3402 Waves and Wave Scattering Notes
Based on the spring 2013 lectures by
Prof V Smyshlyaev.

I have made every effort to copy down all the content on the board during lectures. I accept no responsibility what so ever for mistakes on the notes nor changes to the syllabus for the current year. I highly recommend that the reader attends all lectures, making their own notes and to use this document as a reference only.

Chris Owen.

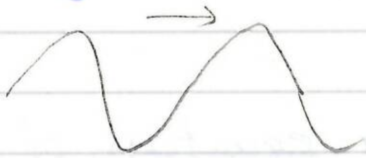
7/1/13

MATH 3402 Waves and Waves Scattering.
Valery S'MYSHLYAEV.

$10 \times 3 = 30$ classes $\rightarrow$ 100% Exams.

0. Introduction

The couse is about maths of waves (sound/acoustic, electromagnetic), and their 'scattering', waves:



- water waves
- sound (acoustics)
- radio/light (electromagnetic)

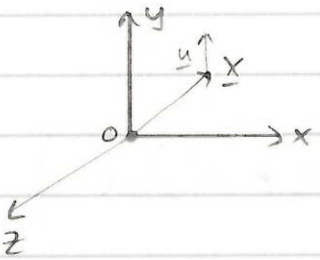
Common: waves propagate (hence transfer energy/information; they scatter when hit an "obstacle"; "Disturbances" not "matter" moves!.

1. Governing equations

1.1 Acoustic (sound) waves

Acoustic (sound) waves are small amplitude mechanics disturbances propagating in a "fluid", typically a compressible gas (e.g air) or liquid (e.g. water).

The governing eqn are derived from basic conservation laws for an "inviscid fluid".



$\underline{x} = (x, y, z)$, t time; $\mapsto$
$\rho = \rho(\underline{x}, t)$ fluid density
$\underline{u} = \underline{u}(\underline{x}, t)$ fluid velocity
$p = p(\underline{x}, t)$ pressure.

Conservation of mass: (cf MATH 2301)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = 0 \qquad (1)$$

Conservation of momentum (≡ Newton's 2nd law for continuum media, neglecting gravity, viscosity etc.)

$$\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u} = -\frac{1}{\rho} \underline{\nabla} p \qquad (2)$$

Additionally, physically, an equation of state is held, connecting one unknown to another, e.g. pressure p to density ρ.

Under the "adiabatic" assumption (no heat exchange), let

$$p = P(\rho) \qquad (3)$$

We will be considering small disturbances (= perturbations) of a uniform state at rest:

$$p = p_0 + \hat{p}, \quad \rho = \rho_0 + \tilde{\rho}, \quad \underline{u} = \underbrace{\cancel{\underline{u}_0}}_{=0} + \hat{\underline{u}}$$

p_0, ρ_0 are constant; $\hat{p}, \hat{\rho}, \hat{u}$ are "small".

Substituting into (1) - (3)

$$(1) \Rightarrow \frac{\partial}{\partial t}\left(\cancel{\rho_0} + \tilde{\rho}\right) + \nabla \cdot \left((\rho_0 + \tilde{\rho}) \tilde{\underline{u}} \right) = 0. \quad (1')$$

$(2) \Rightarrow \dfrac{\partial \underline{\tilde{u}}_0}{\partial t} + \underline{\tilde{u}} \cdot \cancel{\nabla \underline{\tilde{u}}} = - \dfrac{1}{\rho_0 + \cancel{\tilde{\rho}}} \nabla (\cancel{p_0} + p)$ (2')

$(3) \Rightarrow p_0 + \tilde{p} = \underline{P}(\rho_0 + \tilde{\rho})$ (3')

Since $\tilde{p}, \tilde{\rho}, \tilde{u}$ are all small, neglecting in (1') & (2') the "higher order smallness" terms (like $\nabla \cdot (\tilde{\rho} \tilde{u})$) $\Rightarrow$

$(1') \Rightarrow \dfrac{\partial \tilde{\rho}}{\partial t} + \rho_0 \nabla \cdot \underline{u} = 0$ (4)

$(2') \Rightarrow \dfrac{\partial \tilde{u}}{\partial t} = -\dfrac{1}{\rho_0} \nabla \tilde{p}$ (5)

For (3'), using Taylor expansion

$$\underline{P}(\rho_0 + \tilde{\rho}) \approx \underline{P}(\rho_0) + \frac{d\underline{P}}{d\rho}(\rho_0)\tilde{\rho} \left(+ O(\tilde{\rho}^2) \right)$$

$$\Rightarrow \cancel{p_0} + \tilde{p} \approx \underbrace{\cancel{\underline{P}(\rho_0)}}_{= p_0} + \frac{\partial \underline{P}}{\partial \rho}(\rho_0)\tilde{\rho} \Rightarrow$$

Physically $\underline{P}' = \dfrac{\partial \underline{P}}{\partial \rho}(\rho_0) > 0$, so denote

$$\underline{P}' = c^2 \left(\Leftrightarrow c = (\underline{P}')^{1/2} \right)$$

where the physical constant $c > 0$, the "wave speed" (will later see describes the speed of waves). $\Rightarrow$ $\tilde{p} = c^2 \tilde{\rho}$ $\Rightarrow$

$$\tilde{\rho} = \frac{1}{c^2}\tilde{p} \qquad (6).$$

So, eliminating $\tilde{\rho}$ in (4)-(5) via (6):

$$\boxed{\frac{1}{c^2}\frac{\partial \tilde{p}}{\partial t} + \rho_0 \nabla\cdot\tilde{\underline{u}} = 0} \qquad (7)$$

(5) $\Leftrightarrow$ $$\boxed{\frac{\partial \tilde{\underline{u}}}{\partial t} = -\frac{1}{\rho_0}\nabla \tilde{p}} \qquad (8)$$

(7)-(8) are main equations of ("linear") acoustics.

To start analysing (7)-(8), differentiate (7) in t and use (8) $\Rightarrow$

$$\frac{1}{c^2}\frac{\partial^2 \tilde{p}}{\partial t^2} + \rho_0 \nabla\cdot\frac{\partial \tilde{\underline{u}}}{\partial t} = 0 \quad \Rightarrow$$

$$\frac{1}{c^2}\frac{\partial^2 \tilde{p}}{\partial t^2} + \cancel{\rho_0}\, \nabla\cdot\left(-\frac{1}{\cancel{\rho_0}}\nabla \tilde{p}\right) = 0. \quad \Leftrightarrow$$

$$\boxed{\frac{1}{c^2}\frac{\partial^2 \tilde{p}}{\partial t^2} - \nabla^2 \tilde{p} = 0} \qquad (9)$$

Here $\nabla^2 p := \nabla\cdot(\nabla \tilde{p}) = \frac{\partial^2 \tilde{p}}{\partial x^2} + \frac{\partial^2 \tilde{p}}{\partial y^2} + \frac{\partial^2 \tilde{p}}{\partial z^2}$

$=: \Delta p$ (= Laplacian of $\tilde{p}$),

(9) is the <u>wave equation</u> (a partial differential eqn, PDE, for $\tilde{p}$).

For solutions of (9), $\tilde{p} = \tilde{p}(t, x, y, z)$ seek first p depending on t and x only (not on y & z) $\Rightarrow$

$$\frac{1}{c^2}\frac{\partial \tilde{p}}{\partial t^2} - \frac{\partial^2 \tilde{p}}{\partial x^2} = 0. \qquad (9')$$

$\Rightarrow$ $\tilde{p} = f(x - ct) + g(x + ct)$ solves (9') $\forall f, g \in C^2$ (twice differentiable):

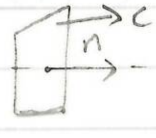
$$\frac{1}{\cancel{c^2}}\left((\cancel{-c})^2 f'' + \cancel{c^2} g'' \right) - \left(f'' + g'' \right) = 0 \quad \checkmark$$

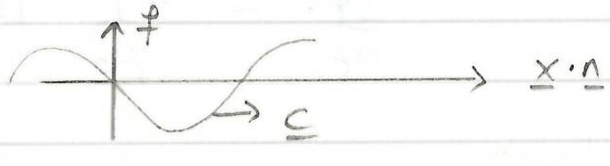
More generally, for any "direction" vector $\underline{n}$, $|\underline{n}| = 1$, a solution of (9) is:

$$\tilde{p} = f(\underline{x}.\underline{n} - ct) + g(\underline{x}\cdot\underline{n} + ct) \qquad (10)$$

In (10), $f(\underline{x}\cdot\underline{n} - ct)$ describe a plane "wave" travelling in positive $\underline{n}$ direction with a "phase speed" c;

$g(\underline{x}\cdot\underline{n} + ct)$ describes a <u>plane wave</u> travelling in negative $\underline{n}$ direction phase speed c:
$p = f(\underline{x}\cdot\underline{n} - ct)$ is constant $\forall$ plane perpendicular ($\perp$) to $\underline{n}$, which plane moves in $\underline{n}$ with speed c:





Returning to (7) & (8):

$$\frac{1}{c^2}\frac{\partial \tilde{p}}{\partial t} + \rho_0 \nabla\cdot\tilde{u} = 0 \qquad (7)$$

$$\frac{\partial \tilde{u}}{\partial t} = -\frac{1}{\rho_0}\nabla\tilde{p} \qquad (8)$$

Let at time $t=0$ $\nabla\times\underline{u} = 0$ (i.e. flow is initially irrotational)
$\Rightarrow$ Taking curl of (8) $\Rightarrow$

$$\frac{\partial}{\partial t}\left(\nabla\times\tilde{u}\right) = -\frac{1}{\rho_0}\nabla\times(\nabla\tilde{p}) = 0.$$

$$\Rightarrow \frac{\partial}{\partial t}(\nabla\times\tilde{u}) \equiv 0 \Rightarrow \nabla\times\tilde{u} \equiv \text{const} = 0 \ (\text{at } t=0)$$

$\Rightarrow \nabla\times\tilde{u} \equiv 0 \quad \forall t \Rightarrow \exists\, \phi(\underline{x},t)$ a (scalar) <u>velocity potential</u>, such that $\tilde{\underline{u}} = \nabla\phi$. Then,
(8) $\Leftrightarrow \nabla\left(\frac{\partial\phi}{\partial t}\right) = \nabla\left(-\frac{1}{\rho_0}\tilde{p}\right) \Rightarrow$

$$\nabla\left(\frac{\partial\phi}{\partial t} + \frac{1}{\rho_0}\tilde{p}\right) = 0 \Rightarrow \frac{\partial\phi}{\partial t} + \frac{1}{\rho_0}\tilde{p} = A(t).$$

$A(t)$ constant of integration, must be taken zero for finiteness of perturbation $\tilde{p}$, $\tilde{u} = \nabla\phi \Rightarrow$

$$\frac{\partial\phi}{\partial t} = -\frac{1}{\rho_0}\tilde{p} \qquad (11)$$

Differentating (11) in t, via (7):

$$\frac{\partial^2 \phi}{\partial t^2} = -\frac{1}{\rho_0}\left(\frac{\partial \tilde{p}}{\partial t}\right) = +\frac{1}{\cancel{\rho_0}}\left(+c^2 \cancel{\rho_0}\, \nabla.(\nabla \phi)\right) \Rightarrow$$

$$\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi = 0 \qquad (12)$$

i.e ϕ solves the same wave eqn as $\tilde{p}$ cf (9).

<u>Spherically symmetric waves</u>.
Seek for solutions of wave eqn (9)/(12) in spherical coordinates $\underline{x} = (r, \theta, \phi) \Rightarrow$

$$\nabla^2 \tilde{p} = \Delta \tilde{p} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \tilde{p}}{\partial r}\right) +$$

$$+ \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \tilde{p}}{\partial \theta}\right)$$

$$+ \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \tilde{p}}{\partial \phi^2}$$

Seek for <u>spherically symmetric</u> solns of (9)
i.e. $\tilde{p} = \tilde{p}(t, r, \cancel{\theta}, \cancel{\phi}) = \tilde{p}(t, r) \Rightarrow$ (9) reads:

$$\frac{1}{c^2}\frac{\partial^2 \tilde{p}}{\partial t^2} - \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \tilde{p}}{\partial r}\right) = 0.$$

Try $\tilde{p} = \frac{q}{r} \Rightarrow \frac{\partial \tilde{p}}{\partial r} = \frac{1}{r}\frac{\partial q}{\partial r} - \frac{1}{r^2} q \Rightarrow$

$$r^2 \frac{\partial \tilde{p}}{\partial r} = r\frac{\partial q}{\partial r} - q \Rightarrow$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \tilde{p}}{\partial r} \right) = r \frac{\partial^2 q}{\partial r^2} + \cancel{\frac{\partial q}{\partial r}} - \cancel{\frac{\partial q}{\partial r}} \Rightarrow$$

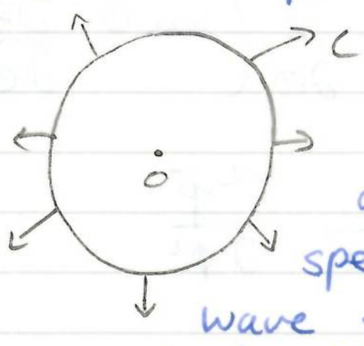
$$\frac{1}{c^2} \cancel{\frac{1}{r}} \frac{\partial^2 q}{\partial r^2} - \cancel{\frac{1}{r}} \frac{\partial^2 q}{\partial r^2} = 0 .$$

$\Rightarrow$ (1-dim wave equ for q, as before).

$$q = f(r-ct) + g(r+ct) \Rightarrow$$

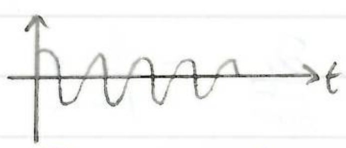
$$\tilde{p} = \frac{f(r-ct)}{r} + \frac{g(r+ct)}{r} . \qquad (13)$$

(13) is general form for a spherically symmetric soln of wave equation (9).



f - part of (13) describes a spherical wave travelling away from centre/origin O, with speed c; g - part describes a wave traveling towards O (f "outgoing", "g" incoming). $\forall f, g \in C^2$

<u>Time-harmonic waves.</u>
Seek for solutions of (9) with following time-dependence: $\tilde{p}(\underline{x}, t) = |\hat{p}(\underline{x})| \cos(\omega t - \varphi)$



i.e "pure tone" solution with <u>angular frequency</u> ω,

$\Rightarrow T = \frac{2\pi}{\omega}$ time period; phase φ, amplitude $|\hat{p}(x)|$

$$\Leftrightarrow \quad \tilde{p}(x,t) = Re\left\{\hat{p}(x)e^{-i\omega t}\right\}, \qquad (14)$$

where $\hat{p}(x) := |\hat{p}(x)|e^{i\varphi}$ is called <u>complex amplitude</u> i.e incorporating the phase.

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Two hours next Monday.

14/1/13

$\underline{u}, p, \rho,$

$$\frac{1}{c^2}\frac{\partial^2 \tilde{p}}{\partial t^2} - \nabla^2 \tilde{p} = 0 \qquad (9)$$

T-H waves

$$\tilde{p}(\underline{x},t) = |\hat{p}|\cos(\omega t - \varphi)$$

$$= \mathrm{Re}\left\{\hat{p}(x)e^{-i\omega t}\right\} \qquad (14)$$

$$\hat{p}(x) = |\hat{p}|e^{i\varphi}$$

Plugging (14) into (9):

$$\mathrm{Re}\left\{\frac{1}{c^2}(-i\omega)^2\hat{p}e^{-i\omega t} - (\nabla^2\hat{p})e^{-i\omega t}\right\} = 0.$$

$$\Leftrightarrow -\mathrm{Re}\left\{\underbrace{\left(\nabla^2\hat{p} + \Big(\overbrace{\frac{\omega}{c}}^{k}\Big)^2\hat{p}\right)}_{=S(\underline{x})}e^{-i\omega t}\right\} = 0, \ \forall t, x.$$

$$\Leftrightarrow (\mathrm{Re}\, S)\cos\omega t + (\mathrm{Im}\, S)\sin\omega t \equiv 0 \quad \forall t, \forall \underline{x}$$

$\Rightarrow \mathrm{Re}\, S \equiv 0, \ \mathrm{Im}\, S \equiv 0$ (varying t for fixed $\underline{x}$).

$$\Leftrightarrow S \equiv 0. \Rightarrow \boxed{\nabla^2\hat{p} + k^2\hat{p} = 0}, \quad k := \frac{\omega}{c} \qquad (15)$$

(15) is called Helmholtz equ ("reduced wave equ" : for T–H waves); $k := \omega/c$ is <u>wave number</u>.

For velocity $\underline{\tilde{u}}(\underline{x}, t)$; (dropping "~" hence forth), for T–H case, also seek

$$\underline{u}(\underline{x}, t) = Re\left(\underline{\hat{u}}(\underline{x})\, e^{-i\omega t}\right)$$

$$\Rightarrow \text{via } (8) \iff \frac{\partial \underline{u}}{\partial t} = -\frac{1}{\rho_0} \nabla p$$

$$\Rightarrow Re\left(-i\omega \underline{\hat{u}}\, e^{-i\omega t}\right) = -\frac{1}{\rho_0} Re\left(\nabla \hat{p}\, e^{-i\omega t}\right)$$

$$\iff \cancel{Re}\left\{\left(-i\omega\hat{u} + \frac{1}{\rho_0}\nabla\hat{p}\right)\cancel{e^{-i\omega t}}\right\} = 0$$

$$\iff \underline{\hat{u}} = -\frac{i}{\rho_0 \omega} \hat{\nabla}\hat{p}. \qquad (16)$$

<u>Plane time-harmonic (T–H) waves:</u>

For a plane wave travelling in positive $\underline{n}$ direction which is also T–H:

$$p(x, t) = f(\underline{x}\cdot\underline{n} - ct) = Re\left\{\hat{p}(\underline{x})\, e^{-i\omega t}\right\}$$

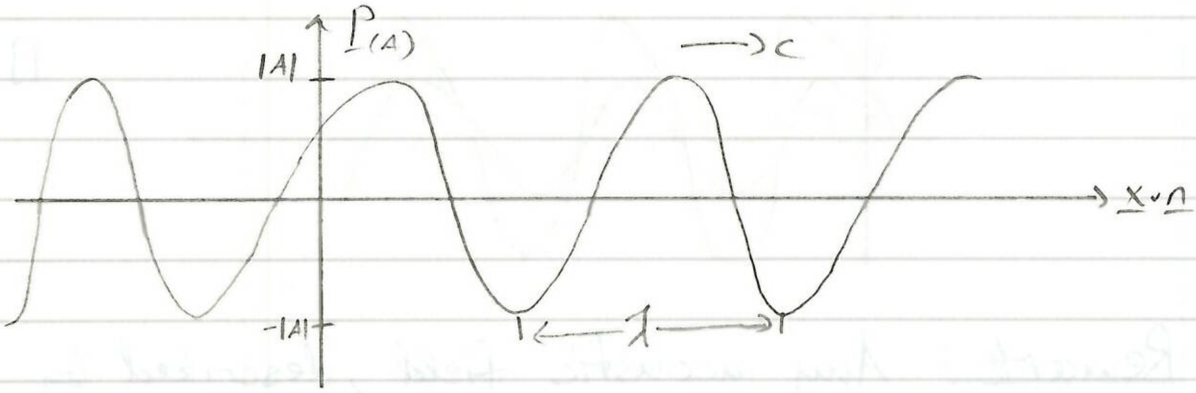
$$= Re\left(e^{i\frac{\omega}{c}(-ct + \underline{x}\cdot\underline{n})} A\right), \quad A \in \mathbb{C} \text{ (complex constant)}$$

$$p = \mathrm{Re}\Big(\underbrace{Ae^{ik\underline{x}\cdot\underline{n}}}_{=\hat{p}}\, e^{-i\omega t}\Big), \iff \hat{p}\, A e^{i\underline{k}\cdot\underline{x}} \quad (17)$$

where $\underline{k} := k\underline{n}$ is called <u>wave vector</u>;
$A = |A|e^{i\varphi}$ complex amplitude

So, (1.7) $\iff$

$$p(\underline{x},t) = |A|\cos(\omega t - k\underline{n}\cdot x + \varphi).$$



$$\lambda = \frac{2\pi}{k} \text{ is } \underline{\text{wavelength}} \quad (= \text{spatial period})$$

$$\left(T = \frac{2\pi}{\omega}, \quad f = \frac{1}{T} = \frac{\omega}{2\pi}\right)$$

Also, plugging (17) to (16):

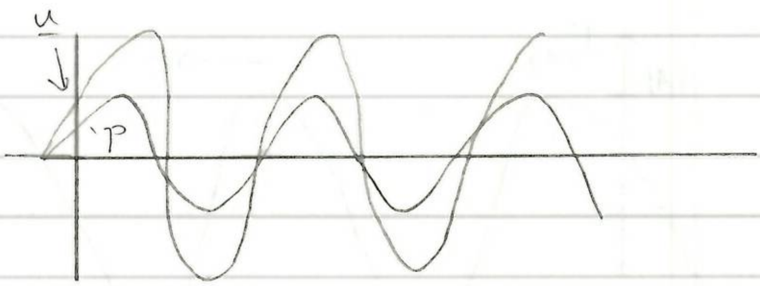
$$\hat{u} = -\frac{i}{\rho_0\omega}\nabla\hat{p} = -\frac{i}{\rho_0\omega}\nabla\left(Ae^{i\underline{k}\cdot\underline{x}}\right)$$

$$= -\frac{i}{\rho_0\omega}Aik\,e^{i\underline{k}\cdot\underline{x}} = \frac{A}{\rho_0}\underbrace{\frac{k}{\omega}}_{=\frac{1}{c}}\underline{n}\,e^{i\underline{k}\cdot\underline{x}}$$

$$= \frac{A}{\rho_0 c} \underline{n} e^{i\underline{k}\cdot\underline{x}}$$

$$\Rightarrow \underline{u} = \mathrm{Re}\left\{ \frac{A\underline{n}}{\rho_0 c} e^{i\underline{k}\cdot x - i\omega t} \right\} = \frac{p}{\rho_0 c} \underline{n} \qquad (18)$$

i.e. for plane T-H waves, velocity $\underline{u}$ and pressure p are "in phase" and move in the $\underline{n}$-direction.



□.

<u>Remark</u>: Any acoustic field, described by wave eqn (9) can be expressed as a superposition of T-H waves via <u>Fourier Transform</u> (FT) in time.

Assuming sufficient decay of $p(\underline{x}, t)$ for $t \to \pm\infty$, seek $p(x,t)$ as FT of $p^*(\underline{x}, \omega)$:

$$p(\underline{x}, t) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} p^*(\underline{x}, \omega) e^{-i\omega t} \, d\omega. \quad (*)$$

$\Rightarrow$ the inverse F.T. gives:

$$p^*(x, \omega) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} p(\underline{x}, t) e^{i\omega t} \, dt.$$

Then (9) $\Rightarrow$

$$0 = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = (2\pi)^{-1/2}\int_{-\infty}^{\infty}\left[\frac{1}{c^2}(-i\omega)^2 p^* - \nabla p^*\right]e^{-i\omega t}\,d\omega$$

$$\underset{\left(\frac{\omega}{c}=k\right)}{=} -(2\pi)^{1/2}\int_{-\infty}^{\infty}\left(\nabla^2 p^* + k^2 p^*\right)e^{-i\omega t}\,d\omega = 0, \ \underline{\forall t}.$$

$\Rightarrow$ (by FT inversion of zero)

$$\nabla^2 p^* + k^2 p^* = 0.$$

i.e. p^* solves Helmholtz eqn (1.5). So $(*)$ is indeed a superposition of $p^* e^{-i\omega t} \longleftrightarrow$ T-H waves.

$\square$

<u>Causality and the Sommerfeild Radiation condition</u>

<u>Causality</u>: the effect / consequency can only follow the cause / source. In the context of acoustic waves, solutions of wave eqn (9) / (12), this means their solution are required to be zero for $t \leq t_0$ for some time t_0 the "source" switches on.

Consider spherically symmteric waves (13):

$$p(\underline{x}, t) = \frac{f(r-ct)}{r} + \frac{g(r+ct)}{r}, \quad f, g \in C^2$$

We require: $\exists\, t_0$ such that, outside a "source region ($r \leq r_0$), i.e $\forall r \geq r_0$, $\forall t < t_0$ holds $p \equiv 0$ $\Rightarrow$ (Exercise) $g(z) = $ constant, so could be "added" to f $\Rightarrow$

$$p(x,t) = \frac{f(r-ct)}{r}$$

is a general "causal" solution of (9), i.e keeping only outgoing (and removing "incoming") wave, consider now radially-symmetric T-H wave:

$$(B) \Rightarrow p(r,t) = \mathrm{Re}\left\{ A\,\frac{e^{\frac{i\omega}{c}(-ct+r)}}{r} + B\,\frac{e^{-\frac{i\omega}{c}(ct+r)}}{r} \right\}$$

$$= \mathrm{Re}\left\{ \frac{A}{r} e^{ikr - i\omega t} + \frac{B}{r} e^{-ikr - i\omega t} \right\} \quad A, B \in \mathbb{C}.$$

We expect for physical solns not to contain the incoming part, i.e to make sure $B = 0$.

This is achieved by the Sommerfeld's radiation condition:

$$r\frac{\partial p}{\partial r} - ikp \to 0, \text{ as } r \to \infty \qquad (19)$$

<u>uniformly</u> as $r \to \infty$. Check

$$r\frac{\partial p}{\partial r} = \left(\cancel{ikAe^{ikr}} - ikBe^{-ikr} - \frac{Ae^{ikr}}{r} - \frac{Be^{-ikr}}{r}\right) e^{-i\omega t}$$

$\ominus$

$$ikrp = \left(\cancel{ikAe^{ikr}} + ikBe^{-ikr}\right)e^{-i\omega t}$$

$$r\frac{\partial p}{\partial r} - ikrp = \left(\underbrace{-2ikBe^{-ikr}}_{\not\to 0 \text{ unless } B=0} - \underbrace{\frac{Ae^{ikr}+Be^{-ikr}}{r}}_{\to 0,\ r\to\infty}\right)e^{-i\omega t}$$

$\Rightarrow B = 0$, as desired. $\square$

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$$r\frac{\partial p}{\partial r} - ikrp \to 0, \quad r \to \infty \qquad (19)$$

Hence $p = \dfrac{A}{r} e^{ikr - i\omega t}$

Remark: Sommerfeld radiation condition (19) is required to hold for any T-H waves, solns of (15). It ensures physically T-H waves are outgoing; mathematically it ensures uniqueness of solutions to boundary-value problems for (15).

One can see that FTs of causal solution to (9) satisfy (19). ∎

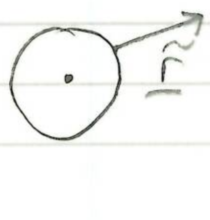
Back to symmetric T-H outgoing waves; under (19)

$$p = \frac{A}{r} e^{ikr - i\omega t} = \hat{p} e^{-i\omega t}, \quad \hat{p} = \frac{A}{r} e^{ikr}$$

$$(16) \equiv \hat{\underline{u}} = \frac{-i}{\rho_0 \omega} \nabla \hat{p} = \frac{-i}{\rho_0 \omega} A \nabla\left(\frac{e^{ikr}}{r}\right)$$

$$= -\frac{iA}{\rho_0 \omega}\left(\frac{ik}{r} - \frac{1}{r^2}\right) e^{ikr} \underline{\tilde{r}}, \quad \underline{\tilde{r}} := \frac{\underline{x}}{r} = \frac{\underline{x}}{|\underline{x}|}$$

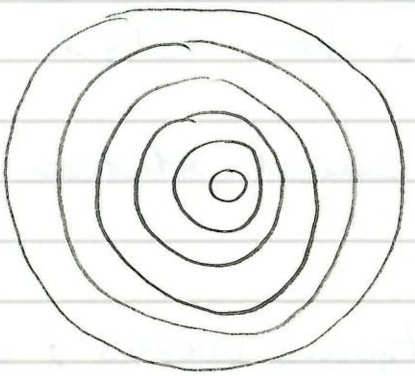
- unit vector in the radial direction.



$$\Rightarrow \hat{\underline{u}} = \hat{u}_r \underline{\tilde{r}} \quad , \quad \hat{u}_r = \frac{\hat{p}}{\rho_0 C}\left(1 + \frac{i}{kr}\right), \quad (20)$$

$$\left(\frac{\omega}{k} = c\right)$$

Notice: The 1st term in (20) is same as for plane T-H waves, see (18), and is "in-plane" with pressure p; 2nd terms out of phase with p by $\pi/2$ $(i = e^{i\pi/2})$;



If kr "large", $kr \gg 1$, the 2nd term is negligable $\Leftrightarrow$ $k = 2\pi/\lambda$, $\Rightarrow r/\lambda \gg 1 \Leftrightarrow r \gg \lambda$ $\Leftrightarrow$ distances of "many" wavelength.

Acoustic energy, Intensity (Kirchhoff 1876)
Going back to main acoustics eqns (7) & (8)

$$(7) \Rightarrow \frac{1}{c^2}\frac{\partial p}{\partial t} + \rho_0 \nabla \cdot \underline{u} = 0 .$$

$$(8) \Rightarrow \underline{u} \cdot \rho_0 \frac{\partial \underline{u}}{\partial t} = -\nabla p$$

Take dot product of (8) with $\underline{u}$ $\Rightarrow$

$$\underline{u} \cdot \rho_0 \frac{\partial \underline{u}}{\partial t} = -\underline{u} \cdot \nabla p = -\nabla(p\underline{u}) + p\nabla \cdot \underline{u}$$

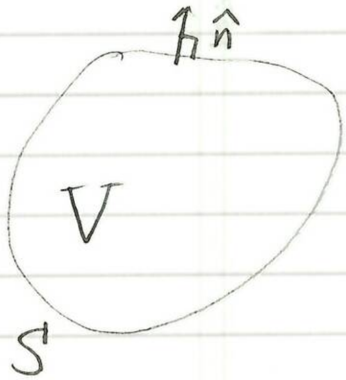
$$= -\nabla\cdot(p\underline{u}) - \frac{p}{\rho_0 c^2}\frac{\partial p}{\partial t}$$

$$\Rightarrow \rho_0 \underline{u}\cdot\frac{\partial \underline{u}}{\partial t} + \frac{1}{\rho_0 c^2}\, p\frac{\partial p}{\partial t} + \nabla\cdot(p\underline{u}) = 0 \iff$$

$$\boxed{\frac{\partial}{\partial t}\Big(\underbrace{\tfrac{1}{2}\rho_0\,\underline{u}\cdot\underline{u} + \tfrac{1}{2}\frac{p^2}{\rho_0 c^2}}_{w}\Big) + \nabla\cdot(\underbrace{p\underline{u}}_{\underline{I}}) = 0} \quad (21)$$

$$\iff \quad \frac{\partial W}{\partial t} + \nabla\cdot\underline{I} = 0.$$

This "conservation law" is interpreted as follows! Integrate over a volume V with boundary S:



$$0 = \int_V \frac{\partial}{\partial t} W \, dV + \int_V \nabla\cdot\underline{I}\, dV$$

$$= \frac{\partial}{\partial t}\int_V W\, dV + \int_S \underline{I}\cdot\underline{n}\, dS = 0.$$

(by Gauss / Divergence Theorem)

Interpretation:

$$w = \underbrace{\tfrac{1}{2}\rho_0 |\underline{u}|^2}_{\text{kinetic energy density}} + \underbrace{\frac{p^2}{2\rho_0 c^2}}_{\text{asoustic (cf "elastic") Pontential energy density}}$$

Hence $\underline{I} = p\underline{u}$ = acoustic energy flux or acoustic intensity (vector):

$\underline{I}$ repersent energy transported per unit area at unit time.

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$$\frac{\partial}{\partial t} w + \nabla \underline{I} = 0, \quad \underline{I} = p\underline{u}.$$

For plane T-H waves:

$$(18) \Rightarrow \underline{u} = \underline{n} \frac{p}{\rho_0 c} \Rightarrow \underline{I} = p\underline{u} = \underbrace{\frac{p^2}{\rho_0 c}}_{\geq 0} \underline{n}$$

i.e the energy is transported in the direction of propagation $\underline{n}$ (= phase velocity).

For spherically symmetric T-H waves:

$$p = Re\Big(\underbrace{\frac{A}{r} e^{ikr}}_{=\hat{p}} e^{-i\omega t}\Big) = \frac{|A|}{r}\cos(\omega t - kr - \psi),$$

$A = |A|e^{i\psi}$ (complex amplitude)

$$(20) \Rightarrow \underline{u} = Re(\hat{\underline{u}} e^{-i\omega t}) = Re(\hat{u}_r \underline{\hat{r}} e^{-i\omega t})$$

$$= u_r \underline{\hat{r}}.$$

$$u_r = Re\left[\frac{\hat{p}}{\rho_0 c}\left(1 + \frac{i}{kr}\right)e^{-i\omega t}\right]$$

$$= Re\left[\frac{A}{r\rho_0 c} e^{ikr}\left(1 + \frac{i}{kr}\right) e^{-i\omega t}\right]$$

$$= \frac{|A|}{r\rho_0 c}\cos(\omega t - kr - \psi) + \frac{|A|}{kr^2\rho_0 c}\sin(\omega t - kr - \psi);$$

$$\Rightarrow \underline{I} = p\underline{u} = pu_r\underset{\sim}{\underline{r}} = I_r \underset{\sim}{r}, \text{ where.}$$

$$I_r = pu_r = \frac{|A|^2}{\rho_0 c r^2}\cos^2(\omega t - kr - \varphi)$$

$$+ \frac{|A|^2}{k r^3 \rho_0 c}\cos(\omega t - kr - \varphi)\sin(\omega t - kr - \varphi)$$

For time-period average $\langle I_r \rangle := \frac{1}{T}\int_0^T I_r \, dt$.

($T = 2\pi/\omega$ time period), since:

$$\int_0^T \cos^2(\omega t - kr - \varphi)\, dt = \frac{T}{2} \quad \text{(exercise)}$$

$$\int_0^T \cos(\omega t - kr - \varphi)\sin(\omega t - kr - \varphi) = 0. \text{ (check)}$$

$$\Rightarrow \langle I_r \rangle = \frac{1}{\cancel{T}}\frac{|A|^2}{\rho_0 c r^2}\frac{\cancel{T}}{2} = \frac{|A|^2}{2\rho_0 c r^2}$$

$$\text{So } \langle I_r \rangle = \frac{|A|^2}{2\rho_0 c r^2} = \frac{\langle p^2 \rangle}{\rho_0 c} \qquad (22).$$

$$\left(\langle p^2 \rangle = \frac{1}{T}\int_0^T \frac{|A|^2}{r^2}\cos^2(\omega t - kr - \varphi) = \frac{|A|^2}{2r^2} \right)$$

So the 2nd term in u_r doesn't contribute.

1.2 Electromagnetic (EM) waves

The EM fields described by the Maxwell's equations (MEs). For electric field $\underline{E}(\underline{x},t)$, magnetic field $\underline{B}(\underline{x},t)$, given electric charge density ρ and electric current density j in a uniform medium, the MEs are

$$\nabla . \underline{E} = \frac{\rho}{\varepsilon} \quad (23), \quad \nabla . \underline{B} = 0 \quad (24)$$

$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} \quad (25), \quad \nabla \times \underline{B} = \mu\varepsilon \frac{\partial \underline{E}}{\partial t} + \mu j \quad (26)$$

where ε, μ are electric permittivity and magnetic permeability of the medium.

[Remark: ME's are generalisations, for continuum EM media, of Coulomb's law = (23), Faraday's law of EM induction = (25), Ampere's law = (26), (24) $\equiv$ $\underline{B}$ forms closed loop.]

In an EM medium without "sources" ($\rho \equiv 0$, $j \equiv 0$), MEs are:

$$\nabla . \underline{E} = 0 \quad (27), \quad \nabla . \underline{B} = 0 \quad (28)$$

$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} \quad (29), \quad \nabla \times \underline{B} = \mu\varepsilon \frac{\partial \underline{E}}{\partial t} \quad (30).$$

To elimintate $\underline{B}$, apply curl to (29), and use (30):

$$\nabla \times \nabla \times \underline{E} = -\frac{\partial}{\partial t} \nabla \times \underline{B} = -\mu \varepsilon \frac{\partial^2 \underline{E}}{\partial t}$$

Also (from vector calculus)

$$\nabla \times \nabla \times \underline{A} = \nabla(\nabla \cdot \underline{A}) - \overbrace{\nabla \cdot (\nabla A)}^{= \nabla^2 \underline{A}}.$$

$$\nabla \times \nabla \times E = \nabla \underbrace{(\nabla \cdot E)}_{= 0 \text{ by } (27)} - \nabla^2 \underline{E}$$

$$-\nabla^2 E = -\mu \varepsilon \frac{\partial^2 \underline{E}}{\partial t^2}$$

$$\iff \frac{1}{c^2} \frac{\partial^2 \underline{E}}{\partial t^2} - \nabla^2 \underline{E} = 0. \quad (31)$$

$c := (\varepsilon \mu)^{-1/2}$ wave speed.

Similarly (Excerise) $\nabla \times (30) + (29) \implies$

$$\frac{1}{c^2} \frac{\partial^2 \underline{B}}{\partial t^2} - \nabla^2 \underline{B} = 0. \quad (32)$$

So both $\underline{E}$ and $\underline{B}$ satisfy (vector) wave equations (31), (32), cf (9), (12) in acoustics. For vacuum, $c = c_0 = (\varepsilon_0 \mu_0)^{-1/2} \approx 3 \times 10^8 \, ms^{-1}$; if any other media it can only be smaller, physically.

Similarly to acoustics, (31)-(32) admit time-harmonic (TH) solutions:

$$\underline{E}(\underline{x},t) = Re\left(\hat{\underline{E}}(\underline{x})e^{-i\omega t}\right) \quad (33).$$

$$\underline{B}(\underline{x},t) = Re\left(\hat{\underline{B}}(\underline{x})e^{-i\omega t}\right) \quad (34)$$

where $\hat{\underline{E}}(x)$, $\hat{\underline{B}}(x)$ are complex-valued vector fields. As in acoustics (33) $\xrightarrow{\text{sub}}$ (31), (34) $\longrightarrow$ (32) yields Helmholz equ. for $\hat{E}$, $\hat{B}$, cf (15).

$$\nabla^2\hat{\underline{E}} + k^2\hat{\underline{E}} = 0 \quad (35), \qquad \nabla^2\hat{\underline{B}} + k^2\hat{\underline{B}} = 0. \quad (36)$$

where $k := \omega/c$ the wavenumber.

Plane T-H EM waves.

Similarly to acoustics, the solns of MEs admit plane T-H (EM) waves, propagating in any direction $\underline{n}$. Choosing $\underline{n}$ to coincide with z-direction, no generality loss. they have the form:

$$\underline{E}(\underline{x},t) = Re\left(\underline{E}_0 e^{ikz-i\omega t}\right), \quad (37)$$

$$\underline{B}(\underline{x},t) = Re\left(\underline{B}_0 e^{ikz-i\omega t}\right) \quad (38)$$

where $\underline{E}_0$, $\underline{B}_0$ are complex vector constants.

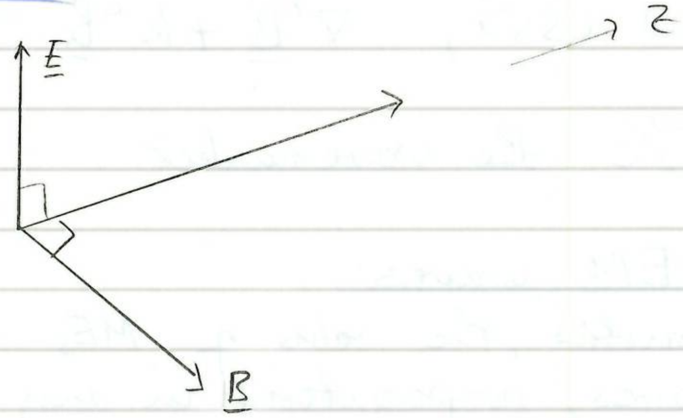
By above construction, $\forall\, \underline{E}_0, \underline{B}_0$ (31) & (32) are satisfied

Now $(27) \equiv \nabla \cdot \underline{E} = 0 \;=\; \mathrm{Re}\left((\underline{E})_z \, ik e^{ikz - i\omega t} \right) \equiv 0.$

$\Rightarrow (\underline{E}_0)_z = 0 \longleftrightarrow \underline{E}_0 = (E_x, E_y, E_z),\ \underline{E_z = 0}$

$(28) \equiv \nabla \cdot \underline{B} = 0 \Rightarrow \underline{B}_0 = (B_x, B_y, B_z),\ \underline{B_z = 0}$

So both E_0 and $\underline{B}_0$ must be perpendicular ($\perp$) to propagation direction (z), i.e. EM waves are <u>transverse</u>.



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Recall: $\nabla \cdot \underline{E} = 0$ (27), $\nabla \cdot \underline{B} = 0$ (28)

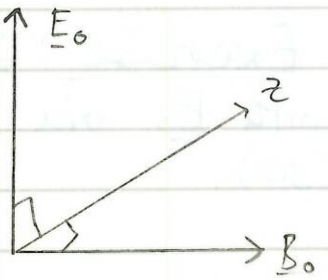
$$\nabla \times \underline{E} = -\frac{\partial B}{\partial t} \quad (29), \quad \nabla \times \underline{B} = \mu\varepsilon \frac{\partial \underline{E}}{\partial t} \quad (30).$$

$$\underline{E} = Re\left(\underline{E}_0 e^{ikz - i\omega t}\right) \quad (37).$$

$$\underline{B} = Re\left(\underline{B}_0 e^{ikz - i\omega t}\right) \quad (38)$$

$\underline{E}_0 = (E_x, E_y, 0)$

$\underline{B}_0 = (B_x, B_y, 0)$



Now plug (37) − (38) to (29):

$$\nabla \times \underline{E} = Re\left[e^{-i\omega t}\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_x e^{ikz} & E_y e^{ikz} & 0 \end{vmatrix}\right]$$

$$= Re\left[e^{-i\omega t}\left(-ikE_y e^{ikz}, ikE_x e^{ikz}, 0\right)\right]$$

$$= \cancel{Re}\left[\cancel{i}k\, \cancel{e^{ikz - i\omega t}}\left(-E_y, E_x, 0\right)\right]$$

$$= \cancel{Re}\left[\cancel{i}\omega\, \cancel{e^{ikz - i\omega t}}\left(B_x, B_y, 0\right)\right]$$

due to (29)

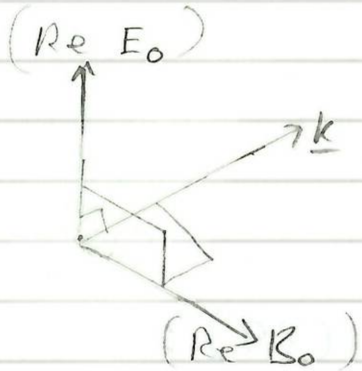
$$\Rightarrow \underline{B}_0 = \frac{k}{\omega}\left(-E_y, E_x, 0\right) = \frac{k}{\omega}\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 0 & 1 \\ E_x & E_y & E_z \end{vmatrix}$$

$$= \frac{k}{\omega} \, \underline{n} \times \underline{E}_0 \, , \quad \underline{n} = (0,0,1) \text{ unit vector in propagation direction,} \Leftrightarrow$$

$$\underline{B}_0 = \frac{1}{\omega} \underline{k} \times \underline{E}_0 \qquad (39)$$

Where $\underline{k} := k\underline{n}$ the <u>wave vector</u>.

So, via (39), $\underline{E}_0$, $\underline{B}_0$, $\underline{k}$ form a right-handed orthogonal triple.



<u>Exercise</u>: Show that $\underline{E}_0 \perp \underline{k}$ with $\underline{B}_0$ via (39) solve also (30).

So Plane T-H EM waves may have different <u>"polarisation"</u>. <u>Polarisation plane</u> is the plane containing $\underline{E}$ and $\underline{k}$.

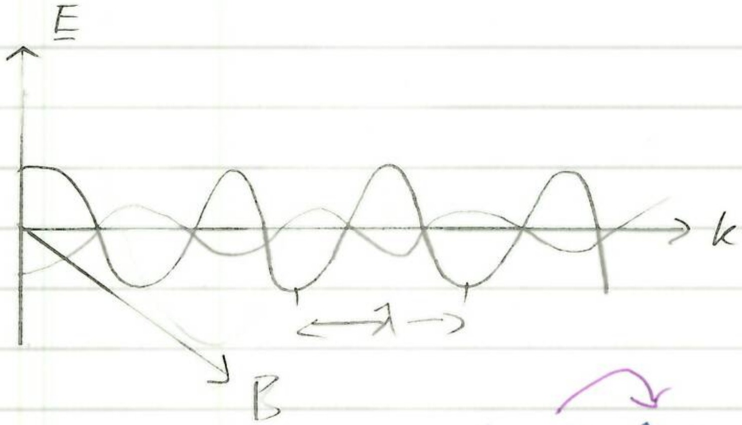
Remember $\underline{E}_0$, $\underline{B}_0$ are generally complex. If $\underline{E}_0$ is proportional to a real vector (i.e. $\underline{E}_0 = \tilde{\underline{E}}_0 e^{i\varphi}$, $\tilde{\underline{E}}_0$ real component vector) then polarisation plane is unchanged; more generally, polarisation will "rotate", however any plane T-H EM wave can be decomposed in sum of two polarised waves:

$$\underline{E}(x,t) = Re\left(\underline{E}_0 e^{i\underline{k}\cdot\underline{x} - i\omega t}\right)$$

$$= (Re\ \underline{E}_0)\cos(\omega t - \underline{k}.\underline{x})$$

$$+ (Im\ \underline{E}_0)\sin(\omega t - \underline{k}.x) =: \underline{E}^{(1)} + \underline{E}^{(2)},$$

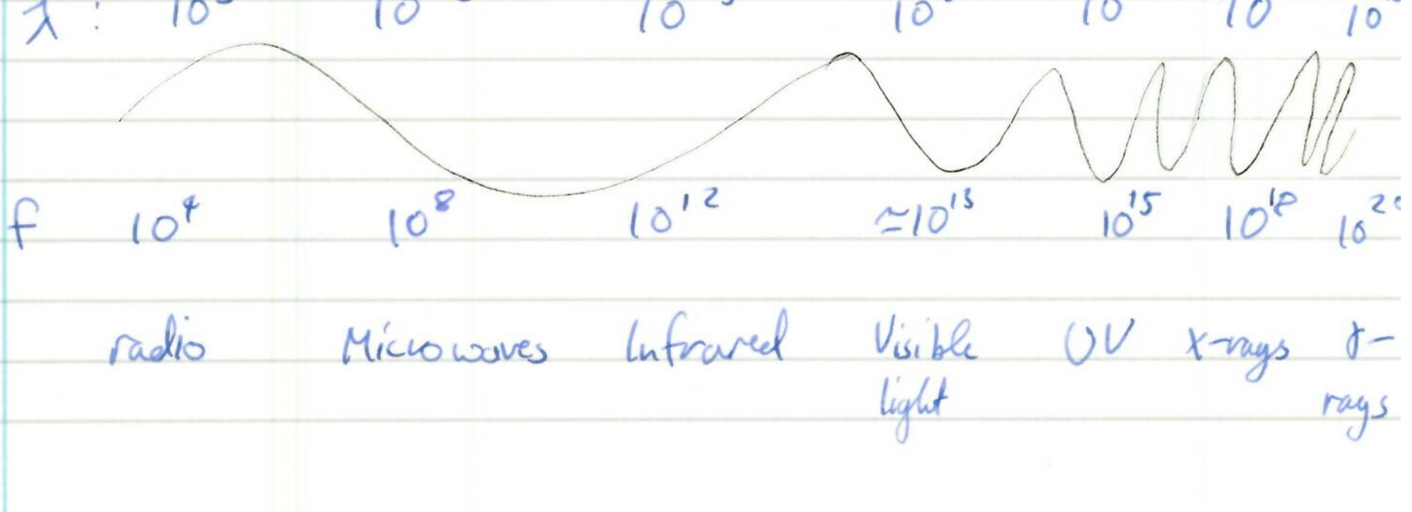
where $\underline{E}^{(1)}$ & $\underline{E}^{(2)}$ are polarised.



wavelength; $\lambda = \frac{2\pi}{k} = \frac{2\pi}{\omega}c = \frac{c}{f}$

$f = \frac{\omega}{2\pi}$. frequency

Depending on f, λ. EM waves found in different parts of the "spectrum".



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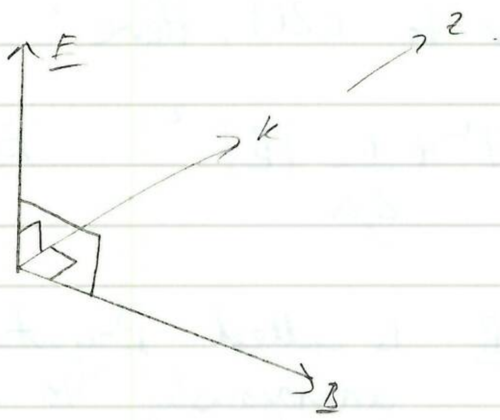


Poynting's vector. Similarly to acoustics of (21), using ME's (29), (30), we can derive a conservation law for EM energy:

$$(29) \Leftrightarrow \underline{B}\cdot\Big|\ \nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$$

$\ominus$

$$(30) \Leftrightarrow \underline{E}\cdot\Big|\ \nabla \times \underline{B} = \mu\varepsilon \frac{\partial \underline{E}}{\partial t}.$$

Take dot product of (29) with $\underline{B}$, (30) with $\underline{E}$ and and subtract:

$$\underbrace{\underline{B}\cdot(\nabla\times\underline{E}) - \underline{E}\cdot(\nabla\times\underline{B})} = -\underline{B}\cdot\frac{\partial \underline{B}}{\partial t} - \mu\varepsilon\,\underline{E}\cdot\frac{\partial \underline{E}}{\partial t}$$

From vector calculus (check):

$$\nabla\cdot(\underline{A}\times\underline{B}) = (\nabla\times\underline{A})\cdot\underline{B} - (\nabla\times\underline{B})\cdot\underline{A}.$$

$$\Rightarrow \nabla\cdot(\underline{E}\times\underline{B}) = -\frac{\partial}{\partial t}\left(\frac{1}{2}\underline{B}\cdot\underline{B} + \frac{1}{2}\mu\varepsilon\,\underline{E}\cdot\underline{E}\right)$$

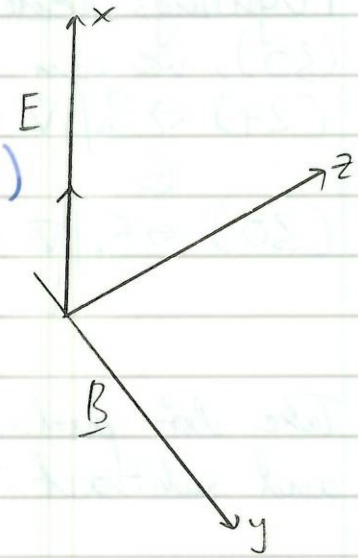
$$\Leftrightarrow \frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon|\underline{E}|^2 + \frac{1}{2\mu}|\underline{B}|^2\right) + \nabla\cdot\left(\frac{\underline{E}\times\underline{B}}{\mu}\right) = 0 \quad (40).$$

which is similar to (21). Here:

$$W := \frac{1}{2}\varepsilon|\underline{E}|^2 + \frac{1}{2\mu}|\underline{B}|^2$$ EM field's energy density.

$\underline{S} := \frac{1}{\mu}\underline{E}\times\underline{B}$ is called Poynting's vector which analogously to acoustic intensity $\underline{I}$ describes energy transfer.

For Plane T-H EM waves.



e.g. $\underline{E}(x,t) = (E_x, 0, 0)\cos(kz-\omega t)$

$$\Rightarrow \underline{B}(\underline{x},t) = \frac{1}{\omega}(\underline{k}\times E_0)$$

$$= \left(0, \frac{E_x}{c}, 0\right)\cos(\omega t - kz)$$

$$\Rightarrow \underline{S} = \frac{1}{\mu}\underline{E}\times\underline{B} = \frac{1}{\mu c}\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ E_x & 0 & 0 \\ 0 & E_x & 0 \end{vmatrix}\cos^2(\omega t - kz)$$

$$= \frac{|E_x|^2}{\mu c}(0,0,1)\cos^2(\omega t - kz)$$

so EM flows in the z-direction ≡ direction of propagation, as expected.

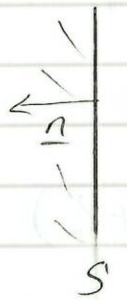
1.3 (Surface) Boundary and Interface conditions.

Both in acoustics and EM, boundary / interface conditions are required e.g. at a solid surface or interface between two media (e.g air and water). Mathematically, a PDEs boundary conditions (BC) typically are:

- Dirichlet BC, when the unknown functionals value is specified on the boundary S.
- Neumann's BCs, for normal derivative $\partial p / \partial n$ of the function p given on S.
- Mixed / Robin / Impedance BC's, with a combination of p and $\partial p / \partial n$ .. on S.

13.1 Acoustic BC's.

i) An acoustically hard surface doesn't yield to the acoustic wave $\leftrightarrow$ is rigid $\leftrightarrow$ $\underline{u}$ velocity, $\underline{u} \cdot \underline{n} = 0$.
(no motion of S)



(Assuming irrotational motion)
For ϕ velocity potential $\underline{u} = \nabla \phi$
$\Rightarrow \nabla \phi \cdot \underline{n} = 0 \Leftrightarrow \partial \phi / \partial n = 0 \leftrightarrow$ Neumann's BC for ϕ.

ii) Acoustically soft surface freely yields to the wave (e.g water surface) $\rightarrow$ Physically on S, $p = 0$, i.e. Dirichlet BC for p.

iii) Impedance BCs are intermediate between (i) and (ii): a surface specific impedence, Z, measures to what extent the surface motion is "impeded" by the resisting pressure.

Physically, Z should depend on frequency, $Z = Z(\omega)$, so for T-H acoustics $p = \mathrm{Re}(\hat{p}e^{-i\omega t})$, $\underline{u} = \mathrm{Re}(\hat{\underline{u}}e^{-i\omega t})$

$$Z := \frac{\hat{p}}{\hat{u}_n}, \quad \hat{u}_n := \hat{\underline{u}} \cdot \underline{n} \text{ (normal velocity)}.$$

$$\Leftrightarrow \hat{p} - Z\hat{u}_n = 0 \Leftrightarrow \hat{p} - Z\frac{\partial \hat{\phi}}{\partial n} = 0.$$

$$\left(\phi = \mathrm{Re}\left(\hat{\phi}e^{-i\omega t}\right)\right)$$

From (11) $\longleftrightarrow \frac{\partial \phi}{\partial t} = -\frac{1}{\rho_0} p \overset{TH}{\Longleftrightarrow}$

$$\cancel{-}i\omega\hat{\phi}\cancel{e^{-i\omega t}} = \cancel{+}\frac{1}{\rho_0}\hat{p}\cancel{e^{-i\omega t}} \Rightarrow \hat{p} = i\omega\rho_0\hat{\phi} \qquad (41)$$

Hence

$$i\omega\rho_0\hat{\phi} - Z(\omega)\frac{\partial \hat{\phi}}{\partial n} = 0 \qquad (42)$$

which is the impedence / mixed BC for $\hat{\phi}$. (Same (42) $\hat{\phi} \to \hat{p}$)

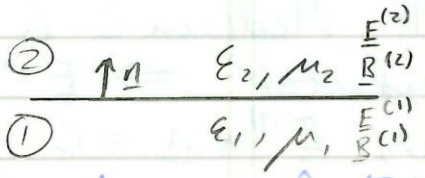
$$i\omega\rho_0\hat{p} - Z(\omega)\frac{\partial \hat{p}}{\partial n} = 0 \qquad (42')$$

Notice: if $Z \to 0 \Rightarrow \hat{p} = \hat{\phi} = 0$ (soft BCs (ii)); $Z \to \infty$ ($\Leftrightarrow 1/Z \to 0$) $\Rightarrow \partial\hat{\phi}/\partial n = 0$. (hard BC's (i)).

Generally, Z is complex; physically $\mathrm{Re}\, Z \geq 0$, with $\mathrm{Re}\, Z > 0$ for energy absorbing surfaces, $\mathrm{Re}\, Z = 0$ for neutral, c.f.

Excercise - 1 Qn 6(ii) to §2.1.

1.3.2 EM BCs / Interface condns:



At an interface between two EM media (ε_1, μ_1 medium 1; ε_2, μ_2 medium 2) the physics of EM implies the following interface conditions:

(i) Tangential components of $\underline{E}$ are continuous (cts): $\Leftrightarrow$

$$\underline{E}^{(1)} \times \underline{n} = \underline{E}^{(2)} \times \underline{n}. \qquad (43)$$

(ii) Normal components of $\underline{B}$ are cts $\Leftrightarrow$

$$\underline{B}^{(1)} \cdot \underline{n} = \underline{B}^{(2)} \cdot \underline{n}. \qquad (44).$$

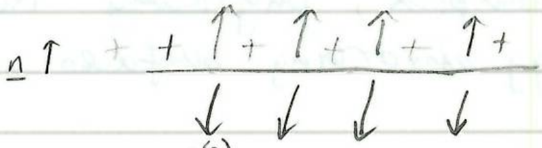
Also, if there exist a surface charge distribution ρ_s and/or surface current distrn j_s, then

(iii) $$\varepsilon_2 \underline{E}^{(2)} \cdot \underline{n} - \varepsilon_1 \underline{E}^{(1)} \cdot \underline{n} = \rho_s \qquad (45)$$

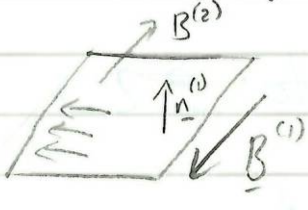
(iv) $$\frac{1}{\mu_1} \underline{B}^{(1)} \times \underline{n} - \frac{1}{\mu_2} \underline{B}^{(2)} \times \underline{n} = j_s \qquad (46).$$

(In (45), (46) $\underline{n}$ points into medium 2.)

[Check :

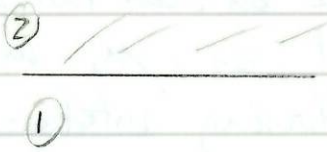


$\rho_s > 0 \Rightarrow \underline{E}^{(2)} \cdot n > 0$,
$\underline{E}^{(1)} \cdot n < 0$



+ "right hand rule" from Physics $\rightarrow$ OK] .

Perfectly conducting BCs : If Medium 2 is a "pefect conductor" $\Rightarrow \underline{E}^{(2)} = 0 \Rightarrow$ from (43), $\underline{E}^{(1)} \times \underline{n} = 0$ (47)



i.e zero normal component of $\underline{E}$ along a perfectly conducting surface.

(End of Chapter 1) .

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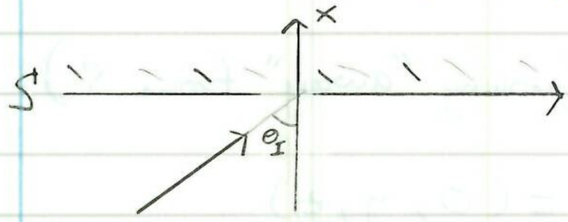
2. Canonical Cases:

2.1 Reflection by a plane (acoustics).



Consider a T-H acoustic plane wave incident upon a plane boundary:

$$S = \{(x,y,z) \mid x=0\}.$$

$$p_I = A_I e^{-i\omega t - ik_I \cdot x}$$
$$= e^{-i\omega t}\hat{p}_I,$$
$$\hat{p}_I = A_I e^{ik_I \cdot x};$$

assume the wave has "angle of incidence" $\theta_I \Rightarrow$

$$\underline{k}_I = k(\cos\theta_I, 0, \sin\theta_I),$$

$$k := |\underline{k}_I| = \omega/c$$

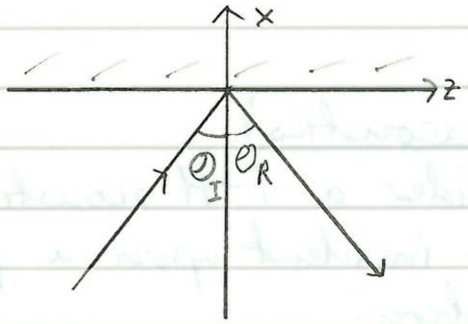
$0 \leq \theta_I < \pi/2$ (moving "towards" S).

The boundary S will "reflect" the incident wave p_I. What is the reflected wave p_R? This depends on BCs on S.

Consider acoustically soft BCs: $p=0$ on S (§1.31)

$\Leftrightarrow p = p_I + p_R = 0$ on S.

Seek p_R also a plane T-H wave:



$$p_R = A_R e^{-i\omega_R t + i\underline{k}_R \cdot \underline{x}},$$

$\underline{k}_R = (k_x, k_y, k_z)$, $|\underline{k}_R| = \omega_R / c = k_R$

$k_x < 0$ (moving "away" from S')

BCs $\Rightarrow$ $\underline{x} \in S' \Leftrightarrow x = 0$, $\underline{x} = (0, y, z)$

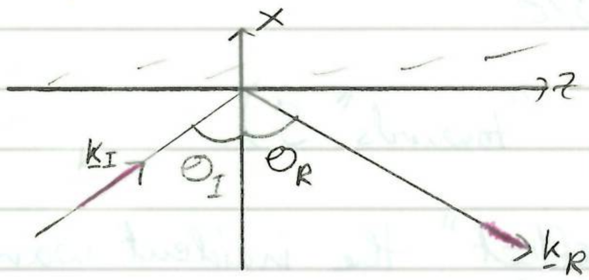
$p = p_R + p_I = 0 \Leftrightarrow p_R = -p_I$

$\Leftrightarrow A_R e^{-i\omega_R t + i(k_y y + k_z z)} = -A_I e^{-i\omega t + ik\sin\theta_I z}$ $\forall t, y, z$.

$\Rightarrow \omega_R = \omega$ (varying t); $k_y = 0$ (varying y);
$k_z = k\sin\theta_I$ (varying z). $\Rightarrow k_R = \omega_R / c = \omega / c = k$;

$A_R = -A_I$; $k_x^2 + k_z^2 = k^2 \Rightarrow k_x = \pm\sqrt{k^2 - k_z}$
$\Rightarrow k_x = -\sqrt{k^2 - k^2\sin^2\theta_I}$ since $k_x < 0$.
$= -k\cos\theta_I$, so
$k_R = k(-\cos\theta_I, 0, \sin\theta_I)$;

Interpretation.



$\theta_R = \theta_I$ (Specular reflection law)

$A_R = -A_I$ (Phase "shifts" for π, $-1 = e^{-i\pi}$)

Excercise(4 i) Show that for acoustically hard plane boundaries ($\partial\phi/\partial n = 0 \Leftrightarrow \partial p/\partial n = 0$)

$$p_R = A_I e^{-i\omega t + ik(-x\cos\theta_I + z\sin\theta_I)}$$

ii) Find the reflected wave for a general impedance BC (1.41)
(For same geometry and incident wave).

$$i\omega\rho \cdot \hat{\phi} - Z\frac{\partial\hat{\phi}}{\partial n}$$

Conclude that $|A_R| \leq |A_I| \Leftrightarrow \mathrm{Re}\, Z \geq 0$ (absorbing / neutral boundary).

$\boxtimes$

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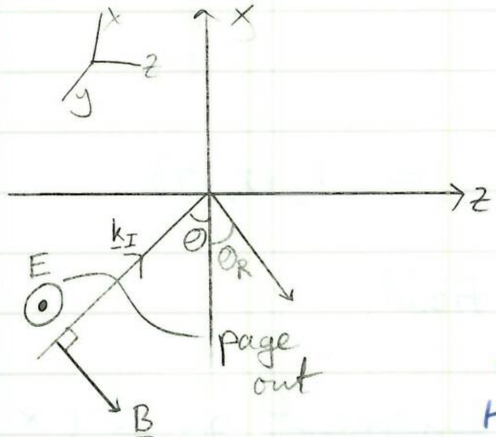
2.2 Reflectcon of plane T-H EM waves by perfectly conducting planes:



Consider a plane T-H EM wave propagating within "insulator" medium. (no currents j) medium occupying half - space $x<0$; with inscidence angle θ, unknown angle of reflection θ_R, ϵ, μ given $\mapsto c = (\epsilon\mu)^{-1/2}$ wave speed.

Let the incident wave be polarised so that $\underline{E}_I$ is parallel to y - axis, pointing out of the plane, i.e.

$$\underline{E}_I = (0, E_I, 0)e^{-i\omega t + ik(x\cos\theta + z\sin\theta)} \qquad (1)$$

$$\underline{B}_I = \frac{E_I}{c}(-\sin\theta, 0, \cos\theta)e^{-i\omega t + ik(x\cos\theta + z\sin\theta)} \qquad (2)$$

$$\underline{k}_I = k(\cos\theta, 0, \sin\theta)$$

The reflected wave is also sought as plane wave:

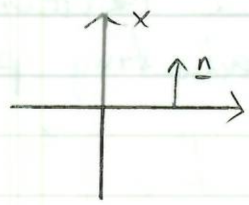
$$\underline{E}_R = (E_x, E_y, E_z)e^{-i\omega_R t + ik_R(-x\cos\theta_R + z\sin\theta_R)} \qquad (3)$$

$$\underline{B}_R = (B_x, B_y, B_z)e^{-i\omega_R t + ik_R(-x\cos\theta_R + z\sin\theta_R)} \qquad (4)$$

$k_R = \omega_R/c$, $E_x, E_y, E_z, B_x, B_y, B_z$ unknown coeff.

From $\underline{\text{prefectly conducting}}$ BCs (1.47) on

$$S = \{x = 0\} \quad , \quad \underline{E} \times \underline{n} = 0 .$$



$\underline{n}$ normal vector to S' $\Leftrightarrow$ $\underline{n} = (1, 0, 0)$

$\underline{E} = \underline{E}_I + \underline{E}_R$ = total electonic field.

$\Leftrightarrow$ $\underline{E}$ has zero y & z - component on S $(x = 0)$

$\underline{y\text{-component}}$:

$$E_y e^{-i\omega_R t + ik_R z \sin\theta_R} = -E_I e^{-i\omega t + ikz\sin\theta} \qquad \forall t, z$$

$$\Rightarrow \omega_R = \omega \Rightarrow k_R = k \,(= \omega/c) \Rightarrow \theta_R = \theta$$

(i.e. specular reflectcon law, same as in acoustics)

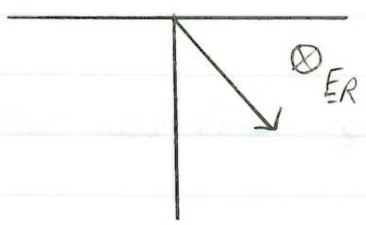
$$E_y = -E_I .$$

$\underline{z\text{-component}}$: $\Rightarrow E_z = 0$.

Also, since $(\underline{E}_R, \underline{B}_R)$ is a plane wave $\Rightarrow$

$\underline{E}_R \perp \underline{k}_R \Leftrightarrow \underline{E}_R \cdot \underline{k}_R = 0 \Leftrightarrow -E_x \cos\theta_R + 0 + 0 = 0$

$\Rightarrow E_x = 0$.



So $\underline{E}_R = (0, -E_I, 0) e^{-i\omega t + i\underline{k}_R \cdot \underline{x}}$

and points in the plane.

Finally, from (1.39)

$$\underline{B} = \frac{1}{\omega}\underline{k}_R \times \underline{E}_R = \frac{k}{\omega}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\cos\theta & 0 & \sin\theta \\ 0 & -E_I & 0 \end{vmatrix} e^{-i\omega t + i\underline{k}_R\cdot\underline{x}}$$

$$= \frac{E_I}{c}\left(\sin\theta, 0, \cos\theta\right) e^{-i\omega t + i\underline{k}_R \underline{x}}$$

Finally, lets us find "induced electric current" on S:

Notice $\underline{E} \equiv 0$ in $x > 0 \Rightarrow B \equiv 0$ in $x > 0$ from ME. (1.25).

$$\frac{\partial \underline{B}}{\partial t} = 0 \Rightarrow \underline{B} = 0 \quad (\text{since } \underline{B} \text{ T-H})$$

Now from interface condition (1.46):

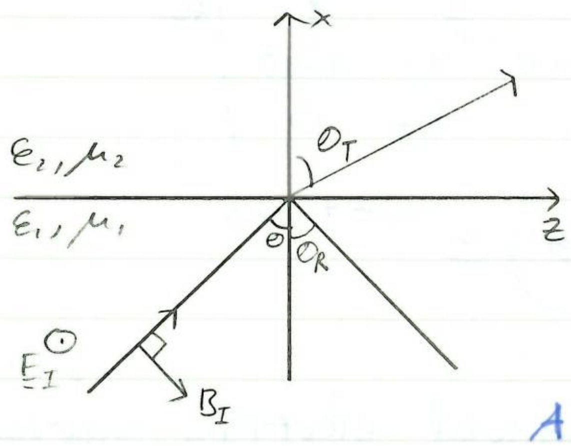
$$\frac{1}{\mu}\underline{B}^{(1)} \times \underline{n} - \cancel{\frac{1}{\mu}\underline{B}^{(2)} \times \underline{n}}^{\,0} = \underline{j}s$$

$$\Rightarrow \underline{j}_s = \frac{1}{\mu}(\underline{B}_I + \underline{B}_R) \times (1,0,0)$$

$$= \frac{E_I}{c\mu}\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 0 & 2\cos\theta \\ 1 & 0 & 0 \end{vmatrix} e^{-i\omega t + i\underline{k}_R\cdot x}$$

$$= \frac{2\cos\theta E_I}{c\mu} e^{-i\omega t - i\underline{k}_R\cdot x}$$ the induced current ☒

2.3 Reflection and refraction by a plane interface between two insulator media.



Consider two insulator EM media with interface $S' = \{x = 0\}$, ε_1, μ_1 medium 1 $(x<0)$, ε_2, μ_2 medium 2 $(x>0)$.

A plane wave incident from medium 1 same as in §2.2, i.e describible by (1) & (2):

(1) $\rightarrow$ $\underline{E}_I = E_I(0,1,0)e^{-i\omega t + ik(x\cos\theta + z\sin\theta)}$

(2) $\rightarrow$ $\underline{B}_I = \frac{E_I}{c}(-\sin\theta, 0, \cos\theta)e^{-i\omega t + ik(x\cos\theta + z\sin\theta)}$

Seek the reflected wave also as in (3) – (4),

(3) $\rightarrow$ $\underline{E}_R = (E_x, E_y, E_z)e^{-i\omega_R t + ik_R(-x\cos\theta_R + z\sin\theta_R)}$

(4) $\rightarrow$ $\underline{B}_R = (B_x, B_y, B_z)e^{-i\omega_R t + \text{ —"— —"—}}$

with (different) unknowns $E_x, E_y, \ldots, B_z$

Additionally, a "transmitted" wave is sought in $x>0$:
$(k_R = \omega_R/c_1, \; k_T = \omega_T/c_2)$

$$\underline{E}_T = (\hat{E}_x, \hat{E}_y, \hat{E}_z)e^{-i\omega_T t + ik_T(x\cos\theta_T + z\sin\theta_T)} \quad (5)$$

$$\underline{B}_T = (\hat{B}_x, \hat{B}_y, \hat{B}_z)e^{-i\omega_T t + ik_T(x\cos\theta_T + z\sin\theta_T)} \quad (6)$$

$(\underline{n} \times \underline{n} = 0)$

From (1.43) $\Longleftrightarrow \underline{E}^{(1)} \times \underline{n} = \underline{E}^{(2)} \times \underline{n} \Longleftrightarrow$ continuity of y and z - components of $\underline{E}$ $\Rightarrow$

<u>y - component</u>: $\Rightarrow \omega_R = \omega_T = \omega \Rightarrow k_R = k = \frac{\omega}{c_1}$, $k_T = \frac{\omega}{c_2}$

$c_1 = (\varepsilon_1 \mu_1)^{-1/2}$, $c_2 = (\varepsilon_2 \mu_2)^{-1/2}$

$\Rightarrow \theta_R = \theta$ (as before).

and <u>additionally</u> $k_T \sin\theta_T = k \sin\theta$

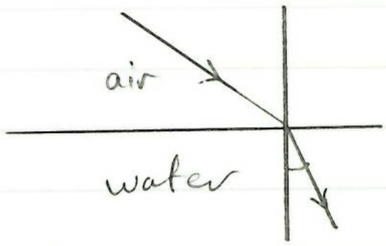
$$\Longleftrightarrow \boxed{\frac{\sin\theta_T}{c_2} = \frac{\sin\theta}{c_1}} \qquad \left(\frac{\sin\theta_1}{c_1} = \frac{\sin\theta_2}{c_2}\right) \qquad (7)$$

Let $n := c_0/c$, c_0 speed of light (vacuum wave speed)

$\Rightarrow n \geq 1$ ($n = 1 \Longleftrightarrow$ vacuum) $\Rightarrow$ (7) $\Longleftrightarrow$

$$n_1 \sin\theta_1 = n_2 \sin\theta_2, \quad n \sin\theta = \text{const} \qquad (8).$$

(7) or (8) is the <u>Snell's law of reflaction</u>



(y-component)

$$\text{Further,} \quad E_I + E_y = \hat{E}_y \qquad (9)$$

$$E_z = \hat{E}_z \qquad (10)$$

(z-component)
(of $\underline{E}$)

Also, $\underline{E}_R \cdot \underline{k}_R = 0$, $\underline{E}_T \cdot \underline{k}_T = 0$

$$\Rightarrow \; -E_x \cos\theta + E_z \sin\theta = 0 \qquad (11)$$

$$\hat{E}_x \cos\theta_T + \hat{E}_z \sin\theta_T = 0 \qquad (12)$$

Further, from (1.46), and no surface currents for the two insulators,

$$\frac{1}{\mu_1}\left(\underline{B}_I + \underline{B}_R\right) \times \underline{n} = \frac{1}{\mu_2}\underline{B}_T \times \underline{n} \, . \qquad (*)$$

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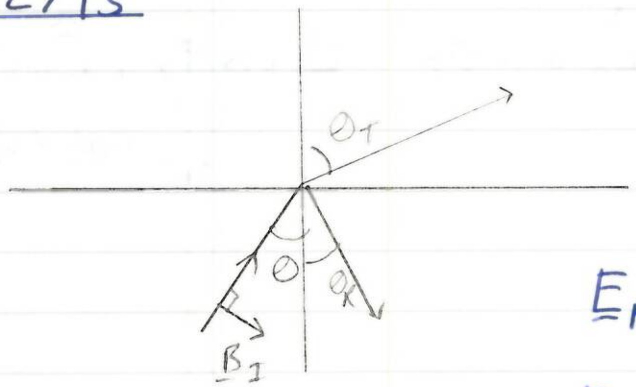

$$\frac{\sin\theta}{c_1} = \frac{\sin\theta_T}{c_2} \quad (7)$$

$$\underline{E}_R = (E_x, E_y, E_z)e^{\cdots}$$

$$\underline{E}_T = (\hat{E}_x, \hat{E}_y, \hat{E}_z)e^{\cdots}$$

$$\underline{E}_I = (0, E_I, 0)$$

$$E_I + E_y = \hat{E}_y \quad (9)$$

$$\ldots \quad (10, 11, 12)$$

$$\frac{1}{\mu_1}(\underline{B}_I + \underline{B}_R) \times \underline{u} = \frac{1}{\mu}\hat{B}_T \times \underline{u}. \quad (*)$$

—/—

$$(1.39) \Rightarrow \underline{B}_R = \frac{1}{\omega}\underline{k}_R \times \underline{E}_R$$

$$= \frac{k}{\omega}\begin{vmatrix} i & j & k \\ \cos\theta & 0 & \sin\theta \\ E_x & E_y & E_z \end{vmatrix} e^{-i\omega t + i\underline{k}_R \cdot \underline{x}}$$

$$= \frac{1}{c_1}(-E_y\sin\theta, E_z\cos\theta + E_x\sin\theta, -E_y\cos\theta)e^{-i\ldots}$$

$$\underline{B}_T = \frac{1}{\omega}\underline{k}_T \times \underline{E}_T = \frac{k_T}{\omega}\begin{vmatrix} i & j & k \\ \cos\theta_T & 0 & \sin\theta_T \\ \hat{E}_x & \hat{E}_y & \hat{E}_z \end{vmatrix}$$

$$= \frac{1}{c_2}\left(-\hat{E}_z \sin\theta_T\,,\ \hat{E}_x \sin\theta_T - E_z\cos\theta_T\,,\ \hat{E}_y\cos\theta_T\right)\cdot$$

$$\cdot e^{-i\omega t + i\underline{k}_T\cdot\underline{x}}$$

where $\underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

y-component of (✱):

$$\frac{1}{\mu_1 c_1}\left(E_z\cos\theta + E_x\sin\theta\right) = \frac{1}{\mu_2 c_2}\left(\hat{E}_x\sin\theta_T - \hat{E}_z\cos\theta_T\right) \quad (13)$$

z-comp of (✱):

$$\frac{1}{\mu_1 c_1}\left(E_I - E_y\right)\cos\theta = \frac{1}{\mu_2 c_2}\hat{E}_y\cos\theta_T \qquad (14)$$

So (9)-(14) are 6 equs for $E_x, E_y, \ldots, \hat{E}_y, \hat{E}_z$.
Notice (10), (11), (12), (13) are equs for E_x, E_z, $\hat{E}_x$, $\hat{E}_z$ only.

$$(11) \rightarrow = E_x$$

$$(13) \Rightarrow \frac{1}{\mu_1 c_1}\left(E_z\cos\theta + \frac{E_z\sin\theta}{\cos\theta}\sin\theta\right)$$

$$= \frac{1}{\mu_2 c_2}\left(-\frac{\hat{E}_z\sin^2\theta_T}{\cos\theta_T} - E_z\cos\theta_T\right)$$

$$\Leftrightarrow \frac{1}{\mu_1 c_1\cos\theta}E_z = -\frac{1}{\mu_1 c_2\cos\theta_T}E_z \Rightarrow E_z = 0.$$

$\Rightarrow E_x = \hat{E}_x = \hat{E}_z = E_z = 0$.

So, like the incident electric field $\underline{E}_I$, both $\underline{E}_R$ and $\underline{E}_T$ have only y-component (same polarisation)

$\leftrightarrow$ "transverse - electric" $\leftrightarrow$ TE wave.

So we have left (9), (14) for $E_y, \hat{E}_y$:

$$(9) \rightarrow E_I + E_y = \hat{E}_y$$

$$\frac{1}{\mu_1 c_1}(E_I - E_y)\cos\theta = \frac{1}{\mu_2 c_2}\hat{E}_y \cos\theta_T . \quad (14)$$

$(9) \rightarrow (14) \Rightarrow$

$$\underbrace{\boxed{\frac{\cos\theta}{\mu_1 c_1}}}_{m_1}(E_I - E_y) = \underbrace{\boxed{\frac{\cos\theta_T}{\mu_2 c_2}}}_{m_2}(E_I + E_y)$$

$$m_1 E_I - m_1 E_y = m_2 E_I + m_2 E_y$$

$$\Rightarrow E_y = \frac{m_1 - m_2}{m_1 + m_2} E_I, \quad m_1 = \frac{\cos\theta}{\mu_1 c_1}, \quad m_2 = \frac{\cos\theta_T}{\mu_2 c_2}$$

$$\hat{E}_y = E_I + E_y = \frac{2m_1}{m_1 + m_2} E_I .$$

If both media 1 and 2 are "non-magnetic" (as often physically the case) i.e $\mu_1 = \mu_2 = \mu_0$

(but $\epsilon_1 \neq \epsilon_2$) $\Rightarrow$ ($n_1 = c_0/c_1$, $n_2 = c_0/c_2$)

$$E_y = \frac{n_1 \cos\theta - n_2 \cos\theta_T}{n_1 \cos\theta + n_2 \cos\theta_T} E_I \quad (15)$$

$$\hat{E}_y = \frac{2n_1 \cos\theta}{n_1 \cos\theta + n_2 \cos\theta_T} E_I \quad (16)$$

Now via Snell's law (8), $n_1 \sin\theta = n_2 \sin\theta_T$

$$E_y = \frac{\sin\theta_T \cos\theta - \sin\theta \cos\theta_T}{\sin\theta_T \sin\theta + \sin\theta \cos\theta} E_I$$

$$= \frac{\sin(\theta_T - \theta)}{\sin(\theta_T - \theta)} E_I \quad (17)$$

$$\hat{E}_y = \frac{2\sin\theta_T \cos\theta}{\sin(\theta_T + \theta)} E_I \quad (18)$$

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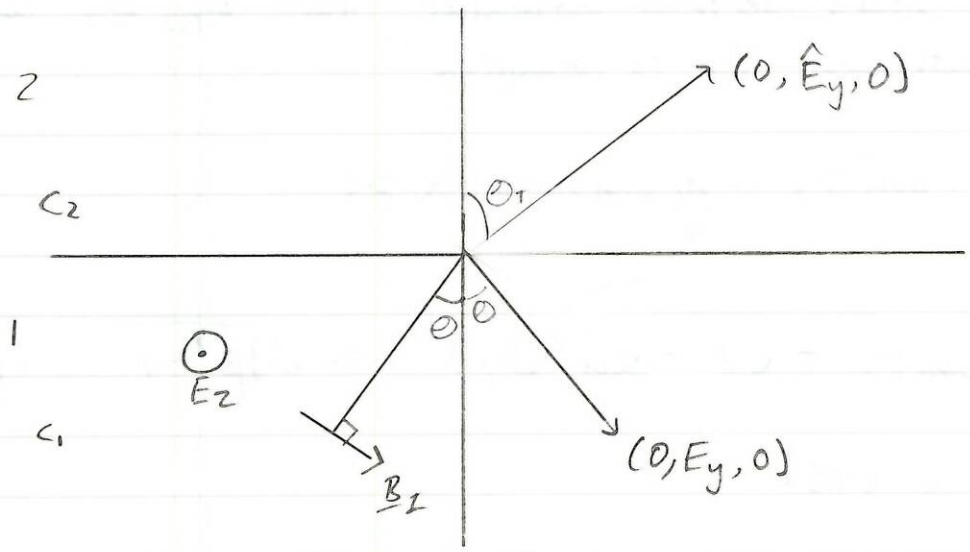


$$\frac{\sin\theta}{c_1} = \frac{\sin\theta_T}{c_2} \qquad (7)$$

$(E_y, \hat{E}_y) \longleftrightarrow (17) - (18) \; / \; (15) - (16)$

After $\underline{E}_R, \underline{E}_T$ found $\Rightarrow$ by (1.39)

$$\underline{B}_R = \frac{1}{\omega} \underline{k}_R \times \underline{E}_R \,, \quad \underline{B}_T = \frac{1}{\omega} \underline{k}_T \times \underline{E}_T \,.$$

Total internal reflection

Snell's law (7) $\Rightarrow \sin\theta_T = \dfrac{c_2}{c_1}\sin\theta = \dfrac{n_1}{n_2}\sin\theta$.

($n_j = c_0/c_j$ $\; j = 1, 2$ refractive index).

Let $c_2 > c_1$ and $c_2/c_1 \sin\theta > 1 \iff$

$$\theta > \sin^{-1}\left(\frac{c_1}{c_2}\right) =: \theta_c = \text{"the critical angle"}.$$

All the above derived formulae for $\underline{E}_R$, $\underline{E}_T$ are still valid, however θ_T is complex:

$$\sin\theta_T = \frac{n_1}{n_2}\sin\theta =: \alpha > 1 .$$

$$\Rightarrow \cos\theta_T = (1-\sin^2\theta_T)^{1/2} = i\sqrt{\beta^2-1} = i\beta, \; \beta > 0 .$$

$\Rightarrow$ from (15)

$$E_y = \frac{n_1\cos\theta - n_2\cos\theta_T}{n_1\cos\theta + n_2\cos\theta_T} E_I = \frac{n_1\cos\theta - i\beta n_2}{n_1\cos\theta + i\beta n_2} = E_I .$$

$\Rightarrow |E_y| = |E_I| \leftrightarrow$ reflected & incident amplitudes same
$\Rightarrow$ All the energy is <u>reflected</u>. (phases differ $\rightarrow$ phase change upon the reflection).

$\langle$ For $\theta < \theta_c$, (15) $\Rightarrow |E_y| < |E_I|$
$\leftrightarrow$ reflected energy less than transmitted. $\rangle$.

For $\theta > \theta_c$, for $x > 0$:

$$\underline{k}_T = k_T(\cos\theta_T, 0, \sin\theta_T)$$

$$= \frac{\omega}{c_2}(i\beta, 0, \alpha)$$

$$\underline{E}_T = Re\left\{\underline{\hat{E}} e^{i\underline{k}_T\cdot\underline{x} - i\omega t}\right\}$$

$$= Re\left\{(0, \hat{E}_y, 0) e^{ik_T\alpha z - k_T\beta x - i\omega t}\right\}$$

$E_y = |E_y| e^{i\varphi}$

$$= \left(0, |E_y| e^{-k_T\beta x}\cos(\omega t - k_T\alpha z - \varphi), 0\right)$$

(so $\underline{E}_T$ exponentially decays as $x \to +\infty$)

$$(1.39) \Rightarrow \underline{B}_T = \frac{1}{\omega}\underline{k}_T \times \underline{E}_T$$

$$= \mathrm{Re}\left\{\frac{k_T}{\omega}\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ i\beta & 0 & \alpha \\ 0 & \hat{E}_y & 0 \end{vmatrix} e^{ik_T\alpha z - k_T\beta - i\omega t}\right\}$$

$$= \frac{e^{-k_T\beta x}}{c_2}\left(-\alpha|\hat{E}_y|\cos(\underbrace{\omega t - k_T\alpha z - \varphi}_{\phi}), 0, \beta|\hat{E}_y|\sin(\omega t - k_T\alpha z - \varphi)\right)$$

Now for Poyting vector $\underline{S}_T$ (§1.2) for transmitted wave,

$$\underline{S} = \frac{1}{\mu}\underline{E}_T \times \underline{B}_T$$

$$= \frac{e^{-2k_T\beta x}}{c_2\mu}|\hat{E}_y|^2 \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & \cos\phi & 0 \\ -\alpha\cos\phi & 0 & \beta\sin\phi \end{vmatrix}$$

$$(\phi := \omega t - k_T\alpha z - \varphi)$$

$$\Rightarrow \langle \underline{S} \rangle_T = \frac{e^{-2k_T\beta x}|\hat{E}_y|^2}{\mu c_2}\left(\beta\overbrace{\langle \sin\phi\cos\phi\rangle_T}^{0}, 0, \alpha\overbrace{\langle\cos^2\phi\rangle_T}^{\frac{1}{2}}\right)$$

$$\left(\langle S\rangle_T := \frac{1}{T}\int_0^T S(\underline{x}, t)\, dt,\ T = \frac{2\pi}{\omega},\ \text{Time averaged } \underline{S}\right)$$

$$= \frac{\alpha e^{-2\beta k_T x}|\hat{E}_y|^2}{2\mu c_2}(0, 0, 1)$$

$\left(\langle \sin\phi \cos\phi \rangle_T = 0 \ , \ \langle \cos^2\phi \rangle = \frac{1}{2} \right)$

So the EM energy (time averaged one) flows along $z \Leftrightarrow$ parallel to the interface S'. (So for "transmitted" wave. No energy transport away from S'.

Exercise: Similar analysis can be done for incident plane wave polarised "in x, z-plane" $\Leftrightarrow \underline{B}_I(0, B_I, 0) \Leftrightarrow$ "transverse magnetic"/TM (cf TE/transverse electric case above).

All same affects / $\theta_R = \dot{\theta}$, θ_T from Snell's law (7), total interval refln for $\theta > \theta_c$)

Difference: (17) − (18) →

$$B_y = \frac{\sin 2\theta - \sin 2\theta_T}{\sin 2\theta + \sin 2\theta_T} B_I.$$

$$\hat{B}_y = \frac{2\sin 2\theta}{\sin 2\theta + \sin 2\theta_T} B_I,$$

$\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$

For $\theta = \theta_B$ (the "Brewter angle")

$B_y = 0$, $(\Leftrightarrow \sin 2\theta_B = \sin 2\theta_T \Leftrightarrow$

$$2\sin\theta\cos\theta = 2\sin\theta_T\cos\theta_T \ \text{etc} \ldots$$

using (7) $\theta_B = \tan^{-1}\left(\frac{c_1}{c_2}\right)$

So for $\theta = \theta_B$ No Reflected wave! (Everything is transmitted); Exam 2010 Q2

$\square$

Remark 2: Any other plane T-H EM wave incident upon S can be decomposed into combination of above TE & TM waves. (Since TE & TM have different reflection/transmission rules, e.g. for $\theta \simeq \theta_B$ $\longleftrightarrow$ "polarisation filtering effect.

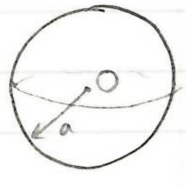
By (7), θ_T depends on C; often C depends on ω, $c = c(\omega) \Rightarrow \theta_T = \theta_T(\omega) \longleftrightarrow$ "dispersion" effect; $\leftrightarrow n = n(\omega)$ this is how a rainbow or a prism works.

$\boxtimes$

2.4 Waves due to spherical sources (acoustics)

Here some fundamental example of radially propagating sound are examined, due to "spherical sources".

2.4.1 A radially pulsating sphere



Consider a sphere centred at origin O. which "pulsates": its radius a changes with time, $a = a(t)$.

We assume $|a(t) - a_0| \ll a_0$, a_0 "original" radius radius, i.e. the perturbations are small (c.f. §1.1)

We then expect that an acoustic wave will be produced for $|x| > a_0$ (the sphere is surrounded by acoustic medium with wavespeed c, density ρ_0). Assume also $a(t) \equiv a_0$ for $t \leq t_0$, cf Causality (§1.11). The solution is then expected to be radially-symmetric (& causal) $\Rightarrow \underline{u}(x,t) = \nabla \phi$, with velocity potential $\phi = \phi(r, t)$

$$\phi(r,t) = \frac{f(t - \frac{r}{c})}{r},$$

$f = f(\xi)$ to be found from the boundary conditions at $r = a(t) \approx a_0$, $r \gtrsim a_0$.

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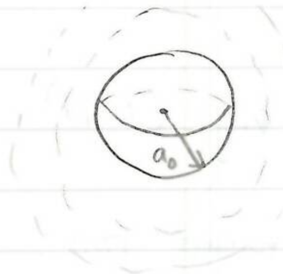

$$a(t) \rightarrow V_s = a'(t)$$

$$\rightarrow \quad \phi = \frac{f(t - \frac{r}{c})}{r}$$

$$\Rightarrow \underline{u} = \nabla \phi = \left(-\frac{f(t-\frac{r}{c})}{r^2} - \frac{f'(t-\frac{r}{c})}{cr} \right) \underset{\sim}{r}$$

(unit radical vector)

Seek:

$$f(t - \tfrac{r}{c}) = -ca_0 \Psi(t - \tfrac{r}{c} + \tfrac{a_0}{c})$$

(Ψ still an arbitrary function, so far)

Boundary condns (BCs): $r = a_0 \Rightarrow \underline{u} = a'(t)\underset{\sim}{r}$

$$\Longleftrightarrow \quad \frac{c}{a_0}\Psi(t) + \Psi'(t) = a'(t) = V_s(t) \qquad (19)$$

$e^{\frac{c}{a_0}t} \times$

$\Rightarrow$ Solving (19) for $\Psi(t)$, using integreting factor $e^{c/a_0 t}$

$$\Rightarrow \frac{d}{dt}\left(e^{\frac{c}{a_0}t}\Psi(t)\right) = e^{\frac{c}{a}t} V_s(t)$$

$$\Rightarrow e^{\frac{c}{a_0}t}\Psi(t) = \int_{-\infty \text{ or } t_0}^{t} e^{\frac{c}{a}\tau} V_s(\tau)\, d\tau$$

(V_s = $\Psi(t) = 0$, $t < t_0$ by causality)

$V_s(t) = 0$

$$\Rightarrow \Psi(t) = \int_{t_0}^{t} e^{-\frac{c}{a}(t-\tau)} V_s(\tau)\, d\tau .$$

$\Psi \to \phi \to \underline{u}$ can be found $\mapsto$ by (1.11)

$p = -\rho_0 \frac{\partial \phi}{\partial t}$ pressure can be found too.

For T-H case, $V_s(t) = Re\left\{ \hat{V}_s e^{-i\omega t} \right\}$ $\hat{V}_s \in \mathbb{C}$.

Seek $\Psi(t) = Re\left\{ B e^{-i\omega t} \right\}$ $B \in \mathbb{C}$ to be found.

Plugging to (19):

$$\frac{c}{a_0}\Psi + \Psi' = V_s(t) \Rightarrow \frac{c}{a} B \cancel{e^{-i\omega t}} - i\omega B \cancel{e^{-i\omega t}} = \hat{V}_s e^{-i\omega t} \cancel{e^{-i\omega t}}$$

$$\Rightarrow B = \frac{a_0 \hat{V}_s}{c\left(1 - \frac{i\omega}{c} a_0\right)} = \frac{a_0 \hat{V}_s}{c(1 - ika_0)}$$

$$\phi = Re\left\{ \frac{f\left(t - \frac{r}{c}\right)}{r} \right\} = Re\left\{ -\frac{ca_0}{r} \Psi\left(t - \frac{r}{c} + \frac{a_0}{c}\right) \right\}$$

$$= Re\left\{ -\frac{ca_0}{r} B e^{-i\omega\left(t - \frac{r}{c} + \frac{a_0}{c}\right)} \right\}$$

$$= Re\left\{ -\frac{a_0^2 \hat{V}_s}{1 - ika_0} \times \frac{e^{ikr - ika_0 - i\omega t}}{r} \right\}$$

From (1.41) $\Leftrightarrow$ $p = -i\omega\rho_0 \phi$

$$= \mathrm{Re}\left\{ -\frac{i\omega a_0^2 \hat{V}_s \rho_0}{(1-ika_0)r} e^{ik(r-a_0)-i\omega t} \right\} \qquad (20)$$

$$p = \mathrm{Re}\left\{ \frac{A}{r} e^{ikr-i\omega t} \right\}, \quad A = \frac{-i\omega a_0^2 \hat{V}_s \rho_0 e^{-ika_0}}{1-ika_0}$$

So, for time-averaged intensity,

$$\langle \underline{I} \rangle_T = \langle I_r \rangle_T \hat{\underline{r}}, \quad (1.22)$$

$$\Rightarrow \langle I_r \rangle = \frac{|A|^2}{2\rho_0 c r^2}$$

$$\Rightarrow \langle I_r \rangle = \frac{\omega^2 a_0^4 |\hat{V}_s|^2 \rho_0 c}{(1+(ka_0)^2) 2c^2 r^2}$$

$$= \frac{(ka)^2 a_0^2 |\hat{V}_s|^2 c\rho_0}{2(1+k^2a_0^2) r^2}$$

$\Rightarrow$ The average radiated power:

$$P = \langle I_r \rangle \times 4\pi r^2$$

$$= \frac{2\pi (ka_0)^2 a_0^2 |\hat{V}_s|^2 c\rho_0}{1+(k\cdot a_0)^2}$$

In "low frequency" regime $\Leftrightarrow$ $ka_0 \ll 1$

$$P_{av} \approx 2\pi (ka_0)^2 a_0^2 |\hat{V}_s|^2 c\rho_0 \qquad (21)$$

i.e "of order" $(ka_0)^2 = \left(\frac{\omega}{c} a_0\right)^2$ w.r.t ω.

$\square$

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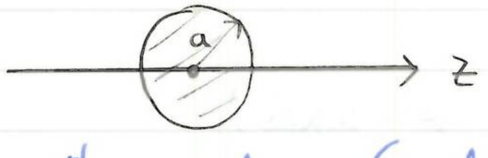
2.4.2 Transversely oscillating rigid sphere



Let sphere of radius a be rigid (i.e. a=constant) and oscillate along z-axis, i.e its centre (and any other point) has velocity

$$\underline{v}_c(t) = V_c(t)\,\hat{\underline{z}} \quad (\hat{\underline{z}} \text{ unit vector along } z)$$

Oscillations are assumed small. Seek, for $r>a$, acoustic velocity $\underline{u} = \nabla\phi$.

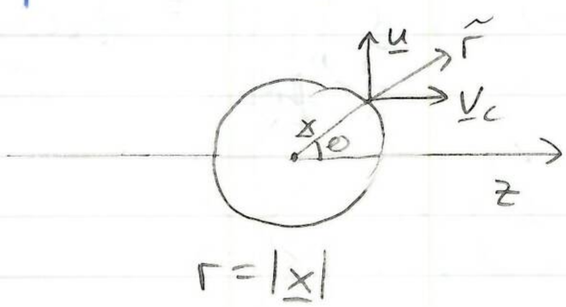
$$\phi = \frac{\partial}{\partial z}\left[\frac{1}{r}\,\psi\left(t - \frac{r}{c} + \frac{a}{c}\right)\right] \qquad (22)$$

$\langle$Check: $\frac{1}{r}\psi(t - r/c + a/c)$ solves wave eqn, (1.11)

$$\Rightarrow \frac{\partial}{\partial z}(\nabla^2 + k^2)\left(\frac{1}{r}\psi\right) = 0 .$$

$$\Rightarrow (\nabla^2 + k^2)\left[\frac{\partial}{\partial z}\left(\frac{1}{r}\psi\right)\right] = 0 \Leftrightarrow \phi \text{ also sloves (1.11)}\rangle$$

Boundary conditions (BCs): At $r=a$ (on the sphere's surface) acoustic medium's velocity relative to the sphere must be parallel to S:



$r=|\underline{x}|$

$$(\underline{u} - \underline{v}_c)\cdot\tilde{\underline{x}} = 0$$

$$\Rightarrow \underline{u} \cdot \hat{\underline{x}} = \underline{V}_c \cdot \hat{\underline{x}}$$

$$\Leftrightarrow \frac{\partial \phi}{\partial r} = \nabla \phi \cdot \hat{\underline{r}} = V_c \cos\theta$$

(θ angle between $\underline{x}$ and z-axis)

$$\Rightarrow \frac{\partial \phi}{\partial r} = V_c(t) \cos\theta \quad , \; r = a \qquad (23)$$

is the BC.

Notice that $\frac{\partial}{\partial z} f(r) = \frac{\partial r}{\partial z} \frac{\partial f}{\partial r}$,

$$r = (x^2 + y^2 + z^2)^{1/2} \Rightarrow \frac{\partial r}{\partial z} = \frac{z}{r} = \cos\theta$$

$$\Rightarrow \frac{\partial f}{\partial z} = \cos\theta \frac{\partial}{\partial r} f$$

$$\Rightarrow \phi = \cos\theta \frac{\partial}{\partial r} \left[\frac{1}{r} \varphi\left(t - \frac{r}{c} + \frac{a}{c} \right) \right]$$

From (23)

$$\Rightarrow \cancel{\cos\theta} \frac{\partial^2}{\partial r^2} \left[\frac{1}{r} \varphi\left(t - \frac{r}{c} + \frac{a}{c} \right) \right]_{r=a} = V_c(t) \cancel{\cos\theta} \qquad (23')$$

Now $\frac{\partial}{\partial r} \left(\frac{1}{r} \varphi\left(t - \frac{r}{c} + \frac{a}{c} \right) \right)$

$$= -\frac{1}{r^2}\varphi - \frac{1}{cr}\varphi'\left(t - \frac{r}{c} + \frac{a}{c}\right)$$

$$\Rightarrow \frac{\partial^2}{\partial r^2}\left(\frac{1}{r}\varphi\right) = \frac{2}{r^3}\varphi\left(t - \frac{r}{c} + \frac{a}{c}\right)$$

$$+ \frac{2}{cr^2}\varphi' + \frac{1}{c^2 r}\varphi''$$

So (23')

$$\Rightarrow \frac{2}{a^3}\varphi(t) + \frac{2}{ca^2}\varphi'(t) + \frac{1}{c^2 a}\varphi'' = V_c(t)$$

$$2c^2\varphi + 2ca\varphi' + a^2\varphi'' = c^2 a^3 V_c(t) \quad (24)$$

which is an ODE for $\varphi(t)$.

Restricting to T-H case, Let

$$V_c(t) = Re\left\{\hat{V}_c e^{-i\omega t}\right\}, \quad \hat{V}_c \in \mathbb{C}.$$

Seek $\varphi(t)$ also as $\varphi(t) = Re\left[Ae^{-i\omega t + ika}\right]$, $A \in \mathbb{C}$

Then from (24)

$$\Rightarrow \left(2c^2 A - 2i\omega ca A - \omega^2 a^2 A\right)e^{\cancel{-i\omega t}+ika}$$

$$= c^2 a^3 \hat{V}_c \cancel{e^{-i\omega t}}$$

$$\Rightarrow A = \frac{c^2 a^3 \hat{v}_c e^{-ika}}{2c^2 - 2i\omega c a - \omega^2 a^2}$$

$(\frac{\omega}{c} = k)$

$$\Rightarrow A = \frac{a^3 \hat{v}_c e^{-ika}}{2 - 2ika - (ka)^2} \quad (25)$$

$\Rightarrow \psi(t) = Ae^{-i\omega t + ika} \Rightarrow$ from (22)

$$\phi = (\mathrm{Re}) \cos\theta \frac{\partial}{\partial r}\left[\frac{A}{r} e^{-i\omega(t - \frac{r}{c} + \frac{a}{c}) + ika}\right]$$

$$= \mathrm{Re} \cos\theta \frac{\partial}{\partial r}\left[\frac{A}{r} e^{-i\omega t + ikr \cancel{-ika} \cancel{+ika}}\right]$$

$$\phi = (\mathrm{Re}) \underbrace{\cos\theta \frac{\partial}{\partial r}\left[\frac{A}{r} e^{ikr}\right.}_{:= \hat{\phi}} \left. e^{-i\omega t}\right]$$

$$\phi = \mathrm{Re}(\hat{\phi} e^{-i\omega t}), \quad \hat{\phi} = \cos\theta \frac{\partial}{\partial r}\left(\frac{A}{r} e^{ikr}\right) \quad (26)$$

$$\hat{p} = i\omega\rho_0 \hat{\phi} = i\omega\rho_0 \cos\theta \frac{A}{r} e^{ikr}\left(\frac{-1}{r} + ik\right)$$

$$= -k\omega\rho_0 \cos\theta \frac{A}{r} e^{ikr}\left(1 + \frac{i}{kr}\right)$$

$\Rightarrow$ In the "far field" ($\Leftrightarrow kr \gg 1$) :

$$\hat{p} = -k^2 c\rho_0 \cos\theta \frac{A}{r} e^{ikr} \quad (28)$$

For radial velocity $u_r := \underline{u}\cdot\hat{\underline{r}} = \frac{\partial \phi}{\partial r}$

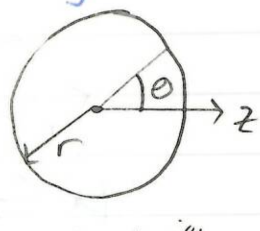
$$= \mathrm{Re}\left[\cos\theta \frac{\partial^2}{\partial r^2}\left(\frac{A}{r}e^{ikr}\right)\right];$$

In the far field:

$$u_r = \mathrm{Re}\left\{\cos\theta \frac{A}{r}e^{ikr}\left(-k^2 + \frac{2}{r^2} - \frac{2ik}{r}\right)\right]$$

$$\approx -\frac{k^2}{r}\cos\theta\, \mathrm{Re}\left(Ae^{ikr - i\omega t}\right)$$

For time-averaged radial acoustic intensity



$$\langle I_r\rangle = \langle p u_r\rangle$$

$$\approx \frac{k^4 c\rho_0 \cos^2\theta\, |A|^2}{r^2}\langle\cos^2(\omega t - kr - \varphi)\rangle_T$$

$A = |A|e^{i\varphi}$

$$= \frac{k^4 c\rho_0 \cos^2\theta\, |A|^2}{2r^2}$$ (in the far field, $kr \gg 1$)

The average radiated power: (S_r; sphere of large radius r)

$$P_{av} = \iint_{S_r} \langle I_r\rangle \, dS_r$$

$$\underset{(\theta,\varphi)}{=} \frac{k^4 c\rho_0 |A|^2}{2\cancel{r^2}}\int_0^{2\pi}\int_0^{\pi}\cos^2\theta\, \underbrace{\cancel{r^2}\sin d\theta\, d\varphi}_{dS}$$

$$= \pi k^4 c\rho_0 |A|^2 \left[-\frac{1}{3}\cos^3\theta \right]_0^\pi$$

$$= \frac{2}{3}\pi c\rho_0 |A|^2 k^4$$

$$(25) \Rightarrow |A|^2 = \frac{a^6|\hat{V}_c|^2}{(2-(ka)^2)^2 + 4k^2a^2}$$

$$= \frac{a^6|\hat{V}_c|^2}{4+(ka)^4}$$

$$\Rightarrow P_{av} = \frac{2}{3}\pi c\rho_0 k^4 \frac{a^6|\hat{V}_c|^2}{4+(ka)^4}$$

$$= \frac{2}{3}\pi \frac{(ka)^4 c\rho_0 a^2 |\hat{V}_c|^2}{4+(ka)^4} \qquad (29)$$

In "low-frequency" regime, $ka \ll 1$

$$P_{av} \approx \frac{\pi}{6}(ka)^4 c\rho_0 a^2 |\hat{V}_c|^2 \qquad (30)$$

So the radiated powers is "order 4" with respect to frequency.

Notice also $(25) \rightarrow$ (low frequency, $ka \ll 1$)

$$A = \frac{1}{2}a^3\hat{V}_c$$

So (28)

$$\Rightarrow \hat{p} \approx -k^2 \rho_0 c A \cos\theta \, \frac{e^{ikr}}{r}$$

$$\hat{p} \approx -\frac{1}{2} \rho_0 c \hat{V}_c (ka)^2 a \cos\theta \, \frac{e^{ikr}}{r} \quad (31)$$

$\boxtimes$

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3. Green's function & Multipole expansion.

3.1 Monopoles & free-space Green's function

$(a_0, \hat{V}_s)$ - We've seen that any spherically-symmetric TH source produces a (sph-symmetric) outgoing TH wave of the form $p = \mathrm{Re}(\hat{p}(r) e^{-i\omega t})$,

$$\hat{p} = \hat{S} \frac{e^{ikr}}{r}, \quad r > a_0 ; \qquad (1)$$

$\hat{S} \in \mathbb{C}$

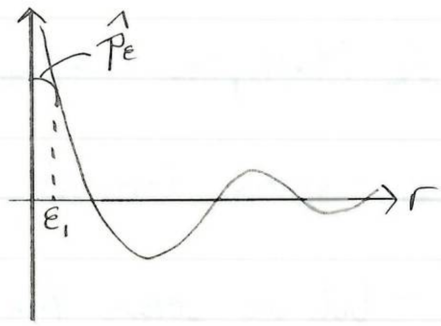
<u>Definition</u>: $\hat{S}$ is referred to as <u>Monopole amplitude</u>.

For radially pulsating sphere (2.20) $\Rightarrow$

$$\hat{S} = -\frac{i\omega \rho_0 a_0^2 \hat{V}_s}{1 - ika_0} e^{-ika_0}$$

Let $a_0 \to 0$ and $\hat{V}_s$ increases, so that $\hat{S}$ remains constant. In the "limit" we have $\hat{p}(r)$ defined by (1) everywhere except $r=0$, with a "singularity" at $r=0$.

Such $\hat{p}$ is called a "point source". (In reality a source can be regarded as a point source if
(i) $ka \ll 1$ ($\Leftrightarrow a_0 \ll \lambda$)
(ii) observation point $r \gg a_0$)



For a small $\varepsilon > 0$, let $\hat{p}_\varepsilon(r) := \hat{p}(r)$ for $r \geqslant \varepsilon$, but $\hat{p}_\varepsilon$ is "smoothed out" for $r < \varepsilon$. $\Rightarrow$

As $(\nabla^2 + k^2)\hat{p} = 0$, $\underline{r \neq 0}$, $\left(p = \hat{S}\,\frac{e^{i\omega r}}{r}\right)$

$$(\nabla^2 + k^2)\hat{p}_\varepsilon = -\hat{S}\,\underbrace{\Delta_\varepsilon(r)}_{} \qquad (2)$$

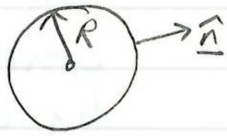
$= 0$ for $r > \varepsilon$.

where the RHS $\Delta_\varepsilon(r) \neq 0$ only for $r < \varepsilon$ ($\Leftrightarrow$ has support in ball B_ε). Let $\varepsilon \to 0$, fix $R > 0$ and integrate (2) over ball centred at O of radius R, B_R:

$$\int_{B_R} \nabla^2 \hat{p}_\varepsilon \, dV + \int_{B_R} k^2 \hat{p}_\varepsilon \, dV = -\hat{S} \int_{B_R} \Delta_\varepsilon(r)\, dV$$

$\Leftrightarrow$ $I_1 + I_2 = I_3$

Divergence theorem



$$I_1 = \int_{B_R} \nabla\cdot(\nabla \hat{p}_\varepsilon)\, dV = \int_{S_R} \nabla \hat{p}_\varepsilon \cdot \underline{n}\, dS$$

sphere of radius R

$$= \int_{S_R} \underbrace{\frac{\partial}{\partial r}}_{=\frac{\partial}{\partial s}} \hat{p}_\varepsilon \, dS = \int_{S_R} \frac{\partial}{\partial r}\left(\hat{S}\,\frac{e^{ikr}}{r}\right) dS$$

$$= \hat{S}\int e^{ikr}\left(\frac{ik}{r} - \frac{1}{r^2}\right) dS = \hat{S}\, e^{ikR}\left(\frac{ik}{R} - \frac{1}{R^2}\right) \times 4\pi R^2$$

$$= 4\pi \hat{S} e^{ikR}\left(ikR - 1\right)$$

$$I_2 = \int_{B_R} k^2 \hat{p}_\varepsilon \, dV \xrightarrow{\varepsilon \to 0} \int_{B_R} k^2 \hat{p} \, dV$$

$$= \hat{S} k^2 \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \frac{e^{ikr}}{r} \underbrace{r^2 \sin\theta \, dr \, d\theta \, d\varphi}_{dV}$$

$$\iint_0^{2\pi}\int_0^{\pi}\int_0^{R} \sin\theta \, d\theta \, d\varphi = 4\pi \quad (= \text{area of sphere of radius } 1)$$

$$\Rightarrow I_2 = 4\pi k^2 \hat{S} \int_0^R \frac{\partial}{\partial r} e^{ikr} r \, dr$$

$$= -4\pi i k \hat{S} \int_0^R \frac{\partial}{\partial r}\left(e^{ikr}\right) r \, dr$$

$$= -4\pi i k \hat{S} e^{ikR} R + 4\pi i k \hat{S} \int_0^R \frac{\partial}{\partial r}\left(e^{ikr}\right) dr$$

$$= -4\pi i \hat{S} e^{ikR} kR + 4\pi \hat{S}\left(e^{ikR} - 1\right)$$

$$= 4\pi \hat{S} e^{ikR}\left(1 - ikR\right) - 4\pi \hat{S}$$

$$\Rightarrow I_1 + I_2 \xrightarrow{\varepsilon \to 0} 4\pi \hat{S} e^{ikR}\left(\cancel{ikR} - \cancel{1} + \cancel{1} - \cancel{ikR}\right) - 4\pi \hat{S}$$

$$= -4\pi \hat{S}$$

$$\Rightarrow I_3 = -\cancel{\hat{S}}\int_{B_R} \Delta_\varepsilon \, dV \xrightarrow[\varepsilon\to 0]{} -4\pi\cancel{\hat{S}}.$$

Hence, denoting $\delta_\varepsilon := \frac{1}{4\pi}\Delta_\varepsilon$, we observe:

(i) $\int \delta_\varepsilon \, dV \longrightarrow 1$ as $\varepsilon \to 0$.

(ii) $\delta_\varepsilon \neq 0$ only within $r < \varepsilon$.

This implies

$$\lim_{\varepsilon\to 0} \delta_\varepsilon(\underline{x}) = \delta(\underline{x})$$

called "Dirac delta-function", with the "limit" understood appropriately.

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$$\hat{p} = \hat{S}\,\frac{e^{ikr}}{r}, \quad r>0 \quad (1)$$

$$(\nabla^2 + k^2)\,\hat{p}_\varepsilon = \Delta_\varepsilon$$

$$\delta_\varepsilon := \frac{1}{4\pi}\Delta_\varepsilon, \quad \int_{\mathbb{R}^3} \delta_\varepsilon \to 1$$

$$\lim_{\varepsilon \to 0} \delta_\varepsilon = \delta(\underline{x})$$

—/—

So $\delta(\underline{x})$ can be viewed as a "function", such that

(i) $$\delta(\underline{x}) = \begin{cases} 0 & \underline{x} \neq 0 \\ +\infty, & \underline{x} = 0. \end{cases}$$

(ii) $$\int_{\mathbb{R}^3} \delta(\underline{x})\, dV(\underline{x}) = 1$$

<Rigorous defn requires more advanced Analysis $\longleftrightarrow$ "distribution theory">

One imporant property of δ-function, often called "sifting property": $\forall$ "reasonable" (e.g. continuous) function f

$$\int_{(\mathbb{R}^3)} \delta(\underline{x} - \underline{x}')\, f(\underline{x}')\, dV(\underline{x}') = \int_{(\mathbb{R}^3)} \delta(\underline{x}' - \underline{x})\, f(\underline{x}')\, dV(\underline{x}') = f(\underline{x}) \quad (3)$$

(follows from "definition" of δ)

Hence, for $\hat{p}$ given by (1), as $\epsilon \to 0$,

$$(\nabla^2 + k^2)\hat{p} = -4\pi \hat{S}\,\delta(\underline{x})$$

More generally, for a point source at $\underline{x}_s$,

$$\hat{p}(\underline{x}) = \hat{S}\,\frac{e^{ikR}}{R}, \quad R := |\underline{x} - \underline{x}_s| \quad (1')$$

$$\Rightarrow (\nabla^2 + k^2)\hat{p} = -4\pi \hat{S}\,\delta(\underline{x} - \underline{x}_s) \quad (4)$$

<u>Green's functions</u>: The solution of (4) with RHS $\delta(\underline{x} - \underline{x}_s)$ i.e. $\hat{S} = -1/4\pi$, is called free-space Green's function.

$G(\underline{x}, \underline{x}') = G_f(\underline{x}, \underline{x}')$:

$$(\nabla^2 + k^2)\,G(\underline{x}, \underline{x}') = \delta(\underline{x} - \underline{x}'). \quad (5)$$

Hence, for Helmholz

$$G_f(\underline{x}, \underline{x}') = -\frac{e^{ik|\underline{x} - \underline{x}'|}}{4\pi|\underline{x} - \underline{x}'|} \quad (6)$$

(Free space Green fns for general linear differential operators are often also called "fundamental solutions")

3.2 Green's functions.

Green's fns paly fundamental role mathematically. Consider a field due to arbitrary "distributed" source f:

$$\mathcal{L}p = (\nabla^2 + k^2)p = f(\underline{x}) \qquad (7).$$

(In (7), $\mathcal{L}$ is Helmholt "differential operator" $\mathcal{L}: p \to f$; generally $\mathcal{L}$ could be wave eqn, Maxwell's system related operator.

Normally (7) is suppilemented by boundary /initial etc (e.g Sommerfield radiation condn) to form boundary-value problem (BVP), to determine p from f uniquely: sympolically $f = \mathcal{L}p \to p = \mathcal{L}^{-1}f$. Assume $\mathcal{L}$ is invertible. Since $\mathcal{L}$ is differential, its natural to seek $\mathcal{L}^{-1}$ as an "integral operator"

$$p(\underline{x}) = (\mathcal{L}^{-1}f)(\underline{x}) = \int_{\Omega} K(\underline{x}, \underline{x}')\, f(\underline{x}')\, dV(\underline{x}')$$

Ω is "domain" of $\mathcal{L}$.

The <u>kernel</u> $K(\underline{x}, \underline{x}')$ is to be found. Now;

$$f(\underline{x}) = (\mathcal{L}p)(\underline{x}) = \mathcal{L}(\mathcal{L}^{-1}f)(\underline{x})$$

$$= \mathcal{L}_x \int K(\underline{x}, \underline{x}')\, f(\underline{x}')\, dV(\underline{x}')$$

$$= \int \overbrace{\mathcal{L}_x \underbrace{K(\underline{x}, \underline{x}')}_{= G(\underline{x}, \underline{x}')}}^{\delta(\underline{x} - \underline{x}')} f(\underline{x}')\, dV(\underline{x}') \overset{(*)}{=} f(\underline{x}).$$

Notice that taking $K(\underline{x}, \underline{x}') = G(\underline{x}, \underline{x}')$ serves the purpose: by (5) $\mathcal{L}_x G(\underline{x}, \underline{x}') = \delta(\underline{x} - \underline{x}')$, and by (3) the (*) then holds

$$\Rightarrow (\mathcal{L}^{-1}f)(\underline{x}) = \int G(\underline{x}, \underline{x}')\, f(\underline{x}')\, dV(\underline{x}') \,. \quad (8)$$

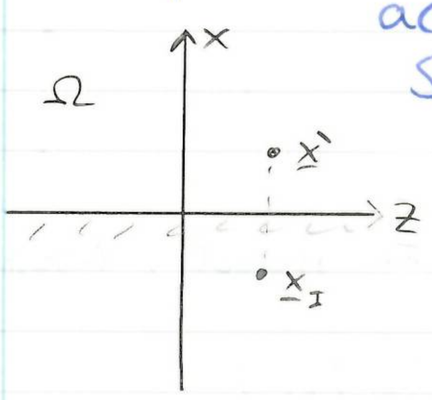
For Helmholtz eqn in <u>domains with boundaries</u> the Green's function is generally of the form:

$$G(\underline{x}, \underline{x}') = G_f(\underline{x}, \underline{x}') + G_B(\underline{x}, \underline{x}')$$

with G_f the free-space Green's fn (Gf) given by (6), and G_B solves:

$(\nabla^2 + k^2) G_B = 0$ in Ω, and $G_f + G_B$ satisfy the boundary conditions (BCs).

<u>Example</u>: Ω half-space $x \geq 0$, with acoustically-soft boundary $S' = \{x = 0\}$.



Let $\underline{x}' = (x', y', z')$, $x' > 0$, a "source" point. Consider an "image" $x_I = (-x', y', z')$
$\Rightarrow \underline{x}_I \notin \Omega$; and take $G_B(\underline{x}, \underline{x}') = -G_f(\underline{x}, \underline{x}_I)$
$\Rightarrow$ (i) $(\nabla^2 + k^2) G_B = -\delta(\underline{x} - \underline{x}_I) = 0$ in Ω;

$\underline{BC}$; $x \in S'$ $(\Leftrightarrow \underline{x} = (0, y, z))$

$$\Rightarrow G(\underline{x}, \underline{x}') = G_f(\underline{x}, x') - G_f(\underline{x}, \underline{x}_I)$$

$$= G_f(|x - x'|) - G_f(|x - x_I|) = 0$$

Since $|x - x'| = |\underline{x} - \underline{x}_I| \quad \forall \underline{x} \in S'$

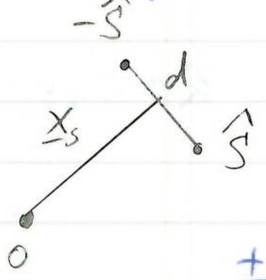
$\Leftrightarrow$ acoustically soft BC.

Excercise: For acoustically hard BC', $\frac{\partial p}{\partial n} = \underline{n} \cdot \nabla p = 0$ on S', show that

$$G(\underline{x}, \underline{x}') = G_f(\underline{x}, \underline{x}') + G_f(\underline{x}, \underline{x}_I)$$

Exam 2012 Q4(b) $\boxtimes$.

3.3 Dipoles and Quadrupoles.



Take two monopole (= point) sources a small distance d apart of amplitudes $\hat{S}$ and $-\hat{S}$, i.e oscillating "in anti-phase". Let $\pm\hat{S}$ be at $\underline{x}_s \pm \frac{1}{2}\underline{d}$, $|\underline{d}| = d$.

$\Rightarrow$ Resulting acoustic field is

$$\hat{p}(x) = -4\pi \hat{S}\, G_f(\underline{x}, \underline{x}_s + \tfrac{1}{2}\underline{d})$$

$$+ 4\pi \hat{S}\, G_f(\underline{x}, \underline{x}_s - \tfrac{1}{2}\underline{d})$$

Let $kd \ll 1$; using Taylor Series:

$$\hat{p}(x) = -4\pi \hat{S} G_f(\underline{x}, \underline{x}_s) - 4\pi \hat{S}\, \underline{\nabla}_s G_f(\underline{x}, \underline{x}_s) \cdot \tfrac{1}{2}\underline{d}.$$

$$+ 4\pi \hat{S} G_f(\underline{x}, \underline{x}_s) - 4\pi \hat{S}\, \underline{\nabla}_s G_f(\underline{x}, \underline{x}_s) \cdot \tfrac{1}{2} \underline{d}$$

$$+ O(d^2)$$

$$= -4\pi \hat{\underline{D}} \cdot \underline{\nabla}_s G_f(\underline{x}, \underline{x}_s) + O(d^2)$$

where $\hat{\underline{D}} := \hat{S}\underline{d}$ is the <u>dipole amplitude vector</u>.
Since $G_f(\underline{x}, \underline{x}_s) = G_f(\underline{x} - \underline{x}_s)$, see (6),
$\Rightarrow \underline{\nabla}_s G_f = -\underline{\nabla} G_f$

$\Rightarrow$ To main order in small d,

$$\hat{p}(\underline{x}) = 4\pi \hat{\underline{D}} \cdot \nabla G_f(\underline{x}, \underline{x}_s)$$

$$= \cancel{4\pi} \hat{\underline{D}} \cdot \nabla \left(-\frac{1}{\cancel{4\pi}} \frac{e^{ik|\underline{x}-\underline{x}_s|}}{\underbrace{|\underline{x}-\underline{x}_s|}_{=R}} \right)$$

$$= -\hat{\underline{D}} \cdot \frac{d}{dR}\left(\frac{e^{ikR}}{R} \right) \hat{\underline{R}}, \quad R := |\underline{x} - \underline{x}_s|$$

$$\hat{\underline{R}} = \frac{\underline{x} - \underline{x}_s}{R}$$

$$\Rightarrow \hat{p} = -\hat{\underline{D}} \cdot \hat{\underline{R}} \frac{d}{dR}\left(\frac{e^{ikR}}{R} \right)$$

<u>Exercise</u>: Show that transversely oscilling rigid sphere (§ 2.4.2) produces a dipole field, and determine related $\hat{\underline{D}}$. Hint: let $\underline{x}_s = 0$,

$$\hat{\underline{\mathcal{D}}} = \mathcal{D}\hat{\underline{z}}$$

$$(*) \Rightarrow \hat{p} = -\mathcal{D}\, \underbrace{\hat{\underline{z}} \cdot \underline{r}}_{\cos\theta} \frac{d}{dr}\left(\frac{e^{ikr}}{r}\right)$$

$$= -\mathcal{D}\cos\theta \frac{d}{dr}\left(\frac{e^{ikr}}{r}\right) \quad \text{cf } (2.27) \quad \square$$

Morale so far : while radially pulsating sphere produces a monopole field, transversely oscillating one produces a dipole field.

$\square$

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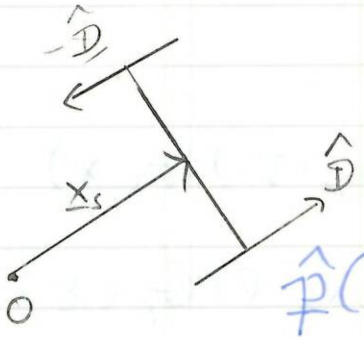
Quadrupoles



Consider now two dipoles small distance $d = |\underline{d}|$ apart, with oppsite dipole vectors $\pm \underline{\mathcal{D}} \Rightarrow$ (for d small $\Leftrightarrow kd \ll 1$)

$$\hat{p}(\underline{x}) = -4\pi \hat{\underline{\mathcal{D}}} \cdot \nabla_s G_f(\underline{x}, \underline{x}_s + \tfrac{1}{2}\underline{d})$$

$$+ 4\pi \hat{\underline{\mathcal{D}}} \cdot \nabla_s G_f(\underline{x}, \underline{x}_s - \tfrac{1}{2}\underline{d})$$

Taylor series

$$\approx -4\pi(\underline{d}\cdot\nabla_s)(\hat{\underline{\mathcal{D}}}\cdot\nabla_s) G_f(\underline{x}, \underline{x}_s)$$

$\nabla_s = -\nabla$

$$= +\cancel{4\pi}(\underline{d}\cdot\nabla)(\hat{\underline{\mathcal{D}}}\cdot\nabla)\left(+\frac{1}{\cancel{4\pi}} \frac{e^{ikR}}{R}\right)$$

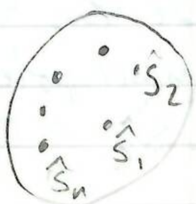
$$= \sum_{\alpha,\beta=1}^{3} Q_{\alpha\beta} \frac{\partial^2}{\partial x_\alpha \partial x_\beta}\left(\frac{e^{ikR}}{R}\right) \qquad (10)$$

where $R = |\underline{x} - \underline{x}_s|$, $\underline{x} = (x_1, x_2, x_3) = (x, y, z)$

$$Q_{\alpha\beta} = d_\alpha \hat{\mathcal{D}}_\beta$$

(10) is a "quadrupole" field $\Leftrightarrow$ combination of 2nd derivatives of "monopole" field e^{ikR}/R.

3.4 Multipole expansions.



Take a cluster of monopole (point) sources:

$$p_n = \frac{\hat{S}_n e^{ik|\underline{x}-\underline{x}_n|}}{|\underline{x}-\underline{x}_n|} = -4\pi \hat{S}_n G_f(\underline{x} - \underline{x}_n)$$

such that $k|\underline{x}_n| \ll 1$ (i.e. close enough to each other), $n = 1, 2, \ldots, N$.

Using Talyor series:

$$G_f(\underline{x} - \underline{x}_n) = G_f(\underline{x}) - (\underline{x}_n \cdot \nabla) G_f(\underline{x}) + \frac{1}{2}(\underline{x}_n \cdot \nabla)(\underline{x}_n \cdot \nabla) G_f(\underline{x}) + \ldots$$

$\Rightarrow$ The resulting field $\hat{p}$:

$$\hat{p} = \sum_{n=1}^{N} p_n = -4\pi \left(\sum_n \hat{\underline{S}}_n\right) \underbrace{G_f(\underline{x})}_{= -\frac{1}{4\pi}\frac{e^{ikr}}{r}} + 4\pi \sum_n \hat{S}_n (\underline{x}_n \cdot \nabla) G_f(\underline{x}) - 4\pi \sum_n \frac{1}{2} \hat{S}_n (\underline{x}_n \cdot \nabla)^2 G_f(\underline{x}) + \ldots$$

$$= \underbrace{\hat{\underline{S}} \frac{e^{ikr}}{r}}_{\text{monopole}} - \underbrace{\hat{\underline{D}} \cdot \nabla\left(\frac{e^{ikr}}{r}\right)}_{\text{dipole}} + \underbrace{\sum_{\alpha,\beta} Q_{\alpha\beta} \frac{\partial^2}{\partial x_\alpha \partial x_\beta}\left(\frac{e^{ikr}}{r}\right)}_{\text{quadrupole}} \quad (11)$$

where $\hat{\underline{S}} := \sum_n \hat{S}_n$, $\hat{\underline{D}} = \sum_n S_n \underline{x}_n$,

$$Q_{\alpha\beta} = \frac{1}{2}\sum_n \hat{S}_n x_{n_\alpha} x_{n_\beta}, \quad \underline{x}_n = (x_{n_1}, x_{n_2}, x_{n_3})$$

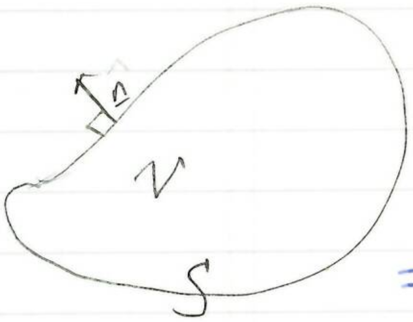
(11) is an example of a <u>multipole expansion</u>: monopole, dipole, quadrupole terms are the first 3 terms in an infinite hierachy of "multipoles". So, monopole term dominates unless $\sum \hat{S}_n = 0$, then dipole term dominates etc.

3.5 Kirchhoff - Helmoltz integral Theorem

This is an imporant integral representatcon for solutcons of Helmholtz eqn.

Based on "Green's formula / identity": Let f, g be two functions in domain / volume V with boundary S.

Apply divergence Thm to $\underline{F} = f\nabla g - g\nabla f$.



$$\int_V \nabla\cdot\underline{F}\, dV = \int_S \underline{F}\cdot\underline{n}\, dS$$

$$\nabla\cdot\underline{F} = \nabla\cdot(f\nabla g) - \nabla\cdot(g\nabla f)$$

$$= \cancel{\nabla f\cdot\nabla g} + f\nabla^2 g - \cancel{\nabla g\cdot\nabla f} - g\nabla^2 f$$

$$\underline{F}\cdot\underline{n} = f\nabla g\cdot\underline{n} - g\nabla f\cdot\underline{n} = f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}$$

$$\Rightarrow \int_V (f\nabla^2 g - g\nabla^2 f)\, dV = \int_S \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right) dS \qquad (12)$$

(12) is the <u>Green's formula / Green's identity</u>:

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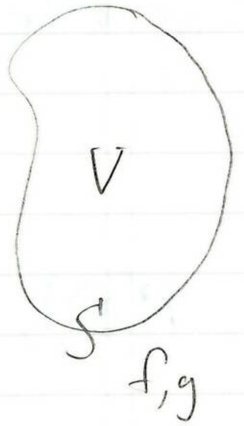
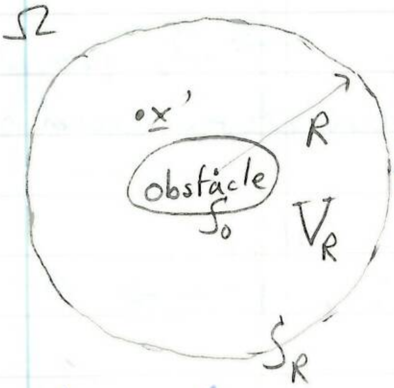


$$\int_V (f\nabla^2 g - g\nabla f)\, dV$$

$$= \int_{S'} \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n} \right) dS' \quad (12)$$

— / —



Choose in (12):

$f(x) = \Psi =$ soln of Helmholtz eqn (1.9),

$$(\nabla^2 + k^2)\Psi = 0.$$

in a domain Ω exterior to a bounded "obstacle" with boundary S_0;

$$g(\underline{x}) = G(\underline{x}, \underline{x}') = e^{i|\underline{x}-\underline{x}'|}/(4\pi\, |\underline{x}-\underline{x}'|) \quad (6),$$

free-space Green's function, $G = G_F$, $\underline{x}' \in \Omega$;

$V = V_R =$ part of Ω within ball B_R of a large radius R, $\underline{x}' \in V_R$, bounded S_R (sphere of radius R)

Plugging into (12):

$$\int_{V_R} \left[\Psi(x)\nabla^2 G(\underline{x}, \underline{x}') - G(\underline{x}, \underline{x}')\nabla^2\Psi(\underline{x}) \right] dV(\underline{x})$$

$$= \int_{S_0 \cup S_R} \left[\Psi(\underline{x}) \frac{\partial G(\underline{x}, \underline{x}')}{\partial n(\underline{x})} - G(\underline{x}, \underline{x}')\frac{\partial \Psi}{\partial n}(\underline{x}) \right] dS(\underline{x})$$

$= RHS'.$

$$\text{LHS}' \overset{(5),(1.9)}{=} \int_{V_R} \left[\Psi(\underline{x})(\cancel{-k^2 G} + \delta(\underline{x} - \underline{x}') + \cancel{G k^2 \Psi}) \right] dV(\underline{x})$$

Recall (5) is $\nabla^2 G + k^2 G = \delta(\underline{x} - \underline{x}')$

$$= \int_{V_R} \Psi(\underline{x}) \delta(\underline{x} - \underline{x}')\, dV(\underline{x})$$

$$\overset{(3)}{=} \Psi(\underline{x}'), \text{ by } (3) \Rightarrow \text{LHS}' = \Psi(\underline{x}');$$

RHS: Show that, as $R \to \infty$, $\int_{S_R} \ldots \to 0$; provided Ψ satisfies also Sommerfield radiation condn (1.19):

$$(1.19) \Leftrightarrow r\left(\frac{\partial \Psi}{\partial r} - ik\Psi\right) \to 0, \quad r \to \infty$$

$$\Leftrightarrow \frac{\partial \Psi}{\partial r} = ik\Psi + o(r^{-1})$$

small "o"

Notice G given by (6) also satisfies (1.19)

$$G = -\frac{e^{ik|\underline{x} - \underline{x}'|}}{4\pi|\underline{x} - \underline{x}'|} \simeq -\frac{e^{ikr}}{4\pi r};$$

capital "O"

Also $G = O(r^{-1})$, $\Psi = O(r^{-1})$, $r \to \infty$ (can be shown to follow from (1.19))

$$\Rightarrow \int_{S_R} \left(\Psi \frac{\partial G}{\partial r} - G \frac{\partial \Psi}{\partial r} \right) dS$$

$(R\to\infty)$ $\qquad\qquad\qquad\qquad = O(R^{-1}) \qquad\qquad\qquad\qquad\qquad = O(R^{-1})$

$$= \int_{S_R} \left[\cancel{\Psi_i k G} + \Psi\, o(R^{-1}) - \cancel{G_i k \Psi} - G\, o(R^{-1}) \right] dS$$

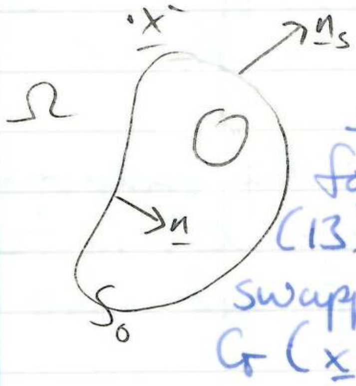
$$= \int_{S_R} O(R^{-1})\, o(R^{-1})\, dS$$

$$= o(R^{-2}) \times 4\pi R^2 \longrightarrow 0 .$$

$R \to \infty$, as required.

So (12) reduces to:

$$\Psi(\underline{x}') = \int_{S_0} \left[\Psi(\underline{x}) \frac{\partial G(\underline{x}, \underline{x}')}{\partial n(\underline{x})} - G(\underline{x}, \underline{x}') \frac{\partial \Psi}{\partial n}(\underline{x}) \right] dS(\underline{x}) \qquad (13)$$



Notice in (13), $\underline{n}$ is exterior to $\Omega \Leftrightarrow$ interior to obstacle O, S_0 for $\underline{n}_s = -\underline{n}$, exterior to O, specialising (13) to $\Psi = \hat{p}$, acoustic pressure, swapping $\underline{x}' \to \underline{x}$, $\underline{x} \to \underline{x}_s$, also $G(\underline{x}', \underline{x}) = G(\underline{x}, \underline{x}')$, (6).

$$\hat{p}(\underline{x}) = \int_{S_0} \left[G(\underline{x}, \underline{x}_s) \frac{\partial \hat{p}}{\partial n}(\underline{x}_s) - \hat{p}(\underline{x}_s) \frac{\partial G(\underline{x}, \underline{x}_s)}{\partial n_s} \right] dS(\underline{x}_s) \qquad (14)$$

(13), (14) are called Kirchhoff - Helmholtz integral theorem

Notice (1.41) $\Leftrightarrow \hat{p} = i\omega\rho_0 \hat{\Phi}$

$$\Rightarrow \frac{\partial \hat{p}}{\partial n} = i\omega\rho_0 \underline{n}\cdot \nabla\phi^{=\hat{u}} = i\omega\rho_0 \hat{V}_n ,$$

$\hat{V}_n$ = "normal component velocity" of S_0;

$$(6) \Rightarrow G = \frac{-e^{ikR}}{4\pi R}, \quad R = |\underline{x} - \underline{x}_s|$$

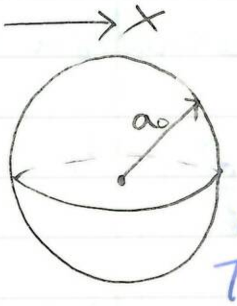
$$\frac{\partial G}{\partial n_s} = \underline{n}_s \cdot \nabla_s G = -\underline{n}_s \nabla G .$$

$$\Rightarrow \hat{p}(\underline{x}) = -\frac{1}{4\pi}\int_{S_0} i\omega\rho_0 \hat{V}_n(\underline{x}_s)\frac{e^{ikR}}{R} + \hat{p}(\underline{x}_s)(\underline{n}_s\cdot\nabla)\left(\frac{e^{ikR}}{R}\right) dS(\underline{x}_s) \quad (15)$$

Notice (15), (14) etc, express the unknown $\hat{p}$ anywhere away from S_0 ($\underline{x} \in \Omega$) in terms of boundary only values of $\hat{p}$ but also $\hat{V}_n$ ($\Longleftrightarrow \frac{\partial \hat{p}}{\partial n}$).

However, we normally know, from BCs, only either $\hat{p}$ or $\hat{V}_n$. Neverless, we normally know, from BCs, only either $\hat{p}$ or $\hat{V}_n$. Nevertheless, if e.g. $\hat{V}_n$ we know, let in (15) $\underline{x} \rightarrow S_0$ ($\underline{x}$ to put on the boundary S_0) $\Rightarrow$ (15) becomes a boundary integral eqn (BIE) which needs to be solved by seperate means. Then (15) constructs $\hat{p}$ (and $\hat{\underline{u}}$) everywhere. ☒

3.6 Example : scattering by air bubbles in water.



Let air bubble of unperturbed radius a_0 in water, be subjected to an incident acoustic field; a_0 "small" : $a_0 < \lambda = 2\pi/k$ the wavelength. Then the "incident field's pressure", p_i is : $p = Ae^{ikx-i\omega t} \approx Ae^{-i\omega t}$ ($|x| = a_0 \Rightarrow |kx| \ll 1$)

The bubble "scatters" p_i : the scattered field p_s, sough as outgoing spherically symmetric (§1)

$$\Rightarrow p_s = \frac{B}{r} e^{ikr-i\omega t}, \quad (B \in \mathbb{C}) \quad (16)$$

Then $(1.16) \Leftrightarrow \underline{\hat{u}} = -\frac{i}{\rho_0 \omega} \nabla \hat{p} \Rightarrow$

$$(17) \quad V_s = \frac{1}{i\rho_0\omega}\frac{\partial p_s}{\partial r} = \frac{B}{i\rho_0\omega r^2}(ikr-1)e^{ikr-i\omega t},$$

V_s = "radial velocity".

We assume the bubble is made of an ideal gas, responding "adiabatically" $\Leftrightarrow PV^\gamma$ = constant. ($\gamma > 1$ adiabadic constant).

Differentiating in time.

$$\frac{dp}{dt}V^\gamma + \gamma P V^{\gamma-1}\frac{dV}{dt} = 0.$$

pressure is not dependent on r since the bubble is so small.

Let $P = P_0 + P'$, $V = \frac{4}{3}\pi(a_0 + a')^3$

$P' \ll P_0$, $a' \ll a_0$ (perturbation small $\Rightarrow$

$$\frac{dP'}{dt} = -\gamma P V^{-1} \frac{4}{3}\pi\, 3 a_0^2 \underbrace{\frac{da'}{dt}}_{= V_s}$$

Now P' time-harmonic T-H $\Rightarrow$ $\frac{dP'}{dt} = -i\omega P'$

$$\Rightarrow -i\omega P' = -\frac{\gamma P_0}{\frac{4}{3}\cancel{\pi} a_0^{\cancel{3}}} \cancel{4\pi} a_0^{\cancel{2}} V_s \qquad \left(V_s = \frac{da'}{dt} : (17)\right)$$

$$P' = \frac{3\gamma P_0}{i\omega a_0} V_s . \qquad (18)$$

On the bubble surface, $r = a_0$, in absence of surface tension, etc, balance of pressures:

$$P_i\Big|_{r=a_0} + P_s\Big|_{r=a_0} = P'$$

$\Leftrightarrow$ via (16)–(18):

$$A\cancel{e^{-i\omega t}} + \frac{B}{a_0} e^{ika_0 \cancel{-i\omega t}} = \frac{3\gamma P_0}{i\omega a_0} \cdot \frac{B(ika_0 - 1)}{i\omega\rho_0 a_0^2} \cdot \ldots$$

$$\cdot\, e^{ika \cancel{-i\omega t}}$$

$$a_0 A = B e^{ika_0}\left[-1 + \frac{3\gamma P_0}{\omega^2 \rho_0 a_0^2}(1 - ika_0)\right]$$

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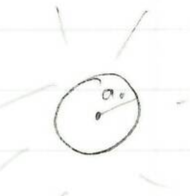


A

$$p = \frac{B e^{ikr - i\omega t}}{r}$$

$$a_0 A = B e^{ika}\left[-1 + \overbrace{\frac{3\gamma P_0}{\omega^2 \rho_0 a_0^2}}^{=:\ \omega_0^2}\left(1 - ika_0\right)\right]$$

—/—

Since $ka_0 \ll 1 \Rightarrow e^{ika_0} = 1 + ika_0 + O((ka_0)^2)$

Also denoting $\omega_0^2 := \dfrac{3\gamma P_0}{\rho_0 a_0^2}$

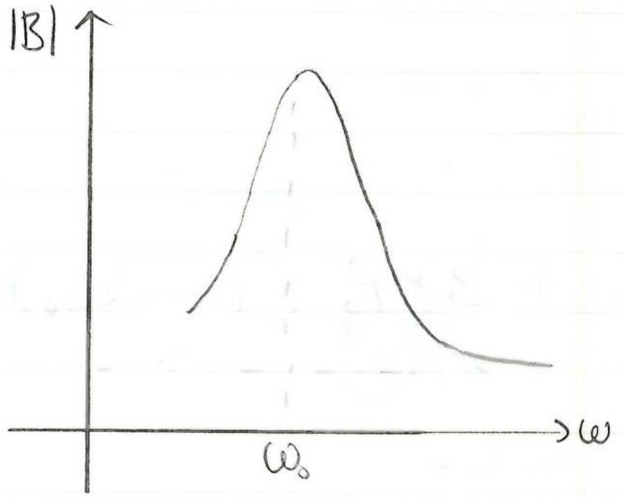
$$\Rightarrow a_0 A = B\left[-1 - ika_0 + \frac{\omega_0^2}{\omega^2}\overbrace{\left(1 + ika_0\right)\times\left(1 + ka_0\right)}^{1 - ika_0}\right.$$

$$\left. + O((ka_0)^2)\right]$$

$$= B\left[\frac{\omega_0^2}{\omega^2} - 1 - ika_0 + O((ka_0)^2)\right]$$

$$\Rightarrow B \approx \frac{a_0 A}{\left(\frac{\omega_0}{\omega}\right)^2 - 1 - ika_0}, \quad B \in \mathbb{C}.$$

So $$|B|^2 = \frac{a_0^2 |A|^2}{\left[\left(\frac{\omega_0}{\omega}\right)^2 - 1\right]^2 + (ka_0)^2}$$



$|B|$ has a peak at $\omega \approx \omega_0$.

Moral: ω_0 is a "resonance frequency" of the bubble: the scattered wave's amplitude surges for $\omega \approx \omega_0$.

$$\omega_0 \approx \left(\frac{3\gamma P_0}{\rho_0 a_0^2} \right)^{1/2} .$$

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4. High frequency waves, WKB method, Waveguides.

4.1 WKB method.

The WKB (Wentzel, Kramers, Brillouin) method is a method for constructing highly-oscillatory solutions of differential eqns containing a "large parameter" (e.g ω, frequency).

Consider an ODE.

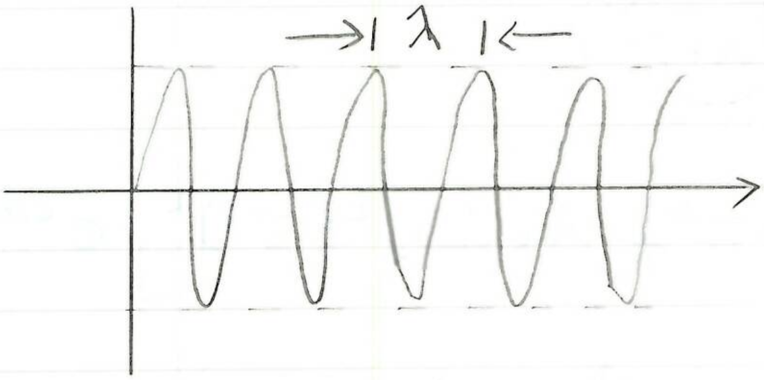
$$\frac{d^2\psi}{dx^2} + q(x)\psi = 0 .$$

where $|q(x)|$ is "large", in particular if $q(x) = k^2(x) > 0$.

$$\frac{d^2\psi}{dx^2} + k^2(x)\psi = 0. \qquad (1)$$

In particular, if $k(x) \equiv k = $ constant, (1) describes plane T-H waves (1-D Helmholz eqn; with solutions $\psi = Ae^{ikx} + Be^{-ikx}$.)

Eg: $\psi_1 = \mathrm{Re}(Ae^{ikx - i\omega t}) = |A|\cos(\omega t - kx - \varphi)$



When k large ($k \gg 1$)
$\lambda = \frac{2\pi}{k} \ll 1$ small
$\rightarrow$ rapid oscillations;

$k = \omega/c$, so $k \gg 1 \Leftrightarrow \omega \gg 1$

$\Leftrightarrow$ High frequency.

Mathematically $k(x)$ "large", can be expressed via $k(x) = \omega\mu(x)$, where $\omega \gg 1$ is a large constant (ω = frequency for waves) and $\mu = \mathcal{O}(1)$ "not large".

Then (1)

$$\Rightarrow \frac{d^2\psi}{dx^2} + \omega^2\mu^2(x)\psi = 0 \qquad (2).$$

[For waves $k(x) = \omega / c(x)$, corresponding to "inhomogeneous media" with varying wave speed $c(x)$; $\Rightarrow \mu(x) = c^{-1}(x)$.

We seek the "WKB approximation" to a solution of (2) in the form.

$$\psi(x, \omega) \approx A e^{i\omega \tau(x, \omega)}, \qquad (3).$$

with A constant, $\omega \gg 1$. Then $(3) \rightarrow (2)$

$$\frac{d\psi}{dx} = A i \omega \underbrace{\frac{d\tau}{dx}}_{= \tau'} e^{i\omega\tau}.$$

$$\frac{d^2\psi}{dx^2} = \left[A i \omega \tau'' + A (i\omega)^2 (\tau')^2 \right] e^{i\omega\tau}.$$

$$\Rightarrow i\cancel{\omega}\tau'' - \omega^2 (\tau')^2 + \omega^2 \mu^2(x) = 0 \qquad (4)$$

Now seek $\tau(x, \omega)$ as so-called "regular perturbation form" with respect to small ω^{-1}:

$$\tau(x, \omega) \sim \tau_0(x) + \omega^{-1} \tau_1(x) + \omega^{-2} \tau_2(x) + \ldots \qquad (5)$$

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$$\psi \approx A e^{i\omega \tau(x)} \quad (3).$$

$$i\omega \tau'' - \omega^2 (\tau')^2 + \omega^2 \mu^2 = 0 \quad (4).$$

$$\tau(x,\omega) \sim \tau_0(x) + \omega^{-1}\tau_1(x) + \ldots \quad (5).$$

Plugging (5) into (4), to main order in ω, $O(\omega^2)$

$$-\cancel{\omega^2}(\tau_0')^2 + \cancel{\omega^2}\mu^2(x) = 0.$$

$$\Rightarrow \tau_0' = \pm \mu(x).$$

$$\tau_0(x) = \pm \int_{x_0}^{x} \mu(x')\,dx' \quad (6).$$

Next, for $O(\omega)$ in (4):

$$i\cancel{\omega}\tau_0'' - \cancel{\omega^2}\, 2\tau_0' \cancel{\omega^{-1}}\tau_1' = 0.$$

$$\Rightarrow \tau_1' = \frac{i}{2}\frac{\tau_0''}{\tau_0'}$$

$$\Rightarrow \tau_1(x) = \frac{i}{2}\ln|\tau_0'| + C_\pm$$

$$= \frac{i}{2}\ln \mu(x) + C_\pm$$

Hence, from (3), the WKB approximation is:

$$\psi(x,\omega) \approx A e^{i\omega(\tau_0 + \omega^{-1}\tau_1)}$$

$$= A e^{i\omega \tau_0 + i\tau_1}$$

$$= A \exp\left\{ \pm i\omega \int_{x_0}^{x} \mu(x')\, dx' - \frac{1}{2} \ln \mu + \tilde{C}_{\pm} \right\}$$

$$= \tilde{A}_{\pm}\, \mu^{-1/2}(x) \exp\left\{ \pm i\omega \int_{x_0}^{x} \mu(x')\, dx' \right\} \quad \text{or}$$

$$\Psi(x,\omega) \approx \mu^{-\frac{1}{2}}(x) \left[A e^{i\omega \int_{x_0}^{x} \mu(x')dx'} + B e^{-i\omega \int_{x_0}^{x} \mu(x')\, dx'} \right]$$

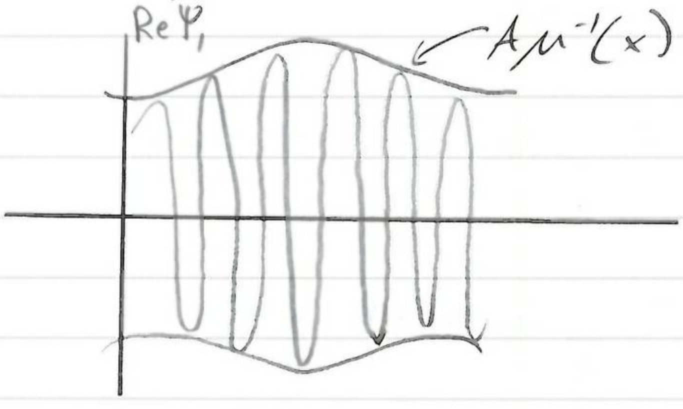
$$=: \tilde{\Psi}(x,\omega) \qquad (7).$$

where A and B arbitrary constants.

The WKB approximatcoi (7) contains two high frequency "waves", travelling to the right and left resp;

$$\tilde{\Psi} = \tilde{\Psi}_1 + \tilde{\Psi}_2$$

e.g for $\tilde{\Psi}_1$:



<u>Remark</u>: If we substitute $\tilde{\varphi}$ into (2), then (Excercise):

$$\frac{d^2\tilde{\varphi}}{dx^2} + \omega^2\mu^2(x)\,\tilde{\varphi}(x) = \left[\frac{3}{4}\,\frac{(\mu')^2}{\mu^2} - \frac{\mu''}{2\mu}\right]\tilde{\varphi}.$$

This suggest the (7) is accurate, provided

$$\frac{3}{4}\,\frac{(\mu')^2}{\mu^2} - \frac{\mu''}{2\mu} \ll \omega^2\mu^2 \iff$$

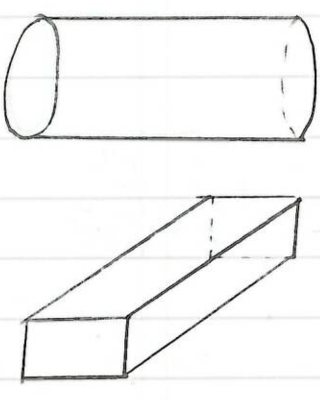
$$\frac{1}{\omega^2}\left[\frac{3(\mu'(x))^2}{4\mu^4(x)} - \frac{\mu''(x)}{2\mu^3(x)}\right] \ll 1 \qquad (8)$$

So this holds provided $\omega \gg 1$ and $\mu(x) \geq \mu_0 > 0$. (not to close to zero) and μ', μ''. "not too large" (slowly enough varying).

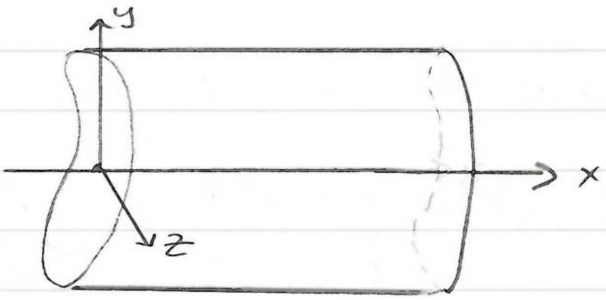
<u>Excer ise</u>: 1. EM wave in layered media (Exam 2011 Qn 5)
2. Exam 2010 Qn 2.

4.2 <u>Waveguides</u>.

eg a cylinder.



A wave guide is typically a tube / pipe of rather arbitrary cross-section, along which waves can propagate. (Physically, the wave bounces / is reflected from the walls, and propagates down the axis.



Choose x along the waveguide's axis, seek for T-H solutcons. So, both in accoutics and EM, we seek solutcons of Helmholz equ (1.15) or (1.35) - (1.36), for $\hat{\Psi}$:

$$(\nabla^2+k^2)\hat{\Psi}=0 \Leftrightarrow \frac{\partial^2\hat{\Psi}}{\partial x^2}+\overbrace{\frac{\partial^2\hat{\Psi}}{\partial y^2}+\frac{\partial^2\hat{\Psi}}{\partial z^2}}^{\nabla_\perp^2}+k^2\Psi=0.$$

$k=\omega/c$. In both cases $\hat{\Psi}$ is sought in the form

$$\hat{\Psi}=\Psi(y,z)e^{i\beta x} \qquad (9)$$

where "propagatcon constant" β is unknown, so (9) looks like a plane wave along x with "amplitude" Ψ depending on y, z.

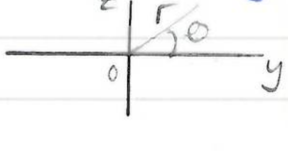
$\Rightarrow \Psi$ solves:

$$\nabla_\perp^2\Psi+(k^2+\beta^2)\Psi=0 \qquad (10).$$

where

$$\nabla_\perp^2=\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

is "cross-sectional Laplacian" on (y, z)-plane (r, θ) polar coordinates ($y=r\cos\theta$, $z=r\sin\theta$).



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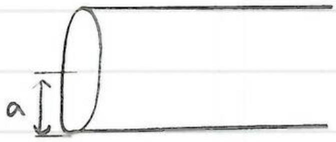


Now specialise to acoustics and <u>circular</u> waveguides : $r \leq a$.

Seek solutions of (10) via separation of variables:

$$\Psi = R(r)\,\Theta(\theta).$$

$$\Rightarrow \frac{r^2}{R\Theta} R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' + (k^2-\beta^2)R\Theta = 0.$$

$$\Leftrightarrow \frac{r^2 R''}{R} + \frac{rR'}{R} + r^2(k^2-\beta^2) = -\frac{\Theta''}{\Theta} = m^2$$

(m^2 separation constant).

$$\Rightarrow \Theta'' + m^2\Theta = 0$$

$$\Rightarrow \Theta = e^{\pm i m\theta}$$

So since Θ must be 2π-periodic in θ

$\Rightarrow m \in \mathbb{Z}$, integer;

$$\Rightarrow r^2R'' + rR' + [\underbrace{r^2(k^2-\beta^2)}_{\tilde{r}^2} - m^2]R = 0 \qquad (11)$$

Via change of variables, $\tilde{r} = r(k^2-\beta^2)^{1/2}$.

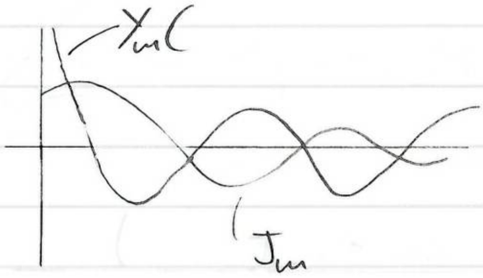
$$\Rightarrow \tilde{r}^2\frac{d^2R}{d\tilde{r}^2} + \tilde{r}\frac{dR}{d\tilde{r}} + [\tilde{r}^2 - m^2]R = 0 \qquad (11').$$

This is known to be Bessel's differ. eqn' of order m, with general soln:

$$R = A J_m(\tilde{r}) + B Y_m(\tilde{r}) = 0 \qquad (12).$$

A, B arbitrary consts, J_m the Bessel functcoin, Y_m (so-called) Neumann functcoin.

The solutcoin has to be a smooth for $0 \leq r \leq a$ in particular for $r \to 0$, which J_m is but Y_m is not.



($Y_m \to \infty$ as $r \to 0$)

$\Rightarrow B = 0$, and our "modal" solutcoins are

$$\hat{\varphi}(x,y,z) = A_m J_m\left(r(k^2-\beta^2)^{1/2}\right)e^{im\theta - i\beta x} \qquad (13)$$

$m \in \mathbb{Z}$, $A_m \in \mathbb{C}$.

Let the waveguide's boundary $r=a$ be e.g acoustically hard i.e:

$$\frac{\partial \hat{\varphi}}{\partial n} = \frac{\partial \hat{\varphi}}{\partial r} = 0, \quad \text{as } r = a.$$

$\Rightarrow$ from (13), $J_m'\left(a(k^2-\beta^2)^{1/2}\right) = 0$, so if α_{mn}, $n = 1, 2, 3, \ldots$ are roots of $J_m(\alpha_{mn}) = 0$.

$$\Rightarrow a(k^2-\beta^2)^{1/2} = \alpha_{mn}.$$

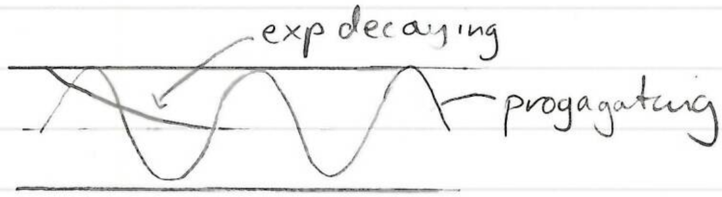
$$\Rightarrow \beta = \beta_{mn} = \pm\left(k^2 - \frac{\alpha_{mn}^2}{a^2}\right)^{1/2}, \quad n=1,2,3,\ldots$$

So $\exists$ infinitely many "modal soln"

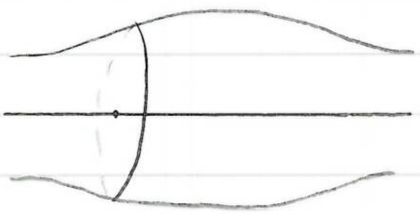
$$\hat{\varphi}_{mn} = A_{mn} J_m\left(\alpha_{mn}\frac{r}{a}\right)e^{im\theta - i\beta_{mn}x}.$$

$m=0,1,2,\ldots;\ n\geqslant 1.$

Notice however that for fixed k, m $\alpha_{mn}\to\infty$ as $n\to\infty$ so β_{mn} becomes complex for $n>(k,m)$: the corresponding modes are exponently decaying in x.



4.3 <u>Slowly varying waveguides</u>.



Let the waveguide's cross-section be circular but with a "slowly varying" radius R. Mathematically, let $\epsilon>0$ be a "small parameter," $\epsilon \ll 1$, and let $R \doteq R(\epsilon x)$: then

$$\frac{dR}{dx} = \epsilon R'(\epsilon x) = \epsilon R'(X) = O(\epsilon).$$

where $X := \epsilon x$ is the so-called "Slow variable".

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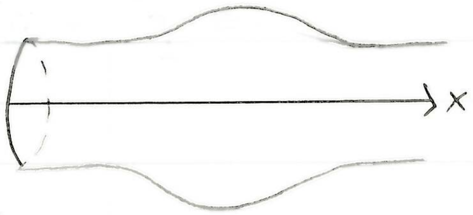
 $\to$ (14)



$R(\varepsilon x)$ $\uparrow$ small , $X = \varepsilon x$

— / —

Seek soln is the form, cf (14).

$$\hat{\phi} = A(X, r)\, e^{im\theta} \exp\left(\frac{i}{\varepsilon}\int_0^X \mu(X')\, dX'\right) \quad (15)$$

Substitute to Helmoltz eqn in (x, r, θ) coord:

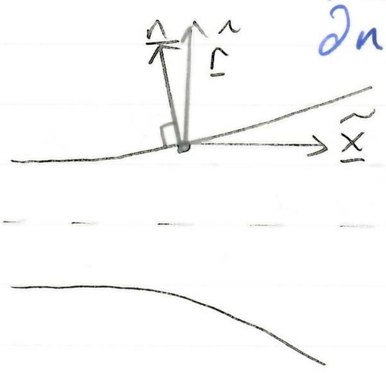
$$\frac{\partial^2 \hat{\phi}}{\partial x^2} + \frac{\partial^2 \hat{\phi}}{\partial r^2} + \frac{1}{r}\frac{\partial \hat{\phi}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \hat{\phi}}{\partial \theta^2} + k^2 \hat{\phi} = 0. \quad (16)$$

$$\Rightarrow \left(X = \varepsilon x, \quad \frac{\partial}{\partial x} = \varepsilon \frac{\partial}{\partial X}\right)$$

$$\Rightarrow -\mu^2(X) A(X, r) + 2\varepsilon i \mu \frac{\partial A}{\partial X} + i\varepsilon \mu'(X) A$$

$$+ \varepsilon^2 \frac{\partial^2 A}{\partial X^2} + \frac{\partial^2 A}{\partial r^2} + \frac{1}{r}\frac{\partial A}{\partial r} - \frac{m^2}{r^2} A + k^2 A = 0 \quad (17)$$

BCs: $0 = \dfrac{\partial \phi}{\partial n} = \underline{n}\cdot\nabla\phi, \quad \underline{n} = \hat{\underline{r}} - \dfrac{dR}{dx}\hat{\underline{x}}$



$$= \hat{\underline{r}} - \epsilon R'(x)\hat{\underline{x}}$$

$$\Rightarrow \frac{\partial\phi}{\partial n} = \frac{\partial\phi}{\partial r} - \epsilon R'(X)\frac{\partial\phi}{\partial x} = 0, \quad r = R(X)$$

$\Rightarrow$ via (15)

$$\frac{\partial A}{\partial r} - \epsilon R'(X)\left(i\mu A + \epsilon\frac{\partial A}{\partial X}\right) = 0 \qquad (18)$$

Seek $A(X, r)$ as perturbation expansion in small ϵ:

$$A = A_0(X, r) + A_1(X, r) + \ldots \qquad (19)$$

Plugging into (17), (18), to leading orders:

$$\mathcal{L}A := \frac{\partial^2 A_0}{\partial r^2} + \frac{1}{r}\frac{\partial A_0}{\partial r} + \left(k^2 - \mu^2(X) - \frac{m^2}{r^2}\right)A_0 = 0 \qquad (20)$$

$$(18) \Rightarrow \frac{\partial A_0}{\partial r} = 0, \quad r = R(X) \qquad (21)$$

(20) - (21) appear identical to § 4.2, $a \to R(X)$, X parameter, cf (11). $\Rightarrow$ cf (14)

$$A_0(X, r) = N(X) J_m(\gamma_{mn}(X) r), \quad (22)$$

where

$$\gamma_{mn}(X) = (k^2 - \mu^2(X)^{1/2}) \overset{(18)}{=} \frac{\alpha_{mn}}{R(X)} \quad J_m'(\alpha_{mn}) = 0$$

$$\Rightarrow \mu = \mu_{mn}(X) = \left(k^2 - \frac{\alpha_{mn}^2}{R^2(X)}\right)^{1/2}$$

$$\Bigg\} (23)$$

Notice $N(X)$ still unknown, and is found by equating next-order terms in (17) – (18):

$$\mathcal{L} A_1 = -2i\mu \frac{\partial A_0}{\partial X} - i\mu' A_0$$

$$= -i \frac{1}{A_0} \frac{\partial}{\partial X}\left(\mu A_0^2\right) \quad (24).$$

$$(18) \Rightarrow \frac{\partial A_1}{\partial r} = i\mu R'(X) A_0 \quad , r = R(X) \quad (25)$$

Take $(24) \times r A_0 - (20) \times A_1 r$, and integrate in A from 0 to $R(X)$:

$$LHS = \int_0^{R(X)} (\mathcal{L} A_1) r A_0 - (\mathcal{L} A_0) r A_1 \, dr \, ,$$

$$\frac{d}{dr} := '$$

$$= \int_0^R \left(\frac{1}{\cancel{r}} (r A_1')' \cancel{r} A_0 - \frac{1}{\cancel{r}} (r A_0')' \cancel{r} A_1 \right) dr$$

$$\stackrel{\text{By parts}}{=} \left[r A_1' A_0 - r A_0' A_1 \right]_0^{R(X)}$$

$$\stackrel{(25)+(21)}{=} R i \mu R'(X) A_0^2(X, R(X))$$

$$\text{RHS} = \cancel{-i} \int_0^{R(X)} \frac{\partial}{\partial x}(\mu A_0^2 r)\, dr = \cancel{i} \mu R' R A_0^2 .$$

$$\Rightarrow \int_0^{R(X)} \frac{\partial}{\partial x}\left(\mu A_0^2 r\right) dr + \mu R'(X) A_0^2 R = 0.$$

$$\Leftrightarrow \frac{d}{dx}\left(\int_0^{R(X)} \mu A_0^2 r \, dr\right) = 0 \text{ (Leibintz's rule).}$$

$$\Rightarrow \mu(X) \int_0^{R(X)} N^2(X) J_m^2\left(\alpha_{mn} \frac{r}{R(X)}\right) r \, dr .$$

$$= \text{const.}$$

$$\Leftrightarrow \left(\frac{r}{R(X)} = \tilde{r}\right) \text{ Change of variable.}$$

$$\Rightarrow \mu(X) N^2(X) R^2(X) \int_0^1 J_m^2(\alpha_{mn} \tilde{r}) \tilde{r} \, d\tilde{r}$$

$$= \text{constant.}$$

$$\Rightarrow \mu N^2 R^2(X) = Q^2 \text{ (= constant).}$$

$$\Rightarrow N(X) = \frac{Q}{R(X)\sqrt{\mu_{mn}(X)}}$$

μ_{mn} given by (23)

<u>Exercise</u> : 2-D channels : Exam 2010 Qn 5,
EM version Exam 2012 . Qn 5 .

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