3401 Mathematical Methods 5 Notes

Based on the 2013 autumn lectures by Dr R I Bolwes

MATH3401 – Mathematical Methods 5.

1 October 2013
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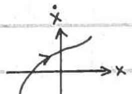
Course Overview.

- Solving ODEs of the form $y'' + P(x)y' + Q(x)y = R(x)$
- Finding solutions of the form $y(x) = \int_c e^{xt} f(t)\,dt$.



- Phase Planes and solving solutions of type $\ddot{x} = g(x, \dot{x})$ – getting a qualitative understanding. / special cases related to oscillations

  e.g. $\ddot{x} + x = 0$ (SHM), $\ddot{x} + \varepsilon f(\dot{x}, x) + x = 0$, $\ddot{x} - \varepsilon\dot{x}(1-\dot{x}^2) + x = 0$.
- Analysis of behaviour of $y(x) = \int_c e^{xt} f(t)\,dt$ as x approaches infinity e.g. $\int_0^1 e^{-xt}\sqrt{1+t^2}\,dt \sim \frac{1}{x}$ as $x \to \infty$.

6-7 sheets of homework, not due every week. Distributed on Moodle.

Proposed Office Hours – Tue 10am, Fri 9am. Recommended text: King, Billingham and Otto – *Differential Equations*.

Chapter 1.
ORDINARY DIFFERENTIAL EQUATIONS.

Consider $y'' + P(x)y' + Q(x)y = R(x)$, with P, Q, R functions of x only, $y(x)$ a function to be found.

This is a linear second order ODE, with solution of form $y = CF + PI$

Complementary function is found by setting $R = 0$; particular integral is any spottable solution.

CF is of the form $y = Ay_1(x) + By_2(x)$ where y_1, y_2 are linearly independent solutions of $y'' + Py' + Qy = 0$. (i.e. $\nexists$ non-zero c_1, c_2 s.t. $c_1y_1 + c_2y_2 = 0\ \forall x$)

Reduction of order method.

Consider $y'' + a_1(x)y' + a_0(x)y = 0$, and presume that one solution is known. Let this be $y = u$, such that $u'' + a_1u' + a_0u = 0$.

We look for a second solution $y(x) = u(x)v(x)$ and find $v(x)$. Substitution gives: $(u''v + 2u'v' + uv'') + a_1(u'v + uv') + a_0uv = 0$. — ⊛

Since u is a solution of the ODE, $u'' + a_1u' + a_0u = 0 \Rightarrow u''v + a_1u'v + a_0uv = 0$, so three terms in ⊛ cancel.

Then ⊛ $\Rightarrow uv'' + (2u' + a_1u)v' = 0 \Rightarrow$ letting $z = v'$, $z' + (\frac{2u'}{u} + a_1)z = 0$. This can be solved with an integrating factor.

$\boxed{\text{Aside}}$ We can also solve non-homogeneous equations of form $y'' + a_1(x)y' + a_0(x)y = r$, in which case we would obtain $z' + (\frac{2u'}{u} + a_1)z = \frac{r}{u}$ by same notation.

$z' + (\frac{2u'}{u} + a_1)z = 0 \Rightarrow \frac{z'}{z} + (\frac{2u'}{u} + a_1) = 0 \Rightarrow \ln z + \ln u^2 + \int^x a_1(t)\,dt = \text{const}$, which is separable.
indefinite integral

$\Rightarrow z = \frac{A}{u^2} e^{-\int^x a_1(t)\,dt} \Rightarrow v = A\int^x \frac{1}{u^2(t)} e^{-\int^t a_1(s)\,ds}\,dt + B$.

However, solutions are $y = uv$, so $y(x) = Au\int^x \frac{1}{u^2(t)} e^{-\int^t a_1(s)\,ds}\,dt + Bu$.

$\boxed{\text{Note}}$ This demonstrates that Bu is part of the complementary function.

$\boxed{\text{Ex}}$ (Legendre's Equation of order 1)

Solve $(1-x^2)y'' - 2xy' + 2y = 0$. [Recall – Legendre's equation of order n is $(1-x^2)y'' - 2xy' + n(n+1)y = 0$].

soln. We know that one solution is the Legendre polynomial of order 1, $P_1(x) = x$. We verify that this is a solution by substituting $y = x$.

Equation is $y'' - \frac{2x}{1-x^2}y' + \frac{2}{1-x^2}y = 0$. We seek a solution $y = xv \Rightarrow y' = xv' + v$, $y'' = xv'' + 2v'$.

$\Rightarrow (1-x^2)(xv'' + 2v') - 2x(xv' + v) + 2xv = 0 \Rightarrow$ all terms with v cancel, as expected. This yields $(1-x^2)(xv'' + 2v') - 2x^2v'$.

On simplification, this gives $x(1-x^2)v'' = (4x^2 - 2)v' \Rightarrow \frac{v''}{v'} = \frac{4x^2-2}{x(1-x^2)} = \frac{A}{x} + \frac{B}{1-x} + \frac{C}{1+x}$ in partial fractions.

Solving the PFs, we get $\frac{v''}{v'} = \frac{-2}{x} + \frac{1}{1-x} - \frac{1}{1+x} \Rightarrow \ln v' = -2\ln x - \ln(1-x) - \ln(1+x)$. Constants of integration do not matter at this point.

$v' = \frac{1}{x^2(1-x^2)} = \frac{A}{x^2} + \frac{B}{x} + \frac{C}{1-x} + \frac{D}{1+x} = \frac{1}{x^2} + \frac{1}{2}\frac{1}{1-x} + \frac{1}{2}\frac{1}{1+x} \Rightarrow v = -\frac{1}{x} + \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)\ [+C]$.

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Thus, $y = \underbrace{-1 + \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right)}_{\text{second solution}} + \underbrace{Cx}_{\text{first solution}}$, Note: $-1 + \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right)$ is $Q_1(x)$, Legendre's second solution, order 1.

Variation of parameters method

Used to solve ODEs of form $y'' + Py' + Qy = R$ (non-homogeneous).

We presume we know y_1 and y_2, which are independent solutions to $y'' + Py' + Qy = 0$. Then, we seek a solution of the type $y(x) = A(x)y_1(x) + B(x)y_2(x)$.

There is obviously too much freedom in the solution, and we use this to our advantage by reducing this freedom, imposing a relationship between A and B.

$y = Ay_1 + By_2 \Rightarrow y' = \underline{A'y_1} + Ay_1' + \underline{B'y_2} + By_2'$. We choose $A'y_1 + B'y_2 = 0$. $\Rightarrow y' = Ay_1' + By_2'$ s.t. differentiating again, we avoid A'' and B''.

Following on, $y'' = A'y_1' + Ay_1'' + B'y_2' + By_2''$. Substitution means we need: $A'y_1' + Ay_1'' + B'y_2' + By_2'' + P(Ay_1' + By_2') + Q(Ay_1 + By_2) = R$ — ⊛.

Note however that $A(y_1'' + Py_1' + Qy_1) = A(0) = 0$ and similarly $B(y_2'' + Py_2' + Qy_2) = 0$ as y_1, y_2 are solutions to homogenous equations.

This reduces ⊛ to $A'y_1' + B'y_2' = R$. We get the system: $\begin{cases} A'y_1' + B'y_2' = R \\ A'y_1 + B'y_2 = 0 \end{cases}$, which are simultaneous equations for A' and B'.

Solve: $\begin{matrix} A'y_1'y_2 + B'y_2y_2' = Ry_2 \\ A'y_1y_2' + B'y_2y_2' = 0 \end{matrix} \Rightarrow A'(y_1'y_2 - y_1y_2') = Ry_2$ and similarly, $B'(y_1y_2' - y_1'y_2) = Ry_1$.

We define the quantity $w = \begin{vmatrix} y_1 & y_1' \\ y_2 & y_2' \end{vmatrix} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ as the Wronskian. Then $A' = -\frac{Ry_2}{w}$ and $B' = \frac{Ry_1}{w}$.

Then $A = -\int^x \frac{R(s)y_2(s)}{w(s)}\,ds + \bar{A}$ and $B = +\int^x \frac{R(s)y_1(s)}{w(s)}\,ds + \bar{B}$. The solution is $Ay_1(x) + By_2(x) = \underbrace{y_2\int^x \frac{R(s)y_1(s)}{w(s)}\,ds - y_1\int^x \frac{R(s)y_2(s)}{w(s)}\,ds}_{\text{PI}} + \underbrace{\bar{A}y_1 + \bar{B}y_2}_{\text{CF}}$

To simplify algebra, we can write particular integral as $\int^x \frac{y_2(x)y_1(s) - y_1(x)y_2(s)}{w(s)} R(s)\,ds$.

$\boxed{\text{Ex}}$ Solve $y'' + y = \sec x$.

Soln. The CF is $y = a\cos x + b\sin x$. Then we seek the PI: let $y = A(x)\cos(x) + B(x)\sin(x) \Rightarrow y' = A'\cos x + A(-\sin x) + B'\sin x + B\cos x$

Choose $A'\cos x + B'\sin x = 0 \Rightarrow y'' = A'(-\sin x) + A(-\cos x) + B'(\cos x) + B(-\sin x)$. Then we need:

$A'(-\sin x) + A(-\cos x) + B'(\cos x) + B(-\sin x) + A\cos x + B\sin x = \sec x \Rightarrow A'(-\sin x) + B'(\cos x) = \sec x$.

Then, $\begin{cases} A'\cos x + B'\sin x = 0 \\ -A'\sin x + B'\cos x = \sec x = \frac{1}{\cos x} \end{cases} \Rightarrow \begin{matrix} A'\cos^2 x + B'\sin x\cos x = 0 \\ -A'\sin^2 x + B'\sin x\cos x = \tan x \end{matrix} \Rightarrow A'(\cos^2 x + \sin^2 x) = -\tan x$

$w = \cos^2 x + \sin^2 x = 1. \Rightarrow A' = -\tan x,\ A = \ln(\cos x)$. Then $B'(\cos^2 x + \sin^2 x) = \frac{\cos x}{\cos x} = 1 \Rightarrow B' = 1 \Rightarrow B = x$.

Thus, solution is $y = a\cos x + b\sin x + \cos x\ln(\cos x) + x\sin x$ //.

The Wronskian

1. If we have linearly dependent functions y_1 and y_2, then $\exists$ non-zero c_1, c_2 s.t. $c_1y_1(x) + c_2y_2(x) = 0$ for all x. Differentiating, $c_1y_1'(x) + c_2y_2'(x) = 0$.

i.e. we get the system $\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ which has solutions if $\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1' = 0$ because matrix is not invertible.

However, if $w \neq 0$, then $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ by invertibility $\Rightarrow c_1 = c_2 = 0 \Rightarrow y_1, y_2$ are linearly independant functions.

2. If $y_1'' + Py_1' + Qy_1 = 0$ ① and $y_2'' + Py_2' + Qy_2 = 0$ ②, then we have $y_1② - y_2①$: $\underbrace{(y_1y_2'' - y_2y_1'')}_{w'} + P\underbrace{(y_1y_2' - y_2y_1')}_{w} + Q\underbrace{(y_1y_2 - y_2y_1)}_{0} = 0$.

$\Rightarrow w' + Pw = 0$. as $w' = y_1'y_2' + y_1y_2'' - y_2'y_1' - y_2y_1'' = y_1y_2'' - y_2y_1''$.

So, the Wronskian of the two linearly independant solutions of the ODE satisfies $w' + Pw = 0$ i.e. $w = Ae^{-\int^x P(s)\,ds}$.

Remark — If $P(x) = 0$, w is constant. Also, if y_1 is known, this gives an equation to find y_2 $\because w = y_1y_2' - y_2y_1'$.

Generalised Transforms.

Recall that for Laplace Transforms we have, for a function $y(t) \to \bar{y}(s)$, then $\overline{y'} = s\bar{y} - y(0)$, $\overline{ty} = -\frac{\partial}{\partial s}\bar{y}(s)$, $y = \frac{1}{2\pi i}\int_\gamma e^{st}\bar{y}(s)\,ds$ [Bromwich integral]

For Fourier Transforms, $y(x) \to \hat{y}(k)$, $\underbrace{\hat{y'} = ik\hat{y}}_{\substack{\text{transform of derivative corresponds to} \\ \text{multiplication by transform variable}}}$, $\underbrace{\widehat{xy} = i\frac{\partial}{\partial k}\hat{y}(k)}_{\substack{\text{transform of multiplication} \\ \text{corresponds to differentiating} \\ \text{transform of function}}}$, $\underbrace{y = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\hat{y}(k)\,dk}_{\substack{\text{integral representation of} \\ \text{original function } y.}}$.

Fourier and Laplace Transforms correspond.

Question: Can we find solutions to differential equations $(a_1x + a_0)y'' + (b_1x + b_0)y' + (c_1x + c_0)y = 0$ [Here polynomial coefficients have degree < order of ODE]

of the type $y = \int_C e^{xt}f(t)\,dt$ where C is some contour in the complex plane? [or if xt is multiplied by i or -1?]

Consider if $y = \int_C e^{xt}f(t)\,dt \Rightarrow \frac{dy}{dx} = \int_C \frac{\partial}{\partial x}e^{xt}f(t)\,dt = \int_C t f(t)e^{xt}\,dt$. Similarly, $y'' = \int_C t^2 f(t)e^{xt}\,dt$.

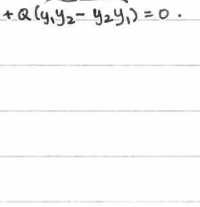
If $y(x) = \int_C e^{xt}f(t)\,dt$ is a solution to $a_0y'' + b_0y' + c_0y = 0$ (i.e. $a_1 = b_1 = c_1 = 0$), then solution requires: $\int_C e^{xt}f(t)[a_0t^2 + b_0t + c_0]\,dt = 0$.

Notice that $a_0t^2 + b_0t + c_0$ is the auxillary equation. Let $\int_C e^{xt}f(t)[a_0t^2 + b_0t + c_0]\,dt = a_0\int_C e^{xt}f(t)[(t-\alpha)(t-\beta)]\,dt$; where α, β are roots of AE.

For $y(x) = \int_C e^{xt}f(t)\,dt$ to be a solution, we must have $a_0\int_C e^{xt}f(t)[(t-\alpha)(t-\beta)]\,dt = 0$. Hence, we must pick a closed contour s.t. integrand has no singularities, by choosing $f(t)$...

because e^{xt}, $(t-\alpha)(t-\beta)$ has no singularities. However, on the interior of C, $\int_C e^{xt}f(t)\,dt$ must have a singularity s.t. $y(x)$ is non-trivial.

We choose: $f(t) = \frac{A}{t-\alpha} + \frac{B}{t-\beta}$, then $\int_C e^{xt}f(t)\,dt \neq 0$ if C surrounds α, β; but $f(t)(t-\alpha)(t-\beta)$ is singularity free.



Then $a_0\int_C e^{xt}f(t)[(t-\alpha)(t-\beta)]\,dt = a_0\int_C e^{xt}[A(t-\beta) + B(t-\alpha)\,dt = 0$.

Depending of curve C, we pick up solutions. Referring to graph on right, we get

- $C \Rightarrow y = 0$
- $C_1 \Rightarrow y = y_1(x) = \int_{C_1} e^{xt}\left(\frac{A}{t-\alpha} + \frac{B}{t-\beta}\right)dt$
- $C_2 \Rightarrow y = y_2(x) = \int_{C_2} e^{xt}\left(\frac{A}{t-\alpha} + \frac{B}{t-\beta}\right)dt$.
- $C_3 \Rightarrow y = y_1(x) + y_2(x)$ [general solution].

By Cauchy's Integral theorem, we calculate integrals using residue formula: $g(t_0) = \frac{1}{2\pi i}\oint_C \frac{g(t_0)}{t - t_0}\,dt$ for some t_0 in interior of C.

Then $y_1(x) = \oint_{C_1} \frac{e^{xt}.A}{t-\alpha}\,dt = 2\pi i\,(Ae^{xt})\big|_{t=\alpha} = \bar{A}e^{\alpha x}$. Similarly $y_2(x) = \int_{C_2} \frac{e^{xt}B}{t-\beta}\,dt = 2\pi i\,(Be^{xt})\big|_{t=\beta} = \bar{B}e^{\beta x}$.

one solution

to get other solution

If $\alpha = \beta$ (repeated roots), then $y = \int_C e^{xt}f(t)\,dt$ is a solution if $\int_C e^{xt}f(t)[(t-\alpha)^2]\,dt = 0$. Then we choose $f(t) = \frac{A}{(t-\alpha)^2} + \frac{B}{(t-\alpha)}$

$\Rightarrow y = \oint_C e^{xt}\left[\frac{A}{(t-\alpha)^2} + \frac{B}{t-\alpha}\right]dt = 2\pi i \operatorname*{res}_{t=\alpha}\left[e^{xt}.\left[\frac{A}{(t-\alpha)^2} + \frac{B}{t-\alpha}\right]\right]$. Recall that residue is coefficient of $\frac{1}{t-\alpha}$ in expansion about $t = \alpha$.

Then we apply a trick. $y = 2\pi i \operatorname*{res}_{t=\alpha}\left[e^{\alpha x} e^{x(t-\alpha)} \cdot \left\{\frac{A}{(t-\alpha)^2} + \frac{B}{t-\alpha}\right\}\right] \overset{\text{small } t-\alpha}{\approx} 2\pi i \operatorname*{res}_{t=\alpha}\left[e^{\alpha x}[1 + x(t-\alpha) + \dots]\left\{\frac{A}{(t-\alpha)^2} + \frac{B}{t-\alpha}\right\}\right] = 2\pi i \cdot e^{\alpha x}(Ax+B) = (\bar{A}x + \bar{B})e^{\alpha x}$

Hence, solution with repeated roots is $y = (Ax+B)e^{\alpha x}$.

In the general case, for $(a_1x + a_0)y'' + (b_1x + b_0)y' + (c_1x + c_0)y = 0$, we try $y = \int_C e^{xt} f(t)\,dt$ and substitute to find $\int_C [x(a_1t^2 + b_1t + c_1) + (a_0t^2 + b_0t + c_0)] f(t) e^{xt} dt = 0$.

Note that $xe^{xt} = \frac{d}{dt}e^{xt}$, which motivates an attempt at solution with integration by parts.

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First however, we shall try to ensure that $\int_C [x(a_1t^2 + b_1t + c_1) + (a_0t^2 + b_0t + c_0)] f(t) e^{xt} dt = 0$ by attempting to write it as:

$\int_C \frac{d}{dt}[e^{xt} g(t)]\,dt = [e^{xt} g(t)]_C = 0$ for a C of choice. (Similar to what we had before)

Now, $\frac{d}{dt}[e^{xt} g(t)] = xe^{xt} g(t) + e^{xt} g'(t)$ by product rule. We identify $g(t) = (a_1t^2 + b_1t + c_1)f$ s.t. $g'(t) = (a_0t^2 + b_0t + c_0)f$

Then, $\frac{g'}{g} = \frac{a_0t^2 + b_0t + c_0}{a_1t^2 + b_1t + c_1}$, which will give us an expression for g.

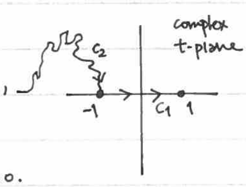
$\boxed{\text{Ex}}$ Solve $xy'' + 4y' - xy = 0$ with $x > 0$.

Soln. Look for a solution $y = \int_C e^{xt} f(t)\,dt$. Substitution gives $0 = \int_C \left(x(t^2-1) + 4t\right) f(t) e^{xt} dt = \int_C (xt^2 + 4t - x) e^{xt} f(t)\,dt$

Then $0 = \int_C \frac{d}{dt}[e^{xt} g(t)]\,dt = [e^{xt} g(t)]_C$ if $\left(x(t^2-1) + 4t\right)f = xg + g' \Rightarrow$

We require $\begin{matrix}(t^2-1)f = g \\ 4t\,f = g'\end{matrix} \Rightarrow \frac{g'}{g} = \frac{4t}{t^2-1} \Rightarrow \log g = 2\log(t^2-1) + c \Rightarrow g(t) = A(t^2-1)^2$. Then by back substitution,

$f = \frac{g}{(t^2-1)} = A(t^2-1)$. Then $y(x) = \int_C e^{xt}(t^2-1)\,dt$ is a solution if $[e^{xt}(t^2-1)^2]_C = 0$.



One possibility, C_1, is to have a contour joining $t=-1$ to $t=1$. Then $[e^{xt}(t^2-1)^2]_{C_1} = e^{xt}(t^2-1)^2\big|_{t=1} - e^{xt}(t^2-1)^2\big|_{t=-1} = 0$.

So $y_1(x) = \int_{C_1} e^{xt}(t^2-1)\,dt = \int_{-1}^{1} e^{xt}(t^2-1)\,dt$ is the first solution.

For the second solution, note that $x > 0 \Rightarrow xt \to 0$ as $t \to -\infty$. Thus, if we pick any contour C_2 running from $t = -\infty$ to $t = -1$, again $[e^{xt}(t^2-1)^2]_{C_2} = 0$

(since $e^{xt} \to 0$ as $\mathrm{Re}(t) \to -\infty$). Then $y_2(x) = \int_{-\infty}^{-1} e^{xt}(t^2-1)\,dt$.

As such, the general solution is $y(x) = A\,y_1(x) + B\,y_2(x)$, as found above.

Aside — some necessary notation:

(a) $f(x) = O(g(x))$ as $x \to 0$ (say, or some kind of limiting process of x), denoted "order" g, if we can find constants K and X s.t. $|f| < K|g|$ for $x > X$. (not necessarily limit)

This means, for instance, $x^2 + x = O(x^2)$ as $x \to \infty$ and $x^2 + x = O(x)$ as $x \to 0$.

(b) $f(x) = o(g(x))$ as $x \to \infty$ (limiting process) if $\left|\frac{f}{g}\right| \to 0$ as $x \to \infty$.

For instance, $f(x) = o(1)$ as $x \to \infty$ as $|f(x)| \to 0$ as $x \to \infty$.

(c) $f(x) \sim g(x)$ as $x \to \infty$ (say) means $\left|\frac{f}{g}\right| \to 1$ as $x \to \infty$. Hence, for instance, $x^2 + x \sim x^2$ as $x \to \infty$, $x^2 + x \sim x$ as $x \to 0$.

Now, we return to our earlier example. Let us examine the behaviour of $y_1(x) = \int_{-1}^{1} e^{xt}(t^2-1)\,dt$ and $y_2(x) = \int_{-\infty}^{-1} e^{xt}(t^2-1)\,dt$ for large positive and small x.

If $x = 0$, then $e^{xt} \equiv 1$ and $y_1(0) = \int_{-1}^{1}(t^2-1)\,dt = -\frac{4}{3}$ (finite). Indeed, if we express $e^{xt} = \sum_{n=0}^{\infty}\frac{x^n t^n}{n!}$ as a Taylor series, $y_1(x) = \sum_{n=0}^{\infty}\frac{x^n}{n!}\int_{-1}^{1} t^n(t^2-1)\,dt$.

However, $y_2(0) = \int_{-\infty}^{-1}(t^2-1)\,dt$ which does not exist. Thus, $y_2(x)$ has a singularity at $x=0$. This is the only singularity derived from equation.

For any non-zero positive x, however small, $y_2(x)$ does actually exist.

As $x \to \infty$, $y_1(x) = \int_{-1}^{1} e^{xt}(t^2-1)\,dt$ becomes exponentially large since e^{xt} becomes exponentially large in the interval $t \in (0,1]$.

The interval for t that gives the greatest contribution to the integral is close to $t=1$.

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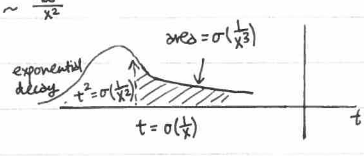
We know that $y_1(x) = \int_{-1}^{1}(t^2-1)e^{xt}\,dt$ is finite at $x=0$. $x \to \infty$?, $y_2(x) = \int_{-\infty}^{-1}(t^2-1)e^{xt}\,dt$ is infinite at $x=0$. $x \to \infty$?

In y_1, we make the substitution $t = 1 - \frac{u}{x}$. $y_1(x) = \int_{2x}^{0}\left(\frac{u^2}{x^2} - \frac{2u}{x}\right)e^x e^{-u}\left(-\frac{du}{x}\right) \sim -\frac{e^x}{x^2}\int_0^{\infty} 2u\,e^{-u}\,du$, as $x \to \infty$. $\sim \frac{-2e^x}{x^2}$

Doing the integral exactly, $y_1(x) = \frac{4}{x^3}(\sinh x - x\cosh x)$.

For y_2, as $x \to 0$, $e^{xt} \to 1$ except where $xt = o(1)$ i.e. $t = o(\frac{1}{x})$. $y_2(x) = \int_{-\infty}^{-1}(t^2-1)e^{xt}\,dt$.

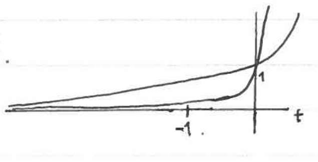


It appears that $y_2 = o(\frac{1}{x^3})$ as $x \to 0$ due to exponential decay. We substitute $xt = -u$. Then we get:

$y_2(x) = \int_{\infty}^{x} e^{-u}\left(\frac{u^2}{x^2} - 1\right)\left(-\frac{du}{x}\right) \sim \frac{1}{x^3}\int_0^{\infty} u^2 e^{-u}\,du$ as $x \to 0$. $\sim \frac{2}{x^3}$.

As $x \to \infty$, make the substitution $t = -1 - \frac{u}{x}$. Then $y_2(x) = \int_0^{\infty} e^{-x}e^{-u}\left(\frac{u^2}{x^2} + \frac{2u}{x}\right)\frac{du}{x} \sim \frac{2e^{-x}}{x^2}\int_0^{\infty} e^{-u}u\,du = \frac{2e^{-x}}{x^2}$.

Or, doing integral by parts, we can obtain that $y_2(x)$ is exactly $\frac{2e^{-x}}{x^3}(1+x)$.



These approximations are consistent and can be checked against values of x.

Ex Let $xy'' + (3x-1)y' - 9y = 0$, $x > 0$. Find solutions of the type $y(x) = \int_C e^{xt} f(t)\, dt$.

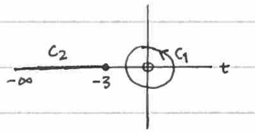
Soln. Substitution gives $0 = \int_C (t^2+3t)\, x\, e^{xt} f(t)\, dt - \int_C (t+9) e^{xt} f(t)\, dt = \int_C (t^2+3t)\left(\frac{d}{dt} e^{xt}\right) f(t)\, dt - \int_C (t+9) e^{xt} f(t)\, dt.$

Using integration by parts, $0 = [e^{xt}(t^2+3t) f(t)]_C - \int_C \left[\frac{d}{dt}\{(t^2+3t) f(t)\} + (t+9) f(t)\right] e^{xt}\, dt$. Thus, we need to choose $f(t)$ s.t.

$\frac{d}{dt}\{(t^2+3t) f(t)\} + (t+9) f(t) = 0$ i.e. $f'(t^2+3t) + f(2t+3+t+9) = 0$ and then choose C s.t. $[e^{xt}(t^2+3t) f(t)]_C = 0$.

$\frac{f'}{f} = -\frac{3t+12}{t^2+3t} = \frac{1}{t+3} - \frac{4}{t}$. $\Rightarrow \log f = \log(t+3) - 4\log t \Rightarrow f(t) = \frac{t+3}{t^4}$. As such, $y(x) = \int_C \frac{t+3}{t^4} e^{xt}\, dt$ is a solution if we have

$[e^{xt}(t^2+3t)\cdot\frac{t+3}{t^4}]_C = [e^{xt}\cdot\frac{(t+3)^2}{t^3}]_C = 0$. One contour we can pick is $C_1 : (-\infty, -3]$ with corresponding solution:



$y_1(x) = \int_{-\infty}^{-3} \frac{(t+3)}{t^4} e^{xt}\, dt$, is a solution. Where could we obtain a second solution?

A second possible C is one that is closed and encircles no branch points of $e^{xt}\frac{(t+3)^2}{t^4}$ (no branch cuts here!)

If C_2 encircles the origin then $y_2(x) = \int_{C_2} e^{xt} \frac{(t+3)}{t^4}\, dt$, is a second solution, since $[e^{xt}\frac{(t+3)^2}{t^3}]_C = 0$ as C is closed.

$y_1(x) = \int_{-\infty}^{-3} \frac{(t+3)}{t^4} e^{-xt}\, dt = \int_3^{\infty} \frac{3-t}{t^4} e^{-xt}\, dt$. At $x=0$, $y_1(x)$ is finite: $y_1(0) = \int_3^{\infty} \frac{3-t}{t^4}\, dt < \infty$. Although $y_1(0)$ is finite, it could still be singular at $x=0$.

$y_1'(x) = \int_3^{\infty} \frac{3-t}{t^4}(-t) e^{-xt}\, dt \Rightarrow y_1'(0) = -\int_3^{\infty} \frac{(3-t)}{t^3}\, dt$. $y_1''(x) = \int_3^{\infty} \frac{3-t}{t^4} t^2 e^{-xt}\, dt \Rightarrow y_1''(0) = \int_3^{\infty} \frac{3-t}{t^2}\, dt$ is infinite as integrand $\sim -1/t$ as $t \to \infty$

$\Rightarrow$ it is not integrable [Note: Functions such as $x^2 \log x$ has a similar property: finite but not analytic].

$y_2(x) = \oint_C \frac{(t+3)}{t^4} e^{xt}\, dt$. Recall Cauchy's integral theorem for derivatives: $f^{(n)}(t_0) = \underbrace{\frac{n!}{2\pi i}}_{\text{const.}} \oint \frac{f(t)}{(t-t_0)^{n+1}}\, dt$. Here, $t_0 = 0$, $n = 3$, $f(t) = (t+3)e^{xt}$.

Consider $\frac{1}{2\pi i}\oint \frac{(t+3)e^{xt}}{t^4}\, dt$ (to keep y_2 real) $= \frac{1}{2\pi i}\frac{2\pi i}{3!} f^{(3)}(0) = \frac{1}{6}\frac{d^3}{dt^3}(t+3)e^{xt}\Big|_{t=0}$, which is a terminating polynomial solution of

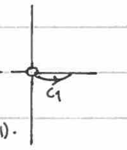
the original equation, $y_2(x) = \frac{1}{6}\left((3)(1)(x^2 e^{xt}) + (t+3)(x^3 e^{xt})\right)\Big|_{t=0}$ by Leibnitz's rule $= \frac{1}{2}(x^2 + x^3)$.

Remark — Tracing working back, notice that polynomial solutions were yielded as a result of integer coefficients of partial fraction expression.

Ex Solve $xy'' + (1-x)y' + ay = 0$; for all values of $a \in \mathbb{Q}$.

Soln. We manipulate algebra to get solutions $y = \int_C \frac{e^{xt}\, t^{a-1}}{(t-1)^a}\, dt$ if $\left[\frac{t^a e^{xt}}{(t-1)^{a-1}}\right]_C = 0$. Consider if $a = \frac{1}{2}$. Then we get

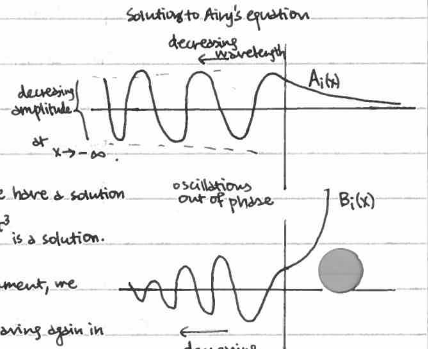
$y(x) = \int_C \frac{e^{xt}}{t^{1/2}(t-1)^{1/2}}\, dt$ if $[t^{1/2}(t-1)^{1/2} e^{xt}]_C = 0$. Then, we need to carefully consider branch cuts.



For instance, we could take C_1 to be running from 0 to 1, running just under the real axis. (or just above). Otherwise, we have problems for $(t-1)$.

Let $t = s$, $y_1(x) = \int \frac{e^{xs}}{\sqrt{s}\sqrt{-1}\sqrt{1-s}}\, ds$. So take $\int_0^1 \frac{e^{xs}}{\sqrt{s}\sqrt{1-s}}\, ds$.

Ex (Airy's Equation)



Solve $y'' - xy = 0$.

Soln. Look for a solution $y = \int_C e^{xt} f(t)\, dt$. Substitution requires $0 = \int_C t^2 f(t) e^{xt}\, dt - \int f(t) \underbrace{x e^{xt}}_{\frac{d}{dt}(e^{xt})}\, dt$

$0 = -[f(t)\, e^{xt}]_C + \int_C (t^2 f + f') e^{xt}\, dt$. So choose f s.t. $f' + t^2 f = 0 \Rightarrow f(t) = e^{-\frac{1}{3}t^3}$ and we have a solution

to the ODE: $y(x) = \int_C e^{xt - \frac{1}{3}t^3}\, dt$ if $[e^{xt-\frac{1}{3}t^3}]_C = 0. \Rightarrow$ any contour joining zeros of $e^{xt-\frac{1}{3}t^3}$ is a solution.

Where do the exponentials have zeros? Since the exponential function has no zeros for finite argument, we need to have contours C coming in from infinity where $e^{xt-\frac{1}{3}t^3}$ is exponentially small, and leaving again in another such direction. If we set $t = Re^{i\theta}$, then $xt - \frac{1}{3}t^3 = xRe^{i\theta} - \frac{1}{3}R^3 e^{3i\theta} \sim -\frac{1}{3}R^3 e^{3i\theta}$ as $R \to \infty$.

and we require θ to be such that $\mathrm{Re}[-\frac{1}{3}R^3 e^{3i\theta}] < 0 \Rightarrow \mathrm{Re}[e^{3i\theta}] > 0 \Rightarrow \cos 3\theta > 0$.

i.e. $-\frac{5\pi}{6} < \theta < -\frac{\pi}{2}$, $-\frac{\pi}{6} < \theta < \frac{\pi}{6}$, $\frac{\pi}{2} < \theta < \frac{5\pi}{6}$.

Thus, we see that $e^{xt-\frac{1}{3}t^3} \to 0$ as $t \to \infty$ if $-\frac{\pi}{6} < \theta < \frac{\pi}{6}$, $-\frac{5\pi}{6} < \theta < -\frac{\pi}{2}$ and $\frac{\pi}{2} < \theta < \frac{5\pi}{6}$.

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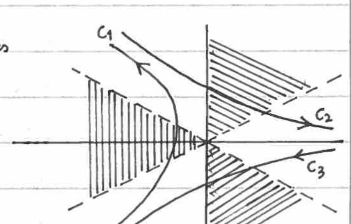
Plotting this on the graph, we can obtain three non-zero solutions y_1, y_2, y_3 from contours C_1, C_2, C_3:



We note that the contours C_1, C_2, C_3 are "joined at infinity" to make 1 closed contour with no singularities of $e^{xt-\frac{1}{3}t^2}$ in it. Hence, $y_1 + y_2 + y_3 = 0$ by Cauchy's Theorem $\Rightarrow$ y_1, y_2, y_3 are linearly dependent

$\Rightarrow$ we have 2 linear independent solutions.

It turns out that the two solutions are $A_i(x) = \frac{1}{2\pi i} y_1(x) = \frac{1}{2\pi i}\int_{C_1} e^{xt-\frac{1}{3}t^3}\, dt$, $B_i(x) = \frac{1}{2\pi}(y_2 - y_3)$.

Of course, we can manipulate the directions of C_1, C_2, C_3 as long as they fit the required angular parameters.

For instance, we can evaluate $A_i(x)$ by choosing C_1 to lie exactly on the imaginary axis. Then we parametrise: $t = is$, $dt = i\, ds$.

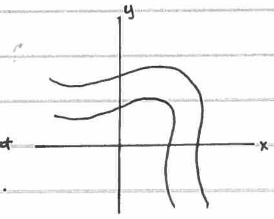
$xt - \frac{1}{3}t^3 = i(xs + \frac{1}{3}s^3) \Rightarrow A_i(x) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{i(xs+\frac{1}{3}s^3)}\, i\, ds = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{\cos(xs+\frac{1}{3}s^3)}_{\text{even}} + \underbrace{i\sin(xs+\frac{1}{3}s^3)}_{\text{odd, cancels}}\, ds = \frac{1}{\pi}\int_0^{\infty} \cos(xs+\frac{1}{3}s^3)\, ds.$

This is not absolutely integrable, but the integral converges.

[To show this, use $t = is - \varepsilon$, then the $e^{-\varepsilon}$ part $\to 0$ as $s \to \infty$].

Chapter 2

PHASE PLANE ANALYSIS OF ODEs.



A non-linear 1^{st} order ODE has the form $\frac{dy}{dx} = f(x,y) = \frac{Q}{P}$. We assume this is reduced, i.e. $Q(x,y)$, $P(x,y)$ have no common factors.

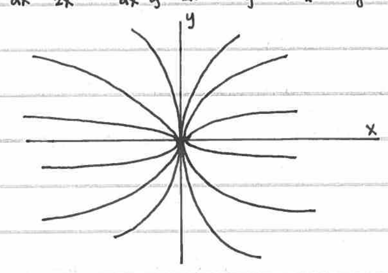
It is not possible to find explicit solutions to all such enquiries, and although numerical methods can help if we have a restricted set of initial conditions, it is still valuable to have techniques that allow us to investigate the qualitative nature of solutions to such ODEs.

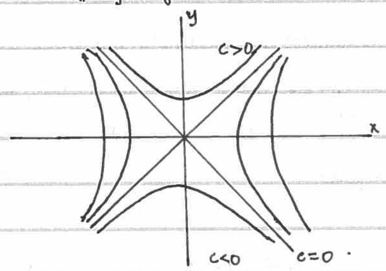
Curves drawn in xy-plane of solutions are called <u>integral curves</u> (or <u>trajectories</u> in some contexts).

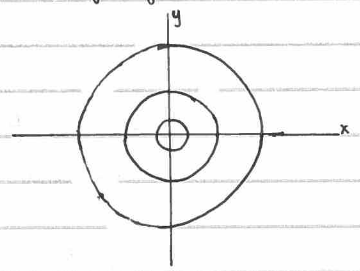
If P and Q are single-valued, trajectories cannot cross, except possibly at points where $P=Q=0$. So points are called singular points of the ODE.

Examples of trajectories —

- $\frac{dy}{dx} = \frac{y}{2x} \Rightarrow \frac{dy}{dx}\frac{1}{y} = \frac{1}{2x} \Rightarrow \frac{dy}{y} = \frac{1}{2}\frac{dx}{x} \Rightarrow y^2 = cx$
- $\frac{dy}{dx} = \frac{x}{y} \Rightarrow y^2 - x^2 = c$
- $\frac{dy}{dx} = -\frac{x}{y} \Rightarrow y^2 + x^2 = c$.







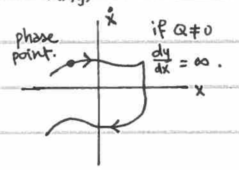
<u>Use for autonomous 2^{nd} order ODE.</u>

Consider the equation $\frac{d^2x}{dt^2} = Q(x, \frac{dx}{dt}, t)$, which is a general 2^{nd} order ODE for $x(t)$.

If Q does not depend explicitly on t (i.e. $\frac{\partial Q}{\partial t} = 0$), then the equation is autonomous and is $\frac{d^2x}{dt^2} = Q(x, \frac{dx}{dt})$. Writing $y = \frac{dx}{dt}$, then $\frac{dy}{dt} = \frac{d^2x}{dt^2} = Q(x,y)$.

Thus, we have the pair of equations: $\frac{dx}{dt} = y$, $\frac{dy}{dt} = Q(x,y)$. Then, considering y as a function of x, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{Q(x,y)}{y} = \frac{Q(x,y)}{P(x,y)}$.

Then, the xy-plane becomes an $x,\dot{x}$-plane, which is known as the <u>phase plane</u>.



In the upper-half, $\dot{x} > 0 \Rightarrow x$ is increasing.

All singular points in this context will be on the x-axis ($y=0$) and they may be referred to as <u>equilibrium points</u>.

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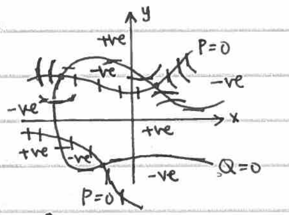
We can identify lines in the phase plane where trajectories have zero slope. These are given by $Q(x,y)$ are known as <u>horizontal nullclines</u>.

Similarly, <u>vertical nullclines</u> are where $P(x,y)=0$ and trajectories are vertical. A <u>critical point</u> is where a vertical nullcline crosses a horizontal nullcline. Then we can populate the regions with signs depending on behaviour of P and Q.



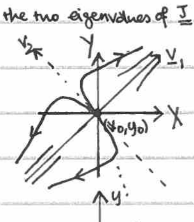
Near the critical points, which are (x_0, y_0) s.t. $P(x_0,y_0) = 0$, $Q(x_0,y_0) = 0$. We can use a Taylor expansion to approximate P and Q.

$P(x,y) = \overbrace{P(x_0,y_0)}^{0} + \frac{\partial P}{\partial x}\Big|_{(x_0,y_0)}(x-x_0) + \frac{\partial P}{\partial y}\Big|_{(x_0,y_0)}(y-y_0)$, $Q(x,y) = \overbrace{Q(x_0,y_0)}^{0} + \frac{\partial Q}{\partial x}\Big|_{(x_0,y_0)}(x-x_0) + \frac{\partial Q}{\partial y}\Big|_{(x_0,y_0)}(y-y_0)$.

Let $X = x - x_0$, $Y = y - y_0$, then $\frac{dY}{dX} = \frac{dy}{dx} = \frac{Q_x X + Q_y Y}{P_x X + P_y Y}$. Consider this as the pair of equations $\frac{dY}{dt} = \overset{C}{\widetilde{Q_x}} X + \overset{D}{\widetilde{Q_y}} Y$, $\frac{dX}{dt} = \overset{A}{\widetilde{P_x}} X + \overset{B}{\widetilde{P_y}} Y$. We introduce new constants.

Then $\frac{d}{dt}\begin{pmatrix} X \\ Y \end{pmatrix} = \underline{\underline{\begin{pmatrix} A & B \\ C & D \end{pmatrix}}}\begin{pmatrix} X \\ Y \end{pmatrix}$. Note here that $\underline{\underline{\begin{pmatrix} A & B \\ C & D \end{pmatrix}}}$ is the Jacobian, J. Letting $\underline{u} = \begin{pmatrix} X \\ Y \end{pmatrix}$, we get $\dot{\underline{u}} = \underline{\underline{J}}\,\underline{u}$.

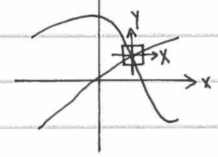
To solve this, look for solutions $\underline{u}(t) = \underline{v}e^{\lambda t}$, $\dot{\underline{u}} = \lambda \underline{v} e^{\lambda t}$ and substitution requires $\lambda \underline{v} e^{\lambda t} = \underline{\underline{J}}\,\underline{v}\,e^{\lambda t} \Rightarrow \lambda \underline{v} = \underline{\underline{J}}\,\underline{v}$. Assume the two eigenvalues of $\underline{\underline{J}}$ are distinct, λ_1 and λ_2, with corresponding eigenvectors are $\underline{v}_1$ and $\underline{v}_2$, then $\underline{u} = \begin{pmatrix} X \\ Y \end{pmatrix} = \bar{A}\underline{v}_1 e^{\lambda_1 t} + \bar{B}\underline{v}_2 e^{\lambda_2 t}$.



Now if λ_1 and λ_2 have the same sign, +ve and $\lambda_1 > \lambda_2 > 0$. As $t \to \infty$, $\underline{u}$ tends to the direction of $\underline{v}_1$. As $t \to -\infty$, they will come out from the direction of v_2. This forms an unstable node. If $0 > \lambda_2 > \lambda_1$, we have a stable node — reverse time in the picture drawn.

If λ_1 and λ_2 are of different signs, WLOG $\lambda_1 < 0 < \lambda_2$. Then as $t \to -\infty$, $\begin{pmatrix} X \\ Y \end{pmatrix} \sim \bar{A}\underline{v}_1 e^{\lambda_1 t}$, as $t \to +\infty$, $\begin{pmatrix} X \\ Y \end{pmatrix} \sim \bar{B}\underline{v}_2 e^{\lambda_2 t}$.

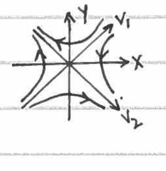
These form <u>saddle points</u>.



For $\dot{\underline{u}} = \underline{\underline{J}}\,\underline{u}$, we attempt to diagonalise $\underline{\underline{J}}$. We do this by forming matrix $\underline{\underline{P}} = (\underline{v}_1 \vdots \underline{v}_2)$ if $\lambda_1, \lambda_2 \in \mathbb{R}$ and so $\underline{v}_1, \underline{v}_2$ are real.

We then switch to coordinates $(\bar{X}, \bar{Y})$ as opposed to (X,Y) by defining $\underline{u} = \underline{\underline{P}}\,\bar{\underline{u}}$ [or $\begin{pmatrix} X \\ Y \end{pmatrix} = (\underline{v}_1 \vdots \underline{v}_2)\begin{pmatrix} \bar{X} \\ \bar{Y} \end{pmatrix} \Rightarrow \bar{\underline{u}} = \underline{\underline{P}}^{-1}\underline{u}$.

We observe that $\underline{\underline{J}}\,\underline{\underline{P}} = (\lambda_1 \underline{v}_1 . \lambda_2 \underline{v}_2) = (\underline{v}_1\ \underline{v}_2)\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \Rightarrow \underline{\underline{J}}\,\underline{\underline{P}} = \underline{\underline{P}}\,\underline{\underline{\Lambda}}$ with $\underline{\underline{\Lambda}} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, so $\underline{\underline{J}} = \underline{\underline{P}}\,\underline{\underline{\Lambda}}\,\underline{\underline{P}}^{-1}$ and $\underline{\underline{J}}$ is similar to $\underline{\underline{\Lambda}}$.

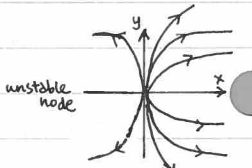


Hence, $\dot{\underline{u}} = \underline{\underline{J}}\,\underline{u} = \underline{\underline{P}}\,\underline{\underline{\Lambda}}\,\underline{\underline{P}}^{-1}\underline{u} \Rightarrow \underline{\underline{P}}^{-1}\dot{\underline{u}} = \underline{\underline{\Lambda}}\,\underline{\underline{P}}^{-1}\underline{u} \Rightarrow \dot{\bar{\underline{u}}} = \underline{\underline{\Lambda}}\,\bar{\underline{u}}$. Then $\dot{\bar{X}} = \lambda_1 \bar{X}$, $\dot{\bar{Y}} = \lambda_2 Y \Rightarrow$ solutions are

$\bar{X}(t) = \bar{X}_0 e^{\lambda_1 t}$, $\bar{Y}(t) = \bar{Y}_0 e^{\lambda_2 t}$. Eliminating t, $\bar{Y} = c\bar{X}^a$.

If λ_1, λ_2 are real, positive, say $\lambda_1 = 2, \lambda_2 = 1$, then $\bar{Y} = C\bar{X}^{1/2}$. We plot these on the right.

In real life however, these can be skewed (or "squished") as eigenvectors might not be orthogonal.



Ex let $\begin{matrix}\dot{x} = 2x + y \\ \dot{y} = y\end{matrix}$ define a system. Draw the trajectories locally in the vicinity of critical point.

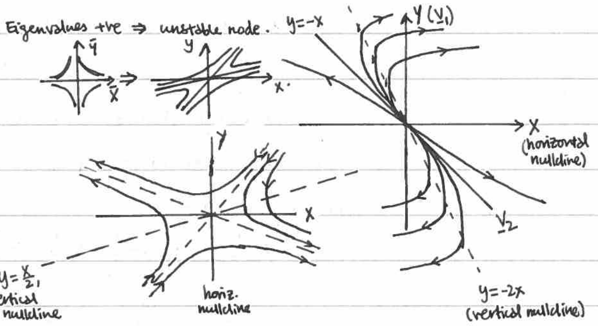
soln. $\underline{\underline{J}} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \Rightarrow \lambda_1 = 2, \lambda_2 = 1, \quad \underline{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \underline{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ are eigenvectors. Eigenvalues +ve $\Rightarrow$ unstable node.

Note — This does not apply in regions which are far from critical point.

If λ_1, λ_2 are of opposite sign, say $\lambda_1 = -2$ and $\lambda_2 = 1$, then $a = \frac{\lambda_2}{\lambda_1} = -\frac{1}{2}$ and $\bar{Y} = C\bar{X}^{-1/2}$.

Ex let $\begin{matrix}\dot{x} = x - 2y \\ \dot{y} = -x\end{matrix}$, do same as above.

soln. $\underline{\underline{J}} = \begin{pmatrix} 1 & -2 \\ -1 & 0 \end{pmatrix} \Rightarrow \lambda_1 = 2, \lambda_2 = -1. \quad \underline{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \underline{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.



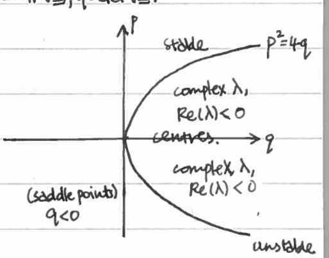
In higher dimensions, it is more difficult to find local eigenvectors.

Instead, we can look for straight line solutions of the local equations, and these correspond to eigenvectors.

For instance, $\frac{\dot{y}}{\dot{x}} = \frac{dy}{dx} = \frac{-x}{x-2y} = \frac{-1}{1-2y/x}$ and we need $m = \frac{-1}{1-2m} \Rightarrow (1-2m)m = -1 \Rightarrow (2m+1)(m-1) = 0 \Rightarrow m = -\frac{1}{2}, 1$, which are straight lines corresponding to the eigenvectors earlier found.

If $\underline{\underline{J}} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, eigenvalues are roots of $(A-\lambda)(D-\lambda) - CB = 0 \Rightarrow \lambda^2 - (A+D)\lambda + (AD-BC) = 0 \Rightarrow \lambda^2 + p\lambda + q = 0$ where $p = -\mathrm{tr}(\underline{\underline{J}}), q = \det(\underline{\underline{J}})$

Then by quadratic formula, $\lambda = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$. Consider the parabola $p^2 = 4q$.



If the eigenvalues are complex, $\lambda_{1,2} = \mu \pm i\omega$. Choose eigenvalue $\lambda_1 = \mu + i\omega$ and corresponding eigenvector $\underline{v}_1 = \underline{f} + i\underline{g}$.

Let $\underline{P} = (\underline{g}, \underline{f}) = (\mathrm{Im}(\underline{v}_1) \ \mathrm{Re}(\underline{v}_1))$. Then $\underline{\underline{J}}\,\underline{P} = (\mathrm{Im}(\underline{\underline{J}}\underline{v}_1), \mathrm{Re}(\underline{\underline{J}}\underline{v}_1)) = (\mathrm{Im}(\lambda_1 \underline{v}_1), \mathrm{Re}(\lambda_1 \underline{v}_1)) = (\mu\underline{g} + \omega\underline{f}, \mu\underline{f} - \omega\underline{g})$.

$(\underline{g} \ \underline{f})\begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix} = \underline{P}\begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}$, and $\underline{P}^{-1}\underline{\underline{J}}\,\underline{P} = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}$. Then $\begin{pmatrix} \dot{\bar{X}} \\ \dot{\bar{Y}} \end{pmatrix} = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}\begin{pmatrix} \bar{X} \\ \bar{Y} \end{pmatrix}$

$\Rightarrow \dot{\bar{X}} = \mu\bar{X} - \omega\bar{Y}, \ \dot{\bar{Y}} = \omega\bar{X} + \mu\bar{Y}$. Then $\ddot{\bar{X}} = \mu\dot{\bar{X}} - \omega(\omega\bar{X} + \frac{\mu}{\omega}(\mu\bar{X} - \dot{\bar{X}})) \Rightarrow \ddot{\bar{X}} - 2\mu\dot{\bar{X}} + (\omega^2 + \mu^2)\bar{X} = 0$

$\Rightarrow \bar{X} = Ae^{-\mu t}\cos(\omega t + \varphi)$.

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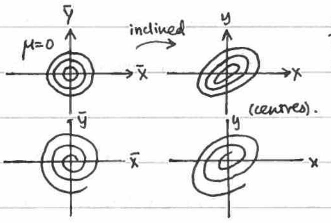
Let $\underline{\underline{J}} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} P_x & P_y \\ Q_x & Q_y \end{pmatrix}$, $\frac{dy}{dx} = \frac{Q}{P}$. Let $p = -\mathrm{tr}(\underline{\underline{J}}) = -(A+D)$, $q = \det(\underline{\underline{J}}) = AD - BC$.



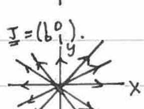
Similar to above, we can establish that $\bar{Y} = Ae^{-\mu t}\cos(\omega t + \bar{\varphi})$.

If $\mu = 0$, we obtain a set of concentric circles in transformed coordinates. If $\mu \neq 0$, we get a continual spiral.

The direction of spiralling motion depends on the sign of μ (i.e. stability of centre).

If $q > 0$, $p^2 = 4q$ then the trajectories are not classifiable just with a knowledge of p and q. For instance, consider $\dot{x} = x, \dot{y} = y$, $\underline{\underline{J}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

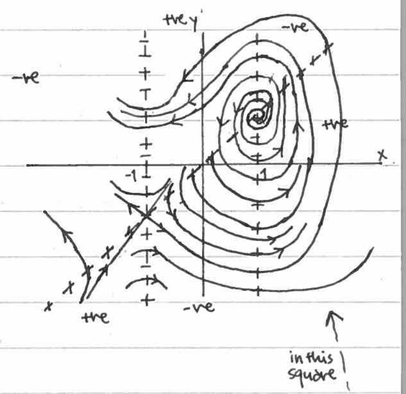
Eigenvalues are 1,1 with corresponding eigenvectors $\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Here, $\frac{dy}{dx} = \frac{y}{x}$, so we get a star:



However, if $\begin{matrix}\dot{x} = x + y \\ \dot{y} = y\end{matrix}$, then $\underline{\underline{J}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, eigenvalues are 1,1, eigenvectors are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow$ we obtain an improper node (see Moodle notes)

Ex solve $\frac{dy}{dx} = \frac{x^2 - 1}{x - y}$ for trajectories.



soln. Horizontal nullclines are at $x^2 - 1 = 0 \Rightarrow x = \pm 1$. Vertical nullclines are at $x - y = 0 \Rightarrow y = x$.

Consider $x = 0$. Then $\frac{dy}{dx} = \frac{1}{y}$, which is positive where $y > 0$. Since no factors are repeated, we can assume that gradients change signs upon crossing all trajectories.

Critical points are where a horizontal and vertical nullcline intersect: (1,1) and (-1,-1).

$\underline{\underline{J}} = \begin{pmatrix} P_x & P_y \\ Q_x & Q_y \end{pmatrix}$ where $P = x - y$, $Q = x^2 - 1 \Rightarrow \underline{\underline{J}} = \begin{pmatrix} 1 & -1 \\ 2x & 0 \end{pmatrix}$ [or: substitute and linearise: $x = 1 + X, y = 1 + Y, \frac{dY}{dX} = \frac{dy}{dx} = \frac{2X}{X - Y}$ valid near critical point.]

At (1,1), $\underline{\underline{J}} = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}$. $p = -\mathrm{tr}(\underline{\underline{J}}) = -1$, $q = \det(\underline{\underline{J}}) = 2$. Then $p^2 - 4q = -7 < 0 \Rightarrow$ unstable spiral point.

At (-1,-1), $\underline{\underline{J}} = \begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix}$. $p = -1$, $q = \det(\underline{\underline{J}}) = -2$. Then $p^2 - 4q = 9 > 0 \Rightarrow$ saddle point.

We write $x = -1 + X$, $y = 1 + Y$ to examine local behaviour of the graph. Then $\frac{dX}{dt} = X - Y$, $\frac{dY}{dt} = -2X$.

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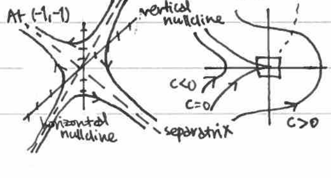
If we look for solutions to $\frac{dY}{dX} = \frac{-2X}{X - Y}$ of the form $Y = mX$, then we need $m = \frac{-2}{1-m} \Rightarrow m^2 - m - 2 = 0$, $m = 2, -1$.

Then the lines $Y = 2X$, $Y = -X$ are called the separatrix.

If $|x| >> 1$, $|y| >> 1$, $\frac{dy}{dx} = \frac{x^2 - 1}{x - y} \approx \frac{x^2}{x - y}$. If $|y| >> |x|$, $\frac{dy}{dx} \approx \frac{x^2}{-y} \Rightarrow \frac{y^2}{2} \approx \frac{x^3}{3} - c$, $|y| \sim |x|^{3/2}$

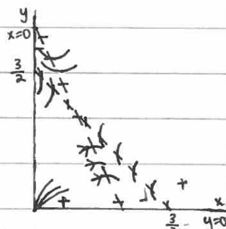
Thus in far field, we observe trajectories by varying c.



Here is an "application" to population dynamics:

Ex. Given two populations of rabbits and foxes (where foxes eat rabbits) or rabbits and sheep (competitors). The rate of growth of these populations is proportional to the number in population (birth rate − death rate). Birth rate may depend on food supply (i.e. linked to population), and death rate may depend on predators.

This modelling, in its simplest form, leads to equations such as $\frac{dx}{dt} = x(A + a_1x + b_1y)$, $\frac{dy}{dt} = y(B + b_2x + a_2y)$, $x \geq 0, y \geq 0$.



Consider, as an example, $\frac{dx}{dt} = x(3-2x-2y) = P(x,y)$, $\frac{dy}{dt} = y(2-2x-y) = Q(x,y)$. Model populations over time.

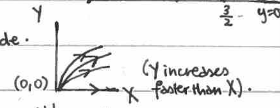
Soln. Vertical nullclines are at $\frac{dx}{dt} = 0 \Rightarrow x = 0$ or $3-2x-2y = 0 \Rightarrow y = \frac{3}{2} - x$.
→ trajectories always lie on nullcline for here.

Horizontal nullclines are at $\frac{dy}{dt} = 0 \Rightarrow y = 0$ or $2-2x-y=0 \Rightarrow y = 2-2x$.

Critical points occur where horizontal and vertical nullclines cross: ① $(0,0)$ ② $(0,2)$ ③ $(\frac{3}{2},0)$ and ④ $(\frac{1}{2},1)$.

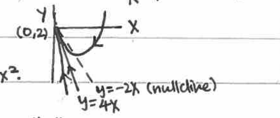
$\underline{\underline{J}} = \begin{pmatrix} P_x & P_y \\ Q_x & Q_y \end{pmatrix} = \begin{pmatrix} 3-4x-2y & -2x \\ -2y & 2-2x-2y \end{pmatrix}$. Then at ①, $\underline{\underline{J}} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow$ 2 real +ve eigenvalues 3 and 2 $\Rightarrow$ unstable node.

Let $x = 0 + X$, $y = 0 + Y$. Then $\frac{dX}{dt} = 3X$, $\frac{dY}{dt} = 2Y$. $\because \frac{d\underline{X}}{dt} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}$. Then $\frac{dY}{dX} = \frac{2Y}{3X} \Rightarrow Y = cX^{2/3}$



②: $\underline{\underline{J}} = \begin{pmatrix} -1 & 0 \\ -4 & -2 \end{pmatrix}$ if $(x,y) = (0,2)$. $\Rightarrow$ Two real −ve eigenvalues −1 and −2 $\Rightarrow$ stable node. Then let $x = 0 + X$, $y = 2 + Y$.

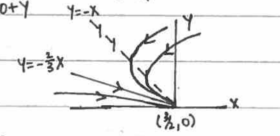
Then $\frac{dX}{dt} = -X$, $\frac{dY}{dt} = -4X - 2Y \Rightarrow \frac{dY}{dX} = \frac{-4X-2Y}{-X} = 4 + \frac{2Y}{X} \Rightarrow \frac{dY}{dX} - \frac{2Y}{X} = 4 \Rightarrow [\frac{Y}{X^2}]' = \frac{4}{X^3} \Rightarrow Y = -4X + cX^2$.



For ③: $(x,y) = (\frac{3}{2}, 0)$. $\underline{\underline{J}} = \begin{pmatrix} -3 & -3 \\ 0 & -1 \end{pmatrix} \Rightarrow$ two real −ve eigenvalues −1 and −3 $\Rightarrow$ stable node. Let $x = \frac{3}{2} + X$, $y = 0 + Y$

$\frac{dY}{dX} = \frac{-Y}{-3X-3Y} = \frac{Y}{3X+3Y} \Rightarrow \frac{dX}{dY} = \frac{3X+3Y}{Y} = 3 + \frac{3X}{Y} \Rightarrow \frac{dX}{dY} - \frac{3}{Y}X = 3 \Rightarrow \frac{d}{dX}[\frac{X}{Y^3}] = \frac{3}{Y^3} \Rightarrow \frac{X}{Y^3} = \frac{-3}{2Y^2} + c$

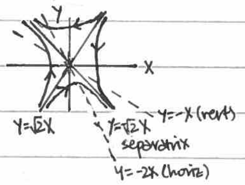
$\Rightarrow X = -\frac{3Y}{2} + cY^3$. Then we plot the local behaviour.



For point ④: $(x,y) = (\frac{1}{2},1)$. $\underline{\underline{J}} = \begin{pmatrix} -1 & -1 \\ -2 & -1 \end{pmatrix}$. Eigenvalues λ satisfy $(-1-\lambda)^2 - 2 = 0 \Rightarrow -1-\lambda = \pm\sqrt{2} \Rightarrow \lambda = -1 \pm \sqrt{2}$.

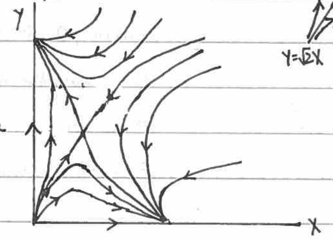
Eigenvalues differ in sign $\Rightarrow$ saddle point. Then $\frac{dY}{dt} = -2X - Y$, $\frac{dX}{dt} = -X - Y \Rightarrow \frac{dY}{dX} = \frac{2X+Y}{X+Y}$, which has solutions

$Y = mX$ for $m = \frac{2+m}{1+m} \Rightarrow m = \pm\sqrt{2} \Rightarrow Y = \pm\sqrt{2}X$ yield our separatrices.



Thus, overall, we get the graph on right:

Depending on where initial conditions determine (X,Y) to be (i.e. on which side of the separatrix, we will be able to predict which species will go exit based on the patterns demonstrated by their trajectories.

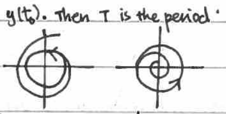


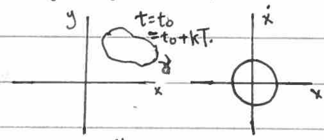
Periodic Solutions

These may arise from solutions to $\frac{dx}{dt} = P(x,y)$, $\frac{dy}{dt} = Q(x,y)$, or from the second order equation $\ddot{x} = f(x,\dot{x})$, written as $\dot{x} = P(x,y) = y$, $\ddot{x} = \dot{y} = Q(x,y) = f(x,y)$

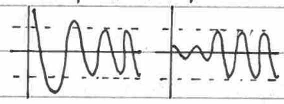
Periodic solutions correspond to closed trajectories. i.e. $x(t_0+T) = x(t_0)$, $y(t_0+T) = y(t_0)$. Then T is the period.

$T = \int_0^T dt = \int_\gamma \frac{dx}{dx/dt} = \int_\gamma \frac{dx}{P(x,y)}$ [or likewise $= \int_\gamma \frac{dy}{Q(x,y)}$].

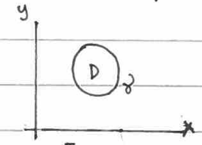




Periodic solutions can be approached as $t \to +\infty$ (or $t \to -\infty$), in which case they are known as limit cycles. To determine the existence of such solutions, apply the following:



Bendixson's Negative criteria for a limit cycle / periodic solution:



Let us suppose that a periodic solution exists and is given by the closed curve γ. Then consider $\iint_D (\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y})\,dx\,dy$, $\frac{dx}{dt} = P$, $\frac{dy}{dt} = Q$.

Note that $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = \underline{\nabla} \cdot \begin{pmatrix} P \\ Q \end{pmatrix} = \underline{\nabla} \cdot \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \underline{\nabla} \cdot \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix}$. Then Stokes's theorem gives $\iint_D P_x + Q_y\,dx\,dy = \oint_\gamma P\,dy - Q\,dx = \int_0^T (P\frac{dy}{dt} - Q\frac{dx}{dt})\,dt = \int_0^T (PQ - QP)\,dt$

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Recall that if $\dot{x} = P$, $\dot{y} = Q$, if $\exists\,\gamma$, then $\iint_D P_x + Q_y\,dx\,dy = \oint_\gamma P\,dy - Q\,dx = \oint_\gamma (PQ - QP)\,dt$. So $P_x + Q_y$ cannot be single-signed in D, by Bendixson's Negative criteria. $\underline{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\underline{v} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} P \\ Q \end{pmatrix}$, $P_x + Q_y = \underline{\nabla} \cdot \underline{v}$. Consider $\frac{dx}{dt} = x(3-2x-2y) = P$, $\frac{dy}{dt} = y(2-2x-y) = Q$. Then $P_x + Q_y = (3-4x-2y) + (2-2x-2y) = 5 - 6x - 4y$, which is zero on $y = -\frac{3x}{2} + \frac{5}{4}$. Any closed orbit must straddle the line $y = -\frac{3x}{2} + \frac{5}{4}$.

However, we have not proven that a closed orbit either exists or does not. Hence, we use Dulac's extension of Bendixson's Negative criteria:

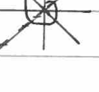
Consider $\underline{\nabla} \cdot (R\underline{v})$ for any R. $\iint_D (RP)_x + (RQ)_y\,dx\,dy = \oint_\gamma RP\,dy - RQ\,dx = \oint (RPQ - RQP)\,dt = 0$. So if we can find an R s.t. $(RP)_x + (RQ)_y$ is single-signed, then we know there is no periodic solution there. If we use $R = \frac{1}{xy}$, then $\frac{\partial}{\partial x}(PR) + \frac{\partial}{\partial y}(QR) = \frac{\partial}{\partial x}(\frac{3}{y} - \frac{2x}{y} - 2) + \frac{\partial}{\partial y}(\frac{2}{x} - 2 - \frac{y}{x}) = -\frac{2}{y} - \frac{1}{x} < 0$ for $x, y > 0$, so no closed orbit.

Ex. (A limit cycle): Consider $\frac{dy}{dx} = \frac{y - x - y(x^2+y^2)}{x + y - x(x^2+y^2)} = \frac{dy/dt}{dx/dt}$. Soln. Critical points require: $y - x = y(x^2+y^2)$, $x + y = x(x^2+y^2) \Rightarrow x(y-x) = y(x+y) \Rightarrow -x^2 = y^2$. So the only critical point is at $x = y = 0$. Close to critical point $(x = X, y = Y, |X|, |Y| \ll 1)$, the linearised form is $\frac{dY}{dX} = \frac{Y-X}{X+Y}$. and $\underline{\underline{J}} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, whose eigenvalues satisfy $(1-\lambda)^2 = -1$, $\lambda = 1 \pm i$.

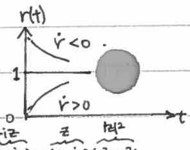
$\Rightarrow$ unstable spiral point at origin. We can solve this equation exactly by switching to polar coordinates, r, θ with $x = r\cos\theta$, $y = r\sin\theta$; $r^2 = x^2 + y^2$, $\tan\theta = \frac{y}{x}$.

Generally, with $\frac{dx}{dt} = P$, $\frac{dy}{dt} = Q$, $r^2 = x^2 + y^2 \Rightarrow 2r\dot{r} = 2x\dot{x} + 2y\dot{y}$, $\dot{r} = \frac{1}{r}(xP + yQ)$. $\tan\theta = \frac{y}{x} \Rightarrow \theta = \arctan\frac{y}{x}$. $\Rightarrow$

$\dot{\theta} = \frac{1}{1 + (y/x)^2} \cdot (\frac{\dot{y}}{x} - \frac{y\dot{x}}{x^2}) = \frac{x^2}{x^2+y^2}(\frac{x\dot{y} - \dot{y}x}{x^2}) = \frac{1}{r^2}(xQ - yP)$. i.e. $\boxed{r\dot{r} = xP + yQ,\ r^2\dot{\theta} = xQ - yP.} \Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{xP + yQ}{xQ - yP}$.



In this particular example, $\frac{1}{r}\frac{dr}{d\theta} = \frac{x[x+y-x(x^2+y^2)]+y[y-x-y(x^2+y^2)]}{x[y-x-y(x^2+y^2)]-y[x+y-x(x^2+y^2)]} = \frac{r^2-r^4}{-r^2} \Rightarrow \frac{dr}{d\theta} = \frac{r^5-r^3}{r^2} = r^3-r$. [or show $\frac{dr}{dt}=r-r^3$, $\frac{d\theta}{dt}=-1$].



Since $\frac{d\theta}{dt}=-1$ is a negative constant, it circles around origin at a constant angular rate. For small r, $\frac{dr}{dt}\sim r$, corresponding to spiral.

Where r is large, $\frac{dr}{dt}\sim -r^3$. Eventually, all trajectories spiral out or in to the circle $r=1$, and $r=1$ is a limit cycle. //

An alternative method: $\frac{dx}{dt} = x+y-x(x^2+y^2)$, $\frac{dy}{dt} = y-x-y(x^2+y^2)$. Then $z=x+iy$; we look for $\frac{dz}{dt}$. Then $\frac{dz}{dt} = \frac{d(x+iy)}{dt} = \underbrace{(x+iy)}_{z}+\underbrace{(y-ix)}_{-iz}-\underbrace{(x+iy)}_{z}\underbrace{(x^2+y^2)}_{|z|^2}$.

$\Rightarrow \frac{dz}{dt} = (1-i)z - z|z|^2$. Then if $z=re^{i\theta}$, $\frac{dz}{dt} = \dot r e^{i\theta} + i\dot\theta r e^{i\theta} = (1-i)re^{i\theta} - re^{i\theta}\cdot r^2 = (1-i)re^{i\theta} - r^3 e^{i\theta}$. $\Rightarrow \dot r + ir\dot\theta = (1-i)r - r^3$. Since r,θ are real,

we can compare real and imaginary components $\begin{cases}\dot r = r-r^3\\ r\dot\theta = -r\end{cases} \Rightarrow \dot\theta = -1$ or $r=0$.

Then we have $\int\frac{dr}{r^3-r} = \int d\theta \Rightarrow$ if $u=r^2$, $\frac{dr}{d\theta} = 2r(r^3-r) \Rightarrow \frac{du}{d\theta} = 2(u^2-u) \Rightarrow \int\frac{du}{u(u-1)} = \int 2\,d\theta \Rightarrow 2\theta = \ln\left(\frac{u-1}{u}\right) + \text{const}$.

$\frac{r^2-1}{r^2} = Ae^{2\theta} \Rightarrow r^2 = \frac{1}{1-Ae^{2\theta}} = \frac{1}{1-\tilde A e^{-2t}}$. If $\tilde A = 0$, $r=1$ is the limit cycle. $\tilde A>0$, $r^2>1$ initially and decreases; $\tilde A<0$, $r^2<1$ initially and increases.

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Maths 707.

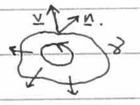
Poincaré-Bendixson's Theorem.

$\boxed{\text{Definition}}$ A closed set of points in the phase plane is said to be positive (or negative) invariant if a trajectory in the set at $t=0$ remains in the set for $t>0$ (or $t<0$).

Examples –

1. At a critical point, $\dot x=0, \dot y=0$ 2. A limit cycle } These are both positively and negatively invariant.

3. Let $\underline{v} = \begin{pmatrix}\dot x\\ \dot y\end{pmatrix}$. If $\underline{n}\cdot\underline{v}>0$ on the edge γ of a region D, D is negatively invariant. If $\underline{n}\cdot\underline{v}<0$ on γ, D is positively invariant.



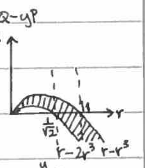
$\boxed{\text{Theorem}}$ (Poincaré-Bendixson Theorem).

If there exists a bounded invariant region of the phase plane with no critical points, then the region contains at least one limit cycle.

$\boxed{\text{Ex}}$ Show that there exists at least one limit cycle for $\begin{cases}\dot x = x-y-2x(x^2+y^2) = P\\ \dot y = x+y-y(x^2+y^2) = Q.\end{cases}$

Soln. $r\dot r = xP+yQ = x(x-y-2xr^2)+y(x+y-yr^2) = r^2-r^2(2x^2+y^2) = r^2-r^4(2\cos^2\theta+\sin^2\theta) = r^2-r^4(1+\cos^2\theta) \Rightarrow \frac{dr}{dt} = r-r^3(1+\cos^2\theta)$. $\frac{d\theta}{dt}$: $r^2\dot\theta = xQ-yP$

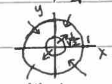
$= x(x+y-yr^2)-y(x-y-2xr^2) = r^2+xyr^2 \Rightarrow \dot\theta = 1+xy = 1+r^2\sin\theta\cos\theta = 1+\frac{1}{2}r^2\sin 2\theta$. Since $\frac{dr}{dt} = r-r^3(1+\cos^2\theta)$, $r-r^3 \geq \dot r \geq r-2r^3$.



Then we know that if $r<\frac{1}{\sqrt2}$, $\dot r>0$ and if $r>1$, $\dot r<0$. Likewise, we see that $1-\frac{1}{2}r^2 \leq \dot\theta \leq 1+\frac{1}{2}r^2$, for $\frac{1}{\sqrt2}\leq r\leq 1$;

$\Rightarrow 1-\frac{1}{2}(1)^2 \leq \dot\theta \leq 1+\frac{1}{2}\left(\frac{1}{\sqrt2}\right)^2 \Rightarrow \frac{1}{2}\leq\dot\theta\leq\frac{5}{4} \Rightarrow \dot\theta$ is not zero for $\frac{1}{\sqrt2}\leq r\leq 1$.

$\Rightarrow$ no critical points in the region $\frac{1}{\sqrt2}\leq r\leq 1 \Rightarrow \exists$ a limit cycle on annulus $\frac{1}{\sqrt2}\leq r\leq 1$ by Poincaré-Bendixson Theorem // q.e.d.

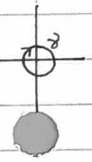


positively invariant region.

We now consider some special cases of ODEs.

1. Consider ODEs of the form $\ddot x + \varphi(\dot x) + f(x) = 0$. If $y=\dot x=P$, then $\dot y = -(\varphi(y)+f(x)) = Q$. As $\dot y = \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = y\frac{dy}{dx}$, we can write $y\frac{dy}{dx} + \varphi(y) + f(x) = 0$ — ⊛.

It turns out that periodic solutions for this equation are not possible in regions of the phase plane where $y\cdot\varphi(y)$ is single-signed.



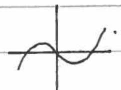
Imagine that there is a periodic solution γ and integrate the equation ⊛ w.r.t. x, around γ. $\oint_\gamma \underbrace{y\frac{dy}{dx}}_{\frac{d}{dx}[\frac{1}{2}y^2]}dx + \oint_\gamma \varphi(y)\,d\underset{\uparrow}{x}_{\dot x} + \oint_\gamma f(x)\,dx = 0$.

$[\frac{1}{2}y^2]_{\text{start of }\gamma}^{\text{end of }\gamma} + \oint_\gamma \varphi(y)\frac{dx}{dt}\,dt$. $\quad 0 + \oint_\gamma y\varphi(y)\,dt + [F]_{\text{start of }\gamma}^{\text{end of }\gamma} = 0 \Rightarrow \int_\gamma y\varphi(y)\,dt = 0$.

So we cannot have such a γ in regions where $y\varphi(y)$ is single-signed, so no periodic solution exists. Physical interpretation: $\underbrace{\dot x\ddot x}_{\frac{1}{2}\frac{d}{dt}[\dot x^2]=KE} + \underbrace{\dot x\varphi(\dot x)}_{\text{work done by damping force}} + \underbrace{\dot x F'(x)}_{PE.} = 0$

$\ddot x + \underbrace{\dot x}_{\varphi(\dot x)} + x = 0 \quad \varphi(y)=y. \quad y\varphi(y) = y^2$, single signed, so no periodic solutions.

2. (Liénhard's equation) This has the form $\ddot x + \dot x f(x) + g(x) = 0$. The



$\boxed{\text{Theorem}}$ (Lienhard's Theorem)

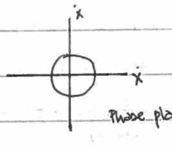
If we have, for the Lienhard's equation, (1) $f(x)$ is even [e.g. $f(x)=x^2-1$], and (2) $g(x)$ is odd [e.g. $g(x)=x$], and

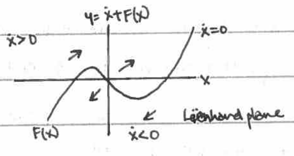
(3) $F(x) = \int_0^x f(s)\,ds$ [e.g. $\frac{1}{3}x^3-x$] has a single positive zero, x_0 [e.g. $\sqrt3$] and $F(x)$ is positive and monotone increasing for $x>x_0$ and $F(x)\to\infty$ as $x\to\infty$.

Then the equation $\ddot x + \dot x f(x) + g(x) = 0$ has a unique periodic solution.

Example – If $g(x)=0$, $f(x) = -\varepsilon(1-x^2)$, we get the Van der Pol equation $\ddot x - \varepsilon(1-x^2)\dot x + x = 0$ which has a unique solution.

Lienhard Plane





$F(x) = \int_0^x f(s)\,ds$.

If $y=\dot x+F(x)$ then $\frac{dy}{dx} = \frac{-g}{y-F}$, $\frac{dy}{dt} = \ddot x + \dot x F' = \ddot x + \dot x f = -g$ $\quad \frac{dy}{dt} = -g(x)$ $\quad \frac{dx}{dt} = y - F(x)$.

We next analyse the van der Pol equation: $\ddot x - \varepsilon(1-x^2)\dot x + x = 0$.

Lienhard's Theorem shows that this equation has a periodic solution (unique for $\varepsilon>0$.

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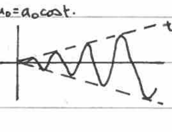
We might look for a periodic solution of the form $x = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \cdots$ This however is not straightforward, as can be demonstrated with the following equation: $\ddot{u} + u + \varepsilon u^3 = 0$.

We seek a periodic solution, $u = u_0(t) + \varepsilon u_1(t) + \varepsilon^2 u_2(t) + \cdots$ and substitute to find $(\ddot{u}_0 + \varepsilon\ddot{u}_1 + \cdots) + (u_0 + \varepsilon u_1 + \cdots) + \varepsilon(u_0^3 + 3\varepsilon u_0^2 u_1 + \cdots) = 0$.

ε^0: $\ddot{u}_0 + u_0 = 0$, $u_0 = A\cos t + B\sin t$ $\Rightarrow$ if we want periodic solutions, we may choose our origin for t approximately, and for example choose $u_0 = a_0 \cos t$.

ε^1: $\ddot{u}_1 + u_1 + u_0^3 = 0 \Rightarrow \ddot{u}_1 + u_1 = -a_0^3\cos^3 t = -\frac{a_0^3}{4}(\cos 3t + 3\cos t)$. For PI try $u = A\cos 3t + B\sin 3t + Ct\cos t + Dt\sin t = \frac{a_0^3}{32}\cos 3t \Rightarrow$



$u = a_0\cos t + \varepsilon(a_1\cos t + b_1 \sin t + \frac{a_0^3}{32}\cos 3t - \frac{3}{8}a_0^3 t\sin t) + \cdots$

This is not periodic, and if $t = O(\frac{1}{\varepsilon})$, $\varepsilon u_1 > O(1)$ the same size as u_0.

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Previously, we derived the result that $\ddot{u} + u + \varepsilon u^3 = 0 \Rightarrow u = u_0(t) + \varepsilon u_1(t) + \cdots$, $u = a_0\cos t + \varepsilon\{\frac{a_0^3}{32}\cos 3t - \frac{3}{8}t a_0^3 \sin t\} + a_1\cos t + b_1 \sin t$.

We notice that $a_0\cos t - \frac{3}{8}\varepsilon t a_0^3 \sin t \approx a_0 \overbrace{\cos[t(1 + \tfrac{3}{8}\varepsilon a_0^2)]}^{\text{bounded}}$. by Taylor series expansion in powers of ε. This is periodic, but the frequency is amplitude-dependent.

Frequency is $1 + \frac{3}{8}\varepsilon a_0^2$, period is $2\pi/1 + \frac{3}{8}\varepsilon a_0^2 \approx 2\pi(1 - \frac{3}{8}\varepsilon a_0^2)$. The method used to deal with this is <u>Lindstedt's method</u>.

We switch to a new variable $s = t(c_0 + \varepsilon c_1 + \varepsilon^2 c_2 + \cdots)$ and then look for a new power series solution $u = u_0(s) + \varepsilon u_1(s) + \varepsilon^2 u_2(s) + \cdots$ and we insist that u is 2π-periodic in s.

$\frac{d}{dt} = \frac{d}{ds}\cdot\frac{ds}{dt} = (c_0 + \varepsilon c_1 + \varepsilon^2 c_2 + \cdots)\frac{d}{ds}$, $\frac{d^2}{dt^2} = (c_0 + \varepsilon c_1 + \varepsilon^2 c_2 + \cdots)^2 \frac{d^2}{ds^2} \sim (c_0^2 + 2c_0 c_1 \varepsilon + \cdots)\frac{d^2}{ds^2}$. Then we get that, taking dashes w.r.t. s,

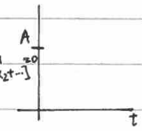
$\ddot{u} + u + \varepsilon u^3 = 0 \Rightarrow (c_0 + \varepsilon c_1 + \varepsilon c_2 + \cdots)^2(u_0'' + \varepsilon u_1'' + \varepsilon^2 u_2'' + \cdots) + (u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots) + \varepsilon(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots)^3 = 0$. Then we get that

ε^0: $c_0^2 u_0'' + u_0 = 0 \Rightarrow u_0 = a_0 \cos(\frac{s}{c_0})$, taking origin appropriately. For this to be 2π-periodic in s requires that $c_0 = 1$.

ε^1: $\overbrace{c_0^2 u_1'' + 2c_0 c_1 u_0''}^{\ddot{u}} + \overbrace{u_1}^{u} + \overbrace{u_0^3}^{\varepsilon u^3} = 0$. $c_0 = 1$, so $u_1'' + u_1 = -2c_1 u_0'' - u_0^3 = -2c_1(-a_0\cos s) - a_0^3(\frac{1}{4}\cos 3s + \frac{3}{4}\cos s)$. We can choose c_1 to ensure that u_1 is periodic, by fixing the forcing to have no component in $\cos s$, which is part of the complementary function for the equation $u_1'' + u_1 = 0$. Then $2c_1 a_0 - \frac{3}{4}a_0^3 = 0$. $\Rightarrow c_1 = \frac{3}{8}a_0^2$.

Then $u_1'' + u_1 = -\frac{a_0^3}{4}\cos 3s$ and $u_1 = a_1\cos(s) + b_1\sin(s) - \frac{a_0^3}{4}\cdot\frac{\cos 3s}{-9+1} \Rightarrow u = a_0\cos s + \varepsilon[a_1\cos(s) + b_1\sin(s) + \frac{1}{32}a_0^3\cos(3s)]$, $s = t(1 + \varepsilon a_0^2\cdot\frac{3}{8} + \cdots)$

<u>Rayleigh's solution:</u> $\ddot{x} - \varepsilon[\dot{x} - \frac{1}{3}\dot{x}^3] + x = 0$, $\varepsilon \ll 1$. We look for periodic solutions using Lindstedt's method and introduce $\theta = nt$, $n = n_0 + \varepsilon n_1 + \varepsilon^2 n_2 + \cdots$ and expand using $x = x_0(\theta) + \varepsilon x_1(\theta) + \varepsilon^2 x_2(\theta) + \cdots$ with $x_0, x_1, x_2, \ldots$ 2π-periodic in θ. We choose initial conditions $x(0) = A$, $\dot{x}(0) = 0$. Then we get $n^2 x'' - \varepsilon[nx' - \frac{1}{3}n^3 x'^3] + x = 0$.



Expanding up to ε^2, $[n_0^2 + 2\varepsilon n_0 n_1 + \varepsilon^2(n_1^2 + 2n_0 n_2) + \cdots][x_0'' + \varepsilon x_1'' + \varepsilon^2 x_2'' + \cdots] - \varepsilon[(n_0 + \varepsilon n_1 + \cdots)(x_0' + \varepsilon x_1' + \cdots) - \frac{1}{3}(n_0^3 + 3n_0^2 n_1 \varepsilon + \cdots)(x_0'^3 + 3\varepsilon x_0'^2 x_1' + \cdots)] + [x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots] = 0$

Also, boundary conditions give $x(0) = A \Rightarrow x_0(0) + \varepsilon x_1(0) + \varepsilon^2 x_2(0) + \cdots = A \Rightarrow x_0(0) = A$, $x_1(0) = x_2(0) = \cdots = 0$. $\dot{x}(0) = nx'(0) = (n_0 + \varepsilon n_1 + \cdots)(x_0' + \varepsilon x_1' + \cdots) = 0$

$\Rightarrow x_0'(0) = 0$, $n_1 x_0'(0) + n_0 x_1'(0) = 0 \Rightarrow x_1'(0) = 0$. We then write down equations at different orders.

ε^0: $n_0^2 x_0'' + x_0 = 0$. ε^1: $n_0^2 x_1'' + x_1 = -2n_0 n_1 x_0'' + n_0 x_0' - \frac{1}{3}n_0^3 x_0'^3$. ε^2: $n_0^2 x_2'' + x_2 = -(n_1^2 + 2n_0 n_2)x_0'' - 2n_0 n_1 x_1'' + [\{n_0 x_1' + n_1 x_0'\} - \frac{1}{3}\{3n_0^3 x_0'^2 x_1' + 3n_0^2 n_1 x_0'^3\}]$

$x_0 = A\cos(\frac{\theta}{n_0})$ and for x_0 to be 2π-periodic in θ, $n_0 = 1$. Then $x_1'' + x_1 = -2n_1(-A\cos\theta) + (-A\sin\theta) - \frac{1}{3}(-A\sin\theta)^3$. We use our flexibility of n_1, A to eliminate $\cos\theta, \sin\theta$ terms on right, to get periodic solutions without $\theta\cos\theta$ etc. terms. $x_1'' + x_1 = 2An_1\cos\theta - A\sin\theta + \frac{A^3}{3}(\frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta)$.

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$\Rightarrow x_1'' + x_1 = 2An_1\cos\theta + (\frac{A^3}{4} - A)\sin\theta - \frac{A^3}{12}\sin 3\theta$. Then fix $n_1 = 0$, $\frac{A^3}{4} - A = 0 \Rightarrow A = 0$ or 2 (ignore -2, since amplitude $= A > 0$). Then $x_1'' + x_1 = -\frac{A^3}{12}\sin 3\theta$.

If $A = 2$, $x_1'' + x_1 = -\frac{2}{3}\sin 3\theta$. $\Rightarrow$ then solving, $x_1 = a_1\cos\theta + b_1\sin\theta + \frac{1}{12}\sin 3\theta$. Apply initial conditions: $x_1(0) = 0 \Rightarrow a_1 = 0$, $x_1'(0) = 0 \Rightarrow b_1 = -1/4$. Thus, only periodic solutions of system have the form $x(t) = 2\cos\theta + \varepsilon(\frac{1}{12}\sin 3\theta - \frac{1}{4}\sin\theta)$ where $\theta = t(1 + O(\varepsilon^2))$. $\Rightarrow$ small non-linearity ($\varepsilon \ll 1$) fixes amplitude of periodic solution.

<u>A general approach.</u>

Consider $\ddot{x} + \varepsilon f(x, \dot{x}) + w^2 x = 0$ and use Lindstedt's procedure to find periodic solutions. $x = x_0(\theta) + \varepsilon x_1(\theta) + \cdots$, $\theta = t(n_0 + \varepsilon n_1 + \cdots)$. Then we have $(n_0^2 + 2\varepsilon n_0 n_1 + \cdots)(x_0'' + \varepsilon x_1'' + \cdots) + \varepsilon f(x_0 + \varepsilon x_1 + \cdots, (n_0 + \varepsilon n_1 + \cdots)(x_0 + \varepsilon x \cdots)) + w^2(x_0 + \varepsilon x_1 + \cdots) = 0$.

ε^0: $n_0^2 x_0'' + w^2 x_0 = 0$ ε^1: $n_0^2 x_1'' + w^2 x_1 = -2n_0 n_1 x_0'' - f(x_0, n_0 x_0')$ ← ignore ε terms as f is multiplied by ε. A periodic solution for x_0 is $x_0 = a\cos(w\theta/n_0)$ by picking a suitable origin. For this to be 2π-periodic, $n_0 = w$. Then $w^2 x_1'' + w^2 x_1 = -2wn_1 x_0'' - f(x_0, n_0 x_0') \Rightarrow x_1'' + x_1 = -\frac{2n_1}{w}(-a\cos\theta) - \frac{1}{w^2}f(a\cos\theta, -aw\sin\theta)$.

We need to expand f in its Fourier series, and we can do this as f is 2π-periodic in θ. Indeed, the whole RHS is 2π-periodic in θ. Non-periodicity only results if RHS has non-zero coefficients for $\cos\theta$, $\sin\theta$ terms $\Rightarrow$ no components proportional to $\cos/\sin\theta$. [Aside: let $g(\theta) = a_0 + a_1\cos\theta + \cdots + a_n\cos n\theta + b_1\sin\theta + \cdots + b_n \sin n\theta$. Then $\int_0^{2\pi}\cos\theta\, g(\theta)\,d\theta = a_1\int_0^{2\pi}\cos^2\theta\,d\theta$.

$\Rightarrow \int_0^{2\pi}\cos g(\theta)\,d\theta = \frac{\pi}{2}\cdot a_1$]. So, x_1 will be periodic if $\int_0^{2\pi}\cos\theta[-\frac{2n_1}{w}(-a\cos\theta) - \frac{1}{w^2}f(a\cos\theta, -aw\sin\theta)]\,d\theta = \int_0^{2\pi}\sin\theta[-\frac{2n_1}{w}(-a\cos\theta) - \frac{1}{w^2}f(a\cos\theta, -aw\sin\theta)]\,d\theta = 0$.

We have two equations for the unknowns n_1 and a. $\frac{2n_1 a}{w}\cdot\frac{1}{2}\cdot 2\pi = \frac{1}{w^2}\int_0^{2\pi}\cos\theta\, f(a\cos\theta, -aw\sin\theta)\,d\theta \Rightarrow n_1 = \frac{1}{2\pi a w}\int_0^{2\pi}\cos\theta\cdot f(a\cos\theta, -aw\sin\theta)\,d\theta$.

$0 = \int_0^{2\pi}\cos\theta\sin\theta\,d\theta = \frac{1}{w^2}\int_0^{2\pi}\sin\theta\, f(a\cos\theta, -aw\sin\theta)\,d\theta$. For the van der Pol equation, $\ddot{x} + \varepsilon\dot{x}(x^2 - 1) + x = 0$, we have $w = 1$, $f(x, \dot{x}) = \dot{x}(x^2 - 1)$.

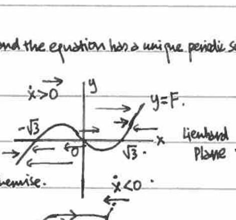
Then we find, for a, $0 = \int_0^{2\pi}\sin\theta[(a^2\cos^2\theta - 1)(-a\sin\theta)]\,d\theta \Rightarrow a^3\int_0^{2\pi}\sin^2\theta\cos^2\theta\,d\theta = a\int_0^{2\pi}\sin^2\theta\,d\theta \Rightarrow a = 0$ or $a^2 = \frac{\int_0^{2\pi}\sin^2\theta\,d\theta}{\int_0^{2\pi}\sin^2\theta\cos^2\theta} = \frac{\int_0^{2\pi}\sin^2\theta\,d\theta}{\frac{1}{4}\int_0^{2\pi}\sin^2\theta\,d\theta} = \frac{1}{1/4} \Rightarrow a = 2$.

$n_1 = \frac{1}{2\pi a}\int_0^{2\pi}\cos\theta[(a^2\cos^2\theta - 1)(-a\sin\theta)]\,d\theta = 0$ as $\sin\theta$ is odd, integrand is odd around $\theta = \pi$ by frameshift

Consider the van der Pol equation $\ddot{x} + \varepsilon\dot{x}(x^2 - 1) + x = 0$, $\varepsilon \gg 1$. This equation is of Liénard type – compare with $\ddot{x} + \dot{x}f(x) + g(x) = 0$, $f(x) = (x^2 - 1)\varepsilon$, $g(x) = x$, and the equation has a unique periodic solution

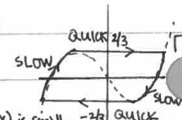
We introduce the Liénard variable $y = \dot{x} + F(x)$, $F'(x) = f(x)$ and $F(0) = 0$. Then $\dot{y} = \ddot{x} + \dot{x}F' = \ddot{x} + \dot{x}f(t) = -g$. Thus $\dot{y} = -g$, $\dot{x} = y - F$. For van der Pol equation,

$F(x) = \int_0^x \varepsilon(s^2 - 1)\,ds = \varepsilon(\frac{1}{3}x^3 - x)$. Then since $\dot{x} = y - F$, $\dot{x} > 0$ if $y > F$, $\dot{x} < 0$ if $y < F$. $\dot{y} = -x$, $\dot{x} = y - \varepsilon(\frac{1}{3}x^3 - x)$.



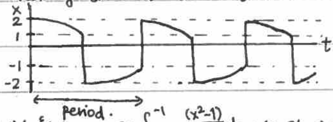
We observe trajectories approximately: if ε is big, $\dot{x}$ is big except where $y \approx \varepsilon(\frac{1}{3}x^3 - x)$, and x quickly increases if $y > \varepsilon(\frac{1}{3}x^3 - x)$, quickly decreases otherwise.

In this time, y does not alter $\Rightarrow$ to first order in $\frac{1}{\varepsilon}$ (small), trajectories in Lienhard plane are horizontal. This gives us periodic solutions (van der Pol oscillator), as pictured. Or, in a more rigorous manner, if we write $y = \varepsilon z$, then $\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = -\frac{x}{y - \varepsilon(\frac{1}{3}x^3 - x)} = \frac{-x}{\varepsilon[z - (\frac{1}{3}x^3 - x)]} = \varepsilon \frac{dz}{dx}$.



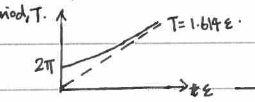
Then $\frac{dz}{dx} = \frac{1}{\varepsilon^2} \cdot \frac{-x}{z - (\frac{1}{3}x^3 - x)}$. So if $\varepsilon >> 1$, then $\frac{dz}{dx} = O(\frac{1}{\varepsilon^2}) << 1 \Rightarrow \frac{dy}{dx} = O(\frac{1}{\varepsilon}) << 1 \Rightarrow$ horizontal trajectories, unless $z - (\frac{1}{3}x^3 - x)$ is small, i.e. $z = \frac{1}{3}x^3 - x$ approximately. The local ~~maximum~~ extrema for F ~~is~~ are at $\frac{d}{dx}(\frac{1}{3}x^3 - x) = 0 \Rightarrow x = \pm 1 \Rightarrow F(\pm 1) = \mp \frac{2}{3}$. The line $y = \frac{2}{3}$ meets $y = \frac{1}{3}x^3 - x$ at $\frac{1}{3}x^3 - x - \frac{2}{3} = 0$.

$\Rightarrow \frac{1}{3}(x+1)^2(x-2) = 0 \Rightarrow$ amplitude is 2. So expressing x in terms of t, on the closed orbit, we get something like the graph on the right. The period is dominated by behaviour in the "slow" component. To find the period, $T = \int_0^T dt = \int_\Gamma \frac{dx}{\dot{x}}$



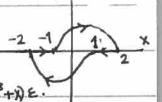
$= \int_\Gamma \frac{dz}{dx} \frac{dt}{dz} dx = 2\left\{ \int_{-2}^{-1} (x^2-1) \frac{dt}{dz} dx + \int_{-1}^{2} 0 \cdot \frac{dt}{dz} dx \right\}$ ($z = \frac{1}{3}x^3 - x$; $z = \text{const} = 2/3$) $= 2\int_{-2}^{-1} (x^2-1)\left(\frac{dz}{dt}\right)^{-1} dx \overset{y = \varepsilon z}{=} 2\int_{-2}^{-1} (x^2-1) \frac{\varepsilon}{dy/dt} dx = 2\int_{-2}^{-1} (x^2-1)(-\frac{\varepsilon}{x}) dx = 2\varepsilon \int_{-2}^{-1} \frac{(x^2-1)}{-x} dx = (3 - 2\ln 2)\varepsilon \approx 1.614\varepsilon$.

The period is about 1.614ε, which is a long time for large ε. These give us "metastable" states.



We do not know what phase plane looks like for $\varepsilon >> 1$. For $\varepsilon << 1$, period is 2π, so we can plot ε against t.

Finally, we produce the phase plane: $\dot{x} = y - F = y - \varepsilon(\frac{1}{3}x^3 - x)$. We sketch it by individual segments.



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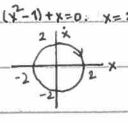
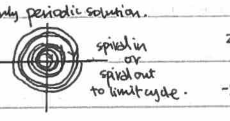
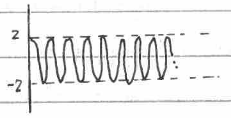
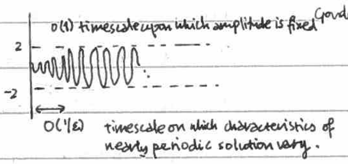
$\ddot{x} + \varepsilon\dot{x}(x^2-1) + x = 0$. $x = 2\cos t$ is the only periodic solution.









We can capture these solutions by explicitly introducing two timescales:

- t, representing the $O(1)$ scale, and
- $T = \varepsilon t$, representing the long (or slow) timescale $O(\frac{1}{\varepsilon})$.

For T to change by an $O(1)$ amount, t must change by $O(\frac{1}{\varepsilon})$. We treat t and T as being independent time variables.

$\boxed{\text{Ex}}$ Solve the general equation $\ddot{x} + \varepsilon f(x, \dot{x}) + \omega^2 x = 0$ to describe its slow variation. Then apply it to the van der Pol equation $\ddot{x} + \varepsilon\dot{x}(x^2-1) + x = 0$.

Soln. Let $x = x(t, T)$ with $T = \varepsilon t$. Then $x = x_0(t, T) + \varepsilon x_1(t, T) + \dots$ as a power series in ε. $\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{dT}{dt}\frac{\partial}{\partial T}$ by the chain rule. $= \frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial T} = \partial_t + \varepsilon \partial_T$.

Then $\frac{d^2}{dt^2} = (\partial_t + \varepsilon\partial_T)^2 = \frac{\partial^2}{\partial t^2} + 2\varepsilon \frac{\partial^2}{\partial T \partial t} + \varepsilon^2 \frac{\partial^2}{\partial T^2}$. We take terms up to order ε, so we get $\left(\frac{\partial^2}{\partial t^2} + 2\varepsilon \frac{\partial^2}{\partial T \partial t} + \dots\right)(x_0 + \varepsilon x_1 + \dots) + \varepsilon \overbrace{f(x_0 + \dots, (\partial_t + \dots)(x_0 + \dots)}^{f(x_0, x_{0t})} + \omega^2(x_0 + \varepsilon x_1 + \dots) = 0$

$\Rightarrow$ comparing ε^0 terms: $(x_0)_{tt} + \omega^2 x_0 = 0$, where x_0 has coefficients as functions of $T \Rightarrow x_0 = \bar{A}(T)\cos(\omega t) + \bar{B}(T)\sin(\omega t) = A(T)\sin\underbrace{(\omega(t) + \Phi(T))}_{\chi}$ ← phase.

Here, A and Φ are functions of slow variable T, so they vary slowly. Then comparing ε^1 terms: $(x_1)_{tt} + \omega^2 x_1 = -f(\overbrace{A\sin\chi}^{x_0}, \overbrace{\omega A\cos\chi}^{(x_0)_t}) - 2(A\sin\chi)_{tT}$

$\Rightarrow (x_1)_{tt} + \omega^2 x_1 = -f(A\sin\chi, \omega A\cos\chi) - 2\frac{\partial}{\partial T}(\omega A\cos\chi) = -f(A\sin\chi, \omega A\cos\chi) - 2\omega\frac{dA}{dT}\cdot\cos\chi + 2\omega A\sin\chi \cdot \frac{d\Phi}{dT}$.

$x_1(t)$ will grow linearly in amplitude, as seen, if the coefficient of either of the terms $\sin(\omega t + \varphi)$, $\cos(\omega t + \varphi)$ in the Fourier series of the RHS is non-zero. We can ensure that this is the case by evaluating these (multiply by $\sin x$, $\cos x$ and integrate over the period in χ), and set them to be equal to 0. For instance, multiplying through by $\sin\chi$

$0 = -\int_0^{2\pi} f(A\sin\chi, \omega A\cos\chi) \cdot \sin\chi \, d\chi - 2\omega \cdot \frac{dA}{dT}\int_0^{2\pi} \cos\chi \sin\chi \, d\chi + 2\omega A \frac{d\Phi}{dT} \overbrace{\int_0^{2\pi} \sin\chi \sin\chi \, d\chi}^{\pi}$. Then, we can rearrange our equations to get:

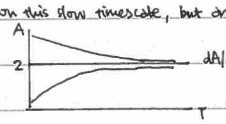
$2\pi\omega A \frac{d\Phi}{dT} = \int_0^{2\pi} f(A\sin\chi, \omega A\cos\chi) \cdot \sin\chi \, d\chi$. —① Similarly, multiplying by $\cos\chi$, $2\pi\omega \frac{dA}{dT} = -\int_0^{2\pi} f(A\sin\chi, \omega A\cos\chi) \cdot \cos\chi \, d\chi$. —②

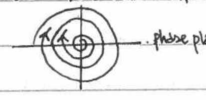
[Note — In older nomenclature, this is known as the "method of averaging". Here, we can describe this instead as the "method of multiple timescales".

For specific question, $\omega = 1$, $f(x, \dot{x}) = \dot{x}(x^2-1)$. Then ①: $2\pi\omega A \frac{d\Phi}{dT} = \int_0^{2\pi} \underbrace{(A\cos\chi)}_{\text{even}}\underbrace{(A^2\sin^2\chi - 1)}_{\text{even}} \underbrace{\sin\chi}_{\text{odd}} \, d\chi = 0$. The other equation is

②: $2\pi \frac{dA}{dT} = -\int_0^{2\pi} (A\cos\chi)(A^2\sin^2\chi - 1) \cdot \cos\chi \, d\chi = A\pi - \int_0^{2\pi} A^3 \cos^2\chi \sin^2\chi \, d\chi = A\pi - \frac{A^3}{4}\int_0^{2\pi} \sin^2(2\chi) = A\pi - \frac{A^3}{4}\pi$. i.e. we get that the phase Φ does not vary on this slow timescale, but amplitude satisfies $\frac{dA}{dT} = \frac{1}{2}[A - \frac{A^3}{4}]$. Indeed $\frac{dA}{dT} = 0$ if $A = 2$, $\frac{dA}{dT} < 0$ if $A > 2$, $\frac{dA}{dT} < 0$ if $A < 2$.

So we get:





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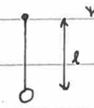
Working a swing (?)



The displacement of a swing satisfies $\ddot{x} + \frac{l}{g}x = 0$, $\frac{l}{g} = \omega^2$. Let us alter the length of the swing periodically (e.g. by changing centre of gravity):

$\frac{l}{g} = \omega^2 + a\cos qt$ (a will be small — analogous to wagging legs back and forth on a relatively long swing), q is close to ω. We will scale t with $(1/q)$.

Then $q^2\ddot{x} + (\omega^2 + a\cos t)x = 0$. If we write $\frac{\omega^2}{q^2} = 1 + \varepsilon^2 k$, $\frac{a}{q^2} = \varepsilon$, then $\ddot{x} + (1 + \varepsilon^2 k + \varepsilon\cos t)x = 0$. Substitute $x = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t)$. Then we compare terms:

ε^0: $\ddot{x}_0 + x_0 = 0$, $x_0 = A\cos t + B\sin t$. ε^1: $\ddot{x}_1 + x_1 = -\cos t \, x_0 = -A\cos^2 t - B\cos t\sin t$ gives forcing $1, \cos 2t, \sin 2t$, not part of complementary function for $\ddot{x}_1 + x_1 = 0$.

$\Rightarrow x_1$ is periodic! ε^2: $\ddot{x}_2 + x_2 = -kx_0 - \cos t \, x_1$. These do cause problems $\Rightarrow$ Fourier series have components in $\cos t$, $\sin t$. i.e. $\ddot{x}_2 + x_2 = -k\underbrace{x_0}_{\sin t, \cos t} - \underbrace{\cos t \, x_0}_{\cos t \sin 2t = \frac{1}{2}(\sin 3t + \sin t)}$.

So it is x_2, which appears at $O(\varepsilon^2)$, which is not periodic. If we introduce a slow timescale $T = \varepsilon^2 t$, then we will get additional $\frac{dA}{dT}$ and $\frac{dB}{dT}$ terms on RHS of equation for x_2, which will lead to equations governing the growth of A and B. Since $d_t = \partial_t + \varepsilon^2\partial_T$, $d^2_{tt} = \partial^2_{tt} + 2\varepsilon^2\partial^2_{Tt} + \varepsilon^4\partial^2_{TT}$. Then $x_{0tt} + x_0 = 0$, $x_{1tt} + x_1 = -\cos t \, x_0$

and $x_{2tt} + x_2 = -kx_0 - \cos t \, x_1 - 2\frac{\partial}{\partial t}x_{0t}$. $x_0 = A(T)\cos t + B(T)\sin t$. Here, we do not combine to amplitude and phase (more general).

$\ddot{x}_1 + x_1 = -A\cos^2 t - B\cos t\sin t$, so $x_1 = CF - \frac{1}{2}A + \frac{1}{6}A\cos t + \frac{1}{6}B\sin 2t$ where $CF = A_1(T)\cos t + B_1(T)\sin t = 0$ (we set it as that — $O(\varepsilon)$ being corrections to original amplitudes A, B. They contain no new physics).

$\ddot{x}_2 + x_2 = -k(A\cos t + B\sin t) = -\cos t(-\frac{1}{2}A + \frac{1}{6}A\cos 2t + \frac{1}{6}B\sin 2t) = -2\frac{\partial}{\partial T}(-A\sin t + B\cos t)$. x_2 will remain bounded, on $x = \underbrace{x_0}_{\text{main term}} + \underbrace{\varepsilon x_1}_{\text{(small) correction to } x_0 \text{ term}} + \underbrace{\varepsilon^2 x_2}_{\text{(really small) overall correction}}$.

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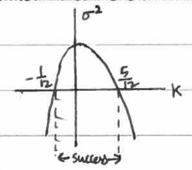
if the coefficient of $\sin t$ and $\cos t$ on RHS is zero. $\cos t$: $-kA+\frac{1}{2}A-\frac{1}{12}A-2\frac{\partial B}{\partial T}=0$ $\quad \sin t$: $-kB-\frac{1}{12}B+2\frac{\partial A}{\partial T}=0$. [use $\cos t\cos 2t=\frac{1}{2}(\cos 3t+\sin t)$, $\cos t\sin 2t=\frac{1}{2}(\sin 3t+\sin t)$]

$\frac{\partial A}{\partial T}=\frac{1}{2}(k+\frac{1}{12})B$, $\frac{\partial B}{\partial T}=\frac{1}{2}(-k+\frac{5}{12})A$. Then $\begin{pmatrix}\partial A/\partial T\\ \partial B/\partial T\end{pmatrix}=\begin{pmatrix}0 & \frac{1}{2}(k+\frac{1}{12})\\ \frac{1}{2}(-k+\frac{5}{12}) & 0\end{pmatrix}\begin{pmatrix}A\\B\end{pmatrix}$. Seek solutions $\begin{pmatrix}A\\B\end{pmatrix}=\underline{u}e^{\sigma T}$, then $\sigma\underline{u}e^{\sigma T}=(\underline{\underline{M}})\underline{u}e^{\sigma T}$, so σ is an eigenvalue of $\underline{\underline{M}}$

$\Rightarrow \sigma^2=\frac{1}{4}(k+\frac{1}{12})(-k+\frac{5}{12})$. We want σ to be real s.t. amplitude increases i.e. $\sigma^2>0$ (since amplitudes A,B will increase only if $\sigma^2>0$). Only element that we can control is $k=\frac{q^2-w^2}{a^2/q^2}$.



Asymptotic expansion of integrals.

Consider the exponential integral $E_i(x)=\int_x^\infty \frac{e^{-t}}{t}dt$. Clearly $x>0$ for this integral to converge. Let $t=xu$, then $E_i(x)=\int_1^\infty \frac{e^{-xu}}{xu}x\,du=\int_1^\infty \frac{e^{-xu}}{u}du$.

$=[e^{-xu}(-\frac{1}{x})\frac{1}{u}]_1^\infty-\int_1^\infty e^{-xu}(-\frac{1}{x})(-\frac{1}{u^2})du=\frac{e^{-x}}{x}-\frac{1}{x}\int_1^\infty\frac{e^{-xu}}{u^2}du$ (as $x>0$ so $e^{-xu}\to 0$ as $u\to\infty$) $=\frac{e^{-x}}{x}-\frac{1}{x}\{[e^{-xu}(-\frac{1}{x})\frac{1}{u^2}]_1^\infty-\int_1^\infty e^{-xu}(-\frac{1}{x})(-\frac{2}{u^3})du\}$

$=\frac{e^{-x}}{x}-\frac{e^{-x}}{x^2}+\frac{2}{x^2}\int_1^\infty e^{-xu}/u^3\,du$. Doing this n times gives $E_i(x)=e^{-x}\sum_{r=1}^n(-1)^{r-1}\cdot\frac{1}{x^r}(r-1)!+R_n(x)$, where $R_n(x)=\frac{(-1)^n(n!)}{x^n}\int_1^\infty\frac{e^{-xu}}{u^{n+1}}du$. Let $xu=t$, $du=\frac{dt}{x}$, $u^{n+1}=\frac{t^{n+1}}{x^{n+1}}$.

$\Rightarrow R_n(x)=(-1)^n(n!)\int_x^\infty\frac{e^{-t}}{t^{n+1}}dt$. As $t\geq x$, $\frac{1}{t^{n+1}}\leq\frac{1}{x^{n+1}}$, $|R_n|<\frac{n!}{x^{n+1}}\int_x^\infty e^{-t}dt=\frac{n!\,e^{-x}}{x^{n+1}}$. Thus, using this estimate, $E_i(x)=e^{-x}[\sum_1^n\frac{(-1)^{r-1}(r-1)!}{x^r}+S_n]$, where $|S_n|<\frac{n!}{x^{n+1}}$.

Recall we said that $f=O(g)$ as $x\to\infty$, $|f/g|\to 0$ as $x\to\infty$; so $E_i(x)=e^{-x}[\sum_1^n\frac{(-1)^{r-1}(r-1)!}{x^r}+O(\frac{1}{x^n})]$ as $x\to\infty$ $\sim e^{-x}[\sum_{r=1}^\infty\frac{(-1)^{r-1}(r-1)!}{x^r}]$. Then, for fixed n, $R_n\to 0$ as $x\to\infty$.

But for fixed x, $R_n\to\infty$ as $n\to\infty$. So there is an optimum value of n for a particular x, for which the expansion $E_i(x)\approx e^{-x}\sum_{r=1}^n\frac{(-1)^{r-1}(r-1)!}{x^r}$ performs best, we write

$E_i(x)\sim e^{-x}(\frac{1}{x}-\frac{1}{x^2}+\frac{2}{x^3}-\frac{6}{x^4}+\dots)$ where it is understood that we take a finite number of terms of this otherwise divergent series for fixed x.

Factorial function.

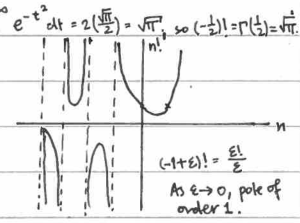
Consider $I(n)=\int_0^\infty e^{-u}u^n du=[-e^{-u}u^n]_0^\infty-\int_0^\infty -e^{-u}nu^{n-1}du=n\,I(n-1)$ if $n>0$. Also, $I(0)=\int_0^\infty e^{-u}du=1$. So $I(n)=n!$ This integral is defined $\forall n>-1$.

Note – The Gamma function $\Gamma(n)=\int_0^\infty e^{-u}u^{n-1}du$, so $\Gamma(n)=(n-1)!$ or $\Gamma(n+1)=n!$

We can use $I(n)$ to extend our definition of $n!$ to any $n>-1$. e.g. $(-\frac{1}{2})!=\int_0^\infty\frac{e^{-u}}{\sqrt{u}}du=\int_0^\infty\frac{e^{-t^2}}{t}2t\,dt$ (set $u=t^2$) $=2\int_0^\infty e^{-t^2}dt=2(\frac{\sqrt{\pi}}{2})=\sqrt{\pi}$, so $(-\frac{1}{2})!=\Gamma(\frac{1}{2})=\sqrt{\pi}$.

Then $(\frac{1}{2})!=\frac{1}{2}(\frac{1}{2}-1)!=\frac{1}{2}\cdot(-\frac{1}{2})!=\frac{\sqrt{\pi}}{2}$. Also, $n!=\frac{(n+1)!}{n+1}\Rightarrow(-\frac{3}{2})!=\frac{(-\frac{3}{2}+1)!}{-\frac{3}{2}+1}=\frac{(-1/2)!}{1/2}=-2\sqrt{\pi}$. (Note that this is negative).

We can plot factorials for all $n\in\mathbb{R}\setminus\mathbb{Z}^-$.



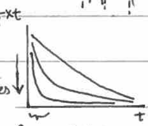
$(-1+\varepsilon)!=\frac{\varepsilon!}{\varepsilon}$

As $\varepsilon\to 0$, pole of order 1.

Watson's Lemma.

Consider integrals of the form $I(x)=\int_0^T e^{-xt}f(t)dt$ as $x\to\infty$. We need $T>0$ (it can be infinity), and f cannot grow quicker than an exponential.

(i.e. $f(t)=e^{1000t}$ is okay, $f(t)=e^{0.0001t^2}$ is not). If we plot e^{-xt} against t, we get the following family of curves:



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As x increases, $f(t)$ affects the function only at small t. i.e. As $x\to\infty$, e^{-xt} becomes exponentially small except where $xt=O(1)\Rightarrow t=O(\frac{1}{x})$. f can only have a significant multiplicative impact here.

As small exponential terms are much smaller than algebraic terms in an expansion, if we are happy with an expansion that neglects the exponential terms, then the only part of the range of integration that matters is where $t=O(\frac{1}{x})$. With the substitution $u=xt$ (in the important region), we have $I(x)=\int_0^{xT}e^{-u}f(\frac{u}{x})\frac{du}{x}$. [Here, $\frac{1}{x}$ is analogous to width of t region that matters]

If $f(t)$ has a Taylor series about $t=0$, we can use this to expand $f(\frac{u}{x})$ as $x\to\infty$. [In fact, this Taylor series can be replaced by an asymptotic expansion for $f(t)$ as $t\to 0$.

Then $I(x)=\int_0^{xT}e^{-u}\sum_{n=0}^\infty\frac{f^{(n)}(0)}{n!}\cdot\frac{u^n}{x^n}\cdot\frac{du}{x}$. We now replace xT by infinity and interchange the integral and summation symbols. Any errors we commit in doing so are exponentially small.

This yields: $I(x)\sim\sum_{n=0}^\infty\frac{f^{(n)}(0)}{n!}\cdot\frac{1}{x}\frac{1}{x^n}\int_0^\infty e^{-u}u^n du$ as $x\to\infty$ $\Rightarrow I(x)\sim\sum_{n=0}^\infty\frac{f^{(n)}(0)}{n!}\frac{1}{x^{n+1}}(n!)=\sum_{n=0}^\infty\frac{f^{(n)}(0)}{x^{n+1}}$. More generally, if $f(t)\sim t^\lambda\sum_{n=0}^\infty a_n t^{\lambda_n}$, $\lambda_0=0$, $\lambda_0<\lambda_1<\lambda_2<\dots$ [e.g. $\sqrt{t}(1+t^{3/2}+t^3+\dots)$]

then $\boxed{I(x)\sim\sum_{n=0}^\infty\frac{a_n(\lambda+\lambda_n)!}{x^{\lambda+\lambda_n+1}}}$

Examples –

1. $E_i(x)=\int_x^\infty\frac{e^{-t}}{t}dt$. Let $t=xu$, then $E_i(x)=\int_1^\infty\frac{e^{-xu}}{xu}x\,du=\int_1^\infty\frac{e^{-xu}}{u}du$. Let $u=1+s$, then $E_i(x)=\int_0^\infty\frac{e^{-x(1+s)}}{1+s}ds=e^{-x}\int_0^\infty\frac{e^{-xs}}{1+s}ds$. (something like a scaling component.) Using Watson's lemma, $f(s)=\frac{1}{1+s}$

$f(s)=1-s+s^2-s^3+\dots\Rightarrow E_i(x)\sim e^{-x}\sum_{n=0}^\infty\frac{(-1)^n n!}{x^{n+1}}$. Alternatively, from $E_i(x)=e^{-x}\int_0^\infty\frac{e^{-xs}}{1+s}ds$, take $q=sx$, then $E_i(x)=e^{-x}\int_0^\infty\frac{e^{-q}}{1+\frac{q}{x}}\frac{dq}{x}\sim e^{-x}\int_0^\infty e^{-q}(1-\frac{q}{x}+\frac{q^2}{x^2}-\dots)\frac{dq}{x}$

$=e^{-x}\sum_{n=0}^\infty\frac{(-1)^n}{n+1}\int_0^\infty e^{-q}q^n dq=e^{-x}\sum_{n=0}^\infty\frac{(-1)^n n!}{x^{n+1}}$.

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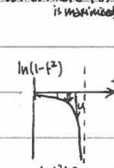
2. $I(x)=\int_0^\infty e^{-xt}\ln(1+t^2)dt$. Then since $\ln(1+\varepsilon)=\varepsilon-\frac{\varepsilon^2}{2}+\frac{\varepsilon^3}{3}$, $\ln(1+t^2)=t^2-\frac{t^4}{2}+\frac{t^6}{3}-\dots$ Then by immediate application of Watson's Lemma, we have

$I(x)\sim\frac{2!}{x^{2+1}}-\frac{1}{2}\frac{4!}{x^{4+1}}+\frac{1}{3}\frac{6!}{x^{6+1}}\dots=\frac{2}{x^3}-\frac{12}{x^5}+\frac{240}{x^7}+\dots$

3. $I(x)=\int_0^{\pi/2}e^{-x\cos\theta}d\theta$. Put $t=\cos\theta$ to get $\int e^{-xt}$. $dt=-\sin\theta\,d\theta=-\sqrt{1-t^2}\,d\theta$. $I(x)=(-1)\int_1^0 e^{-xt}\frac{dt}{\sqrt{1-t^2}}=\int_0^1\frac{e^{-xt}}{\sqrt{1-t^2}}dt$ [Aside – for $\int e^{x\varphi(t)}$, greatest contribution comes where $\varphi(t)$ is maximised]

$(1+t^2)^{-\frac{1}{2}}\sim 1+(-\frac{1}{2})(-t^2)+(-\frac{1}{2})(-\frac{3}{2})(\frac{1}{2})(-t^2)^2+\dots=1+\frac{1}{2}t^2+\frac{3}{8}t^4+\dots$, so $I(x)=\int_0^1\frac{e^{-xt}}{\sqrt{1-t^2}}\sim\frac{1}{x}+\frac{1}{2}\frac{2!}{x^3}+\frac{3}{8}\frac{4!}{x^5}+\dots$

4. $I(x)=\int_0^1(1-t^2)^x\,dt=\int_0^1 e^{x\ln(1-t^2)}dt$. Let $u=-\ln(1-t^2)$ [this is physically represented by distance from x-axis to curve at each point]. $du=\frac{2t}{1-t^2}dt$



$\Rightarrow du=2\frac{\sqrt{1-e^{-u}}}{e^{-u}}dt$ (take +ve root $\because u>0, t>0$) $\Rightarrow I(x)=\int_0^\infty e^{-xu}\cdot\frac{1}{2}\frac{e^{-u}}{\sqrt{1-e^{-u}}}du$. We then examine behaviour of $\frac{e^{-u}}{\sqrt{1-e^{-u}}}$ as $u\to 0$. Then we have

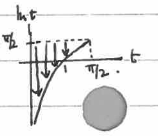
$\frac{e^{-u}}{\sqrt{1-e^{-u}}}$ (no Taylor series at $u=0$ $\because$ singularity in denominator) $\underset{u\to 0}{=}\frac{1-u+\dots}{\sqrt{1-(1-u+u^2/2+\dots)}}=\frac{1-u}{\sqrt{u}(1-u/2)^{1/2}}\sim\frac{1}{\sqrt{u}}\cdot(1-u)(1+\frac{1}{4}u)\sim\frac{1}{\sqrt{u}}-\frac{3}{4}\sqrt{u}$. Then, we have

$I(x)\sim\frac{1}{2}[\frac{1}{x^{-1/2+1}}(-\frac{1}{2})!-\frac{3}{4}\frac{(1/2)!}{x^{1/2+1}}]=\frac{1}{2}[\frac{1}{\sqrt{x}}\sqrt{\pi}-\frac{3}{4}(\frac{\sqrt{\pi}}{2})\frac{1}{x\sqrt{x}}]=\frac{1}{2}\sqrt{\frac{\pi}{x}}[1-\frac{3}{8}\frac{1}{x}]$.

Note – this is consistent as $\ln(1-t^2)$ has zero slope near $t=0$. Moreover, $\ln(1-t^2)$ has its maximum value close to $t=0$. Near $t=0$, $\ln(1-t^2)\sim t^2-\frac{t^4}{2}$. Consider $\int_0^1 e^{x(-t^2-\frac{t^4}{2}\dots)}dt$ (or even ∞ as upper limit).

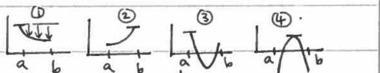
$\sim\int_0^\infty e^{-xt^2}(1+\frac{t^4}{2})dt$. Substitute $u^2=xt^2$ to "blow up" region where $xt^2\sim O(1)$… $\sim\int_0^\infty e^{-u^2}\underbrace{e^{-\frac{1}{2}x\frac{u^4}{x^2}}}_{e^{-\frac{1}{2}\frac{u^4}{x}}\sim 1-\frac{1}{2}\frac{u^4}{x},\ x\to\infty.}\frac{du}{\sqrt{x}}\Rightarrow\frac{1}{\sqrt{x}}\underbrace{\int_0^\infty e^{-u^2}du}_{\frac{\sqrt{\pi}}{2}}-\frac{1}{2}\frac{1}{x^{3/2}}\int_0^\infty u^4 e^{-u^2}du=\frac{1}{\sqrt{x}}\frac{\sqrt{\pi}}{2}-\frac{1}{2}\underbrace{\int_0^\infty v^2 e^{-v}\frac{dv}{2\sqrt{v}}}_{\frac{3}{4}\sqrt{\pi}}$ ($v=u^2$)

5. $I(x)=\int_0^{\pi/2} t^x \sin t\, dt = \int_0^{\pi/2} e^{x\ln t}\sin t\, dt$. Write $u=\ln(\frac{\pi}{2})-\ln t$, $\ln t=\ln(\frac{\pi}{2})-u$, $\sin t=\sin(e^{\ln\frac{\pi}{2}-u})=\sin(\frac{\pi}{2}e^{-u})$. $du=-\frac{1}{t}dt=\frac{-dt}{\frac{\pi}{2}e^{-u}}$



$I(x)=\int_\infty^0 e^{x\ln(\frac{\pi}{2})-xu}\sin(\frac{\pi}{2}e^{-u})(-1)(\frac{\pi}{2}e^{-u})du = e^{x\ln(\frac{\pi}{2})}\cdot\frac{\pi}{2}\int_0^\infty e^{-xu}f(u)\,du$ where $f(u)=e^{-u}\sin(\frac{\pi}{2}e^{-u})$. $f(0)=1$. $\sim(\frac{\pi}{2})^{x+1}\cdot\frac{1}{x}$.

Laplace Integrals.



Laplace integrals have the form $I(x)=\int_a^b x^{\varphi(t)} f(t)\,dt$. The largest contribution comes from the point where $\varphi(t)$ is maximised.

We have a few cases as on right. ①: $\varphi'(t)<0$ in $[a,b]$ and $\varphi'(c)\neq 0$ for any $c\in[a,b]$. Let $u=\varphi(a)-\varphi(t)$, then $I(x)=\int_0^\beta e^{x\varphi(a)}e^{-xu}f(t(u))\frac{du}{\varphi'(t(u))}$.

By Watson's lemma, $I(x)=-e^{x\varphi(a)}\int_0^\beta e^{-xu}\frac{f(t(u))}{\varphi'(t(u))}du$ as $x\to\infty$ has general terms which can be calculated. First term however is given by expression (thinking of t as function of u) $-\frac{e^{x\varphi(a)}}{x}\frac{f(t(0))}{\varphi'(t(0))}$.

This is equivalent to $\frac{e^{x\varphi(a)}}{x}\cdot\frac{f(a)}{|\varphi'(a)|}$ since $\varphi'(a)<0$. ②: Similarly, if $\varphi'(t)>0$, we find $I(x)\sim\frac{e^{x\varphi(b)}f(b)}{x\varphi'(b)}$, $\varphi'(b)>0$. Together, $I(x)\sim\frac{e^{x\varphi(d)}}{x}\frac{f(d)}{|\varphi'(d)|}$ where d is the endpoint a or b where $\varphi(d)$ is biggest.

This result can be obtained in a different way (integration by parts). $I(x)=\int_a^b e^{x\varphi(t)}f(t)\,dt=\int_a^b\varphi'(t)e^{x\varphi(t)}\frac{f(t)}{\varphi'(t)}dt=\int_a^b\frac{1}{x}\frac{d}{dt}e^{x\varphi(t)}\frac{f(t)}{\varphi'(t)}dt=\left[\frac{1}{x}e^{x\varphi(t)}\frac{f(t)}{\varphi'(t)}\right]_a^b-\underbrace{\frac{1}{x}\int_a^b e^{x\varphi(t)}\frac{d}{dt}\left(\frac{f(t)}{\varphi'(t)}\right)dt}$

$I(x)\sim\frac{1}{x}\left[e^{x\varphi(b)}\frac{f(b)}{\varphi'(b)}-e^{x\varphi(a)}\frac{f(a)}{\varphi'(a)}\right]$ and if we choose the exponentially dominant contribution given by whichever is bigger, $\varphi(a)$ or $\varphi(b)$, we get the same result. [$O(\frac{1}{x^2})$ as seen by integrating by parts again.] For case ③, if $c\in(a,b)$ is such that $\varphi'(c)=0$, $\varphi''(c)>0$, we have a minimum then using integration by parts gives $\int_a^b\varphi'(t)e^{x\varphi(t)}\frac{f(t)}{\varphi'(t)}dt$

This integral is indefinite due to the singularity in the integrand at $t=c$. The singularity is integrable however and all goes through. The major contribution to the integral comes from near a or b depending on which is greater. Finally, we consider case ④: if $\varphi'(c)=0$ and $\varphi''(c)<0$, then the dominant contribution to the integral comes from near $t=c$ (where $\varphi'(c)=0$). One way of dealing with this is to split the integral into two parts: $[a,c]$ and $[c,b]$ and use Watson's lemma on each. As $\varphi'(c)=0$, the result is $I(x)\sim e^{x\varphi(c)}f(c)\sqrt{\frac{2\pi}{x|\varphi''(c)|}}$

Or, we can proceed as follows: $I(x)=\int_a^b e^{x\varphi(t)}f(t)\,dt$. If near c where $\varphi'(c)=0$, $\varphi(t)\sim\varphi(c)+(t-c)\varphi'(c)+\frac{(t-c)^2}{2}\varphi''(c)$. Then $x\varphi(t)\approx x\varphi(c)+x\frac{(t-c)^2}{2}\varphi''(c)$

Therefore $e^{x\varphi(t)}\approx e^{x\varphi(c)}e^{\underbrace{-x\frac{(t-c)^2}{2}|\varphi''(c)|}_{u^2}}$, and make the substitution $\frac{x(t-c)^2}{2}|\varphi''(c)|=u^2$ (focusing in on region around c). This behaves like a Gaussian curve. Then $I(x)\sim\int_{-\infty}^\infty e^{x\varphi(c)}e^{-u^2}(1+\dots)(f(c)+\dots)\cdot du\sqrt{\frac{2}{x|\varphi''(c)|}}=e^{x\varphi(c)}f(c)\sqrt{\frac{2\pi}{x|\varphi''(c)|}}\Rightarrow$ decays like $\frac{1}{\sqrt{x}}$ rather than $\frac{1}{x}\Rightarrow$

← infinite limits in u!

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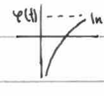
slightly slower decay. Also, if maximum is at an endpoint, we get half the value of this integral. (analogous to going 0 to ∞ or $-\infty$ to 0)

As an example, we evaluate $I(x)=\int_0^{\pi/2}t^x\sin t\,dt=\int_0^{\pi/2}e^{x\overbrace{\ln t}^{\varphi(t)}}\underbrace{\sin t}_{f(t)}\,dt$. φ is maximum at $t=\frac{\pi}{2}$ where $\varphi'(t)=\frac{1}{t}=\frac{2}{\pi}$. $I(x)\sim\frac{e^{x\varphi(c)}f(c)}{x|\varphi'(c)|}=\frac{e^{x\ln\pi/2}\sin(\pi/2)}{x\cdot\frac{2}{\pi}}=(\frac{\pi}{2})^{x+1}\cdot\frac{1}{x}$.

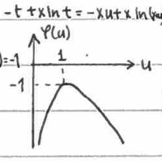


This could have been made far more difficult if $I(x)=\int_0^{\pi/2}t^x\cos t\,dt$, since $\cos(\frac{\pi}{2})=0$. We deal with it by evaluating from first principles!

One example where we proceed from first principles is per the Gamma function:

Gamma Function.

$\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}dt$, then $\Gamma(x+1)=x!$ $\Gamma(x)=\int_0^\infty e^{-t+x\ln t}\frac{dt}{t}$. Here, the maximum is clearly x-dependent, so we perform a scaling. Let $t=xu$, $-t+x\ln t=-xu+x\ln(xu)$ $=-xu+x\ln x+x\ln u$, so $\Gamma(x)=\int_0^\infty e^{x(\ln u-u)}e^{x\ln x}\frac{du}{u}=x^x\int_0^\infty e^{x(\ln u-u)}\frac{du}{u}$. Then let $\varphi(u)=\ln u-u$, $f(u)=\frac{1}{u}$. $\varphi'(u)=\frac{1}{u}-1=0$ at $u=1$, $\varphi(1)=-1$



$\varphi''(u)=-\frac{1}{u^2}$ $\varphi^{(3)}(u)=\frac{2}{u^3}$

so maximum is attained at $u=-1$. So we expand about $u=1$ (or for just one term, we can apply formula immediately). $\varphi''(1)=-1$ $\varphi^{(3)}(1)=2$

$\varphi^{(4)}(u)=-6/u^4$

$\varphi^{(4)}(1)=-6$ and we expand $e^{x(\ln u-u)}\sim e^{x[-1+0+\frac{1}{2}(-1)(u-1)^2+\frac{1}{6}(2)(u-1)^3+\frac{1}{24}(-6)(u-1)^4]}\sim e^{-x}\underbrace{e^{-\frac{x}{2}(u-1)^2}}_{e^{-s^2}}e^{\frac{x}{3}(u-1)^3-\frac{x}{4}(u-1)^4}$ by change of variables,

$s=\sqrt{\frac{x}{2}}(u-1)$, $u-1=\sqrt{\frac{2}{x}}s$. Then $I(x)\sim x^xe^{-x}\int_{-\infty}^\infty e^{-s^2}\frac{e^{\frac{2}{3}\sqrt{\frac{2}{x}}s^3-\frac{1}{4}\frac{4}{x}s^4}}{1+\sqrt{\frac{2}{x}}s}(ds\sqrt{\frac{2}{x}})$. $e^\varepsilon=1+\varepsilon+\frac{\varepsilon^2}{2}\sim1+\underbrace{\frac{2}{3}\sqrt{\frac{2}{x}}s^3-\frac{s^4}{x}}_{\varepsilon}+\frac{1}{2}(\frac{4}{9}\frac{2}{x}s^6\dots)$. Then

$(1+\sqrt{\frac{2}{x}}s)^{-1}=1-\sqrt{\frac{2}{x}}s+\frac{2}{x}s^2\dots$ So noting that $\int_{-\infty}^\infty e^{-s^2}s^{odd}ds=0$, we get $I(x)\sim x^xe^{-x}\sqrt{\frac{2}{x}}\int_{-\infty}^\infty e^{-s^2}[1+\frac{1}{x}\{2s^2-\frac{7}{3}s^4+\frac{4}{9}s^6\}]ds$. We know that $\int_{-\infty}^\infty e^{-s^2}ds=\sqrt{\pi}$.

$\int_{-\infty}^\infty e^{-s^2}s^{2n}ds=2\int_0^\infty e^{-v}\frac{v^n}{2v^{1/2}}dv=(n-\frac{1}{2})!$ ($v=s^2$, $ds=\frac{dv}{2v^{1/2}}$) Then $I(x)\sim x^xe^{-x}\sqrt{\frac{2\pi}{x}}(1+\frac{1}{x}\cdot\frac{1}{12})$. [Stirling's formula, with correction term].

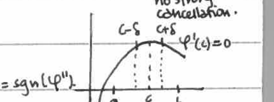
$(\frac{1}{2})!=\frac{1}{2}(-\frac{1}{2}!)=\frac{1}{2}\sqrt{\pi}$ $(\frac{3}{2})!=\frac{3}{4}\sqrt{\pi}$ $(\frac{5}{2})!=\frac{5}{2}\cdot\frac{3}{4}\sqrt{\pi}=\frac{15}{8}\sqrt{\pi}$.

This is an approximation for large x (greater than 6).

Fourier Integrals.

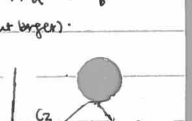
These integrals are of the type $I(x)=\int_a^b e^{ix\varphi(t)}f(t)\,dt$. We use integration by parts: $I(x)=\frac{1}{ix}\int_a^b ix\varphi'(t)e^{ix\varphi(t)}\frac{f(t)}{\varphi'(t)}dt=\left[\frac{e^{ix\varphi(t)}}{ix}\frac{f(t)}{\varphi'(t)}\right]_a^b-\overbrace{\int_a^b\frac{e^{ix\varphi(t)}}{ix}\frac{d}{dt}\left\{\frac{f(t)}{\varphi'(t)}\right\}dt}^{O(\frac{1}{x^2})\text{ by similar integration}}$

$\sim-\frac{i}{x}\left[e^{ix\varphi(b)}\frac{f(b)}{\varphi'(b)}-e^{ix\varphi(a)}\frac{f(a)}{\varphi'(a)}\right]$. Here, we need to keep both contributions. By the Riemann-Lesbegue lemma, we can predict $O(\frac{1}{x})$ decay for large x by considering the rapid oscillation of $e^{ix\varphi(t)}$ for large x. Now, if $\varphi(t)$ has a maximum (or minimum), the rapidly varying part of $e^{ix\varphi(t)}$ varies less quickly.



The method of stationary phase states that the dominant contribution to $I(x)=\int_a^b e^{ix\varphi(t)}f(t)\,dt$ arises from close to where $\varphi'(t)=0$ as $t\to\infty$.

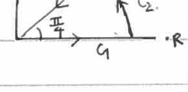
We split the integral: $I(x)=\int_a^{c-\delta}+\int_{c+\delta}^b+\int_{c-\delta}^{c+\delta}e^{ix\varphi(t)}f(t)\,dt$. $\int_a^{c-\delta}$ and $\int_{c+\delta}^b$ contributions are of $O(\frac{1}{x})$, independent of δ.



close to where $t=c$, $\varphi(t)=\varphi(c)+0+\frac{1}{2}\varphi''(c)(t-c)^2+\dots$ Then $e^{ix\varphi(t)}\sim e^{ix\varphi(c)}e^{\underbrace{i\frac{x}{2}\varphi''(c)(t-c)^2}_{iu^2}}$. $f(t)\sim f(c)$. Substitute $\varphi''=s|\varphi''|$ with $s=\mathrm{sgn}(\varphi'')$.

Also, let $u^2=\frac{x}{2}\varphi''(c)(t-c)^2$, s.t. $I(x)\sim e^{ix\varphi(c)}f(c)\int_{-\infty}^\infty e^{isu^2}\sqrt{\frac{2}{x|\varphi''(c)|}}du\sim e^{ix\varphi(c)}f(c)\sqrt{\frac{2}{x|\varphi''(c)|}}J$, with $J=\int_{-\infty}^\infty e^{isu^2}du$ (also $O(\frac{1}{x})$ decay, but larger).

The integral J is not absolutely integrable, but it does converge. Moreover, the integral with $s=-1$ is the complex conjugate of integral with $s=+1$.



$J=2\int_0^\infty e^{iu^2}du$. Define contour γ as on right: $\int_\gamma e^{iu^2}du=0$ by Taylor's theorem. On C_2, write $u=Re^{i\theta}$, $\theta\in(0,\frac{\pi}{4})$. Then $e^{iu^2}=e^{iR^2e^{2i\theta}}$

$=e^{iR^2(\cos2\theta+i\sin2\theta)}\to0$ because modulus of this is $e^{-R^2\sin2\theta}\to0$ as $R\to\infty$, since $\theta\in[0,\frac{\pi}{4}]$. Thus, $\int_{C_1}e^{iu^2}du=-\int_{C_3}e^{iu^2}du$.

So $J=2\int_0^\infty e^{iu^2}du=2\int_0^\infty e^{ie^{i\pi/2}q^2}e^{i\pi/4}dq$ by $u=(e^{i\frac{\pi}{4}})q\Rightarrow J=2e^{i\pi/4}\int_0^\infty e^{-q^2}dq=\sqrt{\pi}e^{i\pi/4}$. $I(x)\sim e^{ix\varphi(c)}f(c)\sqrt{\frac{2\pi}{x|\varphi''(c)|}}e^{i\,\mathrm{sgn}(\varphi''(c))\pi/4}$.

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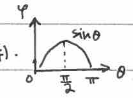

Note however that if extremum is at endpoint, we take half of the value in the estimate of our integral.

Example — 1. $I(x) = \int_0^{\pi} e^{ix\sin\theta}$, $x \to \infty$. $\varphi(\theta) = \sin\theta$, $\varphi'(\theta) = 0$ at $\theta = \frac{\pi}{2}$, $\varphi''(\frac{\pi}{2}) = -1$, $\varphi(\frac{\pi}{2}) = 1 \Rightarrow I(x) \sim e^{i(x)(1)} \cdot 1 \cdot \sqrt{\frac{2\pi}{x|-1|}}\, e^{ix\frac{\pi}{4}} = (\frac{2\pi}{x})^{\frac{1}{2}} e^{i(x-\frac{\pi}{4})}$.

This is relation to Bessel functions of the first time.

2. $I(x) = \int_{-\infty}^{\infty} \cos[xt - \frac{t^3}{3}] = \mathrm{Re}\int_{-\infty}^{\infty} e^{i(xt - t^3/3)}\,dt = \int_{-\infty}^{\infty} e^{i(xt - t^3/3)}$ [e^{t^3} not acceptable] [$\because$ imaginary part is odd over symmetric domain. Take $xt \sim t^3$, $t \sim x^{\frac{1}{2}}$, so $t = x^{\frac{1}{2}}u \Rightarrow$

$I(x) = \int_{-\infty}^{\infty} e^{i(x x^{\frac{1}{2}}u - x^{3/2}u^3/3)}\, x^{\frac{1}{2}}\,du = x^{\frac{1}{2}} \int_{-\infty}^{\infty} e^{ix^{3/2}(u - u^3/3)}\,du$. Use the method of stationary phase, replacing x by $x^{\frac{3}{2}}$. $\varphi(u) = u - \frac{u^3}{3}$, $f(u) = 1 \Rightarrow \varphi'(u) = 0 = 1 - u^2$,

so $u = \pm 1$. Both of them contribute to the integral (large x expansion). $\varphi(1) = \frac{2}{3}$, $\varphi(-1) = -\frac{2}{3}$. $\varphi''(u) = -2u$, so $\varphi''(1) = -2$, $\varphi''(-1) = 2$. So, we will get a sum:

$I(x) \sim x^{\frac{1}{2}}\left[e^{ix^{\frac{3}{2}}(-\frac{2}{3})} \cdot 1 \cdot \sqrt{\frac{2\pi}{x^{3/2}|2|}}\, e^{i\,\mathrm{sgn}(2)\pi/4} + e^{ix^{\frac{3}{2}}(\frac{2}{3})} \cdot 1 \cdot \sqrt{\frac{2\pi}{x^{3/2}|-2|}}\, e^{i\,\mathrm{sgn}(-2)\pi/4} \right]$ [Note the use of $x^{3/2}$ rather than x! this comes from our earlier substitution]

$= \frac{\sqrt{\pi}}{x^{1/4}} \cdot 2\cos\left[\frac{2}{3}x^{\frac{3}{2}} - \frac{\pi}{4}\right]$ //

We now return to an earlier topic — solving equations using multiple scales and Linstedt's method: Here is an example — solve $\ddot{y} + \varepsilon\dot{y} + y + \varepsilon^2 y^3 = 0$, $y(0) = 1$, $\dot{y}(0) = 0$.

Try a solution $y(t) = y_0(\tau, T) + \varepsilon y_1(\tau, T) + \varepsilon^2 y_2(\tau, T)$, $T = \varepsilon t$, $\tau = nt$. $n = n_0 + \varepsilon n_1 + \varepsilon^2 n_2 + \cdots = 1 + \varepsilon \cdot 0 + \varepsilon^2 n_2$ (possibly from initial condition info). $\frac{d}{dt} = n\frac{\partial}{\partial\tau} + \frac{dT}{dt}\frac{\partial}{\partial T}$

$\Rightarrow \frac{d}{dt} = (1 + \varepsilon^2 n_2 + \cdots)\frac{\partial}{\partial\tau} + \varepsilon\frac{\partial}{\partial T}$, $\frac{d^2}{dt^2} = n^2\frac{\partial^2}{\partial\tau^2} + 2n\varepsilon\frac{\partial^2}{\partial\tau\partial T} + \varepsilon^2\frac{\partial^2}{\partial T^2} = (1 + 2\varepsilon^2 n_2 + \cdots)\frac{\partial^2}{\partial\tau^2} + 2\varepsilon(1 + \cdots)\frac{\partial^2}{\partial\tau\partial T} + \varepsilon^2\frac{\partial^2}{\partial T^2}$. So substitution into our equation gives us

$(1 + 2\varepsilon^2 n_2 \cdots)(y_{0\tau\tau} + \varepsilon y_{1\tau\tau} + \varepsilon^2 y_{2\tau\tau} + \cdots) + 2\varepsilon(1)\frac{\partial}{\partial T}(y_{0\tau} + \varepsilon y_{1\tau} + \cdots) + \varepsilon^2(y_{0TT} + \cdots) + \varepsilon(1 + \cdots)(y_{0\tau} + \varepsilon y_{1\tau}) + \varepsilon^2(y_{0T} + \cdots) + (y_0 + \varepsilon y_1 + \varepsilon^2 y_2 \cdots) + \varepsilon^2(y_0^3 + \cdots) = 0$

Initial conditions become $y(0) = 1 \Rightarrow t = 0 \Rightarrow \tau = 0, T = 0 \Rightarrow 1 = y_0(0,0) + \varepsilon y_1(0,0) + \varepsilon^2 y_2(0,0) \Rightarrow y_0(0,0) = 1$, $y_1(0,0) = 0$, $y_2(0,0) = 0$. $\dot{y}(0) = 0 \Rightarrow$ by substitution,

$0 = (1 + \varepsilon^2 n_2)(y_{0\tau}(0,0) + \varepsilon y_{1\tau}(0,0) + \varepsilon^2 y_{2\tau}(0,0) \ldots) + \varepsilon(y_{0\tau}(0,0) + \varepsilon y_{1T}(0,0))$. So $y_{0\tau}(0,0) = 0$, $y_{1\tau}(0,0) + y_{0T}(0,0) = 0$. Overall, we have $O(1)$: $y_{0\tau\tau} + y_0 = 0$.

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$O(\varepsilon)$: $y_{1\tau\tau} + y_1 = -2\frac{\partial}{\partial T}y_{0\tau} - y_{0\tau}$. $O(\varepsilon^2)$: $y_{2\tau\tau} + y_2 = -2n_2 y_{0\tau\tau} - 2\frac{\partial}{\partial T}y_{1\tau} - y_{0TT} - y_{1\tau} - y_{0T} - y_0^3$. Then we solve our equations: $y_0 = A_0(T)\cos\tau + B_0(T)\sin\tau$.

From boundary conditions, $y_0(0,0) = 1$, $y_{0\tau}(0,0) = 0 \Rightarrow A_0(0) = 1$, $B_0(0) = 0$. Then $y_{1\tau\tau} + y_1 = -2\frac{\partial}{\partial T}(-A_0\sin\tau + B_0\cos\tau) - A_0(-\sin\tau) - B_0\cos\tau$. We have seen that y_1 is not

periodic unless the terms proportional to $\cos\tau$ and $\sin\tau$ (the CF of $y_{1\tau\tau} + y_1 = 0$) on the RHS are 0. Then $2A_{0T} + A_0 = 0$ with $A_0(0) = 1$, $A_0 = e^{-\frac{1}{2}T}$; $2B_{0T} + B_0 = 0$ with $B_0(0) = 0$. $B_0 = 0$.

Thus far, $y = e^{-\frac{1}{2}T}\cos\tau + \varepsilon y_1 + \cdots$ since $y_{1\tau\tau} + y_1 = 0$, $y_1 = A_1(T)\cos\tau + B_1(T)\sin\tau$. Apply BC: $y_1(0,0) = 0 \Rightarrow A_1(0) = 0$. $y_{1\tau}(0,0) = -y_{0T}(0,0) = \frac{1}{2} \Rightarrow B_1(0) = \frac{1}{2}$.

$(a = n_2)$

$y_{2\tau\tau} + y_2 = -2a[e^{-\frac{1}{2}T}(-\cos\tau)] - 2\frac{\partial}{\partial T}(-A_1\sin\tau + B_1\cos\tau) - \frac{1}{4}e^{-\frac{1}{2}T}\cos\tau - A_1(-\sin\tau) - B_1\cos\tau + \frac{1}{2}e^{-\frac{1}{2}T}\cos\tau - e^{-\frac{3}{2}T}(\frac{3}{4}\cos\tau + \frac{1}{4}\cos 3\tau)$. If we want the

coefficients of $\cos\tau$, $\sin\tau$ to be 0, $\frac{\partial B_1}{\partial T} + \frac{1}{2}B_1 = -\frac{3}{8}e^{-\frac{3T}{2}} + (a + \frac{1}{8})e^{-\frac{T}{2}}$. $\frac{\partial A_1}{\partial T} + \frac{1}{2}A_1 = 0$ and $A_1(0) = 0 \Rightarrow A_1 \equiv 0$. For the B_1 equation, we have integrating

factor $e^{\frac{1}{2}T}$, then multiply through to get $\frac{\partial}{\partial T}[e^{\frac{T}{2}}B_1] = -\frac{3}{8}e^{-T} + (a + \frac{1}{8}) \Rightarrow e^{\frac{T}{2}}B_1 = \frac{3}{8}e^{-T} + (a + \frac{1}{8})T + B_{1,0}$ which is not a small correction because of T!

$B_1 = \frac{3}{8}e^{-3T/2} + (a + \frac{1}{8})Te^{-\frac{T}{2}} + B_{1,0}e^{-T/2}$. Then solution so far is $y = e^{-T/2}\cos\tau + \varepsilon\sin\tau\,(\frac{3}{8}e^{-3T/2} + (a + \frac{1}{8})\underbrace{Te^{-T/2}}_{=!} + B_{1,0}e^{-T/2}) + \cdots$ so we lose the

asymptotic nature unless $a - \frac{1}{8} = 0 \Rightarrow a = \frac{1}{8}$, to maintain that $\varepsilon y_1 \ll y_0$ i.e. $\varepsilon y_1 = o(y_0)$. Then $B_1 = \frac{3}{8}e^{-3T/2} + B_{1,0}e^{-T/2}$. $B_1(0) = \frac{1}{2} \Rightarrow B_{1,0} = \frac{1}{8}$, and

$y_1 = (\frac{3}{8}e^{-3T/2} + \frac{1}{8}e^{-T/2})(\sin\tau) \Rightarrow y \sim e^{-T/2}\cos\tau + \varepsilon(\frac{3}{8}e^{-3T/2} + \frac{1}{8}e^{-T/2})\sin\tau = e^{-\varepsilon t/2}\cos[t(1 - \frac{1}{8}\varepsilon^2)] + \frac{\varepsilon}{8}[3e^{-3\varepsilon t/2} + e^{-\varepsilon t/2}]\sin[t(1 - \frac{1}{8}\varepsilon^2)]$ //

END OF SYLLABUS

END OF COURSE.