3503 Graph Theory and Combinatorics Notes

Based on the 2011 spring lectures by Dr J Talbot

3503 GRAPH THEORY & COMBINATORICS

John Talbot talbot@math.ucl.ac.uk
Office 812A
Office hour: Fri 11am

Homework: due Wed 12pm

Lots of the course from/inspired by the work of Paul Erdős.

We'll look at: Graph Theory
Extremal Set Theory
Ramsey Theory

1. INTRODUCTION

Notation:

$\mathbb{N} = \{1, 2, 3, \dots\}$
$[n] = \{1, 2, 3, \dots, n\}$

If X is a set and $k \geq 0$ an integer,

Subsets of X of size k $\longrightarrow$ $\binom{X}{k} = \{A \subseteq X : |A| = k\}$ e.g. $\binom{X}{0} = \{\emptyset\}$.

$|X|$ = size of X
$\#$ = no. of

e.g. $\binom{[4]}{2} = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}$

$= \{12, 13, 14, 23, 24, 34\}$

Lemma 1.1 If $|X| = n$ then

$$\left|\binom{X}{k}\right| = \binom{n}{k} := \frac{n!}{k!(n-k)!}$$

Proof: # ordered k-tuple of distinct elements of $X = n(n-1)\cdots(n-k+1)$.
Each k-subset of X occurs $k!$ times
$\Rightarrow$ # k subsets for $X = \dfrac{n(n-1)\cdots(n-k+1)}{k!} = \binom{n}{k}$ $\square$

$\mathcal{P}(X)$ = power set of $X = \{A : A \subseteq X\}$

e.g. $\mathcal{P}([2]) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$
$= \{\emptyset, 1, 2, 12\}$

Lemma 1.2 If $n \geq 1$ is an integer then $|\mathcal{P}([n])| = 2^n$.

Proof (inductive): $\mathcal{P}([1]) = \{\emptyset, 1\}$ $\qquad |\mathcal{P}([1])| = 2 = 2^1$

$n=1$ ✓

Consider $\mathcal{P}([n+1])$

$$= \mathcal{P}([n]) \overset{\text{disjoint union}}{\dot{\cup}} \{A \subseteq [n+1] : n+1 \in A\}$$

$$\{A \subseteq [n+1] : n+1 \in A\} = \{B \cup \{n+1\} : B \in \mathcal{P}([n])\}$$

Now inductive hypothesis $\Rightarrow |\mathcal{P}([n])| = 2^n$

$$\Rightarrow |\{A \subseteq [n+1] : n+1 \in A\}| = 2^n$$

$$\Rightarrow |\mathcal{P}([n+1])| = 2^{n+1}$$

□

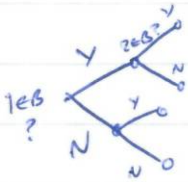
Proof (direct combinatorial proof):

$[n]$ contains n elements

For each $a \in [n]$, ask "does $a \in B$"?
The n answers to these questions specify each $B \in \mathcal{P}([n])$ uniquely.

Number of different answers $= 2^n \Rightarrow |\mathcal{P}([n])| = 2^n$

□



Lemma 1.3 If $0 \leq k \leq n$ then $\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}$

Proof: $\binom{[n+1]}{k+1} = \binom{[n]}{k+1} \dot\cup \{A \subseteq [n+1] : n+1 \in A \text{ and } |A| = k+1\}$

And $\{A \subseteq [n+1] : n+1 \in A \text{ and } |A| = k+1\}$
$= \{B \cup \{n+1\} : B \subseteq [n],\ |B| = k\}$

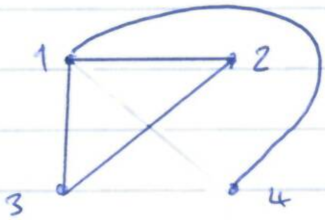
But $|\{B \cup \{n+1\} : B \subseteq [n],\ |B| = k\}| = \left|\binom{[n]}{k}\right| = \binom{n}{k}$.

and $\left|\binom{[n+1]}{k+1}\right| = \binom{n+1}{k+1}$, and $\left|\binom{[n]}{k+1}\right| = \binom{n}{k+1}$ $\square$

2. GRAPH THEORY

A graph (network) is a pair $G = (V, E)$, where V is a set of vertices (nodes) and $E \subseteq \binom{V}{2}$ "a pair of vertices" called edges.

e.g. $V = [4]$, $E = \{12, 13, 23, 14\}$
gives



When confusion could arise, write $V(G)$ and $E(G)$ for the vertices and edges of G

The order of G is the n°. of vertices
The size of G is the n°. of edges.

If $v \in V(G)$, the neighbourhood (nbhd) of v is

$$\Gamma(v) = \{u \in V(G) : uv \in E(G)\}$$

set of neighbours!

e.g. in above example, $\Gamma(1) = \{2, 3, 4\}$
$\Gamma(2) = \{1, 3\}$

$|\Gamma(v)| = d(v)$ is the degree of v. (how many friends you have on Facebook)

If $uv \in E(G)$ then u and v are adjacent

Lemma 2.1 If $G = (V, E)$ is a graph then

$$\sum_{v \in V} d(v) = 2|E|.$$

Proof: Each edge is counted twice, once as part of $d(u)$ and once as part of $d(v)$. $\square$

Lemma 2.2 If $G = (V, E)$ is a graph then the nº. of vertices of odd degree is even.

Proof: If $V = V_{even} \,\dot{\cup}\, V_{odd}$

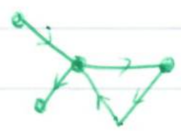
We know $$2|E| = \underbrace{\sum_{v \in V_{even}} \overset{\text{even}}{d(v)}}_{\text{even}} + \underbrace{\sum_{v \in V_{odd}} \overset{\text{odd}}{d(v)}}_{\text{even}}$$

$\Rightarrow |V_{odd}|$ is even. $\square$

2.2 Components & Connectedness

A path is a sequence of distinct vertices $v_0 v_1 \dots v_t$ s.t. $v_{i-1} v_i \in E$ for $1 \leq i \leq t$

A walk is a sequence of vertices $v_0 v_1 \dots v_t$ s.t. $v_{i-1} v_i \in E$ for $1 \leq i \leq t$



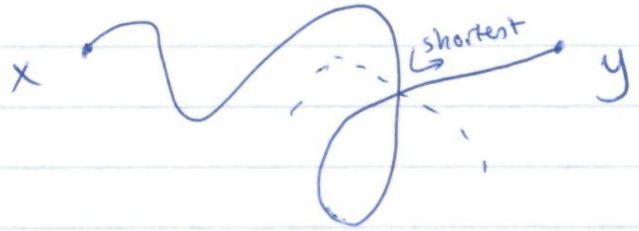
A walk is closed if $v_0 = v_t$

A walk in which no edges are repeated is called a tour.

Lemma 2.3 There is a path from x to y in G iff there is a walk from x to y in G.

Proof: $(\Rightarrow)$ by defⁿ

($\Leftarrow$) Suppose ~~x~~ there is a walk from x to y in G



Take a shortest walk from x to y. This must be a path since otherwise it could be shortened. $\square$

<u>Lemma 2.4</u> Define a relation $\sim$ on V by $v \sim w$ iff there is a walk from v to w. This is an equivalence relation.

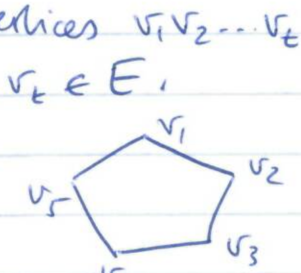
<u>Proof</u> (trivial). $\square$

Let $V = V_1 \cup V_2 \cup \dots \cup V_k$ be the partition into equivalence classes induced by $\sim$. We call the V_i <u>components</u>.
If there is just one component, we say the (whole) graph is <u>connected</u>.

2.3 HAMILTON CYCLES

A <u>cycle</u> in G is a sequence of <u>distinct</u> vertices $v_1 v_2 \dots v_t$ s.t. $v_i v_{i+1} \in E$ for $1 \le i \le t-1$ and $v_1 v_t \in E$.

e.g.



A <u>Hamilton cycle</u> in G is a cycle containing every vertex of G (exactly once)

If $G=(V,E)$ is a graph then the <u>minimum</u> degree of G is $\delta(G) = \min\{d(v) : v \in V(G)\}$

<u>Theorem 2.5 (Dirac's Thm)</u> :

If $G=(V,E)$ is a graph of order $n \geq 3$ and $\delta(G) \geq n/2$, then G has a Hamilton cycle.

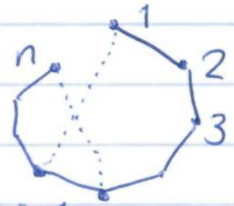
<u>Proof</u>: (by contradiction)
Let G have $n \geq 3$ vertices and $\delta(G) \geq n/2$, but does not have a Hamilton cycle.

If there is any missing edge that we can add to G without creating a Hamilton cycle, then <u>add one</u>. Repeat until there are no such missing edges.

Now G contains a Hamilton cycle with one edge removed.

WLOG let $V(G) = [n]$ and suppose the edges $i\ i+1$ belong to G for $i=1,\dots,n-1$

look at dotted 'possible' edges



For $i=2,\dots,n-2$ consider the possible edge $1\ i+1$ and in.

Now, if <u>both</u> are present, then G contains a Hamilton cycle ($1\ i+1\ i+2 \dots n\ i\ i-1\ i-2 \dots 2\ 1$)

Edges involving 1 and n:

$1n \notin E(G)$

$12 \in E(G)$

pairs of $1i{+}1$, in (i.e. pairs of dotted lines)

$$\left.\begin{array}{l} 13,\ 2n \leftarrow i=2 \\ 14,\ 3n \leftarrow i=3 \\ \vdots \\ 1\,n{-}1,\ (n{-}2)n \end{array}\right\} n-3 \text{ pairs,}$$

$i = n-2$

(picking vals of i)

at most $n-3$ of the edges are in $E(G)$ since we can't have both $1\,i{+}1$ and in for $2 \le i \le n-2$.

i.e. can only pick one from each pair

$(n-1)n \in E(G)$

What is $\underbrace{d(1) + d(n)}_{} \le \underset{12\ (n-1)n}{2} + \underset{\text{max } n-3 \text{ edges in bracket}}{n-3} = n-1$

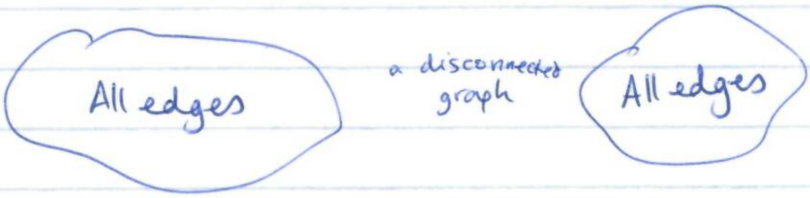
$\Rightarrow \min\{d(1), d(n)\} \le \frac{n-1}{2}$

$\Rightarrow \delta(G) \le \frac{n-1}{2} < \frac{n}{2}$ $\#$.

This result is the best possible for all $n \geq 3$!!

To show this,

if n even, $n = 2k$. Need G with $2k$ vertices $\delta(G) = k-1$ and G has no Hamilton cycle.



a vertices $\qquad$ $n-a$ vertices

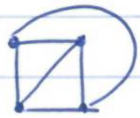
$d(v) = a-1$ $\qquad$ $d(v) = n-a-1$

$\rightarrow \delta(v) = \min\left\{\begin{matrix} n-a-1 \\ a-1 \end{matrix}\right\}$

So choose $a = k \rightarrow \delta(G) = k-1$. No Hamilton cycle.

<u>Def^{n}</u>: The <u>complete graph</u> of order t is K_t

$$V(K_t) = [t]$$
$$E(K_t) = \binom{[t]}{2}$$

e.g. $K_3 =$  $K_4 =$ 

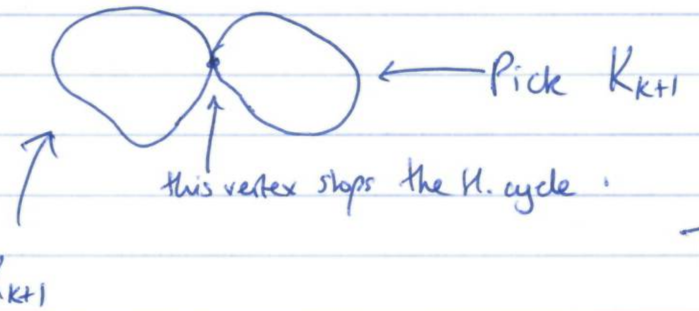
So Graph G described just before was $G = K_k \dot{\cup} K_k$

<u>if n odd</u> $n = 2k+1$. Need G with $2k+1$ vertices and $\delta(G) = k$, with no Hamilton cycle.

Can't use same approach because you <u>could</u> say

but then you have min. degree $k-1$ (in the LHS).

Different way to destroy a Hamilton cycle.



Total vertices
$2(k+1) - 1$ (shared)

So G above is two copies of K_{k+1} with a vertex from each identified: $|V(G)| = 2k+1$

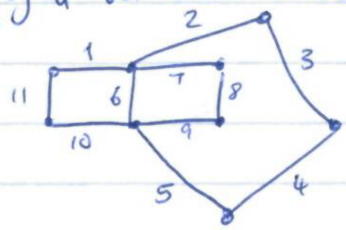
$\delta(G) = k$.

No Ham. cycle

Recall that a <u>tour</u> is a walk with no repeated edges.

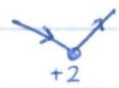
An <u>Euler circuit</u> in a graph is a tour $v_0 v_1 \dots v_t v_0$ that is closed containing every <u>edge</u> of G (exactly) once, and every <u>vertex</u> of G at least once.

no edge repeated

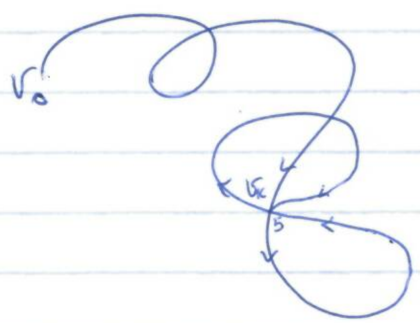


<u>Thm 2.6</u> A graph G has an Euler circuit iff it is connected and the degree of every vertex is even

<u>Proof</u>: ($\Rightarrow$) Let G be a graph with an Euler circuit $v_0 v_1 \dots v_t v_0$. Since this is a walk containing all vertices, then G is connected. If you walk around the circuit counting the contribution of the edge you're on to each vertex degree, we add 2 as we pass any vertex, and the first and last edges add 2 to the degree of v_0. Hence all degrees are even.



($\Leftarrow$) Take a longest tour in G: $T = v_0 v_1 \dots v_k$.
<u>Claim</u>: $v_k = v_0$. If not, consider $j = \#\{i \in [k-1] : v_i = v_k\}$



We have used $\underbrace{2j+1}_{\text{odd.}}$ edges containing v_k.
But all vertices have even edges so there is an ~~...~~ unused edge $v_k x \Rightarrow$
$T' = v_0 v_1 \dots v_k x$ is a longer tour #

$\Rightarrow v_k = v_0$.

Now we need to show it uses every edge in G.
Suppose we have an unused edge $e = uv$.

<u>Case 1</u>: u or v lie in T.

Suppose $v = v_i$. consider $T' = u v_i v_{i+1} \dots \underset{v_0}{v_k} \longrightarrow v_1 \dots v_{i-1} v_i$.

This is longer than T #

<u>Case 2</u>: If u and v do not lie in T.
Since G is connected there is a path from v_0 to u. Consider the edge where this path leaves T.
This is case 1 again. $\square$

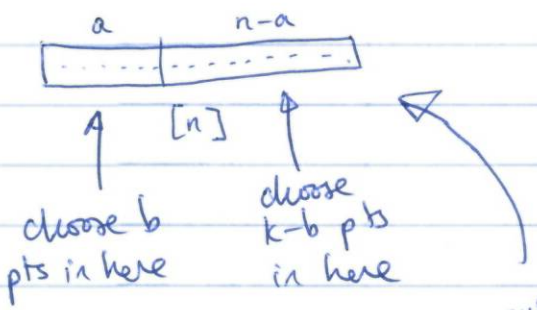
<u>PROBLEM SHEET QNS</u>

2. Prove if $0 \le a,\ k \le n$, then

$$\sum_{b=0}^{k} \binom{a}{b}\binom{n-a}{k-b} = \binom{n}{k}$$

$\binom{c}{d} = 0$ if $\begin{cases} d < 0 \\ \text{or } c < d \end{cases}$

$\underbrace{\text{k-subsets}}_{\text{subsets of size } k}$ of $[n]$



subsets of $[n]$ of size k with b elements in $[a]$
$k-b$ - - - $[n] \setminus [a]$

For $0 \le b \le k$ let $\mathcal{F}_b = \{ F \subseteq [n] : |F \cap [a]| = b,\ |F| = k \}$

$$\binom{[n]}{k} = \mathcal{F}_0 \,\dot\cup\, \mathcal{F}_1 \,\dot\cup \dots \dot\cup\, \mathcal{F}_k$$

$$\left| \binom{[n]}{k} \right| = \sum_i |\mathcal{F}_i|$$

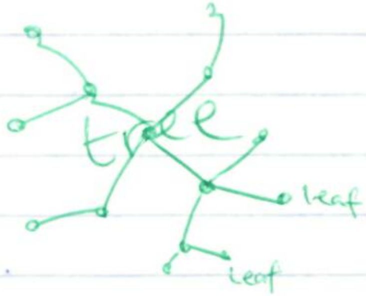
So $|\mathcal{F}_b| = \binom{a}{b}\binom{n-a}{k-b}$.

One-line proof: Partition the k-subsets of $[n]$ according to how many elements they contain from $[a]$.

3. $S_{n,k} = \#\{(x_1, \dots, x_k) : x_1 + \dots + x_k = n, \quad x_i \geq 0 \text{ integers}\}$

$S_{n,k} = \binom{n+k-1}{k-1}$ = n° of $k-1$ subsets from $[n+k-1]$

Think about $\binom{[n+k-1]}{k-1}$.



TREES

A connected graph with no cycles is a <u>tree</u>.

A vertex in any graph of degree 1 is called a <u>leaf</u>

<u>Lemma 2.7 (Leaf lemma)</u>: Any tree with at least two vertices has at least two leaves.

<u>Proof</u>: Take a longest path $P = v_0 v_1 \dots v_t$ in the tree T. (vertices not repeated)
If $d(v_t) \neq 1$ then $d(v_t) \geq 2$ so there is an edge $v_t x$, where $x \notin \{v_0 v_1 \dots v_{t-1}\}$ because T has no cycle.
So $P' = v_0 v_1 \dots v_t x$ is a longer path #

Similarly v_0 is a leaf. $\square$

If G is a graph and $v \in V(G)$ then $G-v$ is the graph with $V(G-v) = V(G) \setminus \{v\}$
and $E(G-v) = E(G) \setminus \{uv : u \in \Gamma(v)\}$
$\Gamma(v)$: neighbourhood

$G-e$: $V(G-e) = V(G)$
$E(G-e) = E(G) \setminus \{e\}$.

connected graph
no cycles

Lemma 2.8 (Tree growing lemma)

If G is a graph with a leaf v then
G is a tree $\iff$ $G-v$ is a tree.

Proof: ($\Rightarrow$) deleting a leaf cannot disconnect the graph or introduce any cycles

($\Leftarrow$) adding a leaf also cannot disconnect the graph or introduce any cycles. $\square$

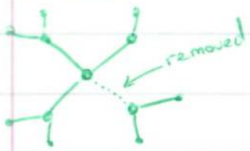
Lemma 2.9 Let G be a graph.
The following are equivalent:

(i) G is a tree

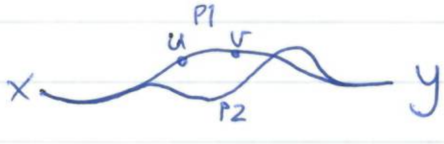
(ii) G is connected and removing any edge makes it disconnected.

(iii) $\forall\, x, y \in V(G)$ there is a unique path from x to y.

Proof (i)$\Rightarrow$(ii) By def^{n} of a tree G is connected.
If G with the edge $e = uv$ removed is still connected then there is a path from u to v in $G-e$. This forms a cycle in G.



(ii)$\Rightarrow$(iii) Since G is connected, there is at least 1 path between any pair of vertices.
So suppose $\exists$ distinct paths P_1, P_2 from x to y.
Let $e = uv$ be an edge of P_1 but not of P_2.



Claim: removing $e = uv$ does not disconnect G.
Proof: $G-e$ is still connected since $\exists$ a walk in $G-e$ from u to v. (follow P1 from u to x then P_2 from x to y and back along P_1 to v.) $\#$

(iii)$\Rightarrow$(i) G is clearly connected since there is a path joining any pair of vertices. If G contains a cycle then there are at least 2 paths joining any pair of adjacent vertices from the cycle. $\square$

Thm 2.10 (Euler's formula for trees)

A connected graph G is a tree iff $|V(G)| = |E(G)| + 1$

Proof: ($\Rightarrow$) Induction on $n = |V(G)|$. True for $n=1$ ✓

If $n \geq 2$ ~~then~~ and G is a tree with $|V(G)| \geq 2$ then G has a leaf v. By Lemma 2.8, $G-v$ is also a tree so our inductive hypothesis $\Rightarrow |V(G-v)| = |E(G-v)| + 1$

e.g. 6 edges 7 verts

Since $|V(G-v)| = n-1$
$|E(G-v)| = |E(G)| - 1$, the result follows.

($\Leftarrow$) Induction on $n = |V(G)|$. True for $n=1$ ✓

Let G be a connected graph with $n \geq 2$ vertices and $|V(G)| = |E(G)| + 1$.

G is connected $\Rightarrow d(v) \geq 1 \; \forall v \in V(G)$.

$$\sum_{v \in V(G)} d(v) = 2|E(G)| = 2(n-1)$$

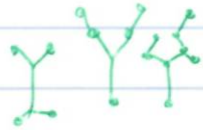
$\Rightarrow$ average degree in $G = \frac{2n-2}{n} < 2$.

$\Rightarrow \exists v \in V(G)$ with $d(v) < 2 \Rightarrow v$ is a leaf!

Apply the inductive hypothesis to $G-v \Rightarrow$ result. $\square$

<u>Def</u>

Def^n: A graph with no cycles is called a <u>forest</u>.



<u>Corollary 2.11</u> If G is a forest with k connected components then $|E(G)| = n-k$

<u>Proof</u>: Let $G_1, \dots, G_k$ be connected components of G.

$$|E(G_i)| = |V(G_i)| - 1,$$

$$\text{Hence } |E(G)| = \sum_{i=1}^{k} |E(G_i)|$$
$$= \sum_{i=1}^{k} \left[|V(G_i)| - 1 \right]$$
$$= n-k$$
$\square$

<u>Def^n</u>: Let G be a graph. If T is a tree with $V(T) = V(G)$ then T is a <u>spanning tree</u> of G

<u>Lemma 2.12</u> If G is connected then it contains a spanning tree.

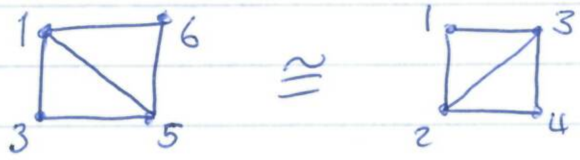
<u>Proof</u>: If G is a tree, let $T = G$.

Else by Lemma 2.9(ii) $\exists\, e \in E(G)$ s.t. $G-e$ is connected. Delete e from G and repeat. Eventually we find a spanning tree. $\square$

How many trees are there with n vertices?

<u>Def^{n}</u>: We say that graphs G and H are <u>isomorphic</u> if $\exists$ a bijection $f: V(G) \to V(H)$ s.t. $\forall\, u,v \in V(G)$ $f(u)f(v) \in E(H)$ iff $uv \in E(G)$.

e.g.



How many labelled trees are there with n vertices using the labels $\{1,2,\ldots,n\} = [n]$?

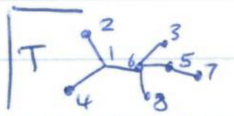
<u>Thm 2.13 (Prüfer)</u>: There are exactly n^{n-2} labelled trees with n vertices and labels $[n]$.

[what is n^k? It is # sequences of length k with elements from $[n]$
so n^{n-2} is # . . . . . . $n-2$ . . . . . .]

<u>Proof</u>: $n^{n-2} = \#\underbrace{\text{sequences of length } n-2 \text{ with elements from } [n]}_{S_n}$.

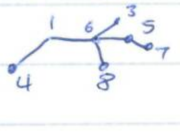
Let $\mathcal{T}_n$ = Family of all labelled trees of order n and labels $[n]$.

To show $|\mathcal{T}_n| = |S_n|$, we need $f: \mathcal{T}_n \to S_n$ a bijection.



$n=8$.

(1)



$(1, 6)$

Find the smallest leaf in T. Delete it and record the label of its neighbour. Repeat until only 2 vertices remain.

eventually get

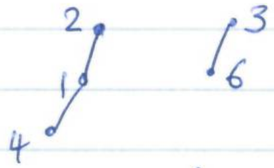
(1, 6, 1, 6, 5, 6)

REVERSING.

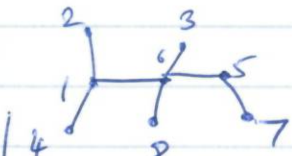
Length of (1, 6, 1, 6, 5, 6) is 6 so
$n=8$.

$\{1, 3, 4, 5, 6, 7, 8\}$
$(\cancel{6}, 1, 6, 5, 6)$

$\{1, \cancel{3}, \cancel{4}, 5, 6, 7, 8\}$
$(\cancel{6}, \cancel{1}, 6, 5, 6)$



⋮



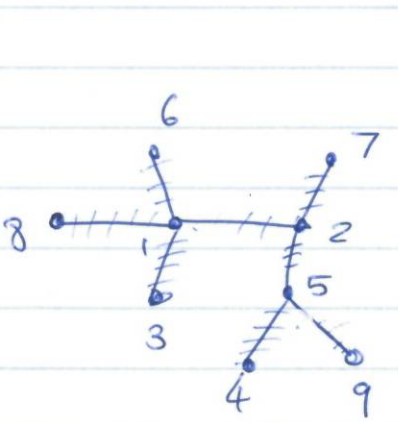
How do you show it's a bijection? By reversing the process.

All the original leaves of T do not appear in the sequence: $(p_1, \dots, p_{n-2})$.

Let $l = \min\left\{ [n] \setminus \{p_1, p_2, \dots, p_{n-2}\} \right\}$

Insert an edge from l to p.
Repeat with $(p_2, \dots, p_{n-2})$ and $[n]\setminus\{l\}$

Another example



$n=9$

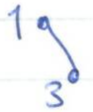
del: 3 4 6 7 8 1 9

"Prüfer code" → (1, 5, 1, 2, 1, 2, 5)   Left with

Let P = labels in code = $\{1, 2, 5\}$

Backwards: say $n = 7+2 = 9$
so use labels $\{1, 2, 3, 4, 5, 6, 7, 8, 9\} = \mathcal{L}$

What is the minimum element of $\mathcal{L} \setminus \mathcal{P}$?
3.
and it's joined to the first thing in the code, 1.

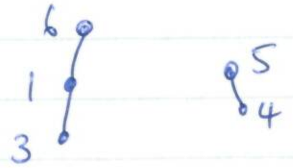


Sequence becomes $(5, 1, 2, 1, 2, 5)$
$\mathcal{L} \to \mathcal{L} \setminus \{3\}$.
$\mathcal{P}$ is still $\mathcal{P}$.

What is the minimum element of $\mathcal{L} \setminus \mathcal{P}$?
4.
And it's joined to the first thing in our new sequence, 5.

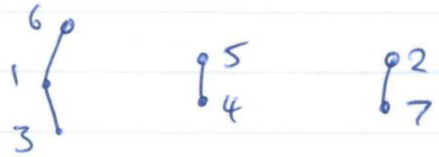
$\mathcal{L} \to \mathcal{L} \setminus \{4\}$, ie. $\mathcal{L} = \{1, 2, 5, 6, 7, 8, 9\}$
Seq. becomes $(1, 2, 1, 2, 5)$
$\mathcal{P}$ is still $\mathcal{P}$.

Min el. of $\mathcal{L} \setminus \mathcal{P}$? 6. Joined to 1.



Seq. becomes $(2, 1, 25)$
$\mathcal{L} \to \mathcal{L} \setminus \{6\} = \{1, 2, 5, 7, 8, 9\}$
$\mathcal{P} = \mathcal{P}$.

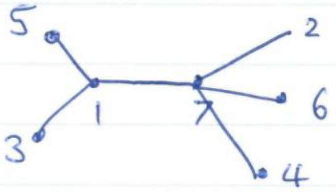
Repeat. Smallest el. of $L \setminus S$ is 7.
joined to 2.



repeat ad infinitum, eventually get an empty sequence
L becomes $\{5, 9\}$.

These last two things are connected.

<u>Another example</u>



$n=7$

becomes $(7,1,7,1,7)$.

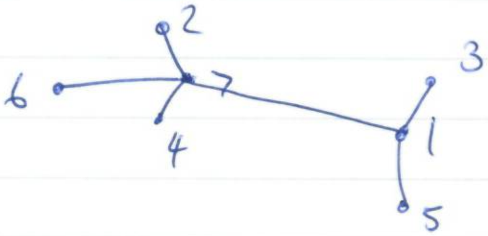
left with.

always add 2

<u>Backwards</u>: Seq is $(\cancel{7},\cancel{1},\cancel{7},\cancel{1},\cancel{7})$ length 5 $\Rightarrow n = 5+2 = 7$

$L = \{\cancel{1}, \cancel{2}, \cancel{3}, \cancel{4}, \cancel{5}, 6, 7\}$

then these two joined.

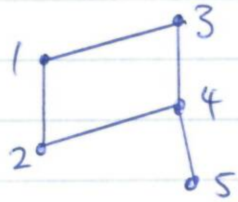


Def^n: A graph is <u>r-regular</u> if $\forall v \in V(G)$, $d(v) = r$

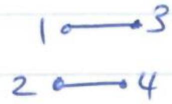
Def^n: If G and H are graphs, we say H is a <u>subgraph</u> of G, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

Def^n: We say a subgraph H of G is <u>induced</u> if $E(H) = E(G) \cap \binom{V(H)}{2}$. ie all the same edges as G.

<u>examples</u>: G

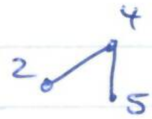


$H_1 =$



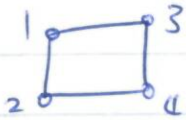
<u>not</u> induced

$H_2 =$



induced

$H_3 =$



induced

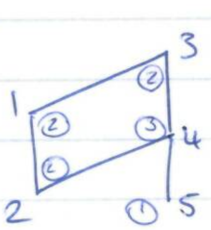
If $G = (V, E)$ is a graph and $A \subseteq V(G)$ then $G[A]$ is the induced subgraph of G <u>induced by</u> A. ie. $V(G[A]) = A$ and $E(G[A]) = E(G) \cap \binom{A}{2}$.

If $G = (V, E)$ is a graph, its <u>degree sequence</u> is

$$\left(d(v_1), d(v_2), \ldots, d(v_n)\right), \text{ where } V = \{v_1, \ldots, v_n\}$$

and $d(v_1) \leq d(v_2) \leq \cdots \leq d(v_n)$

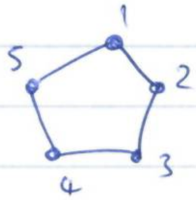
e.g.

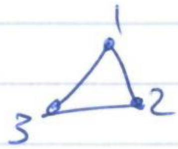


degree sequence $(1, 2, 2, 2, 3)$.

Some graphs : K_n : $V(K_n) = [n]$, $E(K_n) = \binom{[n]}{2}$.

The cycle of length n is C_n : $V(C_n) = [n]$
(n ≥ 3)
$E(C_n) = \{\{i, i+1\} : 1 \le i \le n-1\} \cup \{\{1, n\}\}$

C_5 

C_3 

$= K_3$.

The path of length n is $V(P_n) = \{0, 1, \dots, n\}$
$E(P_n) = \{\{i-1, i\} : 1 \le i \le n\}$

e.g. P_0 : 

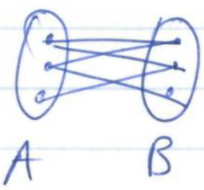
P_5 : 

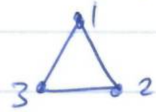
The empty graph E_n of order n is $V(E_n) = [n]$
$E(E_n) = \emptyset$.

If G is a graph, G^c is G complement is the graph of G
with $V(G^c) = V(G)$, and $E(G^c) = \binom{V(G)}{2} \setminus E(G)$

e.g. $E_n = K_n{}^c$. all verts opp. edges

A bipartition of a graph $G = (V, E)$ is a partion $V = A \dot\cup B$
s.t. $E(G) \subseteq \{ab : a \in A, b \in B\}$.

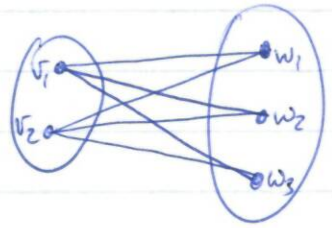


Not every graph can be a bipartitioned,
e.g. $K_3 = C_3$ 

A graph G is bipartite if it has a bipartition.

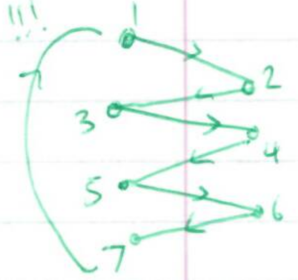
$K_{a,b}$ is the complete bipartite graph
$V(K_{a,b}) = \{v_1, \dots, v_a\} \dot\cup \{w_1, \dots, w_b\}$
$E(K_{a,b}) = \{v_i w_j : i \in [a], j \in [b]\}$

$K_{2,3}$



<u>Theorem 2.14</u> A graph G is bipartite iff it contains no <u>odd</u> cycles.

<u>Proof</u>: (⇒) G contains an odd cycle ⇒ G is not bipartite.



If C_{2t+1} is contained* in G, suppose G has bipartition $V(G) = A \dot\cup B$.

wlog if $V(C_{2t+1}) = \{v_1, \dots, v_{2t+1}\}$ then
$v_1 \in A \Rightarrow v_2 \in B \Rightarrow v_3 \in A \rightarrow \cdots \Rightarrow v_{2t+1} \in A$ ※
as $v_1 v_{2t+1} \in E(G)$.

(⇐) G has no odd cycles ⇒ G is bipartite
Assume G is connected.
Choose a vertex $v_0 \in V(G)$. Let $v_0 \in A$.
Define dist(x,y) = length of shortest path from x to y.

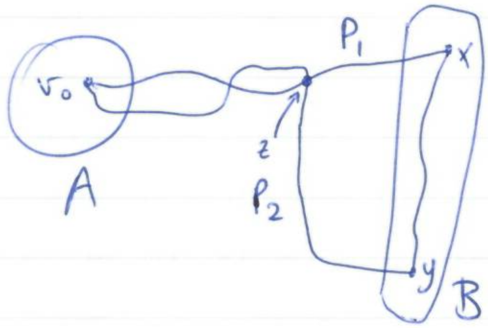
Then $A = \{v \in V(G) : \text{dist}(v, v_0) \text{ is even}\}$
$B = \{v \in V(G) : \text{dist}(v, v_0) \text{ is odd}\}$.

PTO

[* <u>Def^n</u>: we say G <u>contains</u> H if G has a subgraph which is isomorphic to H.]

(contd) Since G is connected, $V(G) = A \dot\cup B$.
Need to show there are no edges inside A or B.

Suppose $x, y \in B$ and $xy \in E(G)$



Let P_1 be the shortest path from v_0 to x of length $2i+1$
" P_2 " " " " v_0 to y " $2j+1$

Let z be the last common vertex on P_1 and P_2
(if they don't intersect then $z = v_0$ so we're OK).

Let $\mathrm{dist}(v_0, z) = d$. Any shortest path from v_1 to v_2 through v_3 then the part of that path from v_1 to v_3 is a shortest path from v_1 to v_3.

Hence the n° of edges on P_1 from z to x $= 2i+1-d$.

Similarly the n° of edges on P_2 from z to y $= 2j+1-d$.

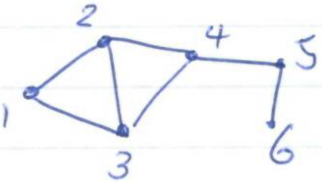
If C is the cycle formed ~~from~~ by following P_1 from z to x, then xy, then P_2 from y to z.

Then C has $(2i+1-d) + 1 + (2j+1-d)$ edges
$= 2(i+j+1-d) + 1$ edges, an <u>odd</u> n°

ie. C is an odd cycle $\# \square$.

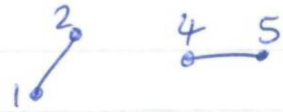
A <u>matching</u> in a graph is a set of disjoint edges

e.g. $G =$



$G = ([6], \{12, 13, 23, 24, 34, 45, 56\})$

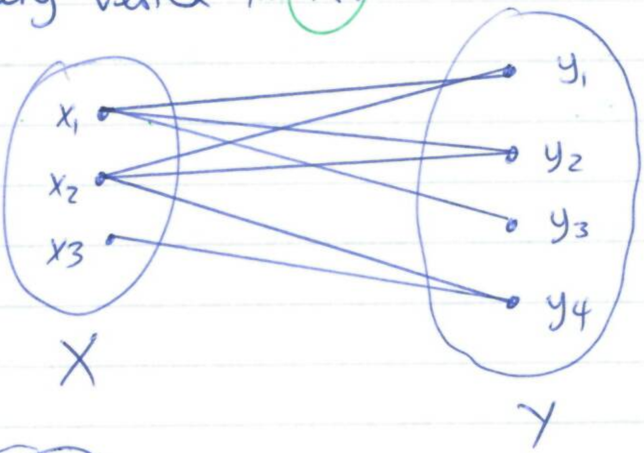
$M_1 =$



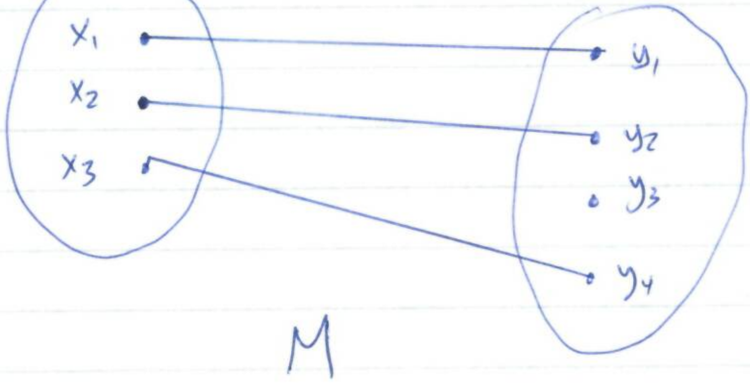
$M_2 =$

If $G = (X, Y; E)$ is a bipartite graph (with bipartition X, Y) then a <u>complete matching</u> is a matching that covers every vertex in X.

e.g.



G



M

For a complete matching in $G = (X, Y; E)$ need:

(1) $|Y| \geqslant |X|$ or better, $|\Gamma(X)| \geqslant |X|$ — nº of jobs ppl want to do

(2) $\forall x \in X \quad |\Gamma(\{x\})| \geqslant 1$ — nº of neighbours of $x \geqslant 1$.

(3) $\forall x_1, x_2 \in X, \ x_1 \neq x_2, \quad |\Gamma(\{x_1, x_2\})| \geqslant 2$

The general condition is:

(*) If $A \subseteq X$ then $|\Gamma(A)| \geqslant |A|$ — necessary!

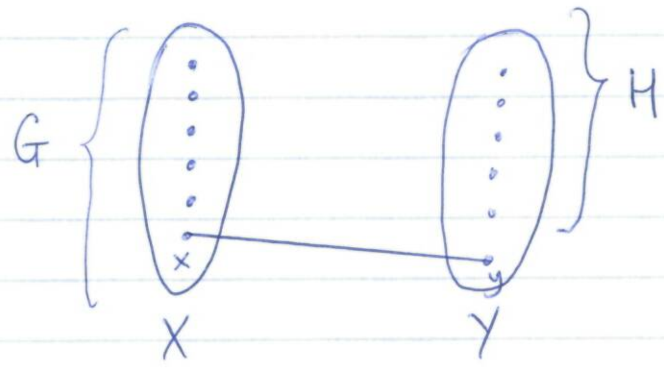
Thm 2.15 (König-Hall)

If $G = (X, Y; E)$ then G has a complete matching iff $\forall A \subseteq X \quad |\Gamma(A)| \geqslant |A|$.

Proof: ($\Rightarrow$) Let M be a complete matching in $G = (X, Y; E)$.
If $A \subseteq X$ then $|\Gamma(A)| \geqslant |\{y : ay \in M, a \in A\}| = |A|$.

($\Leftarrow$) (Induction on $|X|$)
Let $G = (X, Y; E)$ satisfy (*).
(0) True for $|X| = 1$.
(i) Suppose that for every non-empty proper subset $A \subset X$ we have $|\Gamma(A)| \geqslant |A| + 1$

Let $xy \in E$, and let $H = G - x - y$.

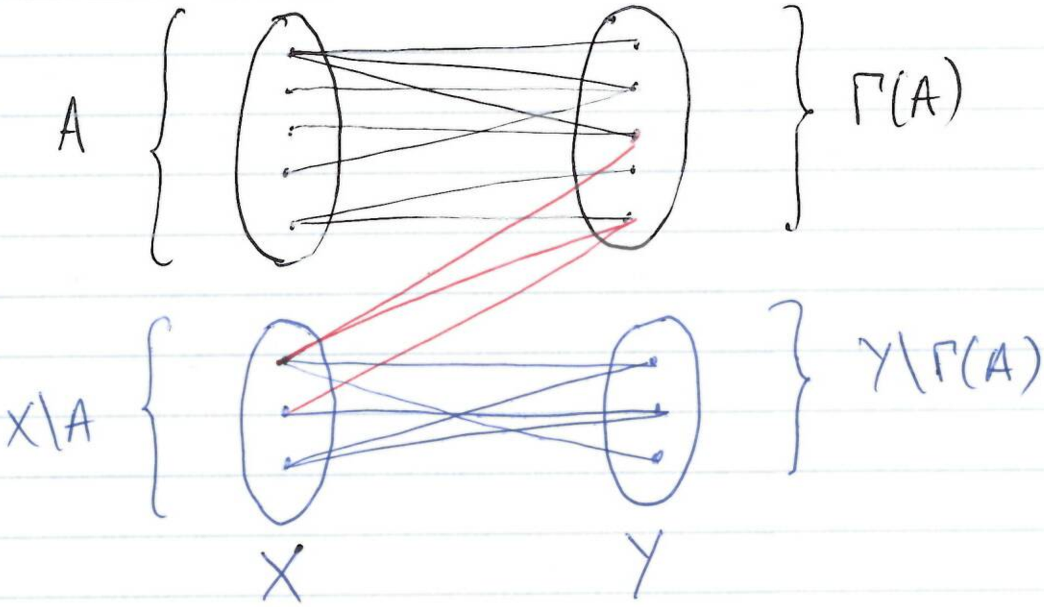


H is bipartite. If $A \subseteq X \setminus \{x\}$ then

$$|\Gamma_H(A)| \geq |\Gamma_G(A)| - 1 \geq |A| + 1 - 1 = |A|$$

By our inductive hypothesis, H has a complete matching, together with xy this gives a complete matching in G.

(2) If (1) does not hold then $\exists\ \emptyset \neq A \neq X$, $A \subset X$ s.t. $|\Gamma(A)| = |A|$.



Let G_{Black} be the bipartite subgraph of G induced by A, $\Gamma(A)$.

Let G_{Blue} be the bipartite subgraph of G induced by $X \setminus A$, $Y \setminus \Gamma(A)$.

Claim: (a) G_{Black} satisfies $(*)$
(b) G_{Blue} satisfies $(*)$.

(a) If $C \subseteq A$, $\Gamma_{G_{Black}}(C) = \Gamma_G(C)$

$$\Rightarrow |\Gamma_{G_{Black}}(C)| = |\Gamma_G(C)| \geqslant |C|.$$

(b) If $D \subseteq X \setminus A$ and $|\Gamma_{G_{Blue}}(D)| \leqslant |D| - 1$

Consider $A \cup D$. $\Gamma_G(A \cup D) = \Gamma_G(A) \cup \Gamma_{G_{Blue}}(D)$

$$\Rightarrow |\Gamma_G(A \cup D)| = |\Gamma_G(A)| + |\Gamma_{G_{Blue}}(D)|$$
$$\leqslant \underbrace{|A|}_{\text{by def}^n \text{ of } A} + |D| - 1$$
$$= |A \cup D| - 1 \qquad \#$$

Hence (b) also holds

Our inductive hypothesis $\Rightarrow \exists\, M_{Black}, M_{Blue}$, complete matchings in G_{Black} and G_{Blue} respectively.

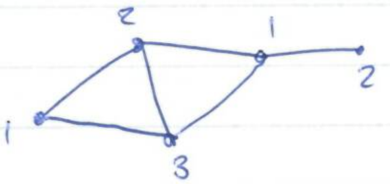
So $M = M_{Black} \cup M_{Blue}$ is a complete matching in G $\square$

Colouring

If G is a graph, $A \subseteq V(G)$ is an independent set if no two vertices in A are joined by an edge.

If $k \geq 1$ is an integer, then k-colouring is a function $c: V(G) \to [k]$ with the property that $c^{-1}(i)$ is an independent set for each $i \in [k]$.

$$c^{-1}(i) = \{ v \in V(G) : c(v) = i \}$$

e.g. 

is a 3-colouring.

If G has a k-colouring it is said to be k-colourable. We say G is k-chromatic if G is k-colourable but not $(k-1)$-colourable.

$\chi(G) = \min\{k : \exists \text{ k-colouring of } G\}$. 'chromatic n°'

$\Delta(G) = \max_{v \in V(G)} d(v)$.

Thm 2.16: $\chi(G) \leq \Delta(G) + 1$

easiest algorithm for colouring a graph

Proof: Let $V(G) = \{v_1, \dots, v_n\}$

$k = \Delta(G) + 1$.

$c(v_1) = 1$

j is a colour

define $c(v_i) = \min\{ j : j \in [k] \setminus \{t : t \text{ is the colour of a neighbour of } v_i\}\}$

Hence have a k colouring of G. $\square$

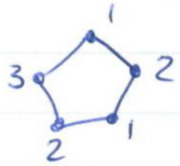
Def^n: A graph G is <u>k-partite</u> if $\exists$ a partition $V = \bigcup_{i=1}^{k} V_i$ s.t. each V_i is an independent set (ie. it contains no edges)

<u>Note</u>: G is k-partite $\equiv$ G is k-colourable.

Some examples: $\chi(K_t) = t$

$$\chi(C_{2t}) = 2$$

$$\chi(C_{2t+1}) = 3$$



If H is a subgraph of G then $\chi(H) \leq \chi(G)$.

<u>Forbidden subgraph problems</u>

If G and H are graphs then we say G is <u>H-free</u> if G has no subgraph isomorphic to H.

The <u>extremal n° of the graph</u> H, $ex(n, H)$ is

$$ex(n,H) = \max\{|E(G)| : G \text{ has } n \text{ vertices and is } H\text{-free}\}$$

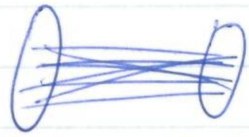
$$ex(n,H) \leq \binom{n}{2}.$$ ← n° edges poss in G.

[If $|V(H)|$ is constant, $ex(n,H) < \binom{n}{2}$.]

This turns out to be a difficult problem.

E.g. $H = K_2$ $\quad ex(n, K_2) = 0$

$H = K_3$



$\lfloor \frac{n}{2} \rfloor$ $\quad \lceil \frac{n}{2} \rceil$

$K_{\lfloor n/2 \rfloor \lceil n/2 \rceil}$

$\Rightarrow$ n° edges $= \lfloor \frac{n^2}{4} \rfloor$

Since $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ is K_3-free (because it is 2-colourable) and has $\lfloor \frac{n^2}{4} \rfloor$ edges so $\quad ex(n, K_3) \geq \lfloor n^2/4 \rfloor$

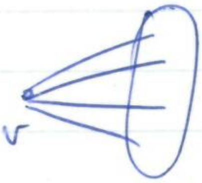
<u>Thm. 2.17 (Mantel's Thm)</u> (1900) : $ex(n, K_3) = \lfloor \frac{n^2}{4} \rfloor$.

<u>Proof</u>: Let G be K_3-free

$|V(G)| = n$

$|E(G)| = ex(n, K_3)$.

Claim: if $v \in V(G)$ then $\Gamma(v)$ is an independent set (since an edge in $\Gamma(v)$ gives a copy of K_3 in G)

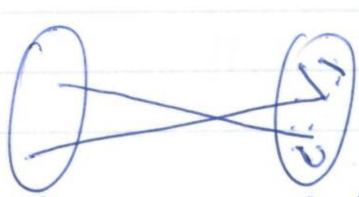


v $\quad \Gamma(v)$

← can you have an edge in there? <u>no</u>

∵ you'd have a triangle

Let $A \subseteq V$ be a largest independent set in $V(G)$.
So if $|A| = a$ then $\forall v \in V(G)$, $d(v) = |\Gamma(v)| \leq a$ (by the claim).



A $\quad V \setminus A$

min $d(v)$ ↓, nº of terms in sum

$$\sum_{v \in V\setminus A} d(v) \leq a(n-a)$$

$$\sum_{v \in V\setminus A} d(v) = \#\text{ edges from } V\setminus A \text{ to } A + 2\#\text{edges inside } V\setminus A$$

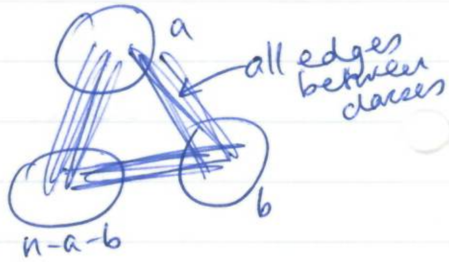
$$\geq |E(G)|$$

$$\Rightarrow |E(G)| \leq a(n-a) \leq \frac{n^2}{4} \qquad \text{(by first year calculus)}$$

$$\underset{\text{integer}}{\uparrow} \Rightarrow \quad \cdots \leq \left\lfloor \frac{n^2}{4} \right\rfloor \qquad \square$$

$H = K_4$. In the same way as with K_3, this time since K_4 is 4-colourable, look for 3-partite graph.

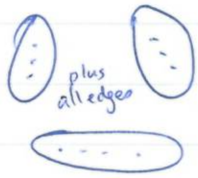
<u>Defn</u>: A k-partite graph is <u>complete</u> if all the edges between classes are present.



<u>Defn</u>: The <u>Turán graph</u> $T_r(n)$ is the complete r-partite graph with n vertices and vertex classes as equal as possible in size.

e.g. $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil} = T_2(n)$.

e.g. $T_3(10) =$



Let $t_r(n) = |E(T_r(n))|$

$t_3(10) = 3\times 3 + 3\times 4 + 4\times 3$
$= 33$.

Thm 2.19 (Turán's Thm) [1941] If $r, n \geq 2$ then (up to isomorphism), $T_r(n)$ is the unique order n graph that is K_{r+1}-free and has $ex(n, K_{r+1})$

Lemma 2.18 Amongst all r-partite graphs of order n, $T_r(n)$ has the most edges.
Moreover, $t_r(n) = t_r(n-r) + (r-1)(n-r) + \binom{r}{2}$

Proof: (by contradiction). Suppose G is r-partite
$|V(G)| = n$
maximal n° of edges
but $G \neq T_r(n)$.

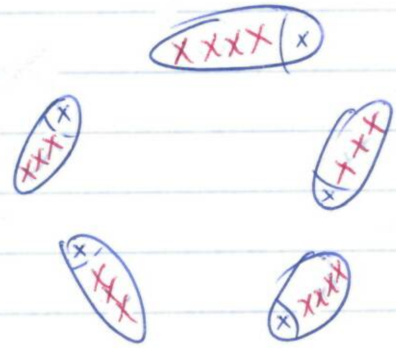
By maximality of $|E(G)|$ we know it is complete r-partite.

Since $G \neq T_r(n)$, $\exists$ vertex classes, say V_1, V_2 of sizes a, b respectively with $a \geq b+2$.

Take a vertex $x \in V_1$ and move it to $x \in V_2$ while keeping G complete r-partite.

The change in # of edges $= -(n-a) + (n-b-1)$.
$= a-b-1$
but since $a \geq b+2$, ≥ 1 $\times\!\!\!\times$.

$T_r(n)$.



want to relate to $T_r(n-r)$.

There is a copy of $T_r(n-r)$ inside $T_r(n)$.
(Simply remove one vertex from each class).

Colour the vertices of $T_r(n-r)$ red and colour the remaining r vertices blue.

$t_r(n) = \#$ blue-blue + # red-red edges + # red-blue edges

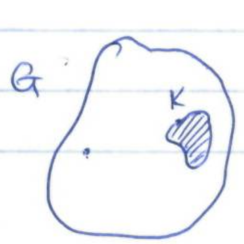
red-red edges $= t_r(n-r)$
blue-blue edges $= \binom{r}{2}$
red-blue edges $= (n-r)(r-1)$ since there are $n-r$ red vertices each joined to every blue vertex except the one in the same class.

<u>Proof of Thm 2.19</u>: Let G be K_{r+1}-free, order n, with $ex(n, K_{r+1})$ edges. Since $T_r(n)$ is K_{r+1}-free and has order n, we know
$ex(n, K_{r+1}) \geq t_r(n)$.

(Induction): If $n \leq r$ then $ex(n, K_{r+1}) = \binom{n}{2} = t_r(n)$.
So assume $n \geq r+1$. So by maximality of $|E(G)|$, $\exists$ a copy K of K_r inside G.

$V(K) = \{v_1, v_2, \dots, v_r\}$. We need to show $|E(G)| \leq t_r(n)$.

Consider $G-K$. It is a K_{r+1}-free graph of order $n-r$. By inductive hypothesis, $|E(G-K)| \leq t_r(n-r)$



Each $v \in V(G-K)$ has at most $r-1$ neighbours in $V(K)$.
$|V(K)| = r$

el's in G−K. things in K it could be joined to

$$\Rightarrow \#\text{edges from } G-K \text{ to } K \leq (n-r)(r-1) \quad \cdots (*)$$

$$|E(K)| = \binom{r}{2}. \qquad \because K = K_r$$

Hence, using Lemma 2.18,

$$|E(G)| = \underbrace{|E(G-K)|}_{\text{ind. hyp. } \Rightarrow \text{ th}} + |E(K)| + \underbrace{\#\text{edges from } G-K \text{ to } K}_{\text{from above}}.$$

$$\leq t_r(n-r) + (n-r)(r-1) + \binom{r}{2} = t_r(n).$$

But $|E(G)| = ex(n, K_{r+1}) \geq t_r(n)$ by setup

Hence $|E(G)| = t_r(n)$ and equality holds in $(*)$.

Hence every vertex $v \in V(G-K)$ has exactly $r-1$ neighbours in K.

Things not connected to v_i

<u>Defⁿ</u>: $V_i = \{ v \in V(G) : vv_i \notin E(G) \}$ (so $v_i \in V_i$)

Since each $v \in V(G-K)$ is not joined to exactly one $v_i \in V(K)$ so $v \in V_i$, for some unique i. Hence $V(G) = V_1 \dot\cup V_2 \dot\cup \cdots \dot\cup V_r$

Remains to prove that G is r-partite, then we can use Lemma 2.18.

<u>Claim</u>: each class V_i is an independent set

Suppose $u, v \in V_i$ and $uv \in E(G)$, then $\{u, v, v_1, v_2, \ldots, \underbrace{\widehat{v_i}}_{\text{omitted}}, \ldots, v_r\}$ form a copy of K_{r+1} #.

Hence G is r-partite, and by maximality of $|E(G)|$

and Lemma 2.18, we must have $G = T_r(n)$.

Result follows by induction. $\square$

So now we know $ex(n, K_{r+1}) = t_r(n)$.

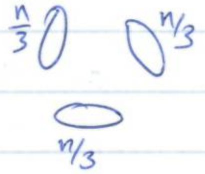
$ex(n, C_4) = ?$
$ex(n, C_5) = ?$

Let H be a graph. Let the <u>Turán density</u> of H,

$$\pi(H) = \lim_{n\to\infty} \underbrace{\frac{ex(n,H)}{\binom{n}{2}}}_{\substack{\text{proportion of edges you could ever have} \\ \text{in a graph without } H.}}$$

e.g. $\pi(K_3) = \frac{1}{2}$

$\pi(K_4) = \frac{2}{3}$ $T_3(n)$



$\#$ edges $\approx 3 \times \left(\frac{n}{3}\right)^2 = \frac{n^2}{3}$

$\pi(C_4) = 0$

$\pi(C_5) = \frac{1}{2}$

$\pi(K_2) = 0$

[density of a graph G, where G has n vertices, is
$$\text{density}(G) = \frac{|E(G)|}{\binom{n}{2}}$$]

<u>Lemma 2.20</u>: $\pi(H)$ is well-defined.

<u>Proof</u>: Let G have $n+1$ vertices and be H-free, with $|E(G)| = ex(n,H)$.

$$\sum_{A \in \binom{V(G)}{n-1}} |E(G[A])| \leq \sum_{A \in \binom{V(G)}{n-1}} ex(n-1, H)$$

since $G[A]$ has $n-1$ vertices and is H-free.

$$\Rightarrow \underbrace{\sum_{A \in \binom{V(G)}{n-1}} |E(G[A])|}_{= (n-2)|E(G)|} \leq n \, ex(n-1, H)$$

since n is nº of terms in the sum.

$$\Rightarrow \frac{2\,ex(n,H)}{n(n-1)} \leq \frac{2\,ex(n-1,H)}{(n-1)(n-2)}$$

$$\Rightarrow \frac{ex(n,H)}{\binom{n}{2}} \leq \frac{ex(n-1,H)}{\binom{n-1}{2}}$$

$\Rightarrow$ sequence is decreasing and bounded below
$\Rightarrow$ converges $\square$

What is $\pi(K_r)$ if ~~K_r~~ $ex(n, K_r) = t_{r-1}(n)$.

$$n = k(r-1) + d, \qquad 0 \leq d < r-1$$

$\Rightarrow$ vertex classes in $T_{r-1}(n)$ have sizes k and $k+1$.

How many edges in the Turán graph?

$$\frac{\binom{r-1}{2}k^2}{\binom{n}{2}} < \frac{\overset{= t_{r-1}(n)}{ex(n, K_r)}}{\binom{n}{2}} \leq \frac{\binom{r-1}{2}(k+1)^2}{\binom{n}{2}}$$

most edges you can have in any class. If you have $(k+1)$ in each, we have $(k+1)^2$.

$k = \frac{n-d}{r-1}$. Substitute in, since $0 \le d < r-1$

Both sides tend to $\frac{r-2}{r-1}$.

e.g. if $r=3$ then $\pi(K_3) = \frac{1}{2}$.

Thm 2.21 (Erdös-Stone) (Fund. Thm of Extremal Graph Theory) If H has chromatic n° $\chi(H)$ then

$$\pi(H) = \frac{\chi(H)-2}{\chi(H)-1} \qquad \neq 1$$

Proof: Erdös ~~is~~/was a much cleverer man. So we won't prove it, but the idea is this:

Need to prove $\geq$ and $\leq$, Greek style!

$\pi(H) \geq \frac{\chi(H)-2}{\chi(H)-1}$: If $\chi(H)=r$ then $T_{r-1}(n)$ is H-free and $\frac{t_{r-1}(n)}{\binom{n}{2}} \to \frac{r-2}{r-1}$

PLANAR GRAPHS

A <u>plane graph</u> is $G(V,E)$ where:

(i) $V \subset \mathbb{R}^2$
(ii) each edge e is a polygonal path (finite union of line segments) between 2 vertices
(iii) Distinct edges have distinct endpoints
(iv) The interior of an edge contains no vertices and meets no other edge.

If we delete a plane graph from $\mathbb{R}^2$, we obtain a collection of connected regions. These regions are called <u>faces</u>.

<u>Thm 2.22</u>: In a plane graph

(i) A cycle has two faces (Jordan Curve Lemma)
(ii) An edge lies in two faces iff it lies in a cycle
(iii) The boundary of a face consists of vertices and whole edges

<u>Proof</u>: none given, too hard.

We say that the plane graph H is a <u>drawing</u> of the graph G iff $\exists$ bijection $f: V(G) \to V(H)$ $uv \in E(G)$ iff $f(u)f(v)$ is joined in H by an edge (polygonal path)

If $\exists$ a plane graph drawing of a graph G then G is a <u>planar</u> graph

Lemma 2.23: Any tree is planar with one face.

Proof: True for $|V(T)| = 1$.
So suppose true for smaller trees, consider T a tree with ≥ 2 vertices. T has a leaf v, and $T-v$ is a tree. By inductive hypothesis $\exists$ a plane graph drawing of $T-v$. Take a very small disc D centred at w, where w is the neighbour of v, so that within D all edges are line segments. Choose a new dir^n for the edge to v. Hence we have a plane graph drawing of T.

By inductive hypothesis, $T-v$ has one face, so if T has ≥ 2 faces then the edge vw is in a cycle # since T is a tree. $\square$

Thm 2.24 (Euler's formula for planar graphs):

If G is a connected planar graph with n vertices, m edges and f faces then $n - m + f = 2$.

Proof (induction on f). True for $f = 1$. (any graph with one face is a tree. $m = n-1$, $f = 1$).

So suppose G is planar with at least 2 faces. Take an edge xy in a cycle. Delete this edge to create one face from the two original faces containing xy.

Hence $m \to m-1$ so $n-m+f$ remains unchanged
$f \to f-1$

But in $G \setminus \{xy\}$ we know the result holds by our inductive hypothesis. Hence $n-m+f=2$. $\square$

<u>Thm 2.25</u> If G is a planar graph with $n \geqslant 3$, then $m \leqslant 3n-6$

<u>Proof</u>: If $m < 3$ then $n \geqslant 3 \Rightarrow$ true.

So can assume that every face of any plane graph drawing of G has $\geqslant 3$ edges in its boundary.

Construct the following (totally unrelated) bipartite graph H, with vertex classes
$E(G)$ and $F(G)$ (faces of G)

$E(G)$ $F(G)$

We have an edge in H from $e \in E(G)$ to $F \in F(G)$ iff e is an edge in the boundary of F.

Double counting:

$$\sum_{e \in E(G)} \#\text{faces containing } e = |E(H)| = \sum_{F \in F(G)} \#\text{edges in } F \quad \ldots (*)$$

Now, # faces containing $e \leq 2$
edges in $F \geqslant 3$

So $(*) \Rightarrow \quad 2m \geqslant 3f \quad \Rightarrow f \leq \frac{2m}{3}$

$m = n + f - 2$

$m \leq n + \frac{2m}{3} - 2$

$m \leqslant 3n - 6$. $\square$

<u>Corollary 2.26</u>: K_5 is not planar

<u>Proof</u>: $|E(K_5)| = \binom{5}{2} = 10$
$|V(K_5)| = 5$

$10 \not\leqslant 3 \times 5 - 6 = 9$. $\square$

<u>Corollary 2.27</u> $K_{3,3}$ is not planar

<u>Proof</u>: $|E(K_{3,3})| = 9$
$|V(K_{3,3})| = 6$
$3n - 6 = 12$. So it doesn't fail the criterion

But $K_{3,3}$ is triangle-free. So if we had a plane graph drawing of $K_{3,3}$ every face would contain $\geqslant 4$ edges

Hence following the proof of Thm 2.25, we would have
$2m \geqslant 4f \Rightarrow f \leq \frac{m}{2}$

$\Rightarrow m = n + f - 2$

$m \leq n + \frac{m}{2} - 2$

$m \leqslant 2n - 4$ (True for any planar triangle-free graph)

So here we have $m = 9$ & $2 \times 6 - 4 = 8$. $\square$

Defn: The girth of a graph G is

$$g(G) = \min\{k \geq 3 : C_k \subseteq G\}$$

Defn: We say that H is a subdivision of G if we can obtain H from G by replacing edges by paths.

Lemma 2.28: If G contains a subdivision of a non-planar graph, it is non-planar.

Proof: Obviously non-planar subgraph $\Rightarrow$ non-planar graph. Any subdivision of a non-planar graph is non-planar. $\square$

Corollary 2.29: Any graph containing a subdivision of K_5 or $K_{3,3}$ is non-planar.

Converse also holds !! (not proved)

Thm 2.30: Any planar graph is 6-colourable

Proof: (Induction on $n = |V(G)|$). $n \leq 6$ ✓

If we find $v \in V(G)$ with $d(v) \leq 5$ then can colour $G - v$ by inductive hypothesis and we can colour v with one of the colours not used in its neighbourhood.

Need $\delta(G) \leq 5$

Suppose $\delta(G) \geq 6$, then $6n \leq \sum_{v \in V(G)} d(v) = 2m \leq 6n - 12$ $\#$

Proof follows by induction $\square$

Thm 2.31 Every planar graph is 5-colourable

Proof (by induction on $n = |V|$): Let $G = (V, E)$ planar.

For $n \leq 5$, this is trivial.

Suppose $n > 5$.

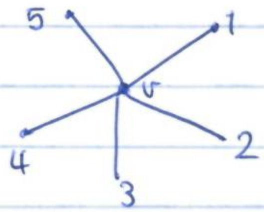
$$\sum_{v \in V} d(v) = 2m \leq \underset{2(3n-6)}{\overset{\shortparallel}{6n-12}}$$

$\Rightarrow \exists\, v \in V : d(v) \leq 5$

By our inductive hypothesis, $\exists$ a five-colouring of $G - v$.

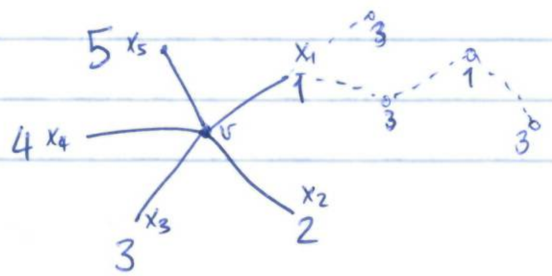
So if $d(v) \leq 4$ or $d(v) = 5$ but not all five colours are used in the neighbourhood ~~the~~ of v then we can colour v legally.

So w.l.o.g. $d(v) = 5$. All five colours are used for $\Gamma(v)$.

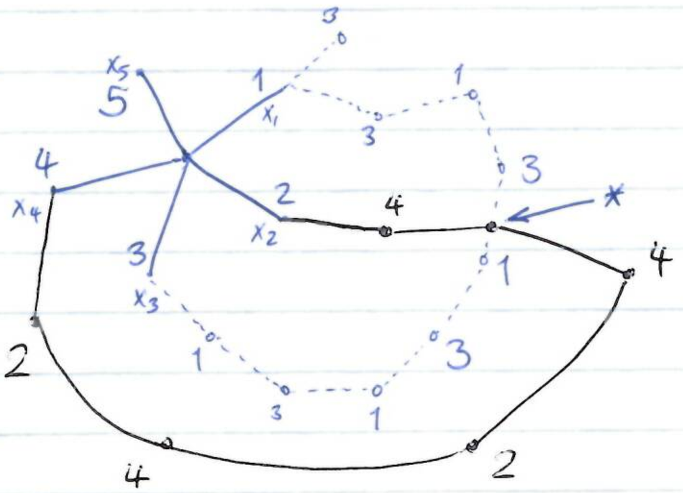


Let $H(i,j)$ denote the subgraph of G induced by vertices with colours i or j.

Consider the connected components of $H(1,3)$ containing vertex x_1.



If x_3 is not in this component of $H(1,3)$ then swap all colours in this component and colour $c(v) = 1$. So WLOG x_3 is in this component



A similar argument for $H(2,4)$ implies that x_2 and x_4 are in the same component of $H(2,4)$

What is the colour of $*$? It has to be a vertex $\because$ planar graph. It isn't anything well-defined ~~so~~ because there are 2 interlocking cycles of $2+4$ and $1+3$. $\#$

$\Rightarrow$ one of them $\nexists$ doesn't exist $\Rightarrow$ you can swap colours in this way.

3. SET SYSTEMS

Recall $\mathcal{P}(X) = \{A : A \subseteq X\}$ 'power set' of X

$\binom{X}{k} = \{A \subseteq X : |A| = k\}$ The 'k-sets' from X.

<u>Defn</u>: A family of sets $\mathcal{C} = \{C_1, C_2, \dots, C_t\}$ is a <u>chain</u> iff $\forall A, B \subset \mathcal{C}$, $A \subset B$ or $B \subset A$.

A family of sets $\mathcal{A}$ is an <u>antichain</u>, if $\forall A, B \subset \mathcal{A}$

- $A \subset B \Rightarrow A = B$

or equivalently
- $A \neq B \Rightarrow A \not\subset B$ and $B \not\subset A$

<u>Lemma 3.1</u> If $\mathcal{A}$ is an antichain and $\mathcal{C}$ is a chain, then $|\mathcal{A} \cap \mathcal{C}| \leq 1$

<u>Proof</u>: If $A, B \in \mathcal{A} \cap \mathcal{C}$ then $A, B \in \mathcal{C} \Rightarrow A \subset B$ or $B \subset A$.

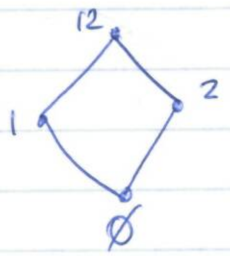
wlog $A \subset B$.

But $A, B \in \mathcal{A} \Rightarrow A \subset B \Rightarrow A = B$

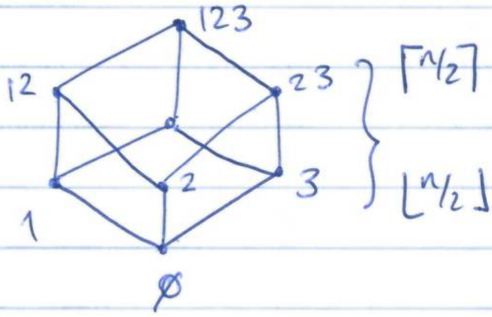
$\Rightarrow |\mathcal{A} \cap \mathcal{C}| \leq 1$ $\square$

Let $X = [n]$. What does $\mathcal{P}[x]$ look like?

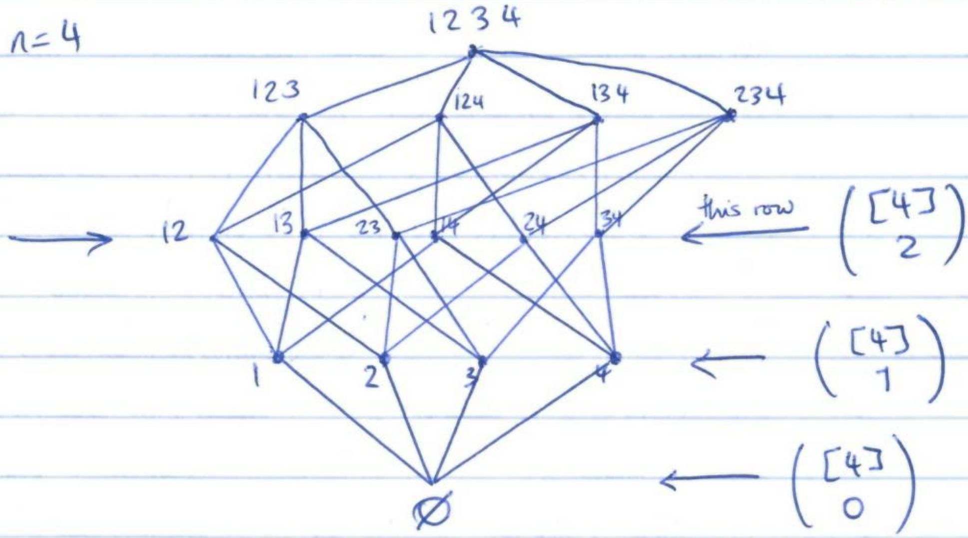
$n=2$: $\mathcal{P}(X) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$



$n=3$ $\quad \mathcal{P}(X) = \{\varnothing, 1, 2, 3, 12, 13, 23, 123\}$



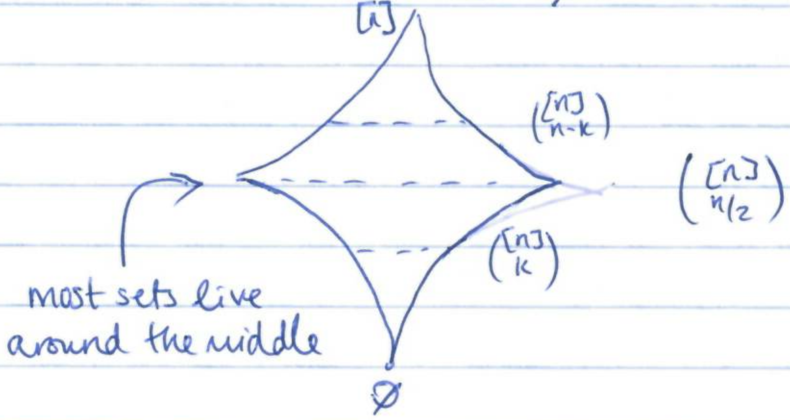
$n=4$



each layer/row is an antichain

Generally, k^{th} row has $\binom{[n]}{k}$.

Middle row has $\binom{[n]}{n/2}$



Prop^n 3.2 Any chain $\mathcal{C}$ in $\mathcal{P}([n])$ satisfies $|\mathcal{C}| \le n+1$.

Proof: Each layer of $\mathcal{P}([n])$ is an antichain so this follows from Lemma 3.1 $\square$

Example of a chain: $\{1, 12, 1245, 12345\} = \mathcal{C}$

Example of an antichain: $\{123, 145, 26, 1278\} = \mathcal{A}$

Example of neither: $\{12, 23, 14, 124\} = \mathcal{F}$

Proof (rewritten): For each $0 \le k \le n$, $\binom{[n]}{k}$ is an antichain, and $\mathcal{P}([n]) = \binom{[n]}{0} \,\dot\cup\, \binom{[n]}{1} \,\dot\cup \cdots \dot\cup\, \binom{[n]}{n}$

So $\mathcal{P}([n])$ can be partitioned into $n+1$ antichains.

If $\mathcal{C} \subseteq \mathcal{P}([n])$ is a chain,

$$\left|\mathcal{C} \cap \binom{[n]}{k}\right| \le 1 \qquad \text{(by L. 3.1)}$$

hence $|\mathcal{C}| \le n+1$.

Question: If $\mathcal{A} \subseteq \mathcal{P}([n])$, how big can $|\mathcal{A}|$ be?

$\binom{[n]}{k}$ $k=0, \dots, n$

Largest is for $k = \lfloor n/2 \rfloor$.

This result will fall out of the following proof.

A chain $\mathcal{C} \subseteq \mathcal{P}([n])$ is <u>symmetric</u> if
$\mathcal{C} = \{C_1, \dots, C_k\}$ with $|C_{i+1}| = |C_i| + 1$ for $i = 1, \dots, k-1$
and $|C_1| + |C_k| = n$.

<u>e.g.</u> $n=6$: $\mathcal{C}_1 = \{\underline{12}, 123, \underline{1235}\}$

$\mathcal{C}_2 = \{\underline{2}, 12, 124, 1234, \underline{12346}\}$

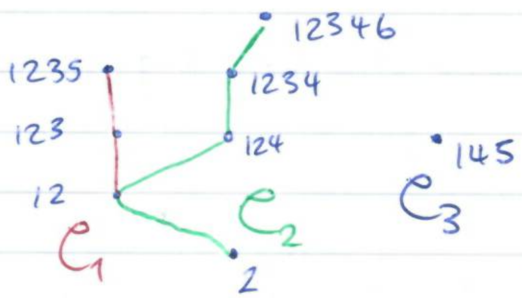
$\mathcal{C}_3 = \{145\}$

<u>Thm 3.3</u> (Sperner): If $\mathcal{A} \subseteq \mathcal{P}([n])$ is an antichain then $|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$

<u>Proof</u>: Suppose we can partition $\mathcal{P}([n])$ into symmetric chains $\mathcal{C}_1, \dots, \mathcal{C}_t$. Since $|\mathcal{A} \cap \mathcal{C}_i| \leq 1$ (by L. 3.1) we know $|\mathcal{A}| \leq t$.

But since every set from $\binom{[n]}{\lfloor n/2 \rfloor}$ lies in some $\mathcal{C}_i$, and each $\mathcal{C}_i$ contains exactly one set from $\binom{[n]}{\lfloor n/2 \rfloor}$, so $t = \binom{[n]}{\lfloor n/2 \rfloor}$

<u>e.g.</u>



$\longleftarrow \binom{[6]}{3}$.

Lemma 3.4: $\mathcal{P}([n])$ can be partitioned into symmetric chains.

Proof: (induction) $n=1$: $\mathcal{P}([1]) = \{\emptyset, 1\}$ ✓

So suppose true for $n-1$, $n \geq 2$.

We have a partition of $\mathcal{P}([n-1])$ into symmetric chains.

$$\mathcal{P}([n-1]) = \mathcal{C}_1 \dot\cup \cdots \dot\cup \mathcal{C}_s$$

Suppose $\mathcal{C}_i = \{C_1, \ldots, C_k\}$

Define $\mathcal{C}_i' = \{C_1, \ldots, C_k, C_k \cup \{n\}\}$

this is a symmetric chain since $|C_1| + |C_k \cup \{n\}| = n-1+1 = n$.

$\mathcal{C}_i'' = \{C_1 \cup \{n\}, \ldots, C_{k-1} \cup \{n\}\}$ also sym.

this is not always defined, say if you input a chain of length 1 (then C_{k-1} is nothing!)

$$\begin{aligned} &|C_1 \cup \{n\}| + |C_{k-1} \cup \{n\}| \\ &= |C_1| + |C_{k-1}| + 2 \\ &= |C_1| + |C_k| - 1 + 2 \\ &= (n-1) + 1 \\ &= n \end{aligned}$$

So $\mathcal{C}_i'$ and $\mathcal{C}_i''$ are symmetric chains in $\mathcal{P}([n])$.

Moreover, $\mathcal{C}_1', \mathcal{C}_1'', \mathcal{C}_2', \mathcal{C}_2'', \ldots, \mathcal{C}_s', \mathcal{C}_s''$ partition $\mathcal{P}([n])$.

$\square$

Let $\mathcal{A} \subseteq \mathcal{P}([n])$ and define $a_k = \left| \mathcal{A} \cap \binom{[n]}{k} \right|$.

If $\mathcal{A}$ is an antichain, what can we say about $a_0, a_1, a_2, \ldots, a_n$?

<u>Thm 3.5</u> (LYM-inequality, actually proven by Bollobás).

If $\mathcal{A} \subseteq \mathcal{P}([n])$ is an antichain
and $a_k = \left| \mathcal{A} \cap \binom{[n]}{k} \right|$ ~~then~~

then $$\sum_{k=0}^{n} \frac{a_k}{\binom{n}{k}} \leq 1.$$

important proof

<u>Proof</u>: Let Π_n be the set of all permutations of $[n]$.
(Method 1) We construct a bipartite graph $G = (\mathcal{A}, \Pi_n; E)$.

Insert an edge from $A \in \mathcal{A}$ to $\pi \in \Pi_n$ iff all the elements of A lie before all the elements of $[n] \setminus A$ in π.

e.g. $1243 = \pi$

$\mathcal{A}$ $\quad$ Π_n

$1 \longrightarrow \pi$
$12 \longrightarrow \pi$
$\emptyset \longrightarrow \pi$
$124 \ —\ \pi$
$1234 \ —\ \pi$

Count the edges in G

$$\sum_{A \in \mathcal{A}} d(A) = |E| = \sum_{\pi \in \Pi_n} d(\pi)$$

Given $\pi \in \Pi_n$, how large can $d(\pi)$ be?

Claim: $d(\pi) \le 1$.

Proof: Suppose $d(\pi) > 1$ then $\exists A_1 \ne A_2$, $A_1, A_2 \in \mathcal{A}$ s.t. $A_1\pi$ and $A_2\pi$ are edges of G. WLOG $|A_1| \le |A_2|$.
Then all the elements of A_i come before all the elements ~~from~~ of $[n] \setminus A_i$ for $i = 1, 2$.
Hence $A_1 \subseteq A_2$ and $A_1 \ne A_2$ #
since $\mathcal{A}$ is an antichain □

So max nº of edges is $\le n!$.

What about $d(A)$ (for $A \in \mathcal{A}$)?

Count # of permutations of $[n]$ starting with the elements of A in some order.

If $|A| = k$ then $d(A) = k!(n-k)!$

$$\pi : \underbrace{\pi_1 \cdots \pi_k}_{A} \underbrace{\pi_{k+1} \cdots \pi_n}_{[n]\setminus A}$$

$$\text{So } \sum_{A \in \mathcal{A}} |A|!(n-|A|)! = |E|$$

$$= \sum_{\pi \in \Pi_n} d(\pi)$$

$$\le |\Pi_n| = n!$$

$\Rightarrow \displaystyle\sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \le 1$ $\Rightarrow$ Result by gathering terms of same $|A|$ □

<u>Proof (Method 2)</u>: Let $\mathcal{A} \subseteq \mathcal{P}([n])$ be an antichain and let $\mathcal{C} \subseteq \mathcal{P}([n])$ be a chain.

Then $|\mathcal{A} \cap \mathcal{C}| \leq 1$.

Let $\mathcal{C}$ be a maximal chain in $\mathcal{P}([n])$ (ie. of size $n+1$), chosen uniformly at random.

So the expected size of $(\mathcal{A} \cap \mathcal{C})$, $\mathbb{E}(|\mathcal{A} \cap \mathcal{C}|)$ we want to compute.

Obviously $1 > \mathbb{E}(|\mathcal{A} \cap \mathcal{C}|)$.

Let $X_A = \begin{cases} 1 & \text{if } A \in \mathcal{C} \\ 0 & \text{if } A \notin \mathcal{C} \end{cases}$ for $A \in \mathcal{A}$.

Let $X = \sum_{A \in \mathcal{A}} X_A = |\mathcal{A} \cap \mathcal{C}|$

$$\mathbb{E}(|\mathcal{A} \cap \mathcal{C}|) = \mathbb{E}\left(\sum_{A \in \mathcal{A}} X_A\right) = \sum_{A \in \mathcal{A}} \mathbb{E}(X_A).$$ by linearity of expectation

And $\mathbb{E}(X_A) = 1 \cdot \Pr(A \in \mathcal{C}) + 0 \cdot \Pr(A \notin \mathcal{C})$
$= \Pr(A \in \mathcal{C})$

Since $\mathcal{C}$ is a maximal chain, it contains exactly one set of size $|A|$.

Moreover, since $\mathcal{C}$ was chosen uniformly at random, all sets of size $|A|$ are equally likely to belong to $\mathcal{C}$.

Hence $\Pr(A \in \mathcal{C}) = \dfrac{1}{\binom{n}{|A|}}$

and this completes the proof since

$$\sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq 1 \qquad \square$$

<u>Def^n</u>. A family of sets $\mathcal{A}$ is <u>intersecting</u> iff $A, B \in \mathcal{A}$ $\Rightarrow A \cap B \neq \emptyset$.

e.g. $\mathcal{A} = \{ A \subseteq [n] : |A \cap [3]| \geq 2 \}$ is intersecting.

$$12 \in \mathcal{A} \quad 13 \in \mathcal{A} \quad 23 \in \mathcal{A}$$

$$\bigcap_{A \in \mathcal{A}} A = \emptyset. \qquad |\mathcal{A}| = 4 \times 2^{n-3} = 2^{n-1}$$

e.g. $\mathcal{B} = \{ B \subseteq [n] : 1 \in B \}$ is intersecting

$$\bigcap_{B \in \mathcal{B}} B = [1] \qquad |\mathcal{B}| = 2^{n-1}$$

<u>Thm 3.6</u>: If $\mathcal{A} \subseteq \mathcal{P}([n])$ is intersecting then $|\mathcal{A}| \leq 2^{n-1}$.

<u>Proof</u>: If $A \in \mathcal{A}$ then $[n] \setminus A \notin \mathcal{A}$. Hence we have at most half of the sets from $\mathcal{P}([n])$. $\square$

<u>Question</u>: If $\mathcal{A} \subset \binom{[n]}{k}$ is intersecting, how large can $|\mathcal{A}|$ be?

If $k > \frac{n}{2}$ then no two k-sets are disjoint.

What about $k \leq \frac{n}{2}$?

Well, say $\mathcal{A}^* = \{ A \in \binom{[n]}{k} : 1 \in A \}$

$|\mathcal{A}^*| = \binom{n-1}{k-1}$ once you've picked one elements you have n-1 left and you want to pick k-1 of them.

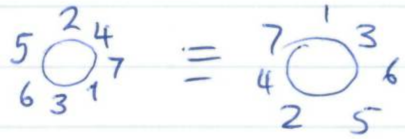
<u>Thm 3.7 (Erdös-Ko-Rado)</u> If $\mathcal{A} \subseteq \binom{[n]}{k}$ is intersecting and $k \leq \frac{n}{2}$ then

$$|\mathcal{A}| \leq \binom{n-1}{k-1}.$$

<u>Proof</u>: Let $\mathcal{A} \subseteq \binom{[n]}{k}$ be intersecting with $n \geqslant 2k$.

Let $\mathcal{C}_n$ be the family of all cyclic permutations of $[n]$.

e.g. $n=7$:



$|\mathcal{C}_n| = (n-1)!$ since $n!$ is # permutations but n ways of arranging each one.

Note: $\binom{n-1}{k-1} = \frac{(n-1)!}{(k-1)!(n-k)!} = \frac{k}{n}\left(\frac{n!}{k!(n-k)!}\right)$

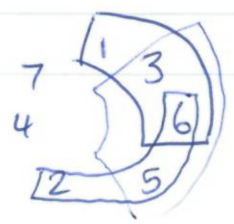
$$= \frac{k}{n}\binom{n}{k}.$$

We say that a set A is an <u>interval</u> in a cyclic permutation C iff the elements of A are all consecutive in C.

e.g. intervals of our $n=7$ cycle above:

136, 356, 256, 245, 247, 147, 137

(reordered numerically - this is OK).



$\boxed{136}$ $\boxed{356}$ $\boxed{256}$

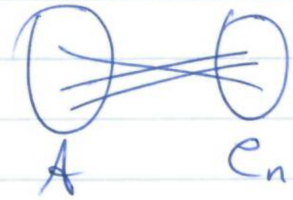
~~245~~ → ~~247~~ → ~~147~~ → ~~137~~

<u>Lemma 3.8</u> If C is a cyclic permutation of $[n]$, $(n \geq 2k)$ and $\mathcal{I} \subseteq \binom{[n]}{k}$ is an intersecting family of intervals from C, then $|\mathcal{I}| \leq k$.

Using this lemma, we complete the proof as follows:

Let $G = (\mathcal{A}, \mathcal{C}_n ; E)$ be a bipartite graph.

There is an edge from $A \in \mathcal{A}$ to $C \in \mathcal{C}_n$ iff A is an interval in C.



$$\sum_{A \in \mathcal{A}} d(A) = |E| = \sum_{C \in \mathcal{C}_n} d(C)$$

Let $C \in \mathcal{C}_n$ and $\mathcal{I}$ be the collection of intervals of C that are sets from $\mathcal{A}$.

$$d(C) = |\mathcal{I}| \leq k \quad \text{by Lemma 3.8.}$$

Hence $|E| = \sum_{C \in \mathcal{C}_n} d(C) \leq (n-1)!\, k$

since $|\mathcal{C}_n| = (n-1)!$

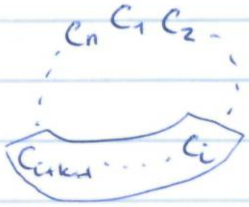
Now, if $A \in \mathcal{A}$, what is $d(A)$?

$$d(A) = \underbrace{k!}_{\text{ways of arranging } A}\ \underbrace{(n-k)!}_{\text{ways of arranging } [n]\setminus A}$$

Hence $|E| = \sum_{A \in \mathcal{A}} d(A) = |\mathcal{A}|\, k!\,(n-k)!$

$$\Rightarrow |\mathcal{A}| \leq \frac{(n-1)!}{(k-1)!\,(n-k)!} = \binom{n-1}{k-1} \qquad \square$$

but still need to prove the lemma.

<u>Proof of Lemma 3.8</u> Let C be 

and suppose $I = \{c_i, c_{i+1}, \dots, c_{i+k-1}\}$ belongs to $\mathcal{F}$.

For any j integer, define $I+j = \{c_{i+j}, c_{i+j+1}, \dots, c_{i+j+k-1}\}$ with subscripts modulo n.

Only possible intervals are $I+j$ for $-(k-1) \le j \le (k-1)$.

But $I-(k-1)$ is disjoint from $I+1$
$I-(k-2)$ " " " $I+2$
$\vdots$
$I-1$ " " " $I+(k-1)$

Hence at most one from each pairs is in $\mathcal{F}$,
so $|\mathcal{F}| \le k-1+1 = k$ $\square$

<u>The Linear Algebra Method</u>

<u>Lemma 3.9</u> If V is a vector space of dimension d and $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_m \in V$ are linearly independent* then $m \le d$.

<u>Proof</u>: FEAJ. $\square$

*recall LI: $\{\underline{v}_1, \dots, \underline{v}_m\}$ are LI iff $\sum_{i=1}^{m} \lambda_i \underline{v}_i = 0 \Rightarrow \lambda_i = 0 \ \forall i$

Thm 3.10 If $\mathcal{A} = \{A_1, A_2, \dots, A_m\} \subseteq \mathcal{P}([n])$ satisfying

(i) $|A_i|$ is odd for $1 \le i \le m$

(ii) $|A_i \cap A_j|$ is even for $1 \le i \ne j \le m$

then $m \le n$

Defn.: For a set $A \subseteq \mathcal{P}([n])$, the incidence vector of A is a zero/one vector with n entries

$$\underline{v}_i = \begin{cases} 1 & i \in A \\ 0 & i \notin A. \end{cases}$$

Proof of Thm: Let $A_i \in \mathcal{A}$ define $\underline{v}_i \in \mathbb{F}_2^n$ $\quad \mathbb{F}_2 = \{0, 1\}$

by $v_{ij} = \begin{cases} 1 & j \in A_i \\ 0 & j \notin A_i. \end{cases}$

Consider
$$\begin{aligned} \underline{v}_i \cdot \underline{v}_i &= \sum_{k=1}^{n} v_{ik} v_{ik} \\ &= \sum_{k=1}^{n} v_{ik}^2 \\ &= |A_i| \mod 2 \\ &= 1 \quad (\text{since } |A_i| \text{ is odd}) \end{aligned}$$

If $i \ne j$,
$$\begin{aligned} \underline{v}_i \cdot \underline{v}_j &= \sum_{k=1}^{n} v_{ik} v_{jk} \\ &= |A_i \cap A_j| \mod 2 \\ &= 0 \quad (\text{since } |A_i \cap A_j| \text{ is even}) \end{aligned}$$

So $\{v_1, v_2, \dots, v_m\}$ satisfy $\underline{v}_i \cdot \underline{v}_j = \begin{cases} 1 & i = j \\ 0 & i \ne j \end{cases}$

So they are orthogonal and hence they are LI.

Thus $m \leq \dim(\mathbb{F}_2^n) = n$ $\square$

Thm 3.11 (Fisher's Inequality):

If $\mathcal{A} = \{A_1, \dots, A_m\} \subseteq \mathcal{P}([n])$
and $|A_i \cap A_j| = k$ for $1 \leq i \neq j \leq m$, $1 \leq k \leq n$
Then $m \leq n$. [Note: if we allow $k=0$ then $m=n+1$ is possible]

Proof: Let $v_i \in \mathbb{R}^n$ be the incidence vector of A_i
So $v_{ij} = \begin{cases} 1 & j \in A_i \\ 0 & j \notin A_i \end{cases}$.

We want to show that $\{v_1, \dots, v_m\}$ is LI.

Know $v_i \cdot v_j = \sum_{k=1}^{n} v_{ik} v_{jk} = \begin{cases} |A_i| & i = j \\ k & i \neq j \end{cases}$

Suppose $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ satisfy

$$\sum_{i=1}^{m} \lambda_i \underline{v_i} = \underline{0}.$$

Consider $0 = \underline{0} \cdot \underline{0}$

$$= \sum_{i=1}^{m} \lambda_i \underline{v_i} \cdot \sum_{j=1}^{m} \lambda_j \underline{v_j}$$

$$0 = \sum_{i=1}^{m} \lambda_i^2 (\underline{v_i} \cdot \underline{v_i}) + \sum_{1 \leq i \neq j \leq m} \lambda_i \lambda_j (\underline{v_i} \cdot \underline{v_j})$$

$$= \sum_{i=1}^{m} \lambda_i^2 |A_i| + \sum_{1 \leq i \neq j \leq m} \lambda_i \lambda_j k.$$

Note $|A_i| \geq k$ for $1 \leq i \leq m$, with equality at most once.

$$0 = \sum_{i=1}^{m} \lambda_i^2 (|A_i| - k) + \sum_{i=1}^{m} k\lambda_i^2 + \sum_{1 \le i \ne j \le m} k \lambda_i \lambda_j$$

$$= \underbrace{\sum_{i=1}^{m} \lambda_i^2 \underbrace{(|A_i| - k)}_{\geq 0}}_{\geq 0} + \underbrace{k \left(\sum_{i=1}^{m} \lambda_i \right)^2}_{\geq 0}$$

$\Rightarrow$ (1) $\sum_{i=1}^{m} \lambda_i^2 (|A_i| - k) = 0$

(2) $k \left(\sum_{i=1}^{m} \lambda_i \right)^2 = 0$.

(1) $\Rightarrow$ At most one λ_i is nonzero

(2) $\Rightarrow$ $\sum_{i=1}^{m} \lambda_i = 0$

But (2) then implies that all the λ_i's are zero.
Hence the set $\{\underline{v}_1, \dots, \underline{v}_m\}$ is LI so $m \le \dim(\mathbb{R}^n)$
$\Rightarrow m \le n$. $\square$

<u>Defn</u>: Let $L \subseteq \{0, 1, \dots, n\}$. We say that $\mathcal{A} \subseteq \mathcal{P}([n])$ is <u>L-intersecting</u> iff $\forall A, B \in \mathcal{A}$, $A \ne B$, $|A \cap B| \in L$

e.g. $\mathcal{A} \subseteq \mathcal{P}([n])$ is intersecting $\Leftrightarrow$ $\mathcal{A}$ is $\{1, 2, \dots, n\}$-intersecting.

In Fisher's Inequality we have: if $1 \le k \le n$ and $\mathcal{A} \subseteq \mathcal{P}([n])$ and $\mathcal{A}$ is $\{k\}$-intersecting. Then $|\mathcal{A}| \le n$.

Let $L = \{0, 1, \dots, s\}$. How large can $\mathcal{A} \subseteq \mathcal{P}([n])$ be if $\mathcal{A}$ is $\{0, 1, \dots, s-1\}$-intersecting?

Well what can we put in there?

$\emptyset, 1, 2, \ldots, n$

$12, 13, \ldots, n-1n$

all sets of size $\leq s$.

In fact, $\#\text{sets} = \sum_{i=0}^{s} \binom{n}{i}$

hardest result in course

<u>Thm 3.12</u> If $\mathcal{A} \subseteq \mathcal{P}([n])$ is L-intersecting and $|L| = s$, then
$$|\mathcal{A}| \leq \sum_{i=0}^{s} \binom{n}{i}$$

<u>Proof</u>: Let $\mathcal{A} = \{A_1, \ldots, A_m\}$ $\quad |A_1| \leq |A_2| \leq \cdots \leq |A_m|$.

Let $L = \{l_1, \ldots, l_s\}$ $\quad l_1 < l_2 < \cdots < l_s$

For each $A_i \in \mathcal{A}$, define $\underline{v_i} \in \mathbb{R}^n$ by $\underline{v_{ij}} = \begin{cases} 1 & j \in A_i \\ 0 & \text{else} \end{cases}$

Recall $\underline{v_i} \cdot \underline{v_j} = \begin{cases} |A_i| & i = j \\ |A_i \cap A_j| & i \neq j \end{cases}$

Define polynomials $p_i(\underline{x})$ over $\mathbb{R}$ in n variables (i.e. $\underline{x} = (x_1, \ldots, x_n)$)

$$p_i(\underline{x}) = \prod_{k : l_k \backslash |A_i|} \left[(\underline{v_i} \cdot \underline{x}) - l_k \right] \quad \cdots\cdots\cdots \quad (*)$$

$$p_i(\underline{v_i}) = \prod_{k : l_k < |A_i|} \left[|A_i| - l_k \right] \neq 0 \text{ by condition } l_k < |A_i|.$$

Let $1 \le j < i$. $|A_j| \le |A_i|$ so $|A_j \cap A_i| = l_k$ for some $l_k \in L$. with $l_k < |A_i|$.

Hence $p_i(\underline{v}_j) = \prod_{k: l_k < A_i} [\underline{v}_i \cdot \underline{v}_j - l_k]$

$= 0$ since one of the terms is zero

What is the maximum possible degree of the $p_i(\underline{x})$?

terms in the product (*) is $\le s = |L|$

So product of $\le s$ terms, each of which is linear
Hence degree $\le s$.

Now use a trick! $0^t = 0 \quad t \geq 2$
$1^t = 1 \quad t \geq 2$

Let $q_i(x)$ be the polynomial obtained from $p_i(x)$ by replacing each x_j^α by x_j, where $\alpha \geq 2$

Note if $\underline{v} \in \{0,1\}^n$ then $q_i(\underline{v}) = p_i(\underline{v})$.
In particular, $q_i(\underline{v}_i) \neq 0$, $q_i(\underline{v}_j) = 0 \quad 1 \le j < i$. . . . . ($\dagger$)

Moreover, $\deg[q_i(x)] \le \deg[p_i(x)] \le s$.

So $q_i(x)$ consists of a sum of terms of the form
$c x_{i_1} x_{i_2} x_{i_3} \cdots x_{i_r}$ where $r \le s$.

Let M be the vector space spanned by
$1, x_1, x_2, \ldots, x_n, x_1x_2, \ldots, x_{n-1}x_n, \ldots, x_1x_2 \overset{\text{cdots}}{—} x_s,$
$\ldots x_{n-s+1} x_{n-s+2} — x_n.$

Then $q_1(x), \dots, q_m(x) \in M$.

Now $\dim(M) \leq \sum_{i=0}^{s} \binom{n}{i}$

So now if we show $q_1, \dots, q_m$ are LI then $m \leq \dim M$.
Suppose they are not LI, i.e. $\exists\ \lambda_1, \dots, \lambda_m$
s.t. $\sum_{i=1}^{m} \lambda_i q_i = 0$ (zero polynomial in n vars)

and not all of the λ_i's are zero.

Let $k = \min\{i : \lambda_i \neq 0\}$.

Evaluate the zero polynomial at v_k.

$$\overset{\text{number } 0}{0} = 0(v_k) = \sum_{i=1}^{m} \lambda_i q_i(v_k)$$

$$= \underset{\neq 0}{\lambda_k}\, \underset{\substack{\neq 0 \\ \text{by } (\dagger)}}{q_k(v_k)} + \sum_{i=k+1}^{m} \underbrace{\lambda_i q_i(v_k)}_{=0 \text{ by } (\dagger)}$$

hence $0 = \lambda_k q_k(v_k) \neq 0$ #

$\Rightarrow q_1, \dots, q_m$ are LI.

Thus $|A| = m \leq \dim(M) = \sum_{i=0}^{s} \binom{n}{i}$ $\square$.

RAMSEY THEORY

A recent result from Ramsey Theory (2007)
<u>Green-Tao:</u> The primes contain arbitrarily long arithmetic progressions.

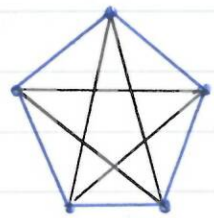
<u>Defn</u>: For $s, t \geq 2$, define <u>$R(s,t)$</u> to be the smallest integer n s.t. whenever the <u>edges</u> of the complete graph K_n are coloured red and blue, there is always either a red K_s, or a blue K_t (or both).

e.g. $R(2,t) = t$.
$R(s,2) = s$

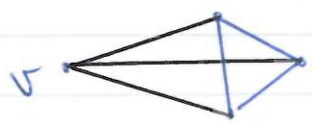
<u>Propn. 4.1</u> $R(3,3) = 6$ (friends and strangers!)

<u>Proof</u>: $R(3,3) > 5$.

This is a red/blue (edge) colouring of K_5 with no red K_3 and no blue K_3.



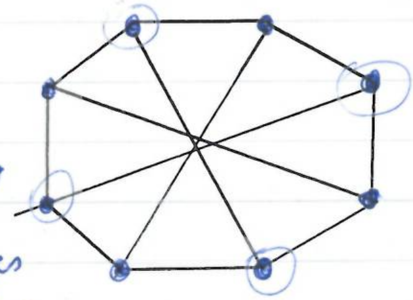
$R(3,3) \leq 6$. Take a red/blue colouring of K_6. Consider $v \in V(K_6)$. Since $d(v) = 5$, one of the colours is used at least three times on edges incident to v. Wlog suppose this is red.



Either a pair $x, y \in \Gamma_R(v)$ are joined by a red edge $\Rightarrow$ red K_3, or all such pairs are joined by blue edges $\Rightarrow$ blue K_3. $\square$

Prop^n 4.2 $R(3,4) = 9$.

Proof: $R(3,4) > 8$.

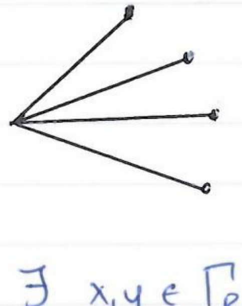


No red K_3.
No blue K_4.

$R(3,4) \leq 9$. Take a red/blue colouring of K_9.

[Let $v \in V(K_9)$. Define
$$\Gamma_R(v) = \{w : vw \text{ is red}\}$$
$$\Gamma_B(v) = \{w : vw \text{ is blue}\}$$
$$d_R(v) = |\Gamma_R(v)|$$
$$d_B(v) = |\Gamma_B(v)|$$
]

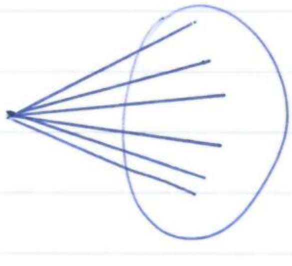
If $d_R(v) \geq 4$ then



either $\exists\ x, y \in \Gamma_R(v)$ with xy red
$\Rightarrow$ red K_3
or $\forall\ x, y \in \Gamma_R(v)$, xy is blue
$\Rightarrow$ blue K_4.

So $\overset{\text{wlog}}{}\ d_R(v) \leq 3 \quad \forall v \in V(K_9)$.

$d_R(v) + d_B(v) = d(v) = 8$.



If $d_R(v) \le 2$ then $d_B(v) \ge 6$. Since $R(3,3) = 6$, $\Gamma_B(v)$ contains a red K_3 or blue K_3 which together with v gives a blue K_4.

Only remaining case is $d_R(v) = 3 \quad \forall v \in V(K_9)$.

$$9 \cdot 3 = \sum_{v \in V(K_9)} d_R(v) = 2 \cdot \#\text{edges} \qquad \#.$$

$\square$

<u>Thm 4.3</u> (Ramsey) If $s, t \ge 2$ then $R(s,t)$ is well-defined and satisfies

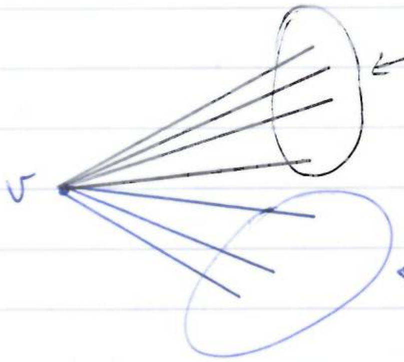
$$R(s,t) \le \binom{s+t-2}{s-1}. \qquad \text{- - - - } (*)$$

<u>Proof</u>: (induction on $s+t$). [proof by Erdős & Szekeres]

Know $R(2,t) = t$
$R(s,2) = s$
and $(*)$ holds for both cases.
So certainly result holds for $s+t = 4$.

Let $n = R(s-1,t) + R(s,t-1)$, which is well defined by our inductive hypothesis.

Consider a red/blue edge colouring of K_n.



$\leftarrow$ looking for red K_{s-1} or blue K_t $\Big\}$ guaranteed if size $\geq R(s-1, t)$

$\leftarrow$ looking for red K_s blue K_{t-1} $\Big\}$ guaranteed if size $\geq R(s, t-1)$

Let $v \in V(K_n)$ so $d(v) = n-1$.

<u>Claim</u>: either $d_R(v) \geq R(s-1, t)$ - - - - - (1)
$\qquad\qquad d_B(v) \geq R(s, t-1)$ - - - - - (2)

<u>Proof of claim</u>: If not then

$$\begin{aligned} n-1 = d(v) &= d_R(v) + d_B(v) \\ &\leq R(s-1,t) - 1 + R(s,t-1) - 1 \\ &= n-2 \qquad \#\end{aligned}$$

So (1) or (2) holds.
If (1) holds, $\Gamma_R(v)$ contains a red K_{s-1} or a blue K_t and so together with v we have a red K_s or blue K_t.

If (2) holds, $\Gamma_B(v)$ contains a red K_s or a blue K_{t-1} and so together with v we have a red K_s or blue K_t.

Hence any red/blue edge colouring of K_n contains a red K_s or a blue K_t. Thus

$$\begin{aligned} \boxed{R(s,t) \leq n = R(s-1,t) + R(s,t-1)} & \\ \leq \binom{s+t-3}{s-2} + \binom{s+t-3}{s-1} & \quad \text{by inductive hypoth.} \\ = \binom{s+t-2}{s-1} & \end{aligned}$$

$\square$

praying

$$R(s,s) \leq \binom{2s-2}{s-1} \approx \frac{4^s}{\sqrt{s}} \quad \leftarrow \text{big } n^{o} \text{ !!}$$

Prop^n 4.4 $\quad R(4,4) = 18$

Proof: There exists a red/blue edge colouring of K_{17} with no red K_4 and no blue K_4 (Google it)

$$\Rightarrow \quad R(4,4) > 17$$

and we know

$$R(4,4) \leq R(3,4) + R(4,3) = 18 \qquad \square$$

What is $R(5,5)$? Noone knows!!

$43 \leq R(5,5) \leq 49$ is best we have.

Why does noone know?

Would need to check every red/blue edge colouring of K_{42}.

How many edges does it have? $\frac{43 \times 42}{2}$ $\left(= \binom{43}{2}\right)$

So $2^{\binom{43}{2}}$ combinations. Not enough computing power in the world! CHALLENGE ACCEPTED!!

Thm 4.5 (Erdös). Let $s \geqslant 2$

$R(s,s) > n$ if n satisfies

$$\binom{n}{s} \frac{2}{2^{\binom{s}{2}}} < 1$$

Proof: Need to show $\exists$ a red/blue edge colouring of K_n with no red K_s or blue K_s.

Colour edges of K_n independently red/blue with probability $1/2$.

Take $A = \{v_1, \ldots, v_s\} \subseteq V(K_n)$.

$\Pr(A \text{ forms a red } K_s) = \left(\frac{1}{2}\right)^{\binom{s}{2}}$

$\Pr(A \text{ forms a blue } K_s) = \left(\frac{1}{2}\right)^{\binom{s}{2}}$

$E(\#\text{ red } K_s\text{'s}) = \binom{n}{s}\left(\frac{1}{2}\right)^{\binom{s}{2}}$

$E(\#\text{ monochromatic } K_s\text{'s}) = \binom{n}{s}\dfrac{2}{2^{\binom{s}{2}}} < 1$

$\Downarrow$

$\exists$ a colouring with no monochromatic K_s.

<u>Corollary 4.6</u> If $s \geqslant 2$ then $R(s,s) > 2^{s/2}$

<u>Proof</u>: True for $s = 2, 3$, so suppose $s \geqslant 4$.

Let $n = 2^{s/2}$, need $\binom{n}{s} < \dfrac{2^{\binom{s}{2}}}{2} = 2^{\frac{s^2}{2} - \frac{s}{2} - 1}$

Well, $\binom{n}{s} \leq \dfrac{n^s}{s!} = \dfrac{2^{s^2/2}}{s!}$

Need to check that $\dfrac{2^{s^2/2}}{s!} < \dfrac{2^{s^2/2}}{2^{s/2+1}}$

ie $2^{\frac{s}{2}+1} < s!$

$\nwarrow$ This is true but boring to prove.

$\square$

So we've proven so far that

$$2^{s/2} < R(s,s) < \frac{4^s}{\sqrt{s}}$$

Thm 4.7 (Fermat's Last Theorem)
There are no non-trivial integer sol^ns to
$x^n + y^n = z^n \quad \forall n \geq 3$

No proof given!! $\square$

Thm 4.8 For every $n \geq 1$ $\exists$ p_n s.t. for any prime $p \geq p_n$, the congruence
$$x^n + y^n \equiv z^n \pmod{p}$$
has a non-trivial sol^n mod p (i.e. $x, y, z \not\equiv 0 \bmod p$)

Equivalently: Fermat's Last Theorem is false in the field $\mathbb{Z}_p$ for sufficiently large primes p.

Thm 4.9 (Schur) For any $k \geq 1$, $\exists$ $S(k)$ s.t. any k-colouring of the integers $\{1, 2, \dots, S(k)\}$ will contain u, v, w of the same colour satisfying $u + v = w$. Proof later

Lemma 4.10 If p is a prime then the multiplicative group $\mathbb{Z}_p^*$ is cyclic.

Proof of Thm 4.8 Let $n \geq 1$ and let p be any prime larger than $S(n)$. Consider $\mathbb{Z}_p^*$ and let $g \in \mathbb{Z}_p^*$ be a generator of $\mathbb{Z}_p^*$ (given by Lemma 4.10).

So for any $x \in \mathbb{Z}_p^*$ we have $x = g^m \pmod p$ so there is a unique $0 \le i \le n-1$ such that $x = g^{nj+i}$.

Define an n-colouring of $\mathbb{Z}_p^*$ by $c(x) = i$
Since $|\mathbb{Z}_p^*| = p-1 \geq S(n)$, by Schur's Thm (4.9), $\exists\ u, v, w \in \mathbb{Z}_p^*$ of the same colour s.t. $u+v=w$ and $c(u) = c(v) = c(w) = i$.

So $\exists\ j_u$ s.t. $u = g^{nj_u+i}$
and " j_v " $v = g^{nj_v+i}$
" j_w " $w = g^{nj_w+i}$

Let $x = g^{j_u}$
$y = g^{j_v}$
$z = g^{j_w}$

Then
$$\begin{aligned} x^n + y^n = g^{nj_u} + g^{nj_v} &= g^{-i}(u+v) \\ &= g^{-i} w \\ &= g^{-i} g^{nj_w+i} \\ &= g^{nj_w} \\ &= z^n \qquad \pmod p \qquad \square \end{aligned}$$

Def^n: For any $k \geq 2$ and $s_1, s_2, \dots, s_k \geq 2$ we can define $R_k(s_1, \dots, s_k)$ to be the smallest integer n s.t. any k-colouring of the edges of the edges of K_n with colours $c_1, c_2, \dots, c_k$ contains a copy of K_{s_i} of colour c_i for some $1 \le i \le s$.

If $s_1 = s_2 = \dots = s_k = s$, we write $R_k(s) = R_k(s, s, \dots, s)$.

<u>Thm 4.11</u> For any $k \geq 2$, $s_1, \dots, s_k \geq 2$, $R_k(s_1, \dots, s_k)$ is well-defined.

<u>Proof</u> (by induction on k). True for $k=2$ by Ramsey's Theorem (4.3).

Assume well-defined for $k-1$.

$$n = R_{k-1}(s_1, s_2, \dots, s_{k-2}, R(s_{k-1}, s_k)).$$

Take a colouring of the edges of K_n with colours $c_1, \dots, c_k$. Either we have a c_i-coloured copy of K_{s_i} for some $1 \leq i \leq k-2$, or we have a copy of $K_{R(s_{k-1}, s_k)}$ coloured with colours c_{k-1} and c_k.

So by Ramsey's Thm (4.3) this contains a c_j coloured K_{s_j} for $j = k-1$ or k. $\square$

<u>Proof of Thm 4.9 (Schur)</u>: Let $n = R_k(3)$.

Now take a k-colouring of $\{1, 2, \dots, n\}$, c.

Define a k-colouring of the edges of K_n by $c^*(ij) = c(|i-j|)$.

$$(V(K_n) = \{1, 2, \dots, n\})$$

So c^* is a k-colouring of the edges of K_n, $n = R_k(3)$
$\Rightarrow \exists$ some colour c_i and 3 vertices i, j, k s.t.
$c^*(i,j) = c^*(ik) = c^*(jk) = c_i$.

Wlog. $i > j > k$. Let $u = i - j$
$v = j - k$
$w = i - k$

$\Rightarrow u + v = w$

and moreover,
$$c(u) = c(i-j) = c(|i-j|) = c^*(ij) = c_i$$
$$c(v) = c(j-k) = c(|j-k|) = c^*(jk) = c_i$$
$$c(w) = c(i-k) = c(|i-k|) = c^*(ik) = c_i$$

$\square$

Some results

Dirichlet (1837) If $hcf(a,d) = 1$ then the arithmetic progression (AP) $a, a+d, a+2d, \ldots$, contains infinitely many primes.

Van der Waerden (1927) If $k, t \geqslant 1$, $\exists$ $W(t,k)$ s.t. any k-colouring of $[W(t,k)]$ contains a monochromatic AP (MAP) of length t

Erdős-Turán Conjecture: If $A \subseteq \mathbb{N}$ and $\sum_{a \in A} \frac{1}{a}$ diverges then A contains arbitrarily long APs

Roth (1956) If $A \subseteq \mathbb{N}$ has positive density then A contains 3-term APs

Szeméredi (1975) If $A \subseteq \mathbb{N}$ has positive density then A contains arbitrarily long APs

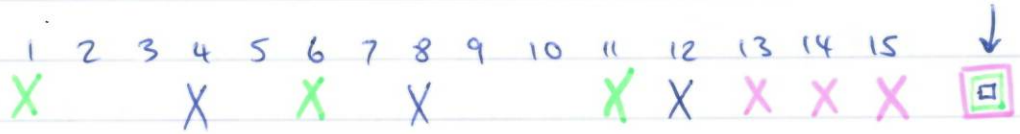
Gowers (2000) (New proof of above)

Green & Tao (2004) The primes contain arbitrarily long APs

We will prove Van der Waerden's Thm, ie.

Thm: $\forall\, k, t \geq 1 \;\exists\, W(t,k)$ s.t. any k-colouring of $[W(t,k)]$ contains a MAP. of length t.

Proof: (Induction on t) $t=1$ ✓ (trivial)
Assume $t \geq 2$ and $W(t-1,k)$ exists of all $k \geq 2$



Def^{n}: If $P_1, \ldots, P_r$ are APs, $P_i = \{a_i, a_i + d_i, \ldots, a_i + (l-1)d_i\}$ all of length l, we say they are focused at $f \in \mathbb{N}$ if for $1 \leq i \leq r$, $a_i + l d_i = f$

Def^{n}: If $P_1, \ldots, P_r$ are monochromatic with distinct colours, we say they are colour-focused at f (CF).

Claim: $\forall\, 1 \leq r \leq k$, $\exists\, n_t(r,k)$ s.t. any k-colouring of $[n_t(r,k)]$ either contains a MAP of length t, or it contains r CFAPs of length $t-1$.

Claim $\Rightarrow W(t,k)$ exists.

Take $W(t,k) = n_t(k,k)$.

k-colour $[W(t, k)]$. Either we have a MAP of length t or we have k CFAPs of length $t-1$, one of which has the same colour as the focus. $\square$

<u>Proof of claim: (induction on r)</u> For $r=1$, we can take

$$n_t(1,k) = W(t-1,k) \checkmark$$

Now suppose $2 \leq r \leq k$ and $n_t(r-1,k)$ exists.
Let $n = n_t(r-1,k)$

Define $N = 2n W(t-1, k^{2n})$. ← bigger than anything astronomical

Consider k-colouring of $[N]$.

Assume there is no MAP of length t.

Define $B_i = [2n(i-1)+1, 2ni]$ ← "integers in this gap"

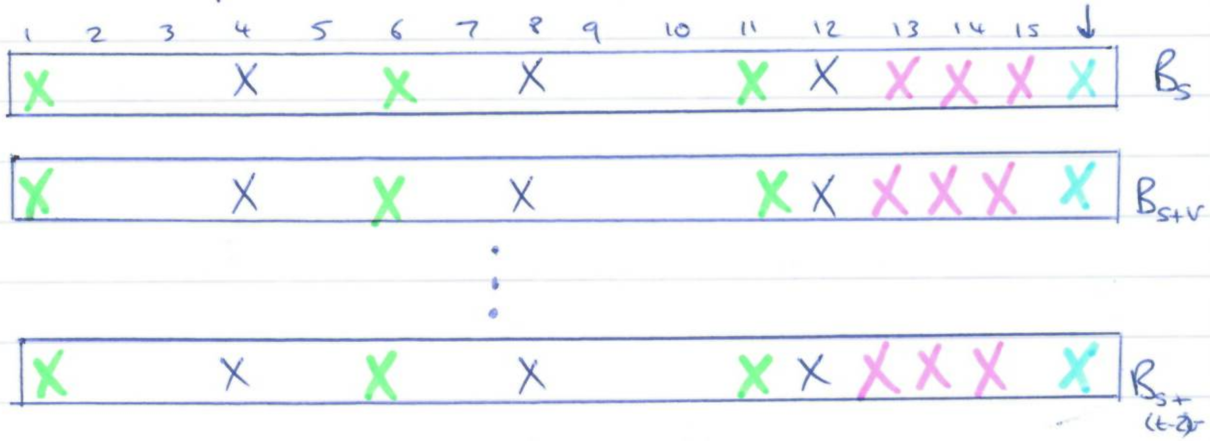
$[N] = B_1 \cup B_2 \cup \dots \cup B_{W(t-1,k^{2n})}$.

If $B_1 = \{1, 2, \dots, 2n\}$, we have k^{2n} different ways to colour a block.

So in fact we can consider this as a k^{2n}-colouring of $[W(t-1, k^{2n})]$ (given by $c(i)$ = "pattern of colours used in block B_i")

By defn of $W(t-1, k^{2n})$, $\exists$ $B_s, B_{s+r}, \dots, B_{s+(t-2)r}$ coloured identically.

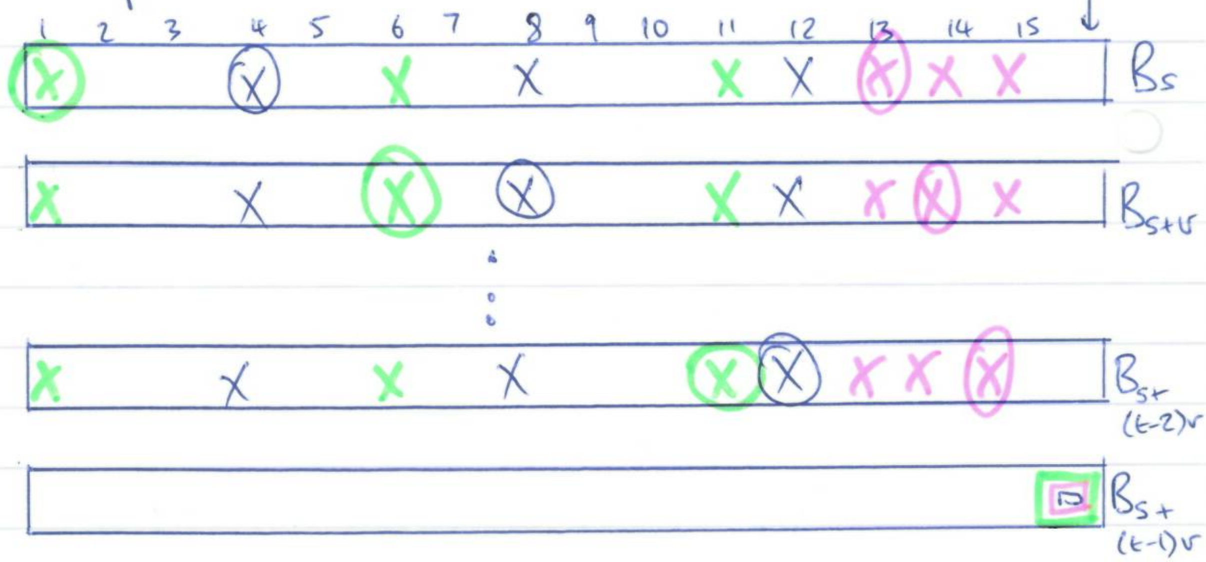
Return to picture:



$|B_s| = 2n = 2n_t(r-1, k)$.

So B_s contains $r-1$ CFAPs of length $\underline{t-1}$, together with their focus.
Say $P_1, \dots, P_{r-1}$ $\quad P_i = \{a_i, a_i + d_i, \dots, a_i + (t-2)d_i\}$

Return to pic:



Define $P_i' = \{a_i, a_i + (d_i + 2nv), a_i + 2(d_i + 2nv), \dots, a_i + (t-2)(d_i + 2nv)\}$

Then $P_1', \dots, P_{r-1}'$ are $r-1$ CFAPs of length $t-1$

with focus $f+(t-1)2nv$. But if we define

$P_r' = \{f, f+2nv, \ldots, f+(t-2)2nv\}$,

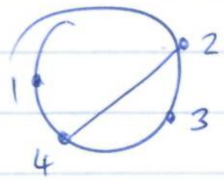
then we have r CFAPs of length $t-1$. $\square$

Two Past Papers in 50 mins

Revise things in this order:

(1) Definitions (making sure noone gets 0)
(2) Statements of thms — Names!
(3) Proofs

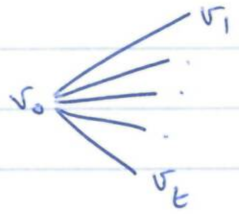
2010

1b 

$1 \to 2 \to 3 \to 4 \to 1$.

1d Proof is same as 1c except for a few points when it's different.

2c Part of a named theorem
2d Unnamed proof
2e



Engage brain.

Proof: If G is S_t-free, $\Delta(G) \leq t-1$

$$\Rightarrow 2m = \sum_{v \in V} d(v) \leq (t-1)n$$

$$\Rightarrow \frac{m}{\binom{n}{2}} \leq \frac{(t-1)n}{2\binom{n}{2}} \longrightarrow 0.$$

Or use Erdös-Stone thm (2.21): If $\chi(H) = k$, then

$$\pi(H) = 1 - \frac{1}{k-1}$$

and $\chi(S_t) = 2$ since it's bipartite.

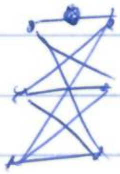
3a $n-m+f=2$

3b

3c $4f \le 2m$

$gf \le 2m$

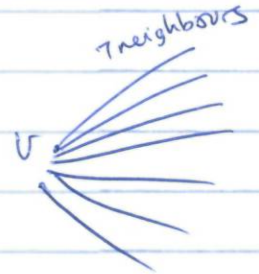
3e Take $K_{3,3}$ and add a vertex



4 All bookwork

5a Ramsey's thm.

5b



no red K_3

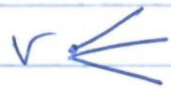
no blue C_4

Something like this

$d_R(v) \geq 4 \Rightarrow$ we get ~~red K~~ blue K_4, contains C_4.

Assume that $d_R(v) \le 3$

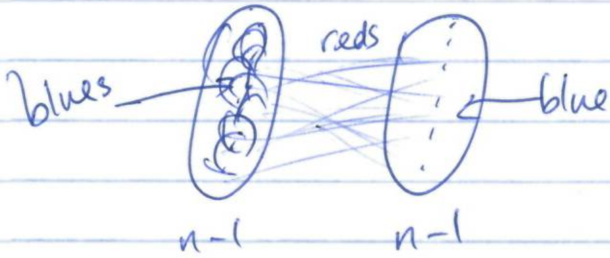
So $d_B(v) \geq 3$



5c K_{2n-2} no red K_3

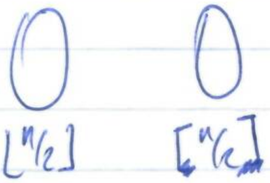
no blue C_n.

Take bipartite graph, of classes with equal size



↑

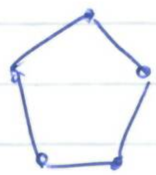
no C_n since $n-1$ el's in here.

2009 2d 

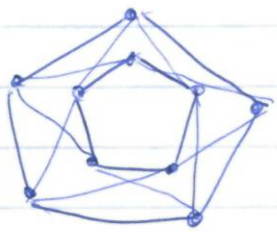
2e Mantel's thm.

2f K_3 free + not bipartite $\Rightarrow$ contains odd cycle of min. length 5 (shortest odd cycle in G in C_5)

Odd n :



Even n



3d Simple thm from notes
3e Prob. sheet

4c $\{A : 1 \in A\}$ $\qquad 2^{n-1}$
$\{A : |A \cap [3]| \geq 2\}$ $\qquad 2^{n-3} \times 4 = 2^{n-1}$