3401 Methematical Methods 5 Notes.
Based on the 2012 autumn by
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2/10/12

Chapter 1 : ODEs.

Consider $y'' + Py' + Qy = R$ for $y(x)$. P, Q, R functions of x. It is a linear 2^{nd} order.

Since the general solution has the form:

$$y = CF + PI$$

Sol^n of $y'' + Py' + Qy = 0$

A solution of $y'' + Py' + Qy = R$

$$y = Ay_1(x) + By_2(x)$$

y_1 and y_2 are linear independant i.e there are no C_1 and C_2 so that $C_1 y_1(x) + C_2 y_2(x) = 0$ for all x.

Reduction of order – Allows us to find the general form of the CF if we spot one solution to $y'' + a_1 y' + a_0 y = 0$, $a_0(x)$, $a_1(x)$

Presume we know one solution $y = u(x)$ i.e $u'' + a_1 u' + a_0 u = 0$.

To find another of the type $g(x) = u(x)v(x)$ substitute and find $v(x)$.

$$(\cancel{u''v} + 2u'v' + uv'') + a_1(\cancel{u'v} + uv') + \cancel{a_0 uv} = 0 \quad \text{(As } u \text{ is a solution)}$$

If $z = v'$ then this is first order equation for z.

$$uz' + (2u' + au)z = 0.$$

$$\Rightarrow \frac{z'}{z} + \frac{2u'}{u} + a_1 = 0.$$

$$\Rightarrow \ln z + \ln u^2 + \int^{x_1} a_1(t)\,dt = \text{Const}$$

$$z(x) = \frac{A}{u^2} e^{-\int a_1(t)\,dt}$$

$$v(x) = A\int^x \frac{1}{u^2(t)} e^{-\int^t a_1(s)\,ds}\,dt + B$$

$$y = uv.$$

$$= Au(x)\int^x \frac{1}{u^2(t)} e^{-\int^t a(s)\,ds}\,dt + Bu(x).$$

B can be set to 0 as we know $u(x)$

is part of C. F.

Example: Legendre's equation of order 1

$$(1-x^2)y'' - 2xy' + 2y = 0.$$

$n(n+1)$ when $n=1$.

One solution is $y = x$.

$\Rightarrow$ $P_1(x)$ - regular/analytic $x = \pm 1$
$Q_1(x)$ - will be singluar at $x = \pm 1$

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$y = u$, $y = uv$.

$$(1-x^2)y'' - 2xy' + 2y = \cancel{0}\ R(x)$$

$y = u = x$ is a solution. Look for a second sol^n,
$y = xv$

$$(1-x^2)(2v' + xv'') - 2x(\cancel{v} + xv') + \cancel{2xv} = \cancel{0}\ R$$

We should have a first order equation for $z = v'$

$$\frac{z'}{z} = \frac{4x^2 - 2}{x(1-x^2)} \quad \text{term from } R$$

$$= \frac{A}{x}^{=-2} + \frac{B}{1-x}^{=1} + \frac{C}{1+x}^{=-1}$$

$$= -\frac{2}{x} + \frac{1}{1-x} - \frac{1}{1+x}$$

$$\Rightarrow \ln z = -2\ln x - \ln(1-x) - \ln(1+x)$$

So

$$v' = z = \frac{1}{x^2(1-x)(1+x)}$$

$$= \frac{A}{x^2}^{=1} + \frac{B}{x}^{=0} + \frac{C}{1-x}^{=+1/2} + \frac{D}{1+x}^{=1/2}$$

$$v = -\frac{1}{x} + \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) + \cancel{\text{const}} \leftarrow \text{No Need.}$$

Second solution is $y = xv$

$$= -1 + \frac{x}{2} \ln\left(\frac{1+x}{1-x}\right)$$

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Variation of Parameters

To solve:

$$y'' + Py' + Qy = R.$$

we presume we know both parts of C.F. i.e y_1 and y_2 so that

$$y_1'' + Py_1' + Qy_1 = 0$$
$$y_2'' + Py_2' + Qy_2 = 0.$$

Look for a solution $y = A(x)y_1(x) + B(x)y_2(x)$ there is a lot of of redundancy in this expression which we use by imposing the condition.

$$A'y_1 + B'y_2 = 0$$

We can find

$$y' = Ay_1' + \underbrace{A'y_1} + By_2' + \underbrace{B'y_2}$$

$$= Ay_1' + By_2'$$

$$y'' = Ay_1'' + A'y_1' + By_2'' + B'y_2'$$

Substitution gives:

$$\cancel{A y_1''} + A' y_1' + \cancel{B y_2''} + B' y_2' + P(\cancel{A y_1'} + \cancel{B y_2'})$$
$$+ Q(\cancel{A y_1} + \cancel{B y_2}) = R$$

$$A' y_1' + B' y_2' = R \quad (\alpha)$$
$$A' y_1 + B' y_2 = 0 \quad (\beta)$$

Solve for A' and B'

$$A'(y_1' y_2 - y_2' y_1) = R y_2 \quad \Big| \quad y_2(\alpha) - y_2'(\beta)$$
$$B'(y_2' y_1 - y_2 y_1') = R y_1 \quad \Big| \quad y_1(\alpha) - y_1'(\beta)$$

$$A'\underbrace{(y_2' y_1 - y_2 y_1')}_{\text{Wronskian}} = -R y_2 .$$

$$W = \begin{vmatrix} y_1 & y_1' \\ y_2 & y_2' \end{vmatrix} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$A(x) = -\int^x \frac{R y_2(s)}{W(s)}\, ds \; + \text{const}$$

$$B(x) = \int^x \frac{R y_1(s)}{W(s)}\, ds \; + \text{const}.$$

General solution is $A y_1(x) + B y_2(x)$
i.e.

$$y = \text{Const}\; y_1(x) + \text{Const}\; y_1(x).$$

i.e

$$y = \text{Const}\, y_1(x) + \text{Const}\, y_2(x) + \int^x \frac{R(s)\left(y_1(s)\, y_2(x) - y_2(s)\, y_1(x)\right)}{y_1(s)\, y_2'(s) - y_2(s)\, y_1'(s)}\, ds$$

<u>Example</u> : Solve $y'' + y = \sec(x)$

CF is $y(x) = A\cos(x) + B\sin(x)$

Look for a solution : $y(x) = A(x)\cos(x) + B(x)\sin(x)$.

where we choose $A'\cos x + B'\sin x = 0$.

Substitution gives $y'' + y$

$$\Rightarrow A'(-s) + \cancel{A(-c)} + B'(c) + \cancel{B(-s)} + \cancel{Ac} + \cancel{Bs} = \sec x$$

$$y' = A(-s) + \cancel{A'c} + B(c) + \cancel{B's}$$

sum to 0.

$$y'' = A'(-s) + A(-c) + B'(c) + B(-s)$$

So :

$$A'(-s) + B'(c) = \sec(x)$$
$$A'(c) + B'(s) = 0.$$

$$B'(\underbrace{c^2 + s^2}_{=1}) = \cos(x)\sec(x) = 1$$

$$A'(-s^2 - c^2) = \sin(x)\sec(x) = \tan(x)$$

So $B = x$

$A = \ln(\cos x)$.

and the general solution is

$$y = A\cos(x) + B\sin(x)$$
$$+ x\sin(x) + \cos(x)\ln(\cos x)$$

If for 3rd ODE's

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

$$\left\{ \begin{array}{l} A'y_1 + B'y_2 + C'y_3 = R \\ A'y_1' + B'y_2' + C'y_3' = 0 \\ A'y_1'' + B'y_2'' + C'y_3'' = 0 \end{array} \right.$$

conditions.

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A property of the Wronskian

$$W = y_1 y_2' - y_2 y_1' = \begin{vmatrix} y_1 & y_1' \\ y_2 & y_2' \end{vmatrix} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

where y_1 and y_2 are so that:

$$y_1'' + P y_1' + Q y_1 = 0$$
$$y_2'' + P y_2' + Q y_2 = 0$$

$\times y_2$ $\quad$ $\times y_1$

$$\underbrace{y_1 y_2'' - y_2 y_1''}_{w'} + P\underbrace{(y_1 y_2' - y_2 y_1')}_{w} + Q\underbrace{(y_1 y_2 - y_2 y_1)}_{0} = 0$$

$$= \cancel{y_1' y_2'} + y_1 y_2'' - \cancel{y_2' y_1'} - y_2 y_1''$$

So $w' + Pw = 0$

So $w = C' e^{-\int^x P(s)\, ds}$

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"<u>Generalised Transforms</u>"

Consider solutions of equation of the type.

$$(a_1 x + a_0) y'' + (b_1 x + b_0) y' + (c_1 x + c_0) y = 0$$

[the coefficient are polynomials of degrees less than the order of the ODE].

of the form:

$$y(x) = \int_C e^{xt} f(t)\, dt \quad \text{or } e^{-xt} \text{ or } e^{ixt}$$

where C is a suitable contour in the complex t plane and f is to be found

Look at the constant coefficient case,

$$a_1 = b_1 = c_1 = 0.$$

If $y(x) = \int_c e^{xt} f(t)\, dt$.

$$\frac{dy}{dx} = \frac{d}{dx} \int_c e^{xt} f(t)\, dt$$

$$= \int_c \frac{\partial}{\partial x} \left[e^{xt} f(t) \right] dt.$$

c is independant of x.

$$= \int_c e^{xt}\, t f(t)\, dt$$

$$y''(x) = \int_c e^{xt} t^2 f(t)\, dt.$$

Substitution requires:

$$\int_c [a_0 t^2 + b_0 t + c_0] e^{xt} f(t)\, dt = 0$$

i.e. $a_0 \int_c \underbrace{[(t-\alpha)(t-\beta)]}_{\text{entire}} \underbrace{e^{xt}}_{\text{entire}} f(t)\, dt = 0.$

Consider the case with C a closed contour.

α, β roots of the auxiliary equation of the ode.

If this is true then $\int_c e^{xt} f(t)\, dt$ is a solution to $a_0 y'' + b_0 y' + c_0 y = 0$.

Choose :

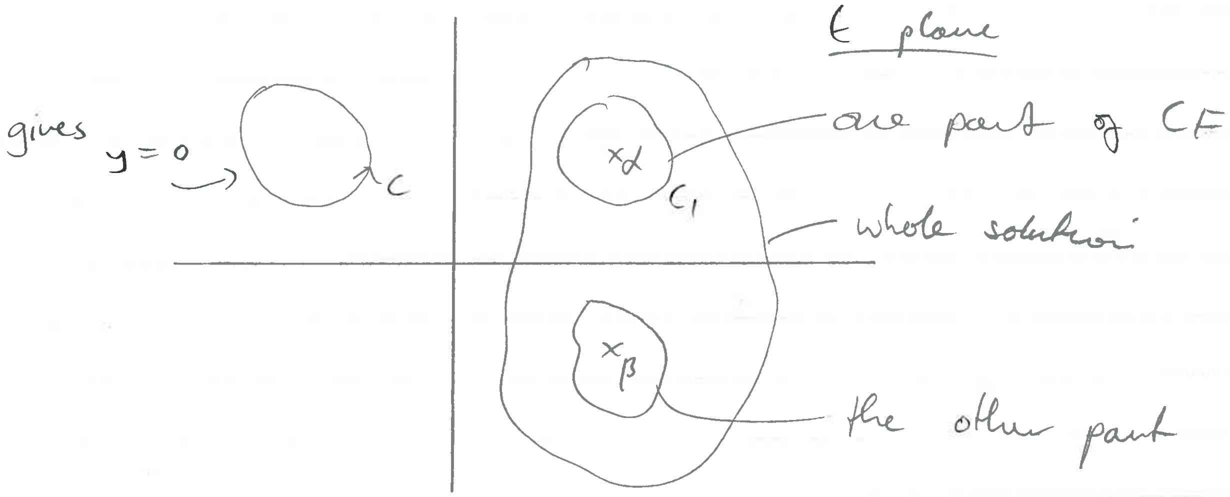
$$f(t) = \frac{A}{t-\alpha} + \frac{B}{t-\beta}$$

A and B are arbitrary constants $f(t)$ has a simple poles at the roots of a.e then
auxiliary eq^n.

$$(t-\alpha)(t-\beta)f(t) = A(t-\beta) + B(t-\alpha)$$

which is entire, but the solution is

$$y(x) = A\int_C \frac{e^{xt}}{t-\alpha}\,dt + B\int_C \frac{e^{xt}}{t-\beta}\,dt$$

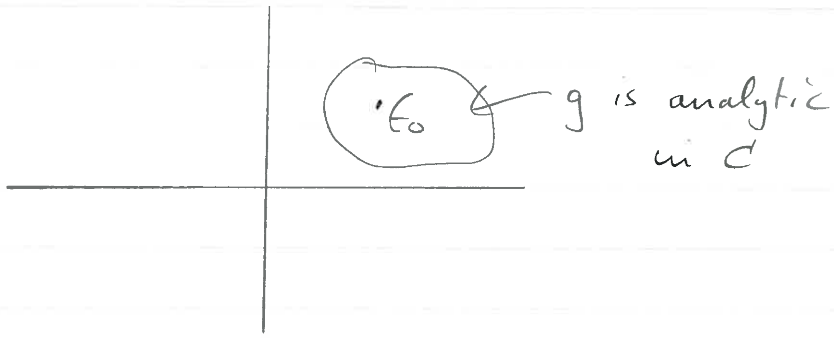


We can evaluate :

$$\int_{C_1} \frac{\overbrace{e^{xt}}^{=g}}{\underbrace{t-\alpha}_{=t_0}}\,dt$$

using the Residue theorem are more directly, Cauchy's integral theorem

$$\frac{1}{2\pi i} \oint \frac{g(\epsilon)}{\epsilon - \epsilon_0} \, d\epsilon = g(\epsilon_0) .$$



to give a solution $y = Ae^{\alpha x}$ $\epsilon_0 = \alpha$ generally we have $y = Ae^{\alpha x} + Be^{\beta x}$

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If $$f(\epsilon) = \frac{p(\epsilon)}{q(\epsilon)}$$

$$q(\epsilon_0) = 0$$
$$q'(\epsilon_0) \neq 0$$

Residue at ϵ_0 is

$$\frac{p(\epsilon_0)}{q'(\epsilon_0)}$$

Since

$$\frac{p(\epsilon)}{q(\epsilon)} = \frac{p(\epsilon_0) + (\epsilon - \epsilon_0) p'(\epsilon_0) + \ldots}{q'(\epsilon_0)(\epsilon - \epsilon_0) + q''(\epsilon_0)\frac{(\epsilon - \epsilon_0)^2}{2} + \ldots}$$

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If the a.e has a repeated root, α we need

$$\int_C (t-\alpha)^2 e^{xt} f(t)\, dt = 0.$$

and solution is $\int_C e^{xt} f(t)\, dt$.

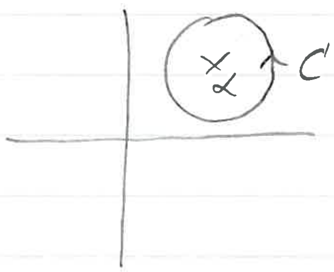
Choose $f(t) = \dfrac{A}{(t-\alpha)^2} + \dfrac{B}{(t-\alpha)}$

then $(t-\alpha)^2 f(t) = A + B(t-a)$ - entire

Choose C so that

$$y(x) = \int_C e^{xt}\left[\frac{A}{(t-\alpha)^2} + \frac{B}{(t-\alpha)}\right] dt$$

is non-zero



$$, \; y(x) = 2\pi i \operatorname*{Res}_{t=\alpha}\left[\underbrace{e^{xt}}_{e^{x(t-\alpha)}e^{\alpha x}}\left(\frac{A}{(t-\alpha)^2} + \frac{B}{(t-\alpha)}\right)\right]$$

$$= 2\pi i e^{\alpha x} \operatorname*{Res}_{t=0}\left[\overbrace{e^{xt}}^{=1+xt+\dots}\left(\frac{A}{t^2} + \frac{B}{t}\right)\right]$$

coeff of $1/t$

$$= 2\pi i e^{\alpha x}(Ax + B).$$

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$$y_2(x) = \int_{-\infty}^{-1} (t^2-1)e^{xt}\,dt.$$

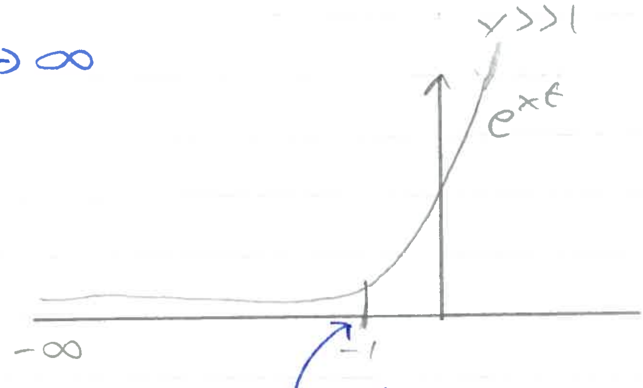
as $x \to 0$, last time we saw it $\sim 1/x^3$

OR write $xt = -u$, $y_2(x) = \int_1^{\infty} \left(\frac{u^2}{x^2}-1\right)e^{-u}\,\frac{du}{x}$

so $y_2(x) \sim \frac{1}{x^3}\int_0^{\infty} u^2 e^{-u}\,du \sim \frac{2}{x^3}$

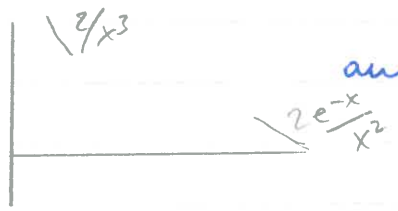
$0 \longleftarrow x \longrightarrow 0.$

as $x \to \infty$



e^{xt} is exponentially small, and we expect the integral to behave like e^{-x}. We can check this by writing:

$t = -1 - \frac{u}{x}$ so that $e^{xt} = e^{-x}e^{-\frac{u}{x}\cdot x}$
$= e^{-x}e^{-u}$



and $y_2(x) = \int_0^{\infty}\left(\frac{2u}{x}+\frac{u^2}{x^2}\right)e^{-u}e^{-x}\,\frac{du}{x}$

$\sim \frac{e^{-x}}{x^2}\,2\int_0^{\infty} ue^{-u}\,du.$

$= \frac{2e^{-x}}{x^2}$

$$y_1(x) = \int_{-1}^{1} (t^2-1)e^{xt}\,dt .$$

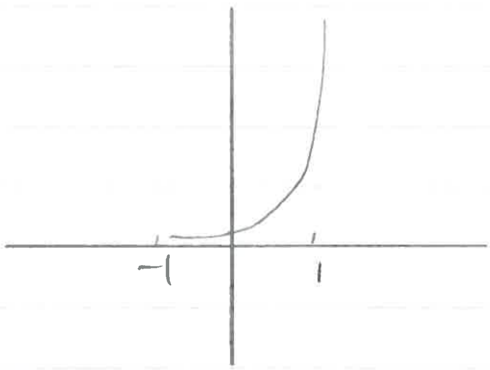
$\underline{x \to 0}$, As the range of integration in t is finite $y_1(0) = \int_{-1}^{1} (t^2-1)\,dt = -4/3$. We could go further and write :

$$e^{xt} = \sum_{n=0}^{\infty} \frac{x^n t^n}{n!} \quad \text{with r.o.c} = \infty .$$

So $$y_1(x) = 2\sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} \frac{x^n}{n!}\left(\frac{1}{n+3} - \frac{1}{n+1}\right)$$

when n odd, the integral is zero.

$\underline{x \to \infty}$:



e^{xt} biggest near $t=1$

Make the change of variable

$$t = 1 - \frac{u}{x}, \quad y_1(x) = \int_{2x}^{0} \left(\frac{u^2}{x^2} - \frac{2u}{x}\right)e^{x}e^{-u}\left(-\frac{du}{x}\right)$$

$$\sim \int_0^{\infty} 2ue^{-u}\,du\, \frac{e^x}{x^2}(-1).$$

$$\ldots = \frac{-2e^{x}}{x^{2}}$$

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To compute this: $y_2(x) = \int_{-\infty}^{-1} (t^2-1)e^{xt}\,dt$

$$\overset{t\to -t}{=} \int_{1}^{\infty} (t^2-1)e^{-xt}\,dt$$

$$= \frac{2e^{-x}}{x^3}(1+x)$$

$$y_1(x) = \int_{-1}^{1} (t^2-1)e^{xt}\,dt$$

$$= \int_{-1}^{0} \ldots + \int_{0}^{1} \ldots$$

$$= \int_{0}^{1} (t^2-1)(e^{-xt}+e^{xt})\,dt.$$

$$= 2\int_{0}^{1} (t^2-1)\cosh xt\,dt$$

$$= \frac{4}{x^3}\left(\sinh x - x\cosh x\right)$$

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Eg: $xy'' + 4y' - xy = 0$, $y = \int_C e^{xt} f(t)\, dt$.

$$0 = \int_C \left\{ x\underline{(t^2-1)} + 4t \right\} \underline{f(t)}\, e^{xt}\, dt.$$

$$\int_C \frac{d}{dt}\left[g(t)e^{xt}\right] dt$$

Note: $x e^{xt} = \frac{d}{dt} e^{xt}$

Purple bit is $\int_C (t^2-1) f(t) \frac{d}{dt} e^{xt}\, dt$ and use parts on this

$$\left[e^{xt}(t^2-1)f(t)\right]_C + \int_C \underbrace{\left\{4tf - \frac{d}{dt}\left[(t^2-1)f\right]\right\}}_{=0} e^{xt}\, dt$$

$\parallel$

0

we make this expression 0 by choosing f to satisfy the ode $4tf = (2t)f + (t^2-1)\frac{df}{dt}$

$$\Rightarrow \frac{1}{f}\frac{df}{dt} = \frac{2t}{t^2-1}$$

So $f(t) = (t^2-1)$

and choose C so that $\left[e^{xt}(t^2-1)^2\right]_C = 0$.

as before

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$$(a_1 x + a_0) y'' + (b_1 x + b_0) y' + (c_1 x + c_0) y = 0$$

$$y = \int_C e^{xt} f(t)\, dt; \quad y' = \int_C e^{xt}\, t f(t)\, dt$$

$$y'' = \int_C e^{xt}\, t^2 f(t)\, dt.$$

Substitution leads to

$$\int_C [x\underbrace{(a_1 t^2 + b_1 t + c_1)}^{g(t)} + \underbrace{(a_0 t^2 + b_0 t + c_0)}^{g'(t)}] e^{xt} f(t)\, dt = 0$$

We can make this zero if we can write it as

$$\int_C \frac{d}{dt}\left[e^{xt} g(t)\right] dt = \left[e^{xt} g(t)\right]_C$$

Now $\frac{d}{dt}\left[e^{xt} g(t)\right] = x e^{xt} g(t) + e^{xt} g'(t)$.

So we can identify

$$g(t) = (a_1 t^2 + b_1 t + c_1) f(t)$$
$$g'(t) = (a_0 t^2 + b_0 t + c_0) f(t)$$

giving: $$\frac{g'}{g} = \frac{a_0 t^2 + b_0 t + c_0}{a_1 t^2 + b_1 t + c_1}$$

which can easily be integrated

We choose C so that $[e^{xt} g(t)]_C = 0$. Remember though that the solution $\int_C e^{xt} f(t)\, dt$ so we must find $f(t)$ also.

Solve $\quad xy'' + 4y' - xy = 0 \quad , \quad x > 0.$

Try $y = \int_c e^{xt} f(t)\, dt$ and substitute to find.

$$\int_c \left[xt^2 + 4t - x\right] e^{xt} f(t)\, dt .$$

Compare with $\displaystyle\int_c \frac{d}{dt}\left[e^{xt} g\right] dt = \int_c (gx + g')e^{xt}\, dt$

$$= \left[e^{xt} g\right]_c$$

and we see we require: $\left.\begin{array}{l}(t^2-1)f = g. \\ 4tf = g'\end{array}\right\} \dfrac{g'}{g} = \dfrac{4t}{t^2-1}$

$\Rightarrow \ln g = 2\ln(t^2-1)$ (Intergrate w.r.t t)

hence $g = (t^2-1)^2$

and as $f = g/(t^2-1) \Rightarrow f = (t^2-1)$.

So a solution is

$$y(x) = \int_{c'} e^{xt}(t^2-1)\, dt$$

with C' choosen so that : $\left[e^{xt}(t^2-1)^2\right]_{c'} = 0$.
We can here choose C' to start and finish at zeros of $e^{xt}(t^2-1)^2$

Example: $xy'' + (3x-1)y' - 9y = 0$.

Try $g(x) = \int_C e^{xt} f(t)\,dt$ and substitute

$$0 = \int_C x(t^2+3t)e^{xt} f(t)\,dt - \int_C (t+9)e^{xt} f(t)\,dt.$$

Integration by parts wrt t.

$$0 = \Big[(t^2+3t) f e^{xt}\Big]_C$$

$$- \int_C \left\{ \frac{d}{dt}\Big((t^2+3t)f(t)\Big) + (t+9)f(t) \right\} e^{xt}\,dt$$

Choose f to satisfy $\{\ldots\} = 0$.

$$(2t+3)f + (t^2+3t)\frac{df}{dt} + (t+9)f = 0.$$

$$\frac{f'}{f} = -\frac{(3t+12)}{t^2+3t} = \frac{1}{t+3} - \frac{4}{t}$$

$$\frac{A}{t} + \frac{B}{t+3}$$

$$\ln f = \ln(t+3) - 4\ln(t)$$

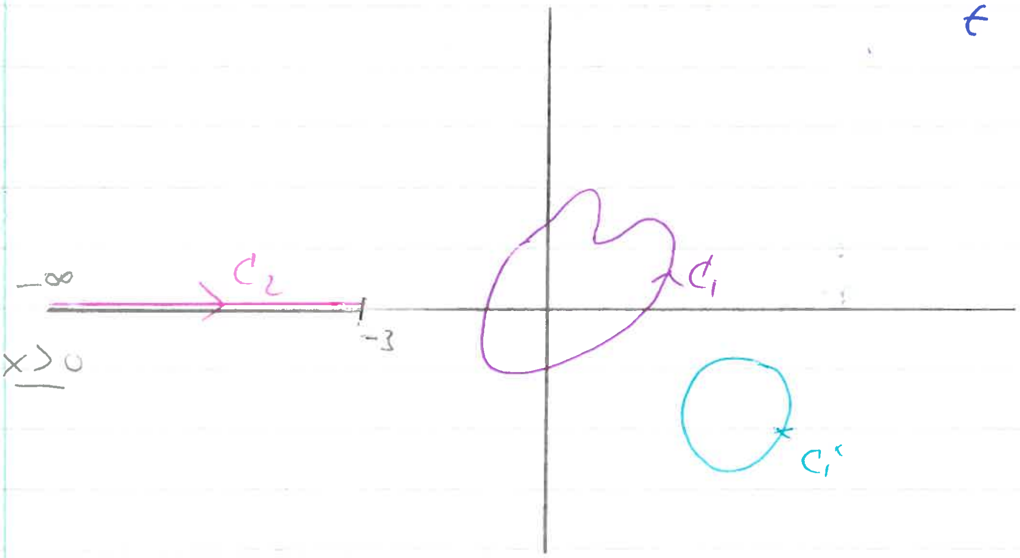
$$f = \frac{t+3}{t^4}$$

So our solution is $y = \int \frac{(t+3)e^{xt}}{t^4}\,dt$ where

C' is chosen so that

$$\left[(t^2+3t)\frac{(t+3)}{t^4} e^{xt} \right]_{C'}$$

$$= \left[\frac{(t+3)^2}{t^3} e^{xt} \right]_{C'} = 0.$$



Solution given by C_2' is $y_2(x) = \int_{-\infty}^{-3} \frac{(t+3)}{t^4} e^{xt}$

C_1^x will make $\left[\frac{(t+3)^2}{t^3} e^{xt} \right]_{C_1^x} = 0$

as $\frac{(t+3)^2}{t} e^{xt}$ is single valued.

But $\int_{C_1^x} \frac{(t+3)}{t^4} e^{xt}\, dt = 0$ as $\frac{(t+3)}{t} e^{xt}$ is analytic within C_1^x

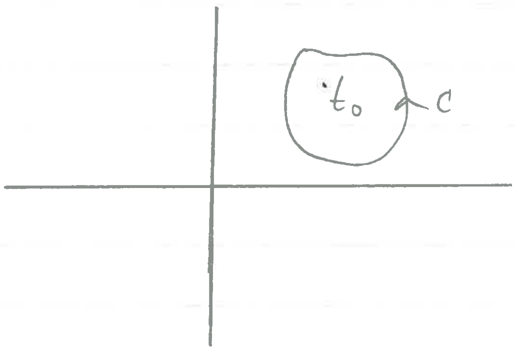
But if we choose a closed contour containing singularities in $\frac{(t+3)}{t^4}e^{xt}$ then we get $[\ ]_C = 0$ and a non zero solution $y(x)$.

$$y_1(x) = \oint \frac{(t+3)}{t^4} e^{xt}\, dt.$$

We can do this integral using Cauchy's integral theorem for derivaties.

$$f^{(n)}(t_0) = \frac{n!}{2\pi i}\oint_C \frac{f(t)}{(t-t_0)^{n+1}}\, dt$$

with f regular inside C.



$(t+3)e^{xt} = f$
$t_0 = 0$
$n = 3$.

$$y_1(x) \underset{\text{lose } 2\pi i}{=} \frac{1}{3!}\frac{d^3}{dt^3}\left((t+3)e^{xt}\right)\Big|_{t=0}$$

$$= \frac{1}{3!}\left(3 \cdot 1\, x^2 e^{xt} + 1(t+3)x^3 e^{xt}\right)\Big|_{t=0}.$$

$$= \frac{x^2}{2} + \frac{x^3}{2}$$ i.e. $y_1(x)$ is a simple polynomial

for $t \to -t$ in $y_2(x) = \int_3^\infty \frac{3-t}{t^4} e^{-xt}\, dt$

We "can see" that this is exponentially small as $x \to \infty$, what about as $x \to 0$.

$$y_2(0) = \int_3^\infty \frac{3-t}{t^4}\, dt .$$

which is some finite number. (The integrand behaves like $1/t^3$ for large t, which is integrable).

$$\frac{dy_2}{dx} = -\int_3^\infty \frac{3-t}{t^3} e^{-xt}\, dt$$

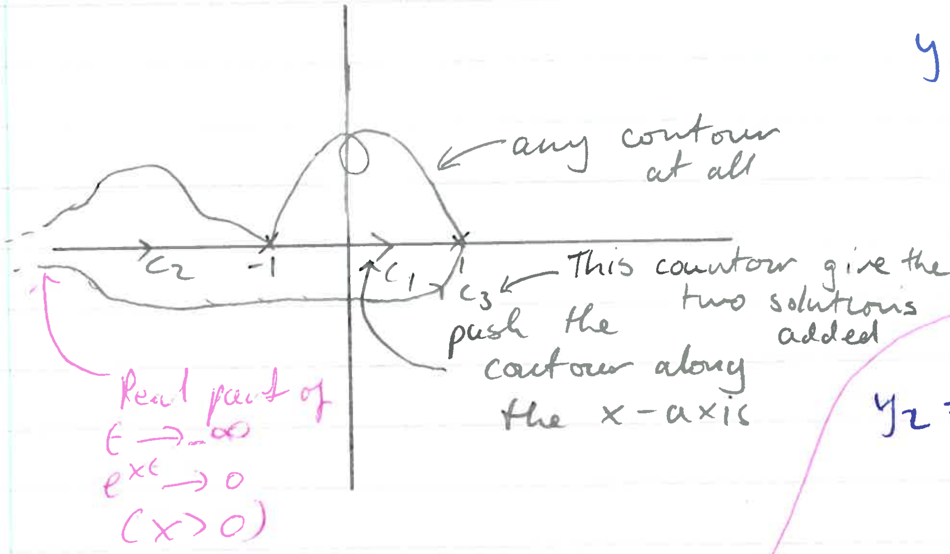
$$y_2'(0) = \int_3^\infty \frac{3-t}{t^3}\, dt$$ which again exists.

$$\frac{d^2 y_2}{dx^2} = \int_3^\infty \frac{(3-t)}{t^2} e^{-xt}\, dt$$

$$y_2''(0) = \int_3^\infty \frac{3-t}{t^2}\, dt$$ which does not exist.

as integrand behaves like $1/t$, $t \to \infty$.

$$\left.\begin{aligned} y_2(0) &= \text{finite} \\ y_2'(0) &= \text{finite} \\ y_2''(0) &= \text{infinite} \end{aligned}\right\} \text{like } x^2 \ln(x).$$



$$y_1 = \int_{C_1} e^{xt}(t^2-1)\,dt$$

$$= \int_{-1}^{1} e^{xt}(t^2-1)\,dt$$

$$y_2 = \int_{C_2} e^{xt}(t^2-1)\,dt$$

$$= \int_{-\infty}^{-1} e^{xt}(t^2-1)\,dt$$

Two linear independant solutions. $\rightarrow$ General solution is $y(x) = Ay_1(x) + By_2(x)$

—/—

Office hour:

Tue:	~~8 – 9~~		Fri	8 – 9
	~~11 – 12~~			9 – 10
	~~12 – 1~~			10 – 11

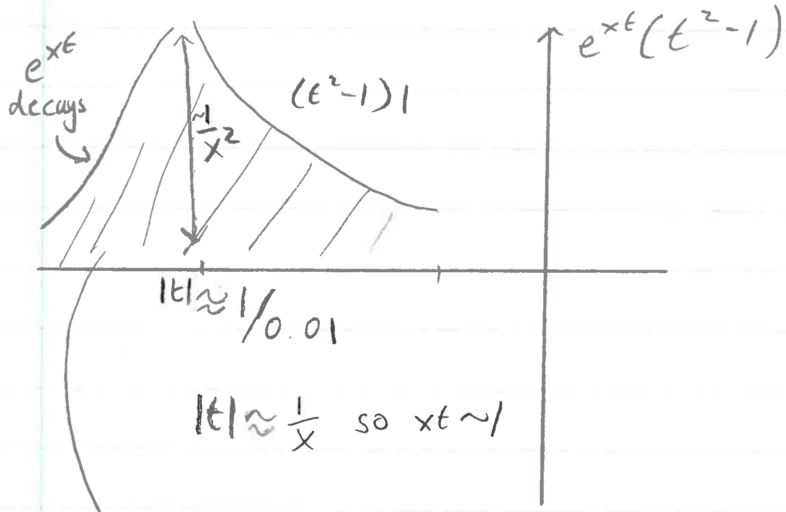
—/—

Let us examine y_1 and y_2 for small values of x.

If we put $x=0$, $y_1(0) = \int_{-1}^{1} (t^2-1)\,dt$, which is finit

On the other hand $y_2(0) \int_{-\infty}^{-1} (t^2-1)\,dt$, which does not exis as the integral diverges at its lower limit.

but $y_2(x)$ (for any small but non-zero x) does exist $y_2(x) = \int_{-\infty}^{-1} e^{xt}(t^2-1)\,dt$.



x is small say 0.01
If $|t|$ is not too large
e^{xt} is ≈ 1

$\frac{1}{x^2} \cdot \frac{1}{x} = \frac{1}{x^3}$, at $x \to 0$, $y_2(x) \sim \frac{1}{x^3}$ and singular at $x = 0$.

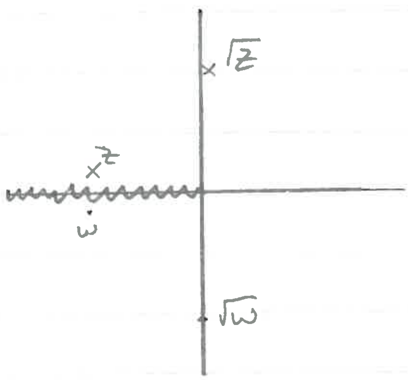
16/10/12

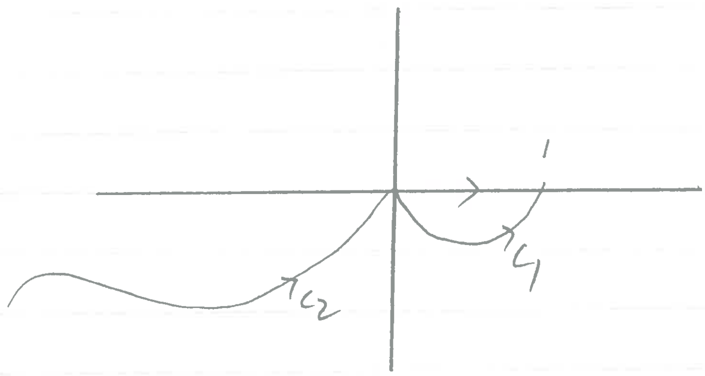
Example $xy'' + (1-x)y' + ay = 0.$

would lead to solutions $y = \int_C e^{xt} \frac{t^{a-1}}{(t-1)} \, dt$ where

$$\left[\frac{t^{a} e^{xt}}{(t-1)^{a-1}} \right] = 0$$

If $a = 1/2$, this becomes $y = \int_c \frac{e^{xt}}{\sqrt{t}\sqrt{t-1}} \, dt$ where $\left[\sqrt{t} e^{xt} \sqrt{t-1} \right]_c =$
with $\sqrt{\ }$ defined so that
the real part is +ve.





$$t = s - i\varepsilon$$

($t = s + i\varepsilon$ gives complex conjugate

$$y_1 = \int_0^1 \frac{e^{sx}}{\sqrt{s}\sqrt{s-1}\,(-i)} \, ds$$ ignore as arbitrary constant

solution: $y_1 = \int_0^1 \frac{e^{sx}}{\sqrt{s}\sqrt{1-s}} \, ds.$

$\sqrt{t} \to \sqrt{s}$ ($\sqrt{s - i\varepsilon}$ gives $\sqrt{s} \cdot 1$)

$\sqrt{t-1} \to \sqrt{s-1-i\varepsilon}$ $\sqrt{\ }$ gives $\sqrt{1-s}\,(-i)$

modulus is $1-s$.

argument is just bigger than $-\pi$ and arguement of $\sqrt{\ }$ is $\frac{-\pi}{2}$

Airy's Equation.

$$y'' - xy = 0.$$

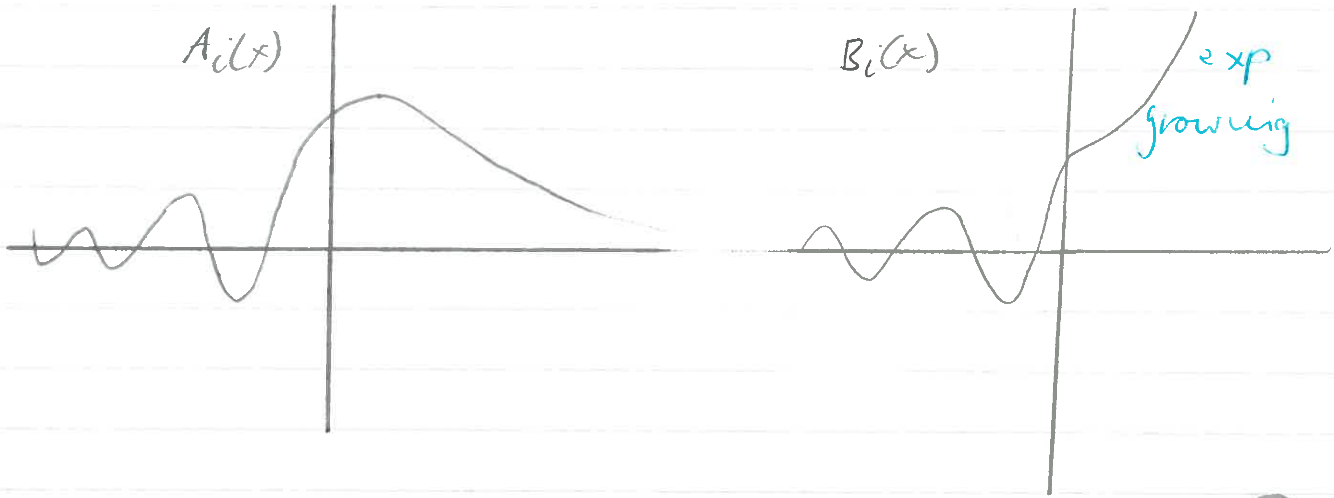
in regions where $x<0$ this is like:

$$y'' + y = 0 \quad \text{i.e oscillatory solutions.}$$

where $x>0$ this is like:

$$y'' - y = 0 \quad \text{i.e exp. growing or decaying solutions}$$

There are two tabulated solutions:



Try $y = \int_C e^{xt} f(t)\, dt$ and substitute:

$$\int_C t^2 e^{xt} f(t)\, dt - \int_C f x e^{xt}\, dt = 0.$$

$$\left[-fe^{xt}\right]_C + \int_C \underbrace{(ft^2 + f')}_{=0} e^{xt}\, dt = 0.$$

So $f' + t^2 f = 0$, $f = e^{-\frac{1}{3}t^3}$

Solutions are:

$$y = \int_c e^{-\frac{1}{3}t^3} e^{xt} \, dt = \int_c e^{xt - \frac{1}{3}t^3} \, dt .$$

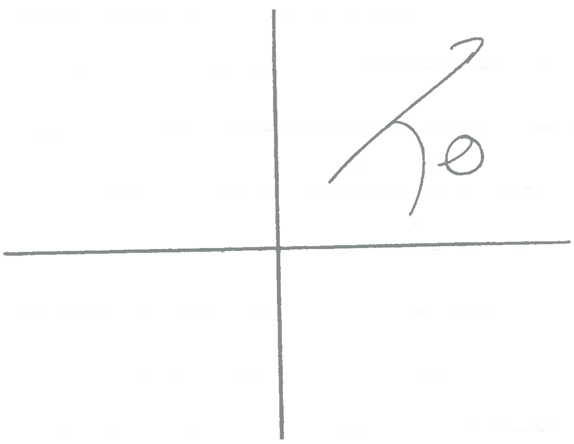
where:

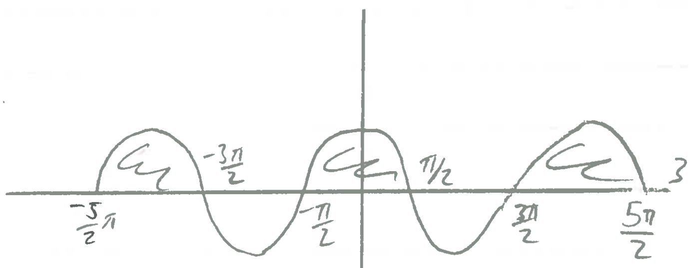
$$\left[e^{xt - \frac{1}{3}t^3} \right]_c = 0 .$$

The only possible zeros of $e^{xt - \frac{1}{3}t^3}$ must be approached as $|t| \to \infty$

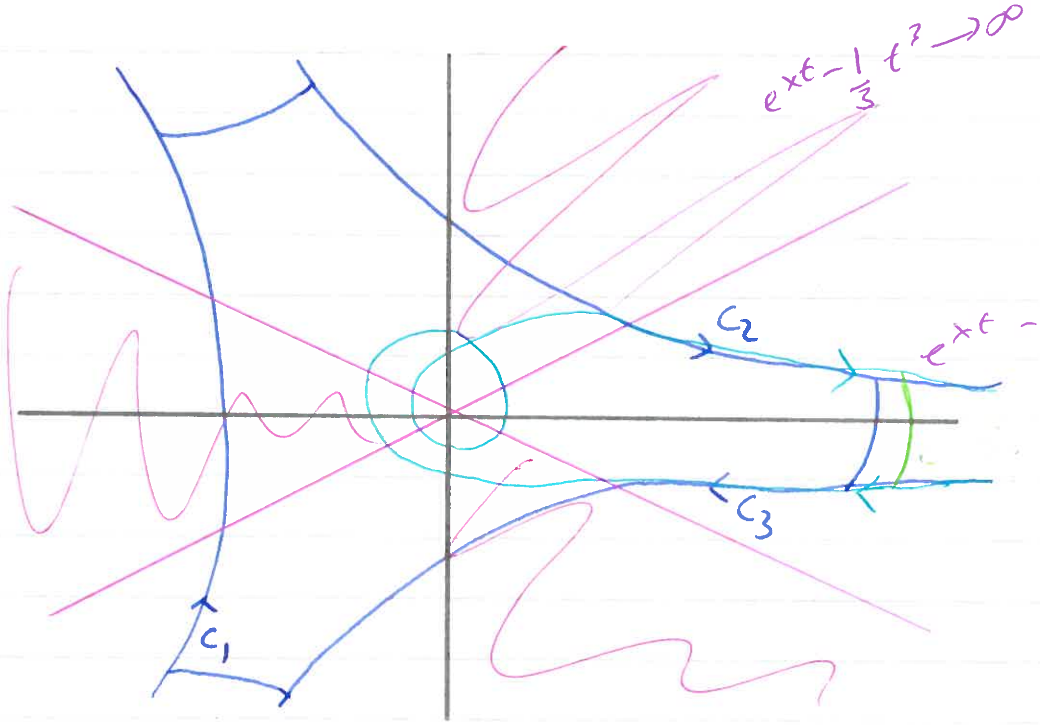
Let us put $t = Re^{i\theta}$ and let $R \to \infty$ then $xt - \frac{1}{3}t^3$ $= xRe^{i\theta} - \frac{1}{3}R^3 e^{3i\theta}$

We can neglect xR term relative to the $\frac{1}{3}R^3$ term and $e^{xt - t^3}$ is exponentially small where $Re(e^{3i\theta}) >$ i.e $\cos 3\theta > 0$.





$-5\pi/6 < \theta < -\pi/2$
$-\pi/6 < \theta < \pi/6$
$\pi/2 < \theta < 5\pi/6.$



$$y = \int_{C_{1,2,3}} e^{xt - \frac{1}{3}t^3}\, dt$$ are non-zero solution

However $\int_{C_1} + \int_{C_2} + \int_{C_3} = 0.$

$$y_1 + y_2 + y_3 = 0$$

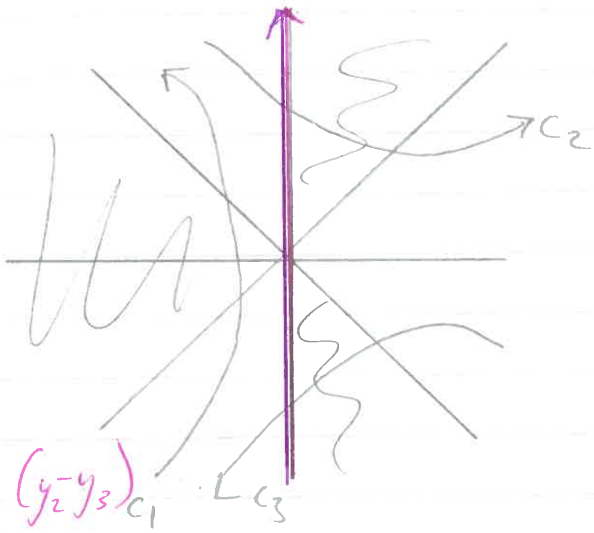
and we have two independant solution's.

<u>19/10/12</u>

$$y'' - xy = 0. \qquad Ai(x),\ Bi(x).$$

$$y = \int_c e^{xt - \frac{1}{3}t^3}\, dt.$$

$$\left[e^{xt - \frac{1}{3}t^3} \right]_c = 0$$



$$Ai(x) = \frac{1}{2\pi i} y_1, \quad Bi(x) = \frac{1}{2\pi}(y_2 - y_3)$$

$t = is$,

$$Ai(x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \cos\left(xs + \frac{1}{3}s^2\right) + \underbrace{i\sin\left(xs + \frac{1}{3}s^3\right)}_{\text{gives zero}}\, i\, ds$$

$$= \frac{1}{\pi} \int_0^{\infty} \cos\left(xs + \frac{1}{3}s^2\, ds\right) ds$$

It is not absolutely integrable, but cancellation means it is integrable.

—/—

<u>Phase plane analysis of ode's.</u>

It is not possible to find explicit solutions to all differential equations. Even if we can find solutions, say in terms of integrals, we don't know what the "graph" of the integral looks like. Numerical may help. But they may be of limited use if we have many possible initial conditions. Phase plane analysis allows us to find relatively easily qualitive aspects of possible solutions.

A non-linear first order equation has the general form

$$\frac{dy}{dx} = f(x,y) = \frac{Q(x,y)}{P(x,y)}$$

curves of solutions in the x,y plane are called solution curves, integral curves, trajectories.

Since $f(x,y)$ is single valued trajectories cannot cross. Exceptions may be where $f(x,y) = Q(x,y)/P(x,y)$ and the ratio P/Q is intermediate. So trajectories may cross at points (x_0, y_0) so that $P(x_0, y_0) = Q(x_0, y_0) = 0$. These are called singular points (or equilibrium points, depending on context).

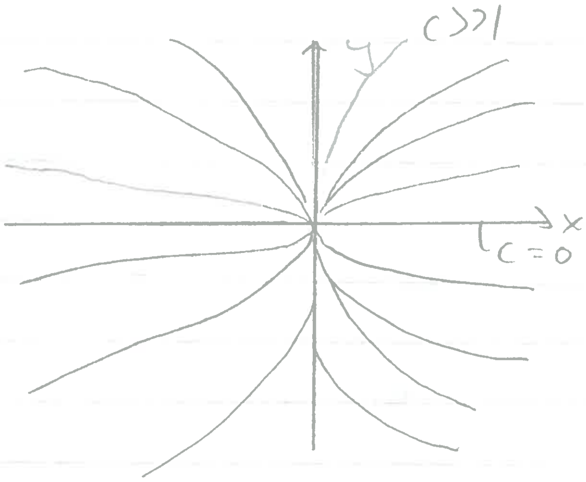
$$\frac{dy}{dx} = \frac{y}{2x}, \quad Q = y, \quad P = 2x.$$

$$2\frac{dy}{y} = \frac{dx}{x} \Rightarrow y^2 = Cx.$$

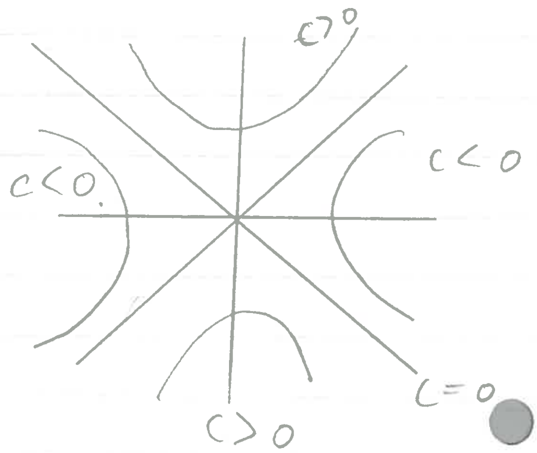
$$\frac{dy}{dx} = \frac{x}{y}$$

$$y\,dy = x\,dx$$

$$y^2 - x^2 = \text{const}.$$



All cross



Only two cross.

Use in solving second order equations

Consider

$$\frac{d^2x}{dt^2} = Q\left(x, \frac{dx}{dt}, t\right)$$

for $x(t)$

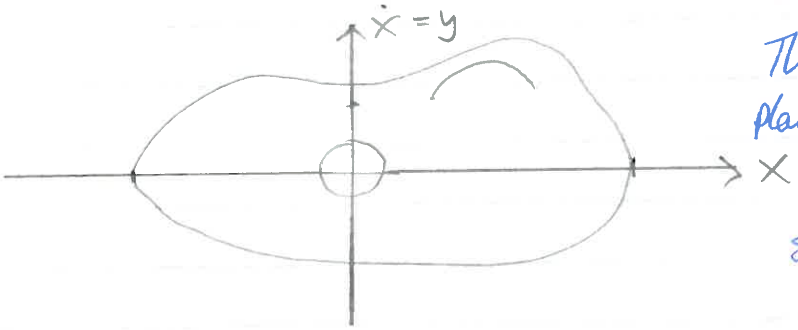
eg. $\ddot{x} + w^2 x = 0$. If Q is such that $\partial Q/\partial t = 0$ i.e $Q = Q(x, \dot{x})$ then the equation is said to be autonomous. In this case we can introduce

$$y = \frac{dx}{dt}, \text{ then } \frac{dy}{dt} = \frac{d\dot{x}}{dt^2} = Q(x, y)$$

$$\frac{dx}{dt} = P(x, y) = y.$$

So we have replaced a second order equation by a pair of first order equations.

$$\frac{dy}{dt} = Q(x, t) \quad , \quad \frac{dx}{dt} = P(x, y)$$
$$= y \text{ here.}$$



This is the phase plane $(x, \dot{x})$-plane. Trajectories show how $\dot{x}$ varies with x.

The slope of the trajectories i.e

$$\frac{d\dot{x}}{dx} = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{Q(x,y)}{P(x,y)}$$

If $P = y$

$$\frac{dy}{dx} = Q/y \, , \quad \frac{dx}{dt} = y .$$

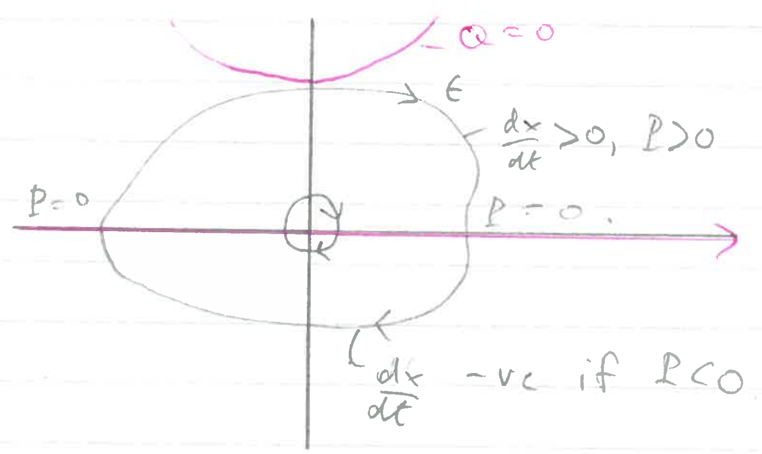
$$\frac{dy}{dt} = Q(x,y)$$

$\frac{dy}{dx} = \infty$ where $y = 0$ i.e the trajectories cut the x-axis at right angle.

$\frac{dy}{dx} = 0$ and the trajectories are horizontal where $Q = 0$.

The lines given by $Q = 0$ are called horizontal nullclines

$P = 0$ " vertical nullclines.



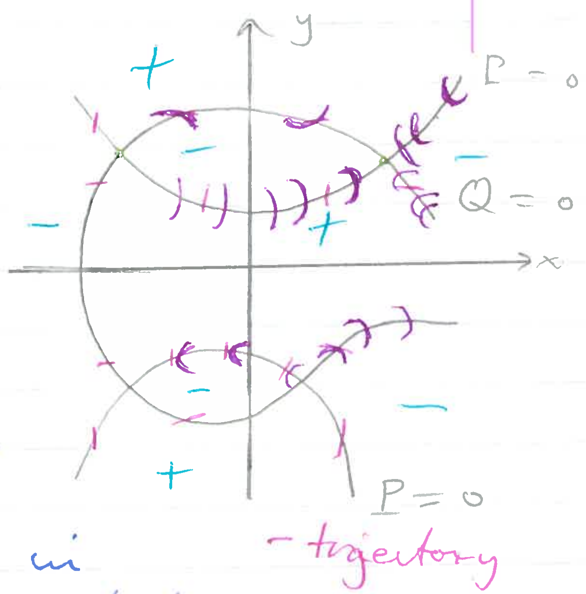
Time may introduced into the problem $\frac{dy}{dx} = Q/P$ by writing this as the pair

$$\frac{dy}{dt} = Q$$

$$\frac{dx}{dt} = P.$$

<u>Importance of nullclines</u>

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}.$$



Usually as one crosses the lines $Q = 0$ or $P = 0$. dy/dx changes sign and the slope of trajectories changes sign.

Let us examine critical points in more detail, let (x_0, y_0) be such that $P(x_0,y_0) = Q(x_0,y_0) = 0$ and use a Taylor expansion

$$P(x,y) = \underbrace{P(x_0,y_0)}_{0} + \left.\frac{\partial P}{\partial x}\right|_{(x_0,y_0)}(x-x_0)$$

$$+ \left.\frac{\partial P}{\partial y}\right|_{(x_0,y_0)}(y-y_0) + \dots$$

$$Q(x,y) = \underbrace{Q(x_0,y_0)}_{0} + \left.\frac{\partial Q}{\partial x}\right|_{(x_0,y_0)} (x - x_0)$$

$$+ \left.\frac{\partial Q}{\partial y}\right|_{(x_0,y_0)} (y - y_0) + \dots$$

Write $x - x_0 = X$
$y - y_0 = Y$

$$\frac{dy}{dx} = \frac{dY}{dX} = \frac{Q_x X + Q_y Y}{P_x X + P_y Y}.$$

$$= \frac{C' X + D Y}{A X + B Y}.$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} P_x & P_y \\ Q_x & Q_y \end{pmatrix} = \underline{\underline{J}}$$, the Jacobian at a singular point.

or : $\dfrac{dY}{dt} = CX + DY$, $\dfrac{dx}{dt} = Ax + By$.

or with $\underline{x} = \begin{pmatrix} X \\ Y \end{pmatrix}$ we have $\dfrac{d\underline{x}}{dt} = \underline{\underline{J}}\,\underline{x}$

Look for solutions $\underline{x} = \underline{u} e^{\lambda t}$, so $\dfrac{d\underline{x}}{dt} = \lambda \underline{u} e^{\lambda t}$

So $\lambda \underline{u} e^{\lambda t} = \underline{\underline{J}}\,\underline{u} e^{\lambda t} \Rightarrow \underline{\underline{J}}\,\underline{u} = \lambda \underline{u}$

So λ is an eigenvalues of $\underline{\underline{J}}$. Presume we have two λ_1 and λ_2 and corresponding eigenvector $\underline{u}_1$ and $\underline{u}_2$, then

$$\underline{x} = A_1 \underline{u}_1 e^{\lambda_1 t} + A_2 \underline{u}_2 e^{\lambda_2 t}$$

A_1 and A_2 define which trajectory we are on:

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = A_1 \begin{pmatrix} u_{11} \\ u_{12} \end{pmatrix} + A_2 \begin{pmatrix} u_{21} \\ u_{22} \end{pmatrix}$$

We investigate further by diagonalising $\underline{\underline{J}}$. Define $\underline{\underline{P}} = (\underline{u}_1, \underline{u}_2)$ and define new variables

$\bar{x}, \bar{y}$ so that $\begin{pmatrix} x \\ y \end{pmatrix} = \underline{\underline{P}} \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$

i.e $\underline{x} = \underline{\underline{P}}\, \underline{\bar{x}}$ or $\underline{\bar{x}} = \underline{\underline{P}}^{-1} \underline{x}$.

Then

$$\underline{\underline{J}}\,\underline{\underline{P}} = \underline{\underline{J}}(\underline{u}_1, \underline{u}_2) = (\lambda_1 \underline{u}_1, \lambda_2 \underline{u}_2)$$

$$= (\underline{u}_1, \underline{u}_2) \underbrace{\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}}_{\underline{\underline{\Lambda}}} = \underline{\underline{P}}\,\underline{\underline{\Lambda}}$$

i.e $\underline{\underline{J}}\,\underline{\underline{P}} = \underline{\underline{P}}\,\underline{\underline{\Lambda}}$

$$\underline{\underline{J}} = \underline{\underline{P}}\,\underline{\underline{\Lambda}}\,\underline{\underline{P}}^{-1}$$

$$\frac{d\underline{x}}{dt} = \underline{\underline{J}}\,\underline{x} = \underline{\underline{P}}\,\underline{\underline{\Lambda}}\,\underline{\underline{P}}^{-1}\underline{x} \Rightarrow \underline{\underline{P}}^{-1}\frac{d\underline{x}}{dt} = \underline{\underline{\Lambda}}\,\underline{\underline{P}}^{-1}\underline{x}$$

$$\Rightarrow \frac{d\bar{\underline{x}}}{dt} = \underline{\underline{\Lambda}}\,\bar{\underline{x}} \qquad \text{where } \bar{\underline{x}} = \underline{\underline{P}}\,\underline{x}$$

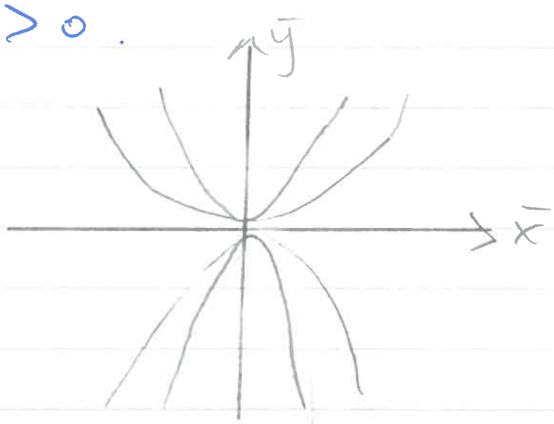
i.e $\frac{d\bar{x}}{dt} = \lambda_1 \bar{x}$. $\Rightarrow \bar{x} = \bar{x}_0 e^{\lambda_1 t}$

$\frac{d\bar{y}}{dt} = \lambda_2 \bar{y}$, $\Rightarrow \bar{y} = \bar{y}_0 e^{\lambda_2 t}$

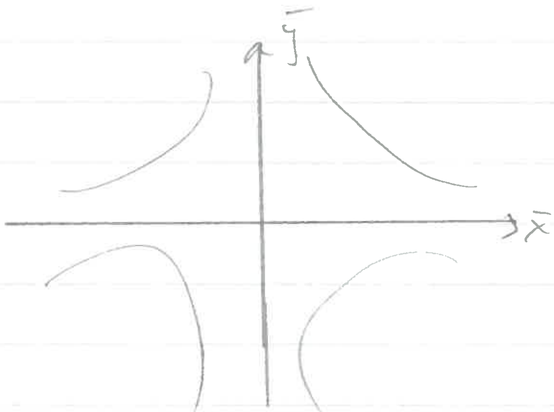
$\Rightarrow \bar{y} = C\bar{x}^{a}$

$a = \lambda_2/\lambda_1$

e.g $a = 2 > 0$.



$a = -1$



λ_1 and λ_2 are eigenvalues of $\underline{\underline{J}} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$

$$= \begin{pmatrix} P_x & P_y \\ Q_x & Q_y \end{pmatrix}$$

and so satisfy the quadratic

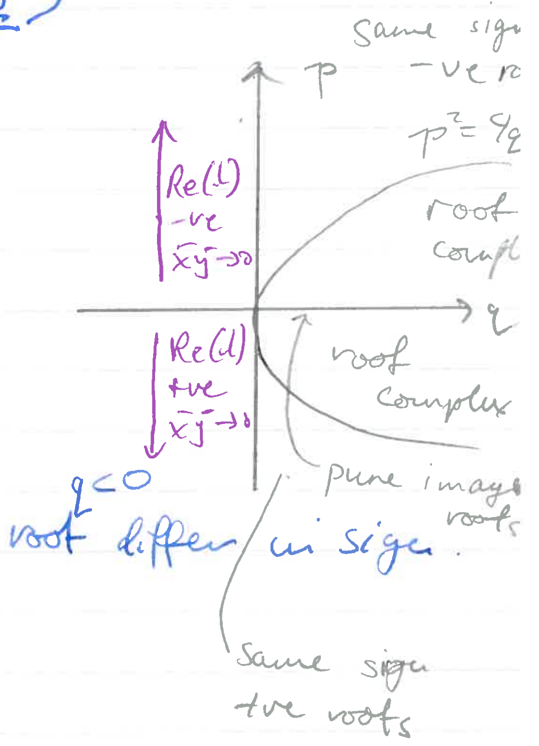
$$(A-\lambda)(D-\lambda) - CD = 0.$$

$$\lambda^2 - \underbrace{(A+D)}_{-p}\lambda + \underbrace{(AD - CD)}_{q = \det \underline{\underline{J}}} = 0$$

$$A + D = -p = -\mathrm{tr}(\underline{\underline{J}})$$

$$\lambda^2 + p\lambda + q = 0.$$

$$\lambda = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$$



Nonlinear differential equations - phase plane analysis

We consider the general first order differential equation for $y(x)$

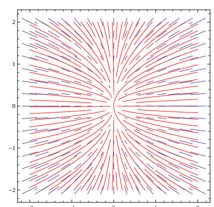
$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y) = \frac{Q(x,y)}{P(x,y)}. \tag{1}$$

1 Revision

Curves in the (x,y)-plane which satisfy this equation are called ***integral curves*** or ***trajectories***. There is a family of such curves, paremterised by the constant of integration associated with solving the equation. The slope of an integral curve that passes through the point (x_0, y_0) is $f(x_0, y_0) = P(x_0, y_0)/Q(x_0, y_0)$ and hence is a unique slope, except perhaps where $f(x_0, y_0)$ is undetermined, i.e. $P(x_0, y_0) = Q(x_0, y_0) = 0$. Hence the only place that the trajectories can intersect is at points where $P = Q = 0$. These are called ***singular points***, or ***equilibrium points***. We will investigate the trajectories in the vicinity of such points below.

Example

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{2x} \Rightarrow \int \frac{2\mathrm{d}y}{y} = \int \frac{\mathrm{d}x}{x} \Rightarrow \ln y^2 = \ln x + C' \Rightarrow y^2 = Cx.$$

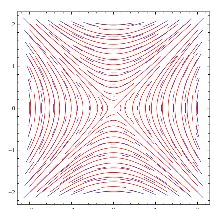


All trajectories cross at $(0,0)$ where $f(x,y) = y/2x$ is undetermined.

```
VectorPlot[{2x,y},{x,-2,2},{y,-2,2},StreamScale->None,
 StreamPoints->Fine,StreamStyle->Red,VectorStyle->Arrowheads[0]]
```

Example

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y} \Rightarrow \int y\,\mathrm{d}y = \int x\,\mathrm{d}x \Rightarrow y^2/2 = x^2/2 + C' \Rightarrow y^2 - x^2 = C.$$



Only two trajectories cross at $(0,0)$ where $f(x,y) = x/y$ is undetermined. These are given by $C = 0$.

```
VectorPlot[{y,x},{x,-2,2},{y,-2,2},StreamScale->None,
 StreamPoints->Fine,StreamStyle->Red,VectorStyle->Arrowheads[0]]
```

2 Second-order equations

The most general form is for a second order equation for $x(t)$ is $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = Q(x, \frac{\mathrm{d}x}{\mathrm{d}t}, t)$. However such an equation is called ***autonomous*** if the coefficients do not depend explicitly on t so that

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = Q\left(x, \frac{\mathrm{d}x}{\mathrm{d}t}\right). \tag{2}$$

For these equations we may introduce

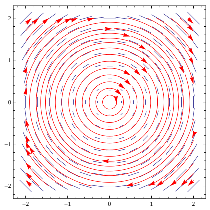
$$y = \frac{\mathrm{d}x}{\mathrm{d}t} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = Q\left(x, \frac{\mathrm{d}x}{\mathrm{d}t}\right) = Q(x,y) \text{ and } \frac{\mathrm{d}x}{\mathrm{d}t} = y = P(x,y) \text{ giving } \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = \frac{Q(x,y)}{P(x,y)} = \frac{Q}{y}.$$

So (2) can be written as a special case of (1). In this case the (x,y)-plane is an $(x,\dot{x})$-plane, known as a ***phase-plane*** and the integral curve/trajectory may also be called a ***phase-trajectory***. The trajectories are solutions of the equations $\dot{x} = y$, $\dot{x} = Q(x,y)$, with t as an effective parameter taking us along a trajectory. The trajectories are therefore traversed in a particular direction as t increases. This direction is easy to identify as it is in the direction of increasing x ($\dot{x} > 0$) in the upper-half plane $y = \dot{x} > 0$. Singular points are more often called equilibrium points in this context since at such a point, $x = x_0$, $y = 0$,say, $P = Q = \dot{x} = \dot{y} = \ddot{x} = 0$ and, if x represents the displacement of a particle, for example, in some physical system, a particle placed exactly at $x = x_0$ so that $y = 0$ will stay there, in equilibrium.

Example

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -x, \quad \text{so} \quad \dot{y} = -x, \quad Q = -x, \quad \dot{x} = y, \quad P = y.$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-x}{y} \Rightarrow \int y \, \mathrm{d}y = -\int x \, \mathrm{d}x \Rightarrow y^2/2 = -x^2/2 + C' \Rightarrow y^2 + x^2 = C.$$



Here no trajectories cross at $(0, 0)$ where $f(x, y) = -x/y$ is undetermined.

```
VectorPlot[{y,-x},{x,-2,2},{y,-2,2},StreamScale->{Full, All, 0.03},
StreamPoints->Fine,StreamStyle->Directive[Red],VectorStyle->Arrowheads[0]]
```

We have seen that the time-dependent system (2) can be rewritten as (1). Similarly (1) can be written as a pair of first order equations for $x(t)$ and $y(t)$, with t as a parameter in describing the solution trajectories. If

$$\frac{\mathrm{d}x}{\mathrm{d}t} = P(x(t), y(t)), \quad \frac{\mathrm{d}y}{\mathrm{d}t} = Q(x(t), y(t)), \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\dot{y}}{\dot{x}} = \frac{Q(x, y)}{P(x, y)}. \tag{3}$$

A direction of travel along the trajectories can then be assigned, moving to the right, in the direction of increasing x in regions of the (x, y)-plane where $P > 0$ and up, in the direction of increasing y in regions where $Q > 0$.

3 Solution near singular points

We examine the solutions to (1) in the vicinity of critical points (x_0, y_0) where $P(x_0, y_0) = Q(x_0, y_0) = 0$. We have seen above that there are several different forms for the trajectories. Expanding about these points we find

$$P(x, y) \approx P(x_0, y_0) + \frac{\partial P}{\partial x}\,|_{(x_0,y_0)}\,(x - x_0) + \frac{\partial P}{\partial y}\,|_{(x_0,y_0)}\,(y - y_0) = P_x X + P_y Y$$
$$Q(x, y) \approx Q(x_0, y_0) + \frac{\partial Q}{\partial x}\,|_{(x_0,y_0)}\,(x - x_0) + \frac{\partial Q}{\partial y}\,|_{(x_0,y_0)}\,(y - y_0) = Q_x X + Q_y Y,$$

where $X = (x - x_0)$, $Y = (y - y_0)$, giving

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{CX + DY}{AX + BY}, \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} P_x & P_y \\ Q_x & Q_y \end{pmatrix}\Bigg|_{(x_0,y_0)} = \mathbf{J}, \tag{4}$$

where $\mathbf{J}$ is called the ***Jacobian*** of the equilibrium point.

Equation (4) is straightforward enough to solve in individual cases, by putting $Y(X) = XZ(X)$.
(see `http://www.ucl.ac.uk/Mathematics/geomath/level2/deqn/MHde.html` and
`http://en.wikipedia.org/wiki/Homogeneous_differential_equation`.)
However it is difficult to undertake a general analysis of the solutions this way. Instead we introduce a time t and use (3) to write

$$\frac{\mathrm{d}X}{\mathrm{d}t} = AX + BY, \quad \frac{\mathrm{d}Y}{\mathrm{d}t} = CX + DY, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}, \qquad \dot{\mathbf{u}} = \mathbf{J}\mathbf{u} \tag{5}$$

with $\mathbf{u} = (X, Y)^T$. We will present two analyses of this system.

As a single second order equation, using brute force

Eliminating $X(t)$ from (5) in favour of $Y(t)$ gives

$$\ddot{Y} = C\dot{X} + D\dot{Y} = C(AX + BY) + D\dot{Y} = A(\dot{Y} - DY) + CBY + D\dot{Y}$$
$$\Rightarrow \quad \ddot{Y} - (A + D)\dot{Y} + (AD - BC)Y = 0. \tag{6}$$

The same equation is derived for X upon eliminating Y in a similar fashion. Note that $A + D = \operatorname{tr}\mathbf{J} = -p$, say and $AD - BC = \det\mathbf{J} = q$, the trace and determinant of $\mathbf{J}$. The auxiliary equation for (6) is

$$\lambda^2 + p\lambda + q = 0, \quad p = -(A + D), \quad q = AD - BC \Rightarrow \lambda = \lambda_{1,2} = (-p \pm \sqrt{p^2 - 4q})/2. \tag{7}$$

This gives

$$Y(t) = \alpha e^{\lambda_1 t} + \beta e^{\lambda_2 t}.$$

This contains two arbitrary constants, which is all we would expect as our original system is a pair of first-order equations. The solution for $X(t)$ can be found corresponding to this $Y(t)$. From (5)

$$\dot{X} - AX = BY \quad \Rightarrow \quad X(t) = B\left(\frac{\alpha e^{\lambda_1 t}}{\lambda_1 - A} + \frac{\beta e^{\lambda_2 t}}{\lambda_2 - A}\right) + \gamma e^{At},$$

but this solution must be consistent with

$$\dot{Y} - DY = \alpha(\lambda_1 - D)e^{\lambda_1 t} + \beta(\lambda_2 - D)e^{\lambda_2 t} = CX = CB\left(\frac{\alpha e^{\lambda_1 t}}{\lambda_1 - A} + \frac{\beta e^{\lambda_2 t}}{\lambda_2 - A}\right) + C\gamma e^{At},$$

which requires, firstly,

$$\gamma = 0,$$

and also

$$(\lambda_{1,2} - A)(\lambda_{1,2} - D) = CB \quad \text{i.e.} \quad \lambda_{1,2}^2 - (A+D)\lambda_{1,2} + (AD - CB) = 0,$$

which we know is true. Hence we have expressions for $X(t)$, $Y(t)$ which we can use the arbitrainess in α and β to write as

$$X(t) = r_1 e^{\lambda_1 t} + r_2 e^{\lambda_2 t}, \quad Y(t) = s_1 e^{\lambda_1 t} + s_2 e^{\lambda_2 t}, \quad \frac{s_1}{r_1} = \frac{\lambda_1 - A}{B} = \frac{C}{\lambda_1 - D}, \quad \frac{s_2}{r_2} = \frac{\lambda_2 - A}{B} = \frac{C}{\lambda_2 - D}. \tag{8}$$

There are two arbitrary constants since, for example choosing r_1 and r_2 fixes s_1 and s_2. These constants determine which trajectory the solution (8) describes in the vicinity of the critical point - we can pick a particular point that the trajectory passes through by, for example evaluating (8) at $t = 0$. We also have an expression for $\frac{dY}{dX}$,

$$\frac{dY}{dX} = \frac{\dot{Y}}{\dot{X}} = \frac{\lambda_1 s_1 e^{\lambda_1 t} + \lambda_2 s_2 e^{\lambda_2 t}}{\lambda_1 r_1 e^{\lambda_1 t} + \lambda_2 r_2 e^{\lambda_2 t}}. \tag{9}$$

The behaviour of the solution depends on the values of $\lambda_{1,2}$ and hence on p and q.

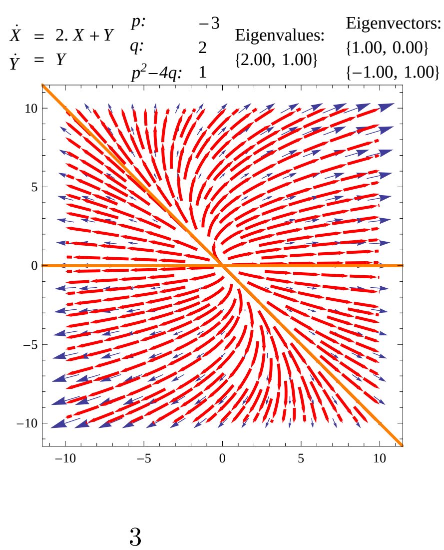
1. If $q > 0$, so that, if real, $\sqrt{p^2 - 4q} < p$
   (a) $q > 0$, $p^2 > 4q$. Here λ_1 and λ_2 are both real. Since $\lambda_1 > \lambda_2$, as $t \to \infty$ $e^{\lambda_1 t} >> e^{\lambda_2 t}$, whereas as $t \to -\infty$, $e^{\lambda_1 t} << e^{\lambda_2 t}$.
      i. $q > 0$, $p^2 > 4q$, $p > 0$. Here $\lambda_2 < \lambda_1 < 0$

$$\text{As } t \to \infty, \quad X \to 0, \quad Y \to 0, \quad Y \approx (s_1/r_1)X.$$

$$\text{As } t \to -\infty, \quad X \to \infty, \quad Y \to \infty, \quad Y \approx (s_2/r_2)X.$$

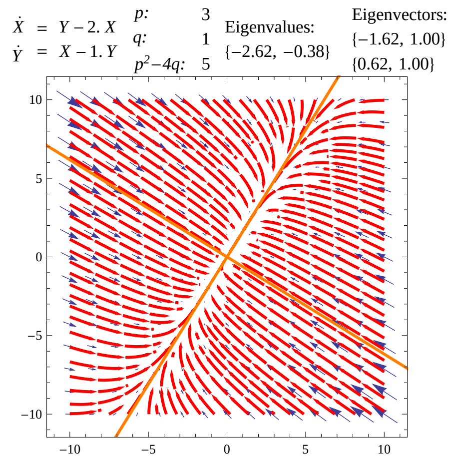
There are special trajectories that are straight lines in the vicinity of the critical point. These are generated by the choices

$$r_1 = s_1 = 0, \quad Y = (s_2/r_2)X, \qquad r_2 = s_2 = 0, \quad Y = (s_1/r_1)X$$



All the trajectories pass through $(0,0)$ and such a point is called a ***stable node***. Note that the straight lines (not shown) $Y = -2X$ and $Y = 0$ delineate regions of increasing/decreasing X and increasing/decreasing Y respectively. The straight lines shown are the special trajectories which are exactly straight lines.

ii. $q > 0$, $p^2 > 4q$, $p < 0$. Here $0 < \lambda_2 < \lambda_1$. The qualitative solution is as above, but with the effects of the limits $t \to \infty$ and $t \to -\infty$ interchanged as the values of λ have changed sign.

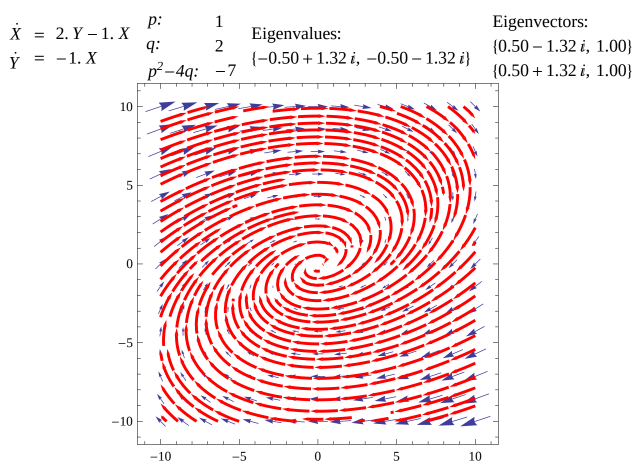


This is known as an ***unstable node***. Again look for the change of direction of the trajectories along $Y = 2X$ and $Y = X$, again not shown.

(b) $q > 0$, $p^2 < 4q$, $p > 0$. In this case the roots are complex, with negative real part. If we write $\lambda_{1,2} = -\mu_1 \pm i\mu_2$, $\mu_{1,2} > 0$. Instead of the exponential solutions given in (8) we have the solutions

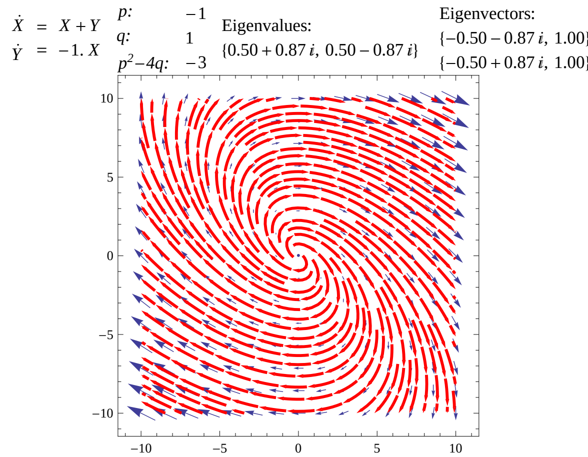
$$X(t) = k_1 \mathrm{e}^{-\mu_1 t} \cos(\mu_2 t + \epsilon_1), \quad Y(t) = k_2 \mathrm{e}^{-\mu_1 t} \cos(\mu_2 t + \epsilon_2).$$

As before, only two of the constants $k_{1,2}$ and $\epsilon_{1,2}$ can be independently chosen. It is clear that the trajectories are spiral, spiraling in towards the origin $(0,0)$ - as t is increased by a value $2\pi/\mu_2$, both X and Y are multiplied by the same factor $\mathrm{e}^{-2\pi\mu_1/\mu_2}$.



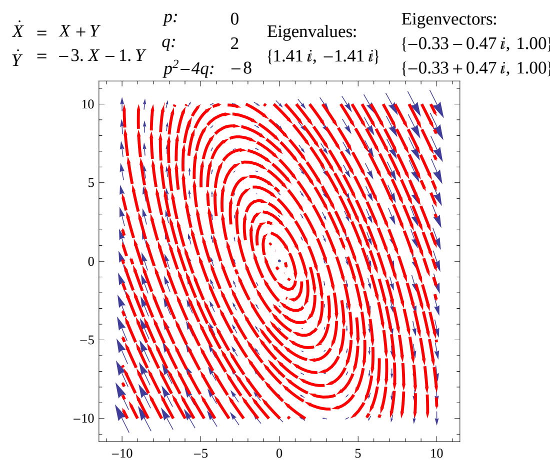
All trajectories approach the origin. The singular point is known as a ***stable spiral point or focus***.

(c) $q > 0$, $p^2 < 4q$, $p < 0$. This case again has imaginary roots, but with a positive real part.



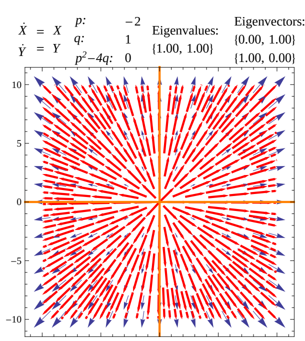
All trajectories depart from the origin. The singular point is known as a ***unstable spiral point or focus***.

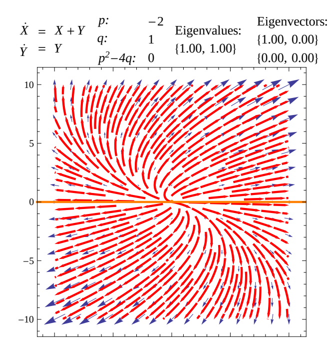
(d) $q > 0$, $p = 0$. This case again has purely imaginary roots, $\mu_1 = 0$ and the trajectories are circles/ellipses. No trajectories pass through $(0, 0)$ except for the trajectory consisting of a single point at $(0, 0)$



The critical point is called a ***centre***. Again it is illustrative to pick out the lines $Y = -3X$ and $Y = -X$ and note that the individual trajectories have turning points on these lines.

(e) $q > 0$, $p^2 = 4q$, $p > 0$. This corresponds to two equal negative roots for λ. The trajectories still form an stable node. However this can be of two types known as a firstly a ***star*** and secondly an ***improper node***. They are indistinguishable simply using the values of p and q



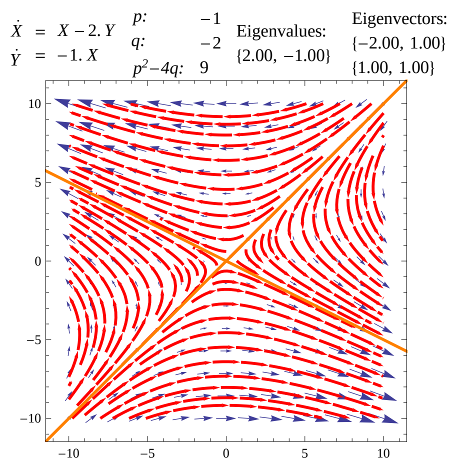


(f) $q > 0$, $p^2 = 4q$, $p < 0$. This corresponds to two equal positive roots for λ. The trajectories form an unstable node, which may be of star type.

2. $q < 0$ so that $\sqrt{p^2 - 4q}$ is real but $\sqrt{p^2 - 4q} > p$ and the roots differ in sign. Here $\lambda_2 < 0 < \lambda_1$

$$\text{As } t \to -\infty, \quad X \approx r_2 \mathrm{e}^{\lambda_2 t} \to \infty \text{ (in modulus)}, \quad Y \approx s_2 \mathrm{e}^{\lambda_2 t} \to \infty \text{ (in modulus)}, \quad Y \approx (s_2/r_2)X.$$

$$\text{As } t \to \infty, \quad X \approx r_1 e^{\lambda_1 t} \to \infty \text{ (in modulus)}, \quad Y \approx s_1 e^{\lambda_1 t} \to \infty \text{ (in modulus)}, \quad Y \approx (s_1/r_1)X.$$

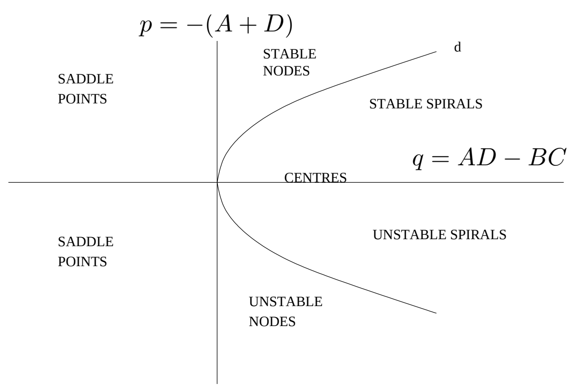


Only the two special straight line trajectories pass through $(0,0)$. The others approach the critical point, from the direction of one of these straight lines and leave the critical point in the direction of the other. The critical point is known as a ***saddle point***. A change in the sign of p interchanges the roles of λ_1 and λ_2 as before.

The figures above have all been generated with the following *Mathematica* commands, varying the coefficients of the matrix `m`.

```
m = {{1,1},{0,1}};{{a,b},{c,d}}=m;p=-(a+d);q=ad-bc;disc=p^2-4q;
Show[VectorPlot[m.{x,y},{x,-10,10},{y,-10,10},StreamPoints->Fine,StreamStyle->{Red,Thick},
ImageSize->{460,310}],Graphics[{Thick,Orange,Map[Line[{-100 #, 100 #}]&,
Select[Eigenvectors[m],(Im[#[[1]]]==0&&Im[#[[2]]]==0)&]]}],
PlotLabel->Row[{Column[{Row[{Column[{Style["\!\(\*OverscriptBox[\"X\",\".\"]\)",Italic],
Style["\!\(\*OverscriptBox[\"Y\", \".\"]\)", Italic]}],Column[{" = ", " = "}],
TableForm[m.{Style["X", Italic], Style["Y", Italic]}//N}]}],"  ",
Column[{Style["p:",Italic],Style["q:",Italic],Style["\!\(\*SuperscriptBox[\"p\", \"2\"]\)-4q:", Italic]}], "  ",Column[{p, q, disc}], "  ",
Column[{"Eigenvalues:",NumberForm[Chop@N@Eigenvalues[m],{4, 2}]}],"  ", ,
Column[{"Eigenvectors:",NumberForm[Chop@N@Eigenvectors[m][[1]],{4, 2}], NumberForm[Chop@N@Eigenvectors[m][[2]], {4, 2}] }]}]]
```

We can summarise what we have found with this diagram



As a first order matrix/vector equation

Equation (5) is $\dot{\mathbf{u}} = \mathbf{J}\mathbf{u}$ for $\mathbf{u}(t)$ with $\mathbf{J}$ a constant matrix. Comparison with a differential equation of the form $\dot{x} = ax$, with solution $x(t) = Ae^{at}$, with A and a constant, suggests we try the solution $\mathbf{u} = \mathbf{v}e^{\lambda t}$. Direct substitution leads to $\lambda \mathbf{v} e^{\lambda t} = \mathbf{J}\mathbf{v}e^{\lambda t}$ or $\lambda \mathbf{v} = \mathbf{J}\mathbf{v}$ so that λ is an eigenvalue of $\mathbf{J}$ and $\mathbf{v}$ the corresponding eigenvector. The general solution is a sum over the possible eigenvalue/eigenvector pairs. The matrix $\mathbf{J}$ is 2×2 so there are a maximum of two and, if they are real, distinct and non-zero, $\lambda_{1,2}$ say,

$$\mathbf{u}(t) = A_1 \mathbf{v}_1 e^{\lambda_1 t} + A_2 \mathbf{v}_2 e^{\lambda_2 t}.$$

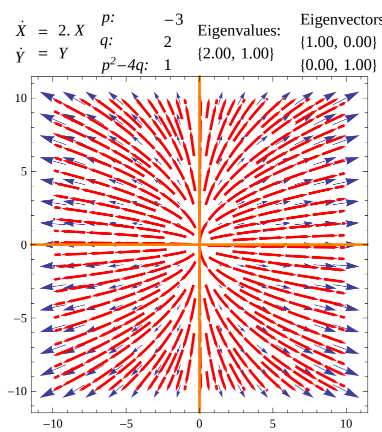
As above we have two degrees of freedom in this solution and $A_{1,2}$ can be found to specify a particular trajectory uniquely. As the eigenvalues are real, distinct and non-zero, then we know the eigenvectors are independent. If we form the matrix $\mathbf{P} = (\mathbf{v}_1, \mathbf{v}_2)$ with the eigenvectors as columns then the transformation to the new variables $(\bar{X}, \bar{Y})$

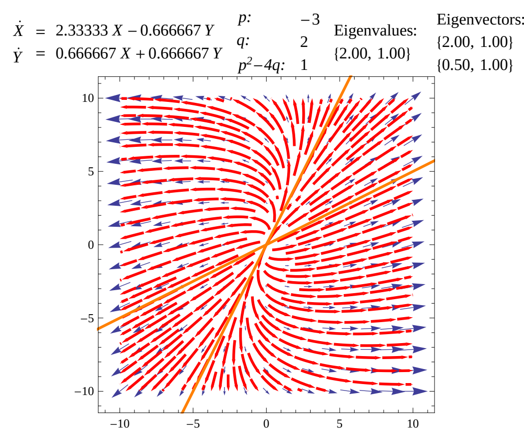
rather than (X,Y) through the definition $\mathbf{u} = \mathbf{P}\bar{\mathbf{u}}$, $\bar{\mathbf{u}} = \mathbf{P}^{-1}\mathbf{u}$, with $\bar{\mathbf{u}} = (\bar{X},\bar{Y})^T$. Also, as $\mathbf{P}$ has columns made of the eigenvectors of $\mathbf{J}$, $\mathbf{JP} = (\lambda_1\mathbf{v}_1, \lambda_2\mathbf{v}_2) = \mathbf{\Lambda}(\mathbf{v}_1,\mathbf{v}_2) = \mathbf{\Lambda P}$, where $\mathbf{\Lambda}$ is a diagonal matrix diag(λ_1,λ_2) with the eigenvalues of $\mathbf{J}$ along its diagonal. We therefore have $\mathbf{J} = \mathbf{P\Lambda P}^{-1}$, or $\mathbf{\Lambda} = \mathbf{P}^{-1}\mathbf{JP}$. (These are standard results on the diagonalisation of matrices.) Therefore

$$\dot{\mathbf{u}} = \mathbf{Ju} = \mathbf{P\Lambda P}^{-1}\mathbf{u}, \quad \Rightarrow \quad \mathbf{P}^{-1}\dot{\mathbf{u}} = \mathbf{\Lambda P}^{-1}\mathbf{u}, \quad \Rightarrow \quad \dot{\bar{\mathbf{u}}} = \mathbf{\Lambda}\bar{\mathbf{u}}, \quad \Rightarrow \quad \dot{\bar{X}} = \lambda_1\bar{X}, \quad \dot{\bar{Y}} = \lambda_2\bar{Y} \quad \Rightarrow$$
$$\bar{X}(t) = \bar{X}_0 \mathrm{e}^{\lambda_1 t}, \quad \bar{Y}(t) = \bar{Y}_0 \mathrm{e}^{\lambda_2 t} \text{ and, eliminating } t, \quad \bar{Y} = C\bar{X}^a, \quad a = \lambda_2/\lambda_1 \tag{10}$$

1. Real, positive eigenvalues. Here, a in (10) is positive. All trajectories pass through $(\bar{X},\bar{Y}) = (0,0)$ (and so the critical point $(X,Y) = (0,0)$. We have an ***unstable node*** as $\lambda_{1,2}$ are positive so $\bar{X}$ and $\bar{Y}$ (and so (X,Y)) tend to infinity as $t \to \infty$. If $a > 1$, i.e. $\lambda_2 > \lambda_1$, then the trajectories have the character of $\bar{Y} = \pm\bar{X}^2$, but if $a < 1$, $\lambda_2 < \lambda_1$, the roles of $\bar{X}$ and $\bar{Y}$ are interchanged with the trajectories looking more like $\pm\bar{Y} = \sqrt{|\bar{X}|}$. This is in terms of the new coordinates. The trajectories in the original (X,Y) coordinates are similar in character but "skewed" so that the $\bar{X}$ and $\bar{Y}$ axes correspond to lines in the (X,Y) plane that point along the eigenvectors of $\mathbf{J}$.

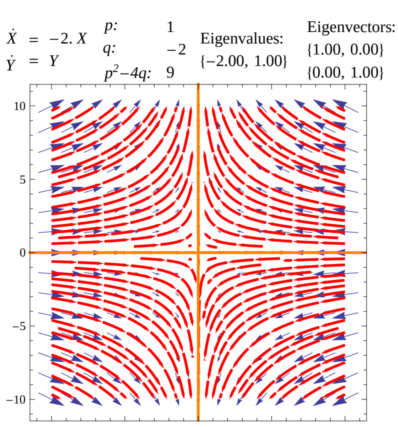
   Choose $\lambda_{1,2} = 2, 1, \quad a = \dfrac{1}{2}, \quad \mathbf{\Lambda} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. Choose $\mathbf{v}_{1,2} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, giving $\mathbf{P} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \mathbf{P}^{-1} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$
   $$\mathbf{J} = \mathbf{P\Lambda P}^{-1} = \begin{pmatrix} \frac{7}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} \end{pmatrix} \tag{11}$$

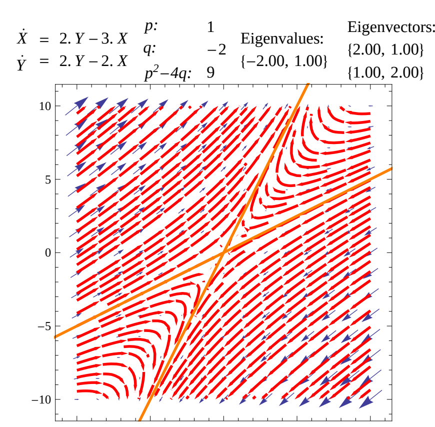




2. Real, negative eigenvalues. This is the same situation as above, but with the direction of t reversed - ***a stable node***.

3. Real eigenvalues, one positive and one negative. Here a is negative and the trajectories generally do not pass through $(X,Y) = (0,0)$. Also as $t \to \infty$ only one of $\bar{X}$ or $\bar{Y}$ approaches zero. The other approaches ∞. As $t- \to -\infty$ the roles are reversed. We have a ***saddle point***.





26/10/12

$\dot{\underline{x}} = \underline{\underline{J}}\,\underline{x}$.

1) Eigenvalues of $\underline{\underline{J}}$ real and of same sign

i) node. $\lambda > 0$ unstable

ii) $\lambda < 0$ stable.

iii) e.v of J real and different sign.
 - saddle

iv) e.v equal - star or improper node

v) Imaginary eigenvalues.

Spiral point

Unstable if real part > 0

Stable " " " < 0 or

centre if real part $= 0$.

Let the eigenvalues be $\lambda = \alpha \pm i\beta$.

$$\underline{x} = e^{\alpha t}\left(A_1 \underline{u}_1 e^{i\beta t} + A_2 \underline{u}_2 e^{-i\beta t}\right)$$

eigenvectors $\underline{u}_1$ and $\underline{u}_2$ are likely to be complex.

To diagonlise, write $\underline{x} = \underline{\underline{P}}\,\bar{\underline{x}}$ where

$\underline{\underline{P}} = (\mathrm{Im}(\underline{u}_1), \mathrm{Re}(\underline{u}_2))$. This means $\underline{\underline{J}}\,\underline{\underline{P}}$

$= (\mathrm{Im}(\underline{\underline{J}}\,\underline{u}_1), \mathrm{Re}(\underline{\underline{J}}\,\underline{u}_2)) = (\mathrm{Im}(\lambda \underline{u}_1), \mathrm{Re}(\lambda \underline{u}_2$

$= (\alpha \mathrm{Im}(\underline{u}_1) + \beta \mathrm{Re}(\underline{u}_1), \alpha \mathrm{Re}(\underline{u}_1) - \beta \mathrm{Im}(\underline{u}_1))$

$\lambda = \alpha + i\beta$

$$\ldots = (\mathrm{Im}(\underline{u}_1), \mathrm{Re}(\underline{u}_1)) \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

so $$\underline{\underline{J}}\,\underline{\underline{P}} = \underline{\underline{P}} \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

or $$\underline{\underline{P}}^{-1}\underline{\underline{J}}\,\underline{\underline{P}} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

$$\dot{\underline{x}} = \underline{\underline{J}}\,\underline{x} = \underline{\underline{P}} \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \underline{\underline{P}}^{-1} \underline{x}$$

$$\Rightarrow \dot{\underline{\bar{x}}} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \underline{\bar{x}}$$

$$\underline{\bar{x}} = \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}, \quad \begin{aligned} \dot{\bar{x}} &= \alpha \bar{x} - \beta \bar{y} \\ \dot{\bar{y}} &= \beta \bar{x} + \alpha \bar{y} . \end{aligned}$$

$$\begin{aligned} \Rightarrow \ddot{\bar{x}} &= \alpha \dot{\bar{x}} - \beta \dot{\bar{y}} \\ &= \alpha \dot{\bar{x}} - \beta(\beta \bar{x} - \alpha \bar{y}) \\ &= \alpha \dot{\bar{x}} - \beta^2 \bar{x} + \alpha \dot{\bar{x}} - \alpha^2 \bar{x} \end{aligned}$$

$$\ddot{\bar{x}} - 2\alpha \dot{\bar{x}} + (\alpha^2 + \beta^2)\bar{x} = 0 .$$

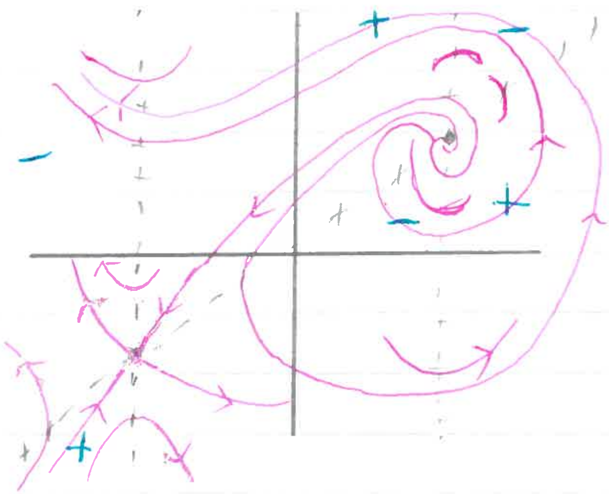
$$\bar{x} = \bar{x}_0 e^{\varphi t}, \quad \varphi^2 - 2\alpha\varphi + (\alpha^2 + \beta^2) = 0 .$$

$$\varphi = \alpha \pm i\beta .$$

$$\bar{x} = \underline{e^{\alpha t}}(A\cos\beta t + B\sin\beta t).$$

Stable if $\alpha < 0$, centre if $\alpha = 0$.
Unstable if $\alpha > 0$

__Example__: $\dfrac{dy}{dx} = \dfrac{x^2-1}{x-y}$, consider $dy/dt = x^2 - 1$
$dx/dt = x - y$.



Horizontal nullclines are at $x^2 - 1 = 0$ ($dy=$
$x = \pm 1$

Vertical nullclines are given by $x - y = 0$ ($dx=0$,
$y = x$.

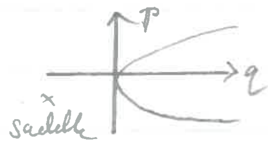
Critical points are where these nullclines cross i.e at $(1,1)$ and $(-1,-1)$.

Near what looks like a saddle, $(-1,-1)$. $x = -1 + X$,
$y = -1 + Y$.

$$\frac{dY}{dX} = \frac{Q_x X + Q_y Y}{P_x X + P_y Y} = \frac{\overbrace{2(-1)}^{2x} X}{\underbrace{1}_{P_x} X - \underbrace{1}_{P_y} Y} = \frac{-2x}{X - Y}.$$

$$\underline{\underline{J}} = \begin{pmatrix} P_x & P_y \\ Q_x & Q_y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix}$$

$p = -tr(J) = -1$
$q = det(J) = -2$



Eigenvalues satisfy $(1-\lambda)(-\lambda) - (-1)(-2) = 0$

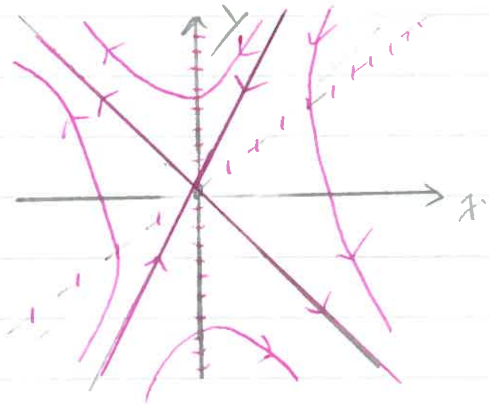
$$\Rightarrow \lambda^2 - \lambda - 2 = 0$$
$$\Rightarrow (\lambda - 2)(\lambda + 1) = 0.$$
$$\Rightarrow \lambda = 2, -1.$$

different signs
$\Rightarrow$ saddle.

Or consider $\frac{dy}{dx} = \frac{-2x}{x-y}$

and look for solution $Y = mX$.

$$m = \frac{-2}{1-m} \Rightarrow m = 2, m = -1.$$



Near $(1,1)$ the Jacobian is

$$\begin{pmatrix} P_x & P_y \\ Q_x & Q_y \end{pmatrix}\Bigg|_{(1,1)}$$

$$= \begin{pmatrix} 1 & -1 \\ 2x & 0 \end{pmatrix}\Bigg|_{(1,1)}.$$

$p = -1$ $\qquad p^2 - 4q = 1 - 8$
$q = 2$ $\qquad\qquad\qquad = -7$
$\qquad\qquad\qquad\qquad < 0$
$\qquad\qquad\qquad \Rightarrow$ spiral point.

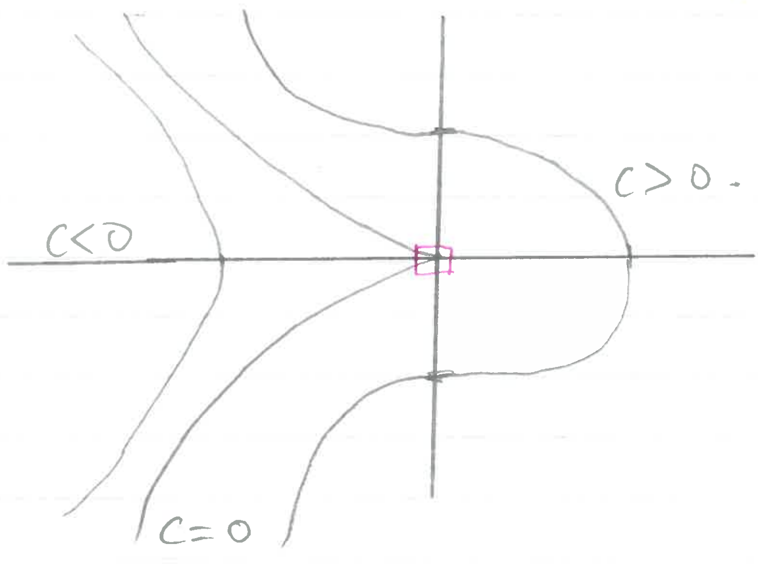
Lets consider the equation for $|x| >> 1$ and $|y| >> 1$

$$\frac{dy}{dx} = \frac{x^2}{-y} \quad \text{if} \quad |y| >> |x| >> 1$$

$$\Rightarrow -y^2/2 = x^3/3 + \text{Constant}.$$

$$\Rightarrow \frac{y^2}{2} = C - \frac{x^3}{3}, \quad \text{i.e.} \quad y \approx x^{3/2}$$

as required by $|y| >> |x| >> 1$



"Application" to population dynamics.

Imagine a population of rabbits ^{r} / foxes ^{f} or of rabbits ^{r} / sheep. ^{s}

The rate of growth of these populations would be proportional to the birthrate − deathrate.

birthrate: food supply, population size

deathrate: number of predators or food supply

This leads to equation of the type:

$$\frac{dx}{dt} = x(A + a_1 x + b_1 y) \qquad x \geq 0$$

$$\frac{dy}{dt} = y(B + b_2 x + a_2 y). \qquad y \geq 0.$$

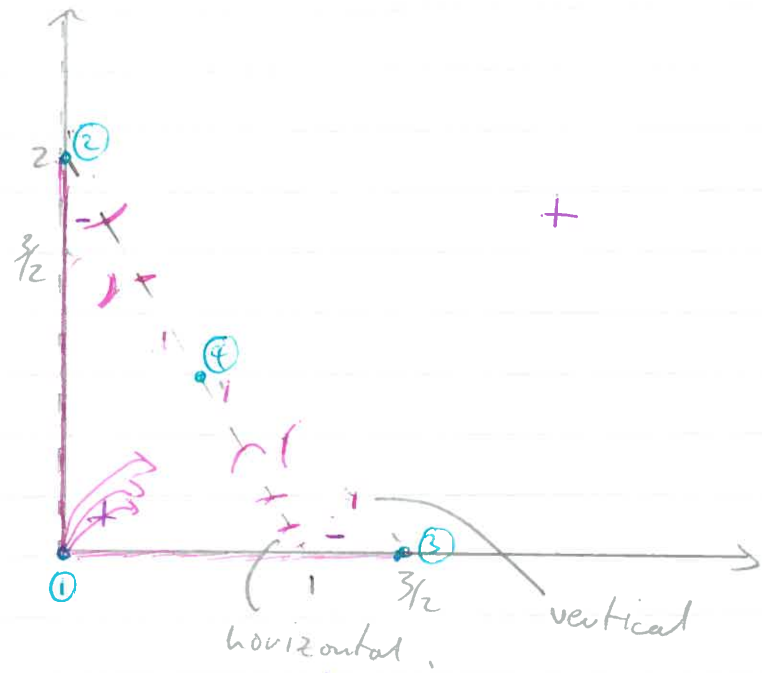
Consider:

$$\frac{dx}{dt} = x(3 - 2x - 2y) = P(x, y).$$

$$\frac{dy}{dt} = y(2 - 2x - y) = Q(x, y).$$

Vertically nullclines, $\frac{dx}{dt} = 0$, $x = 0$, $x = 3/2 - y$

Horizontal nullclines, $\frac{dy}{dx} = 0$, $y = 0$, $y = 2 - 2x$.



$$\frac{dy}{dx} = \left(\frac{x(+2x+2y)}{y(+2x+y)}\right)^{-1}$$

$$\underline{\underline{J}} = \begin{pmatrix} P_x & P_y \\ Q_x & Q_y \end{pmatrix} = \begin{pmatrix} 3-4x-2y & -2x \\ -2y & 2-2x-2y \end{pmatrix}$$

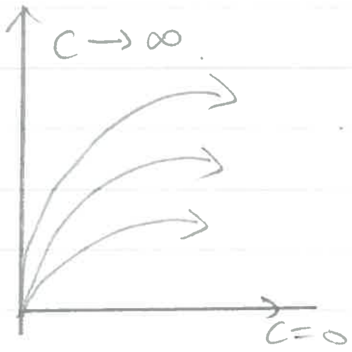
① $x=0$, $y=0$, $J = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$, eigenvalues 3 and 2 both +ve i.e we have an stable node.

Locally with $x = 0 + X$
$y = 0 + Y$.

and $\underline{X} = \begin{pmatrix} X \\ Y \end{pmatrix}$ then $\frac{d\underline{X}}{dt} = \underline{\underline{J}}\,\underline{X}$ so $\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}$

$$\frac{dX}{dt} = 3X, \quad \frac{dy}{dt} = 2Y.$$

So $\frac{dy}{dx} = \frac{2}{3}\frac{y}{x} \Rightarrow y = Cx^{2/3}$



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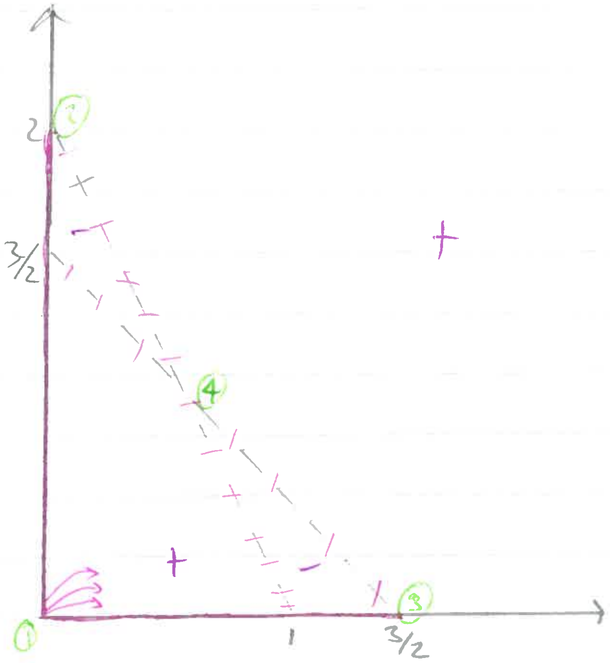
$$\frac{dx}{dt} = x(3-2x-2y) = P \qquad \underline{\underline{J}} = \begin{pmatrix} P_x & P_y \\ Q_x & Q_y \end{pmatrix}$$

$$\frac{dy}{dt} = y(2-2x-y) = Q \qquad = \begin{pmatrix} 3-4x-2y & -2x \\ -2y & 2-2x-2y \end{pmatrix}$$



1) $\underline{\underline{J}} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ $\qquad \frac{dY}{dX} = \frac{3Y}{3X} \iff Y = cX^{2/3}$

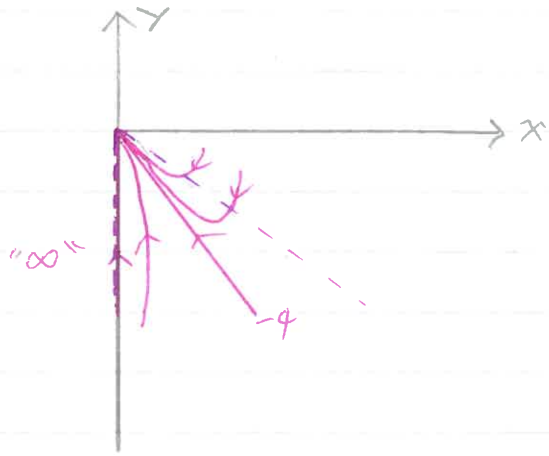
2) $\underline{\underline{J}} = \begin{pmatrix} -1 & 0 \\ -4 & -2 \end{pmatrix}$

two -ve eigenvalue -1 and -2.
$\Rightarrow$ stable node.

$$\frac{dX}{dt} = -X, \quad \frac{dY}{dt} = -4X-2Y.$$

$$\frac{dY}{dX} = 4 + \frac{2Y}{X}$$

Look for special solutions $Y = mX$: $m = 4 + 2m$.
$m = -4$
"$m = \infty$"



$$\frac{dY}{dX} - \frac{2Y}{X} = 4$$

$$\Rightarrow \frac{d}{dx}\left(\frac{Y}{X^2}\right) = \frac{4}{X^2}$$

$$\Rightarrow \frac{Y}{X^2} = -\frac{4}{X} + C.$$

So $Y = -4X + CX^2$.

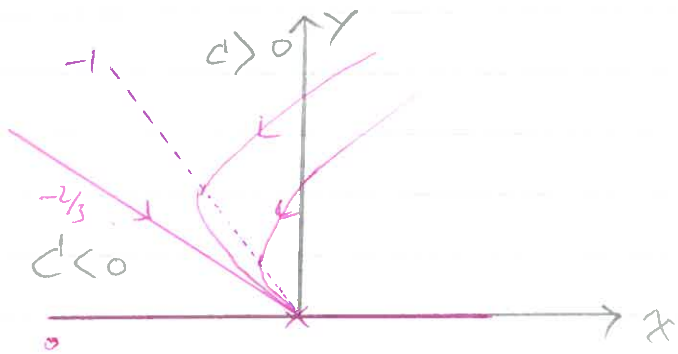
3) $y = 0$, $x = 3/2$.

$\underline{\underline{J}} = \begin{pmatrix} -3 & -3 \\ 0 & -1 \end{pmatrix}$ e.values are real -ve
-3 and -1. - stable node.

Locally $\frac{dX}{dt} = \underbrace{-3X - 3Y}_{\substack{\text{Vertical wall} \\ Y = -X}}$, $\frac{dY}{dt} = \underbrace{-Y}_{\substack{\text{horiz wall} \\ Y = 0}}$.

$$\frac{dY}{dX} = \frac{Y}{3(X+Y)} \quad \text{and if} \quad Y = mX.$$

$$m = \frac{m}{3(1+m)}, \text{ so } m = 0, \; m = -2/3.$$



$$\frac{dX}{dY} = 3 + 3\frac{X}{Y}$$

$$\frac{dX}{dY} - 3\frac{X}{Y} = 3.$$

$$\frac{d}{dY}\left[\frac{X}{Y^3}\right] = \frac{3}{Y^3} \Rightarrow \frac{X}{Y^3} = \frac{-3}{2Y^2} + c$$

$$\Rightarrow X = \frac{-3Y}{2} + cY^3$$

4) $x = 1/2 \ , y = 1$

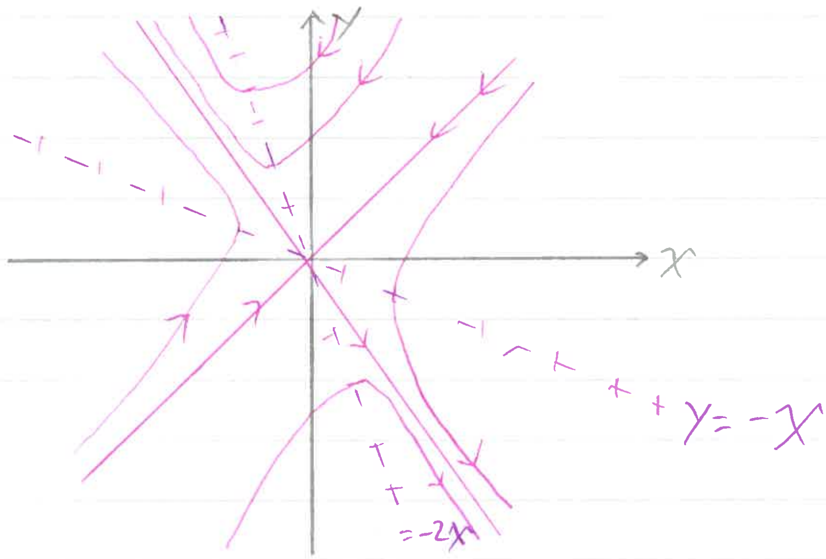
$$\underline{J} = \begin{pmatrix} -1 & -1 \\ -2 & -1 \end{pmatrix}$$

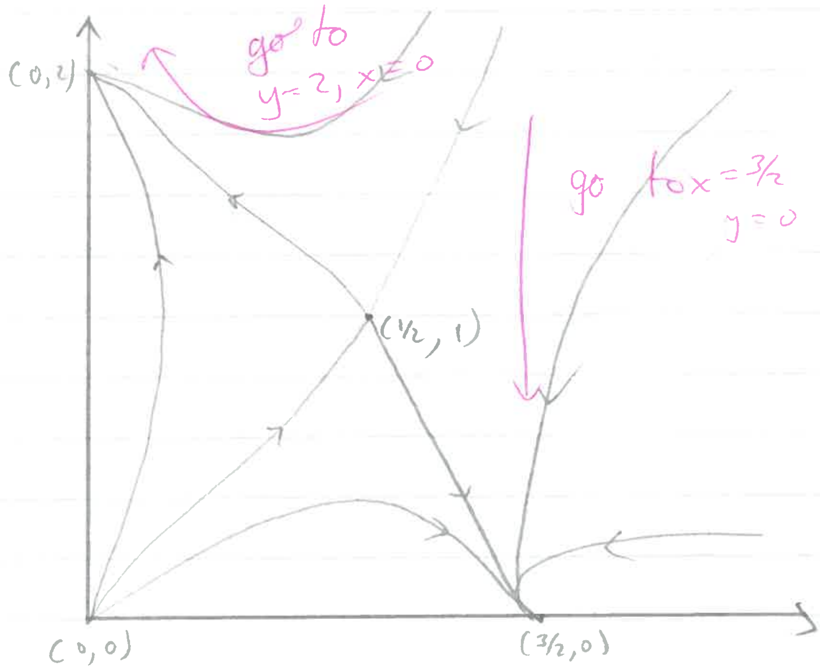
The eigenvalues λ satisfy $(-1-\lambda)^2 = 2 \Rightarrow$
$\Rightarrow \lambda = -1 \pm \sqrt{2}$. Real and different in sign $\Rightarrow$
saddle

$$\frac{dX}{dt} = -X - Y \ , \quad \frac{dY}{dt} = -2X - Y$$

$$\frac{dY}{dX} = \frac{2X+Y}{X+Y}.$$

and if $Y = mX$: $m = \dfrac{2+m}{1+m} \Rightarrow m = \pm\sqrt{2}$





(0,2)
go to
y=2, x=0
go to x=3/2
y=0
(1/2, 1)
(0,0)
(3/2,0)

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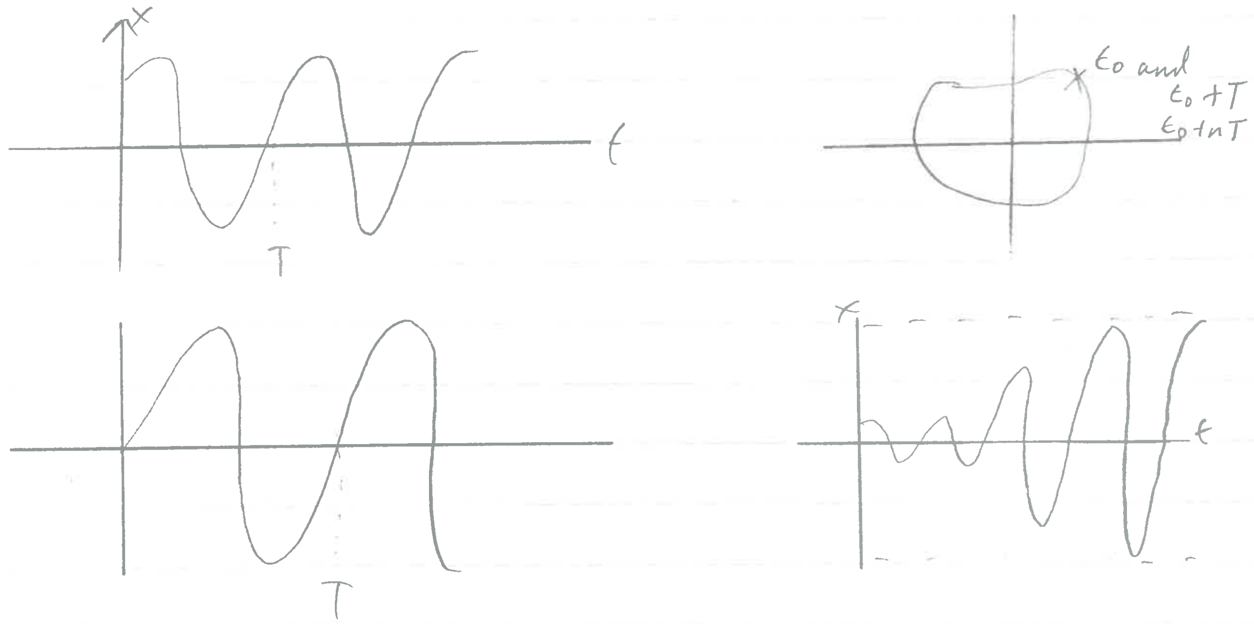
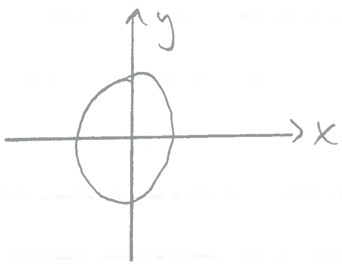
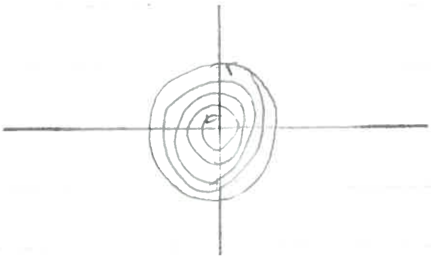
Periodic Solutions and limit cycles.

The pair of equations $dx/dt = P(x,y)$, $dy/dt = Q(x,y)$ may admit periodic solutions, or periodic solutions which emerge as $t \to \pm\infty$, known as limit cycles.



$$\ddot{x} = f(\dot{x}, x)$$
$$\dot{y} = f(y, x)$$
$$\dot{x} = y$$



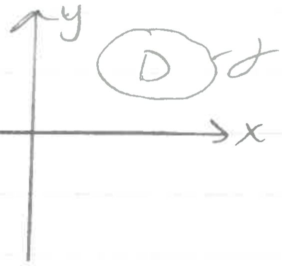


limit cycle

To evaluate the period T

$$T = \int dt = \int_\Gamma \frac{dx}{P} = \int_\Gamma \frac{dy}{Q} \qquad \left(\int_\Gamma \frac{dx}{y} \right)$$

Bendixson's negative criterion for a limit cycle or periodic solution.



Consider $\frac{dx}{dt} = P(x,y)$, $\frac{dy}{dt} = Q(x,y)$ and assume there exists a periodic solution given by curve γ with interior D in phase plane. Consider:

$$\int_D \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \, dx\, dy .$$

$$\begin{aligned} \dot{x} &= P \\ \dot{y} &= Q \end{aligned} \qquad \frac{d}{dx}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} P \\ Q \end{pmatrix} = \underline{F} \quad , \; P_x + Q_y = \operatorname{div} \underline{F} = \underline{\nabla} \cdot \underline{F}$$

By Stoke's theorem this is $\oint_\gamma P\, dy - Q\, dx$.

$$= \int_0^T P \frac{dy}{dt} - Q \frac{dx}{dt} \, dt$$

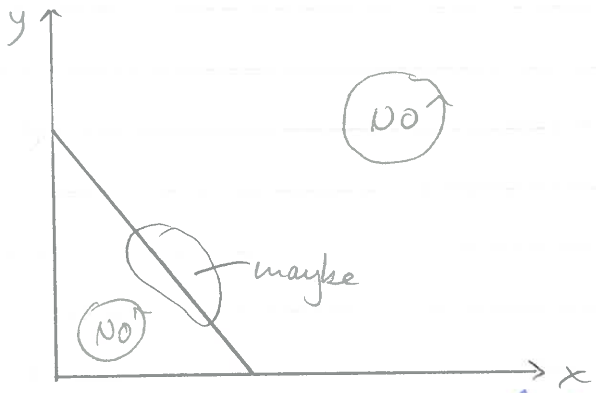
$$= \int_0^T (PQ - QP) \, dt = 0 .$$

So $P_x + Q_y$ must have regions inside D where it is +ve and regions where it is negative. So periodic solutions are impossible in regions of the phase plane where $P_x + Q_y$ is of a single sign.

Consider $\dfrac{dx}{dt} = x(3-2x-2y) = P.$

$$\frac{dy}{dt} = y(2-2x-y) = Q.$$

Then $P_x + Q_y = 3 - 4x - 2y + 2 - 2x - 2y = 5 - 6x - 4y$



zero on $y = \dfrac{5}{4} - \dfrac{3x}{2}.$

Periodic solutions are only possible if they straddle the line $y = \dfrac{5}{4} - \dfrac{3x}{2}.$

<u>Dulac's Extension of Bendixson's negative criterion.</u>

Consider $\iint_D \operatorname{div}(R\underline{F})\,dxdy$ for any function R. This is

$$\iint_D \frac{\partial}{\partial x} R\underline{P} + \frac{\partial}{\partial y} RQ \;dxdy.$$

$$= \oint_\gamma R\underline{P}\,dy - RQ\,dx$$

$$= \int_0^T R\underline{P}Q - RQ\underline{P}\,dt$$

$$= 0$$

$\dfrac{dy}{dt} = Q$

$\dfrac{dx}{dt} = \underline{P}.$

So if we can find any function $R(x,y)$ so that $\operatorname{div}(R\underline{F})$ is single signed in a region, then we can have no periodic solution within that region. Here, if we take $R = \frac{1}{xy}$ and so $R\underline{F} = \begin{pmatrix} \frac{3}{y} - \frac{2x}{y} - 2 \\ 2/x - 2 - y/x \end{pmatrix}$

and the divergence is $-\frac{2}{y} - \frac{1}{x} < 0$ $\quad x > 0,\ y > 0$

and periodic solution are impossible for $x > 0$, $y > 0$.

<u>Example: A limit cycle.</u>

Consider:

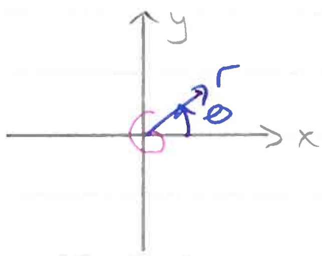
$$\frac{dy}{dx} = \frac{y - x - y(x^2+y^2)}{x+y-x(x^2+y^2)} \qquad \frac{dy}{dt} = y - x - y(x^2+y^2)$$

$$\frac{dx}{dt} = x + y - x(x^2+y^2).$$

Look for critical points requiring $\quad y - x = y(x^2+y^2)$
$\quad x + y = x(x^2+y^2)$

$$\Rightarrow \frac{y-x}{x+y} = \frac{y}{x} \quad \text{i.e} \quad yx - x^2 = yx + y^2 \quad \text{i.e} \quad -x^2 = y^2$$

i.e $x = y = 0$.



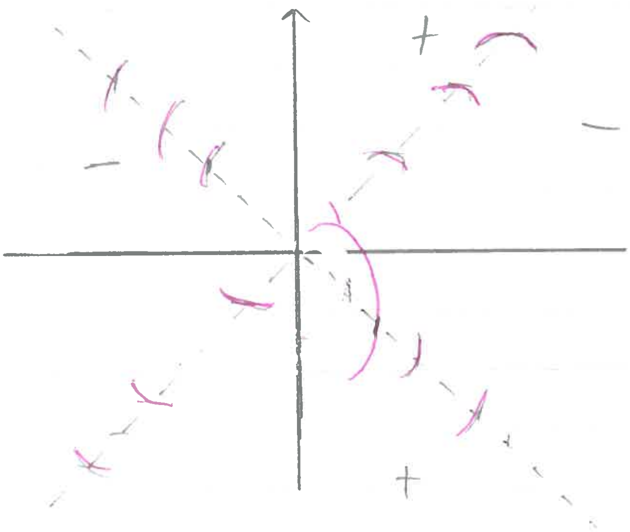
Linearising about $x = y = 0$.

$$\frac{dY}{dX} = \frac{Y - X}{X + Y}, \quad \underline{\underline{J}} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

To find the eigenvalues, λ, st $\det\begin{pmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{pmatrix} = 0$.

$\Rightarrow (1-\lambda)^2 = -1 \Rightarrow \lambda = 1 \pm i$.

which are complex with +ve real part and the origin is therefore an unstable spiral point.



We can swich to polar coordinates to describe a position in the phase plane, $r^2 = x^2+y^2$, $\theta = \tan^{-1} \frac{y}{x}$ and if $\dot{x} = P$, $\dot{y} = Q$.

$$\Rightarrow \dot{r} = \frac{xP + yQ}{r}.$$

$$\dot{\theta} = \frac{1}{1+y^2/x^2} \cdot \left(\frac{\dot{y}}{x} - \frac{\dot{x}y}{x^2} \right)$$

$$= \frac{x\dot{y} - y\dot{x}}{x^2+y^2} = \frac{xQ - yP}{r^2}$$

$$\frac{dy}{dt} = \underbrace{y - x - y(x^2+y^2)}_{Q}$$

$$\frac{dx}{dt} = \underbrace{x+y - x(x^2+y^2)}_{P}$$

$$\frac{dr}{d\theta} = \frac{\dot{r}}{\dot{\theta}} = \frac{r(xP+yQ)}{(xQ-yP)}$$

and here this approach gives $\frac{dr}{dt} = \frac{1}{r}\left(x^2 + \cancel{xy}^{\,0} - x^2(x^2+y^2) + y^2 - \cancel{xy}^{\,0} - y^2(x^2+y^2) \right)$

$$= r - r^3$$

$$\frac{d\theta}{dt} = \frac{1}{r^2}\left(\cancel{xy} - x^2 - xy(x^2+y^2) - \cancel{xy} - y^2 + xy(x^2+y^2) \right) = -1$$

OR use complex numbers.

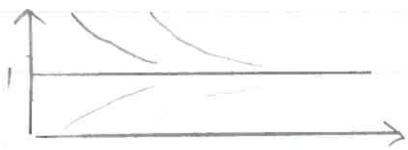
$$\frac{d}{dt}(\underbrace{x+iy}_{z}) = x+iy - i(x+iy) - (x+iy)(x^2+y^2)$$

$$\frac{dz}{dt} = (1-i)z - z|z|^2$$

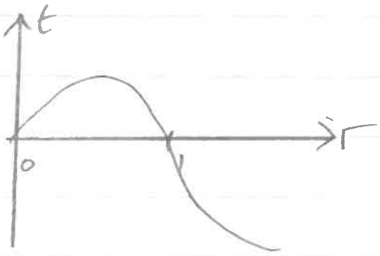
and now proceed to polar form writing $z = re^{i\theta}$.

$$\frac{dz}{dt} = \dot{r}\cancel{e^{i\theta}} + i r\dot{\theta}\cancel{e^{i\theta}} = (1-i)r\cancel{e^{i\theta}} - r\cancel{e^{i\theta}}r^2$$

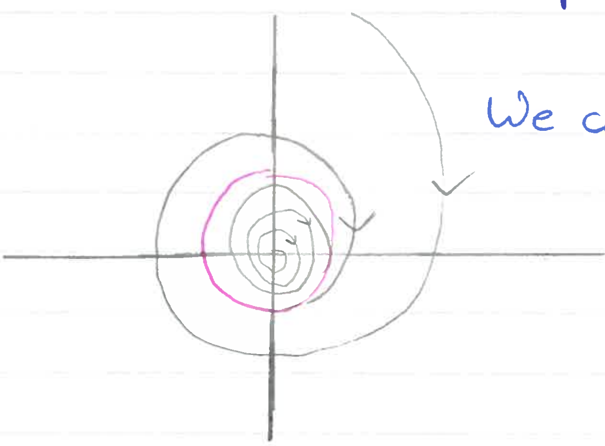
$$\Rightarrow \quad \dot{r} = r - r^3, \quad r\dot{\theta} = -r \quad \text{i.e } \dot{\theta} = -1$$



$r \to 1$ as $t \to \infty$.



$r = 1$ is a limit cycle.



We can slove for $r(\theta)$ exactly

$$\frac{dr}{d\theta} = \frac{\dot{r}}{\dot{\theta}} = r^3 - r \Rightarrow \int \frac{dr}{r^3 - r} = \int d\theta$$

$$\frac{A}{r} + \frac{B}{r+1} + \frac{C}{r-1}$$

OR : If $u = r^2$, then $\frac{du}{d\theta} = 2r\frac{dr}{d\theta} = 2r^4 - 2r^2$

$$= 2u(u-1)$$

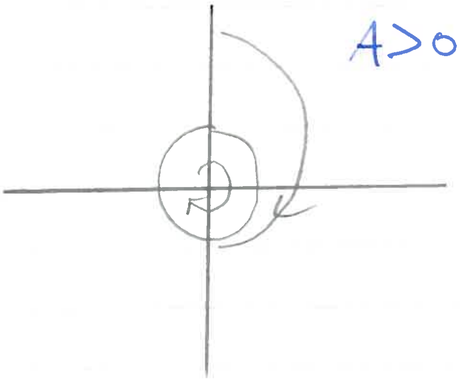
$$\int \frac{du}{u(u-1)} = \int 2 \, d\theta .$$

$$\frac{A}{u} + \frac{B}{u-1}$$

$$2\theta = \ln\left(\frac{u-1}{u}\right) + \text{const} .$$

$$\frac{r^2-1}{r^2} = Ae^{2\theta} , \quad r^2(1-Ae^{2\theta}) = 1$$

$$\Rightarrow r^2 = \frac{1}{1-Ae^{2\theta}}$$



$A>0$

$$= \frac{1}{1-Ae^{-2t}}$$

$\dot{\theta} = -1$

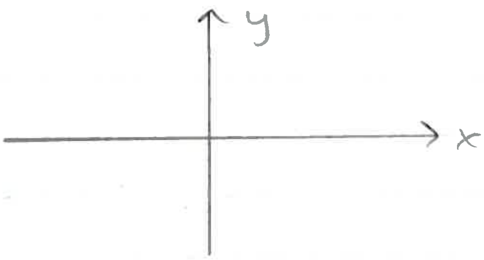
$\theta = \theta_0 - t$.

—/—

Poincare - Bendixson Theorem.

$\dot{x} = x - y - 2x(x^2+y^2)$

$\dot{y} = x + y - y(x^2+y^2)$

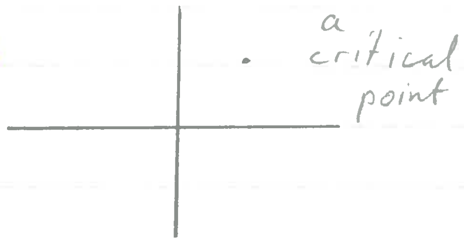


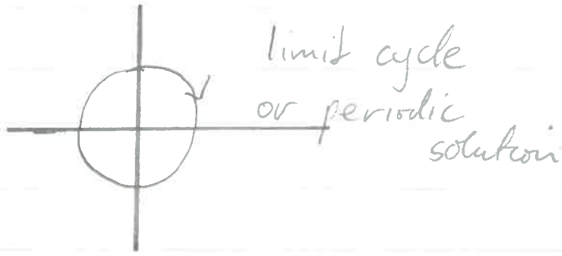
$\dot{y} = Q$

$\dot{x} = P$

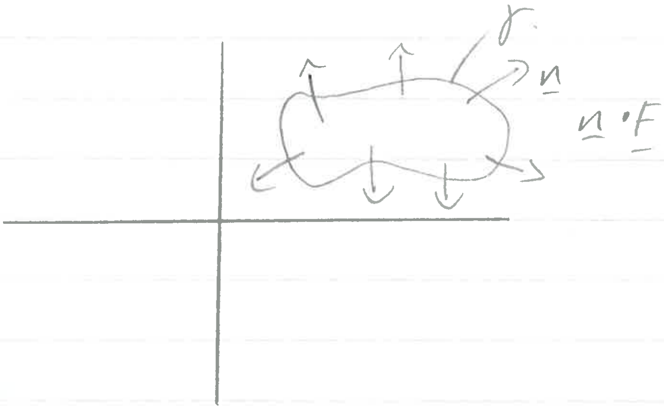
Definition: A ~~region~~ closed set of points in the phase plane is said to be (positive / negative) invariant, if a trajectory in the region at $t=0$ remains in the region for $t>0$, $t<0$.

e.g.:



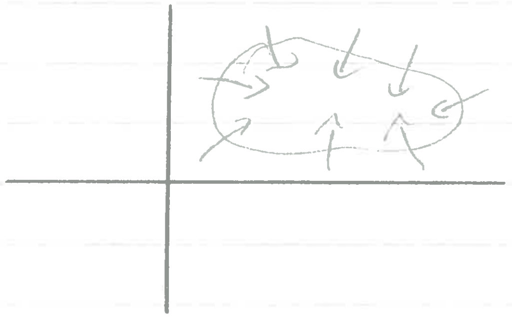


are both +ve / −ve invariant.



$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \underline{F} = \begin{pmatrix} P \\ Q \end{pmatrix}$$

$$\underline{n} \cdot \underline{F} = \underline{n} \cdot \frac{d\underline{x}}{dt}$$



$\underline{n} \cdot \underline{F} > 0$ on γ, the interior of γ is negatively invariant

$\underline{n} \cdot \underline{F} < 0$ on γ, the interior is positively invariant.

The Poincaré–Bendixson theorem states that if there exists a bounded invariant region of the phase plane with no equilibrium points then the region contain's at least one limit cycle.

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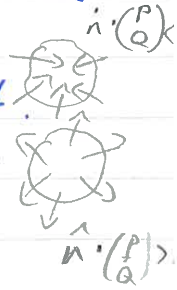
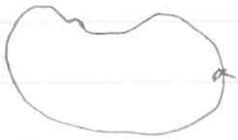
$\dot{x} = P, \quad \dot{y} = Q$

Invariant sets

"in at $t=0$ remains in for $t>0$" +vely invariant

" $t<0$ " -vely invariant



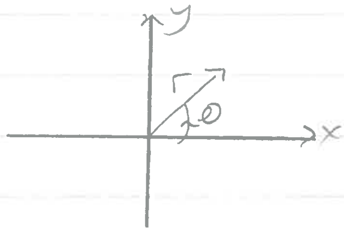


Poincare – Bendixson Theorem:

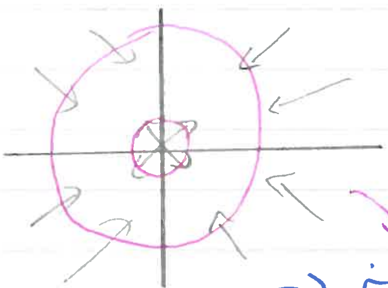
"If a bounded invariant set has no critical points then it contains a limit cycle".

Consider:

$$\dot{x} = x - y - 2x(x^2+y^2) = P$$
$$\dot{y} = x + y - y(x^2+y^2) = Q.$$



$$r^2 = x^2 + y^2 \Rightarrow r\dot{r} = xP + yQ.$$
$$= x(x-\cancel{y}) - 2x^2(x^2+y^2)$$
$$+ \cancel{yx} + y^2 - y^2(x^2+y^2)$$
$$= r^2(1 - 2x^2 - y^2)$$
$$= r^2(1 - 2r^2\cos^2\theta - r^2\sin^2\theta).$$

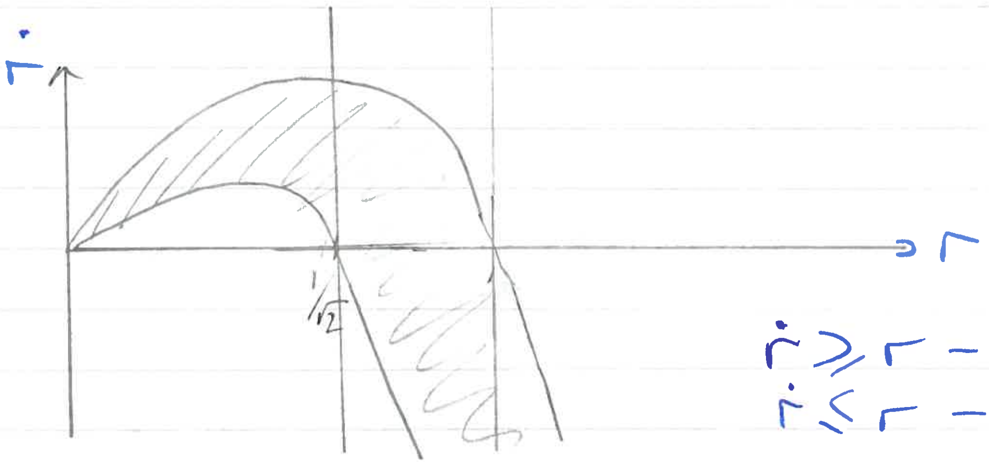


comes from

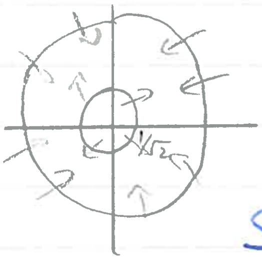
$$\Rightarrow \dot{r} = r - r^3(1+\cos^2\theta).$$

$$\theta = \tan^{-1} y/x \Rightarrow r^2\dot{\theta} = xQ - yP = x(x+y) - xy(x^2+y^2$$
$$- y(x-y) + 2x(x^2+y^2$$

$$\dot{\theta} = 1 + xy = 1 - r^2\sin\theta\cos\theta = 1 + r^2\tfrac{1}{2}\sin 2\theta$$



$\dot{r} \geq r - 2r^3$
$\dot{r} \leq r - r^3$.



So if $r < 1/\sqrt{2}$, $\dot{r} > 0$
$r > 1$, $\dot{r} < 0$

Since $1 - r^2/2 < \dot{\theta} < 1 + r^2/2$.
$r < 1$, $1 - 1/2 < \dot{\theta} < 1 + 1/2 \cdot 1/2$, $r > 1/\sqrt{2}$
$1/2 < \dot{\theta} < 5/4$

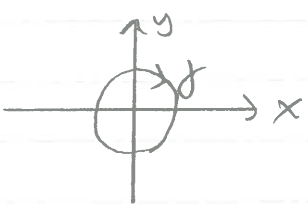
$\dot{\theta}$ is not zero and by the P.B Thm there is a limit cycle in the annulus $1/\sqrt{2} < r < 1$.

—/—

<u>Some special cases</u>.

1) Consider ode's of the form $\ddot{x} + \phi(\dot{x}) + f(x) = 0$.
We can write this as $y = \dot{x}$ and $\dot{y} = -[\phi(y) + f(x)]$.

The second equation is $y \frac{dy}{dx} + \phi(y) + f(\dot{x}) = 0$.
(as $\dot{y} = \frac{dy}{dx} \cdot \frac{dx}{dt} = y \frac{dy}{dx}$) Lets suppose a periodic solution exist.



Integrating around this periodic orbit w.r.t. x gives:

$$\oint_\gamma y \frac{dy}{dx} dx + \oint_\gamma \phi(y)\, dx + \oint_\gamma f(x)\, dx = 0.$$

$$\underbrace{\left[\frac{1}{2}y^2\right]_{start}^{end}}_{=0} + \int_0^T \varphi(\dot{x})\dot{x}\, dt + \underbrace{\left[F(x)\right]_{x\ start}^{x\ end}}_{F'=f}^{=0} = 0.$$

$dx = \frac{dx}{dt}dt$

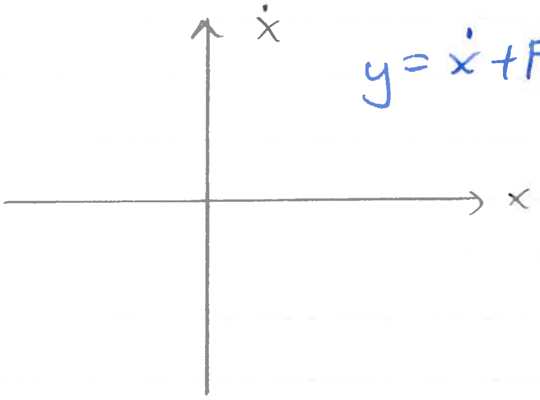
$$\int_0^T \dot{x}\,\varphi(\dot{x})\, dt = 0 \quad \text{if a periodic solution is to exist.}$$

If $y\varphi(y)$ is single-signed this integral cannot be zero and no periodic solution exists.

2) Lienhard's equation

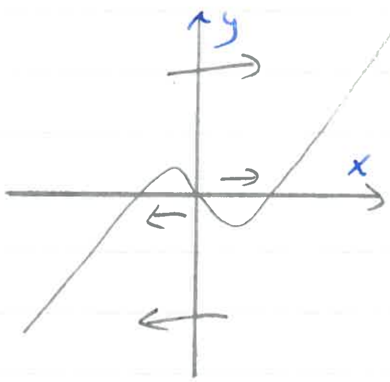
$$\ddot{x} + \dot{x}f(x) + g(x) = 0.$$

Lienhard's theorem states that:
1) If $f(x)$ is even e.g (x^2-1) and
2) $g(x)$ is odd eg x and if $F(x) = \int_0^x f(t)\,dt$ e.g $(\frac{1}{3}x^3 - x)$ and $F(x)$ has a single positive zero, x_0, e.g $\sqrt{3}$, and $F(x)$ is positive and monotonic increasing for $x > x_0$ and $F(x) \to \infty$ as $x \to \infty$ then the equation has a unique periodic solution. Lienhard's Transformation and Liehard plane.



phase plane

$y = \dot{x} + F(x)$.



(*)

(*) $\frac{dy}{dt} = \ddot{x} + F'(x)\dot{x}$

$= \ddot{x} + \dot{x}f(x) = -g(x)$.

So $\left.\begin{aligned} \dot{y} &= -g(x) \\ \dot{x} &= y - F(x) \end{aligned}\right\} \frac{dy}{dx} = \frac{-g}{y-F}$

16/11/12.

The van der Pol equation.

$$\ddot{x} - \varepsilon(1-x^2)\dot{x} + x = 0.$$

Lienhard's theorem shows that this equation has a unique periodic solution.

We will first examine $\varepsilon << 1$, $\varepsilon > 0$.

If $\varepsilon = 0$ then the equation is $\ddot{x} + x = 0$ and we have an infinite number of periodic solutions $x = a\cos t$ for any a. How does this tie in with the prediction of Lienhard's theorem if $\varepsilon > 0$. We can try and answer this by looking for a solution which is a power series in ε, with coefficient dependent on t.

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \ldots$$

However this is not straight forward, which we will demonstrate through a simpler example. Consider

$$\ddot{u} + u + \varepsilon u^2 = 0.$$

and look for a periodic solution

$$u = u_0(t) + \varepsilon u_1(t) + \varepsilon^2 u_2(t) + \ldots$$

Substitution yields $(\ddot{u}_0 + \varepsilon\ddot{u}_1 + \ldots) + (u_0 + \varepsilon u_1 + \ldots)$
$+ \varepsilon(u_0^3 + 3u_0^3(\varepsilon u_1 + \varepsilon^2 u_2 + \ldots) + \ldots) = 0.$

Comes from $\overset{-/-}{\underset{-/-}{(a+b)^3}} = a^3 + 3a^2b + \ldots$

Compare coefficients

$$\ddot{u}_0 + u_0 = 0 \Rightarrow u_0 = a_0 \cos t$$
$$\ddot{u}_1 + u_1 + u_0^3 = 0 \Rightarrow \ddot{u}_1 + u_1 = -a^3 \cos^3 t \ldots$$

As we are looking for periodic solutions we can choose our origin in t. So we can drop $b \sin t$.

$$\Rightarrow \ddot{u} + u_1 = -a_0^3 \cos^3 t$$
$$= -\frac{a_0^3}{4}(\cos 3t + 3\cos t).$$

For PI :

Look for $u_1 = \underset{a_0^3/32}{A \cos 3t} + \underset{0}{B \sin 3t} + \underset{0}{Ct \cos t} + \underset{-\frac{3}{8}a_0^3}{Dt \sin t}$

and $u_1(t) = a_1 \cos t + b_1 \sin t + \frac{a_0^3}{32}\cos 3t - \frac{3}{8}a_0^3 t \sin t$

$\boxed{\ddot{u} + u = -\varepsilon u^2}$

and $u = \underset{\searrow a_0 \cos t}{u_0} + \varepsilon u_1$

But this solution is not periodic. The $\cos 3t$ part is fine but $t \sin t$ is not :

Also the product εt is not a small correction to u_0 when t is of size $1/\varepsilon$.

However $a_0 \cos t \quad -\frac{3}{8} \varepsilon t a_0^3 \sin t$

$$= a_0 \cos\left[t + \frac{3}{8} a_0^2 \varepsilon t\right]$$

The nonlinearity affects the frequency which is now $1 + \frac{3}{8} a_0^2 \varepsilon$.

The method to deal with this is called Linstead t's method.

We switch to a new variable s where $s = t(c_0 + \varepsilon c_1 + \varepsilon^2 c_2 + \dots)$ with c_0, c_1, c_2 to be found and $u = u_0(s) + \varepsilon u_1(s) + \dots$ with u fixed to be 2π-periodic in s. We need to change $\frac{d}{dt}$ to $\frac{d}{ds}$.

$$\frac{d}{dt} = \frac{ds}{dt}\frac{d}{ds} = (c_0 + \varepsilon c_1 + \varepsilon^2 c_2 + \dots)\frac{d}{ds}$$

$$\frac{d^2}{dt^2} = (c_0 + \varepsilon c_1 + \varepsilon^2 c_2 + \dots)^2 \frac{d^2}{ds^2}$$

$$(c_0 + \varepsilon c_1 + \varepsilon^2 c_2 + \dots)^2 (u_0'' + \varepsilon u_1'' + \dots)$$
$$+ (u_0 + \varepsilon u_1 + \dots) + \varepsilon(u_0 + \dots)^3 = 0.$$

So: $\quad c_0^2 u_0'' + u_0 = 0 \Rightarrow u_0 = a_0 \cos(s/c_0)$ — u is 2π periodic gives $c_0 = 1$.

$$\cancel{c_0^2}_1 u_1'' + u_1 + 2\cancel{c_0}_1 c_1 u_0'' + u_0^3 = 0$$

and $u_1'' + u_1 = -a_0^3\left(\frac{1}{4}\cos 3s + \frac{3}{4}\cos s\right)$

$+ 2c_1 a_0 \cos s.$

$u_0'' = -\cos s.$

We can choose c_1 to ensure u_1 is periodic by ensuring the forcing has no component of the CF for $u_1'' + u_1$.

So $-\frac{3}{4}a_0^3 + 2c_1 a_0 = 0 \Rightarrow c_1 = \frac{3}{8}a_0^2.$

and $u_1 = a_1\cos s + b_1 \sin s + \left(-\frac{a_0^3}{4}\frac{\cos 3s}{-9+1}\right)$

So: $u(t) = a_0\cos s + \varepsilon\Big[a_1\cos s + b_1\sin s$

$+ \frac{a_0^3 \cos 3s}{32}\Big] + \dots$

$s = t\left(1 + \varepsilon \tfrac{3}{8} a_0^2 + \dots\right)$

Period is $\frac{2\pi}{1+\varepsilon \frac{3}{8}a_0^2} = 2\pi\left(1 - \frac{3}{8}a_0^2\varepsilon \dots\right)$

Rayleigh's Equation.

$$\ddot{x} - \varepsilon\left[\dot{x} - \frac{1}{3}\dot{x}^3\right] + x = 0 \qquad \underline{\varepsilon \ll 1}$$

We introduce $\theta = nt$, $n = n_0 + \varepsilon n_1 + \varepsilon^2 n_2 + \dots$ and expand $x = x_0(\theta) + \varepsilon x_1(\theta) + \varepsilon^2 x_2(\theta) + \dots$ where $x_0, x_1, \dots$ are 2π periodic in θ and at $t = 0$, $x = A$, and $\dot{x} = 0$.

We have $n^2 x'' + \varepsilon[nx' + 1/3\, n^3 (x')] + x = 0$

$$\left[n_0^2 + 2\varepsilon n_0 n_1 + \varepsilon^2\left(n_1^2 + 2n_0 n_2\right) \dots\right]\left(x_0'' + \varepsilon x'' + \varepsilon^2 x_2'' + \dots\right) - \varepsilon\left[\left(n_0 + \varepsilon n_1 + \varepsilon^2 n_2 \dots\right)\left(x_0' + \varepsilon x_1' + \varepsilon^2 x_2' + \dots\right)\right.$$

$$\left. - \frac{1}{3}\left(n_0^3 + 3n_0^2 \varepsilon n_1 + \underbrace{3\varepsilon^2 n_0 n_1^2 + 3\varepsilon^2 n_0^2 n_2 + \dots}_{\text{all the } \varepsilon^2 \text{ ones}}\right)\left(x_0'^3 + 3x_0'^2 \varepsilon x_1' + 3\varepsilon x_0' x_1'^2 + 3\varepsilon^2 x_0'^2 x_2'\right)\right]$$

$$+ \left[x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots\right] = 0$$

$x(0) = A$, $\dot{x}(0) = 0$.

$O(\varepsilon^0)$:

$n_0^2 x_0'' + x_0 = 0$.

$O(\varepsilon^1)$:

$$n_0^2 x_1'' + x_1 = -2n_0 n_1 x_0'' + \left(n_0 x' - \frac{1}{3} n_0^3 x_0'^3\right)$$

$O(\varepsilon^2)$

$$n_0^2 x_2'' + x_2 = -(n_1^2 + 2n_0 n_2)x_0'' - 2n_0 n_1 x_1''$$

$$+\left[\left\{n_0 x_1' + n_1 x_0'\right\} - \frac{1}{3}\left(n_0^3 3 x_0'^2 x_1 + \right.\right.$$

$$\left.\left. + 3n_0^2 n_1 x_0'^3\right)\right] \to \text{FORGET.}$$

We need to slove these boundary conditions;
$A = x_0(0) + \varepsilon x_1(0) + \ldots$ So $x_0(0) = A$, $x_1(0) = 0$

$\dot{x}(0) = nx'(0) = (n_0 + \varepsilon n_1 + \ldots)(x_0' + \varepsilon x_1' + \ldots) = 0.$

So $n_0 x_0'(0) = 0$ and $n_0 x_1'(0) + n_1 x_0'(0) = 0$. So
$x_0 = A\cos(\theta/n_0)$ and 2π periodic in $\theta \Rightarrow n_0 = 1$
and $x_1'' + x_1 = -2n_1(-A\cos\theta) + (-A\sin(\theta)$
$-1/3(-A\sin\theta)^3)$

$$x_1'' + x_1 = 2n_1 A\cos\theta - A\sin\theta$$

$$+\underbrace{\frac{A^3}{3}\left(\frac{3}{4}\sin\theta - \frac{\sin 3\theta}{4}\right)}_{\sin^3\theta}.$$

We can choose n_1 and A so that $x_1(\theta)$ is periodic. We need to ensure the coefficient of $\cos\theta$ and $\sin\theta$ on the r.h.s is zero:

So $\quad n_1 = 0 \quad , \quad n = 1 - \varepsilon n_2 .$

[frequency independent of ε, to order ε]

$$-A + \frac{1}{4} A^3 = 0 \implies \underline{\underline{A = 2}}$$

So periodic solutions must have amplitude $A = 2$.

$$x_1 = a_1 \cos\theta + b_1 \sin\theta + \frac{1}{3} A^3 \left(\frac{-1}{4}\right) \frac{\sin 3\theta}{-9+1}$$

$$= a_1 \cos\theta + b_1 \sin\theta + \frac{1}{12} \sin 3\theta .$$

a_1 and b_1 found so that

$x_1(0) = 0 \implies a_1 = 0 .$

$x_1'(0) = 0$ too $\implies b_1 = -1/4 .$

$$x = 2\cos\theta + \varepsilon \left(\frac{\sin 3\theta}{12} - \frac{\sin\theta}{4} \right) + \ldots$$

$\theta = t(1 + \ldots) \qquad \ldots = $ terms in ε^2

Consider a general result

$$\ddot{x} + \varepsilon f(x, \dot{x}) + \omega^2 x = 0.$$

<u>20/11/12</u>

A general solution

$$\ddot{x} + \varepsilon f(x, \dot{x}) + \omega^2 x = 0.$$

If we try $x = x_0 + \varepsilon x_1 + \ldots$ to find

$$\ddot{x}_0 + \omega^2 x_0 = 0$$
$$\ddot{x}_1 + \omega^2 x_1 = -f(x_0, \dot{x}_0)$$

$$\Rightarrow x_0 = A\sin(\omega t + \phi)$$
$$= -f(A\sin(\omega t + \phi), \omega A\cos(\omega t + \phi))$$

The r.h.s is periodic with period $2\pi/\omega$ and so we can write it as a Fourier Series.

$$\ddot{x}_1 + \omega^2 x_1 = r_0 + \sum_{n=1}^{\infty} r_n \cos n\omega t + s_n \sin n\omega t$$

where

$$\frac{2\pi r_0}{\omega} = \int_0^{2\pi/\omega} -f(A\sin\chi, \omega A\cos\chi)\, dt, \quad \chi = \omega t + \phi$$

$$\& \; \frac{2\pi}{\omega} \cdot \frac{1}{2} \cdot r_n = \int_0^{2\pi/\omega} (-1)\cos n\omega t\, f(A\sin\chi, \omega A\cos\chi)\, dt$$

$$\& \; \frac{2\pi}{\omega} \cdot \frac{1}{2} s_n = \int_0^{2\pi/\omega} (-1)\sin n\omega t\, f(A\sin\chi, \omega A\cos\chi)\, dt$$

We will not be able to find a periodic solution if $r \neq 0, s \neq 0$. We have seen how to deal with this Lindstedt's method. This involves the substitution $\theta = nt$, $n = n_0 + \varepsilon n_1 + \dots$ and solution 2π periodic in θ.

$x = x_0 + \varepsilon x_1 + \dots$

$$\frac{\partial}{\partial t} \to (n_0 + \varepsilon n_1 + \dots)\frac{\partial}{\partial \theta}$$

$$\frac{\partial^2}{\partial t^2} \to (n_0^2 + 2\varepsilon n_0 n_1 + \dots)\frac{\partial^2}{\partial \theta^2}$$

$n_0^2 x_0'' + \omega^2 x_0 = 0$, $x_0 = a\cos\theta$, $n_0 = \omega$, periodic solution amplitude a.

$$n_0^2 x_1'' + \omega^2 x_1 = -2n_0 n_1 x_0'' - f(a\cos\theta, -n_0 a\sin\theta)$$

$$= 2\omega n_1 a\cos\theta - f(a\cos\theta, -\omega a\sin\theta).$$

$$x_1'' + x_1 = \frac{2an_1}{\omega}\cos\theta - \frac{1}{\omega^2} f(a\cos\theta, -\omega a\sin\theta)$$

We obtain periodic solution for x_1 if the Fourier coefficient of $\cos\theta$ and $\sin\theta$ on the r.h.s are zero.

$\underline{\cos\theta}$: $0 = \frac{2an_1}{\omega} \underbrace{\int_0^{2\pi} \cos^2\theta \, d\theta}_{\pi}$

$$- \frac{1}{\omega} \int_0^{2\pi} \cos\theta \, f(a\cos\theta, -\omega a\sin\theta)\, d\theta$$

$$\boxed{2\pi a n_1 = \int_0^{2\pi} \cos\theta \, f(a\cos\theta, -\omega a\sin\theta)\, d\theta}$$

<u>$\sin\theta$</u>

$$0 = 0 - \int_0^{2\pi} \sin\theta \, f(a\cos\theta, -\omega a\sin\theta)\, d\theta$$

$$\boxed{\int_0^{2\pi} \sin\theta \, f(a\cos\theta, -\omega a\sin\theta)\, d\theta = 0}$$

For example, for the V-dP equation $\ddot{x} - \varepsilon(1-x^2)$ $+ x = 0$.

We have $\omega = 1$, $f(x,\dot{x}) = \dot{x}(x^2 - 1)$

So the above formula give:

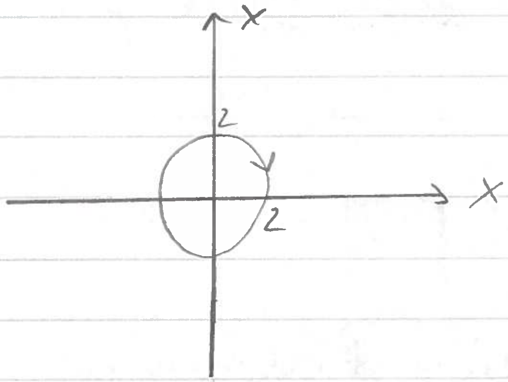
$$2\pi a n_1 = \int_0^{2\pi} \cos\theta(-\sin\theta)(a^2\cos^2\theta - 1)\, d\theta.$$

odd function

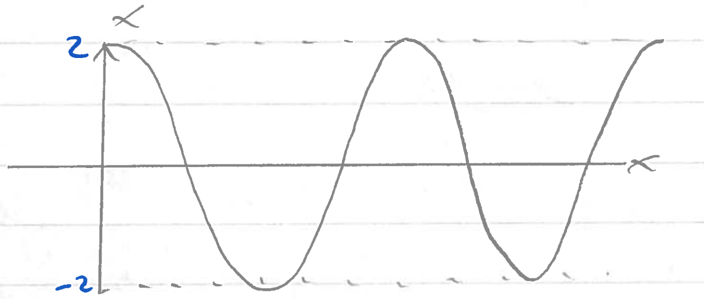
$$= 0, \quad n_1 = 0$$

$$0 = \int_0^{2\pi} \sin\theta(-a\sin\theta)(a^2\cos^2\theta - 1)\, d\theta.$$

$$\Rightarrow a^2 = \int_0^{2\pi} \sin^2\theta \, d\theta \Big/ \int_0^{2\pi} \sin^2\theta\cos^2\theta \, d\theta = 4 \Rightarrow \underline{a = 2}$$

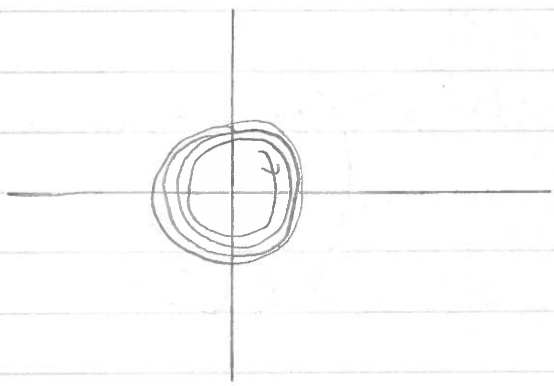


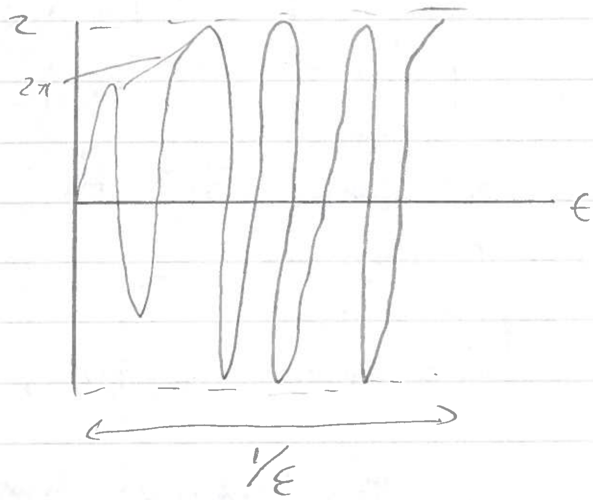
$$\ddot{x} - \varepsilon(1-x^2)\dot{x} + x = 0$$



$x = a\cos t$
$\;\;= 2\cos t$

<u>$\varepsilon \ll 1$</u>





Many oscillation for an appreciable change in amplitude

We have two active timescales in the solution. One is that of the oscillations and is "order one" i.e independant of ε. The second is longer and repersents a slow timescale over which amplitude, or perhaps phase, alters. This is of size $1/\varepsilon$, $\varepsilon \to 0$. We can repersent this by introducing a new variable $T = \varepsilon t$. We look for a solution with $x = (x, T)$ with t and T

considered to be cindepenant of each other.

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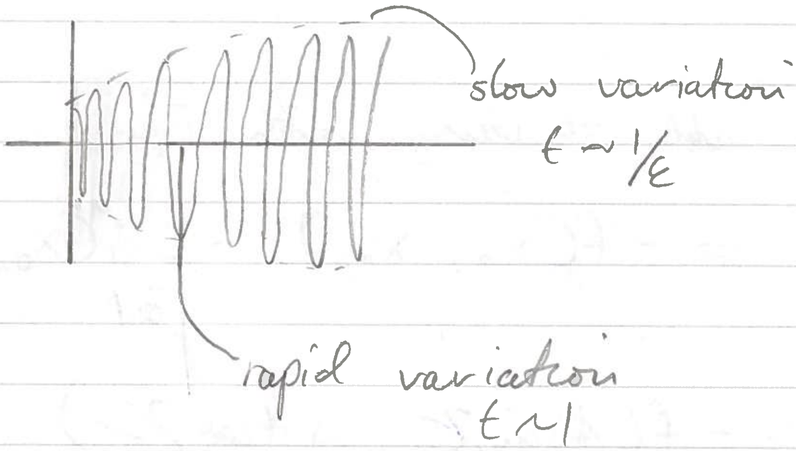
$$\ddot{x} + \varepsilon f(x, \dot{x}) + \omega^2 x = 0$$

Sometimes, solution for small ε are



Set $x = x(t, T)$ with $T = \varepsilon t$

So that

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial T}{\partial t}\frac{\partial}{\partial T}$$

$$= \frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial T}.$$

$$\frac{d^2}{dt^2} = \left(\frac{\partial}{\partial t} + \varepsilon\frac{\partial}{\partial T}\right)\left(\frac{\partial}{\partial t} + \varepsilon\frac{\partial}{\partial T}\right)$$

$$= \underbrace{\frac{\partial^2}{\partial t^2} + 2\varepsilon\frac{\partial}{\partial T}\frac{\partial}{\partial t}} + \varepsilon^2\frac{\partial^2}{\partial T^2}$$

We use this and expand $x = x_0(t, T) + \varepsilon x_1(t, T) + \ldots$

$$\left(\frac{\partial^2}{\partial t^2} + 2\varepsilon\frac{\partial}{\partial T}\frac{\partial}{\partial t} \ldots\right)\left(x_0 + \varepsilon x_1 + \ldots\right) +$$

$$\varepsilon f\left(x_0 + \ldots, \left(\frac{\partial}{\partial t} + \ldots\right)\left(x_0 + \ldots\right)\right)$$

$$+ \omega^2\left(x_0 + \varepsilon x_1 \ldots\right) = 0$$

$$\Rightarrow x_{0_{tt}} + \omega^2 x_0 = 0$$

$$\Rightarrow A(T)\sin(\underbrace{\omega t + \varphi(T)}_{\chi(t,T)})$$

So A & ϕ are able to vary with T.

$$x_{1_{tt}} + \omega^2 x_1 = -f(x_0, x_{0_t}) - 2\frac{\partial}{\partial T}(x_{0_t})$$

$$x_{1_{tt}} + \omega^2 x_1 = -f(A\sin\chi, \omega A\cos\chi) - 2\frac{\partial}{\partial T}(\omega A\cos\chi)$$

$$= -f(A\sin\chi, \omega A\cos\chi) - 2\omega\left(\frac{\partial A}{\partial T}\right)\cos\chi + 2\omega A\left(\frac{\partial \varphi}{\partial T}\right)\sin\chi.$$

We can ensure that x_1 remains bounded by setting the Fourier Coefficient of $\sin\chi$ & $\cos\chi$ on r.h.s to be zero. So multiply by $\sin\chi$ or $\cos\chi$, integrate over the period in t ($2\pi/\omega$ in t or $\underset{(-\pi,\pi)}{2\pi}$ in χ)

$$0 = -\int_0^{2\pi} \sin \chi \, f(A \sin \chi, \omega A \cos \chi)\, d\chi$$

$$- 2\omega \frac{\partial A}{\partial T} \int_0^{2\pi} \cancel{\sin \chi \cos \chi} \, d\chi \quad 0$$

$$+ 2\omega A \frac{\partial \phi}{\partial T} \underbrace{\int_0^{2\pi} \sin^2 \chi \, d\chi}_{\pi}$$

$$2\pi\omega A \frac{\partial \phi}{\partial T} = \int_0^{2\pi} \sin \chi \, f(A \sin \chi, A\omega \cos \chi)\, d\chi$$

and similarly

$$0 = -\int_0^{2\pi} \cos \chi \, f(A \sin \chi, \omega A \sin \chi)\, d\chi$$

$$- 2\omega \frac{\partial A}{\partial T} (\pi) \quad + 2\omega A \frac{\partial \phi}{\partial T} (0)$$

So $\boxed{2\pi\omega \frac{\partial A}{\partial T} = -\int_0^{2\pi} \cos \chi \, f(A \sin \chi, A\omega \cos \chi)\, d\chi}$

If $\ddot{x} - \varepsilon \dot{x}(1 - x^2) + x = 0$.

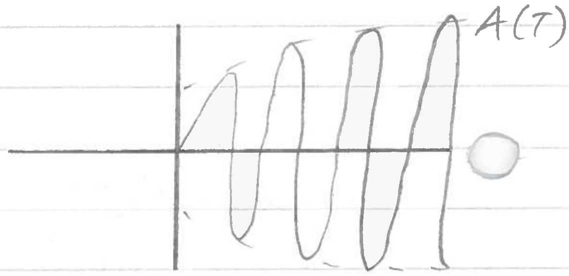
then $\omega = 1$ $f(x, \dot{x}) = \dot{x}(x^2 - 1)$

So $2\pi A \frac{\partial \phi}{\partial T} = \int_{-\pi}^{\pi} \sin \chi \underbrace{(\omega A \cos \chi)}_{\dot{x}} \underbrace{(A^2 \sin^2 \chi - 1)}_{x^2 - 1}\, d\chi = 0$

& $$2\pi \frac{\partial A}{\partial T} = -\int_{-\pi}^{\pi} \cos X \,(A\cos X)(A^2\sin^2 X - 1)\, dX$$

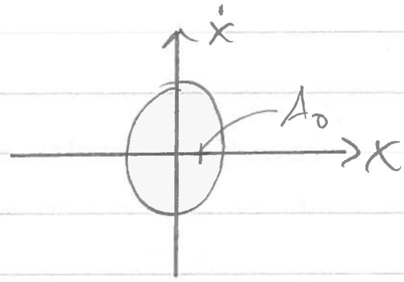
$$= A\underbrace{\int_{-\pi}^{\pi}\cos^2 X \, dX}_{\pi} - A^3\underbrace{\int_{-\pi}^{\pi}\cos^2 X \sin^2 X \, dX}_{\frac{1}{4}\pi}$$

So $$2\frac{\partial A}{\partial T} = A - \frac{1}{4}A^3$$



$$Q = A^2, \quad \frac{\partial Q}{\partial T} = 2A\frac{\partial A}{\partial T}$$

$$\frac{\partial Q}{\partial T} = Q - \frac{1}{4}Q^2 = \frac{Q(4-Q)}{4}$$



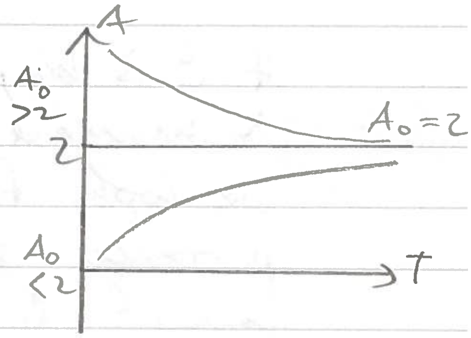
$$\int \frac{4\, dQ}{Q(4-Q)} = \int dT$$

$$\Rightarrow T + \text{Const} = \int \frac{1}{Q} + \frac{1}{4-Q}\, dQ$$

$$= \ln\left(\frac{Q}{4-Q}\right)$$

$$\frac{Q}{4-Q} = Be^T = \frac{A_0^2}{4-A_0^2}e^T$$ At $T=0$, $A = A_0$

$$\Rightarrow A = \frac{2}{\left(1 + \frac{4 - A_0^2}{A_0^2} e^{-T}\right)^{1/2}}$$



$$\& \quad x(t) = \frac{2 \sin t}{\left(1 + \frac{4 - A_0^2}{A_0^2} e^{-\epsilon t}\right)^{1/2}}$$

<u>Vdp equation : $\ddot{x} + \epsilon \dot{x}(x^2 - 1) + x = 0$ for $\epsilon \gg 1$</u>

Compare the Vdp equation with Lienhard's equation $\ddot{x} + \dot{x} f(x) + g(x) = 0$. We have $f(x) = (x^2 - 1)\epsilon$
$g(x) = x$.

We introduce the Lienhard variable

$$y = \dot{x} + F(x) \quad \text{where} \quad F'(x) = f(x) \quad \text{and} \quad F(0) = 0.$$

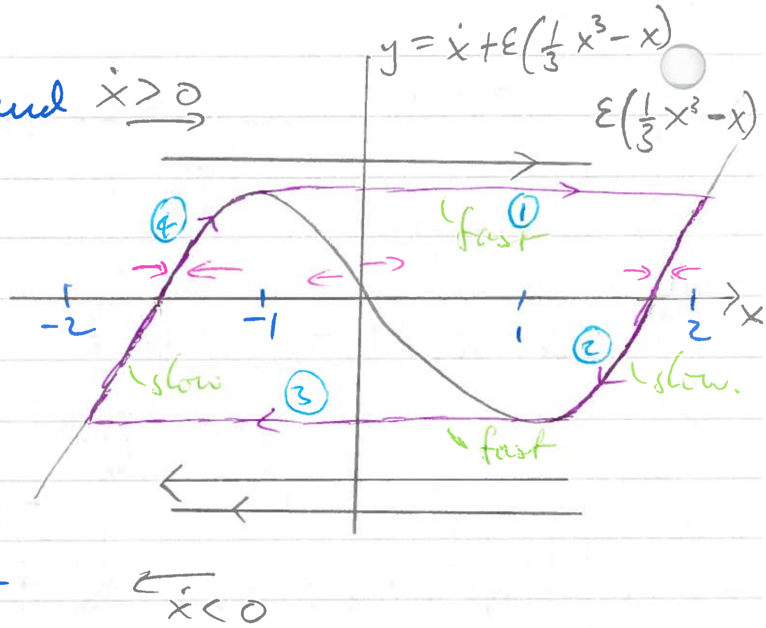
then $\dot{y} = \ddot{x} + F'(x)\dot{x} = \ddot{x} + f(x)\dot{x} = -g(x)$.

$$\dot{x} = y - F.$$

For VdP.

$$\dot{y} = -x, \quad \dot{x} = y - \epsilon\left(\frac{1}{3}x^3 - x\right)$$

If ε is big, $\dot{x}$ is big and x increases rapidly. If we look for bounded periodic solution; x cannot do so long. So x increases rapidly for a short time. In this time y cannot alter much as $\dot{y} = -x$ is not large.



This is not the case if y is close to εF

$$= \varepsilon\left(\frac{1}{3}x^3 - x\right).$$

We can construct a periodic solution as shown. All initial conditions will give solutions that end up following this periodic limit cycle.

OR

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{-x}{y - \varepsilon(\frac{1}{3}x^2 - x)} \quad \text{if } y = \varepsilon z.$$

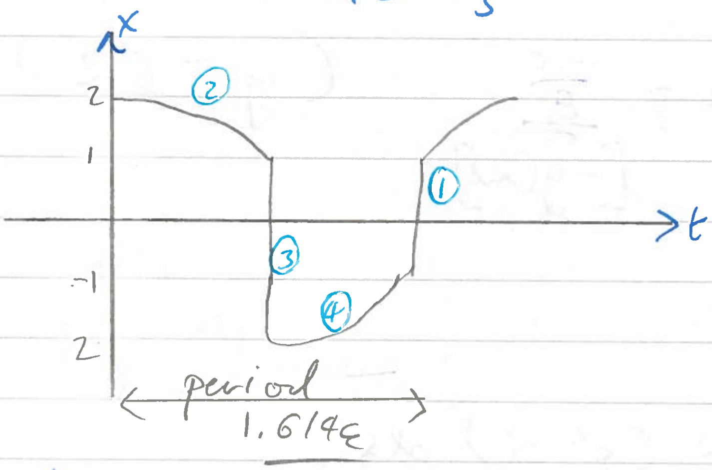
$$\frac{dy}{dx} = \varepsilon\frac{dz}{dx} = \frac{-x}{\underbrace{\varepsilon\left[z - \left(\frac{1}{3}x^3 - x\right)\right]}_{\text{"}z - F\text{"}}}$$

$$\left(z - \left(\frac{1}{3}x^3 - x\right)\right)\frac{dz}{dx} = \frac{-x}{\varepsilon^2}$$

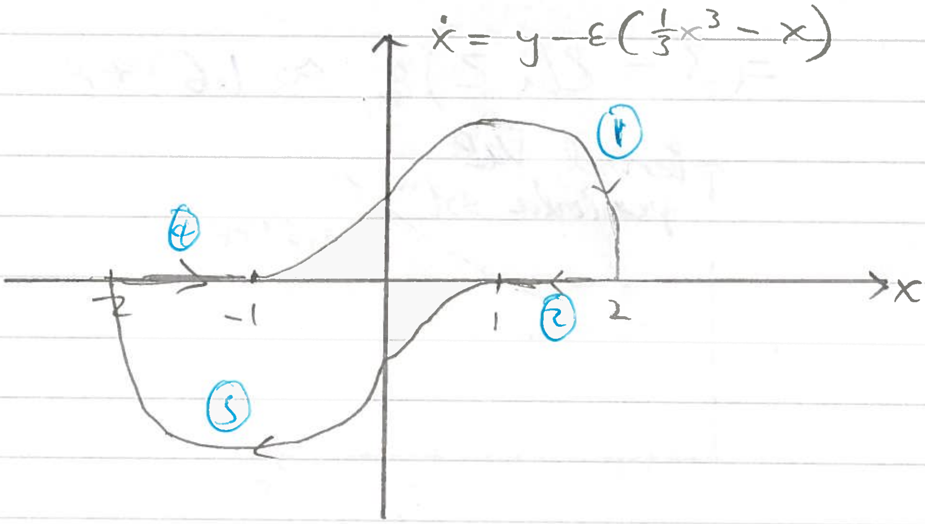
So, if $\varepsilon >> 1$, $\frac{dz}{dx} = 0$ & trajectories horizontal
or $z = \frac{1}{3}x^3 - x$ &
trajectory follows
$z = \frac{1}{3}x^3 - x$

Max/Min of $\frac{1}{3}x^3 - x$ are at $x = -1$ & $+1$
At $x = -1$, $\frac{1}{3}x^3 - x = \frac{1}{3}(-1)^3 - (-1) = \frac{2}{3}$.

Horizontal trajectories given $z = 2/3$ meets
$z = \frac{1}{3}x^3 - x$ at $2/3 = \frac{1}{3}x^3 - x \Rightarrow (x-2)(x+1)^2 = 0$



Phase plane



Period.

$$\left(\dot{x} = \frac{dx}{dz} \cdot \frac{dz}{dt}\right).$$

$$T = \int_0^T dt = \int_\Gamma \frac{dx}{\dot{x}} = \int_\Gamma \frac{dt}{dz}\frac{dz}{dx}\,dx.$$

$$= 2\left\{\int_{-2}^{1} \frac{dt}{dz}(x^2-1)\,dx + \int_{-1}^{2} \frac{dt}{dz}(0)\,dz\right\}$$

④ $z = \frac{1}{3}x^3 - x$ ① $z = 2/3$

Period dominated by slow part of motion.

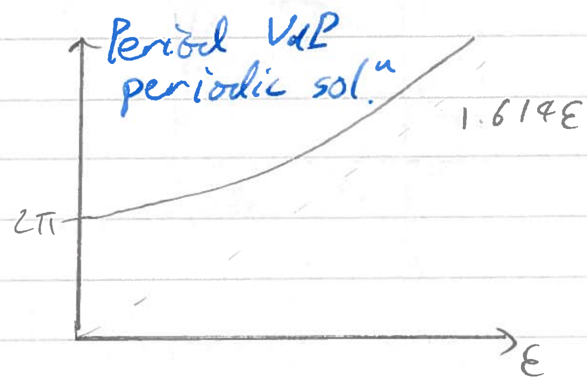
$$\frac{dz}{dt} = \frac{1}{\varepsilon}\frac{dy}{dt} = \frac{-x}{\varepsilon} \qquad (y = \varepsilon z).$$

$[-g(x)]$

Period

$$2\int_{+2}^{+1} -\frac{\varepsilon}{x}(x^2-1)\,dx.$$

$$= (3 - 2\ln 2)\varepsilon \approx 1.614\varepsilon$$



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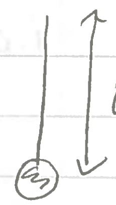
Working a swing

The displacement of a swing obeys



$$\ddot{x} + \frac{l}{g} x = 0$$

Lets us alter the length of the swing periodically & write

$$\frac{l}{g} = \omega^2 + a\cos qt.$$

Scale t with $1/q$, $\quad q^2\ddot{x} + (\omega^2 + a\cos t) = 0.$

$$\ddot{x} + \left(\frac{\omega^2}{q^2} + \frac{a}{q^2}\cos t\right)x = 0$$

Let us write $\frac{\omega^2}{q^2} = 1 + \varepsilon^2 k$, $\quad \varepsilon \ll 1$

Also write $a/q^2 = \varepsilon$

$$\ddot{x} + (1 + \varepsilon^2 k + \varepsilon\cos t)x = 0.$$

Lets us try $x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots$

$$x_0 = A\cos t + B\sin t$$

$\ddot{x}_0 + x_0 = 0$

$\ddot{x}_1 + x_1 = -\cos t\, x_0$

$\ddot{x}_2 + x_2 = -k x_0 - \cos t\, x_1 - 2\frac{\partial}{\partial T}\frac{\partial}{\partial t} x_0$

x_1 is forced by $\cos t \cdot x_0$ i.e terms in $\cos^2 t$ and $\cos t \sin t$ i.e 1, $\cos 2t$, $\sin 2t$.

The CF for x_1 is $\cos t$ & $\sin t$ so x_1 is periodic.
x_2 is forced by $-kx_0$ & $-\cos(t x_1)$
i.e $\underset{\underline{\cos t}}{\sin t}$ & $\cos t \cos 2t$ & $\cos t \sin 2t$.
i.e $\underline{\cos 3t}$ & $\underline{\cos t}$ $\quad$ $\underline{\sin 3t}$ & $\underline{\sin t}$

We introduce a new timescale. $T = \epsilon^2 t$ & consider x_0, x_1, x_2 to be functions of t & T. Then

$$\frac{d}{dt} \to \frac{\partial}{\partial t} + \epsilon^2 \frac{\partial}{\partial T}$$

$$\frac{d^2}{dt^2} \to \frac{\partial^2}{\partial t^2} + 2\epsilon^2 \frac{\partial}{\partial T}\frac{\partial}{\partial t} + \epsilon^4 \frac{\partial^2}{\partial T^2}$$

The equations satisfied by x_0 & x_1 remain unchange

$$x_0 = A(T)\cos t + B(T)\sin t.$$

$$\ddot{x}_1 + x_1 = -A(T)\cos^2 t - B(T)\cos t \sin t.$$

$$x_1 = \underset{\text{set to } 0}{\cancel{CF}} - \frac{1}{2}A + \frac{1}{6}A\cos 2t + \frac{1}{6}B\sin 2t.$$

$$\ddot{x}_2 + x_2 = -k(A\cos t + B\sin t)$$

$$+\frac{1}{2}A\cos t - \frac{1}{6}A\underbrace{\cos t\cos 2t}_{\frac{1}{2}(\cos 3t + \cos t)}$$

$$-\frac{1}{6}B\underbrace{\cos t\sin 2t}_{\frac{1}{2}(\sin 3t + \sin t)}.$$

$$-2\frac{\partial}{\partial T}\left[-A\sin t + B\cos t\right]$$

x_2 will be periodic if

$\underline{\cos t}$: $-kA + \frac{1}{2}A - \frac{1}{12}A - 2\frac{\partial B}{\partial T} = 0$.

$\underline{\sin t}$: $-kB - \frac{1}{12}B + 2\frac{\partial A}{\partial T} = 0$.

i.e $\frac{\partial A}{\partial T} = \frac{1}{2}\left(\frac{1}{12} + k\right)B$,

$\frac{\partial B}{\partial T} = \frac{1}{2}\left(\frac{5}{12} - k\right)A$.

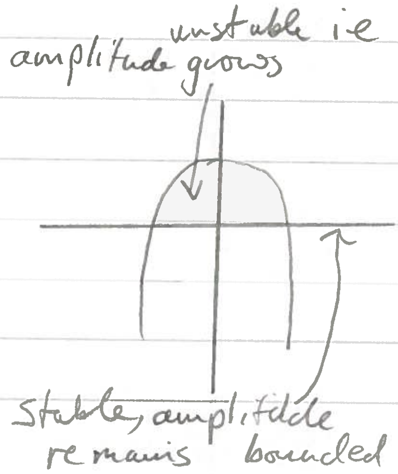
i.e $\frac{\partial}{\partial T}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2}\left(\frac{1}{12}+k\right) \\ \frac{1}{2}\left(\frac{5}{12}-k\right) & 0 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}$

Look for a solution $\underline{u}e^{\sigma T}$

$$\sigma \underline{u}\cancel{e^{\sigma T}} = \begin{pmatrix} \quad \end{pmatrix} \underline{u}\cancel{e^{\sigma T}}$$

i.e σ is a eigenvalue of $\begin{pmatrix} 0 & \frac{1}{2}\left(\frac{1}{12}+k\right) \\ \frac{1}{2}\left(\frac{5}{12}-k\right) & 0 \end{pmatrix}$

i.e $\sigma^2 = \frac{1}{4}\left(\frac{1}{12} + k\right)\left(\frac{5}{12} - k\right)$



$\sigma^2 > 0$ for growing solution.

$$q^2 = \omega^2 - \frac{ka^2}{q^2}$$, a small.

(Notation:
$\mathcal{O}$ – big "O" O – zero
o – small "o")

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Asymptotic Expansion of Integrals

We will develop techniques that allow us to find approximate expressions for integrals of the type

$$I(x) = \int_0^T e^{-xg(t)} f(t)\, dt$$

& similar where x is large.

These approximations are of a type known as "asymptotic approximation". In order to explain some notation is needed.

"order"

a) We say $f(x) = \mathcal{O}(g(x))$ as $x \to \infty, x \to 0, x \to 5$, say if we can find constant K and X such that $|f| < K|g|$ if $x > X$ ($x < X$, $|x-5| < X$ say).

So we can say:

$$x + x^2 = \mathcal{O}(x^2) \quad x \to \infty$$

$$= \mathcal{O}(x) \quad x \to 0.$$

$$= \mathcal{O}(1) \quad x \to 5.$$

since $\dfrac{|f|}{|g|} = \left|\dfrac{x^3+x}{x^2}\right| = \left|1 + \dfrac{1}{x}\right| < \dfrac{3}{2}$, $x > 2$ K X

& $\dfrac{|f|}{|g|} = \left|\dfrac{x^2+x}{x}\right| = |1+x| < 3/2$, $x < 1/2$ K X "little o"

b) $f(x) = \tilde{o}(g(x))$ as $x \to \infty, 0, 5,$

means $\frac{|f|}{|g|} \to 0$

So $f(x) = \tilde{o}(1)$ $x \to \infty$ means $f \to 0$ as $x \to \infty, 0, 5$
as $x \to \infty$

$x = \tilde{o}(x^2)$ as $x \to \infty$ as $\left|\frac{x}{x^2}\right| \to 0$ as $x \to \infty$

$x^2 = \tilde{o}(x)$ as $x \to 0$ as $\left|\frac{x^3}{x}\right| \to 0$ as $x \to 0$

c) $f(x) \sim g(x)$ as $x \to \infty, 0, 5$

means $\frac{|f|}{|g|} \to 1$ as $x \to \infty, 0, 5$.

e.g $x^2 + x \sim x^2$ as $x \to \infty$
$x^2 + 3x + 1 \sim x^2 + \sin(x)$ $x \to \infty$

Example : Consider the Exponential Integral

$$E_i(x) = \int_x^\infty \frac{e^{-t}}{t}\,dt \quad , x > 0.$$

Consider $E_i(x)$ for $x \to \infty$.

Substitute $t = xu$.

$$E(x) = \int_1^\infty \frac{e^{-xu}}{\cancel{x}u}\,\cancel{x}\,du = \int_1^\infty \frac{e^{-xt}}{t}\,dt$$

$$= \left[-\frac{1}{x} e^{-xt} \cdot \frac{1}{t} \right]_1^\infty - \left[\int_1^\infty \left(-\frac{1}{x} e^{-xt}\right)\left(-\frac{1}{t^2}\right) dt \right]$$

integration by parts.

$$= \frac{e^{-x}}{x} - \frac{1}{x} \int_1^\infty \frac{e^{-xt}}{t^2} \, dt.$$

and again

$$= \frac{e^{-x}}{x} - \frac{1}{x} \left\{ \left[-\frac{e^{-xt}}{x} \cdot \frac{1}{t^2} \right]_1^\infty - \int_1^\infty -\frac{e^{-xt}}{x} \cdot \left(\frac{-2}{t^3}\right) dt \right\}$$

$$= \frac{e^{-x}}{x} - \frac{e^{-x}}{x^2} + \frac{2}{x^2} \int_1^\infty \frac{e^{-xt}}{t^3} \, dt. \quad \text{\& so on}$$

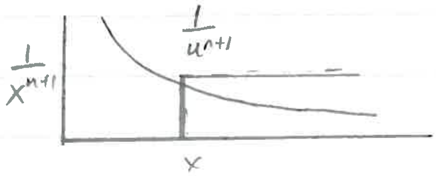
$$= e^{-x} \sum_{r=1}^{n} \frac{(-1)^{r-1}}{x^r} (r-1)! + R_n.$$

$$\text{where } R_n = \frac{(-1)^n n!}{x^n} \int_1^\infty \frac{e^{-xt}}{x^n} \, dt$$

$$R_n = \frac{(-1)^n n!}{x^n} \int_x^\infty \frac{e^{-u}}{\left(\frac{u^{n+1}}{x^{n+1}}\right)} \frac{du}{x}$$

$$\left(\begin{matrix} u = xt \\ t = u/x \end{matrix} \right)$$

$$= (-1)^n n! \int_x^\infty \frac{e^{-u}}{u^{n+1}} \, du.$$



$$|R_n| = n! \int_x^\infty \frac{e^{-u}}{u^{n+1}} \, du < \frac{n!}{x^{n+1}} \int_x^\infty e^{-u} \, du.$$

$$= n! \frac{e^{-x}}{n+1}$$

$$E_i(x) = \int_x^\infty \frac{e^{-t}}{t}\,dt = e^{-x}\left\{\sum_{r=1}^{n} \frac{(-1)^{r-1}}{x^r}(r-1)! + S_n\right\}$$

$$\& \quad S_n < \frac{n!}{x^{n+1}}$$

For fixed n $S_n \to 0$ as $x \to \infty$
For fixed x, $S_n \to \infty$ as $n \to \infty$, i.e you include more terms in the series.
The series diverages.
There is an optimum value of n, for given x, for which the approximation

$$E_i(x) \approx e^{-x} \sum_{r=1}^{n} \frac{(-1)^{r-1}(r-1)!}{x^r}$$

perform best.

We can write $E_i(x) \sim e^{-x}\left\{\frac{1}{x} - \frac{1}{x^2} + \frac{2}{x^3} \dots\right\}$
where it is understood that a finite number of terms are taken in r.h.s.

<u>Factorial Function</u>

Consider $I(n) = \underbrace{\int_0^\infty e^{-u}u^n\,du}_{n>-1} = \left[-e^{-u}u^n\right]_0^\infty + \int_0^\infty e^{-u}nu^{n-1}\,du$

$$= 0 + nI_{n-1}$$

So $I_n = nI_{n-1}$ Also $I_0 = 1$ & $\underline{I_n = n!}$

$$\left(-\frac{1}{2}\right)! = \int_0^\infty e^{-u} \frac{1}{\sqrt{u}}\, du$$

$u = x^2$

$$= \int_0^\infty e^{-x^2} \cdot \frac{2\sqrt{u}}{\sqrt{u}}\, dx\,.$$

$$= 2\int_0^\infty e^{-x^3}\, dx = 2 \cdot \frac{\sqrt{\pi}}{2} = \sqrt{\pi}$$

The Gamma Function is:

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt \quad \& \quad \Gamma(x+1) = x!$$

$x > 0$

$$\Gamma(x) = (x-1)!\,, \quad \Gamma(1/2) = (1/2)! = \sqrt{\pi}$$

For $n < 1$ we can use the recursion relation to find out about $n!$ for $n < -1$. We have

$$n! = n(n-1)! \Rightarrow (n-1)! = \frac{n!}{n}\,, \quad n! = \frac{(n+1)!}{(n+1)}$$

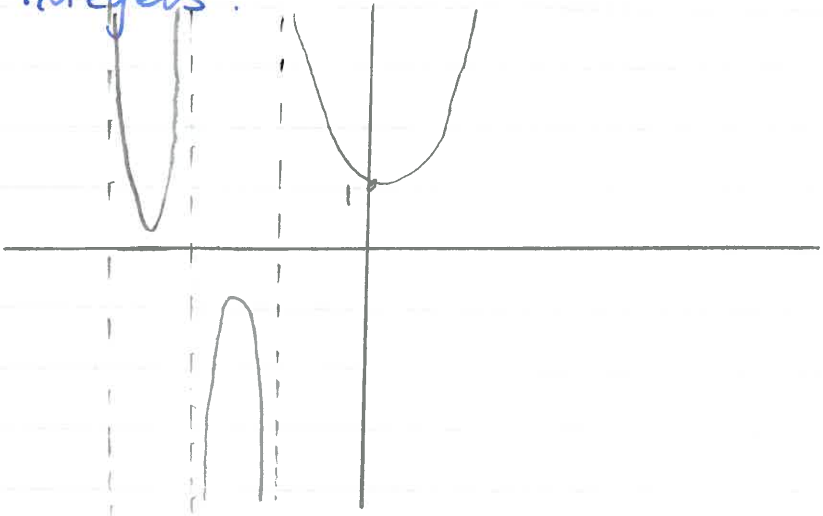
— / —

Eg: $$\left(\frac{-3}{2}\right)! = \frac{(-1/2)!}{-1/2} = \frac{\sqrt{\pi}}{-1/2} = -2\sqrt{\pi}\,.$$

$$\left(-\frac{5}{2}\right)! = \frac{(-3/2)!}{-3/2} = \frac{-2\sqrt{\pi}}{-3/2} = \frac{4}{3}\sqrt{\pi}\,.$$

$$(-3)! = \frac{(-2)!}{-2} = \frac{(-1)!}{(-2)(-1)} = \frac{0!}{(-2)(-1)\,0}\,.$$

and the factorial function has simple poles at the negative integers.



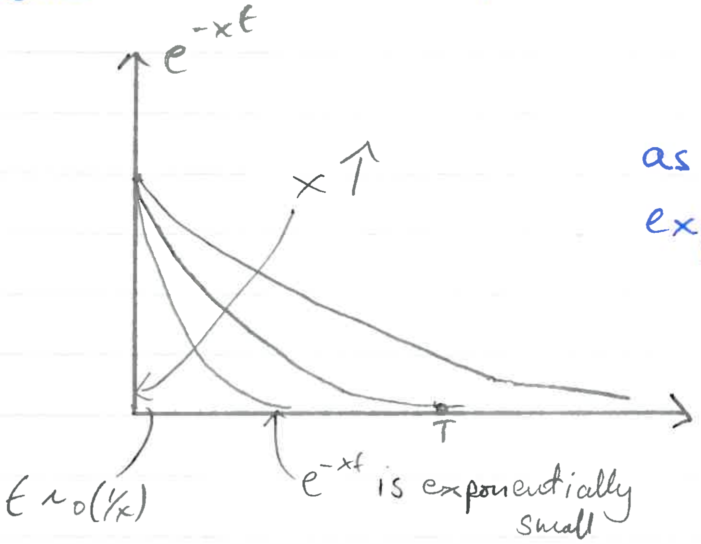
Watson's lemma

Consider

$$I(x) = \int_0^T e^{-xt} f(t)\, dt \qquad T>0$$

& consider $x \to \infty$

Consider e^{-xt} $\quad x \gg 1$



So unless $t = O(1/x)$ as $x \to \infty$, e^{-xt} is exponentially small

$e^{-x} = \tilde{o}(x^n)$ for any $n>0$ <0

little "o"

We see that for an approximation to $I(x)$ which captures algebric behaviour in x as $x \to \infty$ but is happy to neglect exponentially small terms. the range of integration that contributes is only where $xt = O(1)$ i.e $t = O(1/x)$. This assumes that $f(t)$ does not grow faster than any exponential as $t \to \infty$

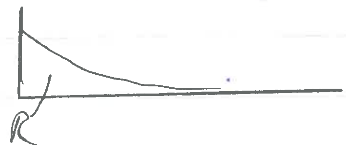
$$I(x) = \int_0^T e^{-xt} f(t)\, dt .$$

So let us therefore make the substitution $xt = u$. The variable u is $O(1)$ in the region that contributes to $I(x)$.

width of region in t that matters.

$$I(x) = \int_0^{xT} e^{-u} f\left(\frac{u}{x}\right) \frac{du}{x} .$$

If $f(x)$ has a Taylor expansion (in fact it also works for an asympototic expansion) we can use it to express



$$f\left(\frac{u}{x}\right) = \sum_{n=0}^{\infty} \left(\frac{u}{x}\right)^n \cdot \frac{f^n(0)}{n!} \quad \&$$

$$I(x) = \int_0^{xT} e^{-u} \sum_{n=0}^{\infty} \left(\frac{u}{x}\right)^n \frac{f^n(0)}{n!} \frac{du}{x}$$

$$\sim \sum_{n=0}^{\infty} \frac{f^n(0)}{x^{n+1}} \underbrace{\int_0^{\infty} u^n e^{-u}\, du}_{=n!} \cdot \frac{1}{n!} = \sum_{n=0}^{\infty} \frac{f^n(0)}{x^{n+1}}$$

$$I(x) = \int_0^{T} e^{-xt} f(t)\, dt$$

$$\sim \sum_{n=0}^{\infty} \frac{f^{n}(0)}{x^{n+1}}, \quad x \to \infty.$$

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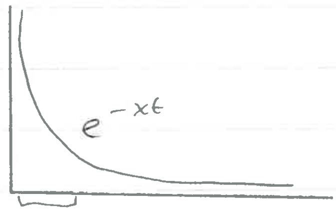
$$E_i(x) = \int_x^\infty \frac{e^{-t}}{t}\,dt.$$

$$\sim \sum_{r=1}^{\infty} \frac{(-1)^{r-1}(r-1)!}{x^n}, \quad x \to \infty$$

—/—

Watson's Lemma:

$$I(x) = \int_0^T e^{-xt} f(t)\,dt.$$



$$\sim \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{x^{n+1}}, \quad x \to \infty$$

More generally if $f(t) \sim t^{\lambda} \sum_{n=0}^{\infty} a_n t^{\lambda_n}$ as $t \to 0$. $\quad \lambda_0 = 0$, $\lambda_0 < \lambda_1 < \lambda_2$

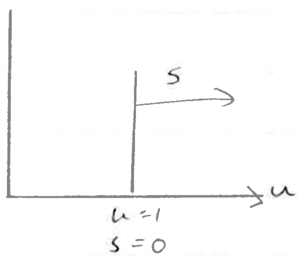
then:

$$I(x) \sim \sum_{n=0}^{\infty} \frac{a_n(\lambda+\lambda_n)!}{x^{\lambda+\lambda_n+1}}$$

—/—

Examples:

1) $$E_i(x) \underset{x>0}{=} \int_x^\infty \frac{e^{-t}}{t}\,dt \underset{t=xu}{=} \int_1^\infty \frac{e^{-xu}}{xu}\,x\,du$$



$$= \int_1^\infty \frac{e^{-xu}}{u}\,du$$

$$\underset{u=1+s}{=} \int_0^\infty \frac{e^{-x(1+s)}}{1+s}\,ds$$

$$= e^{-x}\int_0^\infty \frac{e^{-sx}}{1+s}\,ds \quad \& \quad f(s) = \frac{1}{1+s}$$

$$= 1 - s + s^2 - s^3 + s^4 - \ldots$$

$$\sim e^{-x}\sum_{n=0}^{\infty} \frac{(-1)^n n!}{x^{n+1}}$$

2) $I(x) = \int_0^\infty e^{-xt}\ln(1+t^2)\,dt$.

$$\ln(1+t^2) \sim t^2 - \frac{1}{2}t^4 + \frac{1}{3}t^6 \ldots$$

$$I(x) \sim \frac{2!}{x^{2+1}} - \frac{1}{2}\cdot\frac{4!}{x^{4+1}} + \frac{1}{3}\,\frac{6!}{x^{6+1}} \ldots$$

<u>OR</u>: Make the change of variable suitable for where $xt = O(1)$. So, put $u = xt$

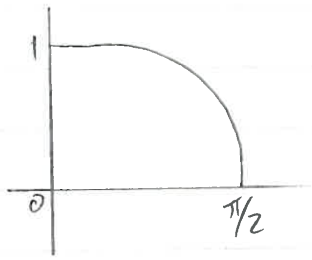
$$I(x) = \int_0^\infty e^{-u}\ln\left(1+\frac{u^2}{x^2}\right)\frac{du}{x}$$

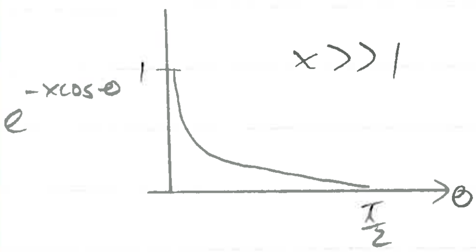
$$\sim \int_0^\infty e^{-u}\left(\frac{u^2}{x^2} - \frac{1}{2}\frac{u^4}{x^4} + \frac{1}{3}\frac{u^6}{x^6}\ldots\right)\frac{du}{x}$$

$$= \frac{2!}{x^3} - \frac{1}{2}\frac{4!}{x^3} + \frac{1}{3}\cdot\frac{6!}{x^7} - \ldots$$

& since $\int_0^\infty u^n e^{-u}\,du = n!$.

c) $I(x) = \int_0^{\pi/2} e^{-x\cos\theta}\, d\theta.$





Make a substitution

$$u = \cos\theta$$
$$du = -\sin\theta\, d\theta$$
$$= -\sqrt{1-u^2}\, d\theta.$$

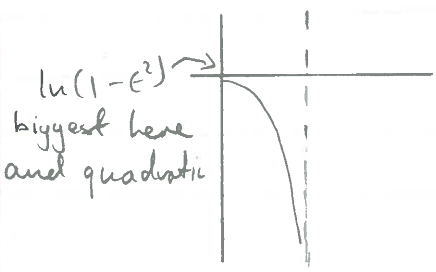
$$I(x) = \int_1^0 \frac{e^{-xu}}{-\sqrt{1-u^2}}\, du = \int_0^1 \frac{e^{-xu}}{\sqrt{1-u^2}}\, du$$

$$\& \ (1-u^2)^{-\frac{1}{2}} = 1 - \frac{1}{2}(-u^2) + \frac{(-\frac{1}{2})(-\frac{3}{2})(-u^2)^2}{2!} + \dots$$

$$= 1 + \frac{u^2}{2} + \frac{3}{8}u^4 + \dots$$

$$I(x) \sim \frac{1}{x} + \frac{1}{2}\cdot\frac{2!}{x^3} + \frac{3}{8}\cdot\frac{4!}{x^5}$$

d) $I(x) \underset{x\to\infty}{=} \int_0^1 (1-t^2)^x\, dt = \int_0^1 e^{x\ln(1-t^2)}\, dt.$



Put $\ln(1-t^2) = -u \Rightarrow 1-t^2 = e^{-u}$

$$\frac{-2t}{1-t^2}\,dt = -du.$$

$t=0,\ u=0$
$t\to 1,\ u\to\infty$

$$I(x) = \int_0^\infty e^{-xu}\frac{(1-t^2)}{2t}\,du.$$

$$= \int_0^\infty e^{-xu}\underbrace{\frac{e^{-u}}{2\sqrt{1-e^{-u}}}}_{f}\,du.$$

Generally $I \sim \frac{f(0)}{x}$, here $f(0) = \infty$.

$e^{-u} \approx 1-u+\ldots$

So $\frac{e^{-u}}{2\sqrt{1-e^{-u}}} \sim \frac{1-u}{2\sqrt{u}} \sim \frac{1}{2\sqrt{u}} = u^{\lambda}a_0$

$\approx \sqrt{1-1+u}$

so $\lambda = \frac{-1}{2}$, $a_0 = 1/2$

$$I(x) \sim \frac{(-1/2)!\cdot 1/2}{x^{-1/2+1}} = \frac{1}{2}\sqrt{\frac{\pi}{x}}$$

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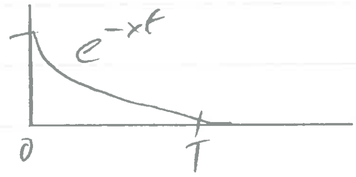
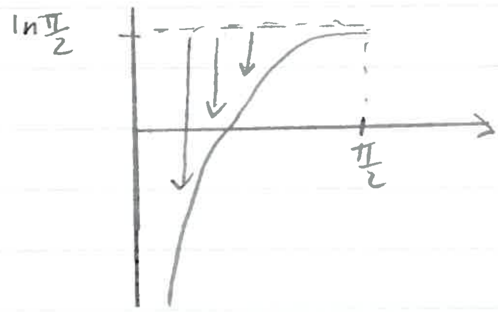
$$\int_0^T e^{-xt} f(t)\, dt \sim \sum_0^\infty \frac{a_n(\lambda+\lambda_n)!}{x^{\lambda+\lambda_n+1}}$$

$$f \sim t^\lambda \sum_0^\infty a_n t^{\lambda_n}$$



example $I(x) = \int_0^{\pi/2} t^x \sin t \, dt = \int_0^{\pi/2} e^{x \ln t} \sin t \, dt$.

Change variable so that



$$u = \ln \pi/2 - \ln t.$$

$$\ln t = \ln \pi/2 - u.$$

$$du = -\frac{1}{t}\, dt$$

$$I(x) = \int_\infty^0 \underbrace{e^{x \ln \pi/2}}_{(\frac{\pi}{2})^x}{}^{- xu} \sin\left(\frac{\pi}{2} e^{-u}\right)\left(-\frac{\pi}{2} e^{-u}\right) du.$$

$$= \left(\frac{\pi}{2}\right)^{x+1} \int_0^\infty e^{-xu} \underbrace{e^{-u} \sin\left(\frac{\pi}{2} e^{-u}\right)}_{f(u)} du.$$

$$, f(0) = 1$$

$$\sim \left(\frac{\pi}{2}\right)^{x+1} \frac{1}{x}.$$

$xu = v$

$$\left(\frac{\pi}{2}\right)^{x+1} \int_0^\infty e^{-v} f(v/x) \frac{dv}{x} \sim \left(\frac{\pi}{2}\right)^{x+1} \frac{f(0)}{x} \int_0^\infty e^{-v}\, dv \quad 1$$

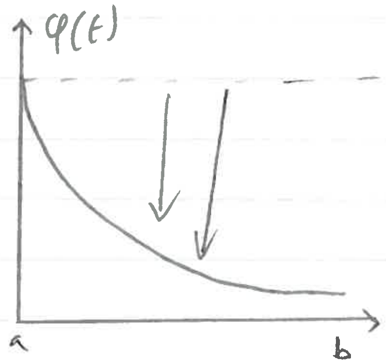
Laplace Integrals.

The previous example is an example of a Laplace Integral

$$D(x) = \int_a^b e^{x\phi(t)} f(t)\, dt.$$

We have several cases to consider:

a) $\varphi'(t) < 0$ in $[a,b]$, $\varphi'(x_0) \neq 0$ for some x_0 in $[a,b]$



Substitute

$$u = \varphi(a) - \varphi(t).$$

$t = a$, $u = 0$.

$t = b$, $u = \varphi(a) - \varphi(b) = \beta > 0$.

(Note: $t(u) = \varphi^{-1}(\varphi(a) - u)$) So $\varphi(t) = \varphi(a) - u$.

$\Rightarrow du = -\varphi'(t)\, dt$.

$$I(x) = \int_0^\beta e^{x\varphi(a)} e^{-xu} \frac{f(t(u))}{-\varphi^{-1}(t(u))}\, du$$

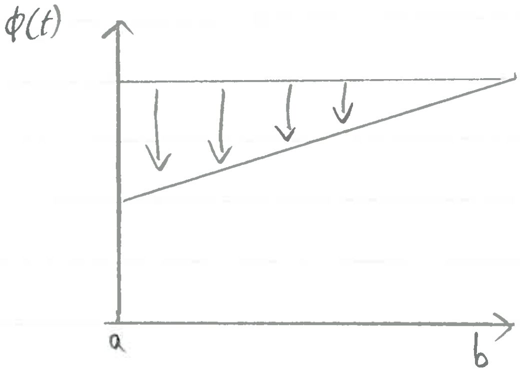
$t(u)$ is a single valued function

$$= -e^{x\varphi(a)} \underbrace{\int_0^\beta e^{-xu} \frac{f(t(u))}{-\varphi'(t(u))}\, du}_{\text{Use Watsons lemma}}.$$

If we only want the first term, this arises from $u = 0$ i.e $t = a$ & is $(f(a)/\varphi'(a))^1/x$. and noting $\varphi'(a) < 0$.

$$I(x) \sim \frac{e^{x\varphi(a)} f(a)}{|\varphi'(a)|} \cdot \frac{1}{x}.$$

b) $\varphi'(t) > 0$ but $\varphi'(x_0) \neq 0$ for x_0 in $[a,b]$



Change variable
$u = \varphi(b) - \varphi(t)$,
$$I(x) \sim \frac{e^{x\varphi(b)} f(b)}{x\varphi'(b)}$$

Together $$I(x) \sim \frac{e^{x\varphi(c)} f(c)}{x|\varphi'(c)|}$$

c is the end point giving largest value of φ.

$\varphi'(x) \neq 0$.

We can get this result by integration by parts.

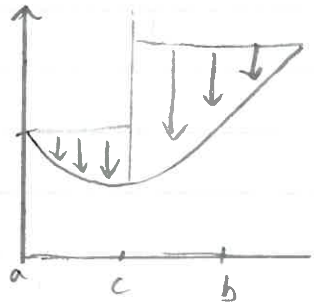
$$I(x) = \int_a^b e^{x\varphi(t)} f(t)\, dt = \int_a^b \underbrace{\varphi'(t) e^{x\varphi(t)}}_{\text{can integrate to give } \frac{1}{x}e^{x\varphi(t)}} \frac{f(t)}{\varphi'(t)}\, dt.$$

$$= \left[\frac{1}{x} e^{x\varphi(t)} \frac{f(t)}{\varphi'(t)}\right]_a^b - \frac{1}{x}\int_a^b e^{x\varphi(t)} \frac{d}{dt}\left(\frac{f(t)}{\varphi'(t)}\right) dt.$$

We could integrate by parts again but the terms obtained would be a factor $1/x$ smaller. The dominant term is $\frac{1}{x}\left[e^{x\varphi(t)} \frac{f(t)}{\varphi'(t)}\right]_a^b$

– the same result if we pick a or b depending on the largest of $e^{x\phi(a)}$ & $e^{x\phi(b)}$.

c) $\varphi'(c) = 0$ but $\varphi''(c) > 0$, $c \in [a,b]$

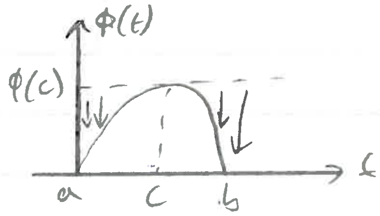


We make a similar change of variables in the variables in the intervals $[a,c]$ & $[c,b]$ as we did in cases (a) & (b). However $\varphi'(c) = 0$, so $\varphi'(t(a)) = 0$ at the end point given by c. The integral is improper, but as we shall see shortly is convergent ($1/r$ sing arises which is integrable)

So as in a & b the dominate contribution as $x \to \infty$ comes from the end point which gives the largest value of $\varphi(a)$ or $\varphi(b)$.

d) $\varphi'(c) = 0$, $c \in [a,b]$ $\varphi''(c) < 0$.

$$I(x) = \int_a^b e^{x\varphi(t)} f(t)\,dt.$$



Split the range of integration at $t = c$ & in $[a,c]$ write $u = \varphi(c) - \varphi(t)$ & in $[c,b]$ write $u = \varphi(c) - \varphi(t)$.

At $t = a$, $u = \varphi(c) - \varphi(a) = \beta > 0$, $du = -\varphi'(t)\,dt$

At $t = b$, $u = \varphi(c) - \varphi(b) = \bar{\beta} > 0$, $du = -\varphi'(t)\,dt$.

$$I(x) = e^{x\varphi(c)} \int_{\beta}^{0} e^{-xu} \frac{f(t(u))}{-\varphi'(t(u))}\, du \quad ①$$

$$+ e^{x\varphi(c)} \int_{0}^{\bar{\beta}} e^{-xu} \frac{f(t(u))}{-\varphi'(t(u))}\, du. \quad ②$$

We can try Watson's lemma on these two integrals. But both integrals have zeros in the denominator at $u=0$ as this where $\varphi'(t) = 0$.

Near $u=0$ i.e $t=c$, we have $-u = \varphi(t) - \varphi(c)$ & we can use Taylor expansion about $t=c$ to examine the local form of the transformation.

$$-u = \cancel{\varphi(c)} + \underbrace{\varphi'(c)}_{0}(t-c) + \frac{1}{2}\varphi''(c)(t-c)^2 + \dots \cancel{- \varphi(c)}$$

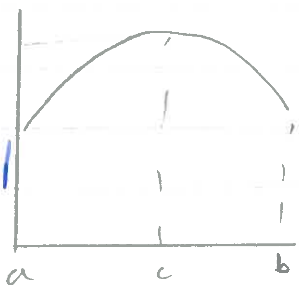
& $\varphi''(c) < 0$ so:

$$(t-c) = \pm\sqrt{\frac{2u}{|\varphi''(c)|}} \qquad \begin{array}{l} t>0 \text{ i.e } [c,b] \\ t<0 \text{ i.e } [a,c] \end{array}$$

$f(t(u)) \approx f(c)$ to first order

& $\varphi'(t) \approx \varphi'(c) + (t-c)\varphi''(c) + \dots$

$$= -|\varphi''(c)| \left(\begin{array}{c}+\\-\end{array}\right) \sqrt{\frac{2u}{|\varphi''(c)|}} = \pm\sqrt{2u|\varphi''(c)|}$$



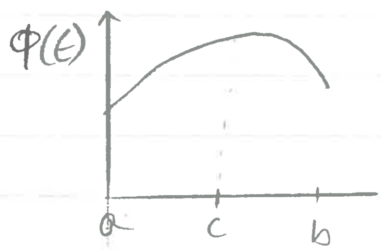
So using the right form for $\varphi'(t(u))$ near $u=0$ in the relevant integral & committing an exponentially small error.

$$I(x) \sim 2e^{x\varphi(c)} \int_0^\infty \frac{e^{-xu} f(c)}{\sqrt{2u|\varphi''(c)|}}\, du.$$

write $v = xu$

$$= \sqrt{\frac{2}{|\varphi''(c)|}}\, f(c) \underbrace{\int_0^\infty \frac{e^{-v}}{v^{1/2}}}_{(-\frac{1}{2})! = \sqrt{\pi}} \cdot \frac{dv}{\sqrt{x}}$$

$$\int_a^b e^{x\varphi(t)} f(t)\, dt \sim \sqrt{\frac{2\pi}{|\varphi''(c)|}}\, \frac{f(c)}{\sqrt{x}}\, e^{x\varphi(c)}$$



$$I(x) = \int_0^{\pi/2} t^x \sin t\, dt \;=\; \int_0^{\pi/2} e^{\overset{\phi(t)}{\ln t\, x}} \frac{\sin t}{f(t)}\, dt$$

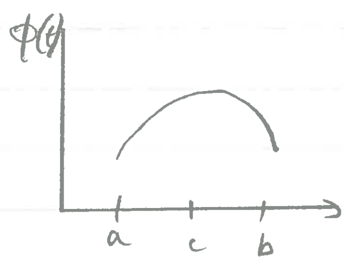
$\varphi(t)$ is a maximum at $t = \pi/2$, where

$$\varphi'(t) = \frac{1}{t} = \frac{2}{\pi} \neq 0 \qquad \left|\; \frac{e^{x\phi(c)} f(c)}{x\,|\phi'(c)|} \right.$$

$$I(x) \sim \frac{e^{x \ln \pi/2} \sin(\pi/2)}{x\,|2/\pi|} = \left(\frac{\pi}{2}\right)^{x+1} \frac{1}{x}$$

An alternative for the case (d) above:

$$\int_a^b e^{x\varphi(t)} f(t)\, dt\,.$$



Focus on the region near the maximum in φ.

$$\varphi(t) \approx \varphi(c) + (t-c)\varphi'(c) + \frac{1}{2}(t-c)^2\varphi''(c) + \left\{ \begin{array}{l} a_1(t-c)^3 \\ + a_2(t-c)^4 \dots \end{array} \right\}$$

$$e^{x\varphi(t)} = e^{x\varphi(c)} \underbrace{e^{-\frac{x}{2}(t-c)^2|\varphi''(c)|}}_{e^{-u^2}} \left\{ e^{a_1 x(t-c)^3} e^{a_1 x(t-c)^4} \dots \right\}$$

$$u^2 = x\frac{|\varphi''(c)|}{2}(t-c)^2, \quad (t-c) = u\sqrt{\frac{2}{x|\varphi''(c)|}}(\pm)$$

$$I(x) \sim e^{x\varphi(c)} \int_{-ve\sqrt{x}}^{+ve\sqrt{x}} e^{-u^2} \left\{ e^{\hat{a}_1 u^3/\sqrt{x}} \right\} \left\{ e^{\hat{a}_2 u^4/x} \right\} \cdot$$

$$\cdot \left[f(c) + \left\{ f'(c)\frac{u}{\sqrt{x}} b_1 + f''(c)\frac{u^2}{x} b_2 \right\} \right] \cdot$$

$$\cdot \sqrt{\frac{2}{x|\varphi''(c)|}}\, du.$$

$$\sim f(c) e^{x\varphi(c)} \sqrt{\frac{2\pi}{x|\varphi''(c)|}} \left[1 + e_1 \cancel{\int_{-\infty}^{\infty} \frac{u e^{-u^2}}{\sqrt{x}} du}^{\,0} \right.$$

$$+ e_2 \int_{-\infty}^{\infty} \frac{u^2 e^{-u^2}}{x} du.$$

$$\left. + e_3 \cancel{\int_{-\infty}^{\infty} \frac{u^3 e^{-u^2}}{x^{3/2}} du}^{\,0} \dots \right\}$$

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Laplace Integrals $I(x) = \int_a^b e^{x\phi(t)} f(t)\,dt \sim \frac{e^{\phi(c)}}{x} \cdot \frac{f(c)}{|\phi'(c)|}$

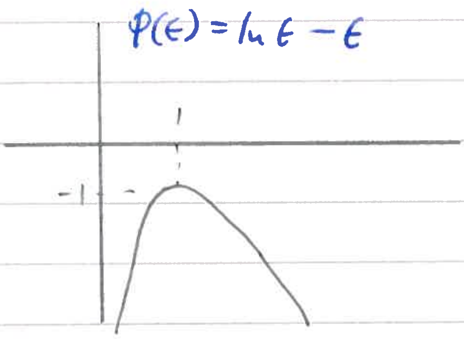
$\phi(c)$ is max value ϕ on $[a,b]$

but if $\phi'(c) = 0$, $I(x) \sim e^{x\phi(c)} f(c) \sqrt{\frac{2\pi}{x|\phi''(c)|}}$ $c \in (a,b)$

Example: Stirlings formula: $x! \sim \sqrt{2\pi}\, e^{-x} x^{x+1/2}$
$x \to \infty$

$I(x) = x! = \int_0^\infty e^{-u} u^x \, du = \int_0^\infty e^{-xt} x^x t^x x \, dt$
$u = xt$

$= x^{x+1} \int_0^\infty e^{x\overbrace{(\ln t - t)}^{\phi}} \, dt$ $f=1$



$\phi'(t) = \frac{1}{t} - 1 = 0$ at $t = 1$

& $\phi(1) = \ln 1 - 1 = -1$.

$\phi''(t) = -\frac{1}{t^2}$ & $\phi''(1) = -1$

So $I(x) \sim x^{x+1} e^{x(-1)} \cdot \sqrt{\frac{2\pi}{x|-1|}} \sim x^{x+\frac{1}{2}} e^{-x} \sqrt{2\pi}$.
$(\phi(1) = -1)$

$\phi''(1) = -1$

Fourier Integrals. These are of the form:

$$I(x) = \int_a^b e^{ix\phi(t)} f(t) \, dt.$$

These integrals are subject to cancellation as $x \to \infty$ as the real and imaginary parts of $e^{ix\phi(t)}$ oscillate rapidly & over a period $f(x)$ does not vary much Riemann-Lesbegue Lemma.

Near a max/min in ϕ, where $\phi'(t) = 0$, the cancellation is less strong & the maximum contribution to the integral is from the vicinity of where $\phi'(t) = 0$.

Lets us consider integration by parts:

$\phi'(t) \neq 0$ this is not proper.

$$I(t) = \int_a^b e^{ix\underbrace{\phi(t)}_{\text{phase}}} f(t) \, dt.$$

$$= \int_a^b \phi'(t) e^{ix\phi(t)} \frac{f(t)}{\phi'(t)} \, dt.$$

$$= \underbrace{\left[\frac{1}{ix} e^{ix\phi(t)} \cdot \frac{f(t)}{\phi'(t)}\right]_a^b}_{\sim O(1/x)} - \underbrace{\int_a^b \frac{e^{ix\phi(t)}}{ix} \left(\frac{f(t)}{\phi'(t)}\right)' dt}_{\sim (1/x^2)}$$

We have to keep both contributions from a & b & $I(x) = O(1/x)$ $x \to 0$.

$$I(x) = \frac{-i}{x}\left[e^{ix\phi(b)} \frac{f(b)}{\phi'(b)} - e^{ix\phi(a)} \frac{f(a)}{\phi'(a)}\right]$$

What if $\phi'(c) = 0$ for $c \in (a,b)$.
As $x\phi(t)$ is the "phase" this method below is called the method of stationary phase.

$$I(x) = \int_{\underline{\underline{a}}}^{c-\delta} + \int_{c-\delta}^{c+\delta} + \int_{\underline{\underline{c+\delta}}}^{b} \left\{ e^{ix\phi(t)} f(t) \right\} dt$$

The contribution from:

$$\int_a^{c-\delta} \; \& \int_{c+\delta}^{b} \quad \text{are} \quad O(1/x).$$

We expect the contribution from $\int_{c-\delta}^{c+\delta}$ to be bigger. As δ can be made small we can approximate $e^{ix\phi(t)} f(t)$ by a Taylor series about $t = c$.

$$f(t) = f(c) + (t-c)f'(c) + \ldots$$

$$ix\phi(t) = ix\phi(c) + ix(t-c)\underbrace{\phi'(c)}_{\to 0} + \frac{ix}{2}(t-c)^2\phi''(c) \ldots$$

$$e^{ix\phi(c)} = \underbrace{e^{ix\phi(c)}}_{\text{const}} \; \underbrace{e^{\frac{ix}{2}(t-c)^2\phi''(c)}}_{e^{iu^2 s}} \; e^{(\ldots)}$$

$$u^2 = (t-c)^2 \frac{|\phi''(c)|}{2} x \Rightarrow t - c = u\sqrt{\frac{2}{|\phi''(c)|x}}$$

$$s = \text{sgn}\ \phi''(c) = \begin{cases} +1 & \phi''(c) > 0 \\ -1 & \phi''(c) < 0 \end{cases}$$

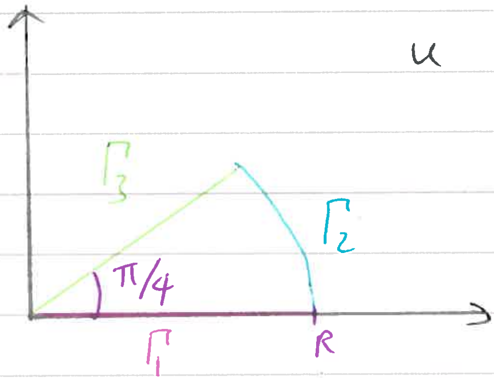
$$I(x) \sim f(x) e^{ix\phi(c)} \int_{-\delta\sqrt{\frac{x\phi''(c)}{2}}}^{+\delta\sqrt{\frac{x\phi''(c)}{2}}} e^{isu^2} \sqrt{\frac{2}{|\phi''(c)x|}}\ du.$$

δ small but x big
& $\sqrt{x}\,\delta$ can be made
as large as we wish.

$$I(x) \sim \frac{f(c)e^{ix\phi(c)}}{\sqrt{x}} \sqrt{\frac{2}{|\phi''(c)|}} \int_{-\infty}^{\infty} e^{isu^2}\ du.$$

We need only consider the case $s=1$ & $s=-1$ is its complex conjugate. Also:

$$\int_{-\infty}^{\infty} e^{iu^2}\ du = 2 \underbrace{\int_0^{\infty} e^{iu^2}\ du}_{I}$$



$$\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3$$

$$0 = \int_\Gamma e^{iu^2}\,du = \underset{\substack{u=t \\ t\in[0,R]}}{\int_{\Gamma_1}} + \underset{\substack{u=Re^{i\theta} \\ \theta\in[0,\pi/4)}}{\int_{\Gamma_2}} + \underset{\substack{u=e^{i\pi/4}t \\ t[R,0]}}{\int_{\Gamma_3}}$$

$$0 = \int_0^R e^{it^2}\,dt + \int_0^{\pi/4} \underbrace{e^{iR^2e^{2i\theta}}}_{\substack{\text{modulus is} \\ e^{-R^2\sin 2\theta} \ \& \ \sin 2\theta \geq 0 \\ \& \text{ as } R\to\infty \text{ this} \to 0}} iRe^{i\theta}\,d\theta + \int_R^0 \underbrace{e^{it^2e^{i\frac{\pi}{2}}}}_{e^{-t^2}} e^{i\pi/4}\,dt$$

As $R\to\infty$

$$\int_0^\infty e^{it^2}\,dt = e^{i\pi/4}\int_0^\infty e^{-t^2}\,dt = e^{i\pi/4}\frac{\sqrt{\pi}}{2}$$

$$\& \int_{-\infty}^{\infty} e^{isu^2}\,ds = e^{is\pi/4}\sqrt{\pi}.$$

$$\int_a^b e^{ix\varphi(t)}\,dt \underset{x\to\infty}{\sim} \cancel{O}(1/x) \quad \text{if } \varphi'(x) \neq 0 \text{ in } [a,b]$$

$$\sim e^{ix\varphi(c)} e^{i\,\mathrm{sgn}(\varphi''(c))\pi/4} f(c) \cdot \sqrt{\frac{2\pi}{|\varphi''(c)|x}}$$

$$\varphi'(c) = 0.$$

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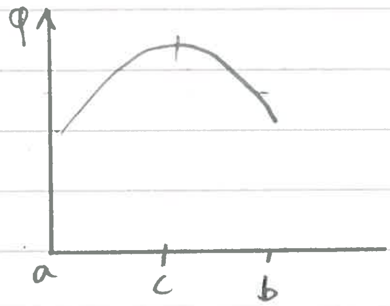
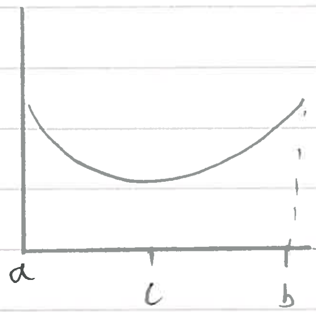
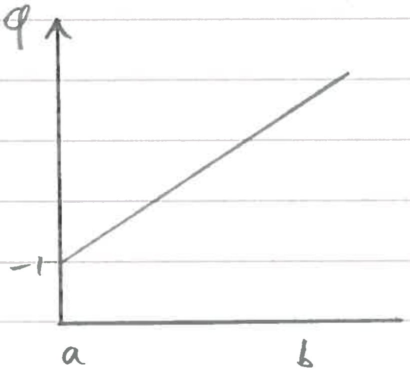
$$\int_a^b e^{ix\varphi(t)} f(t)\,dt \sim e^{ix\varphi(c)} f(c) \sqrt{\frac{2\pi}{x|\varphi''(c)|}}\, e^{i\,\mathrm{sgn}(\varphi''(c))\pi/4}. \quad x \to \infty$$



or



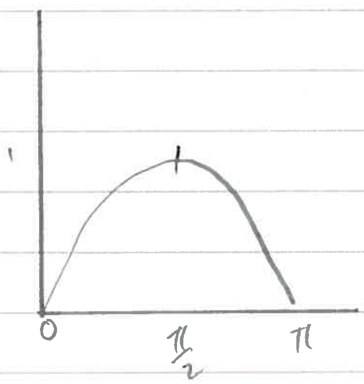
$\varphi''(c) < 0$ $\qquad$ $\varphi''(c) > 0$



Example:

$$I(x) = \int_0^{\pi} e^{ix\underbrace{\sin\theta}_{\varphi}}\,\underbrace{d\theta}_{f=1}$$

cf $\int_0^{\pi} \cos(x\sin\theta)$



$\varphi(\theta) = \sin\theta$

$\varphi'(\theta) = \cos\theta = 0$ at $\theta = \pi/2$

$\varphi''(\theta) = -\sin\theta. = 1$ at $\pi/2$.

$\varphi(\pi/2) = \sin(\pi/2) = 1$

$$I(x) \sim e^{ix1} \; 1 . \sqrt{\frac{2\pi}{x|-1|}} \; e^{i(-1)\pi/4} = e^{i(x-\pi/4)}\sqrt{\frac{2\pi}{x}}$$

$\leftarrow \varphi(\frac{\pi}{2})$ $\quad$ $f(\frac{\pi}{2})$ $\quad$ $\varphi''(\frac{\pi}{2})$ $\quad$ $\operatorname{sgn}(-1)$

$$\int_0^{\pi} \cos(x\sin\theta)\, d\theta \sim Re\left\{ e^{i(x-\frac{\pi}{4})} \sqrt{\frac{2\pi}{x}} \right\}$$

$$= \cos\left(x - \pi/4\right)\sqrt{\frac{2\pi}{x}}.$$

2) $I(x) = \displaystyle\int_{-\infty}^{\infty} \cos(xt - t^3/3)\, dt$

Similarly to $\left.\begin{matrix} A_i(x) \\ B_i(x) \end{matrix}\right\} = \displaystyle\int_C e^{xt - t^3/3}\, dt$

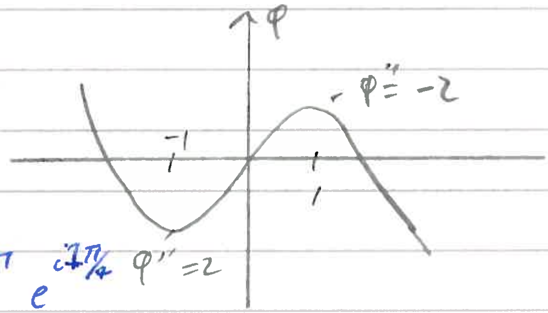
$= Re \displaystyle\int_{-\infty}^{\infty} e^{i[xt - t^3/3]}\, dt$

$xt \equiv t^3$
$x = t^2$
$t = x^{1/2}$

& write $t = x^{1/2} u$.

$$= Re \int_{-\infty}^{\infty} e^{ix^{3/2}\underbrace{[u - u^3/3]}_{\varphi(u)}} x^{1/2}\, du.$$

Both stationary points contribute equally.



$$I(x) \sim x^{1/2} Re\Bigg[e^{i(-\frac{2}{3})x^{3/2}} . 1 . \sqrt{\frac{2\pi}{x^{3/2}|2|}} \; e^{i\pi/4}$$

$$+ e^{i(-\frac{2}{3})x^{3/2}} . 1 . \sqrt{\frac{2\pi}{x^{3/2}|-2|}} \; e^{i(-1)\pi/4} \Bigg]$$

$\varphi'(u) = 1 - u^2$
$\varphi''(u) = -2u$
$\varphi(\pm 1) = \pm 2/3$

$$\ldots \sim \frac{2\sqrt{\pi}}{x^{1/4}} \cos\left[\frac{2}{3}x^{3/2} - \frac{\pi}{4}\right].$$