3402 Waves and Wave Scattering Notes

Based on the 2015 spring lectures by Prof V Smyshlyaev

3402 (Waves and Wave Scattering)

Year:	2014–2015
Code:	MATH3402
Level:	Advanced
Value:	Half unit (= 7.5 ECTS credits)
Term:	2
Structure:	3 hour lectures per week
Assessment:	100% examination
Normal Pre-requisites:	MATH7402
Lecturer:	Prof V Smyshlyaev

Course Description and Objectives

Modelling the propagation and scattering of acoustic and electromagnetic waves has proved a major challenge to mathematicians and physicists for many centuries, and its practical importance can be observed in many applications prevalent throughout our modern world. These include the mitigation of aircraft, rail and traffic noise in urban areas, sonar detection, wireless and fibre optic communication, baggage screening, medical diagnostics and the workings of the cochlea. This course aims to provide an introduction to linear and nonlinear wave theory and the approximate methods used to tackle wave transmission and scattering in inhomogeneous media.

Recommended Texts

(i) Pierce, A.D, *Acoustics: an introduction to its physical principles and applications,* Acoustic Society of America 1989.

(ii) Billingham, J. and King, A.C. *Wave Motion,* CUP 2001.

Detailed Syllabus

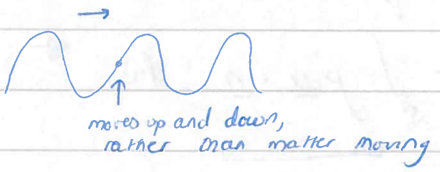
- Acoustic waves - governing equations, plane acoustic waves, spherically symmetric waves, causality and the Sommerfield radiation condition, acoustic energy and intensity.
- Electromagnetic (EM) waves - governing equations, plane EM waves, Poynting's vector.
- Impedance and surface boundary conditions, interfacial boundary conditions.
- Plane wave reflection and transmission at interfaces - reflection by acoustically soft and hard boundaries and by a perfect conductor, reflection and transmission between two insulators.
- Radiation from vibrating bodies - a radially pulsating sphere, a traversely oscillating sphere.
- Green's functions, monopoles, dipoles, quadrupoles, multipole expansions.
- Kirchoff-Helmholtz integral theorem, acoustic scattering by air bubbles in water, acoustic scattering by a fixed rigid sphere.
- Introduction to the WKB approximation, slowly varying waveguides, optic fibres.

MATH3402 - Waves 12/01/14

100% Exam, Office 1-2pm Monday

0. Introduction:

The course is about maths of waves (acoustic/sound, electromagnetic), and their scattering by various "obstacles"



- water waves
- sound (acoustic) - disturbing air & sending disturbances
- radio / mobiles / light (electromagnetic waves
- elastic, etc.

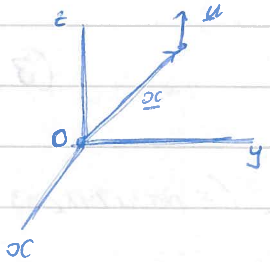
Common feature: waves (more) propagate (hence transfer energy/information; they scatter when hit obstacles; disturbances, not matter, move)
These are described by partial differential equations (PDE)

1. Governing equations

1.1 Acoustic (sound) waves

Acoustic (sound) waves are small amplitude mechanical disturbances propagating in a fluid, typically a compressible gas (e.g. air) or liquid (e.g. water)

The governing equations are derived from basic conservation laws of 'inviscid fluid':



$\underline{x} = (x, y, z)$, t time

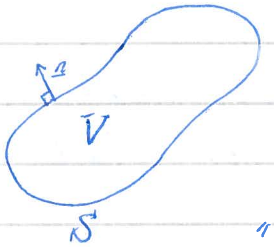
$\rho(\underline{x}, t)$ fluid density

$u(\underline{x}, t)$ fluid velocity

$p(\underline{x}, t)$ fluid pressure

Conservation of mass (of MATH2301) - won't heavily rely on this i.e. will derive

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \underline{u}) = 0 \qquad (1)$$



$$\int_V (1)\, dV$$

$$\underset{\overset{\shortparallel}{\text{rate of change}}}{\frac{d}{dt}} \underbrace{\left(\int_V \rho\, dV\right)}_{\overset{\shortparallel}{M(t),\ \text{mass}}} = -\int_V \nabla\cdot(\rho \underline{u}) =$$

$$= -\int_{S} \underbrace{\rho \underline{u}\cdot\underline{n}}_{\text{mass outflow rate}}\, dS \quad \leftarrow \text{Divergence theorem}$$

Newton's 2nd law holds for continuous mechanics

Conservation of momentum (≡ Newton's 2nd Law for continuum media; neglecting gravity, viscosity etc)

$$\text{nonlinear} \rightarrow \underbrace{\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla \underline{u}} = -\frac{1}{\rho}\nabla p \qquad (2)$$

$$\text{acceleration} \rightarrow \underline{a} = \frac{d\underline{u}}{dt} = \frac{d}{dt}\underline{u}(\underline{x}(t),t) = \sum_j \underset{\overset{\shortparallel}{\nabla u}}{\frac{\partial u}{\partial x_j}} \underset{\overset{\shortparallel}{u_j}}{\frac{\partial x_j}{\partial t}} + \frac{\partial u}{\partial t}$$

acc. of a material point

$$= \frac{\partial u}{\partial t} + u\cdot\nabla u \quad \text{etc.}$$

Additionally, physically, an equation of state is held connecting one unknown to another e.g. pressure p to density ρ

Under the "adiabatic" assumption (no heat exchange) let

$$p = P(\rho) \qquad (3)$$

We will be considering small disturbances (= perturbations) of a uniform state at rest:

$$p = p_0 + \tilde{p}, \qquad \rho = \rho_0 + \tilde{\rho}, \qquad \underline{u} = \overset{0}{\overset{\shortparallel}{\cancel{\underline{u}_0}}} + \tilde{\underline{u}}$$

p_0, ρ_0 constant ; $\tilde{p}, \tilde{\rho}, \tilde{\underline{u}}$ "small"

Plugging in to (1) - (3)

neglect to good approximation

$$(1) \implies \frac{\partial}{\partial t}(\cancel{\rho_0} + \tilde{\rho}) + \nabla \cdot ((\rho_0 + \tilde{\rho})\tilde{\underline{u}}) = 0 \qquad (1')$$

diff. a constant

$$(2) \implies \frac{\partial \tilde{\underline{u}}}{\partial t} + \cancel{\tilde{\underline{u}} \cdot \nabla \tilde{\underline{u}}} = -\frac{1}{\rho_0 + \tilde{\rho}} \nabla(\cancel{p_0} + \tilde{p}) \qquad (2')$$

small so neglect ← small so neglect ← diff. a constant

$$(3) \implies p_0 + \tilde{p} = \underline{P}(\rho_0 + \tilde{\rho}) \qquad (3')$$

Since $\tilde{p}, \tilde{\rho}, \tilde{\underline{u}}$ small, neglecting in (1'), (2') the higher order smallness terms, like, $\nabla \cdot (\tilde{\rho}\tilde{\underline{u}})$

$$(1') \implies \frac{\partial \tilde{\rho}}{\partial t} + \rho_0 \nabla \cdot \tilde{\underline{u}} = 0 \qquad (4)$$

$$(2') \implies \frac{\partial \tilde{\underline{u}}}{\partial t} = -\frac{1}{\rho_0} \nabla \tilde{p} \qquad (5)$$

Use Taylor expansion for (3) (about ρ_0):

$$\underline{P}(\rho_0 + \tilde{\rho}) = \underline{P}(\rho_0) + \frac{d\underline{P}}{d\rho}(\rho_0)\tilde{\rho} + O(\tilde{\rho}^2)$$

← eventually neglect

$\implies$ To leading orders of smallness,

$$\cancel{p_0} + \tilde{p} = \underbrace{\underline{P}(\cancel{\rho_0})}_{= p_0} + \underbrace{\frac{d\underline{P}}{d\rho}(\rho_0)}_{= \underline{P}'}\tilde{\rho}$$

$\implies \tilde{p} = \underline{P}'\tilde{\rho}$, where $\underline{P}' = \frac{d\underline{P}}{d\rho}(\rho_0)$ is a constant

intuition ↓

characterising the medium's physical properties; physically $\underline{P}' > 0$

$\iff \underline{P}' =: c^2$, where $c := (\underline{P}')^{\frac{1}{2}} > 0$ is the <u>wave speed</u> (will soon see describes the speed of waves)

$$\tilde{\rho} = \frac{1}{c^2}\tilde{p} \qquad (6)$$

So eliminating $\tilde{\rho}$ in (4) - (5) via (6):

$$\boxed{\begin{aligned} &\frac{1}{c^2}\frac{\partial \tilde{p}}{\partial t} + \rho_0 \nabla\cdot \underset{\sim}{\tilde{u}} = 0 && (7) \\ &\frac{\partial u}{\partial t} = -\frac{1}{\rho_0}\nabla \tilde{p} && (8) \end{aligned}}$$

could be some other nonlinear equations

(7) - (8) main equations of ("linear") acoustics
(A (vector) PDE system)

To start analyzing (7) - (8);
Dropping 'wiggles' ("~") henceforth, differentiate (7) in t and use (8):

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} + \rho_0 \nabla\cdot\frac{\partial u}{\partial t} = 0 \implies$$

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} + \cancel{\rho_0}\nabla\cdot\left(-\frac{1}{\cancel{\rho_0}}\nabla p\right) = 0$$

$$\boxed{\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = 0} \quad \text{WAVE EQN} \qquad (9)$$

scalar

$$\text{where } \nabla^2 p := \nabla\cdot(\nabla p) = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2}$$
$$=: \Delta p$$

(= Laplacian of p)

(9) is the <u>wave equation</u> (a scalar PDE).

<u>Plane waves</u>: For solutions of (9)
$p(t, x, \cancel{y}, \cancel{z})$, let p does <u>not</u> depend on y & z

in reality depends on only 1 of the variables

$p = p(t, x)$; so (9) $\implies$

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 p}{\partial x^2} = 0 \qquad (9')$$

$$\implies p(x, t) = f(x - ct) + g(x + ct) \quad \text{solves } (9')$$
$$\forall\, f, g \in C^2$$

functions diff. twice continuously

$$\frac{1}{\cancel{c^2}}\left[\cancel{(-c)^2} f'' + \cancel{c^2} g''\right] - (f'' + g'') = 0$$

◻

Decided $p = p(t, x)$, but could choose y, z

More generally, for any direction $\underline{n}$, $|\underline{n}| = 1$,

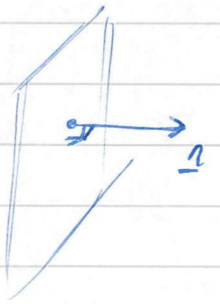
$$p = f(\underline{x} \cdot \underline{n} - ct) + g(\underline{x} \cdot \underline{n} + ct) \qquad (10)$$

Projection i.e. works for any direction

solves (9) $\forall f, g \in C^2$

In (10) $f(\underline{x} \cdot \underline{n} - ct)$ describes a 'plane wave' moving in positive $\underline{n}$ direction

$f(\underline{x} \cdot \underline{n} - ct)$ is constant $\forall$ plane $\perp$ to $\underline{n}$, which plane moves with speed c



2D cross section

Similarly $g(\underline{x} \cdot \underline{n} + ct)$ describes plane wave in $-$ve $\underline{n}$ direction

Thus c is the "phase velocity"

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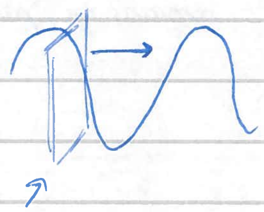
$$\begin{cases} \dfrac{1}{c^2}\dfrac{\partial p}{\partial t} + \rho_0 \nabla\cdot\underline{u} = 0 & (7) \\ \dfrac{\partial \underline{u}}{\partial t} = -\dfrac{1}{\rho_0}\nabla p & (8) \end{cases}$$

$$\Longleftrightarrow \quad \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = 0 \qquad (9)$$

a priori we don't know this but it is true



$\longrightarrow \underline{n}$ $\quad f(\underline{x}\cdot\underline{n}-ct)$

the plane moves with speed c.

Remark: If p solves (9), $\underline{u}$ has to be recovered from (7)-(8), this depends on chosen initial/boundary conditions e.g. if $\forall\, t < t_0$, $\underline{u} \equiv 0$, $p \equiv 0$ $\Longrightarrow$ from (8)

$$\underline{u}(\underline{x},t) = -\frac{1}{\rho_0}\int_{t_0}^{t} \nabla^{\cancel{2}} p(\underline{x},t')\,dt' ;$$

Then (7) is also satisfied (check!) ▨

To eliminate p to end up with equation for $\underline{u}$ is subtle since p scalar, $\underline{u}$ vector

Let at $t=0$, $\overset{\text{irrotational}}{\nabla\times\underline{u} = 0}$ (i.e. fluid is initially <u>irrotational</u>) $\Longrightarrow$ taking curl of (8):

$$\nabla\times\left(\frac{\partial \underline{u}}{\partial t}\right) = \frac{\partial}{\partial t}(\nabla\times\underline{u}) = -\frac{1}{\rho_0}\nabla\times(\nabla p) = \underline{0}$$

can interchange

when curl meets gradient they kill each other

$\Longrightarrow \nabla \times \underline{u} = \underline{c} \longrightarrow \underline{c} = 0 \quad (t=0)$

$\Longrightarrow$ fluid remains irrotational $\forall\, t$

$\Longleftrightarrow \nabla \times \underline{u} = \underline{0}$

$\Longrightarrow \exists\ \phi(\underline{x}, t)$ a (scalar) velocity potential s.t. $\underline{u} = \nabla\phi$

value of integral does not depend on path

curl $= 0$ so circulation is 0

done w.r.t x but t remains a parameter

(ϕ determined up to a constant, $C(t)$)

Then (8) $\Longrightarrow \frac{\partial}{\partial t}\nabla\phi = -\frac{1}{\rho_0}\nabla p$

$$\Longleftrightarrow \nabla\left(\frac{\partial\phi}{\partial t} + \frac{1}{\rho_0}p\right) = 0$$

constant, may depend on t

$$\Longrightarrow \frac{\partial\phi}{\partial t} + \frac{1}{\rho_0}p = A(t)$$

without loss of generality (wlog) $A(t) \equiv 0$

We can do this since φ determined up to a constant

Re-defining $\phi \longrightarrow \phi - \int_0^t A(t')\,dt' = C(t)$

$$\Longrightarrow \frac{\partial\phi}{\partial t} = -\frac{1}{\rho_0}p \qquad (11)$$

Differentiating (11) in t, via (7)

$$\frac{\partial^2\phi}{\partial t^2} = -\frac{1}{\rho_0}\frac{\partial p}{\partial t} = -\frac{1}{\rho_0}\Big(-\rho_0 c^2 \underbrace{\nabla\cdot\overbrace{(\nabla\phi)}^{=u}}_{=\nabla^2\varphi}\Big)$$

$$\Longrightarrow \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi = 0 \qquad (12)$$

i.e. ϕ solves the same wave equation as p, cf (9)

⟨Check 'reverse' argument: ϕ solves (12) $\Rightarrow \underline{u} = \nabla\phi$, and p via (11), $p = -\rho_0\frac{\partial\phi}{\partial t}$ solve (7) and (8)⟩

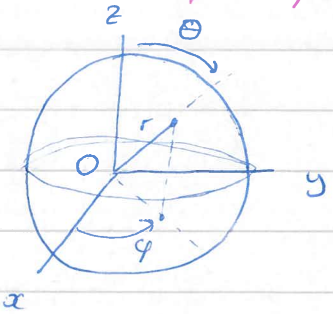
Sphencally symmetric waves

Seek solutions of wave equation (9) / (12) in spherical

$\underline{x} = (r, \theta, \varphi)$

different from vel. potential ϕ

Remember that Laplace is always combination of 2nd derivatives



$$\nabla^2 p = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial p}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial p}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 p}{\partial\varphi^2}$$

Seek sphencally symmetric solutions of ~~(12)~~ (9)

i.e. $p = p(t, r, \cancel{\theta}, \cancel{\varphi}) = p(t, r)$

assume no dependence on spherical coordinates

$$\Rightarrow \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial p}{\partial r}\right) = 0$$

Try $p = \frac{q}{r} \Rightarrow \frac{\partial p}{\partial r} = \frac{1}{r}\frac{\partial q}{\partial r} - \frac{q}{r^2}$

$$\Rightarrow r^2\frac{\partial p}{\partial r} = r\frac{\partial q}{\partial r} - q$$

$$\Rightarrow \frac{\partial}{\partial r}\left(r^2\frac{\partial p}{\partial r}\right) = \frac{\partial q}{\partial r} + r\frac{\partial^2 q}{\partial r^2} - \frac{\partial q}{\partial r} = r\frac{\partial^2 q}{\partial r^2}$$

$$\frac{1}{c^2}\frac{1}{r}\frac{\partial^2 q}{\partial t^2} - \frac{1}{r}\frac{\partial^2 q}{\partial r^2} = 0$$

$\Rightarrow$ q solves 1-dimensional wave equation

$$\frac{1}{c^2}\frac{\partial^2 q}{\partial t^2} - \frac{\partial^2 q}{\partial r^2} = 0$$

$$\Rightarrow q = f(r-ct) + g(r+ct)$$

$$\Rightarrow p = \frac{f(r-ct)}{r} + \frac{g(r+ct)}{r} \qquad (13)$$

Which is a general form for a radially symmetric solution of (9) or (12)

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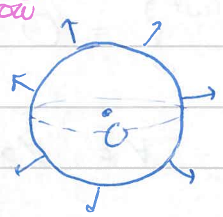
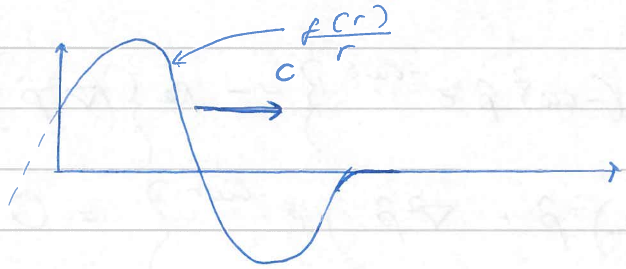
$$p(r,t) = \frac{f(r-ct)}{r} + \frac{g(r+ct)}{r} \qquad (13)$$

$$\forall\ f, g \in C^2$$

In (13), f-part describes a spherical wave travelling away from centre/origin $r=0$, with speed c;

see waves moving away when throw stone →





g-part travels towards O
(f "outgoing", g "incoming")

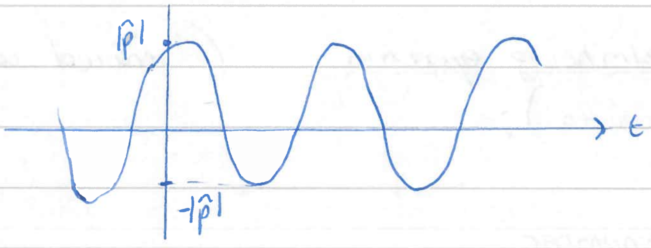
Time-harmonic (T-H) waves

time dep. sinusoidal

Seek solutions of (9) with following time dependence:

$$p(\underline{x}, t) = |\hat{p}(\underline{x})| \cos(\omega t - \psi(\underline{x}))$$

modulus since allowed to be complex



i.e. "pure tone" solution with angular frequency $\omega \Rightarrow$
$T = \frac{2\pi}{\omega}$ time period

Phase $\psi(\underline{x})$, amplitude $|\hat{p}(\underline{x})| \Longleftrightarrow$

$$p(\underline{x}, t) = \mathrm{Re}\left\{ \underbrace{|\hat{p}(\underline{x})| e^{i\psi(\underline{x})}}_{=:\hat{p}(\underline{x})} e^{-i\omega t} \right\}$$

$$p(\underline{x}, t) = \mathrm{Re}\left\{ \hat{p}(\underline{x})\, e^{-i\omega t} \right\} \qquad (14)$$

where $\hat{p}(x) := |\hat{p}(x)| e^{+i\varphi(x)}$ is called "complex amplitude"
i.e. incorporating both the amplitude and the phase.

Plug (14) in to (9)

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = 0 \quad \Longrightarrow$$

can take outside

$$\frac{1}{c^2} Re\left\{(-i\omega)^2 \hat{p}\, e^{-i\omega t}\right\} - Re\left\{\left(\nabla^2 \hat{p}\right) e^{-i\omega t}\right\} = 0$$

$$- Re\left\{\underbrace{\left(\left(\frac{\omega}{c}\right)^2 \hat{p} + \nabla^2 \hat{p}\right)}_{=:\, S} e^{-i\omega t}\right\} = 0 \qquad \forall t \quad \forall x$$

$$(Re\, S')\cos\omega t + (Im\, S')\sin\omega t = 0 \qquad \forall t$$

$\Longrightarrow$ (varying t) $\quad Re\, S = 0, \; Im\, S = 0 \Longrightarrow$

$$S = 0 \iff \boxed{\nabla^2 \hat{p} + k^2 \hat{p} = 0} \qquad k := \frac{\omega}{c} \qquad (15)$$

(15) is called Helmholtz equation ("reduced wave equation" for T-H waves);

$k := \frac{\omega}{c}$ is wavenumber

Time dependence must solve this equation

For velocity $u(x,t)$ ~~for~~ in T-H case also seek in form

$$u(x,t) = Re\left\{\hat{u}(x)\, e^{-i\omega t}\right\} \longrightarrow$$

via (8) $\longleftrightarrow \dfrac{\partial u}{\partial t} = -\dfrac{1}{\rho_0}\nabla p$

p has same form as before

$$\mathrm{Re}\left\{\underbrace{\left(-i\omega\hat{u} + \frac{1}{\rho_0}\nabla\hat{p}\right)}_{=:S} e^{-i\omega t}\right\} = 0 \qquad \forall t$$

Repeating same argument $\Longrightarrow$ $S=0$ $\Longrightarrow$

$$\hat{u} = -\frac{i}{\rho_0\omega}\nabla\hat{p} \qquad (16)$$

Plane T-H waves:

For a plane wave travelling in a positive $\underline{n}$ direction, which is also T-H:

$$p(\underline{x},t) \overset{(10)\,\text{plane}}{=} f(\underline{x}\cdot\underline{n} - ct) \overset{14\ \text{T-H}}{=} \mathrm{Re}\left\{\hat{p}(\underline{x})e^{-i\omega t}\right\}$$

$$= \mathrm{Re}\left\{ e^{i\overset{''k}{\frac{\omega}{c}}(-ct + \underline{x}\cdot\underline{n})} A \right\}$$

$A \in \mathbb{C}$ (a complex constant)

$$p(\underline{x},t) = \mathrm{Re}\left\{ \underbrace{Ae^{ik\underline{x}\cdot\underline{n}}}_{=\hat{p}(x)} e^{-i\omega t} \right\}$$

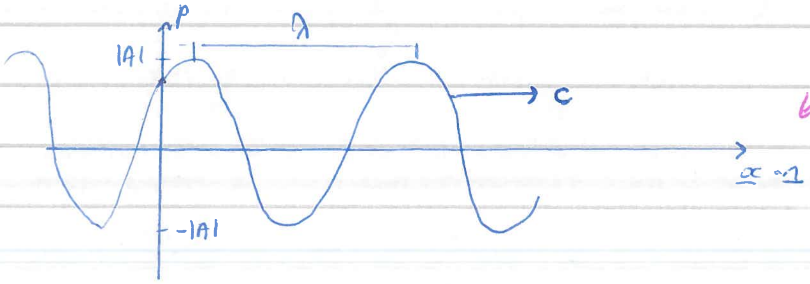
$$\Longleftrightarrow \hat{p}(x) = Ae^{ik\underline{x}\cdot\underline{n}} = Ae^{i\underline{k}\cdot\underline{x}} \qquad (17)$$

where $\underline{k} := k\underline{n}$, $|\underline{n}|=1$

is called wavevector

So (17) $\Longleftrightarrow$ $(A = |A|e^{-i\psi})$ could have + or −

$$p(\underline{x},t) = |A|\cos(k\underline{x}\cdot\underline{n} - \psi - \omega t)$$



Before drew function of t, now function of x

$\lambda = \frac{2\pi}{k}$ is the wavelength (= spatial period)

$\left(T = \frac{2\pi}{\omega}, \quad f = \frac{1}{T} = \frac{\omega}{2\pi} \right.$ frequency, $\left. \underset{\uparrow}{\underline{A}} = |A| \, e^{-i\phi} \right)$

complex amplitude

conventional amplitude

Also plugging (17) in to (16)

$$\hat{\underline{u}} = -\frac{i}{\rho_0 \omega} \nabla \hat{p} = -\frac{i}{\rho_0 \omega} \nabla \left(A e^{i \underline{k} \cdot \underline{x}} \right)$$

$$= \frac{-i}{\rho_0 \omega} A i \underline{k} \, e^{i \underline{k} \cdot \underline{x}} = \frac{A}{\rho_0 \omega} k \underline{n} \, e^{i \underline{k} \cdot \underline{x}}$$

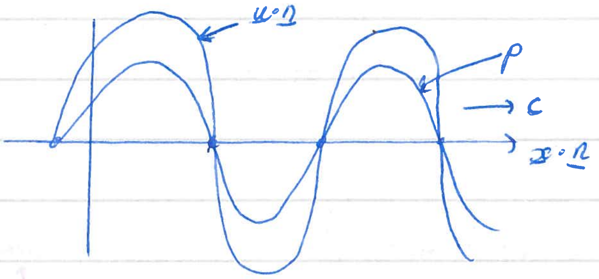
$= \frac{1}{c}$

$$= \frac{A}{\rho_0 c} \underline{n} e^{i \underline{k} \cdot \underline{x}} \implies$$

$$\underline{u} = \mathrm{Re} \left\{ \frac{A \underline{n}}{\rho_0 c} e^{i \underline{k} \cdot \underline{x} - i \omega t} \right\}$$

$$= \frac{p \underline{n}}{\rho_0 c} \qquad (18)$$

i.e. for plane T-H waves, velocity $\underline{u}$ and pressure p are "in phase", and $\underline{u}$ is in the $\underline{n}$-direction (i.e. sound wave "longitudinal")



Remark : Every acoustic field, described by wave equation (9) can be expressed as a 'superposition' of T-H waves, via Fourier Transform (FT) in time:

Assuming sufficient decay of $p(\underline{x},t)$ for $t \to \pm\infty$, seek $p(\underline{x},t)$, a solution to (9), as a FT of $p^*(\underline{x},\omega)$:

$$p(\underline{x},t) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} p^*(\underline{x},\omega)\, e^{-i\omega t}\, d\omega \qquad (19)$$

$\Longrightarrow$ the inverse FT gives:

$$p^*(\underline{x},\omega) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} p(\underline{x},t)\, e^{i\omega t}\, dt$$

Then (9) $\Longrightarrow$

$$0 = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = (2\pi)^{-1/2}\int_{-\infty}^{\infty}\left[\frac{(-i\omega)^2}{c^2}\,p^* - \nabla^2 p^*\right]e^{-i\omega t}\, d\omega$$

$$= -(2\pi)^{-1/2}\int_{-\infty}^{\infty}\left(\nabla^2 p^* + \underset{k=\frac{\omega}{c}}{k^2} p^*\right)e^{-i\omega t}\, d\omega = 0 \qquad \underline{\underline{\forall\, t}}$$

$\Longrightarrow$ (by FT inversion of zero)

$$\nabla^2 p^* + k^2 p^* = 0$$

i.e. p^*, the inverse FT of p, solves Helmholtz equation (15). So (19), for real-valued $p(\underline{x},t)$ gives

$$p(x,t) = \mathrm{Re}\, p(\underline{x},t) = \int_{-\infty}^{\infty} \mathrm{Re}\left\{\underbrace{(2\pi)^{-1/2}\, p^*(\underline{x},\omega)\, e^{-i\omega t}}_{=:\hat{p}}\right\} d\omega$$

i.e. p is a superposition (= integral in ω) of T-H waves

$$\mathrm{Re}\left\{(2\pi)^{-1/2}\, p^*(\underline{x},\omega)\, e^{-i\omega t}\right\} \qquad ▨$$

Causality and Sommerfeld Radiation Condition

Causality: the 'effect' / consequency can only follow the 'cause' / 'source' which can only effect future but not past.

How the wave equation (9) treats future ($t\uparrow$) and past ($t\downarrow$) equally:

$$(9): \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = 0$$

Let $t \to -t$ $\quad (t' := -t) \Longrightarrow$

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial (t')^2} - \nabla^2 p = 0$$

not discriminating between past + future

so an additional mathematical condition needs to be introduced, called causality condition.

It means the solution to (9) is required to be identically zero for $t \leq t_0$ i.e. before some time t_0 when the "source" switches on

Consider spherically symm. waves (13)

$$p(\underline{x}, t) = \frac{f(r-ct)}{r} + \frac{g(r+ct)}{r} \qquad f, g \in C^2$$

twice differentiable

We require: $\exists\ t_0$ s.t. outside a "source" region $r \leq r_0$

i.e. $\forall\ r > r_0,\ \forall\ t \leq t_0$ it holds $p(r, t) \equiv 0$

$\Longrightarrow$ (Exercise: Exam 2013 Q 1 (d))

$g(\xi)$ is a constant so could be added to f $\Longrightarrow$

$$p(\underline{x}, t) = \frac{f(r-ct)}{r}$$

is a general "causal" solution of (9) i.e. only outgoing wave ~~is~~ remains (and no incoming wave)

Consider now radially symmetric T-H waves

this is f(r−ct)
radially symmetric

$$(13) \Longrightarrow \quad p(x,t) \overset{TH}{=} \mathrm{Re}\left\{ A \frac{e^{\frac{i\omega}{c}\overbrace{(-ct+r)}}}{r} + B\frac{e^{\frac{-i\omega}{c}(ct+r)}}{r} \right\}$$

This is our guess

$$= \mathrm{Re}\left\{ \underbrace{\left(A\frac{e^{ikr}}{r} + B\frac{e^{-ikr}}{r}\right)}_{=\hat{p}} e^{-i\omega t} \right\} \qquad A, B \in \mathbb{C}$$

$$\Longleftrightarrow \quad \hat{p} = A\frac{e^{ikr}}{r} + B\frac{e^{-ikr}}{r} \qquad (13')$$

We expect physically (13') <u>not</u> to contain an incoming part $\Longleftrightarrow$ $B = 0$. This is achieved by the Sommerfeld radiation condition:

↑ means same as outgoing

$$r\frac{\partial \hat{p}}{\partial r} - ikr\hat{p} \longrightarrow 0 \quad \text{as} \quad r \to \infty \qquad (20)$$

uniformly with respect to a direction (holds not only for spher. symm solutions but ∀ with 'bounded' sources)

Check: $(13') \Longrightarrow$

$$r\frac{\partial \hat{p}}{\partial r} = Aike^{ikr} - \frac{A}{r}e^{ikr} - Bike^{-ikr} - \frac{B}{r}e^{-ikr}$$

$$-ikr\hat{p} = -ikAe^{ikr} - ikBe^{-ikr}$$

$$r\frac{\partial \hat{p}}{\partial r} - ikr\hat{p} = \underbrace{-2ikBe^{-ikr}}_{\not\to 0 \text{ as } r\to\infty \text{ unless } B=0} - \underbrace{\frac{Ae^{ikr} + Be^{-ikr}}{r}}_{(|e^{\pm ikr}|=1) \ \to 0 \text{ as } r\to\infty}$$

$\Longrightarrow$ $B = 0$ as desired ☒

Remark: Sommerfeld radiation condition (20) ensures, physically, the T-H waves are outgoing; mathematically, it ensures uniqueness of solutions of boundary-value problems for (15)

[Often require additionally

$$p(\underline{x}, t) = O\left(\frac{1}{r}\right) \quad \text{as} \quad r := |\underline{x}| \to \infty \qquad (20')$$

must decay not slower than $\frac{1}{r}$]

One can see that FTs of causal solutions to (9) satisfying (20) (& (20'))

Back to radially-symmetric ~~the~~ T-H outgoing waves

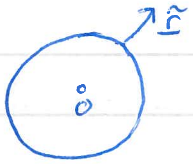
$$\Longrightarrow \quad p = \mathrm{Re}\left\{\hat{p}\, e^{-i\omega t}\right\} = \mathrm{Re}\left\{\underbrace{\boxed{\frac{A e^{ikr}}{r}}}_{=\hat{p}} e^{-i\omega t}\right\}$$

$$(16) \Longrightarrow \quad \underline{\hat{u}} = \frac{-i}{\rho_0 \omega} \nabla \hat{p} = \frac{-i}{\rho_0 \omega} \nabla\left(\frac{A e^{ikr}}{r}\right)$$

$$= \frac{-iA}{\rho_0 \omega}\left(\frac{ik}{r} - \frac{1}{r^2}\right) e^{ikr}\, \underline{\tilde{r}}$$

unit vector in radial direction

$\underline{\tilde{r}} = \frac{\underline{x}}{|\underline{x}|} = \frac{\underline{x}}{r}$ is a unit vector in radial direction



$$\Longrightarrow \quad \underline{\hat{u}} = \hat{u}_r \underline{\tilde{r}}, \qquad \hat{u}_r = \underbrace{\frac{A e^{ikr}}{r}}_{=\hat{p}}\left(ik - \frac{1}{r}\right)\frac{1}{\rho_0 \omega}$$

26/01/15

Sph symm + T-H + outgoing

$$p = Re\left\{ \overbrace{A \frac{e^{ikr}}{r}}^{=\hat{p}} e^{-i\omega t} \right\}$$

$$u = Re\left\{ \hat{u}\, e^{-i\omega t} \right\}$$

connects the two → $(16) \longleftrightarrow \hat{u} = \frac{-i}{\rho_0 \omega} \nabla \hat{p} = \hat{u}_r \tilde{r}$

$$\hat{u}_r = \frac{A}{r} e^{ikr}\left(ik - \frac{1}{r}\right)\frac{-i}{\rho_0 \omega}$$

accidentally missed this factor beforehand

$$= \underbrace{\frac{A}{r} e^{ikr}}_{\hat{p}} \left(1 + \frac{i}{kr}\right) \underset{\frac{1}{c}}{\boxed{\frac{k}{\rho_0 \omega}}}$$

$$\hat{u} = \hat{u}_r \hat{r},\ \hat{u}_r = \frac{\hat{p}}{\rho_0 c}\left(1 + \frac{i}{kr}\right) \qquad (21)$$

unit vector ↗ radial direction

↖ k large, can neglect

Interpretation: The 1st part of (21) is similar to plane T-H waves, $\hat{u}$ is in-phase with p; 2nd term out-of-phase with p by $\pi/2$ $(i = e^{i\pi/2})$

If kr "large", kr >> 1, the 2nd term is negligible

$\Longleftrightarrow k = 2\pi/\lambda$ (λ wavelength)

$\Longrightarrow kr = 2\pi \frac{r}{\lambda} >> 1 \Longleftrightarrow r >> \lambda$

Throwing a stone example →

"many wavelengths away from O" ⟷ far field)

So, far away, spherical waves are close to plane waves

Acoustic energy, Intensity (Kirchhoff, 1876)

Going back main acoustics equations (7) & (8):

(7) $\Longrightarrow \frac{1}{\rho_0 c^2}\frac{\partial p}{\partial t} + \nabla\cdot\underline{u} = 0$ (7')

multiply by p →

(8) $\rho_0\frac{\partial \underline{u}}{\partial t} + \nabla p = 0$ (8')

dot multiply by $\underline{u}$

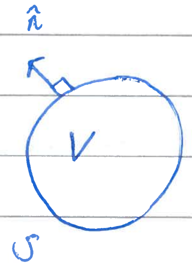
(7')×p + $\underline{u}\cdot$(8') $\Longrightarrow$

$$\frac{1}{\rho_0 c^2}p\frac{\partial p}{\partial t} + \rho_0\underline{u}\cdot\frac{\partial \underline{u}}{\partial t} + \overbrace{p\nabla\cdot\underline{u} + \underline{u}\cdot\nabla p}^{\nabla\cdot(p\underline{u})} = 0$$

$$\frac{\partial}{\partial t}\underbrace{\left(\frac{1}{2}\frac{p^2}{\rho_0 c^2} + \frac{1}{2}\rho_0\underline{u}\cdot\underline{u}\right)}_{W \text{ mechanical (energy)}} + \nabla\cdot\underbrace{(p\underline{u})}_{\underline{I} \text{ intensity}} = 0 \qquad (22)$$

$$\frac{\partial W}{\partial t} + \nabla\cdot\underline{I} = 0$$

Interpretation of 'conservation law' (22):



Integrate (22) over a volume V with boundary S (unit normal $\underline{n}$):

$$0 = \int_V \left(\frac{\partial w}{\partial t} + \nabla\cdot\underline{I}\right)dV \iff$$

flux of vector field $\underline{I}$ over surface

$$\frac{d}{dt}\left(\int_V w\,dV\right) \overset{\text{divergence theorem}}{=} -\int_S \underline{I}\cdot\underline{n}\,dS$$

acoustic intensity
determines where + how
energy flows

Here

$$w = \underbrace{\frac{1}{2}\rho_0|\underline{u}|^2}_{\text{kinetic energy density}} + \underbrace{\frac{p^2}{2\rho_0 c^2}}_{\text{potential energy (elastic energy) density}}$$

= Acoustic energy density (total mechanical energy)

Hence $\underline{I} := p\underline{u} :=$ acoustic energy flux, or acoustic <u>intensity</u>

$\underline{I}$ hence represents energy transported in unit time per unit area

<u>For plane T-H waves</u>

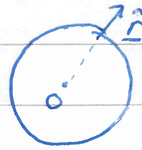
$(18) \Longrightarrow \underline{u} = \frac{p}{\rho_0 c}\underline{n}$ $\left(|\underline{n}| = 1, \text{ direction}\right) \Longrightarrow$

$\underline{I} = p\underline{u} = \underbrace{\frac{p^2}{\rho_0 c}}_{\geq 0}\underline{n}$ i.e. the energy flows in direction of propagation $\underline{n}$, as expected

<u>For spherically symmetric T-H waves:</u>

$$p = Re\left(\frac{A}{r}e^{ikr-i\omega t}\right) = Re\left(\frac{|A|e^{i\varphi+ikr-i\omega t}}{r}\right)$$

$= \frac{|A|}{r}\cos(\omega t - kr - \varphi)$ $\quad A = |A|e^{i\varphi}$ complex amplitude;



$$(21) \Longrightarrow \underline{u} = Re\left\{\hat{u}_r e^{-i\omega t}\right\}\underline{\tilde{r}}$$

$$= Re\left[\frac{|A|}{r\rho_0 c}e^{ikr-i\omega t+i\varphi}\left(1+\frac{i}{kr}\right)\right]\underline{\tilde{r}}$$

$$\frac{|A|}{r\rho_0 c}\cos(\omega t - kr - \varphi)\underline{\tilde{r}} + \frac{|A|}{kr^2\rho_0 c}\sin(\omega t - kr - \varphi)\underline{\tilde{r}}$$

$\Longrightarrow \underline{I} = p\underline{u} = pu_r\underline{\tilde{r}} =: I_r$ where

$$I_r = pu_r = \frac{|A|^2}{r^2\rho_0 c}\cos^2(\omega t - kr - \varphi) + \frac{|A|^2}{kr^3\rho_0 c}\cos(\omega t - kr - \varphi)\sin(\omega t - kr - \varphi)$$

Time harmonic – everything repeats itself i.e. periodic

For time - period average $\langle I_r \rangle := \frac{1}{T}\int_0^T I_r \, dt$

$\left(T = \frac{2\pi}{\omega} \right)$; since

$\int_0^T \cos^2(\omega t - kr - \varphi)\, dt = \frac{T}{2}$ (exercise)

$\int_0^T \cos(\omega t - kr - \varphi)\sin(\omega t - kr - \varphi)\, dt = 0$ (check)

$$\Longrightarrow \quad \langle I_r \rangle = \frac{|A|^2}{\rho_0 c r^2} \, \frac{1}{T} \cdot \frac{T}{2}$$

$$\langle I_r \rangle = \frac{|A|^2}{2\rho_0 c r^2} \geq 0 \qquad (23)$$

silly case when $A=0$

The energy flows in radial direction ($\hat{r}$) away from the origin O, as expected

▨

This finishes the acoustic waves section of chapter 1

1.2 Electromagnetic (EM) waves

The EM fields are described by the Maxwell's equations (ME) For electric field $\underline{E}(\underline{x},t)$, magnetic field $\underline{B}(\underline{x},t)$, given electric charge density ρ and electric current density $\underline{j}$ in a uniform electromagnetic medium, the MEs are:

$$\nabla\cdot\underline{E} = \frac{\rho}{\varepsilon} \quad (24), \qquad \nabla\cdot\underline{B} = 0 \quad (25)$$

current density

Faraday's Law →

$$\nabla\times\underline{E} = -\frac{\partial B}{\partial t} \quad (26); \qquad \nabla\times\underline{B} = \mu\varepsilon\frac{\partial E}{\partial t} + \mu\underline{j} \quad (27)$$

where ε, μ are "electric permitivity" and "magnetic permeability" of the medium (i.e. medium's physical characteristics)

Remark (24) relates to Coulomb's Law / Gauss Law
(25) 'Gauss Law' for $\underline{B}$
(26) = Faraday's Law of EM induction ('−' per Lenz law)
(27) without $\mu\varepsilon\frac{\partial E}{\partial t}$ ('Faraday's term') = Ampere's Law

In an electromagnetic medium without "sources" ($\rho = 0$, $\underline{j} = 0$) MEs are

$$\nabla\cdot E = 0 \quad (28), \qquad \nabla\cdot B = 0 \quad (29)$$

$$\nabla\times\underline{E} = -\frac{\partial B}{\partial t} \quad (30), \qquad \nabla\times B = \mu\varepsilon\frac{\partial E}{\partial t} \quad (31)$$

To eliminate $\underline{B}$, take curl of (30) and use (31)

$$\nabla\times(\nabla\times\underline{E}) = -\frac{\partial}{\partial t}(\nabla\times B) = -\mu\varepsilon\frac{\partial^2 E}{\partial t^2}$$

Also (vector calculus) ↙ gradient of divergence ↙ divergence of gradient = Laplace

$$\nabla\times\nabla\times\underline{E} = \nabla(\nabla\cdot E) - \nabla^2\underline{E}$$

$a\times(b\times c) = (a\cdot c)b - (a\cdot b)c$

$\nabla\cdot E = 0$ by (28)

$$\Longrightarrow -\nabla^2 \underline{E} = -\mu\varepsilon \frac{\partial^2 \underline{E}}{\partial t^2}$$

$$\frac{1}{c^2}\frac{\partial^2 \underline{E}}{\partial t^2} - \nabla^2 \underline{E} = 0 \qquad (32)$$

$c = (\varepsilon\mu)^{-1/2}$ wave speed

Similarly (exercise) $\nabla \times$ (31) & (30)

$$\Longrightarrow \frac{1}{c^2}\frac{\partial^2 \underline{B}}{\partial t^2} - \nabla^2 \underline{B} = 0 \qquad (33)$$

So both $\underline{E}$ and $\underline{B}$ solve (vector) wave equations (32), (33) cf (9)/(12)

↑ vector ↑ scalar

For vacuum $c = c_0 = (\varepsilon_0\mu_0)^{-1/2} \approx 3\times 10^8 \, ms^{-1}$ = speed of light; in any other media it can only be smaller, physically

[Exercise (Exam 2010 Q1(a)): Show that for 'full' MEs (24)-(27), $\underline{E}$ solves

$$\frac{1}{c^2}\frac{\partial^2 \underline{E}}{\partial t^2} - \nabla^2 \underline{E} = -\frac{1}{\varepsilon}\nabla\rho - \mu\frac{\partial \underline{j}}{\partial t} ;$$

Derive similar equation for $\underline{B}$]

Similarly to acoustics (32)-(33) admit T-H solutions:

$$\underline{E}(\underline{x},t) = \mathrm{Re}\left\{\hat{\underline{E}}(\underline{x})e^{-i\omega t}\right\} \qquad (34)$$

$$\underline{B}(\underline{x},t) = \mathrm{Re}\left\{\hat{\underline{B}}(\underline{x})e^{-i\omega t}\right\} \qquad (35)$$

where $\hat{\underline{E}}(\underline{x}), \hat{\underline{B}}(\underline{x})$ are complex-valued vector field.

As in acoustics, (34) $\xrightarrow{\text{subst.}}$ (32), (35) $\rightarrow$ (33)

yields (vector) Helmholtz equations for $\hat{\underline{E}}$, $\hat{\underline{B}}$ of (15):

$$\nabla^2 \hat{\underline{E}} + k^2 \hat{\underline{E}} = 0 \qquad (36)$$
$$\nabla^2 \hat{\underline{B}} + k^2 \hat{\underline{B}} = 0 \qquad (37)$$

$k := \frac{\omega}{c}$ the wavenumber

Plane T-H EM waves

Similarly acoustics, solutions of MEs include plane T-H (EM) waves, in any direction $\underline{n}$ - given $\underline{n}$, choose x, y, z

Choosing WLOG $\underline{n}$ along z-direction they have the form:

$$\underline{E}(\underline{x}, t) = Re\left\{ \underbrace{\underline{E}_0\, e^{ikz - i\omega t}}_{\hat{\underline{E}}} \right\} \qquad (38)$$

$$\underline{B}(\underline{x}, t) = Re\left\{ \underbrace{\underline{B}_0\, e^{ikz - i\omega t}}_{\hat{\underline{B}}} \right\} \qquad (39)$$

where $\underline{E}_0$, $\underline{B}_0$ are complex vector constants

3D space

By the above construction, $\forall\ E_0, B_0 \in \mathbb{C}^3$, (32) and (33) are satisfied

three complex numbers

Now, (28) $\longleftrightarrow$ $\underline{\nabla} \cdot \underline{E} = 0$, $E_0 = (E_x, E_y, E_z)$

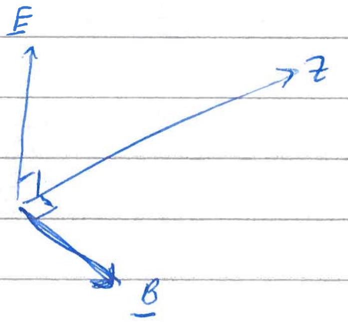
$$\Longrightarrow \underline{\nabla} \cdot \underline{E} = Re\left\{ E_z\, ik\, e^{ikz - i\omega t} \right\} = 0 \qquad \forall z, \ \forall t$$

$$\Longrightarrow E_z = 0$$

Similarly (29) $\longleftrightarrow$ $\nabla \cdot \underline{B} = 0$, $B_0 = (B_x, B_y, B_z)$

$$\Longrightarrow B_z = 0$$

So both $\underline{E}$ and $\underline{B}$ must be perpendicular ($\perp$) to propagation direction (z) i.e. EM waves are (in this sense) transverse (in contrast to acoustics):



$\mathcal{B} = \mathrm{Re}\{ B_0 e^{ikz-i\omega t} \}$

Now plug (38)-(39) in to (30) $\longleftrightarrow$ $\nabla \times \underline{E} = -\frac{\partial B}{\partial t}$:

$$\nabla \times \underline{E} = \mathrm{Re}\left\{ e^{-i\omega t} \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_x e^{ikz} & E_y e^{ikz} & 0 \end{vmatrix} \right\}$$

$$= \mathrm{Re}\left\{ e^{-i\omega t}\left(-ike^{ikz}E_y, \ ike^{ikz}E_x, \ 0 \right) \right\}$$

since nothing depends on x, y

$$= \mathrm{Re}\left\{ \cancel{ike^{ikz-i\omega t}} \left(-E_y, E_x, 0 \right) \right\}$$

$$\overset{(30)}{=} \mathrm{Re}\left\{ \cancel{i\omega e^{ikz-i\omega t}} \underbrace{\left(B_x, B_y, 0 \right)}_{B_0} \right\} \quad \forall t, \ \forall z$$

$$\Longrightarrow B_0 = \frac{k}{\omega}\left(-E_y, E_x, 0 \right) = \frac{1}{\omega} \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 0 & k \\ E_x & E_y & 0 \end{vmatrix}$$

different k

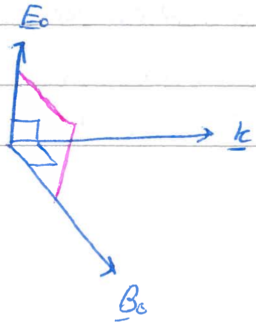
$$= \frac{1}{\omega} \underline{k} \times \underline{E}_0 , \quad \underline{k} = (0,0,k) = k\underline{n} \quad \underline{n} = (0,0,1)$$

$\underline{k}$ is <u>wave vector</u>

So $\underline{B}_0 = \frac{1}{\omega} \underline{k} \times \underline{E}_0$ (40)

So via (40), $\underline{B}_0 \perp \underline{E}_0$ (also $B_0 \perp \underline{k}$, $\underline{E}_0 \wedge \underline{k}$)

i.e. E_0, B_0, $\underline{k}$ form a right-handed orthogonal $\perp$-e triple



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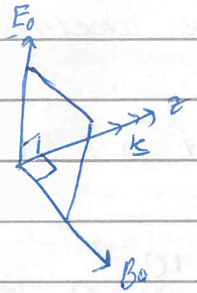
Maxwell - 4 eq^ns, 2 vector unknowns - eliminate one in favour of other



k wavevector along z

E_0, B_0 complex valued vectors

$$\underline{E}(\underline{x},t) = Re\{E_0 e^{ikz-i\omega t}\}$$
$$\underline{B}(\underline{x},t) = Re\{\underline{B}\, e^{ikz-i\omega t}\}$$

Exercise: Show that $\underline{E_0} \perp \underline{k}$ with $\underline{B_0}$ via (40)

$\underline{B_0} = \frac{1}{\omega}\underline{k} \times \underline{E_0}$ (40)

Solve also (31), of Exam 2014 Q1

So plane EM waves may have different 'polarisations';

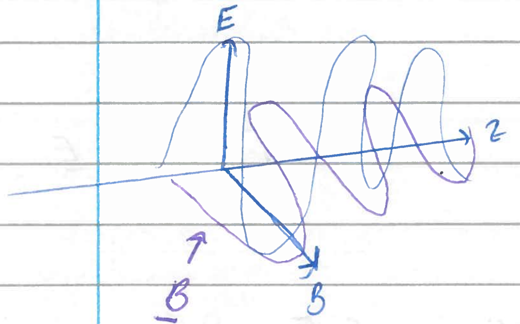
Polarisation plane is the plane containing $\underline{E}$ and $\underline{k}$

Note that $\underline{E_0}$, $\underline{B_0}$ are generally complex, so generally $\underline{E}(x,t)$ will rotate; however ∀ plane T-M EM wave can be decomposed in to sum of two 'polarised' waves:

$$\underline{E}(\underline{x},t) = Re\left\{E_0 e^{ikz-i\omega t}\right\} = \overbrace{(Re\, E_0)\cos(\omega t - kz)}^{\underline{E}^{(1)}} \underbrace{+(Im\, E_0)\sin(\omega t - kz)}_{\underline{E}^{(2)}}$$

sin odd fn. so (-) comes out

where $\underline{E}^{(1)}$, $\underline{E}^{(2)}$ are polarized



3D picture

sinusoidal shapes

pictures move z direction w/ wave speed c

$k = \frac{\omega}{c}$ ω angular frequency $c = (\varepsilon\mu)^{-1/2}$ $\lambda = \frac{2\pi}{k}$ wavelength

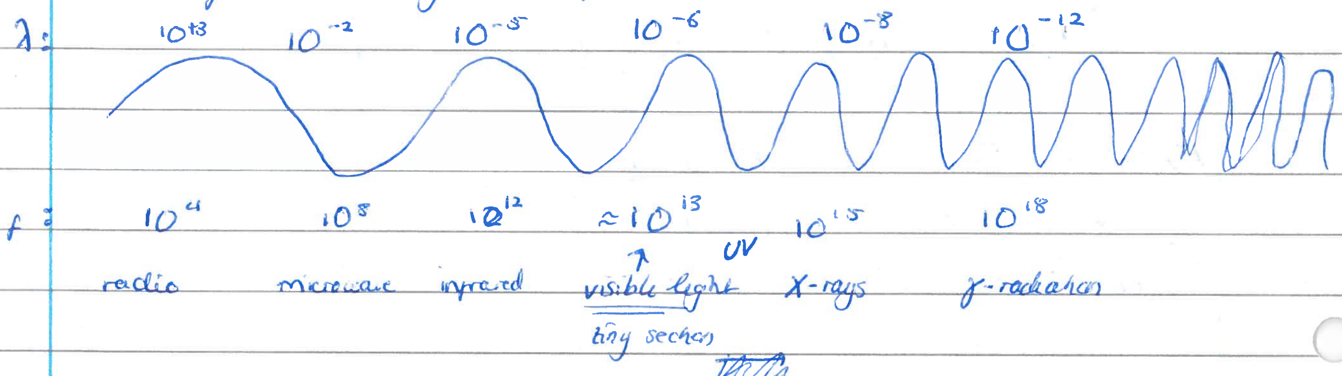
$T = \frac{2\pi}{\omega}$ time period (s)

$f = T^{-1} = \frac{\omega}{2\pi}$ frequency ($Hz = s^{-1}$)

$$\text{frequency} = f = \frac{\omega}{2\pi} = \frac{\omega}{k} \cdot \frac{k}{2\pi} = \frac{c}{\lambda}$$

$$\Longrightarrow \lambda = \frac{c}{f}$$ higher frequency, shorter wavelength

Depending upon f, λ EM waves are found in different parts of electromagnetic spectrum



Certain things go close to acoustic case - parallels

Electromagnetic energy, Poynting's vector

Similarly to acoustics, see (22), we can derive a conservation law for EM energy from MEs (30) & (31):

$$(30) \longleftrightarrow \quad \underline{\nabla} \times \underline{E} = -\frac{\partial B}{\partial t} \qquad \text{Faraday's}$$

$$(31) \longleftrightarrow \quad \nabla \times \underline{B} = \mu\varepsilon \frac{\partial E}{\partial t}$$

Mult. (30) by $\underline{B}$
(31) by $\underline{E}$ and subtract

Take dot product of (30) with $\underline{B}$ and of (31) with $\underline{E}$ and subtract $\Longrightarrow$

$$\underline{B} \cdot (\nabla \times \underline{E}) - \underline{E} \cdot (\nabla \times \underline{B}) = -\underline{B} \cdot \frac{\partial B}{\partial t} - \mu\varepsilon\, \underline{E} \cdot \frac{\partial \underline{E}}{\partial t}$$

From vector calculus (check):

$$\nabla \cdot (\underline{A} \times \underline{B}) = \underline{B} \cdot (\nabla \times \underline{A}) - \underline{A} \cdot (\nabla \times \underline{B})$$

$$\nabla\cdot(\underline{E}\times\underline{B}) = -\frac{\partial}{\partial t}\left(\frac{1}{2}\underline{B}\cdot\underline{B} + \frac{1}{2}\mu\varepsilon\,\underline{E}\cdot\underline{E}\right)$$

$\div\mu \downarrow$

$$\frac{\partial}{\partial t}\overbrace{\left(\frac{1}{2}\varepsilon|\underline{E}|^2 + \frac{1}{2\mu}|\underline{B}|^2\right)}^{W} + \nabla\cdot\underbrace{\left(\frac{1}{\mu}\underline{E}\times\underline{B}\right)}_{\underline{S}} = 0 \qquad (41)$$

which is similar to (22)

Here: $w := \frac{1}{2}\varepsilon|\underline{E}|^2 + \frac{1}{2\mu}|\underline{B}|^2$ EM field's energy density

$\Longrightarrow$ $\underline{S} := \frac{1}{\mu}\underline{E}\times\underline{B}$, called Poynting's vector describes EM flow (per unit time across unit area)

$\uparrow$ has direction + magnitude shows where energy flows

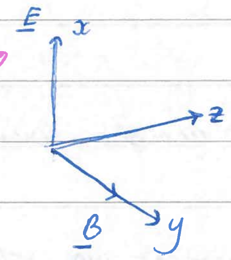
$\uparrow$ integrate (41), divergence theorem

So for plane T-H EM waves, e.g. let $\underline{E}(x,t) = E_x$

$$\underline{E}(x,t) = (E_x, 0, 0)\cos(kz - \omega t) \Longrightarrow$$

by (40)

we choose this $\rightarrow$



$$\underline{B}(x,t) = \frac{1}{\omega}\underline{k}\times\underline{E}_0$$

$$= \frac{1}{\omega}\begin{vmatrix} i & j & k \\ 0 & 0 & k \\ E_x & 0 & 0 \end{vmatrix}\cos(\omega t - kz)$$

$\leftarrow$ different $\underline{k}$ & k vector

$$= \left(0, \overset{\frac{kE_x}{\omega}\downarrow}{\frac{E_x}{c}}, 0\right)\cos(\omega t - kz)$$

$$\Longrightarrow \underline{S} = \frac{1}{\mu}\underline{E}\times\underline{B} = \frac{1}{\mu c}\begin{vmatrix} i & j & k \\ E_x & 0 & 0 \\ 0 & E_x & 0 \end{vmatrix}\cos^2(\omega t - kz)$$

$$= \frac{|E_x|^2}{\mu c}(0,0,1)\cos^2(\omega t - kz)$$

so the EM energy flows in the z-direction $\equiv$ direction of propagation, as expected.

For scattering, need obstacles

We introduce appropriate boundary conditions

1.3 Surface Boundary and Interface conditions

Both in acoustics and in EM, boundary / interface conditions are required e.g. at a solid surface or at an interface between two media (e.g. air and water)

Mathematically, our PDE boundary conditions (BCs) typically are:

- Dirichlet BC: unknown function is specified on the boundary;
- Neumann's BCs: for normal derivative e.g. $\frac{\partial p}{\partial n}$ given on $\underset{\substack{\uparrow \\ \text{boundary}}}{S}$

Can also do hybrid of Dirichlet + Neumann

- Mixed / Robin / Impedance BCs, with a combination of p and $\frac{\partial p}{\partial n}$ given on boundary S

1.3.1 Acoustic BCs

(i) Acoustically hard ~~BF~~ surface doesn't yield to acoustic wave ⟷ is rigid ⟷

can move along surface ↕ but not ↔ through

$\underline{u} \cdot \underline{n} = 0$ ($\Longleftrightarrow$ no normal component in velocity $\underline{u}$ on S); assuming irrotational fields

$$\underline{u} = \nabla \phi \implies \underline{u} \cdot \underline{n} = \underbrace{\nabla \phi \cdot \underline{n}}_{\text{directional derivative}} = \frac{\partial \phi}{\partial n} = 0$$

$\implies$ Zero Neumann BC for ϕ

(ii) Acoustically soft surface freely yields to a wave (e.g. water surface) ⟶ Physically on S, $p = 0$, (on S, pressure = atmospheric pressure) ⟷ $\overset{\text{zero}}{\uparrow}$ Dirichlet BCs

(iii) Impedance BCs are intermediate between (i) and (ii): a surface impedance, Z, measures to what extent the surface resists to the applied pressure; it generally depends on frequency (TH case ⟷ ω); so for T-H acoustics, $p = \mathrm{Re}\{\hat{p} e^{-i\omega t}\}$, $\underline{u} = \mathrm{Re}\{\hat{\underline{u}} e^{-i\omega t}\}$

$\hat{u}_n := \hat{\underline{u}} \cdot \underline{n}$ — higher resistance, more pressure needed to apply

$$z := \frac{\hat{p}}{\hat{u}_n} \iff \hat{p} - z\hat{u}_n = 0 \iff \hat{p} - z\frac{\partial \hat{\phi}}{\partial n} = 0 \quad *$$

$$\left(\phi = \mathrm{Re}\left\{ \hat{\phi}(x) e^{-i\omega t} \right\} \right)$$

$$(11) \longleftrightarrow \frac{\partial \phi}{\partial t} = -\frac{1}{\rho_0} p \overset{T-H}{\iff} -i\omega \hat{\phi} \cancel{e^{-i\omega t}} = -\frac{1}{\rho_0} \hat{p} \cancel{e^{-i\omega t}}$$

$$\Rightarrow \boxed{\hat{p} = i\omega \rho_0 \hat{\phi}} \qquad (42)$$

$$\Rightarrow i\omega\rho_0 \hat{\phi} - z\frac{\partial \hat{\phi}}{\partial n} = 0 \qquad (43)$$

which is the impedance BC.

Also, via (42), ← where did this z come from? ← diff 42 wrt $\hat{\phi}$ & sub in *

$$i\omega\rho_0 \hat{p} - z(\omega)\frac{\partial \hat{p}}{\partial n} = 0 \qquad (43')$$

Notice: if $z \to 0 \Rightarrow \hat{p} = \hat{\phi} = 0$ (soft BC (ii)); ※

if $z \to \infty \iff \frac{1}{z} \to 0 \Rightarrow \frac{\partial \hat{\phi}}{\partial n} = \frac{\partial \hat{p}}{\partial n} = 0$ (hard BC(i))

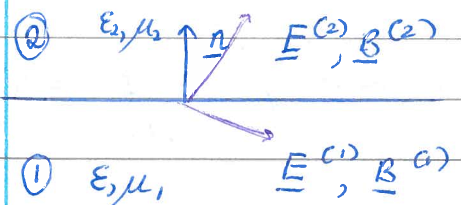
Generally $z(\omega)$ is complex, physically,

$\mathrm{Re}\, z \geq 0$, with $\mathrm{Re}\, z > 0$ for energy absorbing surfaces

$\mathrm{Re}\, z = 0$ for energetically neutral surfaces

(will see in §2)

1.3.2 EM interface conditions / BCs



At an interface between two EM media (ε_1,μ_1 in medium 1, ε_2,μ_2 in medium 2), the EM fields $\underline{E}^{(1)}, \underline{B}^{(1)}$ & $\underline{E}^{(2)}, \underline{B}^{(2)}$ respectively, are physically required to obey the following interface conditions:

i) Tangential components of $\underline{E}$ are continuous (cts) $\Longleftrightarrow$

$$\underline{E}^{(1)} \times \underline{n} = \underline{E}^{(2)} \times \underline{n} \qquad (44)$$

disregards normal component

projections on interface plane are the same

ii) Normal components of $\underline{B}$ are cts $\Longleftrightarrow$

$$\underline{B}^{(1)} \cdot \underline{n} = \underline{B}^{(2)} \cdot \underline{n} \qquad (45)$$

Also, if there exist a surface charge distribution ρ_s and/or surface current distribution $\underline{j}_s$, then (iii)

(iii) $$\varepsilon_2 \underline{E}^{(2)} \cdot \underline{n} - \varepsilon_1 \underline{E}^{(1)} \cdot \underline{n} = \rho_s \qquad (46)$$

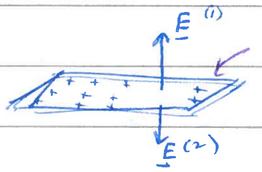
electric permittivity

iv) $$\frac{1}{\mu_1} \underline{B}^{(1)} \times \underline{n} - \frac{1}{\mu_2} \underline{B}^{(2)} \times \underline{n} = \underline{j}_s \qquad (47)$$

magnetic permittivity →

(In (46), (47) the unit normal $\underline{n}$ points in to medium 2)

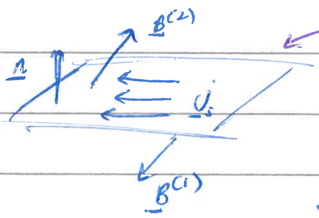
Check:



+vely charged so $\underline{E}$ goes away from plane

$\rho_s > 0 \Longrightarrow \underline{E}^{(2)} \cdot \underline{n} > 0$, $\underline{E}^{(1)} \cdot \underline{n} < 0$

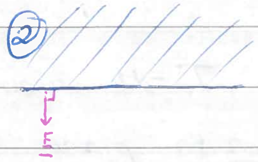
$\Longrightarrow$ (46) OK



current going right to left

"right hand rule" for $\underline{B}^{(1)}, \underline{B}^{(2)}$, due to $\underline{j}_s \Longrightarrow$ signs in (47) OK

Perfectly conducting BCs



If medium 2 is a "perfect conductor"

$\Longrightarrow \underline{E}^{(2)} = 0$

cannot support electric field

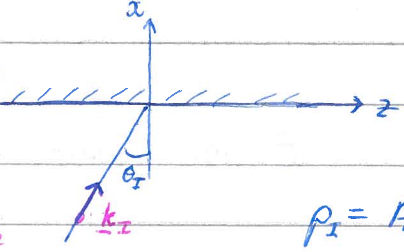
$\Longrightarrow$ from (44) $\underline{E}^{(1)} \times \underline{n} = \underline{0}$ i.e. $\underline{E}$ must have a zero tangential component. (End of chapter 1)

2. Canonical cases

2. 'Canonical' Cases (Reflection & Refraction)

2.1 Reflection by a plane (acoustics)

y is perp. to blackboard



propagating → towards surface

Consider a plane T-H acoustic wave incident upon a plane boundary

$S' = \{(x,y,z) : x = 0\}$

$p_I = A_I e^{-i\omega t + i\underline{k}_I \cdot \underline{x}}$ (dropping "Re_s")

Assume it has 'angle g incidence' θ_I

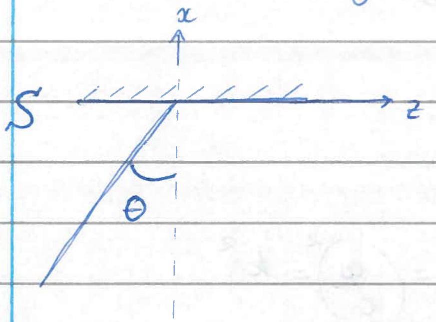
$\underline{k}_I = k(\cos\theta_I, 0, \sin\theta_I)$ wavevector;

$k := |\underline{k}_I| = \frac{\omega}{c}$ the wavenumber

$0 < \theta_I < \frac{\pi}{2}$ (moving 'towards' S')

09/02/15

Canonical - standard, generic



normally write Re, but have dropped it

$$p_I = A_I e^{i k_I \cdot x - i\omega t}$$

$$\underline{k}_I = k(\cos\theta_I,\ 0,\ \sin\theta_I)$$

expect surface bounces wave back

The boundary S will "reflect" the incident wave p_I

What is the reflected wave p_R? Answer depends on BC

This depends on the boundary conditions on S

Consider acoustically sft BC : $p = 0$ on S (§1.3.1)

$\iff$ $p = p_I + p_R = 0$ on S

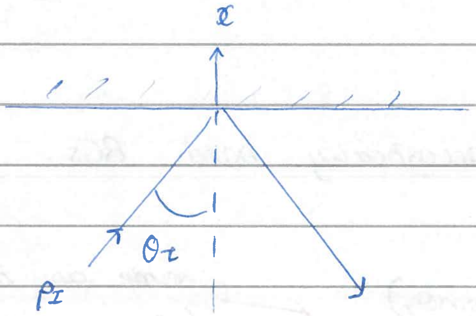
Seek p_R also as a plane T-H wave:

$p_R = A_R e^{i\underline{k}_R \cdot \underline{x} - i\omega_R t}$ will see if T-H can only be with same frequency but let's assume different

$\underline{k}_R = (k_x, k_y, k_z)$

$|\underline{k}_R| =: k_R = \frac{\omega_R}{c}$ ⟶ general rule for T-H waves

← wave speed, c.



$k_x < 0$ (moving 'away' from S)

We have incident wave moving towards, reflected wave moving away

BCs: $\underline{x} \in S \iff x = 0,\ \underline{x} = (0, y, z)$

$p_I + p_R = 0 \iff p_R = -p_I$ for $x = 0 \iff$

$A_R e^{ik_y y + ik_z z - i\omega_R t} = -A_I e^{ik\sin\theta_I z - i\omega t}$ $\forall t, \forall y, \forall z$

⟶ We assume $A_I \neq 0$

- Varying $t \implies \omega_R = \omega$
- " $y \implies k_y = 0$ → since y on LHS, not on RHS
- " $z \implies k_z = k\sin\theta_I$

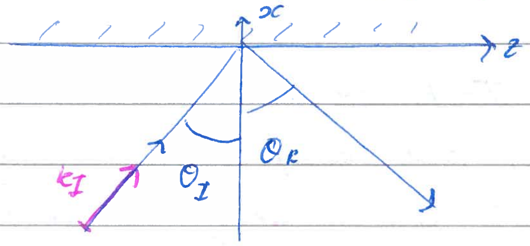
Also $A_R = -A_I$

Now $k_R^2 = k_x^2 + \cancel{k_y^2} + k_z^2 = \left(\frac{\omega_R}{c}\right)^2 = \left(\frac{\omega}{c}\right)^2 = k^2$

$k_x = \pm\left(k^2 - k_z^2\right)^{1/2} = \pm\sqrt{k^2 - k^2\sin^2\theta_I}$

~~$\implies \nvdash$~~ $= -k\cos\theta_I$ (since $k_x < 0$)

So $k_R = k(-\cos\theta_I, 0, \sin\theta_I)$



Interpretation: $\theta_R = \theta_I$ ('Specular reflection law')

$A_R = -A_I \iff$ Phase shift for π upon reflection $(-1 = e^{i\pi})$

Exercises: i) Show that for acoustically hard BCs

$\left(\frac{\partial\phi}{\partial n} = 0 \longleftrightarrow \frac{\partial p}{\partial n} = 0\right)$

$p_R = A_I e^{-i\omega t + ik((-x\cos\theta_I) + z\sin\theta_I)}$ ← same as before but no minus sign

Exam 2012 Q2(b)

2) Find the reflected wave for a general impedance BC (1.42') ← ch. 1, formula 42

~~$\omega\rho_0\phi$~~

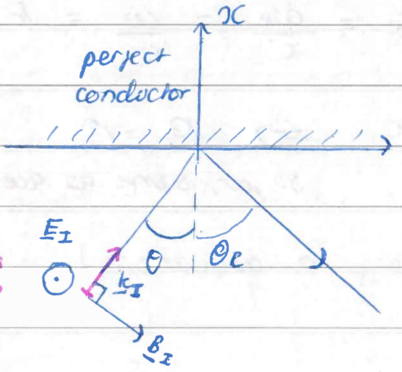
$$i\omega\rho_0 p - z\frac{\partial p}{\partial n} = 0$$

Conclude that $|A_R| \leq |A_I|$
$\Longleftrightarrow$ $\operatorname{Re} z \geq 0$

This is a natural assumption
Reflected signal could only weaken

(Exam 2012 Q2(c))

2.2 Reflection of plane T-H EM waves by perfectly conducting planes



$\underline{E}$ is towards us - this is the tip of the arrow

I for incidence

Consider a plane T-H EM wave propagating in an "insulator" medium (no currents) occupying half-space $x < 0$; with angle of incidence θ, unknown angle of reflection θ_R; ϵ, μ given $\Longrightarrow$ $c = (\epsilon\mu)^{-1/2}$ wave speed

Let the incidence wave be polarised so that $\underline{E}_I$ is parallel to y-axis, pointing out of the plane:

$$\underline{E}_I = (0, E_I, 0)e^{-i\omega t + ik(x\cos\theta + z\sin\theta)} \quad (1)$$

$$\underline{B}_I \overset{(1.40)}{=} \frac{1}{\omega}\underline{k}_I \times \underline{E}_I = \frac{E_I}{c}(-\sin\theta, 0, \cos\theta)e^{-i\omega t + ik(x\cos\theta + z\sin\theta)} \quad (2)$$

$$\underline{k}_I = k(\cos\theta, 0, \sin\theta)$$

The reflected wave is also sought as a plane wave

$$\underline{E}_R = (E_x, E_y, E_z)e^{-i\omega_R t + ik_R(-x\cos\theta_R + z\sin\theta_R)} \quad (3)$$

$$\underline{B}_R = (B_x, B_y, B_z)e^{-i\omega_R t + ik_R(-x\cos\theta_R + z\sin\theta_R} \quad (4)$$

$k_R = \omega_R / c$; $E_x, E_y, E_z, B_x, B_y, B_z, \omega_R, \theta_R$ to be found

From perfectly conducting BCs (1·48) on
$S = \{x = 0\}$, $\underline{E} \times \underline{n} = 0$ $\underline{n} = (1, 0, 0)$
$\underline{E} = \underline{E}_I + \underline{E}_R$ = total electric field

$\underline{E}$ has zero y and z components on S $(x = 0)$

y component:

$$E_y e^{-i\omega_R t + ik_R z\sin\theta_R} = -E_I e^{-i\omega t + ikz\sin\theta}$$

$\Longrightarrow \omega_R = \omega$ (varying t); $\Longrightarrow k_R = \frac{\omega_R}{c} = \frac{\omega}{c} = k$

since $k_R = k$

$\Longrightarrow$ (varying z) $\cancel{k_R}\sin\theta_R = \cancel{k}\sin\theta \Longrightarrow \theta_R = \theta$

so far, same as acoustics

(i.e. specular reflection law, same as in acoustics)

Also $E_y = -E_I$

z-components: $\Longrightarrow E_z = 0$;

Also, since $(\underline{E}_R, \underline{B}_R)$ is a plane EM wave
$\Longrightarrow \underline{E}_R \perp \underline{k}_R \Longleftrightarrow \underline{E}_R \cdot \underline{k}_R = 0 \Longleftrightarrow$

$$k(-E_x\cos\theta_R + \cancel{E_z}\sin\theta_R) = 0 \Longrightarrow \underline{E_x = 0}$$

So $\underline{E}_R = (0, -E_I, 0)e^{-i\omega t + i\underline{k}_R\cdot\underline{x}}$ and so points into the plane;

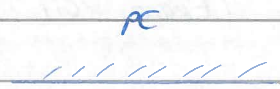
Finally, from (1·40)

$\theta_R = \theta$

$$\underline{B}_R = \frac{1}{\omega}\underline{k}_R \times \underline{E}_R = \frac{k}{\omega}\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -\cos\theta & 0 & \sin\theta \\ 0 & -E_I & 0 \end{vmatrix} e^{-i\omega t + i\underline{k}_R\cdot\underline{x}}$$

$$\frac{E_I}{c}\left(\sin\theta, 0, \cos\theta\right) e^{-i\omega t + i\underline{k}_R\cdot \underline{x}}$$ □

What else can we say about this problem

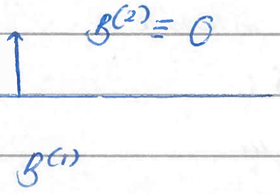


charges can accumulate on boundary
nothing above since perfect conductor (PC)

Finally, lets find induced electric current on S:
Notice that $\underline{E} \equiv 0$ in $x>0$ (in the perfect conductor)
$\Longrightarrow$ $\underline{B} \equiv 0$ in $x>0$ from ME (1·26)

$$\frac{\partial B}{\partial t} = 0 \Longrightarrow \underset{(\underline{B}\ T\text{-}H)}{\underline{B} = 0}$$ (constants ruled out since in T-H case)

Now from interface condition (1·47)



$$\frac{1}{\mu}\underline{B}^{(1)}\times \underline{n} - \overset{0}{\cancel{\frac{1}{\mu_2}\underline{B}^{(2)}\times \underline{n}}} = \underline{j}_s$$

$$\Longrightarrow \underline{j}_s = \frac{1}{\mu}\left(\underline{B}_I + \underline{B}_R\right)\times(1,0,0):$$

$$\underline{B}_I \times \underline{n} = \frac{E_I}{c} e^{-i\omega t + i\underline{k}_I\cdot \underline{x}} \begin{vmatrix} i & j & k \\ -\sin\theta & 0 & \cos\theta \\ 1 & 0 & 0 \end{vmatrix}$$

$$\left(\underline{B}_I + \underline{B}_R\right)\times \underline{n} = \frac{E_I}{c} e^{-i\omega t + i\underline{k}_I\cdot \underline{x}} \begin{vmatrix} i & j & k \\ 0 & 0 & 2\cos\theta \\ 1 & 0 & 0 \end{vmatrix}$$

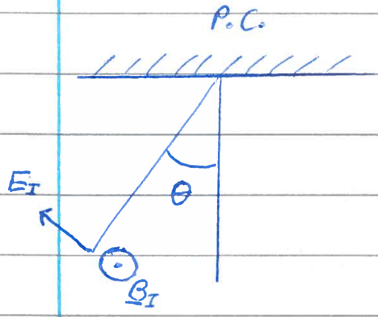
$(x=0)$

$$= \frac{E_I}{c}\, 2\cos\theta\, (0,1,0)\, e^{-i\omega t + i\underline{k}_R\cdot \underline{x}}$$

$$j_s = \frac{2\cos\theta\, E_I}{\mu c} e^{-i\omega t + ikz\sin\theta}(0,1,0)$$

is the induced current □

Exercise

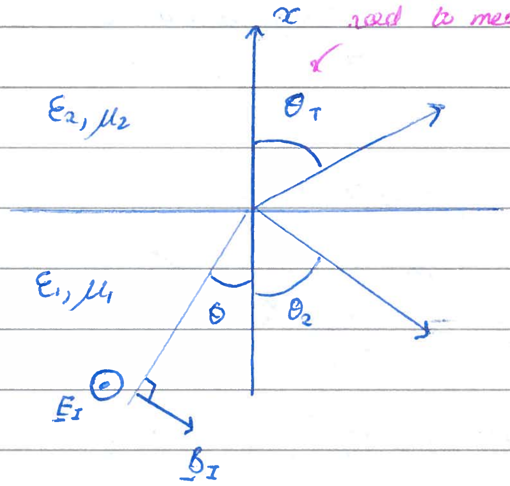


Consider a differently polarised incident wave : $\underline{E}_I$ in-plane

$\underline{B}_I$ out-of-plane

(Exam 2011 Q2)

2·3 Reflection and Refraction by a plane interface between two insulator media



need to measure everything relative to normal ✓

$\theta_T \leftarrow$ transmission

Consider two insulator EM media with interface $S = \{x = 0\}$:

medium 1 : ε_1, μ_1

— " — 2 : ε_2, μ_2

A plane EM wave incident from medium 1 same as in §2·2 i.e. described by ① and ②

$$① \Longrightarrow \underline{E}_I = E_I\,(0, 1, 0)\, e^{-i\omega t + ik(x\cos\theta + z\sin\theta)}$$

$$② \Longrightarrow \underline{B}_I = \frac{E_I}{c}\,(-\sin\theta, 0, \cos\theta)\, e^{-i\omega t + ik(x\cos\theta + z\sin\theta)}$$

Seek the reflected wave also as in ③ & ④

$$③ \Longrightarrow \underline{E}_R = (E_x, E_y, E_z)\, e^{-i\omega_R t + ik_R(-x\cos\theta_R + z\sin\theta_R)}$$

$$④ \Longrightarrow \underline{B}_R = (B_x, B_y, B_z)\, e^{-i\omega_R t + ik_R(-x\cos\theta_R + z\sin\theta_R)}$$

with different unknowns $E_x, E_y, E_z, \dots, B_z$

Additionally a '*transmitted*' wave is sought in $x > 0$ (medium 2):

$$\underline{E}_T = (\hat{E}_x, \hat{E}_y, \hat{E}_z)\, e^{-i\omega_T t + ik_T(x\cos\theta_T + z\sin\theta_T)} \qquad (5)$$

$$B_T = (\hat{B}_x, \hat{B}_y, \hat{B}_z)\, e^{-i\omega_T t + ik_T(x\cos\theta_T + z\sin\theta_T)} \qquad (6)$$

$$\left(k_R = \frac{\omega_R}{c_1},\quad k_T = \frac{\omega_T}{c_2},\quad c_1 = (\varepsilon_1\mu_1)^{-1/2},\quad c_2 = (\varepsilon_2\mu_2)^{-1/2}\right)$$

② From (1.44) $\Longleftrightarrow$ $\underline{E}^{(1)} \times \underline{n} = \underline{E}^{(2)} \times \underline{n}$ $\Longleftrightarrow$ continuity of y and z components of $\underline{E}$ $(x=0)$

① $\Longrightarrow$

y-components:

$$E_y e^{-i\omega t + ikz\sin\theta} + E_y e^{-i\omega_R t + ik_R z\sin\theta_R} = \hat{E}_y e^{-i\omega_T t + ik_T z\sin\theta_T} \qquad (\forall\, t, z)$$

$\Longrightarrow$ (cf §2.2)

$\omega_R = \omega_T = \omega \Longrightarrow k_R = k$

$\Longrightarrow$ (varying z) $k\sin\theta = \underset{\uparrow = k}{k_R}\sin\theta_R = k_T\sin\theta_T$

$\theta_R = \theta$

(as before: specular reflection) and additionally

$k_T\sin\theta_T = k\sin\theta \Longleftrightarrow \dfrac{\omega}{c_2}\sin\theta_T = \dfrac{\omega}{c_1}\sin\theta$

$$\boxed{\frac{\sin\theta_T}{c_2} = \frac{\sin\theta}{c_1}} \qquad (7)$$

will write in equivalent form using refractive index

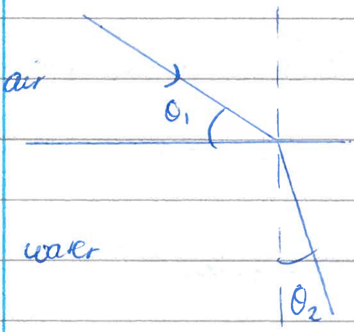
Let $n := \dfrac{c_0}{c}$, c_0 speed of light (in vacuum)

called *refractive index* ($\Longrightarrow n \geq 1$, $n = 1$ in vacuum)

$\Longrightarrow$

$$n_2 \sin\underset{\underset{\theta_T}{\parallel}}{\theta_2} = n_1 \sin\theta_1 \qquad (n\sin\theta = \text{constant}) \qquad (8)$$

(7) or (8) is Snell's law of refraction



wave speed smaller in water

← chooses this angle so it can go faster

Further (from equating the y-components)

$$E_z + E_y = \hat{E}_y \qquad (9)$$

z-components:

$$E_z = \hat{E}_z \qquad (10)$$

Also, $\underline{E}_R \cdot \underline{k}_R = 0, \quad \underline{E}_T \cdot \underline{k}_T = 0$

$$\Longrightarrow \; -E_x \cos\theta + E_z \sin\theta = 0 \qquad (11)$$

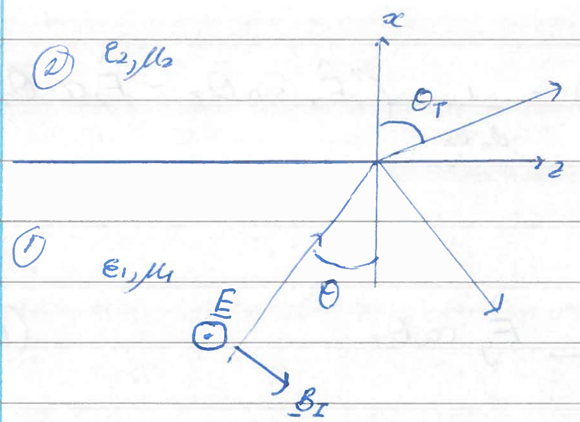
$$\hat{E}_x \cos\theta_T + \hat{E}_z \sin\theta_T = 0 \qquad (12)$$

Further from (1·47) and with no surface currents (since insulators - cannot support currents)

$$\frac{1}{\mu_1}\left(\underline{B}_I + \underline{B}_R\right) \times \underline{n} = \frac{1}{\mu_2}\underline{B}_T \times \underline{n} \qquad \Longleftrightarrow$$

Continuity of y & z components of $\dfrac{B}{\mu}$

23/02/15



Could have boundary w/ appropriate bdry conditions <u>or</u> could have interface

Snell's Law of Refraction: $\dfrac{\sin\theta}{c_1} = \dfrac{\sin\theta_T}{c_2}$ (7)

$$\underline{E}_R = (E_x, E_y, E_z)e^{\cdots}$$
$$\underline{E}_T = (\hat{E}_x, \hat{E}_y, \hat{E}_z)e^{\cdots}$$
$$\underline{E}_I = (0, E_I, 0)e^{\cdots}$$

$$E_I + E_y = \hat{E}_y \qquad (9)$$

(10, 11, 12)

y, z components of $\dfrac{1}{\mu}B$ continuous (*)

$$(1\cdot 40) \Longrightarrow \underline{B}_R = \frac{1}{\omega}\underline{k}_R \times \underline{E}_R$$

← distingoishing between ks

$$= \frac{k}{\omega}\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -\cos\theta & 0 & \sin\theta \\ E_x & E_y & E_z \end{vmatrix} e^{-i\omega t + i\underline{k}_R\cdot\underline{x}}$$

$$= \frac{1}{c_1}\left(-E_y\sin\theta,\ E_z\cos\theta + E_x\sin\theta,\ -E_y\cos\theta\right)e^{\cdots},$$

$$\underline{B}_T = \frac{1}{\omega}\underline{k}_T \times \underline{E}_T = \frac{k_T}{\omega}\begin{vmatrix} i & j & k \\ \cos\theta_T & 0 & \sin\theta_T \\ \hat{E}_x & \hat{E}_y & \hat{E}_z \end{vmatrix} e^{-i\omega t + i\underline{k}_T\cdot\underline{x}}$$

$$= \frac{1}{c_2}\left(-\hat{E}_y\sin\theta_T,\ \hat{E}_x\sin\theta_T - \hat{E}_z\cos\theta_T,\ \hat{E}_y\cos\theta_T\right)e^{\cdots}$$

y component of $(*)$:

$$\frac{1}{\mu_1 c_1}\left(E_z\cos\theta + E_x\sin\theta\right) = \frac{1}{\mu_2 c_2}\left(\hat{E}_x\sin\theta_T - \hat{E}_z\cos\theta_T\right) \qquad (13)$$

z-comp of $(*)$:

$$\frac{1}{\mu_1 c_1}\left(\left(E_I - E_y\right)\cos\theta\right) = \frac{1}{\mu_2 c_2}\hat{E}_y\cos\theta_T \qquad (14)$$

So (9) - (14) are six equations for six unknowns
E_x, E_y, E_z, $\hat{E}_y$, $\hat{E}_z$, $\hat{E}_x$

Notice (10) - (13) equations for
E_x, E_z, $\hat{E}_x$, $\hat{E}_z$ only;

$$(13) \Longrightarrow \frac{1}{\mu_1 c_1}\left(E_z\cos\theta + \overbrace{\frac{E_z\sin\theta}{\cos\theta}}^{\overset{\parallel}{E_x} \text{ by } (11)}\sin\theta\right)$$

$$= \frac{1}{\mu_2 c_2}\left(\underbrace{-\frac{\overbrace{\hat{E}_z}^{E_z \text{ by } (10)}\sin\theta_T}{\cos\theta_T}}_{=\hat{E}_x \text{ by } (12)}\sin\theta_T - \underbrace{E_z}_{\overset{\parallel}{\hat{E}_z} \text{ by } (10)}\cos\theta_T\right)$$

$$\frac{E_z}{\mu_1 c_1\cos\theta} = \frac{-E_z}{\mu_2 c_2\cos\theta_T} \Longrightarrow E_z = 0$$

$$\left(\frac{1}{\mu_1 c_1\cos\theta} + \frac{1}{\mu_2 c_2\cos\theta_T} > 0\right)$$

$$\Longrightarrow \boxed{0 = E_z \underset{\uparrow\,(10)}{=} \hat{E}_z \underset{\uparrow\,(12)}{=} \hat{E}_x \underset{\uparrow\,(11)}{=} E_x}$$

So we have left (9), (14) for E_y, $\hat{E}_y$

Remark: So like the incident field E_I, both reflected and transmitted electric fields have only y-component $\Longleftrightarrow$ "transverse electric" $\longleftrightarrow$ TE wave

(9) $\longrightarrow$ $E_I + E_y = \hat{E}_y$

(14) $\longrightarrow$ $\frac{1}{\mu_1 c_1}\left(E_I - E_y\right)\cos\theta = \frac{1}{\mu_2 c_2}\hat{E}_y \cos\theta_T$

(9) $\rightarrow$ (14) : $\underbrace{\frac{\cos\theta}{\mu_1 c_1}}_{=: m_1}\left(E_I - E_y\right) = \underbrace{\frac{\cos\theta_T}{\mu_2 c_2}}_{=: m_2}\left(E_I + E_y\right)$

$m_1 E_I - m_1 E_y = m_2 E_I + m_2 E_y \implies$

$$\boxed{\begin{aligned} E_y &= \frac{m_1 - m_2}{m_1 + m_2} E_I \\ \hat{E}_y &= E_y + E_I = \frac{2 m_1}{m_1 + m_2} E_I \end{aligned}}$$

$m_1 := \frac{\cos\theta}{\mu_1 c_1}$, $m_2 := \frac{\cos\theta_T}{\mu_2 c_2}$

If both media 1 and 2 are "non-magnetic", i.e. $\mu_1 = \mu_2 = \mu$ (so often the case physically) but $\varepsilon_1 \neq \varepsilon_2$ (so $c_1 \neq c_2$, $c = (\varepsilon\mu)^{-1/2}$)

$\implies$ $n_1 = \frac{c_0}{c_1}$, $n_2 = \frac{c_0}{c_2}$, refractive indices

$$E_y = \frac{n_1\cos\theta - n_2\cos\theta_T}{n_1\cos\theta + n_2\cos\theta_T} E_I \qquad (15)$$

$$\hat{E}_y = \frac{2 n_1 \cos\theta}{n_1\cos\theta + n_2\cos\theta_T} E_I \qquad (16)$$

Now via Snell's law (8), $n_1 \sin\theta = n_2 \sin\theta_T = D$

$$E_y = \frac{\frac{D}{\sin\theta}\cos\theta - \frac{D}{\sin\theta_T}\cos\theta_T}{\frac{D}{\sin\theta}\cos\theta + \frac{D}{\sin\theta_T}\cos\theta_T} E_I$$

$$= \frac{\sin\theta_T\cos\theta - \sin\theta\cos\theta_T}{\sin\theta_T\cos\theta + \sin\theta\cos\theta_T} E_I$$

$$E_y = \frac{\sin(\theta_T - \theta)}{\sin(\theta + \theta_T)} E_I \qquad (17)$$

$$\hat{E}_y = \frac{2\sin\theta_T\cos\theta}{\sin(\theta + \theta_T)} E_I \qquad (18)$$

similarly

After $\underline{E}_R$, $\underline{E}_T$ found $\Longrightarrow$ by (1·40)

$$\underline{B}_R = \frac{1}{\omega}\underline{k}_R \times \underline{E}_R, \qquad \underline{B}_T = \frac{1}{\omega}\underline{k}_T \times \underline{E}_T$$

▨

Total internal reflection

Snell's law (7) $\longleftrightarrow \frac{\sin\theta_T}{c_2} = \frac{\sin\theta}{c_1} \Longrightarrow$

$$\sin\theta_T = \frac{c_2}{c_1}\sin\theta = \frac{n_1}{n_2}\sin\theta$$

$\leftarrow$ notice reverse $\frac{c_2}{c_1} \rightarrow \frac{n_1}{n_2}$

Let $c_2 > c_1$ and $\frac{c_2}{c_1}\sin\theta > 1 \iff \theta > \sin^{-1}\left(\frac{c_1}{c_2}\right) =: \theta_c$ = "critical angle"

$\theta > \theta_c$ is when total internal reflection occurs:
All the above derived formulae still hold, however θ_T is now complex:

real number

$$\sin\theta_T = \frac{c_2}{c_1}\sin\theta =: \alpha > 1 \Longrightarrow$$

~~$\cos\theta_T$~~

$$\cos\theta_T = (1 - \sin^2\theta_T)^{1/2} = (1 - \alpha^2)^{1/2} = i\sqrt{\alpha^2 - 1} =: i\beta$$

$\beta > 0$

$\Longrightarrow$ from (15)

$$E_y = \frac{n_1\cos\theta - n_2\cos\theta_T}{n_1\cos\theta + n_2\cos\theta_T} E_I = \frac{n_1\cos\theta - i\beta n_2}{n_1\cos\theta + i\beta n_2} E_I$$

$\uparrow$

complex conjugates!

so same modulus

$|E_y| = |E_I|$ (since $n_1\cos\theta \pm i\beta n_2$ complex conjugates)

$\longleftrightarrow$ reflected & incident amplitudes are the same

$\Longrightarrow$ all energy reflects (although phases may change upon reflection)

[For $\theta < \theta_c$ (15) $\Longrightarrow |E_y| < |E_I| \longleftrightarrow$ reflected energy less than incident, with the difference going to transmitted energy]

above interface

For $\theta > \theta_c$, for $x > 0$ (in medium 2):

$$\underline{k}_T = k_T(\cos\theta_T, 0, \sin\theta_T)$$
$$= k_T(i\beta, 0, \alpha)$$

$$\underline{E}_T = \mathrm{Re}\left\{(0, \hat{E}_y, 0)\, e^{i\underline{k}_T\cdot\underline{x} - i\omega t}\right\}$$
$$= \mathrm{Re}\left\{(0, \hat{E}_y, 0)\, e^{-k_T\beta x + ik_T\alpha z - i\omega t}\right\}$$

$\hat{E}_y = |\hat{E}_y|e^{i\varphi}$

$$= \left(0, |\hat{E}_y|\underset{\substack{\uparrow\\ \text{exp. decay in } x}}{e^{-k_T\beta x}}\cos(\underbrace{\omega t - k_T\alpha z - \varphi}_{\Phi}), 0\right)$$

so $\underline{E}_T$ exponentially decays as $x \to +\infty$;

$$(1\cdot 40) \Longrightarrow \underline{B}_T = \frac{1}{\omega}\underline{k}_T \times \underline{E}_T$$

$$= \mathrm{Re}\left\{\frac{k_T}{\omega}\begin{vmatrix} i & j & k \\ i\beta & 0 & \alpha \\ 0 & \hat{E}_y & 0 \end{vmatrix} e^{-k_T\beta + ik_T\alpha z - i\omega t}\right\}$$

$$= \frac{e^{-k_T\beta x}}{c_2}\left(-\alpha|\hat{E}_y|\cos(\underbrace{\omega t - k_T\alpha z - \varphi}_{\Phi}), 0, \beta|\hat{E}_y|\sin\Phi\right)$$

energy?

Now for Poynting vector $\underline{S}_T$ (§1.2) for transmitted wave

$$\underline{S}_T = \frac{1}{\mu}\underline{E}_T \times \underline{B}_T = \frac{|\hat{E}_y|^2 e^{-2k_T\beta x}}{\mu c_2}\begin{vmatrix} i & j & k \\ 0 & \cos\phi & 0 \\ -\alpha\cos\phi & 0 & \beta\sin\phi \end{vmatrix}$$

$(\phi := \omega t - k_T\alpha z - \psi)$

$$= \frac{|\hat{E}_y|^2 e^{-2k_T\beta x}}{\mu c_2}\left(\beta\sin\phi\cos\phi,\ 0,\ \alpha\cos^2\phi\right)$$

For time averaged $\langle S_T \rangle = \frac{1}{T}\int_0^T \underline{S}_T(x,t)\,dt$

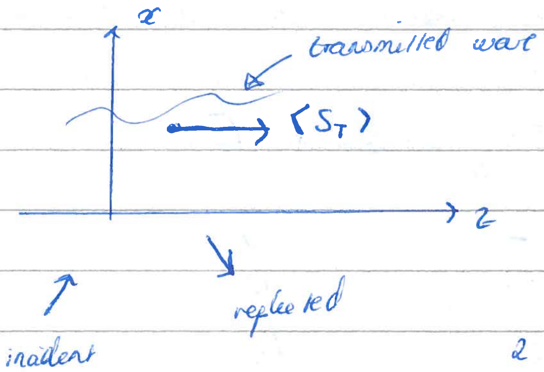
different T to S_T → $T = 2\pi/\omega$

$$\langle \underline{S}_T \rangle = \frac{|\hat{E}_y|^2 e^{-2k_T\beta x}}{\mu c_2}\left(0,\ 0,\ \frac{\alpha}{2}\right)$$

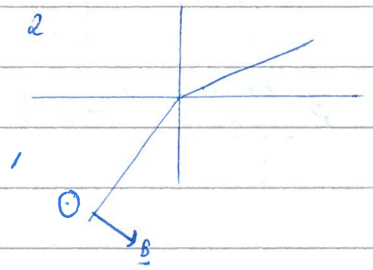
$$= \frac{\alpha|\hat{E}_y|^2 e^{-2k_T\beta z}}{2\mu c_2}(0,\ 0,\ 1)$$ unit vector in z direction

$\left(\text{Since} \quad \langle \sin\phi\cos\phi \rangle = 0, \quad \langle \cos^2\phi \rangle = \frac{1}{2}\right)$

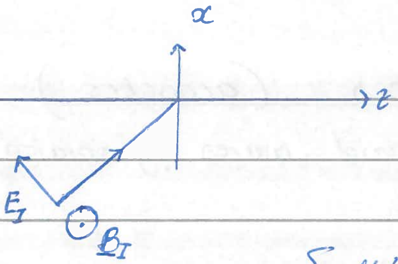
So the (averaged) energy flows z-direction i.e. along the boundary (no energy away from boundary)



▨



Exercise: Similar analysis can be done for incident plane wave polarised in (x,z)-plane



⟵ "transverse magnetic" ⟷ TM

(cf TE case above)

All similar conclusions e.g. $\theta_R = \theta$, Snell's law (7), total internal reflection for $\theta > \theta_c$ if $c_2 > c_1$

Difference: (17) - (18) replaced by

$$B_y = \frac{\sin 2\theta - \sin 2\theta_T}{\sin 2\theta + \sin 2\theta_T} B_I$$

$$\hat{B}_y = \frac{2\sin 2\theta}{\sin 2\theta + \sin 2\theta_T} B_I$$

For $\theta = \theta_B$ (the "Brewster angle") $B_y = 0 \Longleftrightarrow$
$\sin 2\theta = \sin 2\theta_T \Longrightarrow (\theta_T \neq \theta)\ 2\theta_T = \pi - 2\theta$
$\Longleftrightarrow \theta_T = \frac{\pi}{2} - \theta \Longrightarrow$ (using (7))

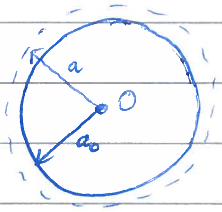
$$\theta_B = \tan^{-1}\left(\frac{c_1}{c_2}\right)$$

So for $\theta = \theta_B$ - No reflected wave present (all is transmitted); Exam 2010 Q2. ▨

2·4 Waves due to spherical sources (acoustics)

Here some fundamental examples of sound waves generated by "spherical sources" are considered

2·4·1 A radially pulsating sphere



Consider a sphere centred at origin O which "pulsates": its radius changes with time

$a = a(t)$

We assume $|a(t) - a_0| << a_0$, a_0 "original" radius i.e. the perturbations in the acoustic medium surrounding the sphere are small

Let the medium have wave speed c, density ρ_0

We expect an outgoing spherically-symmetric acoustic wave generated as a result outside the sphere, i.e.

$r \geq a_0$ $(r := |\underline{x}|)$ r is distance to origin

radially symmetric sol^n of wave eq^n →

$\underline{u} = \nabla\phi$ $\qquad \phi = \frac{f(r-ct)}{r}$

↑ velocity

Seek $f(r-ct) = -ca_0\psi(t - \frac{r}{c} + \frac{a_0}{c})$ ← Looking ahead, when $r = a_0$ last 2 terms cancel

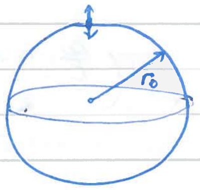
(ψ is still an arbitrary function to be found)

Boundary conditions (BCs): $r = a_0 \implies$

$u = a'(t)\tilde{\underline{r}} =: v_0(t)\tilde{\underline{r}}$ where $v_s(t) = a'(t)$ is 'sphere's radial velocity'

have previously used this notation → $\tilde{\underline{r}} := \frac{\underline{x}}{|\underline{x}|}$ = unit radial vector

02/03/15



$r(t)$

$$\phi = \frac{f(r-ct)}{r} \cancel{4} \underbrace{-ca_0\,\psi\left(t - \frac{r}{c} + \frac{a_0}{c}\right)}_{f(r-ct)}$$

$$\underline{u} = a'(t)\,\underset{\sim}{\hat{r}} = v_s(t)\,\underset{\sim}{\hat{r}}$$

↑ unit radial direction

$r = a_0$

$$\phi = -\frac{ca_0}{r}\,\psi\left(t - \frac{r}{c} + \frac{a_0}{c}\right) \implies$$

$$u = \nabla\phi = \frac{\partial\phi}{\partial r}\,\underset{\sim}{\hat{r}} = u_r\,\underset{\sim}{\hat{r}}$$

$$u_r = \frac{\partial\phi}{\partial r} = \frac{ca_0}{r^2}\,\psi\left(t - \frac{r}{c} + \frac{a_0}{c}\right) + \frac{a_0}{r}\,\psi'\left(t - \frac{r}{c} + \frac{a_0}{c}\right)$$

BCs: at $r = a_0$, $u_r = v_s(t)$

$$\implies \frac{c}{a_0}\,\psi(t) + \psi'(t) = v_s(t) \qquad (19)$$

which is an ODE for $\psi'(t)$

I.F: $e^{\frac{c}{a_0}t}$

Solve (19) using integrating factor $e^{\frac{c}{a_0}t}$

$$\frac{d}{dt}\left(e^{\frac{c}{a_0}t}\,\psi(t)\right) = e^{\frac{c}{a_0}t}\,v_s(t)$$

For causal solutions, $v_s(t) \equiv \psi(t) \equiv 0$ for $t \leq t_0$

$$\implies e^{\frac{c}{a_0}t}\,\psi(t) = \int_{t_0}^{t} e^{\frac{c}{a_0}\tau}\,v_s(\tau)\,d\tau, \qquad t \geq t_0$$

$$\iff \psi(t) = \int_{t_0}^{t} e^{-\frac{c}{a_0}(t-\tau)}\,v_s(\tau)\,d\tau$$

So ψ is found $\longrightarrow \phi \longmapsto \underline{u} = \nabla\phi$ (velocity found);

$$(1.11) \longleftrightarrow \frac{\partial\phi}{\partial t} = -\frac{1}{\rho_0}\,p \implies p = -\rho_0\frac{\partial\phi}{\partial t}$$

▨

For T-H case, $v_s(t) = \mathrm{Re}\left\{\hat{v}_s\,e^{-i\omega t}\right\}$, $\hat{v}_s \in \mathbb{C}$

Seek $\psi(t) = \mathrm{Re}\left\{B e^{-i\omega t}\right\}$, $B \in \mathbb{C}$
to be found

Plugging in to (19):

$$\mathrm{Re}\left\{\frac{c}{a_0}Be^{-i\omega t} - i\omega Be^{-i\omega t}\right\} = \mathrm{Re}\left\{\hat{v}_s\,e^{-i\omega t}\right\} \quad \forall t$$

Holds for $\forall t$ so erase Re parts

$$\Longrightarrow \quad B = \frac{a_0 \hat{v}_s}{c - i\omega a_0} = \frac{a_0 \hat{v}_s}{c(1 - ika_0)} \qquad \left(\frac{\omega}{c} = k\right)$$

$$\Longrightarrow \quad \phi = -\frac{c a_0}{r}\,\varphi\left(t - \frac{r}{c} + \frac{a_0}{c}\right)$$

$$= -\frac{c a_0}{r}\, \mathrm{Re}\left\{ B e^{-i\omega\left(t - \frac{r}{c} + \frac{a_0}{c}\right)} \right.$$

$$= \mathrm{Re}\left\{ -\frac{c a_0}{r} \frac{a_0 \hat{v}_s}{c(1 - ik a_0)} e^{-i\omega t + ikr - ika_0} \right\}$$

$$= \mathrm{Re}\left\{ \underbrace{-\frac{a_0^2 \hat{v}_s e^{-ika_0}}{1 - ika_0} \cdot \frac{e^{ikr - i\omega t}}{r}}_{= \hat{\phi}} \right\}$$

From $(1\cdot 42) \longleftrightarrow \hat{p} = i\omega \rho_0 \hat{\phi} \Longrightarrow$

$$p = \mathrm{Re}\left\{ \hat{p} e^{-i\omega t} \right\} = \mathrm{Re}\left\{ \underbrace{\frac{-i\omega \rho_0 a_0^2 \hat{v}_s e^{-ika_0}}{1 - ika_0}}_{= A} \cdot \frac{e^{ikr - i\omega t}}{r} \right\} \qquad (20)$$

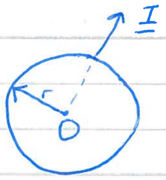
$$= \mathrm{Re}\left\{ A \frac{e^{ikr - i\omega t}}{r} \right\} \qquad A := \frac{-i\omega \rho_0 a_0^2 \hat{v}_s}{1 - ika_0}$$

So for time-averaged intensity $\langle \underline{I} \rangle$,

$$\langle \underline{I} \rangle = \langle I_r \rangle \underset{\sim}{r} \;; \qquad (1\cdot 23) \Longrightarrow \langle I_r \rangle = \frac{|A|^2}{2\rho_0 c r^2}$$

$$= \frac{\omega^2 \rho_0^2 a_0^4 |\hat{v}_s|^2 c}{2(1 + (ka_0)^2)\rho_0 c^2 r^2} \quad \leftarrow \text{multiplied by } \frac{c}{c}$$

$$= \frac{(ka_0)^2 a_0^2 |\hat{v}_s|^2 c \rho_0}{2(1 + t^2 a_0^2) r^2}$$



$\Longrightarrow$ The average radiated power:

$$\underline{P}_{av} = \langle I_r \rangle \times \underbrace{4\pi r^2}_{\text{Area of } S_r \,=\, \text{sphere of radius } r} \leftarrow \text{since } \langle I_r \rangle \text{ constant}$$

$$= \frac{2\pi (ka_0)^2 a_0^2 |\hat{v}_s|^2 c \rho_0}{1 + (ka_0)^2}$$

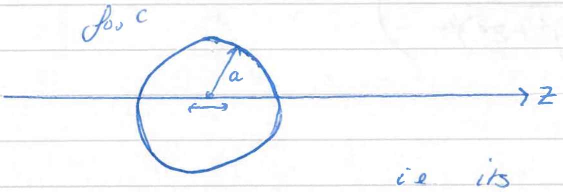
In 'low-frequency' regime $\longleftrightarrow ka_0 << 1$ $\left(\Longleftrightarrow \frac{\omega}{c} a_0 << 1 \Longleftrightarrow \omega << \frac{c}{a_0}\right.$, i.e. ω small$\left.\right)$

$$P_{av} \approx 2\pi (ka_0)^2 a_0^2 |\hat{v}_s|^2 c\rho_0 \qquad (21)$$

i.e. $P_{av} \sim \omega^2$, $k = \frac{\omega}{c}$ i.e. of "order" ω^2 in frequency

□

2.4.2 Transversely oscillating rigid sphere



Let sphere of radius a be rigid (i.e. a = constant) and oscillate along z-axis i.e its centre (and hence any other point) has velocity $\underline{v}_c(t) = v_c(t)\tilde{z}$
($\tilde{z} = \underline{e}_z$ = unit vector in z direction)
Assume the oscillations small.
In the surrounding acoustic medium ($r \geq a$), seek acoustic velocity $\underline{u} = \nabla\phi$, with

$$\phi = \frac{\partial}{\partial z}\left[\overbrace{\frac{1}{r}\psi\left(t-\frac{r}{c}+\frac{a}{c}\right)}^{=F(r)\ \text{(see later)}}\right] \qquad (22)$$

[Check: $\frac{1}{r}\psi\left(t-\frac{r}{c}+\frac{a}{c}\right)$ solves wave equation (1.12)]

$$\Longrightarrow \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2} - \nabla^2 w = 0$$

$$\Longrightarrow \frac{\partial}{\partial z}\left(\frac{\partial^2 w}{\partial t^2} - \nabla^2 w\right) = 0$$

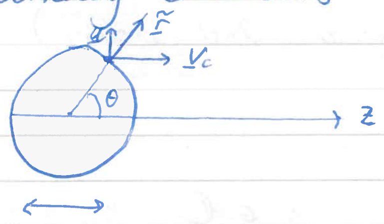
should there be $\frac{1}{c^2}$ here

$$\Longleftrightarrow \frac{\partial^2}{\partial t^2}\left(\frac{\partial w}{\partial z}\right) - \nabla^2\left(\frac{\partial w}{\partial z}\right) = 0$$

$\Longrightarrow \frac{\partial w}{\partial z}$ solves (1.12)

So (22) solves (1.12) ∀ smooth $\psi(\xi)$

Boundary conditions (BCs):



At $r = a$ (on the sphere's surface) acoustic medium velocity $\underline{u}$ relative to the sphere must be parallel to the surface S

$$\Longleftrightarrow (\underline{u} - \underline{v}_c)\cdot\tilde{r} = 0$$

$$\Longrightarrow \underline{u}\cdot\tilde{r} = \underline{v}_c\cdot\tilde{r}$$

$$\Longleftrightarrow \quad \frac{\partial\phi}{\partial r} = \nabla\phi\cdot\underline{r} = \underline{u}\cdot\hat{\underline{r}} = \underline{v}_c\cdot\hat{\underline{r}} = v_c\cos\Theta$$

Θ angle between $\underline{x}$ and z-axis $\Longrightarrow$

$$\frac{\partial\phi}{\partial r} = v_c(t)\cos\Theta \qquad r=a \qquad (23)$$

Notice that for 'radially-symmetric' $F = F(r)$

$$\frac{\partial F}{\partial z} = \frac{\partial r}{\partial z}\frac{\partial F}{\partial r}, \qquad r = (x^2+y^2+z^2)^{1/2}$$

$$\Longrightarrow \quad \frac{\partial r}{\partial z} = \frac{z}{r} \quad \left(= \frac{z}{(x^2+y^2+z^2)^{1/2}}\right)$$

look at diagr- → am to see this

$$= \cos\Theta$$

$$\Longrightarrow \quad \frac{\partial F(r)}{\partial z} = \cos\Theta\,\frac{\partial F}{\partial r} \qquad \text{So} \quad (22) \Longrightarrow$$

$$\Longrightarrow \quad \phi = \cos\Theta\,\frac{\partial}{\partial r}\left[\frac{1}{r}\psi\left(t-\frac{r}{c}+\frac{a}{c}\right)\right] \qquad (22')$$

$\Longrightarrow$ from (23)

$$\cos\Theta\,\frac{\partial^2}{\partial r^2}\left[\frac{1}{r}\psi\left(t-\frac{r}{c}+\frac{a}{c}\right)\right]_{r=a} = v_c(t)\cos\Theta \qquad (23')$$

Now $$\frac{\partial}{\partial r}\left[\left(\frac{1}{r}\psi\left(t-\frac{r}{c}+\frac{a}{c}\right)\right)\right] = -\frac{1}{r^2}\psi(\ldots) - \frac{1}{cr}\psi'\left(t-\frac{r}{c}+\frac{a}{c}\right)$$

$$\frac{\partial^2}{\partial r^2}\left[\frac{1}{r}\psi(\ldots)\right] = \frac{2}{r^3}\psi\left(t-\frac{r}{c}+\frac{a}{c}\right) + \frac{2}{cr^2}\psi'(\ldots) + \frac{1}{c^2 r}\psi''(\ldots)$$

So $(23') \Longrightarrow$ set $r=a$ since this is BC

$$\frac{2}{a^3}\psi(t) + \frac{2}{ca^2}\psi'(t) + \frac{1}{c^2 a}\psi''(t) = v_c(t)$$

$$2c^2\psi + 2ca\psi' + a^2\psi'' = c^2a^3 v_c(t) \qquad (24)$$

which is an ODE for ψ (linear 2nd order)

Returning to T-H case, let

$$v_c(t) = \mathrm{Re}\left\{\hat{v}_c e^{-i\omega t}\right\} \qquad \hat{v}_c \in \mathbb{C}$$

used B previously, changed to A so it cancels later

and seek $\psi(t) = Re\left[Ae^{-i\omega t + ika}\right]$ $\qquad A \in \mathbb{C}$

$$(24) \Longrightarrow (2c^2A - 2i\omega caA - \omega^2a^2A)e^{-i\omega t+ika}$$
$$= c^2a^3\hat{V}_c e^{-i\omega t}$$

$$A = \frac{c^2a^3\hat{V}_c e^{-ika}}{2c^2 - 2i\omega ca - \omega^2a^2}$$

$$A = \frac{a^3\hat{V}_c e^{-ika}}{2 - 2ika - k^2a^2} \qquad \left(\frac{\omega}{c} = k\right) \qquad (25)$$

$$\Longrightarrow \psi(t) = Re\left[Ae^{-i\omega t+ika}\right] \Longrightarrow \text{via } (22')$$

$$\phi = \cos\theta \frac{\partial}{\partial r}\left[\frac{1}{r} Re\left\{Ae^{-i\omega(t-\frac{r}{c}+\frac{a}{c})+ika}\right\}\right]$$

$\uparrow$

replacing t by funny argument

$$= Re\left\{\cos\theta \frac{\partial}{\partial r}\left[\frac{A}{r} e^{-i\omega t+ikr-ika+ika}\right]\right\}$$

$$\phi = Re\left\{\underbrace{\cos\theta \frac{\partial}{\partial r}\left(\frac{A}{r}e^{ikr}\right)}_{\hat{\phi}} e^{-i\omega t}\right\} \qquad (26)$$

(1·42)

$$\hat{p} = i\omega\rho_0\hat{\phi} = i\omega\rho_0\cos\theta \frac{A}{r}e^{ikr}\left(-\frac{1}{r} + ik\right)$$

$$\hat{p} = -k\omega\rho_0\cos\theta\frac{A}{r}e^{ikr}\left(1 + \frac{i}{kr}\right) \qquad (27)$$

In the "far field" ($\Longleftrightarrow kr >> 1 \longleftrightarrow r >> \frac{1}{k}$) 'many wavelengths'

$$\hat{p} \approx -k^2c\rho_0\cos\theta\frac{A}{r}e^{ikr} \qquad (28)$$

For radial component of velocity u_r

$$u_r = \underline{u}\cdot\hat{\underline{r}} = \frac{\partial\phi}{\partial r} = Re\left\{\cos\theta\frac{\partial^2}{\partial r^2}\left(\frac{A}{r}e^{ikr}\right)e^{-i\omega t}\right\}$$

In the far field

In the far field:

$$u_r = Re\left\{-k^2\frac{A}{r}e^{ikr}\left(1+\overbrace{\frac{2i}{kr}-\frac{2}{(kr)^2}}^{\text{using Leibniz rule for differentiation}}\right)\cos\theta\, e^{-i\omega t}\right\}$$

$$u_r \approx \frac{-k^2}{r}\cos\theta\, Re\left\{A e^{ikr-i\omega t}\right\}$$

complex phase

$$A = |A|e^{i\psi}$$

For time-averaged radial acoustic intensity

$$\langle I_r\rangle = \langle p u_r\rangle \approx \frac{k^4 c\rho_0\cos^2\theta\,|A|^2}{r^2}\langle\underbrace{\cos^2(\omega t-kr-\overset{\text{from } A}{\psi})}_{=\frac{1}{2}}\rangle$$

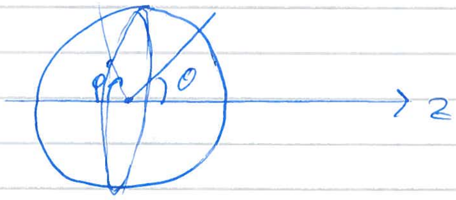
↑ we can see that p will have this form since $\hat{p}$ has e^{ikr} $(kr >> 1)$

$$\langle I_r\rangle = \frac{k^4 c\rho_0|A|^2\cos^2\theta}{2r^2}$$

← not a constant (unlike before) so need to be careful

The average radiated power: take S_r, sphere of a large radius r ⟹

$$P_{ar} = \iint_{S_r}\langle I_r\rangle dS_r$$



$$= \frac{k^4 c\rho_0|A|^2}{2\cancel{r^2}}\int_0^{2\pi}\int_0^{2\pi}\cos^2\theta\,\underbrace{\cancel{r^2}\sin\theta\,d\varphi\,d\theta}_{dS}$$

$$= \frac{\cancel{1}}{\cancel{2}}k^4 c\rho_0|A|^2\cancel{2}\pi\int_0^{\pi}\cos^2\theta\sin\theta\,d\theta$$

$$= -\pi k^4 c\rho_0|A|^2\left[\frac{1}{3}\cos^3\theta\right]_0^{\pi}$$

$$= \frac{2}{3}\pi k^4 c\rho_0|A|^2$$

(25) ⟷ $A = \dfrac{a^3\hat{v}_c e^{-ika}}{2-2ika-(ka)^2}$

since complex we do not know it so take modulus

$$\Longrightarrow |A|^2 = \frac{a^6|\hat{v}_c|^2}{\cancel{2-2ika-(ka)^2}}$$

← woops! written again below.

$$= \frac{a^6|\hat{v}_c|^2}{(2-k^2a^2)^2+4k^2a^2}$$

$$= \frac{a^6 |\hat{v}_c|^2}{4 + (ka)^4} \Longrightarrow$$

$$P_{av} = \frac{2}{3} \pi k^4 c \rho_0 \frac{a^6 |\hat{v}_c|^2}{4 + (ka)^4}$$

$$P_{av} = \frac{2}{3} \pi \frac{(ka)^4 c \rho_0 a^2 |\hat{v}_c|^2}{4 + (ka)^4} \qquad (29)$$

In low frequency regime $\longleftrightarrow$ $ka << 1$

$$P_{av} \approx \frac{\pi}{6} (ka)^4 c \rho_0 a^2 |\hat{v}_c|^2 \qquad (30)$$

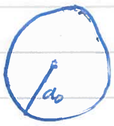
Intuitively transversally osc. sphere produces less energy than pulsating

So the radiated power is "of order 4" with respect to frequency $(ka)^4 = (\omega c a)^4$

End of chapter 2!

3. Green's functions; Multiple expansions 09/08/15

3·1 Monopole & free-space Green's function



radially symmetric T-H

velocity ↓

Consider ~~spherically~~ pulsating sphere (radius a_0; $v_S(t)$

$v_S(t) = \mathrm{Re}\{\hat{v}_S e^{-i\omega t}\}$)

see ~~[illegible]~~ §2·4·1

$\Longrightarrow p = \mathrm{Re}\{\hat{p} e^{-i\omega t}\}$, where; see (2·20),

defined in whole space w/ singularity at origin →

$$\hat{p} = \hat{S}\frac{e^{ikr}}{r} \quad (1)$$

← if have outgoing spherically symm. wave this must be the case

$r \geqslant a_0$, and $\hat{S} = \dfrac{-i\omega\rho_0 a_0^2 \hat{v}_S}{1-ika_0} e^{-ika_0}$

Let $a_0 \longrightarrow 0$ and $\hat{v}_S$ increases, so that $\hat{S}$ remains constant

In the "limit" we have $\hat{p}$ defined by (1) everywhere except $r=0$, with a singularity at $r=0$

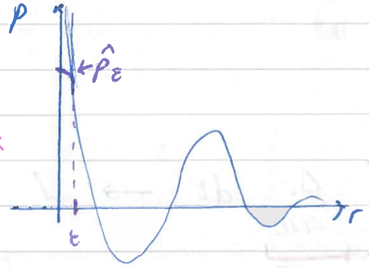
Such $\hat{p}$ is called to be due to a "point source" (in reality, a source can be regarded as a point source if:

(i) $a_0 << \lambda = \frac{2\pi}{k}$ (= wavelength) $\Longleftrightarrow a_0 k << 1$

(ii) Observation point, $r >> a_0$)

So $\hat{p}$ given by (1) solves Helmholtz equation $(\nabla^2 + k^2)\hat{p} = 0$ for $r \neq 0$

should have → looked at Re & Im parts separately since $\hat{p}$ is complex valued



For a small $\varepsilon > 0$, let $\hat{p}_\varepsilon(r) := \hat{p}(r)$ for $r \geqslant \varepsilon$, but $\hat{p}_\varepsilon$ is "smoothed out" for $r < \varepsilon$ $\Longrightarrow$

$$(\nabla^2 + k^2)\hat{p}_\varepsilon = -\hat{S}\Delta_\varepsilon(r) \quad (2)$$

in contrast to $\hat{p}$ is H eq^n

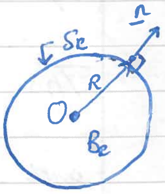
where RHS $\Delta_\varepsilon(r) \neq 0$ only for $r < \varepsilon$ ($\Longleftrightarrow$ has 'support' in ball B_ε of radius ε centred at $r=0$).

Let $\varepsilon \to 0$, fix $R > 0$ and integrate (2) over the ball B_ε:

delta epsilon

$$\int_{B_\varepsilon} \nabla^2 \hat{p}_\varepsilon \, dV + \int_{B_R} k^2 \hat{p}_\varepsilon \, dV = -\hat{S}\int_{B_\varepsilon} \Delta_\varepsilon(r) \, dV$$

$\Longrightarrow I_1 + I_2 = I_3$



divergence thm

$$I_1 = \int_{B_R} \nabla\cdot(\nabla\hat{p}_\varepsilon) \, dV \overset{\downarrow}{=} \int_{S_R} \nabla\hat{p}_\varepsilon \cdot \underline{n} \, dS$$

sphere is surface bounding ball

$$= \int_{S_R} \frac{\partial \hat{p}_\varepsilon}{\partial n} \, dS = \int_{S_R} \frac{\partial}{\partial r}\left(\hat{S}\frac{e^{ikr}}{r}\right) dS$$

$$= \hat{S} \int_{S_R} e^{ikr}\left(\frac{ik}{r} - \frac{1}{r^2}\right) dS$$

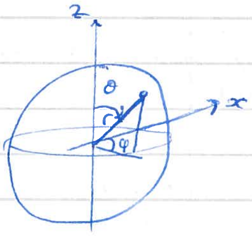
$$= \hat{S}\, e^{ikR}\left(\frac{ik}{R} - \frac{1}{R^2}\right) \times 4\pi R^2$$

$$= \underline{\underline{4\pi \hat{S} e^{ikR}(ikR - 1)}}$$

want to take lim $\varepsilon \to 0$ eventually so can attempt to take it here

$$I_2 = \int_{B_R} k^2 \hat{\rho}_\varepsilon \, dV \longrightarrow \int_{B_R} k^2 \hat{\rho} \, dV$$

$$= \hat{S} k^2 \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \frac{e^{ikr}}{r} \underbrace{r^2 \sin\theta \, dr\, d\theta\, d\varphi}_{dV \text{ in sph. coords}}$$



$$\int_0^{2\pi}\int_0^{\pi} \sin\theta \, d\theta\, d\varphi = 4\pi$$

(r, θ, φ) spherical coordinates

$$I_2 = 4\pi k^2 \hat{S} \int_0^R e^{ikr} r\, dr = -4\pi i k \hat{S} \int_0^R \frac{d}{dr}\left(e^{ikr}\right) r\, dr$$

$$= -4\pi i k \hat{S} e^{ikR} R + 4\pi i k \hat{S} \int_0^R e^{ikr} dr$$

$$= \underline{\underline{-4\pi \hat{S} e^{ikR} ikR + 4\pi \hat{S}\left(e^{ikR} - 1\right)}}$$

since one integral evaluated exactly & other one was limit

$$\Longrightarrow \quad I_1 + I_2 \xrightarrow{\varepsilon \to 0} 4\pi \hat{S} e^{ikR}(ikR - 1 - ikR + 1) - 4\pi \hat{S}$$

$$= -4\pi \hat{S}$$

$$\Rightarrow \quad I_3 \to -4\pi \hat{S} \quad \text{as} \quad \varepsilon \to 0$$

$$-\hat{S} \int_{S_R} \Delta_\varepsilon \, dV \to -4\pi \hat{S} \iff \int_{B_R} \underbrace{\frac{\Delta_\varepsilon}{4\pi}}_{\delta_\varepsilon} dV \to 1$$

Hence denoting $\delta_\varepsilon := \frac{1}{4\pi} \Delta_\varepsilon$ we observe :

(i) $\int_{\mathbb{R}^3} \delta_\varepsilon \, dV \longrightarrow 0$ as $\varepsilon \to 0$

nothing special about radius R - can integ. over whole space

(ii) $\delta_\varepsilon(\underline{x}) = \delta_\varepsilon(|\underline{x}|) = \delta_\varepsilon(r) \neq 0$ only within $r < \varepsilon$

This implies $\lim_{\varepsilon \to 0} \delta_\varepsilon(x) =: \delta(x)$ called 'Dirac delta-function'

with the "limit" understood appropriately

So, $\delta(x)$ can be viewed as a function such that

(i) $\delta(\underline{x}) = \begin{cases} 0 & \underline{x} \neq \underline{0} \\ +\infty & \underline{x} = 0 \end{cases}$

($\mathbb{R}^3$ = whole space)

ii) $\int_{\mathbb{R}^3} \delta(\underline{x})\, dV = 1$

A rigorous definition requires more advanced analysis ⟷ "distribution theory"

One important property of δ-functions often called "sifting property": $\forall$ 'reasonable' (e.g. continuous) functions f

$$\int_{\mathbb{R}^3} \delta(\underline{x}-\underline{x}')\, f(\underline{x}')\, \overset{\text{integrate w/ } x'}{dV(\underline{x}')} = \int \delta(\underline{x}'-\underline{x})\, f(\underline{x}')\, dV(\underline{x}')$$

only 'sifts out' value $\underline{x}$ ⟶ $= f(\underline{x})$ (3)

(Follows from 'definition of $\delta(\underline{x})$: $\delta(\underline{x}-\underline{x}') \neq 0 \Rightarrow \underline{x}' = \underline{x}$ ⟶ only $f(\underline{x})$ matters ⟷ 'sifts in')

Hence for $\hat{p}$ given by (1), as $\varepsilon \to 0$,
$(\nabla^2 + k^2)\hat{p} = -\hat{S}\, 4\pi\, \delta(\underline{x})$;

More generally, for a point source at $\underline{x}_s$,

$$\hat{p}(\underline{x}) = \hat{S}\, \frac{e^{ikR}}{R}, \quad R := |\underline{x} - \underline{x}_s| \qquad (1')$$

← new origin

$$\Longrightarrow (\nabla^2 + k^2)\hat{p} = -4\pi \hat{S}\, \delta(\underline{x} - \underline{x}_s) \qquad (4)$$

Definition $\hat{S} \in \mathbb{C}$ referred to as a "monopole amplitude"

Green's functions: The solution of (4) with RHS $= \delta(\underline{x} - \underline{x}_s)$ i.e. $\hat{S} = -\frac{1}{4\pi}$, is called the free-space Green's function $G_f(\underline{x}, \underline{x}')$:

for dash instead of s

green's free space

$$(\nabla^2 + k^2)\, G_f(\underline{x}, \underline{x}') = \delta(\underline{x} - \underline{x}') \qquad (5)$$

Hence, from (1'), $\hat{S} = -\frac{1}{4\pi}$,

$$G_f(\underline{x}, \underline{x}') = -\frac{e^{ik|\underline{x}-\underline{x}'|}}{4\pi|\underline{x}-\underline{x}'|} \qquad (6)$$

← replacing R by $|\underline{x}-\underline{x}'|$ to be more explicit

(Free-space Green's functions for general linear partial differential equations (PDEs) with constant coefficients are often also called "fundamental solutions")

3·2 Green's functions

Green's fns play a fundamental role mathematically

Consider a field due to an arbitrary distributed source $f(\underline{x})$:

$$\mathcal{L}\hat{p} := (\nabla^2 + k^2)\hat{p} = f(\underline{x}) \qquad (7)$$

where $\mathcal{L}$ is Helmholtz 'differential operator'

$\mathcal{L} : \hat{p} \longrightarrow f$; more generally $\mathcal{L}$ could be *wave eqn, Maxwell's systems related operator etc.

Normally (7) is supplemented by boundary/initial etc. (e.g. Sommerfeld radiation condition) to form a boundary-valued problem (BVP), to determine p from f uniquely, [illegible] symbolically $\hat{p} = \mathcal{L}^{-1} f$

Assume $\mathcal{L}$ invertible. Seek $\mathcal{L}^{-1}$ as an 'integral* operator'

$$\hat{p}(\underline{x}) = (\mathcal{L}^{-1} f)(\underline{x}) = \int_{\Omega} K(\underline{x}, \underline{x}') f(\underline{x}')\, dV(\underline{x}')$$

Ω is domain of p, f;

$K(\underline{x}, \underline{x}')$ "kernel" to be found

Now, $f(\underline{x}) = (\mathcal{L}\hat{p})(\underline{x}) = \mathcal{L}(\mathcal{L}^{-1} f)(\underline{x})$

$$= \mathcal{L}_x \int_{\Omega} K(\underline{x}, \underline{x}') f(\underline{x}')\, dV(\underline{x}')$$

subscript x

$$= \int_{\Omega} \mathcal{L}_x K(\underline{x}, \underline{x}') f(\underline{x}')\, dV(\underline{x}')$$

we assume we can move $\mathcal{L}_x$ inside

The Green's function serves the purpose

Set $K(\underline{x}, \underline{x}') = G(\underline{x}, \underline{x}') \Longrightarrow$

$$\mathcal{L}_x K(\underline{x}, \underline{x}') = \mathcal{L}\, G(\underline{x}, \underline{x}')) \overset{\text{by (5)}}{=} \delta(\underline{x} - \underline{x}')$$

$$\int_{\Omega} \mathcal{L}_x K(\underline{x}, \underline{x}') f(\underline{x}')\, dV(\underline{x}') = \int_{\Omega} \delta(\underline{x} - \underline{x}') f(\underline{x}')\, dV(\underline{x}')$$

$$\overset{\text{by (3)}}{=} f(\underline{x}) \quad \text{as required}$$

$$(\mathcal{L}^{-1} f)(\underline{x}) = \int_{\Omega} G(\underline{x}, \underline{x}') f(\underline{x}')\, dV(\underline{x}') \qquad (8)$$

For Helmholtz equation on domains with boundaries the Green's function is generally in the form:

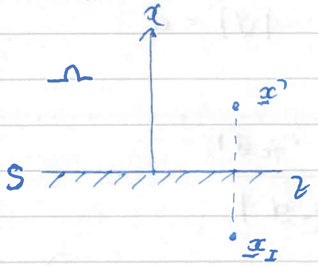
$$G(\underline{x}, \underline{x}') = G_f(\underline{x}, \underline{x}') + G_B(\underline{x}, \underline{x}') \quad \text{with}$$

G_f free-space Green's function (6) and G_B solving:

$(\nabla^2 + k^2) G_B = 0$ in Ω;

$G_f + G_B$ satisfy the boundary conditions (BCs)

Example Ω half space $= \{(x, y, z) : x \geq 0\}$



method of images →

with acoustically soft boundary $S = \{x = 0\}$

Let $\underline{x}' = (x', y', z')$, $x' > 0$, a "source point". Consider an "image" $\underline{x}_I = (-x', y', z') \Longrightarrow x_I \notin \Omega$; and take $G_B(\underline{x}, \underline{x}') = -G_f(\underline{x}, \underline{x}_I)$

(i) $(\nabla^2 + k^2) G_B \overset{(8)}{=} -(\nabla^2 + k^2) G_f(\underline{x}, \underline{x}_I) \overset{(5)}{=} -\delta(\underline{x} - \underline{x}_I) = 0$
in Ω $(\delta(\underline{x} - \underline{x}_I) = 0 \ \forall \ \underline{x} \neq \underline{x}_I \notin \Omega)$

(ii) BC: $\underline{x} \in S \Longleftrightarrow \underline{x} = (0, y, z) \Longrightarrow$
$G(\underline{x}, \underline{x}') = G_f(\underline{x}, \underline{x}') - G_f(\underline{x}, \underline{x}_I)$
dist. is all that matters $(6) \rightarrow = G_f(|\underline{x} - \underline{x}'|) - G_f(|\underline{x} - \underline{x}_I|) = 0$

since $|\underline{x} - \underline{x}'| = |\underline{x} - \underline{x}_I| \Longleftrightarrow G(\underline{x}, \underline{x}') = 0 \ \forall \ \underline{x} \in S \longleftrightarrow$

Acoustically soft (Dirichlet) BC ☐

Neumann rather than Dirichlet

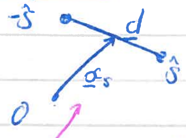
Exercise: For acoustically hard BCs, $\frac{\partial \hat{p}}{\partial n} = \underline{n} \cdot \nabla \hat{p} = 0$ on S. Show that
$G(\underline{x}, \underline{x}') = G_f(\underline{x}, \underline{x}') + G_f(\underline{x}, \underline{x}_I)$

Exam 2012 Q4(b) ☐

Monopole amplitude $\hat{S}$

3.3 Dipoles and Quadrupoles

Dipoles



position vector of midpoint

Take two 'monopole' (= point) sources a small distance d apart, with amplitudes $\hat{S}$ and $-\hat{S}$ i.e. oscillating "in anti phase"

Let $\pm\hat{S}$ be at $\underline{x}_s \frac{\pm 1}{2}\underline{d}$ where $|\underline{d}| = d$

Resulting acoustic field is:

$$\hat{p}(\underline{x}) \underset{(1')}{=} \hat{S}\frac{e^{ik|\underline{x}-\underline{x}_s-\frac{1}{2}\underline{d}|}}{|\underline{x}-\underline{x}_s-\frac{1}{2}\underline{d}|} - \hat{S}\frac{e^{ik|\underline{x}-\underline{x}_s+\frac{1}{2}\underline{d}|}}{|\underline{x}-\underline{x}_s+\frac{1}{2}\underline{d}|}$$

$$\overset{(6)}{=} -4\pi\hat{S}\,G_f(\underline{x},\underline{x}_s+\tfrac{1}{2}d) + 4\pi\hat{S}\,G_f\,\hat{S}G_f\,(\underline{x},\underline{x}_s-\tfrac{1}{2}\underline{d}\ )$$

Let $|\underline{d}| = d$ be small $\longleftrightarrow$ $kd \ll 1$ $\Longrightarrow$

Using Taylor series,

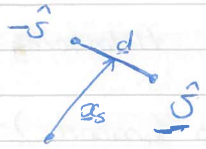
$$\hat{p}(\underline{x}) = -4\pi\hat{S}G_f(\underline{x},\underline{x}_s) - 4\pi\hat{S}\,\nabla_s G_f(\underline{x},\underline{x}_s)\cdot\tfrac{1}{2}\underline{d}$$

$$+ 4\pi\hat{S}G_f(\underline{x},\underline{x}_s) - 4\pi\hat{S}\,\nabla_s G_f(\underline{x},\underline{x}_s)\cdot\tfrac{1}{2}\underline{d} + O(d^2)$$

$$= -4\pi\hat{S}\underline{d}\cdot\nabla_s G_f\ (\underline{x},\underline{x}_s) + O(d^2) \qquad (9)$$

he wrote x' here

16/02/15



(monopole amplitude)

$$\hat{p}(\underline{x}) = -4\pi \underbrace{\hat{S}\underline{d}}_{\hat{\underline{\mathcal{D}}}} \cdot \nabla_s G_f(\underline{x}, \underline{x}_s) + O(d^2)$$

To main order, $\hat{p}(\underline{x}) = -4\pi \hat{\underline{\mathcal{D}}} \cdot \nabla_s G_f(\underline{x}, \underline{x}_s)$ (9)

where $\hat{\underline{\mathcal{D}}} := \hat{S}\underline{d}$ is the dipole amplitude vector

(6) $\implies$

$$G_f(\underline{x}, \underline{x}') = \frac{-e^{ik|\underline{x}-\underline{x}'|}}{4\pi|\underline{x}-\underline{x}'|}$$

Since $G_f(\underline{x}, \underline{x}_s) = G_f(\underline{x} - \underline{x}_s)$, see (6)

$\implies \nabla_s G_f = -\nabla G \implies$

$$\hat{p}(\underline{x}) = 4\pi \hat{\underline{\mathcal{D}}} \cdot \nabla G_f(\underline{x}, \underline{x}_s) \overset{(6)}{=} 4\pi \hat{\underline{\mathcal{D}}} \cdot \left(-\frac{1}{4\pi} \frac{e^{ikR}}{R}\right)$$

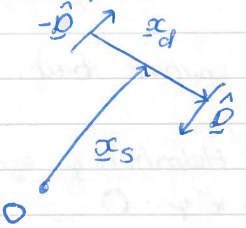
think he meant to put ∇ here

$$\hat{p}(\underline{x}) = -\hat{\underline{\mathcal{D}}} \cdot \nabla\left(\frac{e^{ikR}}{R}\right) \qquad R := |\underline{x} - \underline{x}_s| \qquad (9')$$

$$\hat{p} = -\frac{d}{dR}\left(\frac{e^{ikR}}{R}\right) \hat{\underline{R}} \cdot \hat{\underline{\mathcal{D}}} \qquad \hat{\underline{R}} := \frac{\underline{x} - \underline{x}_s}{|\underline{x} - \underline{x}_s|} \qquad (9'')$$

(9) / (9') / (9'') are 'dipole fields'

Quadrupole



Consider now two dipoles a small distance $d = |\underline{d}|$ apart, with opposite dipole vectors $\pm\hat{\underline{\mathcal{D}}} \implies$ (for small $d \iff kd \ll 1$)

$$\hat{p}(\underline{x}) \overset{(9)}{=} -4\pi \hat{\underline{\mathcal{D}}} \cdot \nabla_s G_f\left(\underline{x}, \underline{x}_s + \frac{1}{2}\underline{d}\right)$$

$$+ 4\pi \hat{\underline{\mathcal{D}}} \cdot \nabla_s G_f\left(\underline{x}, \underline{x}_s - \frac{1}{2}\underline{d}\right)$$

Taylor

$$\approx -4\pi(\underline{d} \cdot \nabla_s)(\hat{\underline{\mathcal{D}}} \cdot \nabla_s) G_f(\underline{x}, \underline{x}_s)$$

$\nabla_s = -\nabla$

$$= \cancel{-}4\pi(\underline{d} \cdot \nabla)(\hat{\underline{\mathcal{D}}} \cdot \nabla)\left(\cancel{-}\frac{1}{\cancel{4\pi}} \frac{e^{ikR}}{R}\right)$$

$$= \sum_{\alpha,\beta=1}^{3} Q_{\alpha\beta} \frac{\partial^2}{\partial x_\alpha \partial x_\beta}\left(\frac{e^{ikR}}{R}\right) \qquad (10)$$

where $Q_{\alpha\beta} = d_\alpha \hat{\mathcal{D}}_\beta$ $\qquad R = |\underline{x} - \underline{x}_s|$ $\qquad \underline{x} = (x_1, x_2, x_3)$

(10) is a 'quadropole' field $\longleftrightarrow$ combination of 2nd derivatives of a 'monopole' field e^{ikR}/R

3.4 Kirchhoff - Helmholtz Integral Theorem

This is an important integral representation for solutions of Helmholtz equations

Based on Green's formula / identity :

Let f, g be two functions in domain/volume V with (smooth) boundary S ; Apply divergence thm to

$$\underline{F} = f\nabla g - g\nabla f \implies$$

$$\int_V \nabla\cdot\underline{F}\, dV = \int_S \underline{F}\cdot\underline{n}\, dS$$

laplacian - divergence of gradient

$$\nabla\cdot\underline{F} = \nabla f\cdot\nabla g + f\nabla^2 g - \nabla g\cdot\nabla f - g\nabla^2 f$$

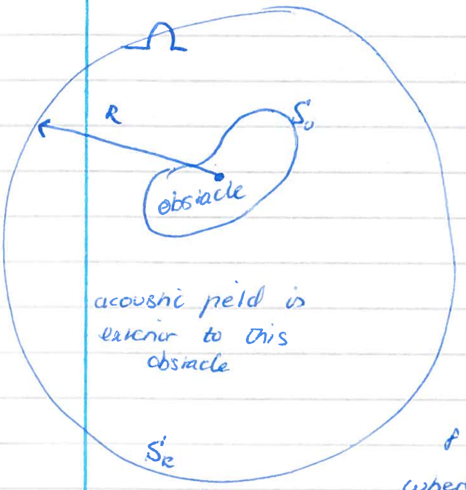
$$\underline{F}\cdot\underline{n} = f\,\underline{\nabla g\cdot\underline{n}} - g\nabla f\cdot\underline{n} = f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n} \implies$$

directional derivative

$$\boxed{\int_V \left(f\nabla^2 g - g\nabla^2 f\right) dV = \int_S \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right) dS} \qquad (11)$$

← Important!

(11) is the Green's identity / formula



this is supposed to be a circle

Consider a T-H, outgoing, acoustic field outside an obstacle with a boundary S_0 in a domain Ω exterior to

Let V_R be a part of Ω within ball B_R of a large radius R

Let ψ be a solution of Helmholtz equation (1.9) or (1.12) $\nabla^2\psi + k^2\psi = 0$

Apply (11) for : $V = V_R$ (part exterior to obstacle)

$f(\underline{x}) = \psi(\underline{x})$, $g(\underline{x}) = G(\underline{x}, \underline{x}')$

where $G(\underline{x}, \underline{x}')$ is the Green's function, $\underline{x}' \in V_R$

$\implies S = S_0 \cup S_R \implies$

wrt $\underline{x}$ ($\underline{x}'$ is parameter)

$$\int_{V_R} \left[\psi(\underline{x})\nabla^2 G(\underline{x},\underline{x}') - G(\underline{x},\underline{x}')\nabla^2\psi(\underline{x})\right] dV(\underline{x})$$

$$= \int_{S_0\cup S_R} \left[\psi(\underline{x})\frac{\partial G(\underline{x},\underline{x}')}{\partial n(\underline{x})} - G(\underline{x},\underline{x}')\frac{\partial\psi(\underline{x})}{\partial n}\right] dS(\underline{x})$$

(5) ⟷ $\nabla^2 G + k^2 G = \delta(\underline{x} - \underline{x}')$

$$LHS \overset{(5),(1.9)}{=} \int_{V_R} \psi(\underline{x})\left(-k^2 G + \delta(\underline{x}-\underline{x}')\right) + G\left(k^2\psi\right) dV$$

$$= \int_{V_R} \psi(\underline{x})\,\delta(\underline{x}-\underline{x}')\, dV(\underline{x}) \overset{(3)}{=} \psi(\underline{x}') ;$$

RHS : Show that, as $R \to \infty$, $\int_{S_R} \ldots \longrightarrow 0$, provided ψ

satisfies also Sommerfeld Radiation Condition (1.20)

$$\longleftrightarrow\ r\left(\frac{\partial\psi}{\partial r} - ik\psi\right) \to 0 \quad \text{as} \quad r = |\underline{x}| \to \infty$$

$\Longleftrightarrow \quad \frac{\partial \psi}{\partial r} = ik\psi + o(\frac{1}{r}) \qquad [\, o(\frac{1}{r})$, 'o-small' $\leftrightarrow o(\frac{1}{r}) \times r \to 0 \quad r \to \infty\,)$

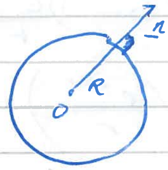
Notice that G, given by (6) also satisfies (1·20):

(6) $\longleftrightarrow$

$$G(\underline{x}, \underline{x}') = -\frac{e^{ik|\underline{x}-\underline{x}'|}}{4\pi|\underline{x}-\underline{x}'|} \overset{|\underline{x}|=r\to\infty}{\approx} -\frac{e^{ikr}}{4\pi r} \qquad (r \to \infty)$$

Also $G = O(\frac{1}{r})$ as $r \to \infty$

Also $\psi = O(\frac{1}{r})$ as $r \to \infty$ $\longleftrightarrow$ (1·20') $\Longrightarrow$



Using also $\frac{\partial}{\partial n} = \frac{\partial}{\partial r}$ on S_R,

$$\int_{S_R} \psi \frac{\partial G}{\partial n} - G \frac{\partial \psi}{\partial n} = \int_{S_R} \psi \frac{\partial G}{\partial r} - G \frac{\partial \psi}{\partial r}$$

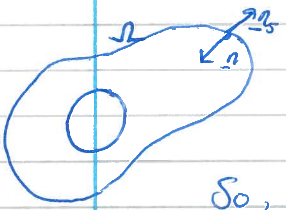
$$= \int_{S_R} \underset{O(r^{-1})}{\underset{\uparrow}{\psi}} (\cancel{ikG} + o(r^{-1})) - \underset{O(r^{-1})}{\underset{\uparrow}{G}} (\cancel{ik\psi} + o(r^{-1}))\, dS$$

$$= \int_{S_R} \underbrace{O(r^{-1}) \times o(r^{-1})}_{o(R^{-2})}\, dS = \int_{S_R} o(R^2)\, dS = o(R^{-2})\, 4\pi R^2$$

$$\to 0 \text{ as } R \to \infty$$

So as $R \to \infty$, (11) reduces to:

$$\psi(\underline{x}') = \int_{S_0} \left[\psi(\underline{x}) \frac{\partial G(\underline{x}, \underline{x}')}{\partial n(\underline{x})} - G(\underline{x}, \underline{x}') \frac{\partial \psi}{\partial n}(\underline{x}) \right] dS(\underline{x}) \qquad (12)$$



Notice in (12), $\underline{n}$ is exterior to Ω $\longleftrightarrow$ interior to obstacle O

So, choosing ψ, a T-H pressure $\hat{p}$, swapping $\underline{x} \longleftrightarrow \underset{\uparrow}{\overset{\text{surface}}{}}\underline{x}' = \underline{x}_s$

$\underline{n} \longrightarrow \underline{n}_s = -\underline{n}$, exterior to O $\Longrightarrow$

$$\hat{p}(\underline{x}) = \int_{S_0} \left[G(\underline{x}, \underline{x}_s) \frac{\partial \hat{p}}{\partial n_s}(\underline{x}_s) - \hat{p}(\underline{x}_s) \frac{\partial G(\underline{x}, \underline{x}_s)}{\partial n_s(\underline{x}_s)} \right] dS(\underline{x}_s) \qquad (13)$$

(12)/(13) are called <u>Kirchhoff-Helmholtz integral theorem</u>

Notice also that (1·42) $\longleftrightarrow$ $\hat{p} = i\omega\rho_0 \hat{\phi}$ $\Longrightarrow$

$$\frac{\partial \hat{p}}{\partial n_s} = i\omega\rho_0 \frac{\partial \hat{\phi}}{\partial n_s} = i\omega\rho_0 \underline{n}_s \cdot \underbrace{\nabla \hat{\phi}}_{\downarrow \atop \hat{v}} = i\omega\rho_0 \underset{\substack{\uparrow \\ \text{normal velocity } \hat{v}\cdot\underline{n}_s}}{\hat{v}_n}$$

$$G = -\frac{e^{ikR}}{4\pi R} \; ; \quad \frac{\partial G}{\partial n_s}(\underline{x}, \underline{x}_s) = \underline{n}_s \cdot \nabla_s G(\underline{x}, \underline{x}_s)$$

$$\overset{\nabla_s = -\nabla}{\overset{\downarrow}{=}} -\underline{n}_s \cdot \nabla G(\underline{x}, \underline{x}_s)$$

$$\Longrightarrow \hat{p}(\underline{x}) = \frac{-1}{4\pi} \int_{S_0} \left[i\omega\rho_0 \hat{v}_n(\underline{x}_s) \frac{e^{ikR}}{R} + \hat{p}(\underline{x}_s)(\underline{n}_s \cdot \nabla)\left(\frac{e^{ikR}}{R}\right)\right] dS(\underline{x}_s)$$

(13')

Let $k|\underline{x}_s| << 1 \longleftrightarrow$ low frequency: $\overset{\text{observation point far away}}{k|\underline{x}| >> k|\underline{x}_s|} \Longrightarrow$

Taylor series about $\underline{x}_s = \underline{0}$: $\quad (R = |\underline{x} - \underline{x}_s|, \quad r = |\underline{x}|)$

$$\frac{e^{ikR}}{R} = \frac{e^{ikr}}{r} - (\underline{x}_s \cdot \nabla)\frac{e^{ikr}}{r} + \frac{1}{2}(\underline{x}_s \cdot \nabla)^2\left(\frac{e^{ikr}}{r}\right) + \dots$$

$$\Longrightarrow \hat{p}(\underline{x}) = \frac{-1}{4\pi}\iint_{S_0}\Big[i\omega\rho_0 \hat{v}_n \left(1 - (\underline{x}_s \cdot \nabla) + \frac{1}{2}(\underline{x}_s \cdot \nabla)^2 + \dots\right)\frac{e^{ikr}}{r}$$
$$+ \hat{p}(\underline{x}_s)(\underline{n}_s \cdot \nabla)\left(1 - (\underline{x}_s \cdot \nabla) + \frac{1}{2}(\underline{x}_s \cdot \nabla)^2 + \dots\right)\frac{e^{ikr}}{r}\Big]\, dS$$

$$\hat{p}(\underline{x}) = \underbrace{\hat{S}\frac{e^{ikr}}{r}}_{\text{monopole term}} - \underbrace{\hat{\underline{D}} \cdot \nabla\left(\frac{e^{ikr}}{r}\right)}_{\text{dipole term}} + \underbrace{\sum_{\alpha,\beta} Q_{\alpha\beta}\frac{\partial^2}{\partial x_\alpha \partial x_\beta}\left(\frac{e^{ikr}}{r}\right)}_{\text{quadrupole term}} \qquad (14)$$

where $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ etc.

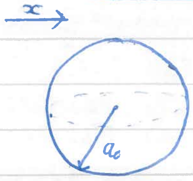
$$\hat{S} = \frac{-i\omega\rho_0}{4\pi}\int_{S_0} \hat{v}_n(\underline{x}_s)\, dS(\underline{x}_s)$$

$$\hat{\underline{D}} = \frac{-1}{4\pi}\iint_{S_0}\left[i\omega\rho_0 \hat{v}_n \underline{x}_s + \hat{p}(\underline{x}_s)\underline{n}_s\right] dS(\underline{x}_s)$$

$Q_{\alpha\beta} = \dots$ (exercise)

(14) is a multipole expansion for $\hat{p}(\underline{x})$

3·5 Scattering example: by air bubbles in water



Let air bubble of initial radius a_0 in water be subjected to an incident acoustic field; a_0 'small':
$a_0 << \lambda = \frac{2\pi}{k}$ = the wavelength

Then for a plane wave incident T-H wave in x-direction
$p_i = Ae^{ikx - i\omega t} \approx Ae^{-i\omega t}$ $(|x| \leq a_0 \Rightarrow kx << 1)$

Then (1·16) $\longleftrightarrow$ $\underline{\hat{u}} = \frac{-i}{\rho_0 \omega} \nabla \hat{p} \Longrightarrow \underline{u_i}$ is small ;

For scattered field, seek p_s is a spherically-symmetric form:

$$p_s = B\frac{e^{ikr - i\omega t}}{r}, \quad B \in \mathbb{C} \quad \text{to be found} \quad (A \in \mathbb{C}, \text{ known}) \tag{15}$$

(1·16) $\Longrightarrow$ $\underline{u_s} = \frac{1}{i\rho_0\omega}\frac{\partial p_s}{\partial r}\underline{\tilde{r}} = v_r \underline{\tilde{r}}$ 'radial velocity'

$$\Longrightarrow v_r = \frac{B}{i\rho_0 \omega r^2}\left(ikr - 1\right)e^{ikr - i\omega t} \tag{16}$$

Assume the bubble is made of an ideal gas responding 'adiabatically'
$\Longleftrightarrow$ $\overset{\substack{\text{press.}}}{p} \overset{\text{volume}}{V^{\gamma}}$ = constant, $\gamma > 1$ 'adiabatic constant'

press. volume

Differentiating in $\overset{\text{time}}{t}$: $\frac{dp}{dt}V^{\gamma} + \gamma p V^{\gamma - 1}\frac{dV}{dt} = 0$

$\Longrightarrow \frac{dp}{dt} = -\frac{\gamma p}{V}\frac{dV}{dt}$;   Now $p = P_0 + \underline{p'}$

atmos. press. 'background thing'

$V = \frac{4}{3}\pi(a_0 + a')^3$, $p' << P_0$ $a' << a_0$ (perturbations small)

↙ neglecting perturbation

$$\Longrightarrow \frac{dp'}{dt} \simeq \frac{-\gamma P_0}{\frac{4}{3}\pi a_0^3} \times \frac{4}{3}\pi \cdot 3(a_0 + a')^2 \underbrace{\frac{da'}{dt}}_{V_b}$$

can neglect to cancel w/ denominator V_b velocity of bubble

$$\approx \frac{-3\gamma P_0}{a_0} V_b$$

Now p' (the perturbed pressure) assumed T-H
$\longleftrightarrow$ $\underline{p'} = \hat{p}e^{-i\omega t} \Longrightarrow \frac{dp'}{dt} = -i\omega \underline{p'}$

don't write Re anymore

$$\Longrightarrow p' = \frac{3\gamma P_0}{i\omega a_0} V_b \tag{17}$$

$V_b := \frac{da'}{dt}$ = radial velocity of bubble vel + press must be same inside + out at surface

On the bubble's surface, must hold:

$$v_b = v_i + v_s \approx v_s = (16);$$ and

$$p' = \left. \underset{\substack{\uparrow \\ \text{incident}}}{p_i} + \underset{\substack{\uparrow \\ \text{scattered}}}{p_s} \right|_{r=a_0}$$

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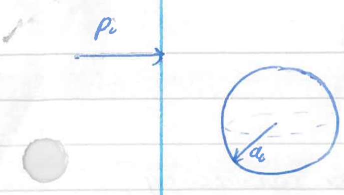


bubble, radius a_0

acoustic wave incident upon bubble, which acts as a scatterer

$ka_0 << 1$ dimensionless parameter

$p_i \approx Ae^{-i\omega t}$

$$p_s = \frac{Be^{ikr-i\omega t}}{r} \qquad (15)$$

$$v_r = \frac{B}{i\rho_0\omega r^2}(ikr-1)e^{ikr-i\omega t} \qquad (16)$$

← he first called this v_s, I think they are the same

(17)

$$p' = \frac{3\gamma P_0}{i\omega a_0} v_b \ , \quad \underbrace{p' = p_i + p_s}_{\uparrow\, r=a_0} \ , \quad \text{also } v_b \approx v_r \text{ (incident negligible)}$$

↓(17), (16)

$$\frac{3\gamma P_0}{i\omega a_0} \times \frac{B(ika_0-1)e^{ika_0 \cancel{-i\omega t}}}{i\rho_0\omega a_0^2} = Ae^{\cancel{-i\omega t}} + \frac{Be^{ika_0\cancel{-i\omega t}}}{a_0}$$

replacing v_b with v_r, and $r=a_0$

$p' \times v_r = p_i + p_s$

$$Be^{ika_0}\left[-1 + \overset{=:\omega_0^2}{\boxed{\frac{3\gamma P_0}{\omega^2\rho_0 a^2}}}\underset{\uparrow \text{ mult. by } i}{(1-ika_0)}\right] = a_0 A$$

Since $ka_0 << 1$, $e^{ika_0} = 1 + ika_0 + O((ka_0)^2)$

$$\Longrightarrow B(1+ika_0 + O((ka_0)^2))\left[-1 + \frac{\omega_0^2}{\omega^2}(1-ika_0)\right] = a_0 A$$

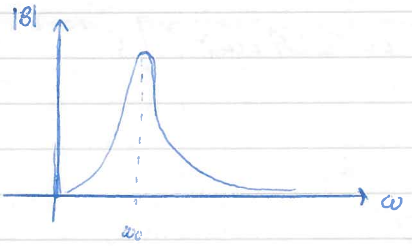
where $\omega_0 := \left(\frac{3\gamma P_0}{\rho_0 a^2}\right)^{1/2}$

$$B\left[-1 - ika_0 + \frac{\omega_0^2}{\omega^2}(1-ika_0)(1+ika_0) + O((ka_0)^2)\right] = a_0 A$$

$$B\left[-1 - ika_0 + \frac{\omega_0^2}{\omega} + O((ka_0)^2)\right] = a_0 A$$

$$B \approx \frac{a_0 A}{\left(\frac{\omega_0}{\omega}\right)^2 - 1 - ika_0} \quad \Longrightarrow \quad |B|^2 = \frac{a_0^2 A^2}{\left[\left(\frac{\omega_0}{\omega}\right)^2 - 1\right]^2 + (ka_0)^2}$$

from Re ↗ ↖ from Im



$|B|$ has a sharp peak at $\omega \approx \omega_0$

Moral: ω_0 is a resonance frequency of the bubble: the scattered wave's amplitude surges for $\omega \approx \omega_0$

Exercise: Exam 2013 Qn 4

4. High frequency waves, WKB method, waveguides

4.1 High frequency (HF) waves, WKB method

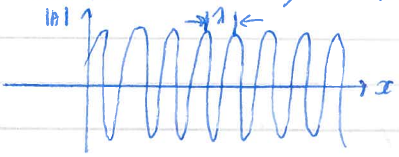
The WKB (Wentzel, Kramers, Brillion) method is a method for constructing highly-oscillatory solutions of differential equations containing a 'large parameter' (e.g. ω, frequency $\Rightarrow$ 'high frequency' regime)

Consider an ODE: $\frac{d^2\psi}{dx^2} + q(x)\psi = 0$, where $|q(x)|$ is 'large'; in particular, if $q(x) > 0$

$$q(x) = k^2(x) \Rightarrow \frac{d^2\psi}{dx^2} + k^2(x)\psi = 0 \qquad (1)$$

If $k(x) \equiv k = \text{constant}$, (1) describes plane T-H waves in x-direction (1-D Helmholtz equation) with solutions $\psi = Ae^{ikx} + Be^{-ikx}$; $A, B \in \mathbb{C}$

e.g. $\psi_1 = \mathrm{Re}(Ae^{ikx - i\omega t}) = |A|\cos(\omega t - kx - \psi)$



when k large $(k \gg 1)$
$\lambda = 2\pi/k \ll 1$ small
$\rightarrow$ rapid oscillations

$k = \frac{\omega}{c}$, $k \gg 1 \iff \omega \gg 1 \iff$ high frequency

Mathematically, $k(x)$ 'large', can be expressed via
$k(x) = \omega\mu(x)$ where $\omega \gg 1$ is a large constant parameter (e.g. ω = frequency)
and $\mu = O(1)$ is fixed. Then

$$\frac{d^2\psi}{dx^2} + \omega^2\mu^2(x)\psi = 0 \qquad (2)$$

$\leftarrow$ ω could not depend on x, but c could in principle

[For waves $k(x) = \frac{\omega}{c(x)}$, corresponding to inhomogeneous media with varying wavespeed $c(x) \Rightarrow \mu(x) = c^{-1}(x)$]

We seek the 'WKB approximation' to a solution of (2) in the form:

$$\psi(x,\omega) = Ae^{i\omega\tau(x,\omega)} \qquad (3)$$

with $A \in \mathbb{C}$, $\omega \gg 1$. Then $(3) \longrightarrow (2)$:

$$\frac{d\psi}{dx} = Ai\omega\frac{d\tau}{dx}e^{i\omega\tau} = i\omega\tau' Ae^{i\omega\tau}$$

$\uparrow$ full derivative since ω just parameter

$$\frac{d^2\psi}{dx^2} = [i\omega\tau''A + i\omega\tau' Ai\omega\tau']e^{i\omega\tau}$$

$$\frac{d^2\psi}{dx^2} = [i\omega\tau'' - \omega^2(\tau')^2]\underbrace{Ae^{i\omega\tau}}_{\psi}$$

$$(2) \Rightarrow i\omega\tau'' - \omega^2(\tau')^2 + \omega^2\mu^2(x) = 0 \qquad (4)$$

Now seek $\tau(x,\omega)$ in a 'regular perturbation* form' wrt small ω^{-1}:

$$\tau(x,\omega) \sim \tau_0(x) + \omega^{-1}\tau_1(x) + \omega^{-2}\tau_2(x) + \ldots \qquad (5)$$

Plugging (5) in to (4), to main order in ω, $O(\omega^2)$:

$$-\omega^2(\tau_0')^2 + \omega^2\mu^2(x) = 0 \Longrightarrow \tau_0 = \pm\mu(x)$$

$$\Longrightarrow \quad \tau_0(x) = \pm\int_{x_0}^{x}\mu(x')\,dx' + C_\pm \qquad (6)$$

Next, $O(\omega)$ in (4):

$$i\omega\,\tau_0'' - \omega^2 2\tau_0'\omega^{-1}\tau_1' = 0$$

$$\Longrightarrow \quad \tau_1' = \frac{i}{2}\frac{\tau_0''}{\tau_0'} = \frac{i}{2}\frac{d}{dx}\ln|\tau_0'| \quad \Longrightarrow$$

$$\tau_1(x) = \frac{i}{2}\ln|\tau_0'| + \tilde{C}_\pm$$

$$= \frac{i}{2}\ln\mu(x) + \tilde{C}_\pm$$

don't have to worry about $C_\pm$ in (6) since taking modulus

Hence, from (3) the WKB approximation is

$$\psi(x,\omega) \approx A e^{i\omega(\tau_0(x) + \omega^{-1}\tau_1(x))}$$

$$= A e^{i\omega\tau_0 + i\tau_1} = A\exp\left\{\pm i\omega\int_{x_0}^{x}\mu(x')\,dx' - \frac{1}{2}\ln\mu(x) + C_\pm\right\}$$

taking linear combination of solutions

$$= \tilde{A}_\pm\,\mu^{-1/2}(x)\exp\left\{\pm i\omega\int_{x_0}^{x}\mu(x')\,dx'\right\}$$

$C_\pm$ comes down as constant

or,

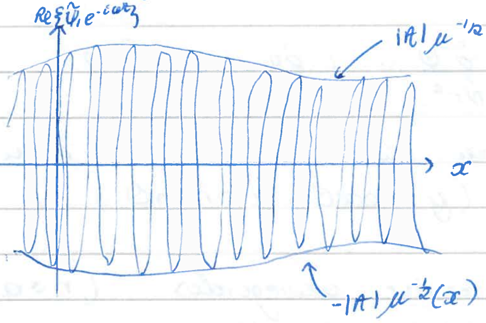
$$\psi(x,\omega) \approx \mu^{-1/2}(x)\left[\cancel{A\exp}\, A e^{i\omega\int_{x_0}^{x}\mu(x')dx'} + B e^{-i\omega\int_{x_0}^{x}\mu(x')dx'}\right] := \tilde{\psi}(x,\omega)$$

where $A, B \in \mathbb{C}$ arbitrary (7)

The WKB approx'n (7) contains two HF 'waves' travelling to the right & left respectively: $\tilde{\psi} = \tilde{\psi}_1 + \tilde{\psi}_2$

e.g. for $\tilde{\psi}_1$

$$Re\{\tilde{\psi}_1 e^{-i\omega t}\} = |A|\mu^{-\frac{1}{2}}(x)\cos\left(\omega t - \omega\int_{x_0}^{x}\mu(x')\,dx' - \varphi\right)$$



Remark: if we substitute (7) in to (2), then (exercise: check!)

if had exact solution would get 0 on RHS

$$\frac{d^2\tilde{\psi}}{dx^2} + \omega^2\mu^2(x)\tilde{\psi}(x) = \left[\frac{3}{4}\frac{(\mu')^2}{\mu^2} - \frac{\mu''}{2\mu}\right]\tilde{\psi}$$

This suggests (7) is accurate, provided

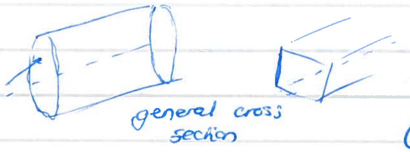
$$\frac{3(\mu')^2}{4\mu^2} - \frac{\mu''}{2\mu} << \omega^2\mu^2 \quad \Longleftrightarrow$$

$$\frac{1}{\omega^2}\left[\frac{3(\mu')^2}{4\mu^4} - \frac{\mu''}{2\mu^3}\right] << 1 \qquad (8)$$

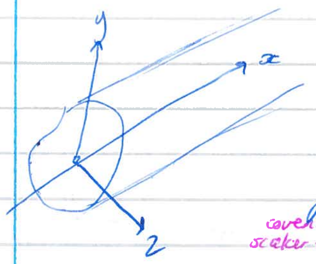
(8) holds provided $\omega >> 1$ and $\mu(x) \geq \mu_0 > 0$ ($\mu(x)$ not too close to zero) and μ', μ'' 'not too large' ($\mu(x)$ varying slowly enough).

Exercises: i. EM waves in layered media ($\varepsilon = \varepsilon(x)$ and/or $\mu = \mu(x)$ but not depending on y or z) $\Rightarrow$? MEs reduce to an ODE (2) or alike: Exam 2011 & 2014 Qn 5

4.2 Waveguides



A waveguide is generally a tube/pipe e.g. a cylinder of a rather general cross section, along which waves can propagate (The wave 'bounces'/is reflected from the walls, and so propagates along the axis.)



Choose x along the waveguide's axis; seek for T-H solutions. So, both in acoustics & EM, we seek solutions of Helmholtz equation (1.15) or (1.35)-(1.36), for $\hat{\varphi}$: $(\nabla^2 + k^2)\hat{\varphi} = 0$

covering T-H scalar & vector case

$$\Longleftrightarrow \quad \frac{\partial^2 \hat{\varphi}}{\partial x^2} + \underbrace{\frac{\partial^2 \hat{\varphi}}{\partial y^2} + \frac{\partial^2 \hat{\varphi}}{\partial z^2}}_{=: \nabla_\perp^2} + k^2 \hat{\varphi} = 0$$

$k = \frac{\omega}{c}$. In both cases $\hat{\varphi}$ is sought in the form

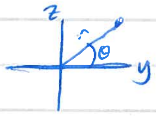
$$\hat{\varphi}(x,y,z) = \varphi(y,z) e^{i\beta x} \tag{9}$$

where β is unknown 'propagation constant', so (9) looks like a plane wave along x with 'amplitude' φ depending on y, z

$\Longrightarrow$ φ solves

$$\nabla_\perp^2 \varphi + (k^2 - \beta^2)\varphi = 0 \tag{10}$$

where $\nabla_\perp^2 := \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = \frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2}$



is the cross-sectional Laplacian on (y,z)-plane; (r, θ) polar coordinates ($y = r\cos\theta$, $z = r\sin\theta$)

Now specialise to acoustics, and circular waveguides ($r \leq a$)
Seek solns of (10) via separation of variables:

$$\psi(r,\theta) = R(r)\,\Theta(\theta)$$

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' + (k^2 - \beta^2)R\Theta = 0$$

$\times \frac{r^2}{R\Theta}$

$$\frac{r^2 R''}{R} + \frac{r R'}{R} + (k^2 - \beta^2)r^2 = -\frac{\Theta''}{\Theta} =: m^2$$

(m^2 separation constant) $\Longrightarrow$

$$\Theta'' + m^2 \Theta = 0 \Longrightarrow \Theta = e^{\pm i m \theta}$$

So since $\Theta(\theta)$ must be 2π-periodic $\Longrightarrow$ $m \in \mathbb{Z}$ (integer);
WLOG $m \geq 0$

[illegible]

$$r^2 R'' + rR' + [\underbrace{(k^2-\beta^2)r^2}_{=:\tilde{r}^2} - m^2]R = 0 \qquad (11)$$

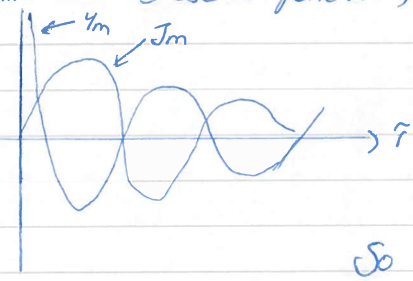
via change of variables, $\tilde{r} = (k^2-\beta^2)^{1/2} r$

$$\tilde{r}^2 \frac{d^2R}{d\tilde{r}^2} + \tilde{r}\frac{dR}{d\tilde{r}} + (\tilde{r}^2 - m^2)R = 0 \qquad (11')$$

which is Bessel's differential equation of order m, with general solutions

$$R = AJ_m(\tilde{r}) + BY_m(\tilde{r}) \qquad (12)$$

J_m the Bessel's function, Y_m (so-called) Neumann function



The solution has to be continuous as $r \to 0$, which J_m is but Y_m is not

$\implies$ must take $B = 0$

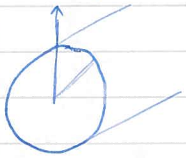
So the solution is of the form

$$\hat{\varphi}(x,y,z) = A_m J_m(r(k^2-\beta^2)^{1/2})\, e^{\pm im\theta + i\beta x} \qquad (13)$$

$m \in \mathbb{Z}$, $m \geq 0$, $A_m \in \mathbb{C}$

Let the waveguide's boundary $r = a$ be e.g. acoustically hard i.e.

$\frac{\partial \hat{\varphi}}{\partial n} = \frac{\partial \hat{\varphi}}{\partial r} = 0$ at $r = a$ $\implies$ from (13)



$J_m'(a(k^2-\beta^2)^{1/2}) = 0$, so if $\{\alpha_{mn}\}_{n=1}^{\infty}$ are roots of $J_m'(\alpha_{mn}) = 0$

$\implies a(k^2-\beta^2)^{1/2} = \alpha_{mn}$

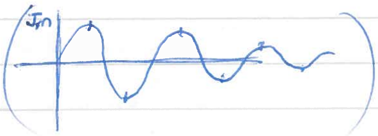
$$\implies \beta = \pm\left(k^2 - \frac{\alpha_{mn}}{a^2}\right)^{1/2} =: \beta_{mn}$$

So $\exists$ infinitely many 'modal solutions'

$$\hat{\varphi}_{mn} = A_{mn} J_m\left(\frac{r}{a}\alpha_{mn}\right) e^{\pm im\theta + i\beta_{mn}x}$$

$m = 0, 1, 2, \ldots$; $n \geq 1$

Notice that, for fixed k and m, $\alpha_{mn} \longrightarrow +\infty$ as $n \longrightarrow \infty$



so β_{mn} becomes complex for $n > n_0(k,m)$: the corresponding modes are exponentially decaying:

$$e^{i\beta_{mn}x} = e^{-|\beta_{mn}|x}$$