decreasing $\quad \phi'(t) < 0$ in $[a,b]$

$\phi'(t) \neq 0$ in $[a,b]$



$$u = \phi(a) - \phi(t)$$

$t = a \Rightarrow u = 0$

$t = b \Rightarrow u = \phi(a) - \phi(b) =: \beta > 0$

$u = \phi(a) - \phi(t) \Rightarrow \phi(t) = \phi(a) - u$

$\Rightarrow du = -\phi'(t)\, dt \qquad$ So consider $t = t(u)$.

$$\Rightarrow I(x) = \int_0^\beta e^{x\phi(a) - xu} f(t(u)) \frac{-1}{\phi'(t(u))}\, du$$

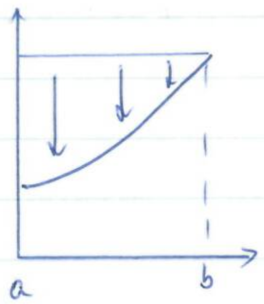
$$= e^{x\phi(a)} \int_0^\beta e^{-xu} \underbrace{\frac{f(t(u))}{|\phi'(t(u))|}}_{h(u)}\, du \qquad \phi'(t) < 0$$

$$\sim e^{x\phi(a)} \frac{h(0)}{x} \qquad u=0 \text{ is } t=a$$

$$= \frac{e^{x\phi(a)}}{x} \frac{f(a)}{|\phi'(a)|}$$

(b) ϕ monotonic increasing $\quad \phi'(t) > 0$ in $[a,b]$

$\phi'(t) \neq 0$ in $[a,b]$



$$u = \phi(b) - \phi(t)$$

$$\Rightarrow \quad I(x) \sim \frac{e^{x\phi(b)}}{x} \frac{f(b)}{|\phi'(b)|}$$

$\phi'(b) > 0 \Rightarrow \phi'(b) = |\phi(b)|$.

Results a and b can be combined to give

$$I(x) \sim \frac{e^{x\phi(c)}}{x} \frac{f(c)}{|\phi'(c)|}$$ where c is the end point giving the largest value of ϕ

This result can be found using integration by parts

$$I(x) = \int_a^b e^{x\phi(t)} f(t)\, dt = \int_a^b \phi'(t) e^{x\phi(t)} \frac{f(t)}{\phi'(t)}\, dt$$

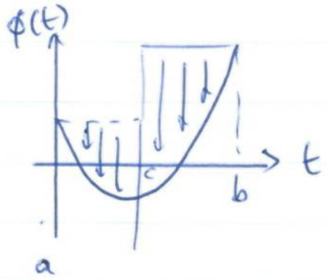
$$= \left[\frac{e^{x\phi(t)}}{x} \frac{f(t)}{\phi'(t)} \right]_a^b - \underbrace{\frac{1}{x} \int_a^b e^{x\phi(t)} \frac{d}{dt}\left(\frac{f(t)}{\phi'(t)} \right) dt}$$

Could repeat but it would be o(first term)

$$\sim \frac{e^{x\phi(b)}}{x}\frac{f(b)}{\phi'(b)} - \frac{e^{x\phi(a)}f(a)}{x\,\phi'(a)}$$

and pick biggest contribution

(c)



$\phi'(c) = 0 \quad c \in (a,b)$
$\phi''(c) > 0$

Split the integral into two parts

$$\int_a^c e^{x\phi(t)} f(t)\,dt + \int_c^b e^{x\phi(t)} f(t)\,dt$$

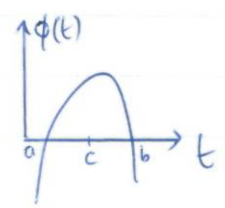
use $u = \phi(a) - \phi(t)$ use $\phi(b) - \phi(t)$

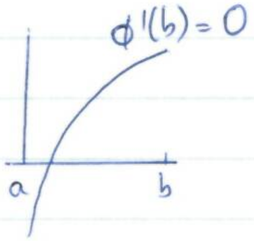
$du = -\phi'(t)\,dt$

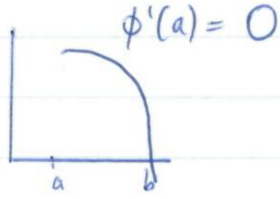
$dt = \dfrac{du}{\phi'(t(u))}$

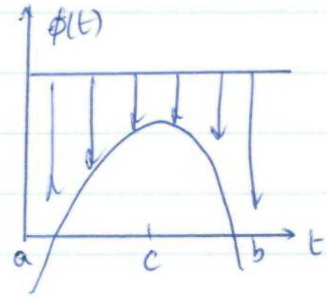
In both cases the integral is improper as the integrand has a zero in its denominator at u corresponding to $t=c$. It is integrable, however, and the result is dominated by the first integral if $\phi(a) > \phi(b)$ and the second if $\phi(b) > \phi(a)$.

(d) $\phi'(c) = 0$
$\phi''(c) < 0$ ie maximum value of ϕ in $[a,b]$ is a turning point.









$$I(x) = \int_a^c e^{x\phi(t)} f(t)\,dt + \int_c^b e^{x\phi(t)} f(t)\,dt$$

in $[a,c]$ use $u = \phi(c) - \phi(t)$

$t=a$, $u = \phi(c) - \phi(a) = \beta > 0$

$t=c$, $u = 0$ $x\phi(t) = x\phi(c) - xu$

$du = -\phi'(t)\,dt$.

In $[c,b]$, use $u = \phi(c) - \phi(t)$.

$t=c$ $u=0$

$t=b$ $u = \phi(c) - \phi(b) > 0 = \tilde{\beta}$

$du = -\phi'(t)\,dt$

$x\phi(t) = x\phi(c) - xu$

$$e^{x\phi(c)}\int_\beta^0 e^{-xu}\frac{f(t(u))}{-\phi'(t(u))}\,du + e^{x\phi(c)}\int_0^{\tilde{\beta}} e^{-xu}\frac{f(t(u))}{-\phi'(t(u))}\,du$$

Both integrals are improper at $u=0$, where $\phi'(t(u))=0$
($\phi'(c)=0$)

However, it turns out that the integrals are convergent.

Look close to $u=0$, $t=c$, $-u = \phi(t) - \phi(c)$

$$-u = \cancel{\phi(c)} + (t-c)\underbrace{\phi'(c)}_{\to 0} + \frac{(t-c)^2}{2}\phi''(c) + \cdots - \cancel{\phi(c)}$$

$\phi''(c) < 0$: $\quad u = (t-c)^2 \dfrac{|\phi''(c)|}{2}$

$$\Rightarrow (t-c) = \begin{cases} \sqrt{\dfrac{2u}{|\phi''(c)|}} & t>c \quad \text{2nd integral} \\ -\sqrt{\dfrac{2u}{|\phi''(c)|}} & t<c \quad \text{1st integral} \end{cases}$$

$f(t(u)) \sim f(c)$ as $u \to 0$

but $\phi'(t) \sim \underbrace{\phi'(c)}_{0} + (t-c)\,\phi''(c)$ as $t \to c$ ie $u \to 0$

$$\sim \mp \sqrt{\frac{2u}{|\phi''(c)|}}\,\phi''(c)$$

$$= \pm \sqrt{\frac{2u}{|\phi''(c)|}}\,|\phi''(c)|$$

$$= \pm \sqrt{2u\,|\phi''(c)|}$$

$$\Rightarrow I(x) \simeq e^{x\phi(c)}\left[\int_0^{\beta} e^{-xu}\underbrace{\frac{f(t(u))}{\phi'(t(u))}}_{\frac{f(c)}{\sqrt{2u|\phi''(c)|}}}\,du - \int_0^{\tilde{\beta}} e^{-xu}\underbrace{\frac{f(t(u))}{\phi'(t(u))}}_{\frac{f(c)}{-\sqrt{2u|\phi''(c)|}}}\,du\right]$$

$$= e^{x\phi(c)} \frac{f(c)}{\sqrt{2|\phi''(c)|}} \sqrt{\frac{\pi}{x}} \cdot 2$$

"are you happy with this?"

$$= e^{x\phi(c)} f(c) \sqrt{\frac{2\pi}{x|\phi''(c)|}}$$

L

Alternative method

Focus on the region near the maximum in ϕ and write

$$\phi(t) = \phi(c) + \overset{0}{\phi'(c)}(t-c) + \frac{1}{2}\phi''(c)(t-c)^2 + \cdots$$

$$\Rightarrow e^{x\phi(t)} \approx e^{x\phi(c)} e^{\underbrace{-\frac{1}{2}|\phi''(c)| \underset{\text{big}}{x} (t-c)^2 + \cdots}_{O(1) \text{ as } x\to\infty}}$$

$$\text{Let } u^2 = \frac{1}{2} x |\phi''(c)| (t-c)^2$$

$$I(x) \sim \int_{-\infty \,(-ve\sqrt{x})}^{\infty \,(+ve\sqrt{x})} e^{x\phi(c)} e^{-u^2} f(c) + \text{h.o.t.} \sqrt{\frac{2}{x|\phi''(c)|}}\, du$$

$$= e^{x\phi(c)} f(c) \sqrt{\frac{2}{x|\phi''(c)|}} \underbrace{\int_{-\infty}^{\infty} e^{-u^2}\, du}_{\sqrt{\pi}}$$

$$= e^{x\phi(c)} f(c) \sqrt{\frac{2\pi}{x|\phi''(c)|}} \qquad \text{as before.}$$

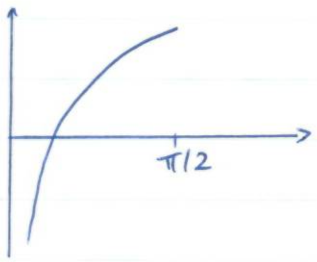
If there is no turning point, $I(x) \sim \dfrac{e^{x\phi(c)} f(c)}{x|\phi'(c)|}$ where $c = a$ or b depending on which is the biggest, $\phi(a)$ or $\phi(b)$.

Example

:) easy exam q! :)

$$I(x) = \int_0^{\pi/2} t^x \sin t \, dt$$

$$= \int_0^{\pi/2} e^{x \overbrace{\ln t}^{\phi(t)}} \overbrace{\sin t}^{f(t)} \, dt$$



maximum at $t = \pi/2$, $\Rightarrow \frac{\pi}{2} = c$.

$$\phi'(t) = \frac{1}{t} = \frac{2}{\pi} \quad \text{at } t = c.$$

$$I(x) \sim \frac{e^{x\ln(\pi/2)} \sin\left(\frac{\pi}{2}\right)}{x \left|\frac{2}{\pi}\right|}$$

$$= \left(\frac{\pi}{2}\right)^{x+1} \frac{1}{x}$$

Example (Stirling's formula)

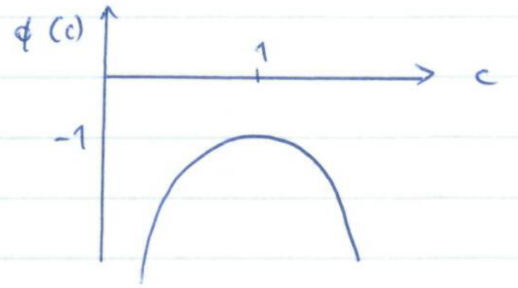
$$x! = I(x) = \int_0^\infty e^{-u} u^x \, du \qquad u = xt$$

$$= \int_0^\infty e^{-xt} x^x t^x x \, dt$$

$$= x^{x+1} \int_0^\infty e^{-xt} e^{x\ln t} \, dt$$

$$= x^{x+1} \int_0^\infty e^{x \underbrace{(-t + \ln t)}_{\phi(t)}} \, dt \qquad f(t) = 1$$

$$\phi'(t) = -1 + \frac{1}{t} \quad = 0 \text{ at } t = 1$$



$c = 1$

$$\phi''(t) = -\frac{1}{t^2} \quad = 1 \text{ at } t = 1.$$

$$\Rightarrow \quad x! \sim x^{x+1} e^{\underbrace{x(-1)}_{\phi(c)}} \underbrace{1}_{f(c)} \sqrt{\frac{2\pi}{x\,\underbrace{|-1|}_{\phi''(c)}}}$$

$$= x^{x+\frac{1}{2}} e^{-x} \sqrt{2\pi}$$

Fourier integrals

These are of the type

$$I(x) = \int_a^b e^{ix\underbrace{\phi(t)}_{\text{real}}} f(t)\, dt$$

Use integration by parts.

$$I(x) = \int_a^b \phi'(t)\, e^{ix\phi(t)} \left[\frac{f(t)}{\phi'(t)}\right] dt$$

assuming $\phi'(t) \neq 0$ for $t \in [a, b]$.

if it is, we split the range and do the same as in the previous type.

$$= \left[\frac{e^{ix\phi(t)}}{ix} \frac{f(t)}{\phi'(t)} \right]_a^b - \underbrace{\int_a^b \frac{e^{ix\phi(t)}}{ix} \left[\frac{f(t)}{\phi'(t)} \right]' dt}_{O\left(\frac{1}{x^2}\right)}$$

$$= \frac{1}{ix} \left[e^{ix\phi(b)} \frac{f(b)}{\phi'(b)} - e^{ix\phi(a)} \frac{f(a)}{\phi'(a)} \right] + O\left(\frac{1}{x^2}\right)$$

We must keep both contributions as $e^{ix\phi(a)}$ remains $O(1)$ as $x \to \infty$.

If $\phi'(c) = 0$ with $c \in [a, b]$, we need to take another approach. As $ix\phi(t)$ is the phase of the integrand, the method we derive is called <u>the method of stationary phase</u>.

We split the range of integration into

$$[a, c-\delta], \quad [c-\delta, c+\delta], \quad [c+\delta, b]$$

$$\Rightarrow \quad I(x) = \underbrace{\int_a^{c-\delta} + \int_{c+\delta}^{b}}_{\substack{\text{no stationary pt} \\ \text{inside them} \Rightarrow \\ O\left(\frac{1}{x}\right)}} + \underbrace{\int_{c-\delta}^{c+\delta} \left(e^{ix\phi(t)} f(t) \right) dt}_{\substack{\text{expand about} \\ t = c.}}$$

$$ix\phi(t) \sim ix\phi(c) + ix\overset{0}{\cancel{\phi'(c)}}(t-c) + ix\phi''(c)\tfrac{1}{2}(t-c)^2 + \cdots$$

$$I(x) \sim \int_{\underset{\to -\infty}{\underset{-ve\cdot\sqrt{x}}{c-\delta}}}^{\overset{\to\infty}{\overset{+ve\cdot\sqrt{x}}{c+\delta}}} e^{ix\phi(c)}\, e^{\underbrace{ix\phi''(c)\frac{1}{2}(t-c)^2}_{iu^2 s}} f(c)\, dt$$

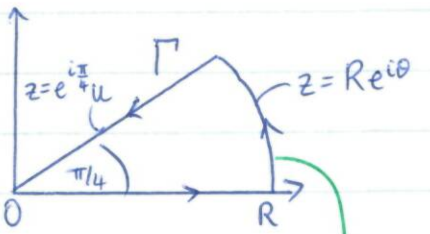
Let $u^2 = x|\phi''(c)|\frac{1}{2}(t-c)^2$ where $s = \mathrm{sgn}[\phi''(c)]$

$$\Rightarrow t = c + u\sqrt{\frac{2}{x|\phi''(c)|}}$$

$$\sim e^{ix\phi(c)} f(c) \sqrt{\frac{2}{x|\phi''(c)|}} \int_{-\infty}^{\infty} e^{isu^2}\, du$$

We need $\int_{-\infty}^{\infty} e^{isu^2}\, du = 2\int_0^{\infty} e^{isu^2}\, du$

Do $s=1$ and find $s=-1$ by taking complex conjugate.



$$\int_\Gamma e^{iz^2}\, dz$$

integral along here $\to 0$ as $R\to\infty$, c.f. Jordan's lemma

$$\int_0^{\infty} e^{iu^2}\, du = \int_0^{\infty} e^{i[(e^{i\pi/4})u]^2} e^{i\pi/4}\, du$$

along real axis

put $z = e^{i\frac{\pi}{4}}u$

$$= e^{i\frac{\pi}{4}} \int_0^{\infty} e^{-u^2}\, du$$

$$= e^{i\pi/4}\frac{\sqrt{\pi}}{2}$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{isu^2}\,du = \sqrt{\pi}\, e^{i\,\mathrm{sgn}[\phi''(c)]\pi/4}$$

$$\Rightarrow \int_a^b e^{ix\phi(t)} f(t)\,dt \sim e^{ix\phi(c)} f(c)\sqrt{\frac{2\pi}{|\phi''(c)|x}} \cdot e^{i\,\mathrm{sgn}(\phi''(c))\pi/4}$$

where $\phi'(c) = 0$.

<u>Note</u>: HALVE it if ~~at~~ c is a or b.