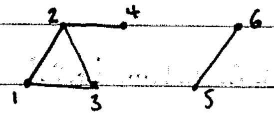
3503 Graph Theory and Combinatorics Notes

Based on the 2013 autumn lectures by Dr J Talbot

Office hour
Friday 704
10.

Graph Theory - lecture 1+2 09/01/2012

Intro to graph theory:



$G = (V, E)$ V - vertices
E - edges

$V = \{1,2,3,4,5,6\}$

$E = \{\{1,2\},\{1,3\},\{2,3\},\{2,4\},\{5,6\}\}$

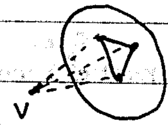
Extremal graph theory Ramsey Theory

number of
#
K for complete

$K_5 =$ 

between nodes (people)
--- strangers
—— friends

no triangles ⇒ situation where you can avoid pairing with stranger



6 people

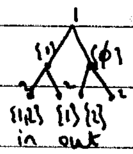
Intro to combinatorics:

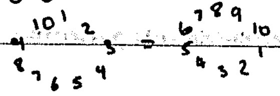
r
number nodes

$\#$ edges in K_5 = # unordered pairs from $\{1,2,3,4,5\} = \binom{5}{2} = \frac{5 \times 4}{2} = 10$

$X = \{1,2,3,\dots,10\}$

How many cyclic permutations of X are there?





cyclic permutations $= 9! = \frac{10!}{10}$

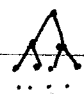
1, 2, 3, ..., 10
2, 3, 4, ..., 10, 1
⋮
9, 10, ..., 8
10, ..., 9

$X = \{1,\dots,n\}$ $\mathcal{A} = \{\{1,2,3\},\{2,4,n\}\}$

$\mathcal{A}$ family of subsets of X is <u>intersecting</u> if

$A, B \in \mathcal{A} \Rightarrow A \cap B \neq \phi$

all subsets of $X = 2^n$



for each element two choices in/out
⋮ n times $\Rightarrow 2^n$

$|\mathcal{A}| \leq 2^{n-1}$ Because have at most one of each complimentary pair: $(B, X \setminus B)$ $B \in \mathcal{A} \Rightarrow X \setminus B = B^c$

$\mathcal{A} = \{A \subseteq [n] : 8 \in A\}$ $(n \geq 8)$

notation for set X
$[n] = \{1,2,\dots,n\}$

$|\mathcal{A}| = 2^{n-1}$ = # subsets of an (n-1) set

<u>lemma 1.1</u>

(i) # k-tuples from $X = [n] = n^k$

<u>Proof</u>: n choices for each of k positions

e.g $X = [4]$ $k = 2$ 2-tuples from X are
(1,1) (2,1) (3,1) (4,1)
(1,2) (2,2) (3,2) (4,2)
(1,3) (2,3) (3,3) (4,3)
(1,4) (2,4) (3,4) (4,4)

(ii) # k-tuples with distinct elements from $X = n(n-1)\dots(n-k+1)$

<u>Proof</u>: n choices for 1^{st} entry
n-1 " —— " 2^{nd} "
etc
n-(k+1) " —— " k^{th} entry □

$\binom{X}{k} = \{A \subseteq X : |A| = k\}$ any ~~set~~ subset A size k in X

$\binom{5}{2} = 10$ $\binom{[5]}{2} = \{12, 13, 14, 15, 23, 24, 25, 34, 35, 45\} = \{\{1,2\}, \{1,3\}, \dots, \{4,5\}\}$

Lemma 1.2

$|X| = n$ and $0 \le k \le n$ then $\left|\binom{X}{k}\right| = \binom{n}{k}$

Each k-set ~~of~~ from X corresponds to k! different k-tuples of distinct elements of X

Hence lemma 1.1 (ii) $\Rightarrow$ $\left|\binom{X}{k}\right| = \frac{n(n-1)\dots(n-k+1)}{k!}$

$= \frac{n!}{k!(n-k)!}$

Probabilistic method

Idea: want an example of some mathematical object

Invent a probabilistic "experiment" where Pr(That the experiment generates a good example) > 0

$0! = 1 \Rightarrow \binom{n}{n} = \binom{n}{0} = 1$

Define $\binom{n}{k} = 0$ if $k < 0, k > n$ integer

$P(X) = \{A : A \subseteq X\}$

Lemma 1.3

(i) $|P(X)| = 2^n$: n elements in or out $\Rightarrow 2^n$

(ii) $\binom{n}{k} = \binom{n}{n-k}$, $\frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!}$

$B \longmapsto X \setminus B$ is a bijection

from $\binom{X}{k}$ to $\binom{X}{n-k}$

(iii) $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$

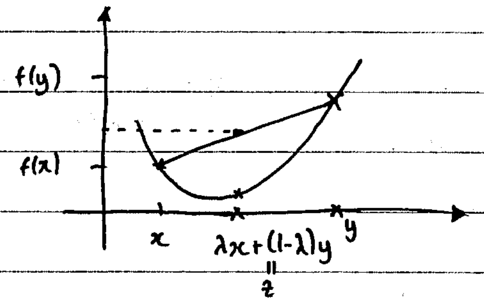
$\binom{n+1}{k} = \left|\binom{[n+1]}{k}\right|$, $\binom{n}{k} =$ # subsets of $[n+1]$ of size k not containing $n+1$, $\binom{n}{k-1} =$ # subsets of $[n+1]$ of size k, containing $n+1$

$x \in \mathbb{R}, s \ge 0$ integer

$$\binom{x}{s} = \begin{cases} \frac{x(x-1)\dots(x-s+1)}{s!} & , x > s-1 \\ 0 & , x \le s-1 \end{cases}$$

$f:(a,b)\to\mathbb{R}$ <u>convex</u> if $\forall x,y\in(a,b)$ $\lambda\in[0,1]$ 09/01/2013

$$f(\lambda x+(1-\lambda)y)\le\lambda f(x)+(1-\lambda)f(y) \quad \text{——} \ ①$$



<u>lemma 1.4</u>

If $f:(a,b)\to\mathbb{R}$ differentiable

$f'(x)$ non-decreasing on (a,b) then f is convex on (a,b)

<u>Proof</u>

let $x,y\in(a,b)$ $\lambda\in[0,1]$, $x<y$

If $z=\lambda x+(1-\lambda)y$ apply mean value Theorem

MVT: there exists $\xi_1\in(x,z)$, $\xi_2\in(z,y)$ s.t. $\dfrac{f(z)-f(x)}{z-x}=f'(\xi_1)$, $\dfrac{f(y)-f(z)}{y-z}=f'(\xi_2)$

Rearrange to give ① using $f'(\xi_1)\le f'(\xi_2)$ $\square$

<u>lemma 1.5</u>

$s\ge1$, $\varphi_s:\mathbb{R}\to\mathbb{R}$ $\varphi_s(x)=\binom{x}{s}$, then $\varphi_s(x)$ is convex

<u>Proof</u>

By induction on s, show $\varphi_s'(x),\varphi_s''(x)\ge0$ for $x\in(s-1,\infty)$

$\varphi_1'(x),\varphi_1''(x)\ge0$ $\therefore$ True for $s=1$

Fact: $s\varphi_s(x)=(x-s+1)\varphi_{s-1}(x)$

Differentiate: $s\varphi_s'(x)=\varphi_{s-1}(x)+(x-s+1)\varphi_{s-1}'(x)\ge0$ (by induction hypothesis on $s-1$)

Similarly for $\varphi_s''(x)$:

$$s\varphi_s''(x)=2\varphi_{s-1}'(x)+(x-s+1)\varphi_{s-1}''(x)\ge0$$

$\therefore$ $\varphi_s'(x),\varphi_s''(x)\ge0\Rightarrow$ (by lemma 1.4) $\varphi_s(x)$ is convex $\square$

<u>Theorem 1.6</u>

If $\varphi:(a,\infty)\to\mathbb{R}$ is convex $x_1,\dots,x_n>a$

$\lambda_1,\dots,\lambda_n\in[0,1]$, $\sum_{i=1}^n\lambda_i=1$ then $\varphi\left(\sum_{i=1}^n\lambda_ix_i\right)\le\sum_{i=1}^n\lambda_i\varphi(x_i)$

Proof

True for $n=1$ ✓ (By induction)

$n=2$ ✓

Now suppose $n \geq 3$, assume $\lambda_{n-1}+\lambda_n > 0$

$$y_i = \begin{cases} x_i, & 1 \leq i \leq n-2 \\ \dfrac{\lambda_{n-1}x_{n-1}+\lambda_n x_n}{\lambda_{n-1}+\lambda_n}, & i=n-1 \end{cases}$$

$$\mu_i = \begin{cases} \lambda_i & 1 \leq i \leq n-2 \\ \lambda_{n-1}+\lambda_n & i=n-1 \end{cases}$$

$y_1,\dots,y_{n-1} > 0$ $\quad \mu_1,\dots,\mu_{n-1} \in [0,1]$ $\quad \sum_{i=1}^{n-1}\mu_i = 1$ $\therefore$ Apply induction hypothesis for $n-1$

$$\Rightarrow \varphi\left(\sum_{i=1}^{n-1}\mu_i y_i\right) \leq \sum_{i=1}^{n-1}\mu_i \varphi(y_i)$$

$$\varphi\left(\sum_{i=1}^{n}\lambda_i x_i\right) \leq \sum_{i=1}^{n-2}\lambda_i\varphi(x_i) + (\lambda_{n-1}+\lambda_n)\varphi\left(\frac{\lambda_{n-1}x_{n-1}+\lambda_n x_n}{\lambda_{n-1}+\lambda_n}\right)$$

convexity $\Rightarrow$ result $\square$

Corollary 1.9

Proof

Directly from Theorem 1.6 by convexity of $f(x)=x^2$ and $f(x)=\binom{x}{s}$ $\square$

lemma 1.8

$$\frac{(n-s+1)^s}{s!} \leq \binom{n}{s} \leq \frac{n^s}{s!}$$

$$\frac{n(n-1)\dots(n-s+1)}{s!}$$ ← s terms ⇒ inequality

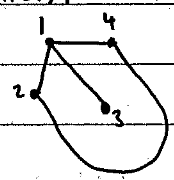
Graphs

$G=(V,E)$

vertices edges

$G = ([4], \{12,13,14,24\})$ — order is 4 (4 vertices), size 3 (3 edges)

$E \subseteq \binom{V}{2}$



Planar graph

$\Gamma(1) = \{2,3,4\}$ neighbourhood of 1

$d(1) = 3$ degree of 1 (number of things it's connected to)

lemma 1.9 (Handshake lemma)

$G=(V,E)$ $\quad \sum_{v \in V} d(v) = 2|E|$ ——— (*)

for above example $3+1+1+1 = 2\times|E| = 6$

Proof

Each edge has two endpoints so is counted twice in the LHS of (*) $\square$

Graph Theory - lecture 3 11/01/2013

$G=(V,E)$ a graph $\qquad \sum_{v\in V} d(v) = 2|E|$

<u>lemma 1.10</u>

In any graph the number of vertices of odd degrees is even

<u>Proof</u>

Take a graph $G=(V,E)$

$V = A \dot{\cup} B$ (← disjoint union) where $A = \{v : d(v) \text{ odd}\}$, $B = \{v : d(v) \text{ even}\}$

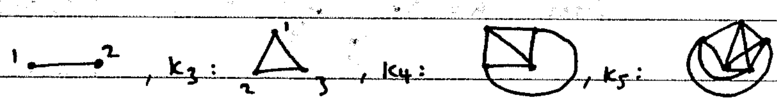
$\sum_{v\in V} d(v) = 2|E|$ is even

$\sum_{v\in B} d(v)$ is even since it is a sum of even numbers

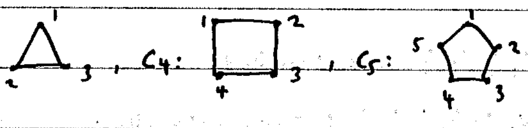
Hence $\sum_{v\in V} d(v) = 2|E| - \sum_{v\in B} d(v)$ is even

Hence $|A|$ is even $\square$

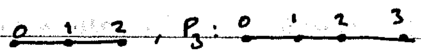
1) complete graph K_n - K_2: , K_3: , K_4: , K_5:



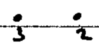
2) cycle C_n - C_3: , C_4: , C_5:



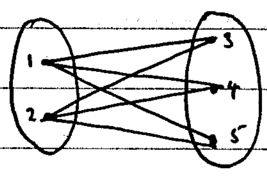
3) path P_n - P_2: , P_3:



4) Empty graph E_n - E_3:

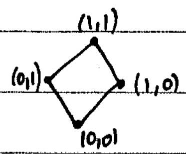


5) Bipartite graph $K_{2,3}$ -

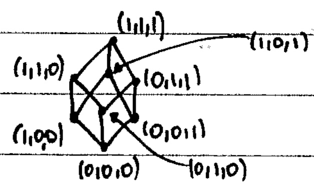


6) $\{0,1\}^n$ $\qquad V(Q_n) = \{0,1\}^n = \{(x_1, x_2, \dots, x_n) : x_i \in \{0,1\}\ \forall i\}$

Q_2:



Q_3:

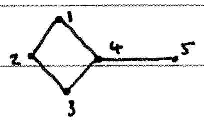


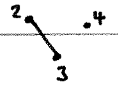
hypercube can relate to powerset

$P([n]) = \{A : A \subseteq [n]\} \leftrightarrow \{0,1\}^n$

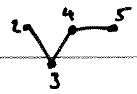
$A \mapsto (x_1, \dots, x_n)$, $x_i = 1$ iff $i \in A$

1.5 Subgraphs

$G=(V,E)$ $G:$ 

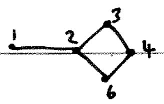
$H_1 =$  is a subgraph of G

not induced (as we haven't taken the edge 3,4)

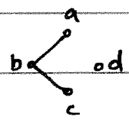
$H_2 =$  is an induced subgraph

$H_2 = G[\{2,3,4,5\}]$

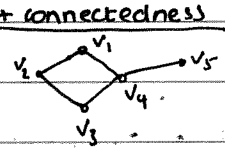
(graph is defined by edges and vertices, not diagram)

$H_3 =$  H_3 and G are isomorphic

can rename labels in H_3 to become same as G

G contains a copy of $H =$ 

1.6 components + connectedness

from $G =$  , $v_1 v_4 v_5$ is a path in G

it is a v_1-v_5-path

$v_1 v_4 v_5 v_4 v_3$ is a walk in G

$v_1 v_4 v_5 v_4 v_3 v_2 v_1$ is a closed walk in G (starts + finishes at same place)

$v_1 v_2 v_3 v_4$ is a tour in G (not allowed to reuse edges)

lemma 1.11

There is an x-y-path in G iff there is an x-y walk in G

Proof

($\Rightarrow$) a path is a walk (trivial)

($\Leftarrow$) Take the shortest walk from x to y

If any vertex is revisited we could shorten this walk

Hence it is a path $\square$

lemma 1.12 11/01/2013

Define a relation $\sim$ on $V(G)$ by $v \sim w$ iff there is a walk from v to w in G

There is an equivalence relation

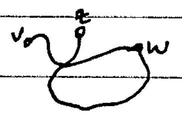
Proof

3 properties of equivalence relation

Reflexive $v \sim v$ take walk v

Symmetric $v \sim w \Rightarrow \exists$ walk v to w, reverse it

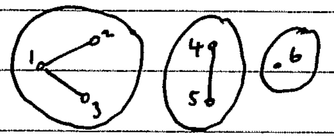
Transitivity $v \sim w$ and $w \sim z$ then concatenate the $v \sim w$ and $w \sim z$ walks to give a $v \sim z$ walk $\square$



$\sim$ induces a ~~path~~ partition of $V(G) = V_1 \dot{\cup} V_2 \dot{\cup} \dots \dot{\cup} V_k$

each V_i is a component

e.g. G:



G is connected iff there is a single component

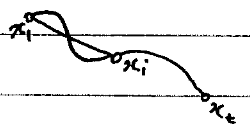
lemma 1.13

$P = x_1 x_2 \dots x_t$ is a path in G

If P is a shortest $x_1 - x_t$-path in G then $x_1 \dots x_i$ and $x_i \dots x_t$ are shortest $x_1 - x_i$ and $x_i - x_t$ paths respectively

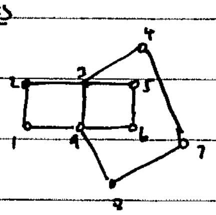
Proof

If not could shorten P

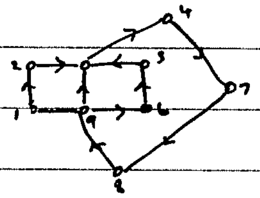


$\square$

Euler circuits



can you find an EC (where we use edge only once)



1.7. Euler Circuits

An Euler circuit in a graph is a closed tour (start - end; no repeated edges) containing all vertices and edges of G

Theorem 1.14

A graph G has Euler circuit iff it is connected and all vertices have even degree.

Proof

($\Rightarrow$) Assume G has an Euler circuit $T = v_0 v_1 \dots v_k$ $(v_0 = v_k)$

So G is certainly connected. Follow T counting the contribution to the degree of each vertex we visit.

Add 2 each time (except at start + end). Hence all degrees are even.

($\Leftarrow$) So suppose G is connected and all vertices have even degree

Take a longest tour $T = v_0 v_1 \dots v_k$ in G.

Claim: $v_0 = v_k$ if not let $j = \#\{i : v_i = v_k\}$

then if $v_0 \neq v_k$ then we ~~could~~ have used $2j - 2 + 1 = 2j - 1$ edges incident to v_k

$\therefore$ An unused edge $v_k v^* \Rightarrow T' = v_0 \dots v_k v^*$ is a longer tour $\#$

Hence $v_0 = v_k$

If there is an unused edge say $e = uv$, there are two cases to consider

Case 1

u or v is in T, say $v = v_i$: $T' = u v_i v_{i+1} \dots v_k = v_0 v_1 \dots v_{i-1}$ is a longer tour $\#$

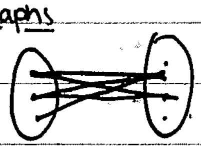
Case 2

$u, v \notin T$. G is connected so $\exists$ a v_0-u-path in G.

Consider the first edge in this path that leaves T but this gives us case 1 again $\#$

All vertices have degree ≥ 2 so they are visited by T $\square$

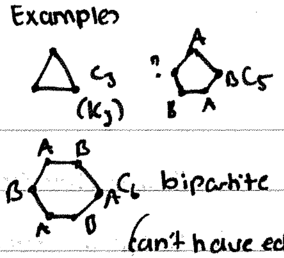
1.8. Bipartite Graphs

incomplete 

Bipartite - $V(G) = A \dot\cup B$ $E(G) \subseteq \{ab : a \in A, b \in B\}$

$G = (A, B; E)$

Examples



cant have bipartite

bipartite (can't have edges in a set)

Theorem 1.15

A graph is bipartite iff it contains no odd cycles

Proof

($\Rightarrow$) Suppose G is bipartite with bipartition $V = A \cup B$. If $C = v_1 \dots v_t$ is a cycle in G and wlog $v_1 \in A$ then

$v_3, v_5, \dots \in A$

$v_2, v_4, \dots \in B$

Hence we must have t is even

($\Leftarrow$) Suppose $G = (V,E)$ is connected (otherwise repeat this argument for each connected component)

For $x,y \in V$ let $d(x,y)$ = length of a shortest x-y-path

Fix a vertex $w \in V$.

Define $A = \{v : d(v,w) \text{ is odd}\}$

$B = \{v : d(v,w) \text{ is even}\}$

Note $V(G) = A \dot\cup B$. Need to check ~~dom~~ A and B do not contain any edges

Suppose there is an edge xy inside A (i.e $x,y \in A$)

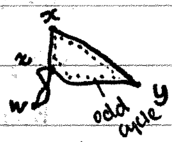
Let P_{wx} be a shortest w-x-path

Let P_{wy} be a shortest w-y-path

Let z be the last common vertex of P_{wx} and P_{wy}

Then the ~~path~~ part of P_{wx} from w to z is a shortest w-z path

" " " P_{wy} " " " w-z path



w-z must have same length.

Both have length $d = d(w,z)$

Now suppose $d(w,x) = 2i+1$, $d(w,y) = 2j+1$ i,j integers.

Then the cycle that follows P_{wx} from z to x, then xy, then P_{wy} from y to z has ~~length~~

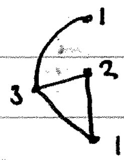
length $= 2i+1-d+1+2j+1-d$

$= 2(i+j+1-d)+1$ is odd #

Hence G is bipartite $\square$

A set $A \subset V$ is <u>independent</u> iff it contains no edges

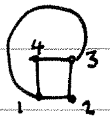
$c : V(G) \to [k]$ $vw \in E \Rightarrow c(v) \neq c(w)$



k-colourable $\equiv$ k-partite

2-colourable $\equiv$ bipartite

$\chi(G) = \min\{k : \exists \text{ k-colouring of } G\}$



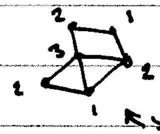
$\chi(C_{2t}) = 2$

$\chi(C_{2t+1}) = 3$

H is a subgraph of G then $\chi(H) \leq \chi(G)$

Theorem 1.16

If G is a graph then $\chi(G) \leq \Delta(G)+1$ $\quad (\Delta(G) = \max\{d(v) : v \in V(G)\})$



start here look at neighbours then colour with diff numbers

Proof

Let $V = \{v_1, \dots, v_n\}$. Let $k = \Delta(G)+1$.

Define a k-colouring $c : V(G) \to [k]$ as follows

$c(v_1) = 1$. If $v_1, \dots, v_{i-1}$ have been coloured

Let $C = \{c \in [k] : \exists\, j \in [i-1]$ s.t. $v_j \in \Gamma(v_i)$ and $c(v_j) = c\}$

Define $c(v_i) = \min [k] \setminus C$. This is well-defined since $|C| \leq d(v_i) \leq \Delta(G) = k-1$.

So $[k] \setminus C \neq \emptyset$ □ "Greedy Algorithm"

1.10 Large girth + chromatic number

If G is a graph containing cycles, then the girth of G is the length of the shortest cycle

Theorem 1.17 (Erdős)

For $k, l \geq 3$ $\exists$ G a graph with $\chi(G) \geq k = 10^6$

$$g(G) \geq l = 10^6$$

$$\alpha(G) = \max\{|A| : A \subseteq V(G) \text{ is independent}\}$$

Lemma 1.18

~~$\alpha(G) = \max\{|A|$~~

For any graph G, $\chi(G) \geq n/\alpha(G)$ $\quad n = |V(G)|$

colouring c

Proof

If $c : V(G) \to [k]$ is a k-colouring of G

then each colour class $c^{-1}(i) = \{v \in V(G) : c(v) = i\}$ is an independent set, so $|c^{-1}(i)| \leq \alpha(G)$

But $V(G) = c^{-1}(1) \,\dot\cup\, c^{-1}(2) \,\dot\cup \dots \dot\cup\, c^{-1}(k)$

so $\sum_{i=1}^{k} |c^{-1}(i)| = n$

Hence ⊕ $\Rightarrow k\alpha(G) \geq n \Rightarrow k \geq n/\alpha(G)$

Thus $\chi(G) \geq n/\alpha(G)$

□

notes pg7

A die (Ω, P_u) $\Omega = \{1,2,3,4,5,6\}$ $P_u(i) = \frac{1}{6}$ $1 \le i \le 6$

$X_1(y) = \begin{cases} 1, & y \in \{1,3,5\} \\ 0, & o/w \end{cases}$ $X_2(y) = \begin{cases} 1, & y \geq 4 \\ 0, & o \end{cases}$

$E[X] = \sum_{z \in O_X} z P(X = z)$ $O_X = \{X(y) \mid y \in \Omega\}$

Lemma 1.19 (linearity of Expectation)

If $X_1, X_2, \dots X_n$ are random variables then

$$E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$$

Proof

Follows from def^n of expectation □

Theorem 1.20

If G has e edges then G ~~has~~ contains a bipartite subgraph with at least $\lceil e/2 \rceil$ edges. ← ⌈ ⌉

(at most $\lfloor e/2 \rfloor$ edges.) ← ⌊ ⌋

Proof

Consider a random bipartition of $V = A \dot\cup B$

For each vertex $v \in V$ flip an independent fair coin, if Heads then put v in A

if Tails then put v in B

For an edge $uv \in E$ let $X_{uv} = \begin{cases} 1 & uv \text{ goes from } A \text{ to } B \\ 0 & o/w \end{cases}$

Let $X = \sum_{uv \in E(G)} X_{uv}$, then $E[X] \underset{\text{L. of E}}{=} \sum_{uv \in E(G)} E[X_{uv}] = \sum_{uv \in E(G)} P(uv \text{ goes from } A \text{ to } B)$

$P(uv \text{ goes from } A \text{ to } B) = \frac{1}{2}$

Hence $E[X] = \sum_{uv \in E(G)} \frac{1}{2} = \frac{e}{2}$

Thus $\exists$ a bipartition $V = A \cup B$ with at least $e/2$ edges between A and B

Hence (since the number of edges is an integer) at least $\lceil e/2 \rceil$ edges between A and B □

Graph Theory + Combinatorics lectures 4+5 08/10/2013

Definition A *path* in a graph G is a subgraph isomorphic to P_t (the path of length t)

i.e. a sequence $v_0 v_1, \dots, v_t$ of distinct values with $v_{i-1}v_i$ an edge for $1 \leq i \leq t$

Definition *x-y path*: start at $x \in V(G)$ end at $y \in V(G)$

$(1 \leq i \leq t)$

Definition A *walk* is a sequence of vertices in G $x_0, x_1, \dots, x_t$ (not necessarily distinct) with $x_{i-1}x_i \in E$

Definition A walk with no repeated edges is called a tour.

Lemma 1.11 There's an x-y path in G iff there is a walk from x to y in G

Proof $(\Rightarrow)$ Trivial

$(\Leftarrow)$ A shortest x-y walk is an x-y path.

Lemma 1.12 Define $\sim$ a relation on $V(G)$ by $x \sim y$ iff there is a walk from x to y

Then $\sim$ is an equivalence relation.

Check conditions of equivalence relation:

Reflexive: $x \sim x$ Take walk from x with no edges

Sym: $x \sim y \Rightarrow y \sim x$

Take an x-y walk and reverse it

Trans: $x \sim y$ and $y \sim z \Rightarrow x \sim z$

Take an x-y walk and a y-z walk and concatenate them $\square$

So $\sim$ partitions $V(G) = V_1 \dot{\cup} V_2 \dot{\cup} \dots \dot{\cup} V_k$. We call each V_i a component.

Definition A graph G is *connected* iff it has a single component.

Lemma 1.13 Let $P = x_1 x_2 \dots x_t$ be a shortest x_1-x_t path in G. Then for any $1 < i < t$, $x_1 x_2 \dots x_i$ and $x_i x_{i+1} x_t$ are shortest x_1-x_i and x_i-x_t paths respectively.

Proof If not could shorten the x_1-x_t-path $\square$

x_1 x_i x_t can go straight without using x_i if needed

1.7 Euler Circuits

start = end.

An Euler circuit is a closed tour that uses all vertices and edges of G with edges used exactly once.

Theorem 1.4 (Euler) A graph G has an Euler circuit iff G is connected and all vertex degrees are even.

Proof : ($\Rightarrow$) Suppose G has an Euler circuit.

The Euler circuit in G has a walk using all vertices hence G is connected.

Follow the circuit, counting the contribution to each vertex degree as we pass through it.

Except for the 1st vertex, we count 2 at each vertex. Since 1st vertex = final vertex, its degree is also even.

($\Leftarrow$) Suppose G is connected and all vertex degrees are even.

Let $T = v_0 v_1 v_2 \dots v_k$ be a longest tour in G.

Claim T is a circuit i.e. $v_0 = v_k$.

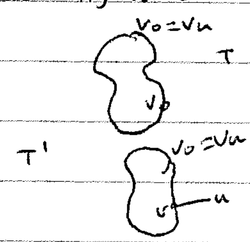
If not $v_0 \neq v_k$ then let $j = \#\{1 \le i \le k-1 : v_i = v_k\}$

So T has used $2j+1$ edges incident to v_k check $v_0 \neq v_k$

Since $d(v_k)$ is even, there is an unused edge say v_k so can extend T ※

We now claim T is an Euler circuit.

Idea original tour



If not, there is an unused edge $uv \in E(G)$

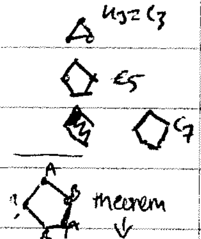
$T = v_0 v_1 \dots v_k$ (now $v_k = v_0$)

Suppose u or v lies in T say $v = v_i$

Define a new tour $T' = u\underset{v}{v_i} v_{i+1} \dots v_k v_1 v_2 \dots v_i$

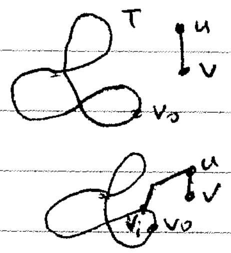
which is a longer tour ※

Aside: odd cycles C_3, C_5, C_7



theorem ↓ can't have connection within set.

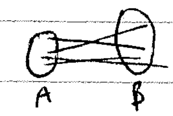
Final case is that neither u nor v lie on T



Since G is connected, there is a v_0-u path, at some v_i this path leaves T.

Thus there is again an unused edge $v_i w$ which we can deal with as in the previous case. □

1.8 Bipartite graph



G is a bipartite iff $V(G) = A \dot\cup B$ and $E(G) \subseteq \{ab : a \in A, b \in B\}$

We say A, B is a bipartition of G

$G = (A, B; E)$ to emphasize a particular bipartition

Theorem 1.15 A graph is bipartite iff it contains no odd cycles

Proof : ($\Rightarrow$) Suppose G is bipartite $G = (A, B; E)$ and $C = v_1 v_2 \dots v_t$ is a cycle, then $v_1, v_3, v_5, \dots$ are all in one class, wlog A, $v_2, v_4, v_6, \dots$ are all in one class B

Then if t is odd, v_t and v_1 are both in A ※

②

($\Leftarrow$) Let G be a graph with no odd cycles. 08/10/2013

Suppose wlog that G is connected (otherwise repeat for each component)

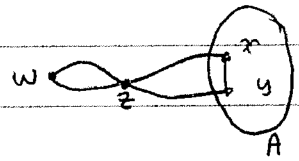
Let $x,y \in V(G)$. Define $d(x,y)$ = length of a shortest x-y path.

Fix $w \in V$, let $A = \{v \in V : d(v,w) \text{ is odd}\}$

$B = \{v \in V : d(v,w) \text{ is even}\}$

So $V = A \dot\cup B$

Need to show that there are no edges inside A or inside B.



Suppose for a contradiction, there is an edge xy inside A

Let P_{wx} be a shortest w-x path

" P_{wy} " ——— " w-y path

would have crossed many times be 4 including at w

Let z be the last common vertex on P_{wx} and P_{wy}

Let $d(w,x) = 2i+1$

Let $d(w,y) = 2j+1$

Following P_{wx} from z to x then x to y then P_{wy} back to z gives a cycle say C

Note: Since P_{wx} and P_{wy} are both shortest paths, the path of each between w and z are both shortest w-z paths of length $d(w,z)=d$

C is a cycle of length $= (2i+1-d)+1+(2j+1-d)$

$= \cancel{}\ 2(i+j+1-d)+1$ is odd #

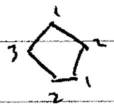
$\therefore$ A contains no edges

Similarly B contains no edges.

$\therefore$ G is bipartite $\square$

1.9 Graph colouring

A set $A \subseteq V(G)$ is independent if it contains no edges.

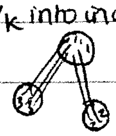


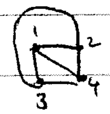
For any $k \geq 1$, a k-colouring of G is $c : V(G) \to [k]$ s.t. if $vw \in E$ (edge) then $c(v) \neq c(w)$

G is k-colourable $\Leftrightarrow$ $\exists$ a k-colouring of G.

G is k-partite iff $\exists$ a partition $V(G) = V_1 \dot\cup V_2 \dot\cup \dots \dot\cup V_k$ into independent sets.

i.e. 2-partite $\equiv$ bipartite 3-partite graph





k-partite $\equiv$ k-colourable.

chromatic number

$\chi(G) = \min\{k : G \text{ is } k\text{-colourable}\}$

$\chi(K_t) = t$, $\chi(C_{2t}) = 2$, $\chi(C_{2t+1}) = 3$

$H \subseteq G$ then $\chi(H) \leq \chi(G)$

$\Delta(G) = \max_{v \in V(G)} d(v)$, $\delta(G) = \min_{v \in V(G)} d(v)$

Theorem 1.16 — If G is a graph, then $\chi(G) \leq \Delta(G)+1$

Proof — Define $k=\Delta(G)+1$

k - number of colours — Define a k-colouring $c: V(G) \to [k]$ as follows

let $V(G) = \{v_1, v_2, \dots, v_n\}$

let $c(v_1) = 1$

Now suppose we have coloured vertices $v_1, v_2, \dots, v_{i-1}$

Γ - neighbourhood of v_i — let $C = \{j \in [k] : \exists v \in \Gamma(v_i) \text{ s.t. } c(v) = j\}$

$d(v_i)$ — $|C| \leq d(v_i) \leq \Delta(G) \leq k-1$

$\therefore [k] \setminus C \neq \phi$

So set $c(v_i) = \min [k] \setminus C$ $\square$

"Greedy colouring algorithm"

Definition — Girth of G = length of a shortest cycle in G

Theorem (Erdös 1959) — $\forall k, l \geq 3 \ \exists G$ a graph with $\chi(G) \geq k$ and $g(G) \geq l$

①

$g(G)$ = length of a smallest cycle in G

$\chi(G) = \min\{k : \exists\ k\text{-colouring of } G\}$

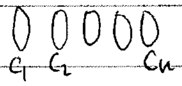
Theorem 1.17 (Erdős) $\forall k, l \geq 3$ $\exists$ graph G s.t. $\chi(G) \geq k$ and $g(G) \geq l$

~~Pf~~

$\alpha(G) = \max\{|A| : A \subseteq V(G)$ is an independent set$\}$

Lemma 1.18 $\chi(G) \geq \dfrac{|V(G)|}{\alpha(G)}$

Proof Take a k-colouring of G



colour of vertices.

$C_i = \{v : c(v) = i\}$

$V(G) = C_1 \dot\cup C_2 \dot\cup \dots \dot\cup C_k$ coloured s.t no adjoining vertices have same colour.

Each C_i is an independent set so $|C_i| \leq \alpha(G)$

$|V(G)| = \sum_{i=1}^{k} |C_i| \leq k\alpha(G)$

Hence $|V(G)| \leq \chi(G)\alpha(G)$ $\square$

Random variable X, $O_X = \{X(y) : y \in \Omega\}$

$$X_1(y) = \begin{cases} 1 & y = 1,3,5 \\ 0 & \text{o/w} \end{cases}$$

$$\mathbb{E}[X] = \sum_{z \in O_X} z\, P(X = z)$$

Lemma 1.19 $X_1, \dots, X_n$ $\mathbb{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbb{E}[X_i]$

Theorem 1.20 G is a graph with e edges, then G has a bipartite subgraph with $\geq \lceil \frac{e}{2} \rceil$ edges

Proof Assign each $v \in V$ to A or B

For $uv \in E$ let $X_{uv} = \begin{cases} 1, & uv \text{ is good} \\ 0, & \text{o/w} \end{cases}$

Say an edge is good if it joins vertices from different classes

Consider the bipartite subgraph H of G given by the bipartition $A \dot\cup B$

X = # edges in the H

$$X = \sum_{uv \in E(G)} X_{uv}$$

$$\mathbb{E}[X] \overset{1.19}{=} \sum_{uv \in E} \mathbb{E}[X_{uv}] = \sum_{uv \in E(G)} P(uv \text{ is good})$$

For any $uv \in E(G)$, $P(uv \text{ is good}) = \frac{1}{2}$

$\mathbb{E}[X] = \frac{e}{2}$

LHS of ineq is integer.

$\therefore \exists$ bipartite subgraph with $\geq \lceil \frac{e}{2} \rceil$ edges. $\square$

Lemma 1.21 (Markov)

X non-negative random variable and $\lambda > 0$ then $P(X \geq \lambda) \leq \frac{\mathbb{E}[X]}{\lambda}$

Proof

Suppose X takes value in O_X

$+\sum_{y<\lambda} yP(X=y)$ +ve

$\mathbb{E}[X] = \sum_{y \in O_X} yP(X=y) \geq \sum_{y \geq \lambda} \lambda P(X=y) = \lambda P(X \geq \lambda)$ $\square$

$\Omega = \{G : V(G) = [n], E(G) \leq \binom{[n]}{2}\}$

$\mathcal{G}(n,p)$: Start with E_n. For each $ij \in \binom{[n]}{2}$

Flip an independent coin, with probability p it is heads in which case insert the edge ij

Want G with $\chi(G) \geq k$, $g(G) \geq l$

Call a cycle short if it has length $\leq l$

Let H be a random graph for $\mathcal{G}(n,p)$

A : "H has $\geq \frac{n}{2}$ short cycles"

B : "H has $\alpha(H) > \frac{n}{2k}$"

PROOF

Claim : If $H \in \mathcal{G}(n,p)$ then

$P(A \text{ holds}) \leq \frac{1}{3}$, $P(B \text{ holds}) \leq \frac{1}{3}$

$P(\text{neither } A \text{ nor } B \text{ holds}) \geq \frac{1}{3}$

Assume this claim is true

Hence $\exists$ a graph H for which neither A nor B hold.

Delete a single vertex from each short cycle in H.

This gives a graph G with no short cycles $\therefore$ $g(G) > l$

Also $\alpha(H) \leq \frac{n}{2k}$ (because B doesn't hold)

so $\alpha(G) \leq \alpha(H) \leq \frac{n}{2k}$

So G has $|V(G)| \geq \frac{n}{2}$, $g(G) > l$, $\alpha(G) \leq \frac{n}{2k} \leq \frac{|V(G)|}{k}$ lemma 1.18 $\Rightarrow \chi(G) \geq k$ $\square$

haven't proved claim yet.

Graphtheory-lecture 7+8 15/10/2013

Claim: If $H \in G(n,p)$ (n,p given below) and

A = "H has $\geq \frac{n}{2}$ short cycles" short $\equiv \leq l$

B = "$\alpha(H) \geq \frac{n}{2k}$"

then $P(A) \leq \frac{1}{3}$ ① and $P(B) \leq \frac{1}{3}$ ②

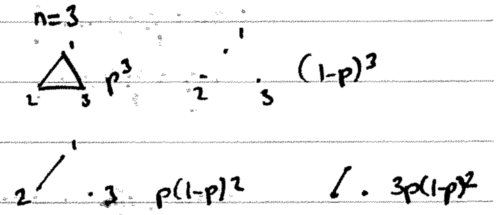
$n \geq$?

$p = \frac{1}{n^{1-1/2l}}$

indpt set: largest set with no vertices.

$G(n,p)$ random graph model.

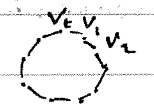
n=3



lemma 1.22 Let $H \in G(n,p)$ and let X_t = # t-cycles in H then

$$\mathbb{E}[X_t] = \frac{n(n-1)\dots(n-t+1)}{2t} p^t$$

Proof For $v_1, v_2, \dots, v_t$ fixed



Probability they form a t-cycle in this order is p^t

$$\begin{cases} v_1, v_2, \dots, v_t & v_t, v_{t-1}, \dots, v_1 \\ v_2, \dots, v_t, v_1 & \text{and reverse} \\ \vdots & \\ v_t, v_1, \dots, v_{t-1} & v_{t-1}, v_{t-2}, \dots, v_t \end{cases}$$

$\therefore$ 2t different t-tuples give rise to the same t-cycle

$$\mathbb{E}[X_t] = \sum_{\substack{\text{potential} \\ \text{t-cycle } T}} P_r(T \text{ occurs}) = \frac{n(n-1)\dots(n-t+1)}{2t} p^t \quad \square$$

Now let $H \in G(n,p)$,

$$X = \text{\# short cycles in } H = \sum_{t=3}^{l} X_t$$

$$\mathbb{E}[X_t] = \sum_{t=3}^{l} \mathbb{E}[X_t] = \sum_{t=3}^{l} \frac{n(n-1)\dots(n-t+1)}{2t} p^t \leq \sum_{t=3}^{l} (np)^t$$

$$\mathbb{E}[X] \leq \sum_{t=3}^{l} n^{t/2l} \leq l n^{\frac{1}{2}} \leq \frac{n}{6} \text{ if } n \geq 36 l^2$$ calculated question mark.

MARKOV: $P(X \geq 3\mathbb{E}[X]) \leq \frac{1}{3}$

so $P(X \geq \frac{n}{2}) \leq \frac{1}{3}$

$(1-p)^{\binom{s}{2}}$

If $W \subseteq V(H)$ $|W| = s$

then $P(W \text{ is independent}) = (1-p)^{\binom{s}{2}}$

"$\alpha(H) \geq s$" = "H has an independent set of size s"

using $P(C \cup D) \leq P(D) + P(C)$

$$P(\alpha(H) \geq s) = P(\exists W \subseteq V(H), |W| = s, W \text{ indpt}) \leq \sum_{\substack{W \subseteq V(H) \\ |W| = s}} P(W \text{ is indpt})$$

not set bigger than s, otherwise you can get a subset of s using $\boxed{1-p \le e^{p}}$ $\qquad$ $\binom{n}{s} \le n^s$

So $P(\alpha(H) \geq s) \leq \binom{n}{s}(1-p)^{\binom{s}{2}} \leq \left(n e^{-p\frac{(s-1)}{2}}\right)^s = \left(n e^{-2\log n}\right)^s = \frac{1}{n^s} < \frac{1}{3}$

let $s = \frac{4}{p}\log n + 1$

$\frac{s-1}{2} = \frac{2}{p}\log n$

want ~~$s \leq \frac{n}{2k}$~~ $s \leq \frac{n}{2k}$ $\qquad$ $1 - \frac{1}{2l} < 1$

$\cancel{\frac{4n^{\frac{1}{2l}}}{n}}$ $\frac{4n}{n^{1/2l}}\log n + 1 \leq \frac{n}{2k}$ true for n large since LHS grows slower than RHS ($n^{1-\frac{1}{2l}}\log n$ vs n)

$\square$

2 Extremal Graph Theory

Definition: A *Hamilton cycle* in a graph G is a cycle containing all the vertices of G. (exactly once)

$\delta(G) = \min_{v \in V} d(v)$

Definition: $u, v \in V(G)$ are *adjacent* iff uv is an edge in G

Theorem 2.1 (Dirac): If G has order $n \geq 3$ and $\delta(G) \geq \frac{n}{2}$ then G has a Hamilton cycle

Theorem 2.2 (Ore): If G has order $n \geq 3$ and $d(u) + d(v) \geq n$ for every pair of non-adjacent vertices.

Proof (2.2): By contradiction

Let G have order $n \geq 3$ satisfy $d(u) + d(v) \geq n$ $\forall u, v$ non-adjacent

But suppose G doesn't contain a Hamilton cycle.

If there is a pair $uv \notin E(G)$ which we can add to $E(G)$ without creating a H-C (hamilton cycle), then add it

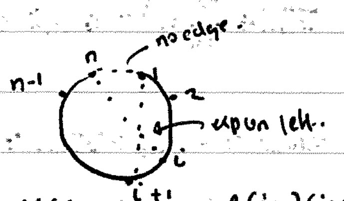
Repeat until no more pairs can be added.

Call the final graph G'

$[n] = 1, \dots, n$

let $V(G') = [n]$

Then wlog G' contains a path $1\,2\,3\,4 \dots n$



$1(i+1)$ from 1 to $i+1$

If $1(i+1)$ and in are both edges in G' then we have a HC: $1(i+1)(i+2)\dots n\,i(i-1)\dots 2\,1$

G' certainly still satisfies the condition that $d(u) + d(v) \geq n$ $\forall u, v \in V(G')$ non-adjacent.

Consider $d(1) + d(n)$

For $i = 2, 3, \dots, n-2$

Have at most one edge from $1(i+1)$ and in

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13, 2n
14, 3n
⋮
1(n-1), (n-2)n } Have at most n-3 edges from there and 2 other edges: 12, (n-1)n
12, (n-1)n

So $d(1)+d(n) \le n-3+2 = n-1$ ## □

PROBLEM CLASS

1

$$\sum_{b=0}^{k} \binom{a}{b}\binom{n-a}{k-b} = \binom{n}{k} \leftarrow \#\text{ k-sets from } [n]$$

$\underline{1\,2\,3 \ldots a}\ a+1 \ldots n$

$\binom{a}{0}\binom{n-a}{k}$ = # k sets from [n] with 0 underlined elements

$\binom{a}{1}\binom{n-a}{k-1}$ = " ______ " 1 " ______ ".

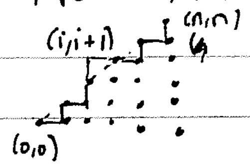
$\binom{a}{b}\binom{n-a}{k-b}$ = " ______ " b " ______ "

2 (a) # paths from (0,0) to (m,n) $\binom{m+n}{n}$ = # subsets of [m+n] of size n

there m+n steps needed to be taken

label steps $\overset{1}{\uparrow} \to \overset{2}{\uparrow} \to \overset{3}{\to} \uparrow \ldots \overset{m+n}{\uparrow}$

(b) # paths (0,0) to (n,n) do not cross $x=y$



$\binom{2n}{n}$

Reflect the path from (i, i+1) to (n,n)

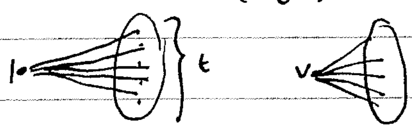
→ gives a path to (n-1, n+1)

bad paths = # paths (0,0) to (n-1, n+1) = $\binom{2n}{n+1}$

Ans = $\binom{2n}{n} - \binom{2n}{n+1} = \frac{1}{n+1}\binom{2n}{n}$ "n^{th} catalan number"

4 $|V(G)| = n \quad |E(G)| = e$

copies of $K_{1,t} \ge \binom{2e/n}{t} n$ $\qquad \frac{2e}{n} = \frac{1}{n}\sum d(v)$



copies of $K_{1,t}$ with v on the left = $\binom{d(v)}{t}$

total copies of $K_{1,t} = \sum_{v \in V} \binom{d(v)}{t} \overset{\geq}{\cancel{\leq}} n \binom{\frac{1}{n}\sum d(v)}{t}$

Jenson for Binomial coefficient

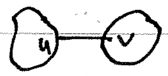
5 $|V(G)| = n \geq 2$

Assume false then degree sequence must be

$$\underbrace{\hat{0}, 1, 2, \ldots, \widehat{n-1}}_{⨳}$$

6 G connected all degree even except two say u, v odd degree $\Rightarrow \exists$ walk using all edges exactly once

consider uv, $\exists\, uv \notin E(G)$ add it

If $uv \in E(G)$  $\square$

(1)

Graph Theory - lecture 9 18/10/2013

<u>Forbidden Subgraphs</u>

Let H be a graph and $n \geq 1$

$$ex(n,H) = \max\{|E(G)| : G \text{ is H-free}, |V(G)| = n\}$$

G is H-free $\Leftrightarrow$ G has no subgraph isomorphic to H

eg G is K_3-free $\Leftrightarrow \nexists\, u,v,w \in V(G)$ st $uv, uw, vw \in E(G)$
distinct

$H = K_3 \quad \chi(H) = 3$

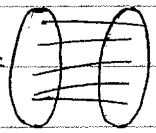
G bipartite ($\chi = 2$)
$\Downarrow$
G is K_3-free

Lemma 2.3 If G and H are graphs and $\chi(G) < \chi(H)$ then G is H-free

Proof If G contains a copy of H then $\chi(G) \geq \chi(H)$ $\divideontimes$

$\square$

Theorem 2.4 If $n \geq 2$ then $ex(n, K_3) = \lfloor \frac{n^2}{4} \rfloor$
(Mantel)

Proof Example: take G =  $= K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$

$\lceil \frac{n}{2} \rceil \quad \lfloor \frac{n}{2} \rfloor$

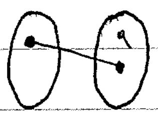
$|E(G)| = \lfloor \frac{n^2}{4} \rfloor$. So $ex(n, K_3) \geq \lfloor \frac{n^2}{4} \rfloor$. G is K_3 free.

remember to show this in exam

Now show $ex(n, K_3) \leq \lfloor \frac{n^2}{4} \rfloor$

Let G have n vertices, be K_3-free with e edges

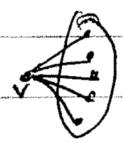
Let $A \subseteq V(G)$ be a largest independent set

G = 

A V/A

If $v \in V(G)$ then $\Gamma(v)$ is an independent set

So if $|A| = a$ then $d(v) \leq a$



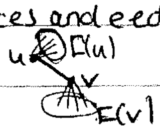
$$e \leq \sum_{v \in V \setminus A} d(v) \leq \underset{|V \setminus A|}{(n-a)}\underset{d(v) \leq a}{a} \leq \max_{0 \leq b \leq n} (n-b)b = \frac{n^2}{4}$$

because every edge has at least one vertex in $V \setminus A$

So $e \leq \frac{n^2}{4}$ $\therefore e \leq \lfloor \frac{n^2}{4} \rfloor$

2nd proof: $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ K_3-free $\Rightarrow ex(n, K_3) \geq \lfloor \frac{n^2}{4} \rfloor$

Let G K_3-free, n vertices and e edges

Let $uv \in E(G)$ 

then G is K_3-free
$\Downarrow$
$\Gamma(u) \cap \Gamma(v) = \emptyset$

otherwise we get triangle if overlap.

So $d(u)+d(v) \leq n-2+2=n$

$$\sum_{uv \in E(G)} (d(u)+d(v)) \leq e$$

Fix a vertex $x \in V(G)$. How many times does "$d(x)$" occur in the sum?

$$\sum_{x \in V(G)} d(x)^2 \leq en$$

$$\sum_{x \in V(G)} d(x) = 2e$$

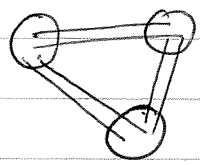
Cauchy-Schwarz: $\frac{1}{n}\left(\sum_{x \in V} d(x)\right)^2 \leq \sum_{x \in V} d(x)^2 \leq en$

$$\frac{4e^2}{n} \leq en \Rightarrow e \leq \frac{n^2}{4}$$

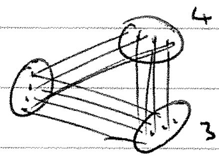
■

H: $ex(n,H)=?$ $\quad H=K_{r+1}$

Definition A graph G is r-partite iff $V(G)=V_1 \dot\cup V_2 \dot\cup \dots \dot\cup V_r$, each V_i is independent set

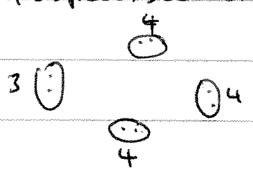
eg 

$H=K_4$, $n=11$



Definition Turán graph: $T_r(n)$ is the complete r-partite graph with n vertices and vertex classes as equal as possible in size.

$T_4(15)$

$T_4(15)$ 

$(3 \times 4) \times 3 + (4 \times 4) \times 3$

$= (9 \times 4) + (16 \times 3)$

$= 84 = t_4(15)$

Theorem (Turán) $ex(n, K_{r+1}) = |E(T_r(n))| = t_r(n)$

Lemma 2.5 Amongst all r-partite graphs of order n

$T_r(n)$ has the most edges

Moreover $t_r(n) = t_r(n-r) + (r-1)(n-1) + \binom{r}{2}$

Proof If G is r-partite order n with $|E(G)|$ maximal then G is complete r-partite

If $G \neq T_r(n)$ then $\exists$ V_1, V_2 vertex classes st $|V_1|=a$, $|V_2|=b$ $\quad a \geq b+2$

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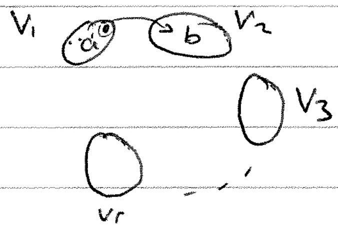
$a \geq b+2$



Remove a vertex from V_1 and insert it into V_2

Keep G complete r-partite

Change in $|E(G)| = -b + (a-1)$

≥ 1 ※

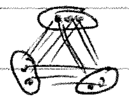
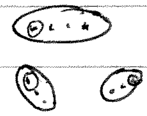
can connect to different vertices since it's moved set.

Lemma 2.5 Among all r-partite graphs on n vertices, $T_r(n)$ has the most edges, moreover $(n \geq r)$

r subscript.

(b) $|E(T_r(n))| = t_r(n) = t_r(n-r) + (r-1)(n-r) + \binom{r}{2}$

Proof (b) Take $G = T_r(n)$

eg $T_3(7)$  $T_3(10)$ 

like removing a clique from 3-partite graph

G contains a copy of $T_r(n-r)$ given by removing a vertex from each class.

Colour one vertex in each vertex class of $T_r(n)$ red, colour the rest black. Count edges according to colour of end vertices:-

$$\# \text{ edges red-red} = \binom{r}{2}$$

$$\# \text{ edges black-black} = t_r(n-r)$$

$$\# \text{ edges red-black} = (n-r)(r-1)$$

(since each black vertex is joined to every red vertex except the one in the same class) □

$T_r(n)$ is K_{r+1}-free. $\chi(T_r(n)) = r < \chi(K_{r+1})$

Theorem 2.6 (Turán) If $2 \leq r \leq n$ and G is K_{r+1}-free with $ex(n, K_{r+1})$ edges and n vertices then G is $T_r(n)$

Proof By induction on n

$n \leq r$ then $ex(n, K_{r+1}) = \binom{n}{2}$ and $T_r(n) = K_n$ eg taking 4 edges avoids K_5 so can take all edges.

So suppose $n \geq r+1$

Let G have n vertices and $ex(n, K_{r+1})$ edges and be K_{r+1}-free

if we had many edges as possible then if we don't have a K_4 we would definitely find K_3

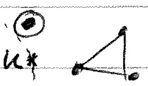
By maximality of $|E(G)|$, G contains a copy of K_r call this K.

$K = \{v_1, v_2, \ldots, v_r\}$. Consider $G-K$

$|V(G-K)| = n-r$, $G-K$ is K_{r+1}-free

Inductive hypothesis $\Rightarrow |E(G-K)| \leq t_r(n-r)$

fix one vertex in V

Also if ~~random~~ $v \in V(G-K)$ is joined to at most $(r-1)$ vertices in K 

(otherwise we have a K_{r+1})

can't join to all as we would get a clique i.e. K_4 (copy of)

from lemma (b) above.

$$|E(G)| \leq \underbrace{\binom{r}{2}}_{\substack{\# \text{edges} \\ \text{in } K}} + \underbrace{t_r(n-r)}_{\substack{\# \text{edges} \\ \text{in } G-K}} + \underbrace{(n-r)(r-1)}_{\substack{\# \text{edges } K \\ \text{to } G-K}} = t_r(n)$$

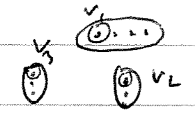
equality

{know $\underline{t_r(n)} \leq ex(n, K_{r+1}) = |E(G)| \leq \underline{t_r(n)}$

Since $|E(G)| = t_r(n)$

[So every $v \in V(G-K)$ is joined to exactly $r-1$ vertices in K. (using * statement)

Need to show G is r-partite.



v_i different coloured vertex

$$V_i = \{v \in V(G) : vv_i \notin E(G)\}$$

Since each $v \in V(G-K)$ is joined to $r-1$ vertices in $K = \{v_1, v_2, \dots, v_r\}$ for each $v \in V(G-K)$ there is a unique i st $vv_i \in E(G-K)$

$\therefore$ $V_1 \dot\cup V_2 \dot\cup \dots \dot\cup V_r = V(G)$ is a partition

Claim: each class V_i is independent

Proof: if $u, v \in V_i$, $uv \in E(G)$

then u, v are both adjacent to every vertex in $\{v_1, v_2, \dots, \hat{v}_i, \dots, v_r\}$

omitted

So $\{u, v, v_1, v_2, \dots, \hat{v}_i, \dots, v_r\}$ forms a copy of K_{r+1} ⋇

□

Hence G is r-partite, so by lemma 2.5 G is $T_r(n)$ □

Theorem 2.7 If G is a graph of order n and vertex degree $d_1, d_2, \dots, d_n$

then $\alpha(G) \geq \sum_{i=1}^{n} \frac{1}{d_i+1}$

Proof $V(G) = [n]$

Let S_n = permutations of $[n]$

Let $\pi \in S_n$ be chosen uniformly at random

A_i = "$\pi(j) > \pi(i)$ $\forall j \in \Gamma(i)$"

$\underbrace{\Gamma(i) \cup \{i\}}_{d_i+1 \text{ (size of set)}}$ considering if i comes before its neighbours.

i.e A_i holds iff vertex i comes before all of its neighbours in the ordering of $[n]$ given by π.

$$\Pr[A_i] = \frac{1}{d_i+1}$$

since in a random permutation the chance that

$\pi(i) < \pi(j)$ $\forall j \in \Gamma(i)$, is simply $\frac{1}{d_i+1}$

U is set of vertices which come before its neighbours

$U = \{i : A_i \text{ holds}\}$ is an independent set

Since if $x, y \in U$, $xy \in E(G)$ then $A_x \Rightarrow \pi(x) < \pi(y)$
$A_y \Rightarrow \pi(y) < \pi(x)$ ⋇

So $\mathbb{E}[|U|] = \sum_{i=1}^{n} \Pr[A_i] = \sum_{i=1}^{n} \frac{1}{d_i+1}$

$\therefore$ $\exists$ an independent set of size $\geq \sum_{i=1}^{n} \frac{1}{d_i+1}$

$\Rightarrow \alpha(G) \geq \sum_{i=1}^{n} \frac{1}{d_i+1}$

□

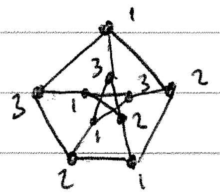
extra step: $U = \{i \in [n] : A_i \text{ holds}\}$ $X_i = \begin{cases} 1, & A_i \text{ holds} \\ 0, & \text{o/w} \end{cases}$

$|U| = \sum_{i=1}^{n} X_i \Rightarrow \mathbb{E}[|U|] = \mathbb{E} \sum_{i=1}^{n} X_i$
$= \sum_{i=1}^{n} \mathbb{E} X_i$
$= \sum_{i=1}^{n} \Pr[A_i]$

②

Problem Class Q7-11 22/10/2013

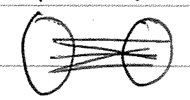
7)



$\Rightarrow \chi(G) \le 3$

$C_5 \subset G \Rightarrow \chi(G) \ge \chi(C_5) = 3$

8) X Y



$$\sum_{v \in X} d(v) = |E| = \sum_{v \in Y} d(v)$$

$$x|X| = y|Y|$$

9) $\chi(G) = k \Rightarrow |E(G)| \ge \binom{k}{2}$

Take a k-colouring

c(v) Let $V_i = \{v : c(v) = i\}$

If there is no edge from V_i to V_j for some $i \ne j$ $i,j \in [k]$

then can colour all vertices in V_j with colour i

$\Rightarrow \chi(G) \le k-1$ ※

All such edges are distinct $\therefore \ge \binom{k}{2}$ such edges.

10) G has e edges $\Rightarrow \exists$ k-partite subgraph with $\lceil \frac{(k-1)e}{k} \rceil$ edges.

<u>Proof</u> For $v \in G$ assign v i.u.a.r to $V_1, V_2, \dots$ or V_k independently uniformly and at random

Call an edge $uv \in E(G)$ "good" if it goes between two class, "bad" otherwise

Fix $uv \in E(G)$

$\Pr(uv \text{ is bad}) = \frac{1}{k}$

$\mathbb{E}[\#\text{ bad edges}] = \sum_{uv \in E} \Pr(uv \text{ is bad}) = \frac{e}{k}$

$\mathbb{E}[\#\text{good edges}] = \frac{e(k-1)}{k}$

$\therefore \exists$ a k-partition with $\ge \lceil \frac{e(k-1)}{k} \rceil$ good edges □

$n \ge 3$

11) $\delta(G) \ge \frac{n}{2} \Rightarrow G$ has Hamilton cycle

$n = 2k+1$

$\lceil \frac{n}{2} \rceil - 1$ $\lfloor \frac{n}{2} \rfloor + 1$ $a = k$ $\lceil \frac{n}{2} \rceil - 1, \lceil \frac{n}{2} \rceil + 1$

k k+1

k-1 k+1 $n = 2k$

(need to alternate when going from vertex to vertex (set to set)

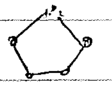
to have Hamilton cycle the set sizes have to be equal.

$ex(n,K_{r+1}) = t_r(n)$ ← # edges in $T_r(n)$

~~Then~~ Given H, what is $ex(n,H)$?

$ex(n,H) = \max\{|E(G)| : G \text{ is } H\text{-free}, |V(G)| = n\}$

$ex(n, C_5)$



odd cycle has χ, 3

to get graph without C_5, take bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ has $\lfloor \frac{n^2}{4} \rfloor$ edges.

no copy of C_5

$ex(n, C_5) \geq \lfloor \frac{n^2}{4} \rfloor$

If H has $\chi(H) = 3$ then $ex(n,H) \geq \lfloor \frac{n^2}{4} \rfloor = t_2(n)$

" — $\chi(H) = r$ " — $\geq t_{r-1}(n)$

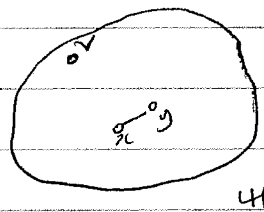
For a graph H, Turán density of H is

$$\pi(H) = \lim_{n \to \infty} \frac{ex(n,H)}{\binom{n}{2}} \qquad 0 \leq \frac{ex(n,H)}{\binom{n}{2}} \leq 1$$

Lemma 2.8 (a) For a graph $\pi(H)$ is well defined

(b) $\pi(K_{r+1}) = 1 - \frac{1}{r}$

Proof a) $\left\{ \frac{ex(n,H)}{\binom{n}{2}} \right\}_{n=1}^{\infty}$ is bounded



G

Let G be H-free with n vertices and $ex(n,H)$ - max edges without H

$$(n-2)ex(n,H) = \sum_{v \in V(G)} |E(G-v)| \leq \sum_{v \in V(G)} ex(n-1,H) = n\, ex(n-1,H)$$

(4): don't count x-y two times as $G-x$ wouldn't count x-y, $G-y$ " " y.

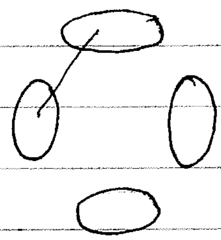
divide through.

$$\frac{ex(n,H)}{\binom{n}{2}} = \frac{ex(n,H)}{n(n-1)} \leq \frac{ex(n-1,H)}{(n-2)(n-1)} = \frac{ex(n-1,H)}{\binom{n-1}{2}}$$

⇒ sequence is monotone decreasing ∴ converges

b) $ex(n, K_{r+1}) = t_r(n)$

$$\frac{\left(\frac{n-r}{r}\right)^2 \binom{r}{2}}{\binom{n}{2}} \leq \frac{t_r(n)}{\binom{n}{2}} \leq \frac{\left(\frac{n+r}{r}\right)^2 \binom{r}{2}}{\binom{n}{2}}$$



$T_r(n)$ r classes.

$$\left(\frac{r-1}{r}\right)\frac{(n-r)^2}{n(n-1)} \leq \frac{t_r(n)}{\binom{n}{2}} \leq \frac{(r-1)}{r}\frac{(n+r)^2}{n(n-1)}$$

$\frac{r-1}{r} \to 1-\frac{1}{r}$, $\frac{(n-r)^2}{n(n-1)} \to 1$, $\frac{r-1}{r} \to 1-\frac{1}{r}$, $\frac{(n+r)^2}{n(n-1)} \to 1$

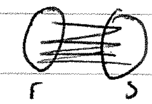
so $\pi(K_{r+1}) = 1-\frac{1}{r}$

Theorem 2.11 (next week lecture) If $\chi(H)=r$ then $\pi(H) = 1 - \frac{1}{r-1}$

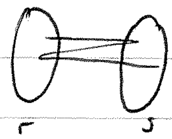
Theorem 2.9 If $r,s \geq 2$, n is large

$$ex(n,K_{r,s}) \leq \frac{1}{2}(r-1)^{\frac{1}{s}} n^{2-\frac{1}{s}} + \frac{1}{2}(s-1)n$$

$K_{r,s}$



B



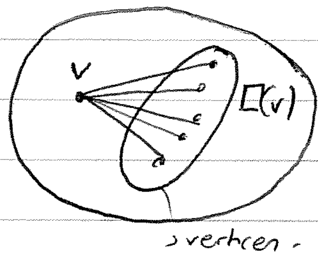
$0 \leq \pi(B) \leq \pi(K_{r,s}) = 0$
$\quad\;\; \overset{\shortparallel}{0}$

Proof Let G be $K_{r,s}$-free with n vertices $\underbrace{ex(n,K_{r,s})}_{e}$ edges

Say a vertex $v \in V(G)$ covers an s-set $\{v_1, v_2, \ldots, v_s\}$

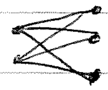
if $vv_i \in E(G)$ for $1 \leq i \leq s$

$\sum_{v \in V(G)} \binom{d(v)}{s}$, counts pairs (v,S) where v covers S



Since no s-set is covered by more than $r-1$ vertices (since G is $K_{r,s}$-free)

eg



$s=3$
$r=2$

$$n\binom{\sum \frac{d(v)}{n}}{s} \leq \sum_{\forall v \in V(G)} \binom{d(v)}{s} \leq (r-1)\binom{n}{s}$$

Jensens (convexity of binomial coefficients)

$$n\binom{2e/n}{s} \leq (r-1)\binom{n}{s}$$

$e \leq \frac{n^2}{2}$ so we can say $e = n^{2-\alpha}$

$$n\binom{2n^{1-\alpha}}{s} \leq (r-1)\binom{n}{s} \leq (r-1)\frac{n^s}{s!}$$

$$\frac{n(2n^{1-\alpha}-s+1)^s}{s!} \leq$$

$$(2n^{1-\alpha}-s+1) \leq (r-1)^{\frac{1}{s}} n^{1-\frac{1}{s}}$$

$$2n^{1-\alpha} \leq (r-1)^{\frac{1}{s}} n^{1-\frac{1}{s}} + s - 1$$

$$e = n^{2-\alpha} \leq \frac{1}{2}(r-1)^{\frac{1}{s}} n^{2-\frac{1}{s}} + \frac{1}{2}(s-1)n$$

(2)

Corollary 2.10 $X \subset \mathbb{R}^2$, $|X| = n$, then at most 25/10/2013

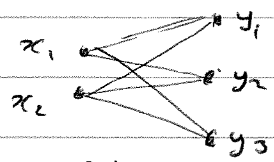
$\frac{n^{3/2}}{\sqrt{2}} + \frac{n}{2}$ pairs of points are at unit distance

Proof Take $X \subset \mathbb{R}^2$, $|X| = n$

Form a graph G with $V(G) = X$

$E(G)$ = pairs from X at unit distance.

G is H-free $\Rightarrow E(G) \leq ex(n, H)$



all distances are 1

take $H = K_{2,3}$

should be $K_{3,2}$ but doesn't make diff

i.e. if $K_{2,3} \subseteq G$

circles of radius 1 at x_1, x_2 should have y_1, y_2, y_3 but circles can't meet at 3 points

we have circles centre x_1, x_2 that meet in 3 points $\#$

$\therefore |E(G)| \leq ex(n, K_{3,2})$

$r = 3, s = 2$ in theorem $\Rightarrow$ result

$\square$

theorem 2.11 (Erdős-Stone) If $\chi(H)=r\geq 2$ then $\pi(H)=1-\frac{1}{r-1}$

we know
$$\pi(H)=\lim_{n\to\infty}\frac{ex(n,H)}{\binom{n}{2}}$$

$\geq$

$\pi(H)\geq 1-\frac{1}{r-1}$ Because $\chi(H)=r \Rightarrow H\not\subseteq T_{r-1}(n)$

$\therefore\ ex(n,H)\geq t_{r-1}(n)$

$\Rightarrow \pi(H)\geq \pi(K_r)=1-\frac{1}{r-1}$

Need to show $\pi(H)\leq 1-\frac{1}{r-1}$

Note if $\chi(H)=r$, take an r-colouring of H and suppose no colour is used more than t times, then

$H\subseteq K_r(t)=T_r(rt)$

$K_r(t)$ = the complete r-partite graph with vertex classes all of size t.

So it is enough to show $\pi(K_r(t))\leq 1-\frac{1}{r-1}$

need to prove this lemma before we finish proof.

Lemma 2.12: let $0<c,\varepsilon<1$, $n>\frac{2}{\varepsilon}\left(1+\frac{1}{c}\right)$

If G is a graph of order n with at least $(c+\varepsilon)\binom{n}{2}$, then G has a subgraph G' of order $n'>\varepsilon^{\frac{1}{2}}n$ with $\delta(G')\geq cn'$

Proof. We find G' as follows. let $G_n=G$

If $\delta(G_n)\geq cn$ then set $G'=G$, otherwise delete a vertex of min degree to give G_{n-1}

If $\delta(G_{n-1})\geq c(n-1)$ then set $G'=G_{n-1}$, otherwise repeat: gives sequence of graphs $G_n, G_{n-1},\dots,G_k$, where G_k has k vertices and G_{k-1} is obtained from G_k by deleting a vertex of min degree

We claim process terminates at some $k\geq \varepsilon^{\frac{1}{2}}n$

Lets set $s=\lceil \varepsilon^{\frac{1}{2}}n\rceil$

Note: $\sum_{k=s+1}^{n}k=\binom{n+1}{2}-\binom{s+1}{2}$

$|E(G_s)|\geq |E(G)|-c(n+(n-1)+\dots+(s+1))$

$|E(G_s)|>(c+\varepsilon)\binom{n}{2}-c\binom{n+1}{2}+c\binom{s+1}{2}$

$$>\varepsilon\binom{n}{2}-cn+c\binom{s+1}{2}$$

$r \geq 2$ for these theorems to make sense.

by stating assumption

$s = \lceil \varepsilon^{\frac{1}{2}} n \rceil \qquad \binom{s+1}{2} > \frac{s^2}{2} \geq \frac{\varepsilon n^2}{2} > \left(1+\frac{1}{c}\right)n$

$|E(G_s)| > \varepsilon \binom{n}{2} + n$

But $|E(G_s)| \leq \binom{s}{2} \leq \frac{(\sqrt{\varepsilon}n)(\sqrt{\varepsilon}n+1)}{2} = \frac{\varepsilon n^2}{2} + \frac{\sqrt{\varepsilon}n}{2} \leq \varepsilon\binom{n}{2} + \frac{n}{2}(\sqrt{\varepsilon}-\varepsilon) < \varepsilon\binom{n}{2} + n$ ⋕ ▢

Theorem 2.13 (Erdős - Stone min deg version)

let $r \geq 2$, $t \geq 1$, $0 < \varepsilon < \frac{1}{r}$

There is $n_0(r,t,\varepsilon)$ s.t. if G has $n \geq n_0$ ~~then contains $K_r(t)$~~ vertices and $\delta(G) \geq \left(1 - \frac{1}{r-1} + \varepsilon\right)n$ then G contains $K_r(t)$

Proof of Thm 2.11

know $\pi(H) \geq 1 - \frac{1}{r-1}$

Need to show $\pi(K_r(t)) \leq 1 - \frac{1}{r-1}$ (*)

If (*) fails to hold then $\exists\, \varepsilon > 0$ s.t

$k_r(t) \longrightarrow$

$\pi(K_r(t)) = \lim_{n\to\infty} \left(\frac{ex(n,K_r)}{\binom{n}{2}}\right)$

$$\pi(K_r(t)) > 1 - \frac{1}{r-1} + 2\varepsilon$$

Since $\left\{\frac{ex(n,K_r(t))}{\binom{n}{2}}\right\}_{n=1}^{\infty}$ is mono decreasing

we have $ex(n, K_r(t)) > \left(1 - \frac{1}{r+1} + 2\varepsilon\right)\binom{n}{2} \quad \forall n$

Let $n \geq \frac{n_0(r,t,\varepsilon)}{\varepsilon^{\frac{1}{2}}}$ (n_0 given by Thm 2.13)

and let G be $K_r(t)$-free with n vertices and at least $\left(1 - \frac{1}{r-1} + 2\varepsilon\right)\binom{n}{2}$ edges.

lemma 2.12 $\Rightarrow \exists$ a subgraph G' of G with $n' \geq \varepsilon^{\frac{1}{2}} n$ vertices and $\delta(G') \geq \left(1 - \frac{1}{r-1} + \varepsilon\right)n'$

(with $\varepsilon = \varepsilon \quad c = 1 - \frac{1}{r-1} + \varepsilon$)

Since $n' \geq \varepsilon^{\frac{1}{2}} n \geq n_0(r,t,\varepsilon)$

Thm 2.13 $\Rightarrow$ there is a copy of $K_r(t)$ in G' ⋕

Proof of Thm 2.13

(Induction on r)

from last lecture (KST) Thm 2.9

$r=2$: $K_2(t) = K_{t,t}$, $ex(n, K_{t,t}) \leq \frac{1}{2}(t-1)^{\frac{1}{t}} n^{2-\frac{1}{t}} + \frac{1}{2}(t-1)n$
$< t n^{2-\frac{1}{t}}$

Given $\varepsilon > 0$ and $t \geq 1$

let $n_0 = \left\lceil \left(\frac{2t}{\varepsilon}\right)^t \right\rceil$, then for $n \geq n_0$ we have $\varepsilon > \frac{2t}{n^{1/t}}$

If G has $n \geq n_0$ vertices, and $\delta(G) \geq \varepsilon n$ then $|E(G)| \geq \frac{\varepsilon n^2}{2} > t n^{2-1/t} \geq ex(n, K_{t,t})$ $\therefore K_2(t) \subseteq G$

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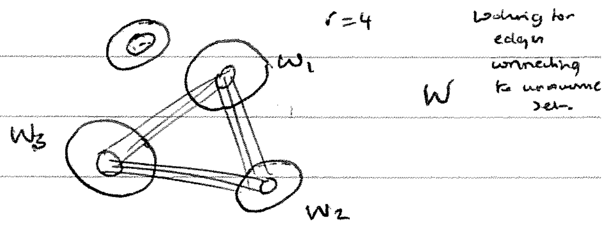
Let $r \geq 3, t \geq 1$, $0 < \varepsilon < \frac{1}{r}$ be given

Let G has n vertices $\delta(G) \geq \left(1 - \frac{1}{r-1} + \varepsilon\right)n$

Need to show if n is sufficiently large then $K_r(t) \subset G$

Let $w = \left\lceil \frac{2t}{\varepsilon} \right\rceil$, $n \geq n_0(r-1, w, \varepsilon)$

Since $1 - \frac{1}{r-1} + \varepsilon > 1 - \frac{1}{r-2} + \varepsilon$



our ind hypothesis $\Rightarrow K_{r-1}(w) \subseteq G$

Call the vertex set of $K_{r-1}(w)$, W

$$W = W_1 \dot\cup \dots \dot\cup W_{r-1}$$

each $|W_i| = w$, so $|W| = (r-1)w$

* $S = \{v \in V \setminus W : v \text{ has} \geq (r-2)w + t \text{ neighbours in } W\}$

Want $|S|$ to be big

Claim: $|S| \to \infty$

So in particular for n sufficiently large. $|S| > (t-1)\binom{w}{t}^{r-1}$

Proof of thm assuming claim: Let H be a copy of $K_{r-1}(t)$ in W

we say $v \in S$ is "good" for H if v is adjacent to every vertex in H.

Assume G is $K_r(t)$-free.

Then each copy of $K_{r-1}(t)$ has at most $(t-1)$ good vertices in S

$$\# \text{copies of } K_{r-1}(t) \text{ in } W = \binom{w}{t}^{r-1}$$

Every vertex in S is good for at least one copy of $K_{r-1}(t)$ in W

(Since each $v \in S$ has $\geq t$ neighbours in each W_i)

Hence $|S| \leq (t-1)\binom{w}{t}^{r-1}$ $\#$ by Claim

Proof of the claim: Count edges from W to $V \setminus W$

Denote this $e(W, V \setminus W)$

$$e(W, V \setminus W) = \sum_{v \in W} d(v) - 2e(W)$$

$$\delta(G) \geq \left(1 - \frac{1}{r-1} + \varepsilon\right)n, \text{ and } e(W) \leq \binom{|W|}{2} < \frac{|W|^2}{2}$$

$$e(W, V \setminus W) \geq |W| n \left(1 - \frac{1}{r-1} + \varepsilon\right) - |W|^2$$

If $v \in (V \setminus W) \setminus S$ then v has $< (r-2)w + t$ neighbours in W

If $v \in S$ then v has $\leq |W|$ neighbours in W

* So $e(V \setminus W, W) < ((r-2)w+t)(n-|W|-|S|) + |W||S|$

$$= n((r-2)w+t) - |W|^2 + (w-t)|W| + |S|(w-t)$$

(using the fact $|W| = (r-1)w = (r-2)w + t + (w-t)$)

$$|W|n\left(1 - \frac{1}{r-1} + \varepsilon\right) - \cancel{|W|^2} < n((r-2)w+t) - \cancel{|W|^2} + (w-t)|W| + |S|(w-t)$$

show working out

$$\hookrightarrow |S| > n\left(\frac{\varepsilon(r-1)w - t}{w-t}\right) - (r-1)w$$

using $w = \left\lceil \frac{2t}{\varepsilon} \right\rceil$

$w \geq \frac{2t}{\varepsilon}$ $\quad \varepsilon(r-1)w \geq 2t(r-1)$

$$\geq 4t$$

So $|S| > \frac{3tn}{w-t} - (r-1)w \to \infty$ as $n \to \infty$

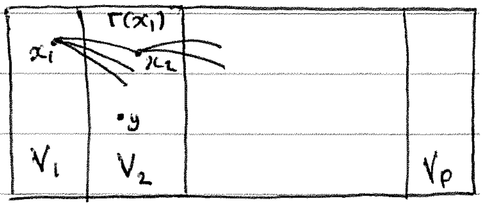
$|S| > (t-1)\left(\frac{w}{t}\right)^{r-1}$ for n suff large ▨

①

MATH3503 Graph Theory – lecture 15 01/11/2013

Thm 2.14 (Füredi 2010) If G is K_{r+1}-free, order n, with at least $ex(n, K_{r+1}) - t$ edges for some $t \geq 0$ then $\exists H \subseteq G$ s.t. $\chi(H) \leq r$ and $|E(H)| \geq |E(G)| - t$

← proof using the diagram.



Proof: Let $x_1 \in V = V(G)$ have maximum degree ✓

Let $V_1 = V \setminus \Gamma(x_1)$

[ ] – induced subgraph.

Next find a vertex $x_2 \in V \setminus V_1$ that has max degree in $G[V \setminus V_1]$

$V_2 = V \setminus (V_1 \cup \Gamma(x_2))$

have two type of edges in V_2: $|+|$ $|-|$

Continue in this way finding $x_{i+1} \in V \setminus (V_1 \cup \dots \cup V_i)$ of max degree in $G[V \setminus (V_1 \cup \dots \cup V_i)]$

set $V_{i+1} = V \setminus (V_1 \cup \dots \cup V_i \cup \Gamma(x_{i+1}))$

Suppose this process finishes with $V_1, V_2, \dots, V_p$ and $x_1, \dots, x_p$

By construction we know $\{x_1, \dots, x_p\}$ forms a copy of K_p

Hence, since G is K_{r+1}-free, $p \leq r$.

Let $K(V_1, \dots, V_p)$ be the complete biparhite graph with vertex classes, $V_1, \dots, V_p$.

Let $G_i = G\{V_i \cup V_{i+1} \cup \dots \cup V_p\}$ $d_1 = |V_2| + \dots + |V_p|$

$d_{G_i}(x_i) = d_i$ → So if $d_{G_i}(x_i) = d_i$, then if $y \in V_i$, $d_{G_i}(y) \leq d_i$ $d_2 = |V_3| + \dots + |V_p|$

$$\sum_{i=1}^{p} \sum_{y \in V_i} d_{G_i}(y) \leq \sum_{i=1}^{p} d_i |V_i| = |E(K(V_1, \dots, V_p))| \leq t_p(n) \leq t_r(n)$$

Let k = # edges in G that ~~the~~ lie inside a single V_i for ~~some~~ some i

$$|E(G)| + k = \sum_{i=1}^{p} \sum_{y \in V_i} d_{G_i}(y) \leq t_r(n) = ex(n, K_{r+1})$$

Now let $H \subseteq G$ be obtained by deleting the k "internal" edges

Since $|E(G)| \geq ex(n, K_{r+1}) - t$

and $|E(G)| \leq ex(n, K_{r+1}) - k$

so $k \leq t$ and $|E(H)| \geq |E(G)| - t$

Note: H is p-partite so $\chi(H) = p \leq r$ □

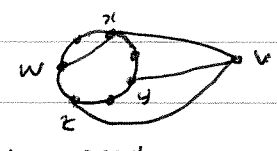
Theorem 2.15 (Andrásfai – Erdős – Sós) If G is K_3-free and $\delta(G) > \frac{2n}{5}$, then G is bipartite

ex  $n/5$ consider blob of edges between vertices then $\deg = \frac{2n}{5}$ shows theorem is as strong as it could be

Proof: Suppose G is K_3-free, n vertices, $\delta(G) > \frac{2n}{5}$, not bipartite

Let C be a shortest odd cycle in G with n vertices $\{v_1, v_2, \dots, v_{2k+1}\}$, $k \geq 2$ (K_3-free)

Claim: $v \in V(G)$ has at most 2 neighbours in C



Proof: Let $v \in V(G) \setminus C$ have 3 neighbours x, y, z in C

Since C has odd length wlog x, y are odd distance apart

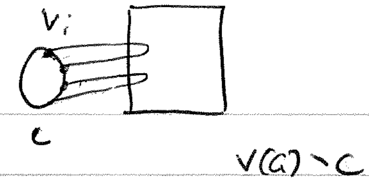
∴ form a shorter odd cycle using v

Same is true for $w \in C$

Let E^* be the edges of G from C to $V(G) \setminus C$

By the claim $|E^*| \leq (n-2k-1)\cdot 2$

shown this before. $\longleftarrow$ $\leq 2n-10 \quad (k \geq 2)$



$$2n-10 \geq |E^*| = \sum_{i=1}^{2k+1} (d(v_i)-2) > \left(\frac{2n}{5}-2\right)(2k+1) \geq 2n-10 \quad \# \quad \square$$

①

MATH3503 Graph theory - lectures 16+17 12/11/2013

We will be working with

$[n]=\{1,\dots,n\}$, $|[n]|=n$

Power set

either in or out. $P([n])=\{A: A\subseteq[n]\}$, $|P([n])|=2^n$ 2 sets of same size neither can be subset of other

$0\le k\le n$ $\binom{[n]}{k}=\{A: A\subseteq[n], |A|=k\}$

$\mathcal{A}\subseteq P([n])$ $\mathcal{A}=\{123, 45, 37\}$

(same as)

$\mathcal{A}=\{\{1,2,3\},\{4,5\},\{3,7\}\}$

3·1 Chains and Antichains

$\mathcal{A}\subseteq P[n]$ is a chain if and only if

$\forall A,B\in\mathcal{A}$ $A\subseteq B$ or $B\subseteq A$

For example

$\mathcal{A}=\{\phi,1,13,13457\}$

$\mathcal{A}\subseteq P([n])$ is an antichain, $\forall A,B\in\mathcal{A}$ $A\subseteq B \Rightarrow A=B$

For example

$\mathcal{A}=\{123,145,245,136\}$

Lemma 3·1 If $\mathcal{A}$ is antichain and $\mathcal{C}$ is a chain then

$|\mathcal{A}\cap\mathcal{C}|\le 1$

Proof If $A,B\in\mathcal{A}\cap\mathcal{C}$ then $A,B\in\mathcal{C}$

so w.l.o.g $A\subseteq B$, then $A,B\in\mathcal{A}\Rightarrow A=B$

Hence $|\mathcal{A}\cap\mathcal{C}|\le 1$

Question. If $\mathcal{C}\subseteq P([n])$, how large can $|\mathcal{C}|$ be?

Proposition 3·2 If $\mathcal{C}\subseteq P([n])$ is a chain then $|\mathcal{C}|\le n+1$

Proof $P([n])=\binom{[n]}{0}\dot\cup\binom{[n]}{1}\dot\cup\binom{[n]}{2}\dot\cup\dots\dot\cup\binom{[n]}{n}$

can be partitioned into $n+1$ antichains

Hence by lemma 3·1, $\mathcal{C}$ can contain at most one set from each antichain

Therefore $|\mathcal{C}| \leq n+1$

Theorem 3.3 (Sperner) If $\mathcal{A} \subset P([n])$ is an antichain then

$$|\mathcal{A}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} = \binom{n}{\lceil \frac{n}{2} \rceil}$$

For example we can achieve this bound with

$$\mathcal{A} = \binom{[n]}{\lfloor \frac{n}{2} \rfloor} \quad \text{or} \quad \mathcal{A} = \binom{[n]}{\lceil \frac{n}{2} \rceil}$$

Consider

$P([3]) = \{\emptyset, 1, 2, 3, 12, 13, 23, 123\}$

Partition into three chains

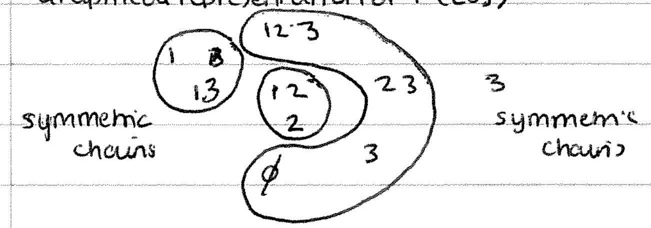
$\{\emptyset, 1, 12, 123\}, \{2, 23\}, \{3, 13\}$ we would like to do this process for any n.

A chain $\mathcal{C} \subseteq P([n])$ is symmetric

(1) $\mathcal{C} = \{C_1, C_2, \dots, C_k\}$ $\quad |C_{i+1}| = |C_i| + 1 \quad$ for $i = 1, 2, \dots, k-1$

(2) $|C_1| + |C_k| = n$

Graphical representation of $P([3])$



Lemma 3.4 $P([n])$ can be ~~positive~~ partitioned into symmetric chains

Proof (Sperner) If $\mathcal{C} = \{C_1, C_2, \dots, C_k\}$ is a symmetric chain then

$$|C_1| \leq \left\lfloor \frac{n}{2} \right\rfloor, \quad |C_k| \geq \left\lfloor \frac{n}{2} \right\rfloor \qquad \text{(from condition 2 of defn)}$$

Hence $\exists\ 1 \leq i \leq k$ such that $|C_i| = \lfloor \frac{n}{2} \rfloor$

Let $P([n]) = \mathcal{C}_1 \dot\cup \mathcal{C}_2 \dot\cup \dots \mathcal{C}_t$ be a partition into symmetric chains

Each symmetric chains contains exactly one set from $\binom{[n]}{\lfloor \frac{n}{2} \rfloor}$ and conversely

each $A \in \binom{[n]}{\lfloor \frac{n}{2} \rfloor}$ belongs to exactly one of the symmetric chains

Hence $t = \binom{n}{\lfloor \frac{n}{2} \rfloor}$

So if $\mathcal{A} \subseteq P([n])$ is an antichain

$$|\mathcal{A}| = |\mathcal{A} \cap P([n])| = \sum_{i=1}^{t} |\mathcal{A} \cap \mathcal{C}_i| \leq t = \binom{n}{\lfloor \frac{n}{2} \rfloor} \quad \blacksquare$$

(split into symmetric chains, we know it must pass through the middle layer chain meets antichain once }, There is a one to one correspondence between which meets which)

Proof (lemma 3.4) By induction on n

For $n=1$, $P([1]) = \{\phi, 1\}$ is a symmetric chain

Let $n \geq 2$ and suppose $P([n-1])$ is partitioned into symmetric chains $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_t$

(if $|\mathcal{C}_i| \geq 2$) Let $\mathcal{C}_i = \{C_1^i, C_2^i, \ldots, C_{k_i}^i\}$ then $|C_1^i| + |C_{k_i}^i| = n-1$

Recall

$P([n]) = P([n-1]) \cup \{A \cup [n] : A \in P([n-1])\}$ (Partition into set that contains n, and doesn't contain n)

[n] might be {n} Check online notes.

Let $\mathcal{C}_i' = \{C_1^i \cup [n], C_2^i \cup [n], \ldots, C_{k_i-1}^i \cup [n]\}$

Note $|C_1^i \cup [n]| + |C_{k_i-1}^i \cup [n]| = |C_1^i| \underbrace{+1}_{\text{added } n} + \underbrace{|C_{k_i}^i| - 1}_{\text{stopped one before the last.}} \underbrace{+1}_{\text{added } n}$

Therefore $\mathcal{C}_i'$ is a symmetric chain

Let $\mathcal{C}_i'' = \{C_1^i, C_2^i, \ldots, C_{k_i}^i, C_{k_i}^i \cup \{n\}\}$

$$|C_1^i| + |C_{k_i}^i \cup \{n\}| = |C_1^i| + |C_{k_i}^i| + 1$$
$$= n-1+1 = n$$

Therefore $\mathcal{C}_i''$ is a symmetric chain

Since $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_t$ form a partition of $P([n-1])$ so $\mathcal{C}_1', \mathcal{C}_2', \ldots, \mathcal{C}_t', \mathcal{C}_1'', \ldots, \mathcal{C}_t''$ form a partition of $P([n])$ into symmetric chains

Example from $n=1$ to $n=2$

$P([1]) = \{\phi, 1\} = \mathcal{C}_1 \qquad \{2\} = \mathcal{C}_1'$

$\{\phi, 1, 12\} = \mathcal{C}_1''$

Theorem 3.5 (LYM-inequality) If $\mathcal{A} \subseteq P([n])$ is an antichain then

$$\sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq 1$$

If $a_k = |\mathcal{A} \cap \binom{[n]}{k}|$

Then $\displaystyle\sum_{k=0}^{n} \frac{a_k}{\binom{n}{k}} \leq 1$

Recall $\binom{n}{k} \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$

$$\Rightarrow \sum_{k=0}^{n} \frac{a_k}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} \le \sum_{k=0}^{n} \frac{a_k}{\binom{n}{k}} \le 1$$

$$\Rightarrow |\mathcal{A}| = \sum_{k=0}^{n} a_k \le \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

(So LYM-inequality proves Sperner in exam, could prove LYM then deduce Sperner)

Proof Let S_n be the permutation on $[n]$

Let $\mathcal{A} \subseteq P([n])$ be an antichain

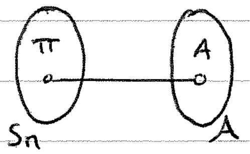
Form a bipartite graph $G = (S_n, \mathcal{A}; E)$

There is an edge in E from $A \in \mathcal{A}$ to $\pi \in S_n$ if and only if the elements of A all appear before the elements of A^c in π.

Example $A = 125$ $n = 6$

$\pi_1 = \boxed{512}346$ $\pi_1 A \in E$

$\pi_2 = \boxed{\boxed{126}5}34$ $\pi_2 A \notin E$



Now (double counting argument)

$$\sum_{\pi \in S_n} d(\pi) = |E| = \sum_{A \in \mathcal{A}} d(A)$$

How large can $d(\pi)$ be?

Can $d(\pi) \ge 2$?

If $d(\pi) \ge 2$, let A, B be neighbours of π, $A \ne B$

Then without loss of generality $|A| \le |B|$, then the first $|A|$ elements of π form the set A and this is a subset of B which forms the first $|B|$ elements of π.

Hence $A \subset B$. This is a contradiction because antichain

Hence $d(\pi) \le 1$ therefore $|E| \le n!$

Fix $A \in \mathcal{A}$, $|A| = k$

$$\underbrace{x\ x\ x\ x\ x}_{\substack{k \\ A}} \qquad \underbrace{x\ x\ x\ x\ x\ x}_{\substack{n-k \\ A^c}}$$

k is fixed but we choose where elements go $k!$ choices same with $n-k$, $(n-k)!$ choices.

$d(A) = k!(n-k)! = |A|!(n-A)!$

Hence $n! \ge |E| = \sum_{A \in \mathcal{A}} |A|!(n-|A|)!$

$$\Rightarrow \sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq 1$$

PROBLEM CLASS SHEET 3

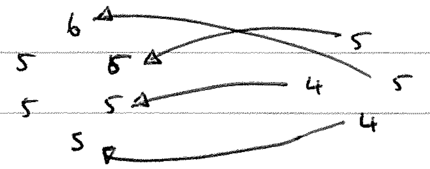
13) a) $T_6(32)$

6
5 6
5 5 5

3 cases 5-5 : 6x25

5-6 : 8x30

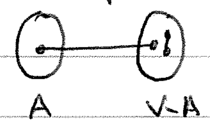
6-6 : 1x36

b) 

$T_4(18)$

Isomorphic as each ~~set~~ class of $T_6(32)$ is larger than $T_4(18)$

14) Take an independent set of max size A



$$\sum_{v \in V \setminus A} d(v) = |E| + \text{internal}$$

$$|E| + \text{internal} = \sum_{v \in V \setminus A} d(v) \leq a(n-a) \leq \left\lfloor \frac{n^2}{4} \right\rfloor$$

$\downarrow$

$\left\lfloor \frac{n^2}{4} \right\rfloor - t + \text{internal}$ with $|A| = a$

$\Rightarrow$ internal $\leq t$

$$|E| = \left\lfloor \frac{n^2}{4} \right\rfloor - t$$

Therefore $G[A, V \setminus A]$ is the graph G induced by A and V\A is a bipartite subgraph with

$\geq \left\lfloor \frac{n^2}{4} \right\rfloor - 2t$ edges

Induced bipartite induced by bipartition A and V\A

15) $$\pi(H) \leq 1 - \frac{1}{\Delta(H)}$$

$$\chi(H) \leq \Delta(H) + 1$$

$$\pi(H) = 1 - \frac{1}{\chi(H)-1} \qquad (E\text{-}S)$$

16) $\pi(K_{1,t}) = 0$

If G is $K_{1,t}$ free then $\Delta(G) \leq t-1$

$$2|E| = \sum_{v \in G} d(v) \leq 2(t-1)n$$

$$\Rightarrow \frac{|E|}{\binom{n}{2}} \leq \frac{(t-1)n}{\binom{n}{2}} \longrightarrow 0$$

3.2 Intersecting Families

$\mathcal{A}$ is intersecting iff $A, B \in \mathcal{A} \Rightarrow A \cap B = \phi$

eg $\mathcal{B} = \{12, 13, 23\}$ $\quad \bigcap_{B \in \mathcal{B}} B = \phi$

Theorem 3.6 If $\mathcal{A} \subseteq P([n])$ is intersecting then $|\mathcal{A}| \le 2^{n-1}$

Proof If $\mathcal{A} \subseteq P([n])$ is intersecting and $A \in \mathcal{A}$ then $A^c \notin \mathcal{A}$ $\square$

Fix an integer, eg 1

e.g. $\mathcal{A}^* = \{A \subseteq [n] : 1 \in A\}$

n odd $\mathcal{B} = \{B \subseteq [n] : |B| > \frac{n}{2}\}$ $\quad \binom{n}{k} = \binom{n}{n-k}$

$\mathcal{C} = \{C \subseteq [n] : |C \cap [3]| \geq 2\}$ $\quad \{12, 13, 23, 123\}$

sets in C st $C \cap [3] = \{12\}$ is 2^{n-3} $\quad 2^{n-3} + 2^{n-3} + 2^{n-3} + 2^{n-3} = 2^{n-1}$

$= \{13\}$ " "

$\mathcal{C}_t = \{C \subseteq [n] : |C \cap [2t-1]| \geq t\}$

$\frac{2^{2t-1}}{2} 2^{n-(2t-1)} = 2^{n-1}$

If $\mathcal{A} \subseteq \binom{[n]}{k}$ is intersecting then: $n < 2k \Rightarrow$ all such families are intersecting

so $|\mathcal{A}| \le \binom{n}{k}$

$n = 2k \Rightarrow |\mathcal{A}| \le \frac{1}{2}\binom{n}{k} = \binom{n-1}{k-1}$ if $n = 2k$.

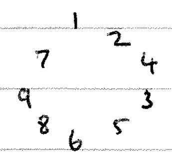
Theorem 3.7. If $\mathcal{A} \subseteq \binom{[n]}{k}$ is intersecting and $n \geq 2k$ then $|\mathcal{A}| \le \binom{n-1}{k-1} = |\{A \in \binom{[n]}{k} : 1 \in A\}|$

Proof Let $\mathcal{C}_n$ the set of all cyclic permutations of $[n]$.

i.e. $\mathcal{C}_n$ consists of permutations of $[n]$ where two permutations are considered equal if when written around a circle we can obtain one from the other by rotation.

$\pi = 124356897$ $\quad \tau = 568971243 5$ $\quad$ eg. interval: 234, 568

$n = 9$



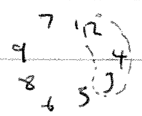
$\pi = \tau$ in $\mathcal{C}_n$ $\quad$ not interval: 389

$|\mathcal{C}_n| = (n-1)!$

Given $\pi \in \mathcal{C}_n$ and $A \subseteq [n]$, we say A is an interval in π if the elements of A appear consecutively in π

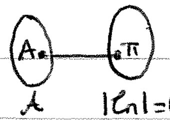
eg $\quad$ 1 24 fixed

can only have
217 or 435
243 or 127



243
127 or 345
124 or 356

$$\Rightarrow \frac{k}{n}\binom{n}{k} = \binom{n-1}{k-1}$$



$|\mathcal{C}_n| = (n-1)!$

Define a bipartite graph with $\mathcal{A} \subseteq \binom{[n]}{k}$ our intersecting family, as one vertex class and $\mathcal{C}_n$ as the other. Insert an edge from $A \in \mathcal{A}$ to $\pi \in \mathcal{C}_n$ iff A is an interval in π.

$$\sum_{A \in \mathcal{A}} d(A) = |E| = \sum_{\pi \in \mathcal{C}_n} d(\pi)$$

Lemma 3.8 If $\pi \in \mathcal{C}_n$, $n \geq 2k$ and $\mathcal{I} \subseteq \binom{[n]}{k}$ is an intersecting family of intervals in π then $|\mathcal{I}| \leq k$

Lemma 3.8 $\Rightarrow \forall \pi \in \mathcal{C}_n$, $d(\pi) \leq k$. So $|E| \leq k.(n-1)!$

If $A \in \mathcal{A}$, what is $d(A)$?

k! ways of ordering the k element any order

(n-k) other element to rearrange

$$d(A) = k!(n-k)!$$

$$\therefore |E| = |\mathcal{A}|k!(n-k)! \leq k((n-1)!)$$

$$\Rightarrow |\mathcal{A}| \leq \binom{n-1}{k-1}$$

$\square$

Lemma 3.8 If $\pi \in \mathcal{C}_n$, $n \geq 2k$ and $\mathcal{I} \subseteq \binom{[n]}{k}$ is an intersecting family of intervals in π then $|\mathcal{I}| \leq k$

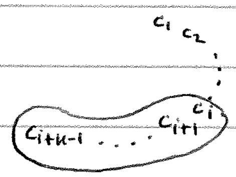
Proof Let $\pi = c_1 c_2 \dots c_n$. Let $I \in \mathcal{I}$

So $I = \{c_i, c_{i+1}, \dots, c_{i+k-1}\}$

Define $I+j = \{c_{i+j}, c_{i+j+1}, \dots, c_{i+j+k-1}\}$ where all subscripts are modulo n

I is disjoint from all but $2k-2$ other intervals from π

Namely: $I+j$, $-(k-1) \leq j \leq (k-1)$ $j \neq 0$



Interval

But $I+j$ and $I+j-k$, $j = 1, 2, \dots, k-1$ are pairwise disjoint

Therefore have at most one interval in $\mathcal{I}$ from each pair: $I+1$, $I+1-k$

$I+2$, $I+2-k$

$\vdots \qquad \vdots$

$I+k-1$, $I-1$

Therefore $|\mathcal{I}| \leq 1 + (k-1) = k$ $\square$

Compressions

$A \subseteq [n]$ $1 \leq i < j \leq n$, ij^{th} compression of A is: $C_{ij}(A) = \begin{cases} A \setminus \{j\} \cup \{i\}, & \text{if } j \in A, i \notin A \\ A & \text{otherwise} \end{cases}$

Examples

$C_{14}(235) = 235$

$C_{14}(145) = 145$

$C_{14}(245) = 125$

If $\mathcal{A} \subseteq \mathcal{P}([n])$ then $C_{ij}(\mathcal{A}) = \{C_{ij}(A) : A \in \mathcal{A}\} \cup \{A \in \mathcal{A} : C_{ij}(A) \in \mathcal{A}\}$

$\mathcal{A}$ We get $C_{ij}(\mathcal{A})$ as follows: Take each set $A \in \mathcal{A}$ in turn

Now apply C_{ij} to A. If $C_{ij}(A) \in \mathcal{A}$ then take it, otherwise take A.

Example

$\mathcal{A} = \{135, 235, 125, 346, 146\}$

$C_{13}(\mathcal{A}) = \{135, 235, 125, 346, 146\}$

$C_{23}(\mathcal{A}) = \{135, 235, 125, 246, 146\} = \mathcal{A}'$

$C_{12}(\mathcal{A}') = \{135, 235, 125, 246, 146\} = \mathcal{A}'$

$C_{26}(\mathcal{A}') = \{135, 235, 125, 246, 124\} = \mathcal{A}''$

Lemma 3.9 $\mathcal{A} \subseteq \binom{[n]}{k}$, ~~mnay~~ $1 \le i < j \le n$

(i) $C_{ij}(\mathcal{A}) \subseteq \binom{[n]}{k}$

(ii) $|C_{ij}(\mathcal{A})| = |\mathcal{A}|$

(iii) If $\mathcal{A}$ is intersecting then so is $C_{ij}(\mathcal{A})$

(iv) If we repeatedly apply ij-compressions to a family $\mathcal{A}$, for all $1 \le i \le j \le n$, we will eventually obtain a left-compressed family, where $\mathcal{A}$ is left-compressed $\Leftrightarrow C_{ij}(\mathcal{A}) = \mathcal{A}$ $\forall 1 \le i < j \le n$

Proof (i) Obvious from defn

(ii) Obvious from defn.

(iii) $A, B \in C_{ij}(\mathcal{A})$

Case 1: If $A, B \in \mathcal{A}$ then $A \cap B \neq \emptyset$

Case 2: If $A, B \in C_{ij}(\mathcal{A}) \setminus \mathcal{A}$ then $i \in A \cap B$

Case 3: So suppose $A \in C_{ij}(\mathcal{A}) \setminus \mathcal{A}$ and $B \in \mathcal{A}$

For a contradiction, suppose $A \cap B = \emptyset$

Since $A \in C_{ij}(\mathcal{A}) \setminus \mathcal{A}$, $\exists D \in \mathcal{A}$ s.t $A = D \setminus \{j\} \cup \{i\}$

Now $D, B \in \mathcal{A} \Rightarrow D \cap B \neq \emptyset$ but $A \cap B = \emptyset$

Therefore $D \cap B = \{j\}$

So $j \in B$ and $i \notin B$ since $i \in A$

Since $B \in C_{ij}(\mathcal{A})$ and $j \in B$, $i \notin B$, $B \setminus \{j\} \cup \{i\} = E$ must also belong to $\mathcal{A}$

But then $D \cap E = \emptyset$ ※ since $D, E \in \mathcal{A}$ ~~is~~ are an intersecting family.

(iv) Define weight (potential) function

$$s(\mathcal{A}) = \sum_{A \in \mathcal{A}} \sum_{a \in A} a$$

If $C_{ij}(\mathcal{A}) \neq \mathcal{A}$ then $s(C_{ij}(\mathcal{A})) \le s(\mathcal{A}) - (j-i)$

$$\le s(\mathcal{A}) - 1$$

$s(\mathcal{A}) \ge 0$ and originally $s(\mathcal{A}) < \infty$, so this ends after a finite number of steps.

2nd proof of E-K-R Proof by induction on $n \ge 2$

Trivial if $n = 2$

If $n = 2k$, let $\mathcal{A} \subseteq \binom{[n]}{k}$ be intersecting then for each $A \in \mathcal{A}$, $A^c \in \binom{[n]}{k} \setminus \mathcal{A}$

Therefore $|\mathcal{A}| \le \frac{1}{2}\binom{n}{k} = \frac{1}{2}\binom{2k}{k} = \binom{n-1}{k-1}$

Therefore we can suppose $n \ge 2k+1$ and the result holds for smaller n and k

By lemma 3.9, we can prove the result for $\mathcal{A}$ left-compressed.

So let $\mathcal{A} = \binom{[n]}{k}$ be intersecting and left-compressed

$\mathcal{B} = \{B \in \mathcal{A} : n \notin B\}$ ← sets not containing n.

$\mathcal{C} = \{C \setminus \{n\} : C \in \mathcal{A}, n \in C\}$ ← sets containing n.

So $|\mathcal{A}| = |\mathcal{B}| + |\mathcal{C}|$

$\mathcal{B} \subseteq \binom{[n-1]}{k}$ intersecting

$\binom{n-1}{k-1}$ Since $n-1 \geq 2k$ our ~~an~~ inductive hypothesis $\Rightarrow$ $|\mathcal{B}| \leq \binom{n-2}{k-1}$

Claim: $\mathcal{C}$ is intersecting

Assuming this, we have $\mathcal{C} \subseteq \binom{[n-1]}{k-1}$ is intersecting

Therefore $|\mathcal{C}| \leq \binom{n-2}{k-2}$ by our inductive hypothesis

So $|\mathcal{A}| \leq \binom{n-2}{k-1} + \binom{n-2}{k-2} = \binom{n-1}{k-1}$ $\square$

Proof (of Claim): Suppose $\exists\, C, D \in \mathcal{C}$ s.t. $C \cap D = \emptyset$

not in A or B

A B A B n

□△○△○△○X

Then if $A = C \cup \{n\}$, $B = D \cup \{n\}$

then $A, B \in \mathcal{A}$, $A \cap B = \{n\}$

$|A \cap [n-1]| = k-1 = |B \cap [n-1]|$

Therefore $A \cup B$ contain $2k-2$ elements from $[n-1]$

$(n-1 \geq 2k)$ Therefore $\exists\, i \in [n-1] \setminus (A \cup B)$

But $\mathcal{A}$ left-compressed $\Rightarrow$ $C_{in}(\mathcal{A}) = \mathcal{A}$

So $E = A \setminus \{n\} \cup \{i\} \in \mathcal{A}$ and $B \cap E = \emptyset$ ⋇

$\square$

③

MATH3503 PROBLEM CLASS 19/11/2013

18. $\mathcal{P}([5])$ $\binom{5}{\lfloor \frac{5}{2} \rfloor} = \binom{5}{2} = \binom{5}{3} = 10$

$\binom{[5]}{2}, \binom{[5]}{3}$

$\{123, 124, 134, 234, 15, 25, 35, 45\}$

19. $P([1]) = \{\phi, 1\}$

$P([2]) = \{2\}, \{\phi, 1, 12\}$

$P([3]) = \{2, 23\}, \{3, 13\}, \{\phi, 1, 12, 123\}$

$P([4]) = \underline{\{24\}}, \{2, 23, 234\}, \{34\}, \{3, 13, 134\}, \{4, 14, 124\}, \{\phi, 1, 12, 123, 1234\}$

smallest + largest = 4.

20. $x_1, \ldots, x_n \geq 1$, $A \subseteq [n]$

$S(A) = \sum_{i \in A} x_i$

$\mathcal{A} \subseteq P([n])$ $|S(A) - S(B)| < 1$

Prove $\mathcal{A}$ antichain

If $A, B \in \mathcal{A}$ and $A \subset B$, $A \neq B$

then $|S(A) - S(B)| = S(\underset{\neq \phi}{B \setminus A}) \geq 1$ ※

$\therefore \mathcal{A}$ antichain □

21. $\mathcal{A} = \{A_1, \ldots, A_t\} \subseteq P([n])$ intersecting

A_1, A_1^c B_1, B_1^c

A_2, A_2^c $\vdots$ things not in $\mathcal{A}$

$\vdots$

A_t, A_t^c $B_{2^{n-1}-t}, B^c_{2^{n-1}-t}$

take $B \notin \mathcal{A} \cup \{A^c : A \in \mathcal{A}\}$

$\therefore B^c \notin \mathcal{A} \cup \{A^c : A \in \mathcal{A}\}$

B, B^c - show can add one of them

suppose can't add either of them.

so suppose $\mathcal{A} \cup \{B\}$ not intersecting

" — " $\mathcal{A} \cup \{B^c\}$ " — "

$\exists\, C, D \in \mathcal{A}$ st $C \cap B = \phi$

$D \cap B^c = \phi \Rightarrow D \subseteq B$

$\therefore C \cap D \subset C \cap B = \phi$ # C and D were in family $\mathcal{A}$.

□

22. $t \geq 1$ $\mathcal{A} \subseteq \binom{[n]}{k}$, $|A \cap B| \geq t$, $\forall A, B \in \mathcal{A}$

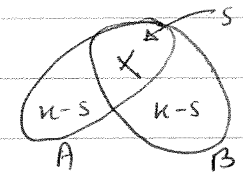
For n large $|\mathcal{A}| \leq \binom{n-t}{k-t}$

$\mathcal{A} = \{A \in \binom{[n]}{k} : [t] \subseteq A\}$ shows sharp.

Let $s = \min_{A,B \in \mathcal{A}} |A \cap B| \geq t$ $\qquad \underline{\binom{n-s}{k-s}} \leq \binom{n-t}{k-t}$

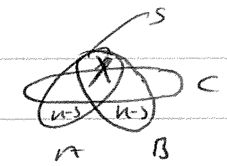
Show $\mathcal{A}$ at most this size (is enough)

Let $A, B \in \mathcal{A}$ st $|A \cap B| = s$, $A \cap B = X$



Either every set in $\mathcal{A}$ contains X or not

ie contain X $\qquad$ If former holds, then $|\mathcal{A}| \leq \binom{n-s}{k-s}$



not contain X $\qquad$ So suppose latter: So $\exists\, C \in \mathcal{A}$ and $|C \cap X| \leq s-1$

For $j = 0, 1, 2, \ldots, s$

Let $\mathcal{A}_j = \{D \in \mathcal{A} : |D \cap X| = j\}$

$|\mathcal{A}| = \sum_{j=0}^{s} |\mathcal{A}_j|$

for both A and B

$$|\mathcal{A}_j| \leq \binom{s}{j}\binom{k-s}{s-j}^2 \binom{n-s}{k-j-2(s-j)} = \binom{s}{j}\binom{k-s}{s-j}^2 \binom{n-s}{k-2s+j}$$

choice of j pts in X $\qquad$ choice of s−j pts from A and B $\qquad$ choice of remaining k−2s+j pts.

$$|\mathcal{A}_j| \leq 2^s\, 2^{2(k-s)} \binom{n-s}{k-2s+(s-1)}$$

$$= 2^s\, 2^{2(k-s)} \binom{n-s}{\underbrace{k-s-1}} \quad \text{for } j = 0, 1, \ldots, s-1 \quad \text{(taking max)}$$

So $\sum_{j=0}^{s-1} |\mathcal{A}_j| \leq s\, 2^{k-s} \binom{n-s}{k-s-1}$ ①

$|\mathcal{A}_s|$?

If $D \in \mathcal{A}_s$ then $|C \cap D| \geq s$

$\Rightarrow D$ meets $C \setminus X$

single choice of element in $C \setminus X$

by defn contain elements of X $\quad |\mathcal{A}_s| \leq k \binom{n-s}{k-(s+1)}$ ②

fix $s+1$ elements

$$|\mathcal{A}| \leq ① + ② \leq (k+s^3 2^{2k-s}) \binom{n-s}{k-s-1} < \binom{n-s}{k-s}$$ if n large.

3.4 The linear algebra method (in combinatorics)

Lemma 3.10 If $v_1, v_2, \ldots, v_t \in V$ are LI (linearly independent) vectors and $\dim(V) = d$, then $t \le d$

Theorem 3.11 If $\mathcal{A} = \{A_1, \ldots, A_m\} \subseteq \mathcal{P}([n])$ satisfy

(i) $|A_i|$ is odd $\forall\ 1 \le i \le m$

(ii) $|A_i \cap A_j|$ is even $\forall\ 1 \le i \ne j \le m$

$\Rightarrow m \le n$ (involves parity so consider $\mathbb{F}_2$ for proof)

$\mathbb{F}_2 = \{0,1\}$ add/mult mod 2

Proof Want to associate to each $A_i \in \mathcal{A}$ a vector $v_i \in \mathbb{F}_2^n$ called the incidence vector of A_i, $v_i = \begin{pmatrix} v_{i1} \\ v_{i2} \\ \vdots \\ v_{in} \end{pmatrix}$

$v_{ij} = \begin{cases} 1 & j \in A_i \\ 0 & o/w \end{cases}$ $\{v_1, v_2, \ldots, v_m\}$ Let $\langle , \rangle$ be the standard inner product.

$$\langle v_i, v_j \rangle = \sum_{k=1}^{n} \underbrace{v_{ik} v_{jk}}_{\substack{=0 \\ \text{unless} \\ v_{ik} = v_{jk} = 1}} = |A_i \cap A_j| = \begin{cases} 1, & i = j \\ 0, & i \ne j \end{cases}$$

$\therefore \{v_1, \ldots, v_m\}$ is orthogonal $\therefore$ LI $\therefore m \le \dim(\mathbb{F}_2^n) = n$ $\square$

Theorem 3.12 (Fisher) $\mathcal{A} = \{A_1, \ldots, A_m\} \subseteq \mathcal{P}([n])$, satisfy $|A_i \cap A_j| = k$ for some fixed $k \ge 1$ and every $1 \le i \ne j \le m$

Then $|\mathcal{A}| = m \le n$

Proof Let $v_i \in \mathbb{R}^n$ be the incidence vector for A_i

So $v_{ij} = \begin{cases} 1, & j \in A_i \\ 0, & o/w \end{cases}$ $\{v_1, \ldots, v_m\}$

$$\langle v_i, v_j \rangle = \sum_{k=1}^{n} v_{ik} v_{jk} = |A_i \cap A_j| = \begin{cases} k, & i \ne j \\ |A_i|, & i = j \end{cases}$$

Suppose $\{v_1, \ldots, v_m\}$ are not LI so $\exists\ \lambda_1, \ldots, \lambda_m \in \mathbb{R}$ not all zero st $\sum_{i=1}^{m} \lambda_i v_i = 0$

$$0 = \langle 0, 0 \rangle = \langle \sum_{i=1}^{m} \lambda_i v_i, \sum_{j=1}^{m} \lambda_j v_j \rangle = \sum_{i=1}^{m} \lambda_i^2 \langle v_i, v_i \rangle + \sum_{1 \le i \ne j \le m} \lambda_i \lambda_j \langle v_i, v_j \rangle$$

$$= \sum_{i=1}^{m} \lambda_i^2 (|A_i|) + \sum_{1 \le i \ne j \le m} \lambda_i \lambda_j k$$

$$= \overset{①}{\sum_{i=1}^{m} \lambda_i^2 (|A_i| - k)} + k \overset{②}{\left(\sum_{i=1}^{m} \lambda_i \right)^2}$$

Since $|A_i \cap A_j| \ge k$, $|A_i| \ge k\ \forall i$ with equality at most once.

So $\lambda_i^2 (|A_i| - k) \ge 0\ \forall i$, clearly ② ≥ 0

① + ② $= 0 \Rightarrow$ ① $= 0$ ② $= 0$

Each $\lambda_i^2(|A_i| - k) = 0$. Since $|A_i| - k = 0$ at most once, we have $\lambda_i = 0$ for all but at most one value of i

But if all but one λ_i are zero then ② $\ne 0$

Thus all λ_i's are zero and $\{v_1, \ldots, v_m\}$ is LI. Hence $m \le \dim(\mathbb{R}^n) = n$ $\square$

Let $L \subseteq \{0,1,\dots,n\}$ and $\mathcal{A} \subseteq P([n])$. We say $\mathcal{A}$ is L-intersecting iff $A,B \in \mathcal{A}$, $A \neq B$ then $|A \cap B| \in L$

e.g. $L = \{1,\dots,n\}$ L-intersecting $\equiv$ intersecting

$L = \{t,\dots,n\}$ L-intersecting $\equiv$ t-intersecting i.e. $|A \cap B| \geq t$ $\forall A,B$

Theorem (Ray-Chaudhuri - Wilson) 1975

If $\mathcal{A} \subseteq P([n])$ is L-intersecting then $|\mathcal{A}| \leq \sum_{i=0}^{|L|} \binom{n}{i}$

Example

$\mathcal{A} = \binom{[n]}{0} \cup \binom{[n]}{1} \cup \dots \cup \binom{[n]}{\ell}$

~~[illegible]~~ $L = \{0,\dots,\ell-1\}$, $|L| = \ell$

$\mathcal{A}$ is L-intersecting

Proof $L = \{\ell_1,\dots,\ell_s\}$ $|L| = s$

Let $\mathcal{A} \subseteq P([n])$ be L-intersecting

$\mathcal{A} = \{A_1,\dots,A_m\}$, $|A_1| \leq |A_2| \leq \dots \leq |A_m|$

Let v_i be the incidence vector of A_i, so $v_{ij} = \begin{cases} 1 & j \in A_i \\ 0 & o/w \end{cases}$

For $x,y \in \mathbb{R}^n$, define $x \cdot y = \sum_{i=1}^{n} x_i y_i$

$v_i \cdot v_j = \sum_{k=1}^{n} v_{ik} v_{jk} = |A_i \cap A_j|$

For $1 \leq i \leq m$, define a polynomial P_i in n variables $(x_1,\dots,x_n)$ over $\mathbb{R}$ by:

dot prod.

$$p_i(x) = \prod_{k : \ell_k < |A_i|} ((v_i \cdot x) - \ell_k)$$

Continuing proof of theorem 3.13...

$\langle , \rangle$ same as normal dot product

For $1 \le i \le m$, define a polynomial in ~~m~~ n variables

$$p_i(x) = \prod_{k: l_k < |A_i|} (\langle v_i, x\rangle - l_k), \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$p_i(v_j) = ?$ for $j<i$

If $j<i$, then $|A_j| \le |A_i|$, $|A_i \cap A_j| \in L$

$|A_i \cap A_j| = l_k < |A_i|$. Corresponding term in $p_i(v_j)$ is:

$\langle v_i, v_j\rangle - l_k = 0$ So $p_i(v_j) = 0$

$p_i(v_i)$, each term is $\langle v_i, v_i\rangle - l_k$, $l_k < |A_i|$

so $(|A_i| - l_k) > 0 \Rightarrow p_i(v_i) \neq 0$

$|L| = s$. Let $q_i(x)$ be the polynomial obtained from $p_i(x)$ by replacing each term x_j^{α}, $\alpha \ge 2$ by x_j. This does not change value on $x \in \{0,1\}^n$. So $q_i(v_j) = 0$ for $j<i$

$q_i(v_i) \neq 0$

eg. L-intersecting (to help understand proof)

$\mathcal{A} = \{13, 123, 124, 345\}$

$L = \{1,2\}$ everything meets at either 1 or 2 pts.

incidence vectors:

$v_1 = \begin{pmatrix}1\\0\\1\\0\\0\end{pmatrix}$, $v_2 = \begin{pmatrix}1\\1\\1\\0\\0\end{pmatrix}$, $v_3 = \begin{pmatrix}1\\1\\0\\1\\0\end{pmatrix}$, $v_4 = \begin{pmatrix}0\\0\\1\\1\\1\end{pmatrix}$

$$p_4(x) = \prod_{k: l_k < |A_4|} (\langle v_4, x\rangle, l_k)$$
$$= (\langle v_4, x\rangle - 1)(\langle v_4, x\rangle - 2)$$
$$= (x_3 + x_4 + x_5 - 1)(x_3 + x_4 + x_5 - 2)$$

$\langle v_i, v_j\rangle = |A_i \cap A_j|$: $p_4(v_1) = 0 \times (-1) = 0$

$p_4(v_2) = 0 \times (-1) = 0$, $p_4(v_3) = 0 \times (-1) = 0$

$p_4(v_4) = 2 \times 1 = 2 \neq 0$

$p_4(x) = x_3^2 + x_4^2 + x_5^2 + 2(x_3x_4 + x_3x_5 + x_4x_5) - 3(x_1 + x_2 + x_3) + 2$

Let Q be the vectorspace over $\mathbb{R}$ spanned by $q_1, \dots, q_m$

<u>Claim</u>: $\{q_1, \dots, q_m\}$ are LI

zeropolynomial $\Rightarrow$ Suppose $\lambda_1 q_1 + \dots + \lambda_m q_m = \underline{0}$

$$\underbrace{\lambda_1 q_1(v_1)}_{\neq 0} + \underbrace{\dots + \lambda_m q_m(v_1)}_{=0} = 0 \qquad \leftarrow q_i(v_1) = 0 \;\; i \ge 2$$

$\lambda_1 q_1(v_1) = 0$

$\Rightarrow q_1(v_1) \neq 0 \Rightarrow \lambda_1 = 0$

Repeat $\lambda_2 q_2(v_2) + \dots + \lambda_m q_m(v_2) = 0 \quad \leftarrow q_i(v_2) = 0 \;\; i \ge 3$

$\lambda_2 q_2(v_2) = 0$, $q_2(v_2) \neq 0 \Rightarrow \lambda_2 = 0$

Repeat, hence $\lambda_1 = \lambda_2 = \dots = \lambda_m = 0$ □ for claim.

Know that $m \le \dim Q$.

Consider $q_i(x)$ expressed as a sum of monomials ← expand out brackets and collect terms and non-distinct products replaced ie $x_1^2, x_2^2, \dots$

$$q_i(x) = \sum_j c_j \underbrace{x_{j_1} x_{j_2} \dots x_{j_r}}_{\text{at most } s \text{ terms}}$$

Define $M_s = \{1, x_1, \dots, x_n, x_1x_2, x_1x_3, \dots, x_{n-1}x_n, x_1x_2x_3, x_{n-2}x_{n-1}x_n, \dots\}$

$$= \left\{ x_{i_1} x_{i_2} \dots x_{i_j} : 1 \le j \le s, \{i_1, i_2, \dots, i_j\} \in \binom{[n]}{j} \right\} \cup \{1\}$$

Q is clearly in the span of M_s $\therefore$ $\dim Q \le |M_s|$

$$|M_s| = \sum_{i=0}^{s} \binom{n}{i} \quad \square$$

$R(2,t)=t$ $\quad R(s,2)=s$

$\uparrow$ $\uparrow$
K_2 K_t

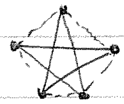
Ramsey Theory

Let $s,t \geq 2$, let $R(s,t)$ be the smallest integer n s.t. every red-blue edge colouring of K_n either contains a red K_s or a blue K_t.

Proposition 4.1 $R(3,3)=6$ $\quad$ (to prove need to show $R(3,3)\leq 6$ and $R(3,3)>5$)

dash-blue
line - red

$R(3,3)>5$:



No red or blue $K_3 \Rightarrow R(3,3)>5$

can use there for general n instead of 6

Take a red/blue colouring of K_6

Let $v \in V(K_6)$, $\Gamma_{red}(v) = \{w : vw \text{ is red}\}$

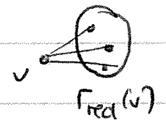
$\Gamma_{blue}(v) = \{w : vw \text{ is blue}\}$

$d_{red}(v) = |\Gamma_{red}(v)|$, $d_{blue}(v) = |\Gamma_{blue}(v)|$

$5 = d(v) = d_{red}(v) + d_{blue}(v)$

Either $d_{red}(v) \geq 3$ or $d_{blue} \geq 3$, so WLOG $d_{red}(v) \geq 3$

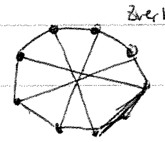
— red



Either there is a red edge inside $\Gamma_{red}(v) \Rightarrow$ red K_3

or $\Gamma_{red}(v)$ has all edges blue $\Rightarrow$ blue K_3 $\square$

Proposition 4.2 $R(3,4)=9$



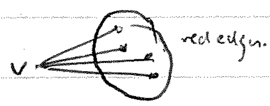
red edges drawn
blue edges missing

Example
$\Rightarrow R(3,4)>8$

No red K_3, no blue K_4

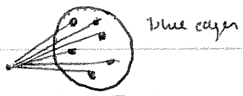
Now show $R(3,4)\leq 9$

Take a red/blue colouring of K_9. Let $v\in V(K_9)$. If $d_{red}(v)\geq 4$



either $\Gamma_{red}(v)$ contains a red edge $\Rightarrow$ red K_3, or $\Gamma_{red}(v)$ has all edges blue $\Rightarrow$ blue K_4. So wlog $d_{red}(v)\leq 3$

If $d_{red}(v)\leq 2 \Rightarrow d_{blue}(v)\geq 6$



$R(3,3)=6 \Rightarrow \Gamma_{blue}(v)$ contains a red K_3 or blue K_3. In latter case have a blue K_4 together with v

Final case: $d_{red}(v)=3$ for all $v\in V(K_9)$

So the $\sum_{v\in V(K_9)} d_{red}(v) = 3\times 9 = 27$

$2\#\text{red edges} = 27$ $\divideontimes$

Theorem 4.3 Let $s,t\geq 2$, then $R(s,t)$ is well defined and satisfies $R(s,t) \leq \binom{s+t-2}{s-1}$

Proof (By induction on $s+t$)

$R(2,t)=t$, $R(s,2)=s$. Theorem holds if $s=2$ or $t=2$ $\therefore$ True for $s+t=4$

Let $s+t>4$ with $s>2, t>2$

Let $n=R(s-1,t)+R(s,t-1)$, which exists by our inductive hypothesis

Want to show $R(s,t)\leq n$

MATH3503 Take a red/blue colouring of K_n 26/11/2013

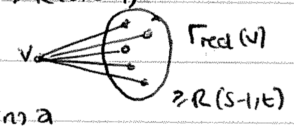
For $v \in V(K_n)$ $\quad \Gamma_{red}(v) = \{w : vw \text{ is red}\}$ $\quad d_{red}(v) = |\Gamma_{red}(v)|$

$\Gamma_{blue}(v) = \{w : vw \text{ is blue}\}$ $\quad d_{blue}(v) = |\Gamma_{blue}(v)|$

$n-1 = d(v) = d_{red}(v) + d_{blue}(v) = R(s-1,t) + R(s,t-1) - 1$

Either $d_{red}(v) \geq R(s-1,t)$ or $d_{blue}(v) \geq R(s,t-1)$

WLOG suppose $d_{red}(v) \geq R(s-1,t)$



By definition of $R(s-1,t)$, $\Gamma_{red}(v)$ contains a red K_{s-1} or a blue K_t, in the latter case we are done, in the former case the red K_{s-1} together with v forms a red K_s.

Hence $R(s,t) \leq n = R(s-1,t) + R(s,t-1)$ (*)

and so $R(s,t) \leq \binom{s+t-3}{s-2} + \binom{s+t-3}{s-1} = \binom{s+t-2}{s-1}$ □

$R(s,s) = \binom{2s-2}{s-1} \leq 2^{2s-2} \simeq 4^s$

Proposition 4.4 $\quad R(4,4) = 18$

Proof $\quad$ Theorem $\Rightarrow R(4,4) \overset{(*)}{\leq} \underset{9}{R(3,4)} + \underset{9}{R(4,3)} = 18$ □

$R(4,4) > 17$ ... Google.

Theorem 4.6 (Erdős 1947) $\quad$ If $n \geq s \geq 2$ then $R(s,s) > n$ if $\binom{n}{s} 2^{1-\binom{s}{2}} < 1$

Proof $\quad$ Let n,s satisfy $\binom{n}{s} 2^{1-\binom{s}{2}} < 1$

Take K_n and colour each edge red or blue independently uniformly at random. (i.u.a.r)

Fix $v_1, v_2, \ldots, v_s \in V(K_n)$ distinct vertices

$P(v_1, \ldots, v_s \text{ form } K_s) = \frac{1}{2^{\binom{s}{2}}}$

$E[\# \text{red } K_s \text{ in } K_n] = \binom{n}{s} \frac{1}{2^{\binom{s}{2}}}$ (linearity of expectation)

" blue " = " "

$E[\# \text{monochromatic } K_s \text{ in } K_n] = \binom{n}{s} 2^{1-\binom{s}{2}} < 1$

$\therefore \exists$ a colouring of K_n with no red or blue K_s

$\therefore R(s,s) > n$

Corollary 4.7 $\quad$ For $s \geq 2$, $R(s,s) \geq 2^{\frac{s}{2}}$ $(= \sqrt{2}^s)$

Proof $\quad R(2,2) = 2$ works, $R(3,3) = 6$ works

So suppose $s \geq 4$

Let $n = \lfloor 2^{\frac{s}{2}} \rfloor$

For $s \geqslant 4$, $s! \geqslant 2^s$

$$\binom{n}{s} \frac{2}{2^{\binom{s}{2}}} < \frac{n^s}{2^s} \frac{2}{2^{\frac{s^2-s}{2}}} \leq \frac{2^{s^2/2+1}}{2^{s^2/2+s/2}} = \frac{1}{2^{s/2-1}} \leq \frac{1}{2} < 1$$

$\therefore$ Theorem 4.6 $\Rightarrow R(s,s) \geqslant 2^{s/2}$ $\square$

we end up with $(\sqrt{2})^s \leq R(s,s) \leq 4^s$

Theorem 4.8 (Fermats last theorem) There are no trivial integer solution to $x^n+y^n=z^n$ for $n\geq 3$

Theorem 4.9 For every $n\geq 1$, $\exists$ prime p_n st for all primes $p\geq p_n$, $x^n+y^n\equiv z^n \bmod p$ has a non trivial soln.

we've defined $R(s,t)$

now we have $R(s_1,\dots,s_k)$ - this is the smallest integer n st every colouring of the edges of K_n with the colours $c_1,\dots,c_k$ contains a c_i-coloured copy of K_{s_i} for some $1\leq i\leq k$

Theorem 4.12 $R(s_1,\dots,s_k)$ is well defined.

Proof $R(s_1,s_2,s_3)\leq R(s_1,R(s_2,s_3))=N$

Take a c_1,c_2,c_3 colouring of K_N

Either have c_1-coloured K_{s_1} or have c_2 and c_3 coloured $K_{R(s_2,s_3)}$ which contains a c_2-coloured K_{s_2} or a c_3-coloured K_3

By the same argument $R(s_1,\dots,s_k)\leq R(s_1,R(s_2,\dots,s_k))$ is well defined by induction ☒

Theorem 4.10 (Schür) For any $k\geq 1$, $\exists$ $S(k)$ st for any k-colouring of the integers $\{1,2,\dots,S(k)\}$ we have $x,y,z\in[S(k)]$ st $x+y=z$ and $c(x)=c(y)=c(z)$

If $s_1=s_2=\dots=s_k$ then $R(s_1,\dots,s_k)=R_k(s)$

Proof (of thm 4.10) Given k, let $N=R_k(3)$

Now suppose $[N]$ are k-coloured with colour of x as $c(x)$. (j-i at least one)

Define a k-colouring of the edges of K_N with $V(K_N)=[N]$ by $\underbrace{\hat{c}(ij)}_{ij\text{-edge}}=c(j-i)$, $j>i$

Since $N=R_k(3)$, $\exists i<j<k$ st $\hat{c}(ij)=\hat{c}(ik)=\hat{c}(jk)$

$\therefore \underbrace{c(j-i)}_{y}=\underbrace{c(k-i)}_{z}=\underbrace{c(k-j)}_{x}$

$x=(k-j)$, $y=(j-i)$, $z=(k-i)$

$x+y=z$ and $c(x)=c(y)=c(z)$

Hence define $S(k)=R_k(3)$

☒

Lemma 4.11 If p is prime then $\mathbb{Z}_p^*$ is cyclic

eg $p=7$ $g=3$

$3^1 \equiv 3$ $3^4 \equiv 4$
$3^2 \equiv 2$ $3^5 \equiv 5$
$3^3 \equiv 6$ $3^6 \equiv 1$

Proof (of thm 4.9)

Let $n \geq 1$ be given

Let $p_n \geq S(n)+1$

Let p be a prime $\geq S(n)$, with generator g

If $1 \leq x \leq p-1$, then $x = g^{a_x n + b} \bmod p$

[~~...~~ $x = g^{c_x}$, $c_x = a_x n + b_x$, $0 \leq b_x \leq n-1$]

Call b_x the colour of x

Note there are n different possible ~~values~~ colours

Since $p-1 \geq S_n$, we have coloured at least $S(n)$ integers with n colours.

So by Schur's theorem, $\exists\, x, y, z$ that all recieve the same colour b and $x+y=z$

$x = g^{a_x n + b}$, $y = g^{a_y n + b}$, $z = g^{a_z n + b}$ $(x+y=z)$

$x+y = (g^{a_x n} + g^{a_y n})\, g^b = g^{a_z n} g^b = z$

Take $X = g^{a_x}$

$Y = g^{a_y}$

$Z = g^{a_z}$

then $X^n + Y^n = Z^n \bmod p$ ∎

F.L.T Theorem 4.8 There are no trivial integer solutions to $x^n+y^n=z^n$ for $n \geq 3$

Theorem 4.9 For every $n \geq 1$ $\exists$ prime p_n st for all primes $p \geq p_n$, $x^n+y^n \equiv z^n \bmod p$ has a non trivial soln.

$R(s,t)$; $R(s_1,\dots,s_k)$ is smallest integer n st every colouring of the edges of K_n with the colours $c_1,\dots,c_k$ contains a c_i-coloured copy of K_{s_i} for some $1 \leq i \leq k$

Theorem 4.12 $R(s_1,\dots,s_k)$ is well defined.

Proof $R(s_1,s_2,s_3) \leq R(s_1, R(s_2,s_3)) = N$

Take a c_1,c_2,c_3 colouring of K_N. Either have c_1-coloured K_{s_1} or have c_2 and c_3 coloured $K_{R(s_2,s_3)}$ which contains a c_2-coloured K_{s_2} or a c_3-coloured K_{s_3}

By the same argument $R(s_1,\dots,s_k) \leq R(s_1, R(s_2,\dots,s_k))$ is well defined by induction $\square$

Theorem 4.10 (Schur) For any $k \geq 1$ $\exists S(k)$ st for any k-colouring of the integers $\{1,2,\dots,S(k)\}$ we have $x,y,z \in [S(k)]$ s.t $x+y=z$ and $c(x)=c(y)=c(z)$

If $s_1=s_2=\dots=s_k=s$ then $R(s_1,\dots,s_k)=R_k(s)$

(Schur) Proof Given k, let $N=R_k(3)$

Now suppose $[N]$ are k-coloured with ~~where~~ colour of x is $c(x)$

$j-i$ all least one.

Define a k-colouring of the edges of K_N ~~by~~ with $V(K_N)=[N]$ by $\overset{\swarrow \text{edge}}{\hat{c}(ij)} = c(j-i)$, $j>i$

i,j,k distinct vertices

Since $N=R_k(3)$, $\exists$ ~~ijk~~ $\underset{i<j<k}{ijk}$ st $\hat{c}(ij)=\hat{c}(ik)=\hat{c}(jk)$

i.e. $\underset{y}{\underline{c(j-i)}} = \underset{z}{\underline{c(k-i)}} = \underset{x}{\underline{c(k-j)}}$

$x=k-j$, $y=j-i$, $z=k-i$

$x+y=z$ and $c(x)=c(y)=c(z)$

Hence define $S(k)=R_k(3)$ $\square$

Lemma 4.11 If p is prime then $\mathbb{Z}_p^*$ is cyclic

e.g. $p=7$, $g=3$

$3^1=3$
$3^2=9\equiv 2$
$3^3\equiv 6$
$3^4\equiv 4$
$3^5\equiv 5$
$3^6\equiv 1$

Thm 4.9 Proof

all elements are mod p

Let $n \geq 1$ be given

Let $p_n \geq S(n)+1$ given by prev theorem. (4.10)

Let p be a prime $\geq S(n)$, with generator g

If $1 \leq x \leq p-1$, then $x = g^{a_x n + b} \bmod p$

$[x = g^{c_x}$, $c_x = a_x n + b_x$, $0 \leq b_x \leq n-1)$

Call b_x the colour of x. Note there are n different possible colours.

Since $p-1 \geq S_n$, we have coloured at least $S(n)$ integers with n colours.

So by Schurs theorem, $\exists x, y, z$ that all recieve the same colour b and $x+y=z$.

$x = g^{a_x n + b}$, $y = g^{a_y n + b}$, $z = g^{a_z n + b}$ $\quad x+y=z$

$x+y = \left(g^{a_x n} + g^{a_y n}\right) g^b = g^{a_z n} g^b = z$

Take $X = g^{a_x}$
$Y = g^{a_y}$
$Z = g^{a_z}$

then $X^n + Y^n = Z^n \bmod p$ $\square$

lectures 25+26 were missed because of strike

26) $\mathcal{A} = \{A_1, \ldots, A_m\} \subseteq P([n])$

① $\forall i \quad 6 \nmid |A_i|$

② $\forall i \neq j \quad 6 \mid |A_i \cap A_j|$

$\Rightarrow m \leq 2n$

① $\Rightarrow \forall i$ either $2 \nmid |A_i$ or $3 \nmid |A_i|$

② $\Rightarrow \forall i \neq j \quad 2 \mid |A_i \cap A_j|$ and $3 \mid |A_i \cap A_j|$

by ~~thm 3.11~~ lemma 3.10, for $\geq \frac{m}{2}$ sets either $2 \nmid |A_i|$ or $3 \nmid |A_i|$

Former case apply theorem

Latter case, theorem holds for $\mathbb{F}_3$

$R(3,5) \leq R(2,5) + R(3,4)$

$= 5 + 9 = 14$

$R(5,5) \leq R(4,5) + R(5,4)$

$= 2R(4,5)$ shown $R(4,5) \leq 32$ here

$\leq 2(R(3,5) + R(4,4))$

$\leq 2(14+18) = 32 \times 2 = 64$

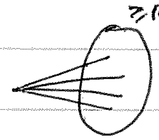
$R(5,5) \leq 63$

proof: take K_{63}, 2-colour edges

take $v \in V(K_{65})$, $d(v) = 62$

$d_R(v) + d_B(v) = 62$

either $d_R(v) \geq 32$ $\Rightarrow$



$\geq R(4,5)$ red lines

there's a red K_4 or a blue K_5 in there

$\Rightarrow$ done

or $d_B(v) \geq 32$

$\forall v \in V(K_{63})$, $d_R(v) = d_B(v) = 31$. ~~$31 \times 63 \sum_{v \in V}$~~ $31 \times 63 = \sum_{v \in V} d_R(v) = 2\,\#$ red edges $\times$

Red/Blue colour edges of K_n, either Red or Blue graph is connected.

Suppose red isn't connected

Let $V(K_n) = V_1 \dot\cup \ldots \dot\cup V_t$

Decompose into Red connected components so $t \geq 2$

Blue graph contains $K(V_1, \ldots, V_t)$ - the complete t-partite graph with parts $V_1, \ldots, V_t$

Conjecture (Erdos - Turan)

If $\{a_n\} \subseteq \mathbb{N}$ and $\sum_{n=1}^{\infty} \frac{1}{a_n}$ diverges then $\{a_n\}$ contains arbitrarily long arithmetic progressions (AP's)

Theorem (Green-Tao 2004)

The primes contain arbitrarily long AP's (24 is longest one found)

lead up up to thm 4.13

Theorem (Szemeredi 1970's)

If $\delta > 0$, $k \geq 1$ then $\exists N_{\delta,k} \in \mathbb{N}$ st if $A \subseteq [1, 2, \dots, N_{\delta,k}]$ and $|A| \geq \delta N_{\delta k}$ then A contains a k-length AP

Thm 4.13 (Van der Waerden 1927)

If $k, t \geq 1$ $\exists$ $W(t,k)$ st if $[W(t,k)]$ is k-coloured there is a monochromatic arithmetic progression (MAP) of length k

AP $a, a+d, a+2d, \dots, a+(t-1)d$ is an AP length t

Coloured $[N]$ with k-colours then a MAP $\equiv$ monochromatic AP $\equiv$ AP with all elements same colour

$P_1 = \{1,4,7\}$ 10
$P_2 = \{7,8,9\}$ 10
$P_3 = \{4,6,8\}$ 10

We say AP's $P_1, P_2, \dots, P_r$ are focused at $f \in \mathbb{N}$ if the 'next' potential member of each P_i is f

eg

1 4 6 8 11 12 13 14 15

1 6 11 red
13 14 15 blue
4 8 12 black

We say that AP's $P_1, P_2, \dots, P_r$ are colour focused if each P_i is monochromatic, all the colours are different and they are focused.

Proof

Induction on t

$t=1$: $W(1,k)=1$

Assume for $t \geq 2$, $W(t-1,k)$ exists for all $k \geq 1$

Claim: For all $1 \leq r \leq k$ $\exists$ $n_r(t,k)$ st if $[n_r(t,k)]$ is k-coloured then either (A) $\exists$ a ~~map~~ MAP of length t

(B) $\exists$ r colour-focused AP's of length t-1

Assume claim holds. Let $W(t,k) = 2n_k(t,k)$

Now if we k-colour $[W(t,k)]$, well claim $\Rightarrow$ $[n_k(t,k)]$ contains either (A) MAP length t

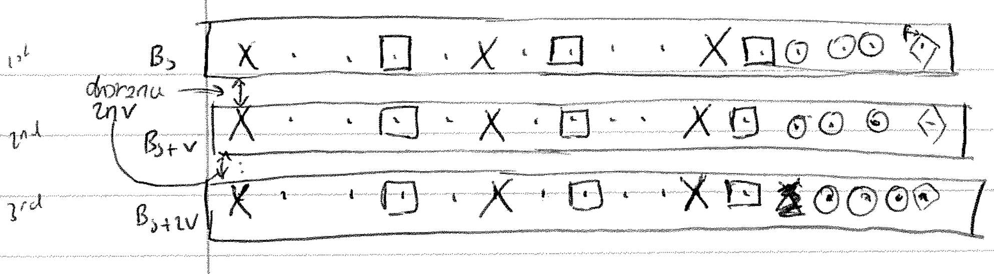
or (B) contains k colour focused AP's of length t-1; $P_1, \dots, P_k$ with common focus f.

~~Now since~~ Wlog assume colour of P_i is colour i,

Then whichever P_i has the same colour as f can be extended (using f) to give a MAP length t.

$\therefore$ $W(t,k)$ is well-defined.

Shapes
X - black
□ - red
○ - blue
colour
◇ - colour dot green



} for proof.

Proof of claim (Induction on r)

$n_1(t,k) = W(t-1,k)$ exist by ind hyp for t

Now suppose $2 \le r \le k$ and $n_{r-1}(t,k)$ exists

let $n = n_{r-1}(t,k)$

Set $n_r(t,k) = W(t-1, k^{2n})2n$

So suppose k-colour $[n_r(t,k)]$

$[n_r(t,k)] = B_1 \dot\cup B_2 \dot\cup B_3 \dot\cup \dots \dot\cup B_{W(t-1,k^{2n})}$

each B_i is an interval of length $2n$ eg: $B_1 = \{1, \dots, 2n\}$
$B_2 = \{2n+1, \dots, 4n\}$

How many different ways can a block be coloured? k^{2n} — 2n size of each block, k different ways of colouring each one

So by defn of $W(t-1, k^{2n})$, $\exists B_s, B_{s+1}, B_{s+2}, \dots, B_{s+(t-2)v}$ coloured identically. (now look at diagram above)

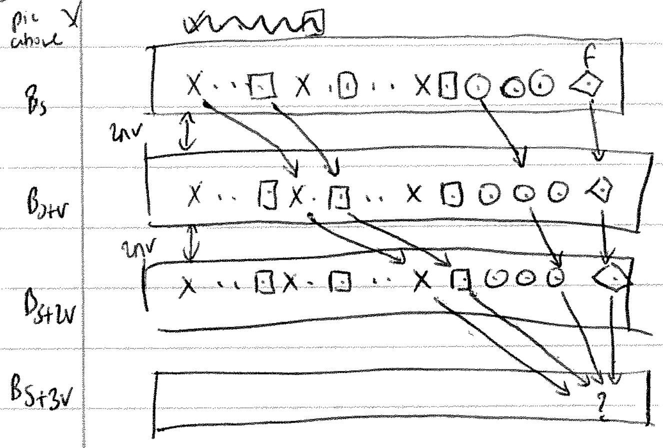
Now B_s has size $2n = 2n_{r-1}(t,k)$

By defn of $n_{r-1}(t,k)$, this either contains MAP length t or $\exists$ r-1 colour focused APs length t-1,

if in 1st case we're done

$P_1, \dots, P_{r-1}$ with focus f, $P_i = \{a_i, a_i+d_i, \dots, a_i+(t-2)d_i\}$

finish proof by annotating pic above



$P_1' = \{a_i, a_i + (d_i+2nv), a_i + 2(d_i+2nv), \dots, a_i + (t-2)(d_i+2nv)\}$

$P_1', \dots, P_{r-1}'$ are colour focused APs length t-1 with focus $f + (t-1)2nv$

Moreover $P_r' = \{f, f+2nv, f+4nv, \dots, f+(t-2)2nv\}$ is another MAP length t-1 and has a different colour to $P_1', \dots, P_{r-1}'$ (otherwise we would already have MAP of length t). $\therefore$ Have r colour focused APs length t-1

LAST LECTURE - NO LECTURE 30!