3503 Graph Theory and Combinatorics Notes
Based on the spring 2013 lectures
by Dr J Talbot.

To those who understands and accepts that the way and only way to learn mathematics is to slove mathematics problems and to do them honestly and faithfully.

[illegible]

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Graph theory + Combinatorics

Introduction; an example:

$G = (V, E)$

$V = \{1, 2, 3, 4, 5, 6\}$.

$E = \{\{1,2\}, \{1,3\}, \{2,3\}, \{2,4\}, \{5,6\}\}$.

Notation:

Ramsey Theory.

$\#$ = "the number of"

So $K_5 =$ 

$\#$edges in K_5
$= \#$ unordered pairs from $\{1, 2, 3, 4, 5\}$

$$= \binom{5}{2} = \frac{5 \times 4}{2} = 10.$$

Def: If X is a set then $|X|$ is the size or cardinality of X.

Def: For any $k \geq 1$ we define k factorial to be $k! = k(k-1) \cdot \ldots \cdot 2 \cdot 1$. We define $0! = 1$.

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$\mathcal{X} = \{1, 2, 3, 4, \ldots, 10\}$.

How many cyclic permutation of $\mathcal{X}$ are there?

$\#$ cyclic perms $= 9! = \dfrac{10!}{10}$.

1 2 3 4 5 . . . 10
2 3 4 . . . 10 1
3 4 10 2 1
⋮
10 1 . . . 9

— / —

$\underline{\mathcal{A}}$ family of subset of $\mathcal{X}$.

eg. $\mathcal{A} = \{\{1,2,3\}, \{2,4,n\}\}$.

$\mathcal{A}$ family of subset $\mathcal{X}$ is <u>intersecting</u> if $A, B \in \mathcal{A} \Rightarrow A \wedge B \neq \emptyset$.

$\#$ subset of $\mathcal{X} = 2^n$

n times.

$B \in \mathcal{A} \Rightarrow \mathcal{X} - B = B^c$.

$|\mathcal{A}| \leq 2^{n-1}$ Because have at most one of each complementary pairs : $(B, \mathcal{X} - B)$.

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$[x] = \{1, 2, \ldots, n\}$

$\lfloor x \rfloor$ $\lceil x \rceil$
floor ceiling

—/—

$\mathcal{A} = \{A \subseteq [n] : 8 \in A\}$.
" $\mathcal{X}$

$|\mathcal{A}| = 2^{n-1}$ = # subset of an $(n-1)$ - set.

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$|\mathcal{X}|$.

$4! = 4 \times 3 \times 2 \times 1 = 24$
$0! = 1$.

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<u>Lemma 1.1</u>

(i) # k - tuples from $\mathcal{X} = [n] = n^k$.

(ii) # k - tuples with distinct elements from $\mathcal{X}$ is $n(n-1) \ldots (n-k+1)$.

<u>Proof</u> :

(i) n choices for each k positions

(ii) n choices for 1st entry
$n-1$ " " 2nd entry etc.

$n-(k-1)$ choices for k^{th} entry $\square$

<u>Def</u>: $\binom{X}{k} := \{ A \subseteq X : |A| = k \}$

eg. $\binom{5}{2} = 10$

$$\binom{[5]}{2} = \{12, 13, 14, 15, 23, 24, 25, 34, 35, 45\}.$$

$$= \{\{1,2\}, \{1,3\}, \ldots, \{4,5\}\}.$$

<u>Lemma 1.2</u>: $|X| = n$, and $0 \le k \le n$ then

$$\left| \binom{X}{k} \right| = \binom{n}{k}$$

Each k-set from X correspond to $k!$ different k-tuples of distinct. Hence, Lemma 1.1 (ii)

$$\Rightarrow \left| \binom{X}{k} \right| = \frac{n(n-1)\ldots(n-k+1)}{k!}$$

$$= \frac{n!}{k!(n-k)!} \qquad \square$$

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Probabilistic Method:

Idea: Want an example of some mathematical object. Invent a probabilistic "experiment" where Pr(that the experiment generates a good example).

$0! = 1 \Rightarrow \binom{n}{n} = 1$ and $\binom{n}{0} = 1$ (there is only one way to choose a set with no elements). By convention we define $\binom{n}{k} = 0$ for $k \in \mathbb{Z} \setminus \{0, 1, \dots n\}$. i.e define $\binom{n}{k} = 0$ if $k < 0$ or $k > n$ integer.

Def: The powerset of a set X is:

$$\mathcal{P}(X) = \{ A : A \subseteq X \}$$

Lemma 1.3. If $|X| = n \geq 0$ and $0 \leq k \leq n$ then

i) $|\mathcal{P}(X)| = 2^n$.

ii) $\binom{n}{k} = \binom{n}{n-k}$

iii) $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$

Proof:

i) n elements in or out

ii) Observe that $\frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!}$

So $B \to X - B$ is a bijection.

iii) $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k+1}$ = # subset of $[n+1]$ of size k, containing $n+1$

$\left|\binom{[n+1]}{k}\right|$ // // # subset of $[n+1]$ of size k! not containing $n+1$

— (—

$x \in \mathbb{R}$, $s \geq 0$ integer

$$\binom{x}{s} = \begin{cases} \dfrac{x(x-1)\cdots(x-s+1)}{s!}, & x \geq s-1 \\ 0, & x \leq s-1. \end{cases}$$

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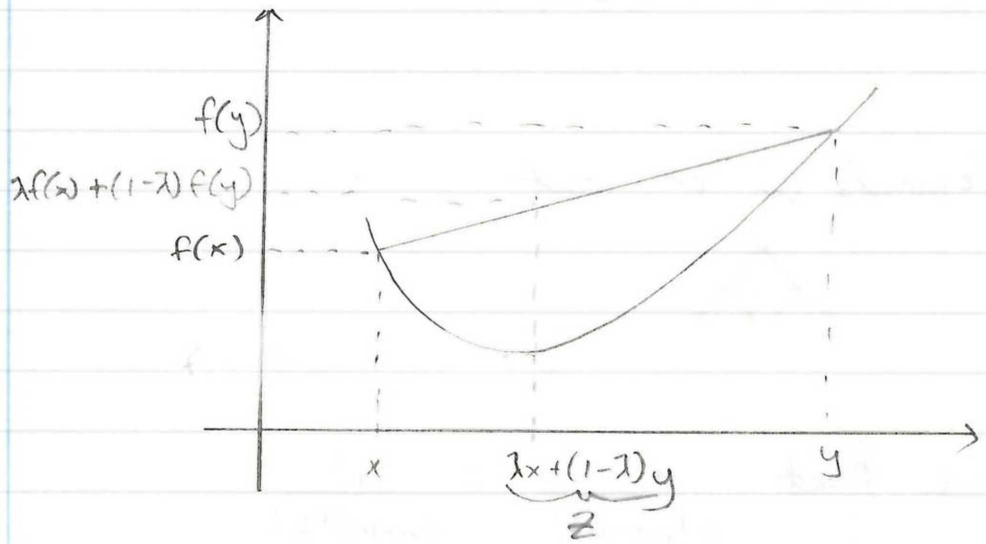
<u>Def</u>: $f:(a,b) \to \mathbb{R}$ <u>convex</u> iff $\forall x, y \in (a,b)$ $\lambda \in [0,1]$

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y). \quad (1)$$



Lemma 1.4 : If $f(a,b) \to \mathbb{R}$ diff^{ble} $f'(x)$ non-decreasing on (a,b) then f is convex on (a,b)

Proof: Let $x, y \in (a,b)$, $\lambda \in [0,1]$, $x<y$ If $z = \lambda x + (1-\lambda) y$ apply Mean-Value theorem there exist $\xi_1 \in (x,z)$, $\xi_2 \in (z,y)$ st

$$\frac{f(z)-f(x)}{z-x} = f'(\xi_1) \quad , \quad \frac{f(y)-f(z)}{y-z} = f'(\xi_2)$$

... Rearrange to give ① using $f'(\xi_1) \leq f'(\xi_2)$.

Lemma 1.5 : $s \geq 1$, $\varphi_s : \mathbb{R} \to \mathbb{R}$, $\varphi_s(x) = \binom{x}{s}$, then $\varphi_s(x)$ is convex.

Proof : By induction on s, show $\varphi_s'(x), \varphi_s''(x) \geq 0$ for $x \in (s-1, \infty)$. $\varphi_1'(x), \varphi_1''(x) \geq 0$

$\therefore$ True for $s=1$ $\varphi_s(x) = \dfrac{x(x-1)\ldots(x-s+1)}{s!}$

Fact $s\varphi_s(x) = (x-s+1)\varphi_{s-1}(x)$

Differentiate $s\varphi_s'(x) = \varphi_{s-1}(x) + (x-s+1)\varphi_{s-1}'(x) \geq 0$ (by induction step on $s-1$)

Similarly for $\varphi_s''(x)$.

$$s\varphi_s''(x) = 2\varphi_{s-1}'(x) + (x-s+1)\varphi_{s-1}''(x) \geq 0.$$

$\therefore \varphi_s'(x), \varphi_s''(x) \geq 0 \Rightarrow$ (by lemma 1.4), $\varphi_s(x)$ is convex $\square$

Thm 1.6 (Jensen's Inequality): If $\varphi : (a, \infty) \to \mathbb{R}$ is convex, $x_1, \dots, x_n > a$, $\lambda_1, \dots, \lambda_n \in [0, 1]$, $\sum_{i=1}^n \lambda_i = 1$ then $\varphi(\sum_{i=1}^n \lambda_i x_i) \leq \sum_{i=1}^n \lambda_i \varphi(x_i)$

Proof: True for $n=1$, $n=2$ (by induction).

Now suppose $n \geq 3$, assume $\lambda_{n-1} + \lambda_n > 0$

$$y_i = \begin{cases} x_i & 1 \leq i \leq n-2 \\ \dfrac{\lambda_{n-1} x_{n-1} + \lambda_n x_n}{\lambda_{n-1} + \lambda_n} & i = n-1. \end{cases}$$

$y_1, \dots, y_{n-1} > a$, $\mu_1, \dots, \mu_{n-1} \in [0, 1]$, $\sum_{i=1}^{n-1} \mu_i = 1$
$\therefore$ Apply induction hypothesis for $n-1$.

$\Rightarrow \varphi(\sum_{i=1}^{n-1} \mu_i y_i) \leq \sum_{i=1}^{n-1} \mu_i \varphi(y_i)$

$$\Rightarrow \varphi(\sum_{i=1}^n \lambda_i x_i) \leq \sum_{i=1}^{n-2} \lambda_i \varphi(x_i) + (\lambda_{n-1} + \lambda_n)\varphi\left(\frac{\lambda_{n-1} x_{n-1} + \lambda_n x_n}{\lambda_{n-1} - \lambda_n}\right)$$

Convexity $\Rightarrow$ result.

$\square$

Corollary 1.7: Let $s \geq 1$ be an integer
$\lambda_1, \dots, \lambda_n \in [0,1]$ with $\sum_{i=1}^n \lambda_i = 1$ and $x_1, \dots, x_n \geq 0$

(Simple Cauchy-Schwarz): $\frac{1}{n}\left(\sum_{i=1}^n x_i\right)^2 \leq \sum_{i=1}^n x_i^2$

(Bin Coeff Convexity) $\binom{\frac{1}{n}\sum_{i=1}^{n} x_i}{s} \leq \frac{1}{n}\sum_{i=1}^{n}\binom{x_i}{s}$

Proof: Directly from Th^{m} 1.6 by convexity of $f(x) = x^2$ and $f(x) = \binom{x}{s}$

$\square$

Lemma 1.8: $\frac{(n-s+1)^s}{s!} \leq \binom{n}{s} \leq \frac{n^s}{s!}$

$\frac{n(n-1)\ldots(n-s+1)}{s!}$

Graphs

vertices edges

Def: A graph is a pair $G = (V, E)$ of sets, with $E \subseteq \binom{V}{2}$. The elements of V are vertices and the elements of E are edges.

We denote the vertices and edges of a graph G by $V(G)$ and $E(G)$ respectively

Def: The order of a graph is the number of vertices $|V|$. The size of a graph is the number of edges $|E|$.

Def: If G is a graph and $v \in V(G)$ then the neighbourhood (or nbhd) of v is

$$\Gamma(v) = \{u \in V(G) : uv \in E(G)\}$$

The degree of a vertex $v \in V$ is the size of its neighbourhood $d(v) = |\Gamma(v)|$.

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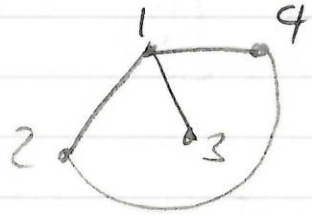

planar graph

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Eg: $G([4], \{12, 13, 14\})$

order is 4

size 3

$\Gamma(1) = \{2, 3, 4\}$.

$d(1) = 3$.

Lemma 1.9 (Handsake lemma): For any graph $G = (V, E)$.

$$\sum_{v \in V} d(v) = 2|E|. \qquad (*)$$

Proof: Each edge has two endpoints so to count twice in the LHS of $(*)$

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Last time : $G=(V, E)$, $\sum_{v\in V} d(v) = 2|E|$.

a graph.

Lemma 1.10 : In any graph the number of vertices off odd degree is even.

Proof : $G=(V,E)$, $V = A \dot{\cup} B$.

disjoint union.

$A : \{v : d(v) \text{ odd}\}$, $B=\{v: d(v) \text{ even}\}$

$\sum_{v\in V} d(v) = 2|E|$ is even (via Handshake Lemma)

$\sum_{v\in B} d(v)$ is even since it is a sum of even numbers.

Hence $\sum_{v\in A} d(v) = 2|E| - \sum_{v\in B} d(v)$ is even.

Hence $|A|$ is even $\Box$

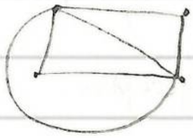
1.4 Special Graphs.

We define $[n] = \{1, 2, \dots, n\}$.

1) The complete graph ~~of order~~ $n \geq 2$: K_n

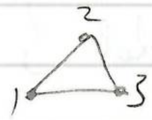
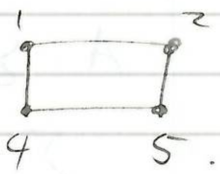
$$V = [n], \quad E = \binom{[n]}{2}$$

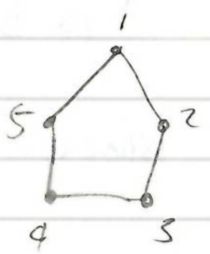
Eg: K_n; K_1:  , K_3: 

K_4:  K_5: 

2) The cycle of length $n \geq 3$: C_n.

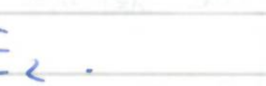
$V = [n]$, $E = \{\{i, i+1\} : i = 1, 2, \dots n-1\} \cup \{\{1, n\}\}$

Eg: C_n; C_3:  , C_4: 

C_5: 

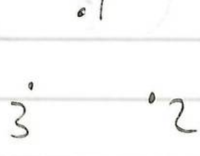
3) The path of length n: P_n (n edges and $n+1$ vertices).

$V = \{0, 1, 2, \dots, n\}$, $E = \{\{i-1, i\} : i \in [n]\}$.

Eg: P_n: P_2:  , P_3: 

4) The empty graph of order n: E_n.

$V = [n]$, $E = \emptyset$.

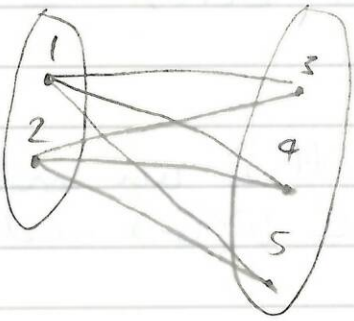
Eg: E_3: 

5) The <u>complete bipartite graph</u> with classes a and b is $K_{a,b}$:

$$V = \{1, 2, \ldots, a\} \cup \{a+1, a+2, \ldots, a+b\}$$

$$E = \{\{i,j\} : 1 \leq i \leq a,\ a+1 \leq j \leq a+b\}.$$

Eg: $K_{a,b}$; $K_{2,3}$:



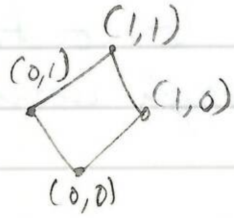
6) The <u>(discrete) hypercube</u> of dimension n: Q_n.

$$V(Q_n) = \{0,1\}^n$$

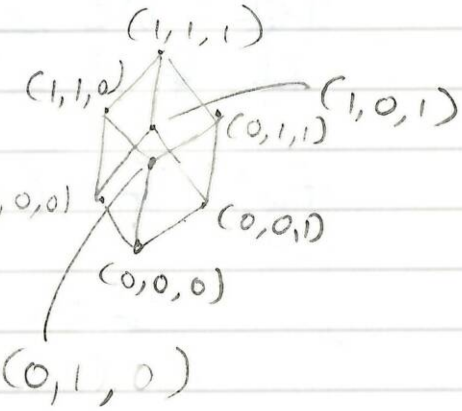
where $\{0,1\}^n = \{(x_1, \ldots, x_n) : x_i \in \{0,1\}\ \forall i\}$

$E(Q_n) = \{xy : x$ and y differ in exactly one coordinate$\}$.

Eg: Q_n; Q_2:



Q_3:



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Note : $\mathcal{P}([n]) = \{A : A \subseteq [n]\} \xleftrightarrow{\text{bijection}} \{0,1\}^n$

$$A \to \{x_1, \dots, x_n\} \quad x_i = 1 \text{ iff } i \in A.$$

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1.5 Subgraphs.

Def : If G and H are graphs satisfying $V(H) \subset V(G)$ and $E(H) \subset E(G)$ then H is a subgraph of G.

Def : We say that H is an induced subgraph of G if $V(H) \subset V(G)$ and $E(H) = E(G) \cap \binom{V(H)}{2}$

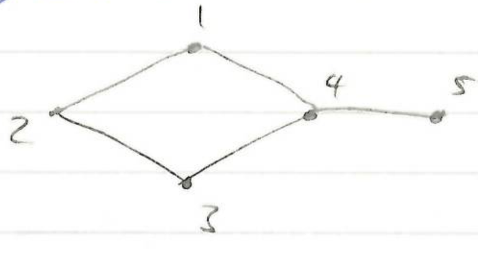
Def : If $G = (V, E)$ is a graph and $A \subset V$ then $G[A]$ is the subgraph induced by A : its vertex set is $V(G[A]) = A$ and the edge set is $E(G[A]) = \binom{A}{2} \cap E(G)$.

Def : Graphs G and H are isomorphic iff there is a bijection $f : V(G) \to V(H)$ such that $vw \in E(G) \iff f(v)f(w) \in E(H)$.

Def : We say that G contains a copy of H if G has a subgraph isomorphic to H.

Eg: $G = (V, E)$.

G :

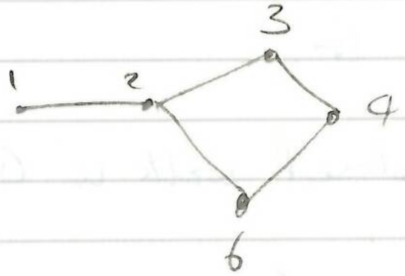


H_1: 

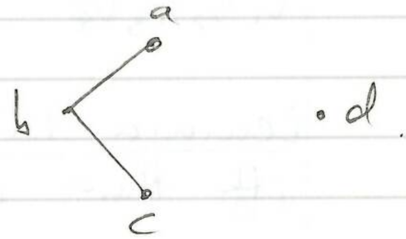
is a subgraph of G, not induced.

H_2: 

is an induced subgraph.

H_3: 

H_3 and G are isomorphic.

G contains a copy of H = 

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1.6: Components + connectedness.

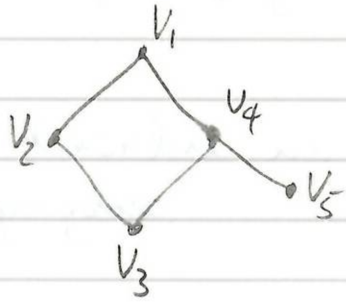
Def: A *path* in a graph G is a subgraph to P_t for some $t \geq 0$.

Def: An *x-y path* in G is simply a path that starts at x and ends at y.

Def: A *walk* in G is a sequences of vertices (not necessarily distinct) $v_0 v_1 \dots v_t$ such that $v_{i-1} v_i \in E$ for all $i \in [t]$. The walk is *closed* if $v_0 = v_t$.

Def: A walk in edge is used more than once (but vertices may be revisited) is called a *tour*.

Eg:

$G =$ 

$v_1 v_4 v_5$ is a path in G.

$v_1 v_4 v_5 v_4 v_3 v_2 v_1$ is a closed walk in G.

$v_1 v_2 v_3 v_4$ is a tour in G.

Lemma 1.11: There is an $x-y$ path in G iff there is an $x-y$ walk in G

Proof: $(\Rightarrow)$ A path is a walk.

$(\Leftarrow)$ Take a shorest walk from x to y. If any vertex is revisited we could shorten this walk. Hence it is a path $\square$

Lemma 1.12: Define a relation $\sim$ on $V(G)$ by $v \sim w$ iff there is a walk from v to w in G. This is an equivalence relation.

Proof: Reflexive $v \sim v$ take walk v.
Symmetric $v \sim w \Rightarrow \exists$ walk v to w, reverse it.
Transitivity $v \sim w$ and $w \sim z$ then concatenate the $v \sim w$ and $w \sim z$ walks to give a $v \sim z$ walk

$\square$

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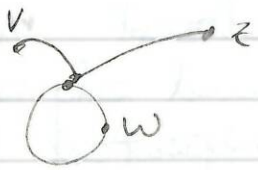
Note; this lemma does not work for a path, take:



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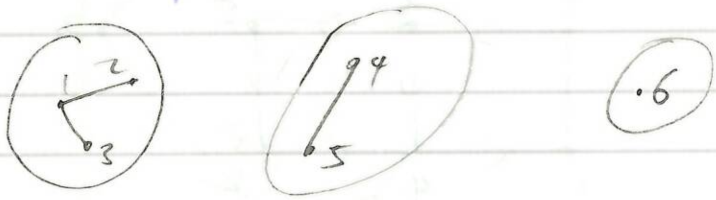
Def: $\sim$ induces a partition of $V(G) = V_1 \cup V_2 \cup \dots \cup V_k$ each V_i is a component

Eg! G:

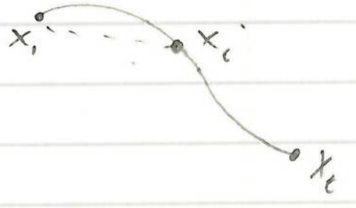


Def: G is connected iff there is a single component.

Lemma 1.13: $P = x_1 x_2 \dots x_t$ is a path in G. If P is the shortest $x_1 - x_t$ path in G then $x_1 \dots x_i$ and $x_i \dots x_t$ are shortest $x_1 - x_i$ and $x_i - x_t$ paths in G for each $1 < i < t$.

Proof: If not; could shorten P

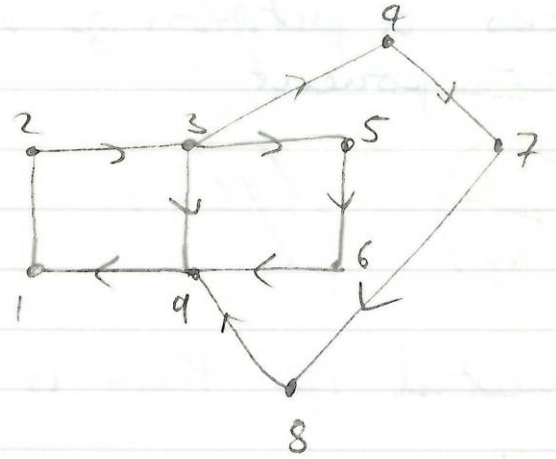
$\square$



1.7 Euler circuits.

<u>Def</u>: An <u>Euler circuit</u> in a graph G is a closed $v_0 v_1 \cdots v_t v_0$ containing all vertices and edges of G, the vertices may be repeated but each edge is used exactly once.

<u>Eg</u>:



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1.7. Euler Circuits

start = end   no repeated edges

An Euler Circuit in a graph is a closed tour containing all vertices and edges of G.

Th^m 1.14. A graph G has Euler circuit iff it is connected and all vertices have even degree.

Proof ($\Rightarrow$) Assume G has an Euler circuit $T = v_0 v_1 \dots v_k$. $v_0 = v_k$ So G is certainly connected. Follow T counting the contribution to the degree of each vertex we visit. Add 2 each time (except at start + end). Hence all degree are even.

($\Leftarrow$) So suppose G is connected and all vertices have even degree. Take a longest tour $T = v_0 v_1 \dots v_k$ in G.

Claim: $v_0 = v_k$, if not let $j = \#\{i : v_i = v_k\}$ then if $v_0 \neq v_k$ then we have used $2j - 2 + 1 = 2j - 1$ edges incident to v_k.

$\therefore$ An unused edge $v_k v^* \Rightarrow T' = v_0 \dots v_k v^*$ is a longer tour $\#$ Hence $v_0 = v_k$

If there is unused egde say $e = uv$, there two cases to consider.

Case ① u or v is in T, say $v = v_i$ $\therefore T' = u v_i \dots v_k \dots$ $\dots v_0 v_1 \dots v_{i-1}$ is a longer tour. $\#$

Case ② $u, v \notin T$. G is connected so $\exists$ a v_0-u-path in G. Consider the first edge in this path that leaves T but this gives us case ① again ※

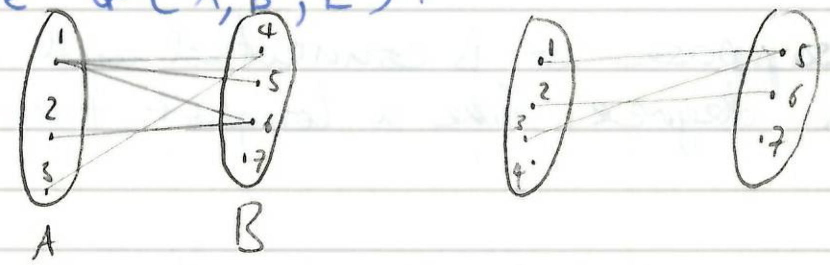
All vertices have degree ≥ 2 so they are visited by T.

$\square$

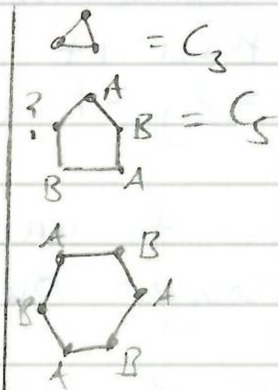
1.8 Bipartite Graphs

Def: A graph G is bipartite if $V(G) = A \,\dot\cup\, B$ and $E(G) \subseteq \{ab : a \in A, b \in B\}$. We say that A, B is a bipartition and sometimes write $G = (A, B; E)$ to emphasise this.

Eg: $G = (V, E)$, $V(G) = A \,\dot\cup\, B$, $E(G) \subseteq \{ab : a \in A, b \in B\}$
i.e. $G = (A, B; E)$.



Th^{m} 1.15: A graph is bipartite iff it contains no odd cycles



Proof: ($\Rightarrow$) Suppose G is bipartite with bipartition $V = A \,\dot\cup\, B$. If $C = v_1 \dots v_\ell$ is a cycle in G and w.l.o.g $v_1 \in A$ then $v_3, v_5, \dots \in A$, $v_2, v_4, v_6, \dots \in B$. Hence we must ℓ is even.

($\Leftarrow$) Suppose $G = (V, E)$ is connected (otherwise repeat this arguement for each connected component)

For $x, y \in V$, let $d(x,y)$ = length of a shortest $x-y$ path.

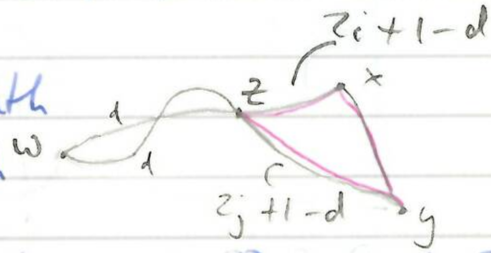
Fix a vertex $w \in V$

Define $A = \{ v : d(v,w) \text{ is odd} \}$

$B = \{ v : d(v,w) \text{ is even} \}$

Note $V(G) = A \dot{\cup} B$. Need to check A and B do not contain any edges. Suppose there is an edge xy inside A (i.e. $x, y \in A$).

Let P_{wx} be a shortest $w-x$-path
" P_{wy} " " " $w-y$-path



Let z be the last common vertex of P_{wx} and P_{wy}

Then the part of P_{wx} from w to z is a shortest $w-z$ path
" " " " P_{wy} " " " " " " $w-z$ path

Now suppose $d(w,x) = 2i+1$, $d(w,x) = 2j+1$, i, j integers. Then the cycle that follows P_{wx} from z to x then xy, then P_{wy} from y to z has length $= 2i+1-d+1+2j+1-d$
$= 2(i+j+1-d)+1$ is odd ※.

Hence G is bipartite

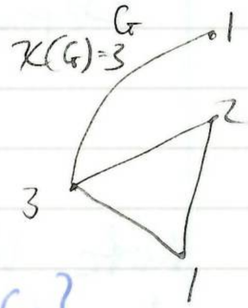
$\square$

1.9 Graph colouring.

A set $A \subset V$ is <u>independent</u> iff it contains

$c : A \quad [k] \quad vw \in E \Rightarrow c(v) \neq c(w)$

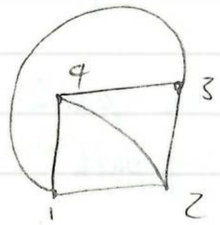
k-colourable $\equiv$ k-partite
2-colourable $\equiv$ bipartite



Chromatic number of G

$$\chi(G) = \min\{k : \exists\ k\text{-clouring of } G\}$$

Eg:



$\chi(K_t) = t$.
$\chi(C_{2t}) = 2$
$\chi(C_{2t-1}) = 3$

<u>Def</u>: If G is a graph then $A \subseteq V(G)$ is an <u>indepenant set</u> iff there are no edges with both endpoints in A

<u>Def</u>: For $k \in \mathbb{N}$ a <u>k-colouring</u> graph G is $c : V(G) \to [k]$ such that if $vw \in E$ then $c(v) \neq c(w)$

<u>Def</u>: A graph G is said to be <u>k-colourable</u> iff it has a k-colouring. Note that a graph is bipartite iff it is 2-colourable.

<u>Def</u>: A graph G is said to be <u>k-partite</u> iff there is a partition $V(G) = V_1 \cup V_2 \cup \dots \cup V_k$, of $V(G)$ into independent sets. Note that a graph is k-partite iff it is k-colourable.

<u>Def</u>: We define the <u>chromatic number</u> of G to be

$$\chi(G) = \min\{k \geq 1 : G \text{ is } k\text{-colourable}\}.$$

If H is a subgraph of G then $\chi(H) \leq \chi(G)$.

<u>Def</u>: We define the <u>maximum degree</u> of G to be

$$\Delta(G) = \max\{d(v) : v \in V(G)\}.$$

Th^m 1.6 : If G is a graph then $\chi(G) \leq \Delta(G)+1$
$(\Delta(G) = \max\{d(v) : v \in V(G)\})$

Proof : Let $V = \{v_1, \dots, v_n\}$. Let $k = \Delta(G)+1$
Define a k-colouring $c : V(G) \to [k]$ as follows
$c(v_1) = 1$. If $v_1, \dots, v_{i-1}$ have been coloured.

Let $C = \{c \in [k] : \exists\, j \in [i-1]$ st $v_j \in \Gamma(v_i)$ and $c(v_j) = c\}$

Define $c(v_i) = \min [k] \setminus C$. This is well-defined
since $|C| \leq d(v_i) \leq \Delta(G) = k-1$.
So $[k] \setminus C = \emptyset$ ⊠

"Greedy Algorthim".

1.10 Large girth + Chromatic number.

Def: If G is a graph containing cycles then girth of G is the length of the shortest cycle. We denote this by $g(G)$. If G contains no cycles then we denote $g(G) = \infty$.

Thm 1.7. (Erdö's) For $k, \ell \geq 3$ $\exists\, G$ a graph with $\chi(G) \geq k$, $g(G) \geq \ell$.

Def : For a graph G we define the independence number of G to be :

$$\alpha(G) = \max\{|A| : A \subset V(G) \text{ is indepenant}\}$$

<u>Lemma 1.18</u> For any graph G, $\chi(G) \geq n/\alpha(G)$. $n = |V(G)|$.

<u>Proof</u>: If $c: V(G) \to [k]$ is a k-colouring of G, then each colour class $c^{-1}(i) = \{v \in V(G) : c(v) = i\}$ is an independent set, so $|c^{-1}(i)| \leq \alpha(G)$. $(*)$

But $V(G) = c^{-1}(1) \cup c^{-1}(2) \cup \ldots \cup c^{-1}(k)$ so $\sum_{i=1}^{k} |c^{-1}(i)| = n$.

Hence $(*) \Rightarrow k\alpha(G) \geq n \Rightarrow k \geq n/\alpha(G)$.

Thus $\chi(G) \geq n/\alpha(G)$ $\square$

—/—

We will give a probabilistic of Theorem 1.7, we will only be interested in the simplested type of probability space: finite (and discrete).

A probability space is a pair $(\Omega, \mathbb{P})$ where Ω is a finite set of outcomes (e.g $\{H, T\}$ or $\{1, 2, \ldots, 6\}$) and $\mathbb{P}: \Omega \to [0,1]$ st $\sum_{y \in \Omega} \mathbb{P}[y] = 1$. For $A \subset \Omega$ we define $\mathbb{P}[A] = \sum_{y \in A} \mathbb{P}[y]$.

A <u>random variable</u> is a function $X: \Omega \to \mathbb{R}$. For example, if our probability space is $(\{1, 2, \ldots, 6\}, \mathbb{P}_U)$, where $\mathbb{P}_U(y) = 1/6$ for all $y \in [6]$ then we could have

$$X_1(y) = \begin{cases} 1, & y = 1, 3, 5 \\ 0, & \text{otherwise.} \end{cases}$$

$$X_2(y) = \begin{cases} 1, & y \geq 4 \\ 0, & \text{otherwise.} \end{cases}$$

Def: The expectation of a random variable is simply its average value. If $O_x = \{X(y) \mid y \in \Omega\}$ is the set of values taken by X then

$$E[X] = \sum_{z \in O_x} z P(X = z)$$

Eg: A die (Ω, P_u), $\Omega = \{1, 2, 3, 4, 5, 6\}$.
$P_u(i) = 1/6 \quad 1 \leq i \leq 6$.

$$X_1(y) = \begin{cases} 1, & y \in \{1, 3, 5\} \\ 0, & o/w \end{cases}$$

$$X_2(y) = \begin{cases} 1, & y \geq 4 \\ 0, & o/w. \end{cases}$$

$$E[x] = \sum_{z \in O_x} z P(X = z), \quad O_x = \{X(y) \mid y \in \Omega\}$$

Lemma 1.19 (Linearity of Expectation). If $X_1, \ldots, X_n$ are random variables then:

$$E\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n E[X_i]$$

Proof: Follows from def^n of expectation.

Thm 1.20. If G has e edges then G contains a bipartite subgraph with at least $\lceil e/2 \rceil$ edges

Proof: Consider a random bipartition of $V = A \dot{\cup} B$
For each vertex $v \in V$ flip an indepenant fair coin, if Heads then put v in A, if Tails then put v in B.

For an edge $uv \in E$ let $X_{uv} = \begin{cases} 1, & uv \text{ goes from } A \text{ to } B \\ 0, & \text{o/w}. \end{cases}$

L of E

Let $X = \sum_{uv \in E(G)} X_{uv}$, then $E[X] = \sum_{uv \in E(G)} E[X_{uv}]$

$= \sum_{uv \in E(G)} P(uv \text{ goes from } A \text{ to } B)$

$P(uv \text{ goes from } A \text{ to } B) = 1/2$.

Hence $E[X] = \sum_{uv \in E(G)} \frac{1}{2} = \frac{e}{2}$.

Thus $\exists$ a bipartition $V = A \dot{\cup} B$ with a least $e/2$ edges between A and B. Hence (since the number of edges is an integer) at least $\lceil e/2 \rceil$ edges between A and B. $\square$

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Thm 1.17: For all $k, \ell \geq 3$ $\exists$ graph G with $\chi(G) \geq k$ and $g(G) \geq \ell$.

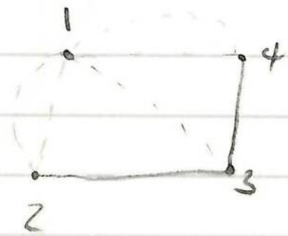
For a graph G we define the independence number of G to be $\alpha(G) = \max\{|A| : A \subset V(G) \text{ is an indepenant set}\}$

Lemma 1.18: $\chi(G) \geq \dfrac{|V(G)|}{\alpha(G)}$

$\mathcal{G}(n, p)$ = random graphs on $[n]$ (Erdős - Renyi)

$V(G) = [n]$, For each ij $(1 \leq i < j \leq n)$ flip an indepenant coin with prob(Heads) $= p$. Insert the edge ij in $E(G)$ iff the coin is Heads.

$n = 4$. $H =$



$G = \in \mathcal{G}(4, p)$

$\mathbb{P}(G = H) = p^2(1-p)^4$

Lemma 1.21: (Markov): If X is non-negative, $\lambda > 0$

$$\mathbb{P}[X \geq \lambda] \leq \frac{\mathbb{E}[X]}{\lambda}$$

Proof: Let X take values of O_X

$$\mathbb{E}[X] = \sum_{y \in O_X} y \mathbb{P}[X = y] \geq \sum_{y \geq \lambda} \lambda \mathbb{P}[X = y]$$

$$= \lambda \sum_{y \geq \lambda} \mathbb{P}[X = y]$$

$$= \lambda \mathbb{P}[X \geq y].$$

$\square$

The probability space we will consider is called $G(n,p)$: the space of <u>Erdős - Renyi random graphs</u>. The underlying set of outcomes is:

$$\Omega = \left\{ G \mid V(G) = [n], E(G) = \binom{[n]}{2} \right\}$$

For a graph $H \in \Omega$ the probability, $\mathbb{P}[H]$, is simply the probability that the following random process produces the graph H.

<u>Generating a random in $G(n,p)$:</u>
Start with the empty graph E_n. For each pair of vertices $ij \in \binom{[n]}{2}$ toss a coin C_{ij} that has probability p of being Heads. If the coin Heads then insert the edge ij otherwise do not insert the edge ij. Repeat with indepenant coins for each pair of vertices.

Lemma 1.22 Let $G \in \mathcal{G}(n,p)$ and $X_t =$ # t-cycles in G. then

$$E[X_t] = \left(\frac{n(n-1)\dots(n-t+1)}{2t} \right) p^t.$$

Proof: Fix t-cycle C, let $Y_C = \begin{cases} 1, & C \text{ is in } G \\ 0, & \text{o/w.} \end{cases}$

$$X_t = \sum_{C \text{ a } t\text{-cycle}} Y_C \Rightarrow E[X_t] = \sum_{C \text{ a } t\text{-cycle}} E[X]$$

$$= \sum_{C \text{ a } t\text{-cycle}} P[C \text{ in } G]$$

But $P[C \text{ is in } G] = p^t$ for any t-cycle C.
$E[X_t] = p^t \times$ # possible t-cycle in G.
Any t-tuple of distinct vertices $v_1, \dots, v_t$ gives rises to t-cycle.
such t-tuples $= n(n-1)(n-t+1)$.

$$\underbrace{\begin{array}{ll} v_1 \dots v_t & v_t v_{t-1} \dots v_1 \\ v_2 \dots v_t v_1 & v_{t-1} v_{t-2} \dots v_t \\ \vdots & \vdots \\ v_t v_1 \dots v_{t-1} & v_1 v_t v_{t-1} \dots v_2 \end{array}}$$

$2t$ different t-tuples gives the same t-cycle.

v_t v_1 v_2
$\dots$

Hence # possible t-cycle $= \frac{n(n-1)\ldots(n-t+1)}{2t}$.

(1) $\Rightarrow$ result. $\square$.

<u>Proof</u> of <u>Theorem 1.17</u> Let k, ℓ be given. Call a cycle <u>short</u> if it has length $\leq \ell$.

<u>Claim</u>: If $\exists$ a graph G with n vertices and at most $n/2$ short cycles with $\alpha(G) < n/2k$ then $\exists$ G' with $\chi(G') > k$ and $g(G') > \ell$.

<u>Proof</u>: Remove a vertex from each short cycle to give G'.

$|V(G')| \geq n/2$, $g(G') > \ell$, $\alpha(G') \leq \alpha(G) < n/2k$

Thus $\chi(G') \underset{\text{lemma 1.8.}}{\geq} \frac{|V(G')|}{\alpha(G')}$

So $\chi(G') > \frac{n}{2} \cdot \frac{2k}{n} = k$ $\square$

Now need to need $\exists$ G with $V(G) = n$, at most $n/2$ short cycles and $\alpha(G) < n/2k$.

Let $\underset{(1)}{n \geq 36\ell^2}$, $\frac{n^{1/2}}{8\log n} \geq 2k$.

Let $p = \frac{1}{n^{1-1/2l}}$

Let $G \in \mathcal{G}(n,p)$. Let X_t be the number of t-cycles in G.

Lemma 1.22 $\Rightarrow E[X_t] = \frac{n(n-1)\dots(n-t+1)}{2t} p^t$

$X = \sum_{t=3}^{l} X_t$ be the number of short cycles in G.

$$E[X] \overset{\text{L of E}}{=} \sum_{t=3}^{l} \frac{n(n-1)\dots(n-t+1)}{2t} p^t.$$

$$\leq \sum_{t=3}^{l} \frac{n^t}{2t\, n^{t(1-1/2l)}} \leq ln^{1/2} \leq \frac{n}{6} \text{ by (1)}$$

$$P(X \geq n/2) \underset{\text{Markov}}{\leq} \frac{E[X]}{(n/2)} \leq \frac{1}{3}.$$

So we have $P(G \text{ has} \leq n/2 \text{ short cycles}) \geq 2/3$.

Next: need to show $P(\alpha(G) \geq n/2k) \leq 1/3$.
Because then $P(\alpha(G) < n/2k) \geq 2/3$
And then $P(G \text{ satisfies conditions of the Claim}) \geq 1/3$

$\therefore \exists$ a graph G with those properties. $\square$

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Proof of Thm 1.17:

Last time: Need to show $\exists G$ with $\leq n/2$ short cycles and $\alpha(G) \leq n/2k$.

Let $G \in \mathcal{G}(n,p)$. $n \geq 36\ell^2$, $\frac{n^{1/2\ell}}{8\log n} \geq 2k$.

$$p = \frac{1}{n^{1-1/2\ell}}$$

A = "G has $\geq n/2$ short cycles"

B = "$\alpha(G) \geq n/2k$"

$\mathbb{P}(A) \leq 1/3$. If we show $\boxed{\mathbb{P}(B) \leq 1/3}$ then $\mathbb{P}(\text{not } A \text{ and not } B) \geq 1/3 > 0$.

Let $s = (4/p)\log n + 1$

$$n/2k \overset{②}{\geq} \frac{8n\log n}{n^{1/2\ell}} = \frac{8}{p}\log n \geq s$$

$$\mathbb{P}(B) \leq \mathbb{P}(\alpha(G) \geq s) = \mathbb{P}(\exists \text{ an ind. set of size } s)$$

For a set $T \subset V(G)$, $|T| = s$ let E_T = "T is an ind. set"

If $E_1, \dots, E_t$ are events

$$\mathbb{P}\left(\bigcup_{i=1}^{t} E_i\right) \leq \sum_{i=1}^{t} \mathbb{P}(E_i)$$

$$\mathbb{P}(B) = \mathbb{P}\left(\bigcup_{T\in\binom{V}{s}} E_T\right) \leq \sum_{T\in\binom{V}{s}} \mathbb{P}(E_T) = \binom{n}{s}(1-p)^{\binom{s}{2}}$$

$$\leq n^s e^{-p\binom{s}{2}}$$

$$\mathbb{P}(B) \leq n^s e^{-p\binom{s}{2}} = \left(n e^{-p\left(\frac{s-1}{2}\right)}\right)^s$$

$$= \left(n e^{-2\log n}\right)^s = \frac{1}{n^s} \leq \frac{1}{3} \quad \text{for } n \text{ large}$$

$\square$

—/—

$\alpha(G) \geq s \iff$ the max size of an ind set in G is $\geq s$.

—/—

2. Extremal Graph Theory

2.1 Hamilton Cycles: A Ham cycle in a graph is a cycle containing all vertices of G (exactly once)

$$\delta(G) = \min\{d(v) : v \in V(G)\}.$$

minimum degree of G

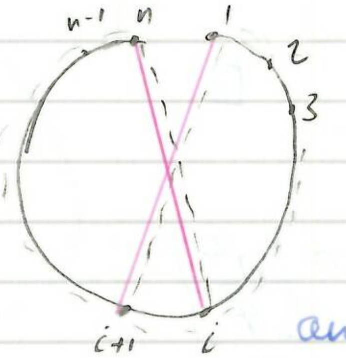
u, v adjacent iff $uv \in E(G)$ otherwise they are non-adjacent.

Thm 2.1 (Dirac 1952): If G has $n \geq 3$ vertices and $\delta(G) \geq n/2$ then G contains a Ham. cycle.

Thm 2.2. (Ore 1960) If G has $n \geq 3$ vertices and every pair of non-adj. vertices u, v satisfy $d(u) + d(v) \geq n$ then G has a Ham. cycle.

Proof: (By contradiction) Assume G satisfies the conditions of Thm 2.2 but does not contain a Ham. cycle. If there is an edge that can be added to G without creating a Ham. cycle then do so, repeat until can't add any more edges.

Now know that G contains a Ham cycle with one edge removed.



So wlog let $V(G) = [n]$ and $12, 23, 34, \ldots, n\!-\!1n \in E(G)$, $1n \notin E(G)$. Note from any $i = 2, \ldots, n-1$ we cannot have both $1(i+1) \in E(G)$ and $in \in E(G)$, because otherwise we would have Ham. cycle: $1(i+1)(i+2)\ldots n i (i+1)(i+2)\ldots 2$.

Consider $d(1) + d(n)$ since we have at most no edge from each pair.

$$\left.\begin{array}{l} 13\ ,\ 2n \\ 14\ ,\ 3n \\ \vdots \\ 1n\!-\!1\ ,\ n\!-\!2n \end{array}\right\} \text{gives} \leq n-3 \text{ edges.}$$

Hence (since $12 \in E(G)$ and $n\!-\!1n \in E(G)$) we have $1n \notin E(G)$.

$$d(1) + d(n) \leq n - 3 + 2 = n + 1 \qquad ⨳$$

2.2. Forbidden subgraphs.

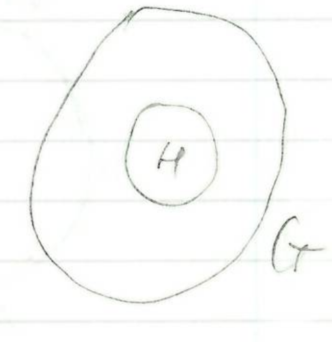
Given G and H, we say G is H-free if G has no subgraph isomorphic to H.

$$ex(n, H) = \max\{|E(G)| : G(V,E),\ |V| = n,\ G \text{ is } H\text{-free}\}.$$

<u>Lemma 2.3</u> If G and H are graphs $\chi(H) > \chi(G)$ then G is H-free.

<u>Proof</u>: If G contains H then any colouring of G gives a colouring of H hence $\chi(H) \leq \chi(G)$.

$\square$



<u>Thm 2.4</u> (Mantel 1907)
If $n \geq 1$ then $ex(n, K_3) = \left\lfloor \frac{n^2}{4} \right\rfloor$

floor ceiling

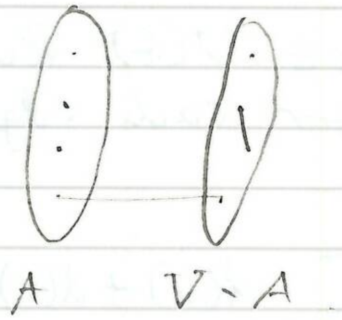
<u>Proof</u>: Take $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ the complete bipartite graph with vertex classes of size $\lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n}{2} \rceil$

This is K_3-free (by Lemma 2.3) and has $\lfloor n^2/4 \rfloor$ edges hence $ex(n, K_3) \geq \lfloor n^2/4 \rfloor$

Now let G be K_3-free of order n.

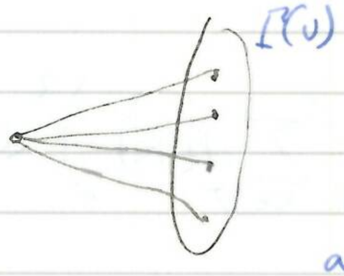
Need to show that $|E(G)| \leq \lfloor n^2/4 \rfloor$

Let $A \subseteq V(G)$ be a largest independent set in G. $|A| = a$.



Consider $\sum_{v \in V \setminus A} d(v) \geq |E(G)|$

Since we count every at least once (infact we count those in $V \setminus A$ twice).



Since G is K_3-free, $\Gamma(v)$ is an indepenant set for each $v \in V$. Hence $d(v) = |\Gamma(v)| \leq a$. Since no indepenent set is larger than a.

$$|E(G)| \leqslant \sum_{v \in V \setminus A} d(v) \leqslant (n-a)a \leqslant \frac{n^2}{4} \text{ (by basic calculus)}$$

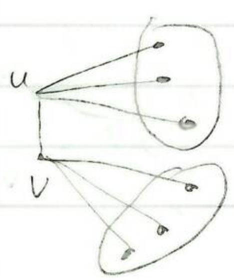
$|V \setminus A| = n - a$.

So $|E(G)| \leqslant n^2/4$. Since $|E(G)|$ is an integer we have $|E(G)| \leqslant \lfloor n^2/4 \rfloor$. Thus $ex(n, K_3) \leqslant \lfloor n^2/4 \rfloor$ $\square$

<u>Proof [2nd]</u>: $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ is K_3-free $\Rightarrow ex(n, K_3) \geqslant \lfloor n^2/4 \rfloor$.

Let G have order n, be K_3-free.

Let $|E(G)| = e$. If $uv \in E(G)$ then $\Gamma(u) \cap \Gamma(v) = \emptyset$ Since G is K_3-free. So $d(u) + d(v) \leqslant n - 2 + 2 = n$.



So $\sum_{uv \in E(G)} (d(u) + d(v)) \leqslant en$.

Note, if we fix a vertex $x \in V(G)$ then "$d(x)$" occurs once in this sum for each edge containing x, i.e it occurs $d(x)$ times.

So $$\sum_{x \in V(G)} (d(x))^2 = \sum_{uv \in E(G)} d(u) + d(v) \leq en.$$

—/—

Know $\sum_{x \in V} d(x) = 2e$.

Cauchy-Schwarz: $$\frac{1}{n}\left(\sum_{x \in V} d(x)\right)^2 \leq \sum_{x \in V} (d(x))^2$$

So $$\frac{4e^2}{n} \leq en.$$

$$\Rightarrow e \leq n^2/4$$

$\square$.

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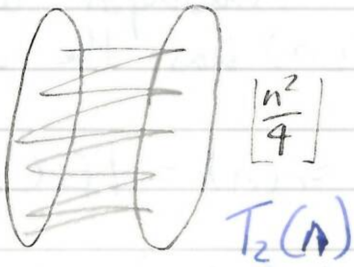
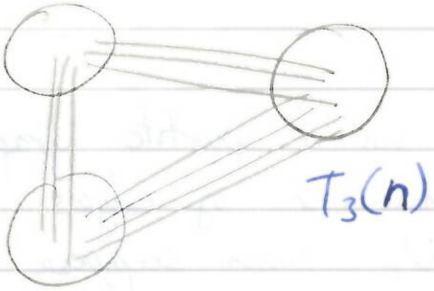
K_3 - free $\longrightarrow$



K_4 - free

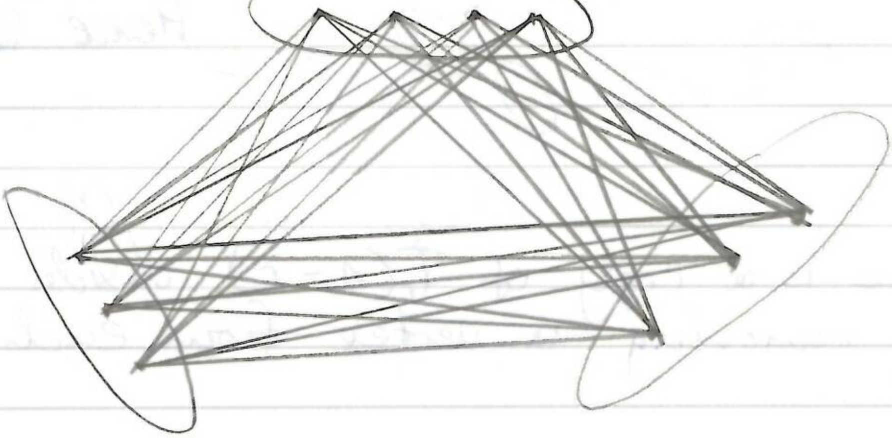


G is r-partite : $V(G) = V_1 \dot\cup V_2 \dot\cup \ldots \dot\cup V_r$

$$E(G) = \{vw : v \in V_i,\ w \in V_j\ i \neq j\}$$

Turán graph : $T_r(n)$ is the complete r-partite graph, of order n with vertex classes as equal possible $\equiv$ all vertex class sizes differ by at most one.

Eg $T_3(10)$



Lemma 2.5: Amongst all r-patite graphs with n vertices $T_r(n)$ has the most edges.

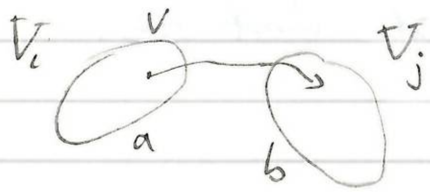
Moreover if $t_r(n) = |E(T_r(n))|$

then $t_r(n) = t_r(n-r) + (r-1)(n-r) + \binom{r}{2}$

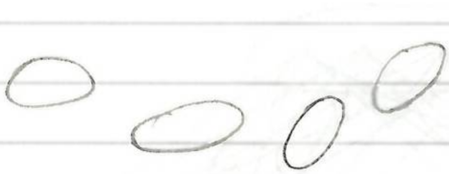
Proof: Take an r-partite graph G of order n, with maximum number of edges. Suppose vertex classes are $V_1, \ldots, V_r$. Can suppose G is complete r-partite.

If $G \neq T_r(n)$ then $\exists$ V_i, V_j vertex classes with $|V_i| = a$, $|V_j| = b$ and $a \geq b+2$.

Remove a vertex v from V_i and add a vertex to V_j, and take the complete r-patite graph on these new vertex classes.



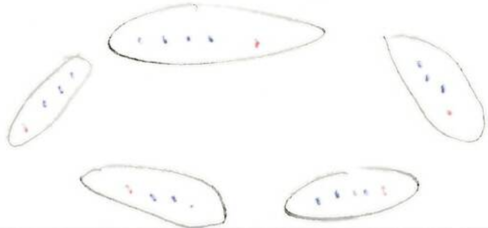
lose: $n-a$
Gain: $n-(b+1)$
Change: $(n-(b+1))-(n-a)$
$= a-b-1 \geq 1$



Hence $G = T_r(n)$

#

There is a copy of $T_r(n-r)$ inside $T_r(n)$ given by removing a vertex from each class.



Colour the r vertices in $T_r(n) \setminus T_r(n-r)$ red.
Colour the rest blue.
Number of blue-blue edges $= t_r(n-r)$
" " red-red " $= \binom{r}{2}$
" " blue-red " $= (n-r)(r-1)$

Since each blue vertex is joined to every red vertex except the one in 'the vertex' class

$\square$

<u>Th^{m} 2.6.</u> If $2 \leq r \leq n$ and G is K_{r+1} free, of order n with $ex(n, K_{r+1})$ edges then G is $T_r(n)$

<u>Proof</u>: (Induction on n). If $n \leq r$ then $ex(n, K_{r+1}) = \binom{n}{2}$ and $T_r(n) = K_n$ so result holds. So suppose $n \geq r+1$.

Let G have n vertices and $ex(n, K_{r+1})$ edges. Then by maximality of $|E(G)|$ there is a copy K of K_r $V(K) = \{v_1, \dots, v_r\}$. By our ind. hyp. $G - K$ has $t_r(n-r)$ edges and each $v \in V(G-K)$ has at most $r-1$ neighbours in $V(K)$

So $|E(G)| \leqslant \binom{r}{2} + t_r(n-r) + (n-r)(r-1)$

\# edges in K — \# edges in $G \setminus K$ — \# edges from $G \setminus K$ to K

So $|E(G)| \leqslant t_r(n)$ (by lemma 2.5)

By maximality of $|E(G)|$ must have equality above.

For equality to hold each $v \in V(G - K)$ must have exactly $r-1$ vertices neighbour in $V(K)$

For $1 \leq i \leq r$ let $W_i = \{ v \in V(G) : vv_i \notin E(G) \}$. Note $v_i \in W_i$ for each i, and $v_i \notin W_j$ for $i \neq j$. If $v \in V(G \setminus K)$ then v has exactly $r-1$ neighbours in $V(K)$ and hence there is a unique $1 \leq i \leq r$ such that $vv_i \notin E(G)$, hence $v \in W_i$ for some unique i. Thus $W_1 \dot\cup W_2 \dot\cup \dots \dot\cup W_r$ is a partition of $V(G)$.

omitted

If $u, v \in W_i$ and $uv \in E(G)$ then $u, v, v_1, v_2, \dots, \tilde{v}_i, v_{i+1}, \dots, v_r$ from a K_{r+1} $\#$

Hence G is an r-partite graph with vertex classes $W_1, \dots, W_r$. Then by lemma 2.5 and maximality of $E(G)$ we must have $G = T_r(n)$

$\square$

30/11/12

Homework due next wed 12pm.

Def: If $G = (V, E)$ is a graph then the complement of G, is $G^c = (V, \binom{V}{2} - E)$

Theorem 2.7 (Caro and Wei) If G is a graph of order n with vertex degree $d_1, \ldots, d_n$ then

$$\alpha(G) \geq \sum_{i=1}^{n} \frac{1}{d_i + 1}$$

In particular if all vertices have degree d then $\alpha(G) \geq n/d+1$.

—/—

$$a+b, \quad a, b \mapsto \frac{a+b}{2}, \frac{a+b}{2}$$

$$\frac{1}{a} + \frac{1}{b} \geq \frac{2}{a+b} + \frac{2}{a+b}$$

—/—

Proof: $V(G) = [n]$ choose $\pi \in S_n$ uniformly at random. Let A_i be the event "$\pi(i) < \pi(j)$ for every $j \in \Gamma(i)$".

i.e A_i holds iff amongst $\{i\} \cup \Gamma(i)$, i is "first" under the ordering given by π.

Let $U = \{i \in V(G) : A_i \text{ holds}\}$.

Suppose $a, b \in U$ and $ab \in E$ so $a \in \Gamma(b)$ and $b \in \Gamma(a)$.

But $A_a \Rightarrow \pi(a) < \pi(b)$

While $A_b \Rightarrow \pi(b) < \pi(a)$ ※ Hence U is an indepenant set.

$\mathbb{P}(A_i \text{ holds}) = \mathbb{P}_r($In a random ordering of $\{i\} \cup \Gamma(i)$ the elements i is first$)$.

$= \frac{1}{d_i+1}$ (since each of the d_i+1 elements is equally likely to be first).

Since U is an indepenant set

$$\mathbb{E}\ \alpha(G) \geq \mathbb{E}|U|$$

$$\text{so } \alpha(G) \geq \mathbb{E}|U| = \sum_{i=1}^{n} \mathbb{P}(A_i \text{ holds})$$

$$= \sum_{i=1}^{n} \frac{1}{d_i+1} \qquad \square$$

—/—

$C_5 =$  $\qquad ex(n, C_5^*)$, $ex(n, H)$

—/—

Turán density of H is $\pi(H) = \lim_{n\to\infty} \dfrac{ex(n,H)}{\binom{n}{2}}$

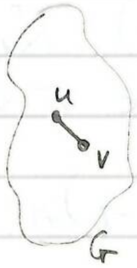
<u>Lemma 2.8</u> : For any graph H, $\pi(H)$ is well-defined. Also $\pi(K_{r+1}) = 1 - 1/r$, $r \geq 2$.

<u>Proof</u>: N.T.S $\left\{ \dfrac{ex(n,H)}{\binom{n}{2}} \right\}_{n=1}^{\infty}$ is monotonic decreasing, is bounded below by zero, it must converage.

Let G be H-free, order n, with $ex(n,H)$ edges:

$$\underbrace{\sum_{v \in V(G)} |E(G - v)|}_{= (n-2)ex(n,H)} \leq ex(n-1, H)$$

since $G-v$ has order $n-1$ and is H-free.



$$\frac{2ex(n,H)}{n(n-1)} \leq \frac{2ex(n-1,H)}{(n-2)(n-1)}$$

i.e. $\dfrac{ex(n,H)}{\binom{n}{2}} \leq \dfrac{ex(n-1,H)}{\binom{n-1}{2}}$ $\square$

Now show $\pi(K_{r+1}) = 1 - \frac{1}{r}$ is true.

Turán Th^m $\Rightarrow ex(n, K_{r+1}) = t_r(n)$

edges in a complete r-partite graph with vertex classes of size $\lceil \frac{n}{r} \rceil$ or $\lfloor \frac{n}{r} \rfloor$

$$\binom{r}{2} \left\lfloor \frac{n}{r} \right\rfloor^2 \leq t_r(n) \leq \binom{r}{2} \left\lceil \frac{n}{r} \right\rceil^2$$

$$\frac{\binom{r}{2}\left(\frac{n-r}{r}\right)^2}{\binom{n}{2}} \leq \frac{t_r(n)}{\binom{n}{2}} \leq \frac{\binom{r}{2}\left(\frac{n-r}{r}\right)^2}{}$$

$$\left(\frac{r-1}{r}\right)\underbrace{\frac{(n-r)^2}{n(n-1)}}_{\to 1} \leq \frac{t_r(n)}{\binom{n}{2}} \leq \frac{(r-1)}{r}\underbrace{\frac{(n+r)^2}{n(n-1)}}_{\to 1}$$

as $n \to \infty$, since r is fixed

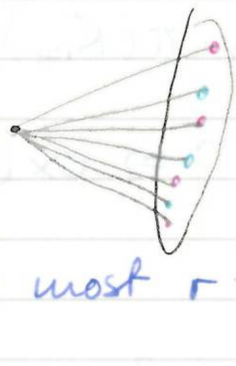
Hence $\lim_{n\to\infty} \frac{t_r(n)}{\binom{n}{2}} = \pi(K_{r+1}) = 1 - \frac{1}{r}$. $\square$

Th^m 2.9: (Kövari – Sós – Turán 1954). $K_{r,s}$ = complete bipartite graph with class size r and s.

$$ex(n, K_{r,s}) \leq \frac{1}{2}(r-1)^{1/s} n^{2-1/s} + \frac{1}{2}(s-1)n.$$

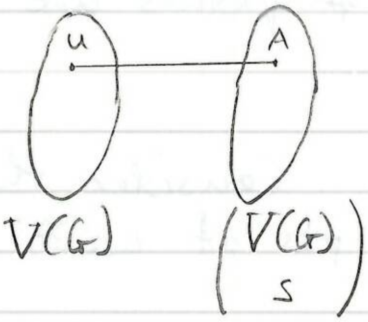
Proof: Let G be $K_{r,s}$-free, order n with e edges. If $u \in V(G)$ and $A = \{v_1, \dots, v_s\} \in \binom{V(G)}{s}$ then u covers A if $uv_1, uv_2, \dots, uv_s \in E(G)$. So u covers $\binom{d(u)}{s}$, s-sets

How many different vertices can cover the same s-set A?



Since G is $K_{r,s}$-free at most $r-1$ vertices can cover the same s-set.

Form a bipartite graph H.



Edge from $u \in V(G)$ to $A \in \binom{V(G)}{s}$ iff u covers A.

Now count the number of edges in H.

$$|E(H)| = \sum_{u \in V(G)} d_H(u) = \sum_{u \in V(G)} \binom{d_G(u)}{s}$$

$$|E(H)| = \sum_{A \in \binom{V(G)}{s}} d_H(A) \leq \sum_{A \in \binom{V(G)}{s}} (r-1)$$

Thus $\displaystyle\sum_{u \in V(G)} \binom{d(u)}{s} \leq (r-1)\binom{n}{s}$

$\displaystyle\sum_{u \in V(G)} d(u) = 2e$. By convexity of binomial coefficients and Jensen's Inequality:

$$n\binom{2e/n}{s} \leq (r-1)\binom{n}{s}$$

Let $\alpha \geq 0$ be defined by $e = n^{2-\alpha}$

So $n\binom{2n^{1-\alpha}}{s} \leq (r-1)\binom{n}{s}$

Use:

$$\frac{(a-b+1)^b}{b!} \leq \binom{a}{b} \leq \frac{a^b}{b!}$$

$$(2n^{1-\alpha} - s + 1)^s \leq (r-1)n^{s-1}$$

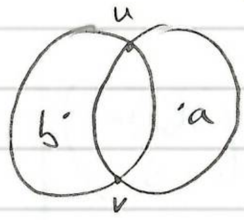
$$2n^{1-\alpha} - s + 1 \leq (r-1)^{1/s} n^{1-1/s}$$

$$e = n^{2-\alpha} \leq \frac{1}{2}(r-1)^{1/s} n^{2-1/s} + \frac{(s-1)n}{2}$$

$\square$

Corollary 2.10. (Erdös) $X \subseteq \mathbb{R}^2$, $|X| = n$, then at most $n^{3/2}/\sqrt{2} + n/2$ pairs of points are at unit distance.

Proof: Let X be as above. Consider the graph on X formed by pairs of pts at unit distance



Claim: this graph is $K_{3,2}$-free

Proof: Two unit circles meet at most twice. #

So # pairs of pts at unit distance

$$= |E(G)|$$

$$\leq ex(n, K_{3,2})$$

$$\leq \frac{\sqrt{2}}{2} n^{3/2} + \frac{n}{2}. \qquad \square$$

Thm 2.11 (Erdös-Stone) If $\chi(H) = r$ then $\pi(H) = 1 - 1/_{r-1}$ (e.g $\pi(C_5) = 1/2$).

Proof: Let H be given.

Suppose $\chi(H) = r \geq 2$. So H is r-partite, so $T_{r-1}(n)$ is H-free. So

$$ex(n,H) \geq |E(T_{r-1}(n))| = t_{r-1}(n).$$

$$\frac{ex(n,H)}{\binom{n}{2}} \geq \frac{t_{r-1}(n)}{\binom{n}{2}} \longrightarrow 1 - \frac{1}{r-1}$$

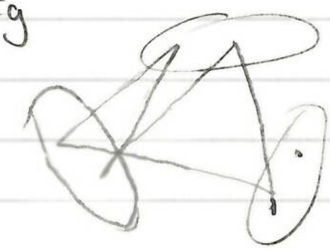
Hence $\pi(H) \geq 1 - \frac{1}{r-1}$

Let $K_r(t)$ is the complete r-partite graph with t vertices in each class (it has rt vertices).

If $t \geq |V(H)|$ then $K_r(t)$ contains a copy of H.

Hence $\pi(H) \leq \pi(K_r(t))$
So sufficient to prove that
$\pi(K_r(t)) \leq 1 - 1/r-1$

Eg



$C_5 \subset K_3(2)$

1/2/13.

Thm 2.11 (Erdös - Stone) If $\chi(H) = r$ then $\pi(H) = 1 - 1/r-1$.

Lemma 2.12 : If $0 < c, \varepsilon < 1$ and $n > \frac{2}{\varepsilon}(1 + 1/c)$. If G is a graph with n vertices and at least $(c+\varepsilon)\binom{n}{2}$ edges then G contains a subgraph G' of order $\varepsilon^{1/2} n$. with $\delta(G') \geq cn'$

Theorem 2.13 : Let $r, t \geq 1$, $0 < \varepsilon < 1/r$. Then $\exists n_0(r, t, \varepsilon)$ st if G has $n \geq n_0(r,t,\varepsilon)$ vertex and $\delta(G) \geq (1 - \frac{1}{r-1} + \varepsilon)n$, then G contains $K_r(t)$.

—/—

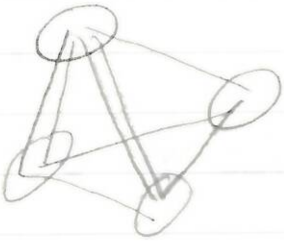
$$\text{density} \equiv \frac{|E(G)|}{\binom{n}{2}}$$

—/—

Proof of Th^m : 2.11 : Know $T_{r-1}(n)$ is H-free so $\pi(H) \geq \pi(K_r) = 1 - 1/r-1$. Also if $t \geq |V(H)|$ then $H \subseteq K_r(t)$. so $\pi(H) \leq \pi(K_r(t))$

$\chi(H) = r \Rightarrow H =$

Then $K_r(t)$ contains H



no edges inside classes

So need to show $\pi(K_r(t)) \leq 1 - 1/r-1$. Suppose this fails to hold then $\exists \varepsilon > 0$ st $\pi(K_r(t)) > 1 - \frac{1}{r-1} + 3\varepsilon$

given by Th^{m} 2.13

Let $n \geqslant \frac{n_0(r,t,\varepsilon)}{\varepsilon^{1/2}}$. and let G be a free graph of order n and at least $(1 - \frac{1}{r-1} + 2\varepsilon)\binom{n}{2}$ edges.

By lemma 2.12 with $c = 1 - \frac{1}{r+1} + \varepsilon$, G contains a subgraph G' of order $n' \geqslant \varepsilon^{1/2} n \geqslant n_0(r,t,\varepsilon)$ and $\delta(G') \geqslant (1 - \frac{1}{r-1} + \varepsilon)n'$ vertices. So Th^{m} 2.13 $\Rightarrow$ $K_r(t) \subset G'$. Since $G' \subseteq G$ is $K_r(t)$ - free. $\square$

<u>Proof of lemma 2.12</u>. We find G' as follows. Let $G_n = G$. If the $\delta(G_n) \geqslant cn$ then let $G' = G_n$. Otherwise $\delta(G_n) < cn$, so remove a vertex of min. degree to give G_{n-1}. If $\delta(G_{n-1}) \geqslant c(n-1)$ then $G = G_{n-1}$ otherwise repeat. Construct a sequence $G_n, G_{n-1}, \dots, \dots,$ where G_k has order k and G_{k-1} from G_k by deleting a vertex of min. degree. We claim this process terminates at some $k \geqslant \varepsilon^{1/2} n$. Since otherwise if $s = \lceil \varepsilon^{1/2} n \rceil$ then:

$$|E(G_s)| > |E(G)| - \underbrace{\sum_{k=s+1}^{n} ck}. \geqslant$$

$$\geqslant (c+\varepsilon)\binom{n}{2} - \overbrace{c\left(\binom{n+1}{2} - \binom{s+1}{2}\right)}$$

$$\geqslant \varepsilon\binom{n}{2} - cn + c\binom{s+1}{2}$$

$$\sum_{k=1}^{n} k = \binom{n+1}{2}$$

By our choice of $s = \lceil \varepsilon^{1/2} n \rceil$ and n satisfies

$$n > \frac{2}{\varepsilon}(1 + 1/c) \Rightarrow \binom{s+1}{2} > \frac{s^2}{2} \geqslant \frac{\varepsilon n^2}{2} > \left(1 + \frac{1}{c}\right)n = n + \frac{n}{c}$$

Hence $|E(G_S)| > \varepsilon \binom{n}{2} + n$.

so $\varepsilon \binom{n}{2} + n \leq \binom{S}{2} \leq \frac{(\varepsilon^{1/2} n + 1)(\varepsilon^{1/2} n)}{2}$

so $\varepsilon n^2 - \varepsilon n + 2n \leq \varepsilon n^2 + \varepsilon^{1/2} n$.

$2 \leq \varepsilon^{1/2} + \varepsilon < 2$. $\#$

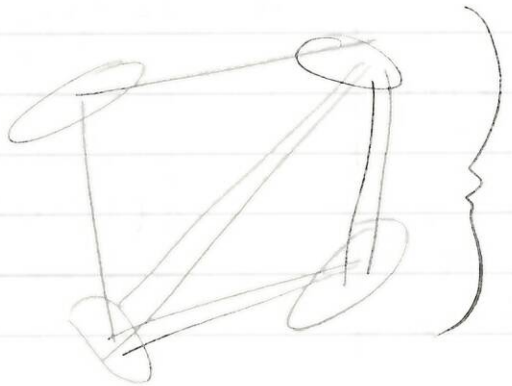
$\square$

6/2/13

Th^m: 2.11 (Erdös – Stone) If $\chi(H) = r$ then $\pi(H) = 1 - 1/r-1$

Th^m 2.13: Let $r \geq 2$, $t \geq 1$ and $0 < \epsilon < 1/r$. There exist $n_0(r, t, \epsilon)$ such that if G has $n \geq n_0$ vertices and $\delta(G) \geq (1 - \frac{1}{r-1} + \epsilon)n$ then G contains a copy of $K_r(t)$

$\delta(G) = \min_{v \in V(G)} d(v)$



r classes
t vertices
in each class all edges between.

Proof (of Thm 2.13) Induction on r.

$r = 2$ $\quad K_2(t) = K_{t,t}$ — Kovari – Sos Turan theorem

So $ex(n, K_2(t)) \leq \frac{1}{2}(t-1)^{1/t} n^{2-1/t} + \frac{1}{2}(t-1)n$

$$< t n^{2-1/t} \quad \oplus$$

Given $\epsilon > 0$ and $t \geq 1$ define $n_0(2, t, \epsilon)$ so that for $n \geq n_0$ we have $\epsilon > 2t/n^{1/t}$ $(*)$. Let G be a graph with $n \geq n_0$ vertices and $\delta(G) \geq \epsilon n$. Then G has at least $\epsilon n^2/2 > t n^{2-1/t}$ by $(*)$ so $|E(G)| > ex(n, K_2(t))$ by $\oplus$, so G contains

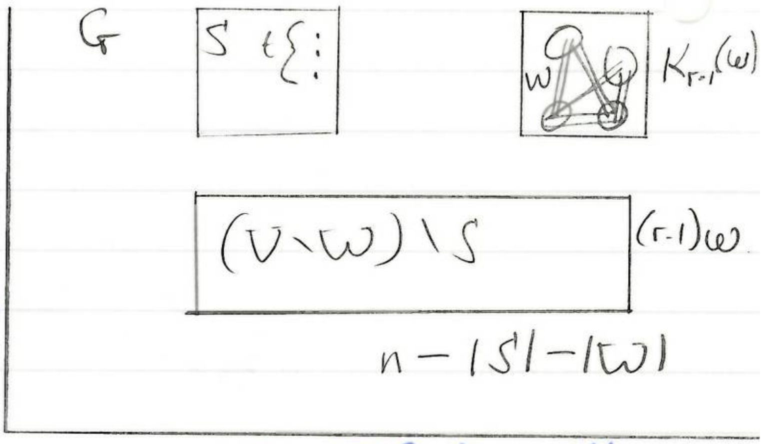
$K_2(t)$.

Now suppose $r \geq 3$, $t \geq 1$ and $0 < \epsilon < 1/r$ is given, and the result holds for $r-1$. Let G have n vertices, $\delta(G) \geq (1 - \frac{1}{r-1} + \epsilon)n$. We need to show that for n sufficiently large, G contains $K_r(t)$.

Let $\omega = \left\lceil \frac{2t}{\epsilon} \right\rceil$ and let $n \geq n_0(r-1, \omega, \epsilon)$

Since $\delta(G) \geq \left(1 - \frac{1}{r-1} + \epsilon\right)n$

$$> \left(1 - \frac{1}{r-2} + \epsilon\right)n$$



We know G contains a copy of $K_{r-1}(\omega)$ with vertex W, $|W| = (r-1)\omega$.

Let $S' = \{ v \in V \setminus W : v \text{ has } \geq (r-2)\omega + t$ neighbour inside $W \}$

Notice if $v \in S$ then v has $\geq t$ neighbours in each vertex class W, so v is adjacent to all the vertices of a copy of $K_{r-1}(t)$.

<u>Claim</u>: $|S| \to \infty$ as $|n| \to \infty$, in particular if n is sufficiently large then $|S| > (t-1)\binom{\omega}{t}^{r-1}$

Call a vertex $v \in S'$ good for a copy $\hat{K}$ of $K_{r-1}(t)$ in W if v is adjacent to every vertex in $\hat{K}$

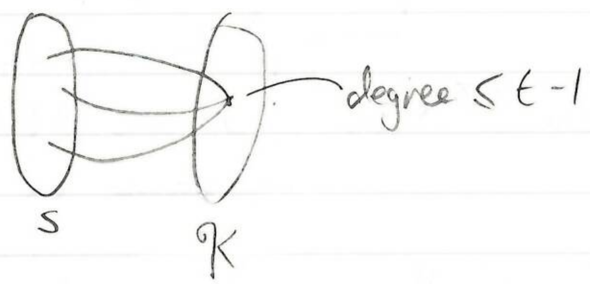
If G is $K_r(t)$-free, then for each copy of $K_{r-1}(t)$ in W there are at most $t-1$ good vertices in S'.

By definition of S', every vertex in S' is good for at least one copy of $K_{r-1}(t)$ in W.

<u>Qu</u>: How many copies of $K_{r-1}(t)$ are there in W?

<u>Ans</u>: $\binom{w}{t}^{r-1}$

So we have the following bipartite graph H:



$\mathcal{K} = \{\hat{K} : \hat{K} \text{ is a copy of } K_{r-1}(t) \text{ in } W\}$
$v \in S'$ is joined by an edge in H to $\hat{K} \in \mathcal{K}$ iff v is good for $\hat{K}$.

$$|S| \leq \sum_{v \in S'} d_H(v) = |E(H)|$$

$$= \sum_{\hat{K} \in \mathcal{K}} d_H(\hat{K}) \leq (t-1)\binom{\omega}{t}^{r-1}$$

So $|S| \leq (t-1)\binom{\omega}{t}^{r-1}$

Contradicting the Claim $\#$

Need to prove the claim. Let $e(W, V/W)$ be the number of edges for W to V/W. We know $\delta(G) \geq (1-\frac{1}{r-1}+\varepsilon)n$.

There are at most $|W|^2/2$ edges inside W.

$$e(W, V\setminus W) = \sum_{v \in W} d(v) - \underbrace{2e(W)}_{\#\text{ edges inside } W.}$$

$$\geq \boxed{|W|n\left(1-\frac{1}{r-1}+\varepsilon\right) - |W|^2} \quad ①$$

Recall $S = \{v \in V \setminus W : v \text{ has} \geq (r-2)\omega + t \text{ neighbours in } W\}$

If $v \in (V\setminus W)\setminus S$ then v has $< (r-2)\omega - t$ neighbours in W.

If $v \in S$ " " " $\leq |W|$

" " "

$$e(W, V\setminus W) < \overbrace{((r-2)\omega+t)}^{|W|-(\omega-t)}(n-|W|-|S|) + |S||W| \quad ②.$$

$|W| = (r-1)\omega.$

$$e(W, V \setminus W) < n((r-2)\omega + t) - |W|^2 + |W|(\omega - t) - \cancel{|S||W|} + \cancel{|S||W|} + |S|(\omega - t).$$

So ① + ② $\Rightarrow$

$$|W|n\left(1 - \frac{1}{r-1} + \varepsilon\right) - \cancel{|W|^2} < n((r-2)\omega + t) - \cancel{|W|^2} + |S|(\omega - \varepsilon) + |W|(\omega - t).$$

$$\omega n(r-2+(r-1)\varepsilon) < n((r-2)\omega + t) + |S|(\omega - t) + \omega(r-1)(\omega - t)$$

$$|S| > \underbrace{n\left(\frac{\varepsilon(r-1)\omega - t}{\omega - t}\right)}_{>0} - (r-1)\omega.$$

Since $r \geq 3$, $\omega \geq 2t/\varepsilon$ this coefficient n is >0

$\square$

8/2/13

$ex(n,H) = \max\{|E| : G=(V,E), |V|=n, G \text{ } H\text{-free}\}$

Turán result

1) Turán's Theorem : $ex(n, K_{r+1}) = t_r(n)$

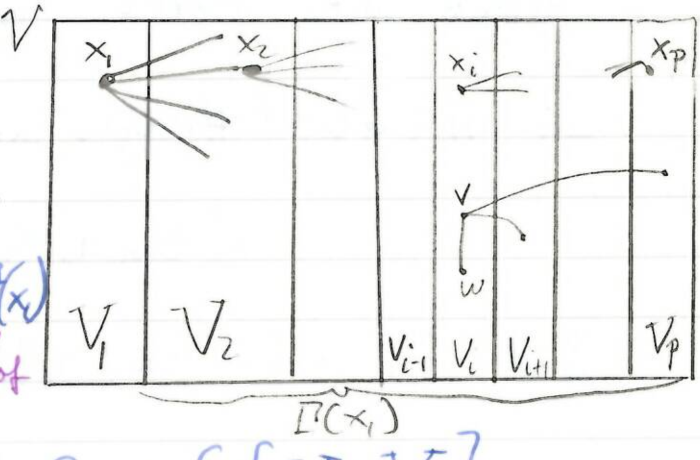
2) $\pi(H) = \lim_{n\to\infty} \frac{ex(n,H)}{\binom{n}{2}}$ exists,

$$\chi(H) = r \geq 2 \Rightarrow \pi(H) = 1 - \frac{1}{r-1} \quad \text{(Erdös – Stone)}$$

3) Stability

Theorem 2.14 (Füredi 2010) : If G is K_{r+1}-free, order n with at least $ex(n, K_{r+1}) - t$ for some $t \geq 0$ then $\exists H \subseteq G$ st $|E(H)| \geq |E(G)| - t$ and $\chi(H) = r$

Proof: Let $G=(V,E)$ be K_{r+1}-free, $|V|=n$ and $|E| = ex(n, K_{r+1}) - t$. Choose $x_1 \in V$ of max degree. Let $V_1 = V \setminus \Gamma(x_1)$
neighbours of x_1



Now consider the graph $G_2 = G[V \setminus V_1]$.

Choose $x_2 \in G_2$ of max degree. Let $V_2 = V(G_2) \setminus \Gamma_{G_2}(x_2)$
Repeat until have no vertices left: suppose we choose $x_1, x_2, \dots, x_p$.

By construction $x_1, x_2, \ldots, x_p$ from a clique (i.e a copy of K_p). Hence $p \leq r$.

Let $d_1 = d(x_1)$, $d = d_{G_2}(x_2)$ etc. to give $d_1, d_2, \ldots, d_p$.

Note that $d_i = |V_{i+1}| + |V_{i+2}| + \ldots + |V_p|$.

Note for $v \in V_i$ define $\vec{d}(v) = \#\{w : vw \in E, w \in V_i \cup V_{i+1} \cup \ldots \cup V_p\}$

If $v \in V_i$ $\overrightarrow{d(v)} \leq d_i$ (by maximality of degree of x_i) in G_i.

$$|E(G)| + \underset{\text{inside classes}}{\#\text{edges}} = \sum_{i=1}^{p} \sum_{v \in V_i} \overrightarrow{d(v)} \leq \sum_{i=1}^{p} d_i |V_i| = \sum_{i=1}^{p} |V_i| (|V_{i+1}| + \ldots + |V_p|)$$
$$= |E(K(V_1, V_2, \ldots, V_p))|$$
$$\text{by lemma 2.5} \quad \leq |E(T_p(n))|$$

where $K(V_1, V_2, \ldots, V_p)$ is the complete p-partite graph with vertex classes $V_1, V_2, \ldots, V_p$.

So $|E(G)| + \#$edges inside classes

$$\leq t_p(n) \leq t_r(n) \quad \text{since } p \leq r$$

But $|E(G)| \geq ex(n, K_{r+1}) - t = t_r(n) - t$.

$\Rightarrow$ #edges inside class $\leq t$.

Let H be G with all edges inside classes removed. So $|E(H)| \geq |E(G)| - t$ and $H \subseteq K(V_1, \ldots, V_p)$ is p-partite. $\square$

22/2/13

3) Set system

$$[n] = \{1, 2, 3, \dots, n\}$$

$$\mathcal{P}(X) = \{A : A \subseteq X\}$$

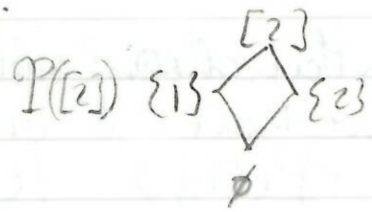
$$\binom{X}{k} = \{A : A \subseteq X, \ |A| = k\}.$$

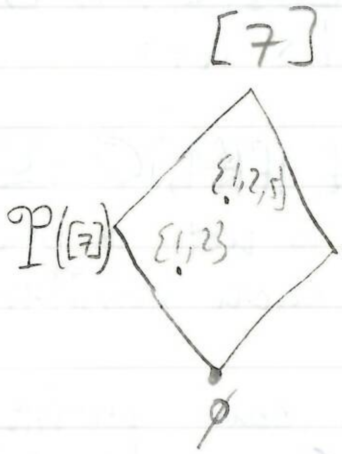
$X = [n]$.

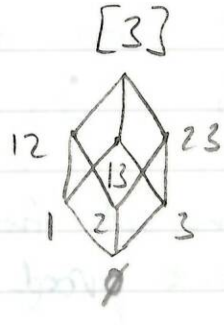
—/—

A family $\mathcal{A} \subseteq \mathcal{P}([n])$ is a chain if $A, B \in \mathcal{A}$ $A \subseteq B$ or $B \subseteq A$.

Eg:







edges $\equiv$ covering relation.

A family $\mathcal{A} \subseteq \mathcal{P}([n])$ is a antichain if $\forall A, B \in \mathcal{A}$
$A \subseteq B \Rightarrow A = B$
or $A \neq B$, $A, B \in \mathcal{A}$ st $A \not\subseteq B$ and $B \not\subseteq A$.

Examples of antichains:

$$\binom{[7]}{3}, \quad \binom{[n]}{k}. \qquad \{123, 45, 1247\}$$

Lemma 3.1: If $\mathcal{A}$ is an antichain and $\mathcal{C}$ is a chain then $|\mathcal{A} \cap \mathcal{C}| \leq 1$.

Proof: If $|\mathcal{A} \cap \mathcal{C}| \geq 2$, let $A, B \in \mathcal{A} \cap \mathcal{C}$, $A \neq B$
Then $A, B \in \mathcal{C}$ is a chain $\Rightarrow$ w.l.o.g $A \subset B$
But then $A, B \in \mathcal{A}$ is an antichain $\Rightarrow$ $A = B$

$\divideontimes$ □

—/—

Lemma 3.2: If $\mathcal{C} \subseteq \mathcal{P}([n])$ is a chain then $|\mathcal{C}| \leq n+1$

Proof If $A, B \in \mathcal{C}$ and $|A| = |B|$ then $A = B$ (otherwise $\mathcal{C}$ is not a chain). Hence we have $\leq$ one set of each possible size from $\mathcal{P}([n])$ $\therefore |\mathcal{C}| \leq n+1$.

[We can partition $\mathcal{P}([n])$ into $n+1$ anti chain

$$\mathcal{P}([n]) = \binom{[n]}{0} \dot\cup \binom{[n]}{1} \dot\cup \dots \dot\cup \binom{[n]}{n}.$$

since $\mathcal{C}$ contains at most one set from each anti-chain, $|\mathcal{C}| \leq n+1$. This is an alternative proof of lemma 3.2

We observe that $\left|\binom{[n]}{\lfloor n/2 \rfloor}\right| = \binom{n}{\lfloor n/2 \rfloor}$

which is the largest of the binomial coefficients raised to power n, then we get...]

$\binom{[n]}{k}$ $n=4$. $\binom{4}{0}=1$ $\binom{4}{3}=4$

$\binom{4}{1}=4$ $\binom{4}{4}=1$

$\binom{4}{2}=6$

$n=5$ $\binom{5}{0}=1 = \binom{5}{5}$

$\binom{5}{1}=5 = \binom{5}{4}$

$\binom{5}{2}=10 = \binom{5}{3}$

$\binom{[n]}{\lfloor n/2 \rfloor}$ = has size $\binom{n}{\lfloor n/2 \rfloor}$

—/—

<u>Theorem 3.3</u> (Sperner): If $\mathcal{A}$ is an antichain in $\mathcal{P}([n])$ then $|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$

<u>Lemma 3.4</u>: If $n \geq 1$ then $\mathcal{P}([n])$ can be partitioned $\binom{n}{\lfloor n/2 \rfloor}$ chains

Lemma 3.4 + Lemma 3.1 $\Rightarrow$ Theorem 3.3.

A chain $\mathcal{C} \subseteq \mathcal{P}([n])$ is symmetric iff
(i) $\mathcal{C} = \{C_1, \ldots, C_k\}$, $|C_{i+1}| = |C_i| + 1$, $i = 1, \ldots, k-1$
(ii) $|C_1| + |C_k| = n \Rightarrow |C_1| \leq n/2$, $|C_k| \geq n/2$
e.g in $\mathcal{P}([3])$ $\{\emptyset, 1, 12, 123\}$, $\{2, 23\}$
in $\mathcal{P}([4])$ $\{1, 12, 124\}$, $\{13\}$.

Note that any symmetric chain $\mathcal{C} \subseteq \mathcal{P}([n])$ meets "the" middle layer $\binom{[n]}{\lfloor n/2 \rfloor}$

Since $\binom{[n]}{\lfloor n/2 \rfloor}$ is itself an antichain, we know that any symmetric chain contains exactly one set from $\binom{[n]}{\lfloor n/2 \rfloor}$ by lemma 3.1.

<u>Proof of Lemma 3.4</u> (Induction on n.) $n=1$
$\mathcal{P}([1]) = \{\emptyset, 1\}$ is a symmetric chain. Now suppose $n \geq 2$ and result holds for $n-1$. So $\exists$ a partition of $\mathcal{P}([n-1])$ into sym chains.

$$\mathcal{P}([n-1]) = \mathcal{C}_1 \,\dot{\cup}\, \mathcal{C}_2 \,\dot{\cup} \ldots \dot{\cup}\, \mathcal{C}_t .$$

$$\mathcal{C}_i = \{C_1^i, C_2^i, \ldots, C_{K_i}^i\}.$$

From two new chains for $\mathcal{C}_i$ (if $K_i \geq 2$)

$$\mathcal{C}_i' = \{C_1^i \cup \{n\}, C_2^i \cup \{n\}, \ldots, C_{K_i - 1}^i \cup \{n\}\}$$

$$\mathcal{C}_i'' = \{C_1^i, C_2^i, \ldots, C_{K_i}^i, C_{K_i}^i \cup \{n\}\}$$

Note that $\mathcal{C}_i'$, $\mathcal{C}_i''$ are both chains and infact both symmettic chain in $\mathcal{P}([n])$. Moreover
$\mathcal{P}([n]) = \mathcal{C}_1' \,\dot\cup\, \mathcal{C}_1'' \,\dot\cup\, \mathcal{C}_2' \,\dot\cup\, \mathcal{C}_2'' \,\dot\cup \dots \dot\cup\, \mathcal{C}_n''$
So the result holds

$\square$

27/2/13

Homework 18, 19, 20, 21 for next Wednesday.

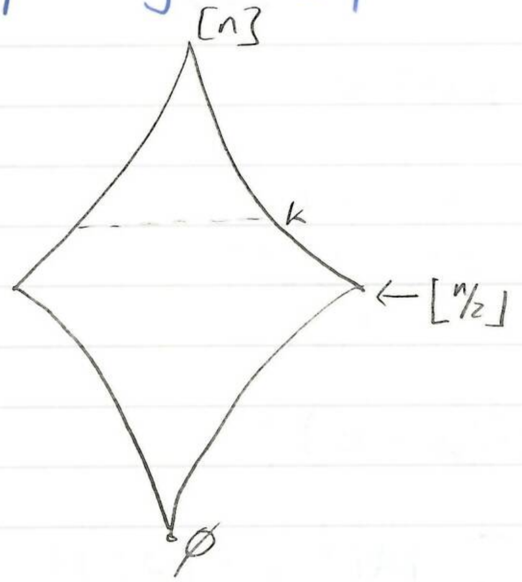
Thm 3.3 (Sperner). If $\mathcal{A} \subseteq \mathcal{P}([n])$ is an antichain then $|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$ (best possible: $\mathcal{A} = \binom{[n]}{\lfloor n/2 \rfloor}$)

$\mathcal{A}$ is antichain $\iff \forall A, B \in \mathcal{A} .\ A \subseteq B \Rightarrow A = B$

Thm 3.5 (LYM) If $\mathcal{A} \subseteq \mathcal{P}([n])$ is an antichain then

$$\sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq 1$$

Note that since $\binom{n}{\lfloor n/2 \rfloor} \geq \binom{n}{k}$ any $0 \leq k \leq n$ LYM-inequality $\Rightarrow$ Sperners Theorem.



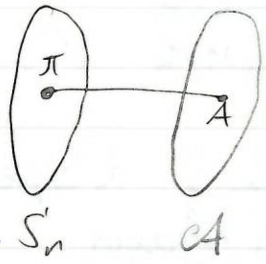
$\mathcal{P}([n])$

—/—

Proof (of LYM). Let $\mathcal{A} \subseteq \mathcal{P}([n])$ be an antichain. S_n = permutations of $[n]$.

Construct a bipartite $G = (S_n, \mathcal{A}; E)$

Where $\pi \in S_n$ is joined by an edge to $A \in \mathcal{A}$ iff all the elements of A appear before the elements of A^c in π.



$n = 8$, $A = 134$, $\pi = 1\,3\,4\,5\,6\,8\,7\,2$

πA is an edge.

$n = 7$, $A = 237$, $\pi = 7\,2\,3\,4\,6\,5\,1$

πA is an edge.

but if $B = 2367$ then πB is not an edge.

<u>Double counting</u>:

$$\sum_{\pi \in S_n} d(\pi) = |E| = \sum_{A \in \mathcal{A}} d(A)$$

If $A \in \mathcal{A}$ and $|A| = k$

$$k! \quad \times \quad (n-k)!$$

$$\underbrace{\cdots\cdots}_{\substack{k \\ A}}\ \underbrace{\cdots\cdots}_{\substack{n-k \\ A^c}}$$

then $d(A) = k!(n-k)!$

Hence $|E| = \sum_{A \in \mathcal{A}} |A|!(n-|A|)!$

Now if $\pi \in S_n$, and πA is an edge and πB is another edge then either $A \subset B$ or $B \subset A$ so $A = B$

$\therefore$ At most one edge from π $\therefore$ $d(\pi) \leq 1$

So $|E| = \sum_{\pi \in S_n} d(\pi) \leq \sum_{\pi \in S_n} 1 = n!$

So $\sum_{A \in \mathcal{A}} \frac{|A|!(n-|A|)!}{n!} \leq 1$

So $\sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq 1$.

$\square$

$\mathcal{A}$ is <u>intersecting</u> $\iff$ $A, B \in \mathcal{A} \wedge B \neq \emptyset$.

e.g $\{12, 13, 23\}$.

<u>Th^m 3.6</u>. If $\mathcal{A} \subseteq \mathcal{P}([n])$ is intersecting then $|\mathcal{A}| \leq 2^{n-1}$.

<u>Proof</u>: Since $A \in \mathcal{A} \Rightarrow A^c \notin \mathcal{A}$, hence $|\mathcal{A}| \leq 2^{n-1}$!

$\square$

<u>Examples</u>: $\mathcal{A}^* = \{A \subseteq [n] : 1 \in A\}$, $|\mathcal{A}^*| = 2^{n-1}$

$\mathcal{B} = \{B \subseteq [n] : |B \cap [3]| \geq 2\}$.

$|\mathcal{B}| = 4 \times 2^{n-3}$
$= 2^{n-1}$

Since B consists of $B = \hat{B} \cup B'$, where
$\hat{B} \in \{12, 13, 23, 123\}$
$B' \subseteq \{4, 5, \ldots, n\}$.

$\mathcal{C} = \{C \subseteq [n] : |C \cap [5]| \geq 3\}$

If $C \in \mathcal{C}$ then $C = \hat{C} \cup C'$

$\hat{C} \in \{123, 124, 125, 134, 135, 145, 234, 235, 245, 345, 1234, 1235, 1245, 1345, 2345, 12345\}$

and $C' \subseteq \{6, 7, \ldots, n\}$

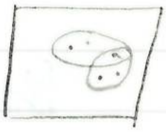
$$\therefore |\mathcal{C}| = 16 \times 2^{n-5} = 2^{n-1}$$

In general $\mathcal{D}_k = \{D \subseteq [n] : |D \cap [2k+1]| \geq k+1\}$

$\mathcal{D}_0 = \mathcal{A}^*$, $\mathcal{D}_1 = \mathcal{B}$, $\mathcal{D} = \mathcal{C}$ | $\mathcal{D}$ is intersecting and $|\mathcal{D}| = 2^{n-1}$

If $\mathcal{A} \subseteq \binom{[n]}{k}$ is intersecting, how large can $|\mathcal{A}|$ be?

If $2k > n$ then $\binom{[n]}{k}$ is intersecting



<u>Th$^{\underline{m}}$ 3.7</u>: (Erdos – Ko – Rado 1961). If $2k \leq n$ and $\mathcal{A} \subseteq \binom{[n]}{k}$ is intersecting then $|\mathcal{A}| \leq \binom{n-1}{k-1}$

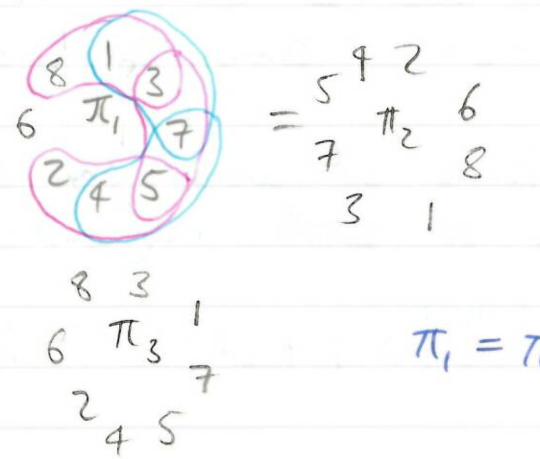
Note: $\mathcal{A}^* = \{A \in \binom{[n]}{k} : 1 \in A\}$. $|\mathcal{A}^*| = \binom{n-1}{k-1}$

<u>Proof</u>: (Katona). Let $n \geq 2k$ and $\mathcal{A} \subseteq \binom{[n]}{k}$ be intersecting.

Let $\mathcal{C}_n$ be the family of cyclic permutations of $[n]$. By this we mean two permutations of $[n]$ are considered the same, if when written around a circle, we can form one to the other by rotation.

e.g: $n=8$,

137 ~~168~~
357 ~~138~~
457
~~245~~
~~246~~
~~268~~



$\pi_1 = \pi_2 \neq \pi_3$.

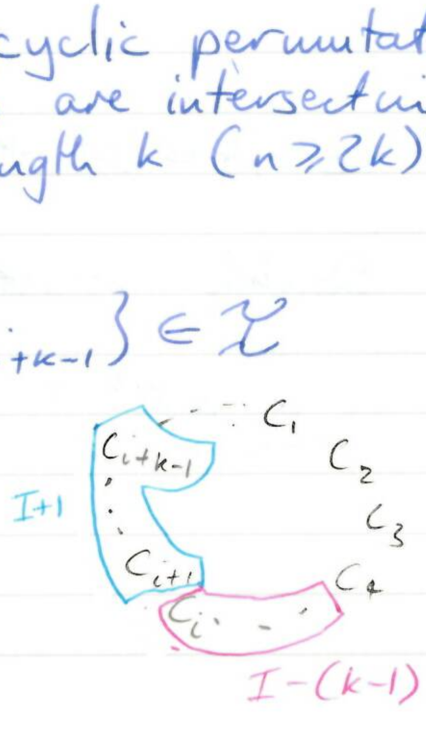
$$|C_n| = \frac{n!}{n} = (n-1)!$$

Given a cyclic permutation π and a set $A \in \mathcal{A}$. Say A is an <u>interval</u> in π if the elements of A appears consectively.

<u>Lemma 3.8</u> If $\pi \in C_n$ is a cyclic permutation of $[n]$ and $\mathcal{I} = \{I_1, \dots, I_t\}$ are intersecting intervals from π each of length k $(n \geq 2k)$ then $t \leq k$.

<u>Proof</u>: Let $I = \{c_i, c_{i+1}, \dots, c_{i+k-1}\} \in \mathcal{I}$

Note I meets at most $2k-2$ other intervals for π.



Namely: $I+1, I+2, \ldots, I+(k-1)$ where $I+j = \{c_{i+j}, c_{i+j+1}, \ldots, c_{i+j+k-1}\}$

$$I-1, I-2, \ldots, I-(k-1)$$

But $I+1$ and $I-(k-1)$ are disjoints as are $I+j$ and $I-(k-1)$ any $1 \le j \le k-1$

Hence there at most one of $I+j$ and $I-(k-j)$ in $\mathcal{I}$ for each $1 \le j \le k-1$

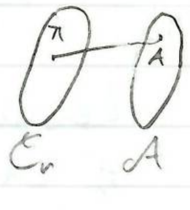
Thus $|\mathcal{I}| \le 1 + (k-1) = k$.

$\square$

<u>Proof of EKR</u>. (ctd). Define a bipartite graph $G = (\mathcal{C}_n, \mathcal{A}; E)$.

Join $\pi \in \mathcal{C}_n$ to $A \in \mathcal{A}$ iff A is an interval in π.

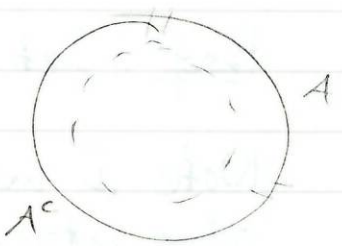
If $\pi \in \mathcal{C}_n$ then $d(\pi) = \#$ intervals of π that belong to $\mathcal{A}$



So $A \in \mathcal{A}$ then $d(A) = k!(n-k)!$

Double Counting:

$$\sum_{\pi \in \mathcal{C}_n} d(\pi) = |E| = \sum_{A \in \mathcal{A}} d(A)$$



$$k|\mathcal{C}_n| \geq |E| = |\mathcal{A}|\,k!\,(n-k)!$$

So $$|\mathcal{A}| \leq \frac{k\,(n-1)!}{k!\,(n-k)!} = \binom{n-1}{k-1} = \frac{k}{n}\binom{n}{k}$$

$\square$

$n > 2k \Rightarrow$ Unique best-family.

1/3/13

(left)

Compressions

$\mathcal{A} \subseteq \mathcal{P}([n])$ If $1 \le i < j \le n$. and $A \in \mathcal{A}$

$$C_{ij} = \begin{cases} (A \setminus \{j\}) \cup \{i\} & j \in A,\ i \notin A. \\ A & \text{otherwise} \end{cases}$$

$A = 246$, $\mathcal{A} = \{236, 246\}$

$$\left.\begin{aligned} C_{34}(246) &= 236 \\ C_{34}(125) &= 125 \\ C_{34}(123) &= 123 \\ C_{34}(134) &= 134 \end{aligned}\right\} \text{different examples}$$

$\mathcal{A} \subseteq \mathcal{P}([n])$

If $1 \le i < j \le n$ then

$C_{ij}(\mathcal{A}) = \{C_{ij}(A) : A \in \mathcal{A}\} \cup \{A : A \in \mathcal{A} \text{ and } C_{ij}(A) \in \mathcal{A}\}$

example: $\mathcal{A} = \{146, 236, 246, 124\}$

$C_{34}(\mathcal{A}) = \{136, 236, 123, 246\} = \mathcal{A}'$

$C_{26}(\mathcal{A}') = \{123, 236, 246, 136\} = \mathcal{A}''$

$C_{16}(\mathcal{A}'') = \{123, 124, 136, 236\} = \mathcal{A}'''$

$C_{46}(\mathcal{A}''') = \{123, 124, 134, 234\} = \tilde{\mathcal{A}}$

If $C_{ij}(\mathcal{A}) = \mathcal{A}$ $\forall 1 \le i < j \le n$, $(\mathcal{A} \subseteq \mathcal{P}([n]))$ then we say $\mathcal{A}$ is compressed.

Lemma 3.9 : $\mathcal{A} \subseteq \binom{[n]}{k}$ and $1 \leq i < j \leq n$.
(i) $C_{ij}(\mathcal{A}) \subseteq \binom{[n]}{k}$
(ii) $|C_{ij}(\mathcal{A})| = |\mathcal{A}|$
(iii) If $\mathcal{A}$ is intersecting then so is $C_{ij}(\mathcal{A})$
(iv) Repeating apply $i-j$-compression we will eventually reach $\tilde{\mathcal{A}}$ st $C_{ij}(\tilde{\mathcal{A}}) = \tilde{\mathcal{A}}$ $\forall 1 \leq i < j \leq n$.
a compressed family.

Proof :
(i) + (ii) Follow instantly from definition of C_{ij}

(iii) Suppose $\mathcal{A}$ is intersecting
Now suppose $\exists A, B \in C_{ij}(\mathcal{A})$ such that $A \cap B = \emptyset$
$\mathcal{A}$ is intersecting $\Rightarrow$ Not both A, B are in $\mathcal{A}$.
Since every "new" set in $C_{ij}(\mathcal{A})$ contains i,
so A, B are not both new. So wlog $A \in \mathcal{A}$ and $B \notin \mathcal{A}$.

So $C = (B \setminus \{i\}) \cup \{j\} \in \mathcal{A}$.

Since $A \cap B = \emptyset$ and $A \cap C \neq \emptyset$ we must hence $j \in A$, $i \notin A$.

Hence, by definition of $C_{ij}(\mathcal{A})$:
$D = C_{ij}(A) \in \mathcal{A}$.
$D = (A \setminus \{j\}) \cup \{i\}$

So $C \cap D \subseteq (B \setminus \{i\}) \cap (A \setminus \{j\})$
$\subseteq A \cap B = \emptyset$ #

Since $C, D \in \mathcal{A}$

□

(iv) Definition $\omega(\mathcal{A}) = \sum_{A \in \mathcal{A}}{}' \sum_{a \in A}{}' a$.

If $C_{ij}(\mathcal{A}) \neq \mathcal{A}$ then $\omega(C_{ij}(\mathcal{A})) \leq \omega(\mathcal{A}) - (j-i)$
$\omega \geq 0$. So apply all i-j-compressions repeatedly we eventually reach a compressed family.

$\square$

<u>Proof of EKR</u>: Induction on $n \geq 2r$ $\quad n = 2$
$n > 2$. Let $\mathcal{A} \subseteq \binom{[n]}{k}$ be intersecting..

If $n = 2k$ then $\binom{n-1}{k-1} = \frac{1}{2}\binom{n}{k}$ and $(A \in \mathcal{A} \Rightarrow A^c \notin \mathcal{A}) \Rightarrow |\mathcal{A}| \leq \frac{1}{2}\binom{n}{k}$

So suppose $n \geq 2k+1$. Now by applying compressions we may suppose $\mathcal{A}$ is compressed.

$\mathcal{A} = \mathcal{B} \cup \mathcal{C}$

Let $\mathcal{B} = \{A \in \mathcal{A} : n \notin A\}$, $\mathcal{C} = \{A \in \mathcal{A} : n \in A\}$
$\mathcal{B} \subseteq \binom{[n-1]}{k}$ Ind. hyp $n \geq 2k+1 \Rightarrow |\mathcal{B}| \leq \binom{n-1-1}{k-1} = \binom{n-2}{k-1}$

Note: $\binom{n-1}{k-1} = \binom{n-2}{k-1} + \binom{n-2}{k-2}$

Consider $\mathcal{D} = \{C \setminus \{n\} : C \in \mathcal{C}\}$.
so $\mathcal{D} \subseteq \binom{[n-1]}{k-1}$. If we show that $\mathcal{D}$ is intersecting then our ind. hyp $\Rightarrow |\mathcal{D}| \leq \binom{n-1-1}{k-1-1} = \binom{n-2}{k-2}$.

$\mathcal{A}$ is compressed. Suppose $D, E \in \mathcal{D}$ st $D \cap E = \emptyset$.

Then $D \cup \{n\}$, $E \cup \{n\} \in \mathcal{A}$.

Since $|D| = k-1 = |E|$ and $n \geq 2k+1$ so $\exists x \in [n-1] \setminus (D \cup E)$.

Since $\mathcal{A}$ is compressed. $C_{xn}(D \cup \{n\}) = (D \setminus \{n\}) \cup \{x\} \in \mathcal{A}$. But $C_{xn}(D) \cap (E \cup \{n\}) = \emptyset$. # Since $\mathcal{A}$ is intersecting.

$\square$

6/3/13.

Homework: Qu 26, 27, 28, 31 next Wed.

The Linear Algebra Method.

Lemma 3.10: If $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_m \in V$, V vector space of dimension d, and $\underline{v}_1, \dots, \underline{v}_m$ are linearly indepenant then $m \leq d$.

—/—

Linear Indepenant:

$\underline{v}_1, \dots, \underline{v}_m \in V$, V a vector space over a field $\mathbb{F}$, are LI iff $\sum_{i=1}^{m} \lambda_i \underline{v}_i = 0 \Rightarrow \lambda_i = 0 \ \forall i$

—/—

Th^m 3.11: If $\mathcal{A} = \{A_1, \dots, A_m\} \subseteq \mathcal{P}([n])$ with $|A_i|$ is odd $\forall i$, and $|A_i \cap A_j|$ is even $\forall i \neq j$ then $m \leq n$.

Proof: For $A_i \in \mathcal{A}$ consider its incidence vector. $\underline{v}_i \in \mathbb{F}_2^n$.

[Recall $\mathbb{F}_2$ is a field with 2 elements]

$$v_{ij} = \begin{cases} 1 & , j \in A_i \\ 0 & , \text{otherwise} \end{cases}$$

e.g $n=6$, $A_2 = \{135\}$

$$\underline{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{matrix} = v_{21} \\ = v_{22} \\ = v_{23} \\ = v_{24} \\ = v_{25} \\ = v_{26} \end{matrix}$$

So we have m vectors $\underline{v}_1, \ldots, \underline{v}_m$.

Consider $\underline{v}_i \cdot \underline{v}_j = \sum_{k=1}^{n} v_{ik} v_{jk} = |A_i \wedge A_j| = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

So $v_1, \ldots, v_m$ are orthogonal.

$\Rightarrow \underline{v}_1, \ldots, \underline{v}_m$ are linear independant.

Lemma 3.10 $\Rightarrow m \leq \dim(\mathbb{F}_2^n) = n$.

$\square$

Th^m: (Fisher Inequality): If $\mathcal{A} = \{A_1, \ldots, A_m\}$ $\subseteq \mathcal{P}([n])$ and $\exists$ $1 \leq k \leq n$ st $\forall i \neq j$ $|A_i \wedge A_j| = k$ then $m \leq n$.

Proof: Let $\mathcal{A}$ be given with the above properties.

For $A_i \in \mathcal{A}$ let $\underline{v}_i$ be its incidence vector

$$v_{ij} = \begin{cases} 1, & j \in A_i \\ 0 & \text{otherwise}. \end{cases}$$

Want to show $\{\underline{v}_1, \ldots, \underline{v}_n\}$ is LI.

Suppose for a contradiction $\exists\ \lambda_1, \ldots, \lambda_m \in \mathbb{R}$ not all zero with $\sum_{i=1}^{m} \lambda_i \underline{v}_i = 0$.

$$0 = \underline{0} \cdot \underline{0} = \left(\sum_{i=1}^{m} \lambda_i \underline{v}_i\right)\left(\sum_{j=1}^{m} \lambda_j \underline{v}_j\right)$$

$$= \sum_{i=1}^{m} \lambda_i^2 \underline{v}_i \cdot \underline{v}_i + \sum_{i \neq j} \lambda_i \lambda_j \underline{v}_i \cdot \underline{v}_j$$

Note: $\underline{v}_i \cdot \underline{v}_j = \begin{cases} |A_i| & i = j \\ k & i \neq j \end{cases}$

$$\ldots = \sum_{i=1}^{m} \lambda_i^2 |A_i| + k \sum_{i \neq j} \lambda_i \lambda_j$$

$$= \underbrace{\sum_{i=1}^{m} \underbrace{\lambda_i^2}_{\geq 0} \underbrace{(|A_i| - k)}_{\geq 0}}_{①} + \underbrace{\underbrace{k}_{\geq 1} \underbrace{\left(\sum_{i=1}^{m} \lambda_i\right)^2}_{\geq 0}}_{②}$$

Since ① + ② = 0, and ① ≥ 0, ② ≥ 0 must have ① = ② = 0.

① = 0 $\Rightarrow$ whenever $|A_i| \neq k$ we must have $\lambda_i = 0$.

Also, since $|A_i \cap A_j| = k\ \forall i \neq j$ we have $|A_i| \geq k\ \forall i$ with equality at most once.

Hence, all but one λ_i must be zero.

② $\Rightarrow \sum_{i=1}^{m} \lambda_i = 0$, this is impossible since exactly one λ_i is non-zero. Hence $\{v_1, \ldots, v_m\}$ is LI.

and lemma 3.10 $\Rightarrow$ $m \leqslant \dim(\mathbb{R}^n) = n$ #

$\square$

Ramsey Theory.

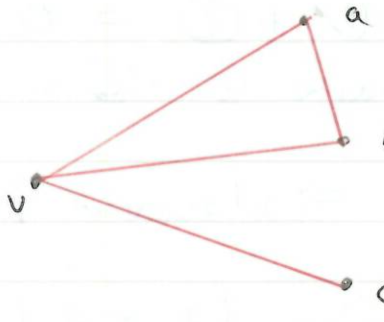
Let $s, t \geqslant 2$ be integers.

$R(s,t) = \min\{n :$ Whenever K_n has its edges coloured red and blue there is always a red K_s or a blue $K_t\}$.

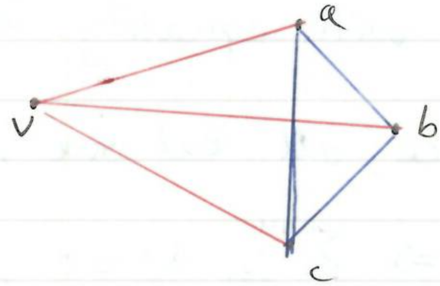
Prop 4.1 $R(3,3) = 6$.

Proof: (1) $R(3,3) \leqslant 6$. Take a red-blue colouring of the edges of K_6

Let $v \in V(K_6)$. Since $d(v) = 5$ w.l.o.g v is incident to at least 3 red edges with endpoints a, b, c. Either one of ab, ac, bc is red or they are all blue.

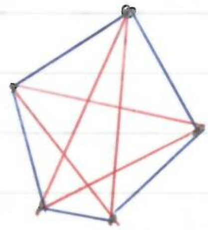


$\Rightarrow$ Either have a red K_3 or blue K_3.



(2) $R(3,3) > 5$.

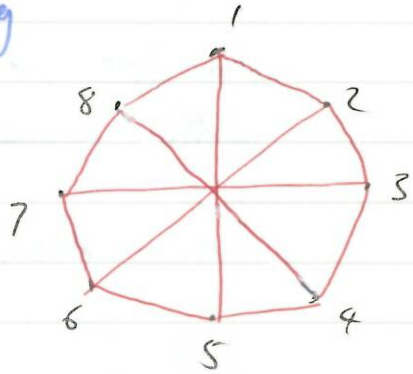
Consider the following colouring, no red K_3 or blue K_3 $\Rightarrow R(3,3) > 5$.



<u>Prop^{n} 4.2</u>. $R(3,4) = 9$,

<u>Proof</u>: (1) $R(3,4) > 8$.

Consider the red-blue edge colouring taking $V(K_8) = [8]$,



Red edges $= \{i\, i+1 : 1 \leq i \leq 8\}$

$= \{i\, i+4 : 1 \leq i \leq 4\}$

No other edges are blue. No red K_3 and no blue K_4.

(2) $R(3,4) \leq 9$.

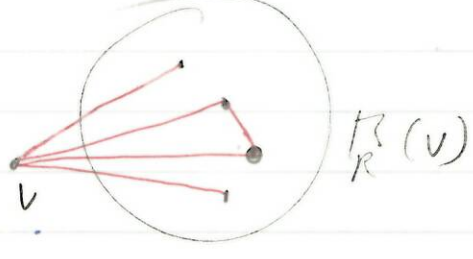
Take a red-blue edges colouring of K_9.

Let $v \in V(K_9)$ $\Gamma_R(v) = \{w : vw \text{ is red}\}, d_R(v) = |\Gamma_R(v)|$

$\Gamma_B(v) = \{w : vw \text{ is blue}\}, d_B(v) = |\Gamma_B(v)|$

So $d_R(v) + d_B(v) = d(v) = 8$.

If $\exists\, v \in V(K_9)$ with $d_R(v) \geq 4$, then either $\Gamma_R(v)$ contains a red edge. So wlog can assume

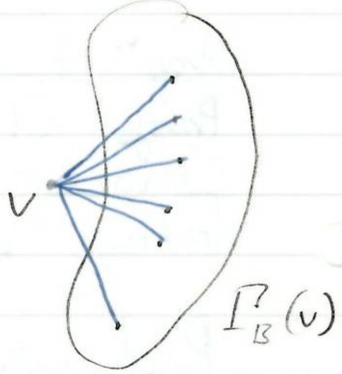


$d_R(v) \leq 3 \quad \forall v \in V(K_9)$

$\Rightarrow d_B(v) \geq 5 \quad \forall v \in V(K_9)$

If $\exists v \in V(K_9)$ st $d_B(v) \geq 6 = R(3,3)$

$\Rightarrow \Gamma_B(v)$ contains a red K_3 or a blue K_3.



In former case have red K_3, in latter case have blue K_4

Only remaining case is if $d_B(v) = 5, \quad \forall v \in V(K_9)$

But $\sum_{v \in V(K_9)} d_B(v) = 2 \times \#$ blue edges

So $\sum_{v \in V(K_9)} d_B(v) = 5 \times 9 = 45$ is impossible

$\square$

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Let $s, t \geqslant 2$.

$R(s,t) = \min\{n \in \mathbb{N}$: Every red-blue colouring of the edges of K_n contains a red K_s or a blue $K_t\}$.

Theorem 4.3 (Ramsey)
If $s,t \geqslant 2$ then $R(s,t)$ is finite and satisfies

$$R(s,t) \leqslant \binom{s+t-2}{s-1}$$

Proof: Induction on $s+t$, $R(2,t) = t$, $R(s,2) = s$.
So result holds if s or t is 2.

So now suppose $s,t \geqslant 3$ and the result holds for smaller $s+t$.

Let $n = R(n-1, t) + R(s, t-1)$. This exists by our inductive hypothesis.

Claim: $R(s,t) \leq n$.

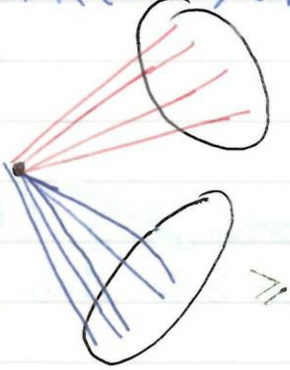
Proof: Take a red-blue colouring of the edges of K_n.

Let $v \in V(K_n)$

Define: $\Gamma_R(v) = \{w : vw \text{ is red}\}$ $d_R(v) = |\Gamma_R(v)|$
and $\Gamma_B(v) = \{w : vw \text{ is blue}\}$ $d_B(v) = |\Gamma_B(v)|$

So $d_R(v) + d_B(v) = d(v) = n-1$. Now since $n = R(s-1,t) + R(s,t-1)$ we must have either $d_R(v) \geq R(s-1,t)$ or $d_B(v) \geq R(s,t-1)$
$\Rightarrow$ w.l.o.g suppose $d_R(v) \geq R(s-1,t)$

Then either $\Gamma_R(v)$ contains a red K_{s-1}, which together with v forms a K_s, or $\Gamma_R(v)$ contains a blue K_t.



either
$\geq R(s-1,t)$

or
$\geq R(s,t-1)$

Hence:

$$R(s,t) \leq n = R(s-1,t) + R(s,t-1)$$

$$\leq \binom{s-1+t-2}{s-1-1} + \binom{s+t-1-2}{s-1}$$

$$= \binom{s+t-2}{s-1} \qquad \square$$

<u>Prop: 4.4</u> : $R(4,4) = 18$.

| x is a quadratic residue mod n. if $\exists\, y$ st $x \equiv y^2 \bmod n$

<u>Proof</u>: $R(4,4) > 17$

Let $n = 17$. Colour the edges of K_{17} as follows: $V(K_{17}) = \{0, 1, 2, \ldots, 16\}$.

Colour xy red iff $x-y$ is a quadratic residue mod 17 (Paley graph).

All other edges are blue. Can check that there is red K_4 and no blue K_4.

$R(4,4) \leq R(3,4) + R(4,3) = 9+9 = 18$ (Using proof of Theorem 4.3 and $R(3,4)=9$.

$\therefore R(4,4) = 18$ $\square$.

—/—

$43 \leq R(5,5) \leq 49$. $n = 45$, K_n

$2^{\frac{(45)^2}{2}}$

—/—

<u>Theorem 4.5</u> (No proof) (Conlon 2009). There exist $c>0$ a constant such that $R(s,s) \leqslant \frac{1}{s^{c\log s - \log\log s}} \binom{2s-2}{s-1}$

$$(\sqrt{2})^s < R(s,s) \leqslant \binom{s+s-2}{s-1} \leqslant \binom{2s}{s-1} < \binom{2s}{s} < 4^s$$

—/—

Let $s_1, s_2, \ldots, s_k \geqslant 2$ define

$R(s_1, s_2, \ldots, s_k) = \min\{n :$ Whenever the edges of K_n are coloured with colours $c_1, c_2, \ldots, c_k$, there is always a c_i-coloured K_{s_i} for some $1 \leq i \leq k\}$.

$$R_2(s_1, s_2) = R(s_1, s_2).$$

—/—

Thm: 4.12 For all $k \geq 2$ and $s_1, s_2, \ldots, s_k \geq 2$, $R_k(s_1, s_2, \ldots, s_k)$ is finite.

Proof: Induction on k. Ramsey's Thm $\Rightarrow$ true for $k = 2$, so let $k \geq 3$. Suppose $s_1, s_2, \ldots, s_k \geq 2$ are given.

Let $n = R_{k-1}(s_1, s_2, \ldots, s_{k-2}, R(s_{k-1}, k))$

Claim: $R_k(s_1, \ldots, s_k) \leq n$.

Take a colouring of the edges of K_n with colours $c_1, c_2, \ldots, c_k$.

Now suppose we cannot distingish between colours c_{k-1} and c_k.

In this way we have a colouring of the edges of K_n with $k-1$ colours: $c_1, c_2, \ldots, c_{k-2}$ and "c_{k-1} or c_k".

By definition of $R_{k-1}(s_1, s_2, \ldots, R(s_{k-1}, s_k))$ we either have c_i-coloured K_{s_i} for some $1 \leq i \leq k-2$ or we have a copy of $K_{R(s_{k-1}, s_n)}$ coloured with colours c_{k-1} and c_k.

But then Ramsey's theorem implies that this contains a c_{k+1}-coloured $K_{s_{k-1}}$ or a c_k-coloured K_{s_k}. $\square$.

$R_k(s) = R_k(\underbrace{s, \ldots, s}_{k})$

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$s, t \geq 2$

$R(s,t) = \min\{n :$ Every colouring of the edges of K_n with red and blue contains a red K_s or a blue $K_t\}$.

—/—

$$\sqrt{2}^{\,s} < R(s,s) \leq \binom{2s-2}{s-1} \leq 4^s$$

—/—

Th^m 4.6 If $n \geq s \geq 2$ satisfy

$$\binom{n}{s} \frac{2}{2^{\binom{s}{2}}} < 1 \qquad (*)$$

then $R(s,s) > n$.

Proof: Let n, s satisfy $(*)$ we need to prove there is a red-blue colouring of the edges of K_n with no monochromatic K_s.

Monochromatic $\equiv$ "all the same clour"

Define a random colouring as follows. Flip indepenant fair coins for each edge.

If coin is Heads colours edge red.
" " Tails " " blue.

Consider $X = \#$ of mono. copies of K_s.

$\boxed{\underline{\text{Claim}}\ \mathbb{E}[X] < 1}$.

$\Rightarrow \exists$ a colouring with no mono. K_s. Hence $R(s,s) > n$.

Fix $A \subset V(K_n)$. $|A| = s$. Let $X_A = \begin{cases} 1, & A \text{ forms mono } K_s \\ 0, & \text{otherwise.} \end{cases}$

$$\mathbb{P}(X_A = 1) = \mathbb{P}\begin{pmatrix}\text{All edges between} \\ \text{vertices in } A \text{ are red}\end{pmatrix} + \mathbb{P}\begin{pmatrix}\text{All edges between} \\ \text{vertices in } A \\ \text{are blue}\end{pmatrix}$$

$$= \frac{2}{2^{\binom{s}{2}}} \quad \left(\text{there are } \binom{s}{2} \text{ edges to consider}\right).$$

$$X = \sum_{\substack{A \subset V(K_n) \\ |A| = s}} X_A \Rightarrow \mathbb{E}[X] = \sum_{\substack{A \subset V(K_n) \\ |A| = s}} \Pr(X_A = 1)$$

$$= \binom{n}{s} \frac{2}{2^{\binom{s}{2}}} < 1$$

by $(*)$. $\square$.

— / —

$$\binom{n}{s} \frac{2}{2^{\binom{s}{2}}} < 1 \qquad — \qquad (*)$$

— / —

Corollary 4.7 If $s \geq 2$. $R(s,s) \geq 2^{s/2}$.

Pf: $R(s,s) = 2$, $R(3,3) = 6 \geq 2^{3/2}$

Let $s \geq 4$ and $n = \lfloor 2^{s/2} \rfloor$ need to show ⊛ holds $s! > 2^s$.

$$\binom{n}{s} < \frac{n^s}{2^s} \cdot \frac{2}{2^{\binom{s}{2}}} \leq \frac{2^{\frac{s^2}{2}+1}}{2^{\frac{s^2}{2}+\frac{s}{2}}} = \frac{1}{2^{\frac{s}{2}-1}} \leq \frac{1}{2} < 1$$

□

— / —

$$2^{s/2} \leq R(s,s) \leq \frac{4^s}{s}.$$

— / —

Fermats last theorem: If $n \geq 3$ there are no non trivial interger solution to $x^n + y^n = z^n$.

Proof (Exercise)

Th^m 4.9 For every $n \geq 1$ there exists p_n such that if $p \geq p_n$ is prime the congruence $x^n + y^n = z^n \mod p$ and non trivial solution.

Th^m 4.10. (Schur) For any $k \geq 1$ $\exists$ $S(k)$ such that in any k-colouring of the integers $\{1, 2, 3, \ldots, S(k)\}$ there is a monochromatic solution to $u + v = w$ (i.e. u, v, w all the same colour).

Proof: Recall $R_k(3) = \min\{n : \text{Every } k\text{-colouring of the edges } K_n \text{ contains a mono } K_3.\}$.

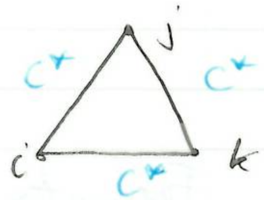
Set $n = R_k(3)$.

Consider a k-colouring of $\{1, 2, \ldots, n\}$ called c.

Define a k-colouring of the edges of K_n (with $V(K_n) = \{1, 2, \ldots, n\}$.

For $ij \in E(K_n)$, $i<j$ $c'(ij) = c(j-i)$

By definition of $R_k(3)$ there is a mono. K_3.



Say with vertices $i<j<k$.

So $c'(ij) = c'(ik) = c'(jk) = c^*$

$\Rightarrow \underbrace{c(j-i)}_{u=j-i} = \underbrace{c(k-i)}_{w=k-i} = \underbrace{c(k-j)}_{v=k-j} = c^*$.

So $u+v=w$ and $c(u) = c(v) = c(w) = c^*$

Hence $S(k)$ is well-defined and satisfies $S(k) \leq n = R_k(3)$

$\square$

Lemma 4.11: If p is prime and $\mathbb{Z}_p^* = \{1, 2, \dots, p-1\}$, then $\mathbb{Z}_p^*$ is a cyclic group. i.e $\exists\, g \in \mathbb{Z}_p^*$ st $\{g^1, g^2, \dots, g^{p-1}\} = \mathbb{Z}_p^*$.

Example: $p = 7,\ g = 3$: 3, 2, 6, 4, 5, 1

Thm 4.9: $\forall n \geq 1\ \exists\, p_n$ st. if $p \geq p_n$ is prime there are non-trivial solutions to $x^n + y^n = z^n \bmod p$

Pf: Let $n \geq 1$ be given. Take $p \geq S(n)$ (given by Schur's Thm) with p prime.

— / —

$u + v = w$.

$u = g^{m_u + c_u}$, $w = g^{m_w + c_w}$

$v = g^{m_v + c_v}$

For any $m\ \exists\, c$ st $m_u = a_u n + c_u$ $0 \leq c_u \leq n-1$.

— / —

By the Lemma 4.11 $\exists$ generator g for $\mathbb{Z}_p^*$

So for any $x \in \mathbb{Z}_p^*\ \exists\, m$ st $x = g^m \bmod p$.

Now define a colour for x, by $c(x) = c$ where $m = an + c$, $0 \leq c \leq n-1$.

So we have an n-colouring of $\{1, 2, \dots, p-1\}$

Since $p - 1 \geq S(n)$. $\exists\ u, v, w$ st $u + v = w$. st $c(w) = c(v) = c(w) = c$.

$\therefore\ u = g^{a_u n + c}$. Let $x = g^{a_u}, y = g^{a_v}, z = g^{a_w}$.

$v = g^{a_v n + c}$ $\quad x^n + y^n = ug^{-c} + vg^{-c}$

$w = g^{a_w n + c}$

$$= g^{-c}(u+v)$$

$$= g^{-c}w$$

$$= g^{a_w n} = z^n. \quad \square$$

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Thm: (Green + Tao 2009):
The primes contains arbitrarily long APs.

Thm: (Van der Waerden):
$\forall t, k \geq 1$ $\exists W(t,k)$ such that every k-colouring of $[W(t,k)]$ contains a MAP of length t.

Proof: Induction on t.
$W(1,k) = 1$
$W(2,k) = k+1$, since if we colour $[k+1]$ with k colours, some colour is used twice $\Rightarrow$ MAP length 2.

So now let $t \geq 3$ suppose $W(t-1,k)$ exists for all choices of k.

Claim: For $1 \leq r \leq k$ $\exists n_r(t,k)$ such that if $[n_r(t,k)]$ are k-coloured $\exists$ either a MAP of length t or $\exists r$ CFAPs of length $t-1$

Take the Claim with $r = k$. If we k-colour $[n_k(t,k)]$ then either we have a MAP length t or have $P_1, \ldots, P_k$ CFAPs length $t-1$.

AP = arithmetic progression.

$$\underbrace{a, a+d, a+2d, \ldots, a+(t-1)d}_{\text{AP length } t}$$

MAP = monochromatic AP.

Eg:
1 2 3 4 5 6 7 8 9
MAP length 3.

1 5 9
3 6 9 $f=9$
7 8 9 9

If $P_1, \ldots, P_r$ are MAPs each of a different colour and with the property that the next term in each P_i is the same, say f. Then we say $P_1, \ldots, P_r$ are colour-focused APs (CFAPs) with focus f.

So one of the P_i's has the same colour as their common focus thus we have a MAP length t. Hence can take $W(t,k) = n_k(t,k)$.

$\square$

<u>Proof of Claim</u>. Induction on r, $r=1$. Take $n_1(t,k) = W(t-1,k)$. Now suppose $2 \leq r \leq k$ and $n_{r-1}(t,k)$ exists, $n = n_{r-1}(t,k)$.

Let $n_r(t,k) = W(t-1, k^{2n})2n$.

Take a k-colouring of $[W(t-1, k^{2n})2n]$, $n = n_{r-1}(t,k)$. Assume there is no MAP length t.

$$[W(t-1, k^{2n})2n] = B_1 \dot\cup B_2 \dot\cup \dots \dot\cup B_i \dot\cup \dots B_{W(t-1,k^{2n})}$$

$$\text{where } B_1 = \{1, \dots, 2n\}$$
$$B_2 = \{2n+1, \dots, 4n\} \text{ etc...}$$

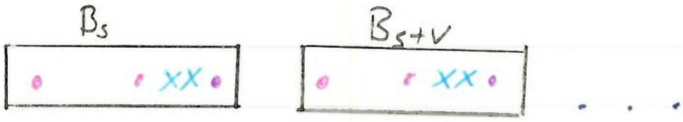
B_i has been coloured with k-colours, $\therefore$ there are k^{2n} different ways a block could be coloured.

By def$^{\underline{n}}$ of $W(t-1, k^{2n})$ have $B_s, B_{s+v}, B_{s+2v}, \dots, B_{s+(t-2)v}$ identically coloured blocks.

Each B_i has length $2n_{r-1}(t,k)$. $\therefore$ Each B_s contains $P_1, \dots, P_{r-1}$ CF APs of length $t-1$. together with their focus.

$$P_i = a_i, a_i+d_i, a_i+2d_i, \dots, a_i+(t-2)d_i \quad 1 \leq i \leq r-1$$

Common focus is f.



Since $B_s, B_{s+v}, \ldots, B_{s+(t-2)v}$ are all coloured identically the following are CFAPs.

$$P_i' = a_i,\ a_i + (d_i + 2nv),\ a_i + 2(d_i + 2nv), \ldots, a_i + (t-2)(d_i + 2nv)$$

Clearly all MAPs length $t-1$ different colours. Focus is $f + (t-1)2nv$.

Moreover $P_r' = f, f+2nv, f+4nv, \ldots, f+(t-2)2nv$ is another MAP of length $t-1$ and a different colour to $P_1', \ldots, P_{r-1}'$.

So $P_1', \ldots, P_t'$ are r colour-focussed APs length $t-1$ with common focus $f + (t-1)2nv$. Thus setting $n_t(r,k) = W(t-1, k^{2n})$ will do $\square$.