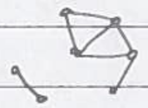
MATH0029 Graph Theory and Combinatorics Notes

Based on the 2019 spring lectures by Dr J Talbot

MATH 0029

08-01-19 Graph Theory and Combinatorics

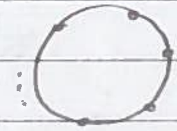
$G = (V, E)$ 

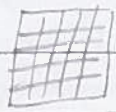
Q: Given a graph G with n vertices and m edges, must G contain a triangle? $(n \geq 3)$

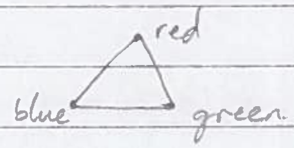
Q: How large can m be?
A: $\binom{n}{2} = \frac{n(n-1)}{2}$

Q: In a complete graph, how many triangles are there?
A: # triangles $= \binom{n}{3}$

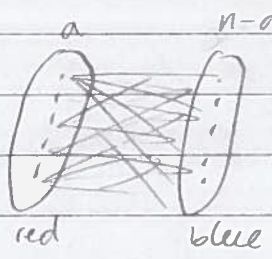
Cycle C_n : n vertices, n edges
proportion of edges: $\frac{n}{\binom{n}{2}} = \frac{2}{n-1}$



 $\sim 2n$ edges



$n = 2k$



$m = a(n-a)$

$a = \frac{n}{2}$

Given a graph G with $n \geq 3$ vertices and $> \frac{n^2}{4}$ edges, G must contain a triangle.

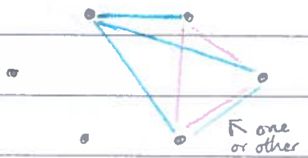
"Turán Problem"

Ramsey problems

In any set of 6 people there are always ≥ 3 mutual friends or ≥ 3 mutual strangers.

Colour edges of K_6 red and blue.
There is always a red K_3 or a blue K_3
$\nwarrow$ complete, 3 vertices

Proof
Consider a vertex v of the K_6.
WLOG ≥ 3 blue edges from v.



In the three target vertices either there is a blue edge $\Rightarrow$ blue triangle or the three edges are red $\Rightarrow$ red triangle.

$R(3,3) = 6$ (Ramsey number)

[Green-Tao (200?) $\Leftarrow$ link to Ramsey theory.
$\forall k \in \mathbb{N}$ $\exists$ an arithmetic progression in the primes of length k.]

Set Systems
$X = [n] = \{1, 2, \ldots, n\}$
$\mathcal{P}(X) = \{A : A \subseteq X\}$

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If $\mathcal{A} \subseteq \mathcal{P}(X)$ we say $\mathcal{A}$ is (pairwise) <u>intersecting</u> iff $\forall A, B \in \mathcal{A}$, $A \cap B \neq \emptyset$.

How large can an intersecting family $\mathcal{A} \subseteq \mathcal{P}(X)$ be?

$|\mathcal{P}(X)| = 2^n$

$$\mathcal{A} = \{\underbrace{1}_{\{1\}}, \underbrace{12}_{\{1,2\}}, 13, \dots\} \text{ etc.} = \{A \subseteq X : 1 \in A\}$$

$|\mathcal{A}| = 2^{n-1} = \frac{1}{2}|\mathcal{P}(X)|$

$\mathcal{B}$, $[B \in \mathcal{B} \Rightarrow B^c \notin \mathcal{B}] \Rightarrow |\mathcal{B}| \leq 2^{n-1}$

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<u>Chapter 1 - Basics</u>

If X is a set, $|X| =$ size of X

For $k \in \mathbb{N}$, $k! = 1 \times 2 \times \dots \times k$, $0! = 1$

<u>Def^n</u>

If X is a set and $k \in \mathbb{N}$ then a k-tuple from X is a sequence of k elements from X.

<u>Lemma 1.1</u>

If $|X| = n$ and $1 \leq k \leq n$ then

1). there are n^k k-tuples from X

2). there are $n(n-1)\dots(n-k+1)$ k-tuples with distinct elements from X

<u>Proof</u>

1). n choices for each entry in a k-tuple $\Rightarrow n^k$

2). n choices for 1st element, $n-1$ for 2nd, etc.

$\square$

If X is a set, $|X| = n$, $0 \le k \le n$, then denote $\binom{X}{k} = \{A \subseteq X : |A| = k\}$, the family of k-sets from X.

Lemma 1.2

If $|X| = n$, $0 \le k \le n$, then $\left|\binom{X}{k}\right| = \binom{n}{k} = \frac{n(n-1)\dots(n-k+1)}{k!}$
$= \frac{n!}{k!(n-k)!}$

Proof

Since a k-set contains k distinct elements, each k-set can be ordered in $k!$ different ways to give a k-tuple.

Hence $\left|\binom{X}{k}\right| = \frac{\#\{k\text{-tuples of distinct elements of } X\}}{k!} \Rightarrow$ result

$\square$

$\mathcal{P}(X) = \{A \mid A \subseteq X\}$ ← Power set

Lemma 1.3

(i) If $|X| = n$, then $|\mathcal{P}(X)| = 2^n$

(ii) If $0 \le k \le n$, then $\binom{n}{k} = \binom{n}{n-k}$

(iii) If $1 \le k \le n$, then $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$

Proof

(i) Induction on n.

$n = 1$, $\mathcal{P}(\{1\}) = \{\emptyset, \{1\}\}$ ✓

Suppose true for $n = t \ge 1$ and let $|X| = t+1$.

WLOG $X = \{1, 2, \dots, t, t+1\}$.

$\dot\cup$ = disjoint union

$\mathcal{P}(X) = \mathcal{P}([t]) \,\dot\cup\, \{A \cup \{t+1\} : A \in \mathcal{P}([t])\}$ disjoint union

So $|\mathcal{P}(X)| = 2|\mathcal{P}([t])| = 2 \times 2^t = 2^{t+1}$

$[t] = \{1, \dots, t\}$

(ii) $\binom{n}{k} = \left|\binom{X}{k}\right|$ since $|X| = n$

$= \left|\binom{X}{n-k}\right|$ since the mapping $A \mapsto X \setminus A$ is a bijection

$= \binom{n}{n-k}$

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(iii) Let $X = [n+1]$

$$\binom{[n+1]}{k} = \binom{[n]}{k} \,\dot\cup\, \left\{ A \cup \{n+1\} \;\middle|\; A \in \binom{[n]}{k-1} \right\} \quad \text{disjoint union}$$

so $\left|\binom{[n+1]}{k}\right| = \left|\binom{[n]}{k}\right| + \left|\binom{[n]}{k-1}\right|$

$\Rightarrow \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$

$\square$

Example

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

$\Updownarrow$

$$|\mathcal{P}([n])| = \left|\binom{[n]}{0}\right| + \left|\binom{[n]}{1}\right| + \dots + \left|\binom{[n]}{n}\right|$$

Chapter 2 - Graphs

A graph is a pair $G = (V, E)$
V is a set of vertices and $E \subseteq \binom{V}{2}$ are edges

(i.e. edges are unordered pairs of vertices)

Sometimes write $V(G)$ and $E(G)$ to denote the vertices and edges resp. of the graph G.

The order of a graph $G = (V, E)$ is $|V|$
The size is $|E|$.

If G is a graph, $v \in V(G)$, the neighbourhood (nbhd) is $\Gamma(v) = \{w \mid vw \in E(G)\}$ [here vw means $\{v, w\}$].

If $v \in V(G)$ then the degree of v is $d(v) = |\Gamma(v)|$

Lemma 2.1 (Handshake Lemma)

If $G = (V, E)$ then $\sum_{v \in V} d(v) = 2|E|$.

Proof

Each edge contains two vertices so contributes to the degree of exactly two vertices. $\square$

Lemma

In any graph $G = (V, E)$ the number of vertices of odd degree is even.

Proof

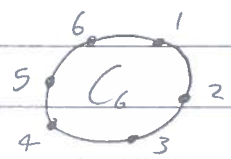
Let $V = V_{even} + V_{odd}$

vertices of even degree ↑ ↖ vertices of odd degree

$$2|E| = \sum_{v \in V} d(v) = \sum_{v \in V_{even}} d(v) + \sum_{v \in V_{odd}} d(v)$$

so mod 2, $0 = 0 + |V_{odd}|$ $\square$

C_n - cycle of length n e.g.



$V(C_n) = [n]$

$E(C_n) = \{ i(i+1) \mid 1 \leq i \leq n-1 \} \cup \{1n\}$

P_n - path of length n e.g.

P_2

(not P_3 ← we count edges not vertices here!)

$V(P_n) = \{0\} \cup [n]$

$E(P_n) = \{ i(i+1) \mid 0 \leq i \leq n-1 \}$

E_n - empty graph of order n

$V(E_n) = [n]$, $E(E_n) = \emptyset$

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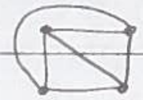
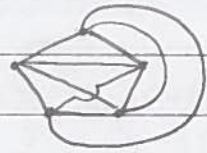
K_n - complete graph of order n

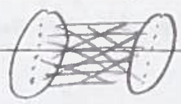

K_4

$V(K_n) = [n]$

$E(K_n) = \binom{[n]}{2}$


K_5

Let $a, b \in \mathbb{N}$, $K_{a,b}$ is the complete bipartite graph



$V(K_{a,b}) = [a] \cup \underbrace{[a+1, a+b]}_{\{a+1,\, a+2,\, \dots,\, a+b\}}$

$E(K_{a,b}) = \{ij \mid 1 \le i \le a,\ a+1 \le j \le a+b\}$

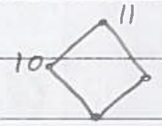
Q_n - discrete hypercube

$V(Q_n) = \{0, 1\}^n$

$E(Q_n) = \{uv \mid u, v \in \{0,1\}^n,\ u \text{ and } v \text{ differ in a single coordinate}\}$

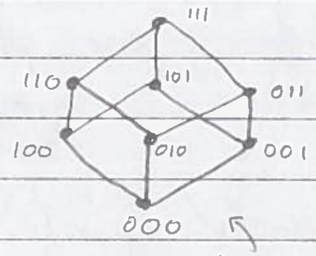
e.g. $n = 2$, $V(Q_2) = \{00, 10, 01, 11\}$ ← binary sequences

sim. to $\{\emptyset, 1, 2, 12\}$ ← sets

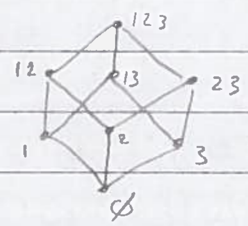


Q_2

Q_3



↑ binary sequences.



← as sets

<u>Subgraphs</u>

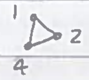
If G, H are graphs with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then H is a <u>subgraph</u> of G.

e.g.

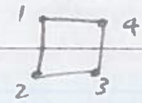
$G = K_4 =$ 

$H_1 = K_3 =$ 

subgraph of G

$H_2 =$ 

subgraph of G

$H_3 = C_4 =$ 

subgraph of G

We say H is an <u>induced subgraph</u> of G if $V(H) \subseteq V(G)$ and $E(H) = E(G) \cap \binom{V(H)}{2}$.

In the previous examples, H_1, H_2 are induced, but C_4 is not.

Graphs G, H are isomorphic iff $\exists f : V(G) \to V(H)$ a bijection s.t. $vw \in E(G) \Leftrightarrow f(v)f(w) \in E(H)$.

We say G <u>contains a copy</u> of H iff G has a subgraph isomorphic to H

We say G is <u>H-free</u> iff G does not contain a copy of H.

Q: Does Q_n ever contain a copy of K_t?

$Q = | = K_2$

$x, y, z \in \{0,1\}^n$

x, y differ in one place

same for x, z, y, z.

Define $w(v) = \#$ 1's in v (weight of v)

$xy \in E(Q_n)$, wlog $w(x) = t$, $w(y) = t+1$

xz, $w(z) = t+1$ or $t-1$ $\Rightarrow$ no edge yz.

§2.3 - Components and connectedness

A <u>path</u> in a graph G is a copy of P_t for some $t \geq 0$.

An $(x\text{-}y)$-path is a path that starts at x and ends at y. If $v_1 v_2 \cdots v_r$ is an x-y-path, $x = v_1$, $y = v_r$, $v_i v_{i+1} \in E(G)$ for $1 \leq i \leq r-1$, and the vertices $v_1, \ldots, v_r$ are distinct.

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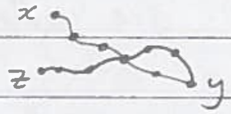
A *walk* in G is a sequence of (not necessarily distinct) vertices $v_0 v_1 \dots v_t$ such that $v_i v_{i+1} \in E(G)$ for $0 \le i \le t-1$.



(not necessarily distinct edges)

A walk is *closed* iff $v_0 = v_t$.

A walk in which no edge occurs more than once is called a *tour*.

Lemma 2.3

There is an x-y-path in G iff there is a walk from x to y in G.

Proof

($\Rightarrow$) trivial since a path is a walk.

($\Leftarrow$) Consider the shortest walk from x to y. Either this is an x-y-path, or there is a repeated vertex, say $W = v_1 v_2 \dots v_i \dots v_t$ and $v_i = v_j$, $i<j$ then $W' = v_1 v_2 \dots v_i v_{j+1} \dots v_t$ is a shorter walk from x to y. $\divideontimes$ $\square$

Lemma 2.4

Define a ~~equivalence~~ relation $\sim$ on $V(G)$ by $v \sim w \Leftrightarrow$ there is a walk from v to w.

This is an equivalence relation.

Proof

$\forall v \in V(G)$, $v \sim v$. (Reflexive)

If $v \sim w$ then reverse the walk to get a walk from w to v, $w \sim v$. (Symmetric)

If $x \sim y$, $y \sim z$, then concatenate walks from x to y and y to z to give a walk from x to z, $x \sim z$.

(Transitive).

$\square$

Let $V(G) = V_1 \dot\cup V_2 \dot\cup \cdots \dot\cup V_k$ as a disjoint union of equivalence classes induced by $\sim$.
Then the V_i are called <u>components</u>.

We say G is <u>connected</u> iff it has a single component i.e. $k = 1$.

 $\leftarrow$ 3 components

C_n, K_n, $K_{a,b}$, Q_n, P_n are all connected.

<u>Lemma 2.5</u>
Let $P = x_1 x_2 \cdots x_i \cdots x_t$ be a path, if P is a shortest $x_1 - x_t$ -path then $x_1 \cdots x_i$ and $x_i \cdots x_t$ are shortest $x_1 - x_i$ and $x_i - x_t$ paths respectively $\forall 1 \leq i \leq t$

<u>Proof</u>
If there were a shorter $x_1 - x_i$ -path for some i, then use this in place of $x_1 x_2 \cdots x_i$ in P to produce a shorter $x_1 - x_t$ -path.
$\square$

15-01-19 <u>Lemma 2.5</u> (correct statement)
If $P = x_1 x_2 \cdots x_t$ is a shortest $x_1 - x_t$ path then $x_1 \cdots x_i$ and $x_i \cdots x_t$ are shortest $x_1 - x_i$ and $x_i - x_t$ paths resp. for $1 < i < t$.

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Euler circits

Path

Walk $x_1 \dots x_t$, $x_i x_{i+1} \in E$ $1 \le i \le t-1$,
vertices allowed to repeat.

Tour $x_1 \dots x_t$, $x_i x_{i+1} \in E$ $1 \le i \le t-1$,
vertices can be repeated but edges are distinct.

An Euler circuit in a graph G is a closed tour $T = v_0 v_1 \dots v_k v_0$ containing all vertices and edges of G.

Theorem (Euler 1735)
A graph G has an Euler circuit iff G is connected and every vertex has even degree.

Proof
($\Rightarrow$) Suppose G contains an Euler circuit T.
Since T is a walk containing all vertices of G, G must be connected.
Now let $T = v_0 v_1 \dots v_k v_0$. We walk along the Euler circuit T starting from v_0 and we keep track of the number of edges we have used adjacent to each vertex.
Whenever we enter a vertex we also leave it again (except in our first and last steps).
Thus we have a contribution of 2 to the degree of each vertex. Since we start and end at v_0, this also has even degree.
So all vertex degrees are even.
($\Leftarrow$) Suppose G is connected and all vertex degrees

are even.

Start by taking $T = v_0 v_1 \cdots v_k$ to be a longest tour in G. Claim: T is closed, i.e. $v_0 = v_k$.

Suppose $v_0 \neq v_k$.

Let $j = \{ i : v_i = v_k, 1 \le i \le k-1 \}$, so have used $2j+1$ edges adjacent to v_k, $d(v_k) = $ even

$\Rightarrow \exists$ unused edge $v_k v'$ s.t. $T' = v_0 v_1 \cdots v_k v'$ is a longer tour ※

Hence $v_0 = v_k$.

To show G is an Euler circuit, need to show it contains all edges (then G connected $\Rightarrow$ it contains all vertices).

Suppose there is an unused edge $uv \in E(G)$.

Either one of u or v is in T, or if not then take a path from v_0 to u, then this path must leave the tour so there is in fact an edge xy that is unused and contains a vertex from T.

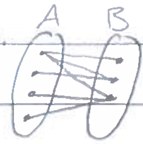
So wlog can suppose there is an unused edge uv, with $u = v_i \in T$.

But this is a longer tour $T'' = v u v_{i+1} \cdots v_k = v_0 v_1 \cdots v_i$.

$\square$

Bipartite Graphs

e.g.

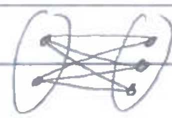


A graph G is *bipartite* if $G = (V, E)$ and there is a partition $V = A \dot\cup B$ s.t. $E \subseteq \{ab : a \in A, b \in B\}$.

We sometimes write $G = (A, B; E)$ to denote the bipartition $V = A \dot\cup B$.

We've seen $K_{a,b}$.

$K_{2,3}$



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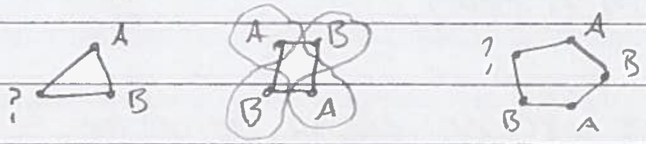
<u>Note</u>: if $n = 2t+1$ then C_n is not bipartite.



etc.

<u>Thm 2.7</u>

A graph G is bipartite iff it contains no odd cycle.

<u>Proof</u>

($\Rightarrow$) Suppose G is bipartite and G contains a copy of C_{2t+1}, with vertices $v_1, v_2, \dots, v_{2t+1}$.
Let $V = A \dot\cup B$ be a bipartition.
Then wlog $v_1 \in A \Rightarrow v_3, v_5, \dots, v_{2t+1} \in A$
$\hookrightarrow v_2 \in B \nearrow$

$\times$ since $v_1 v_{2t+1} \in E$.

($\Leftarrow$) Suppose $G = (V, E)$ contains no odd cycles.

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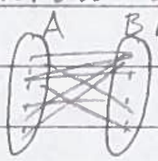
Bipartite graphs



$\triangle$, ⬠ , C_{2t+1} are not bipartite.

<u>Thm 2.7</u>

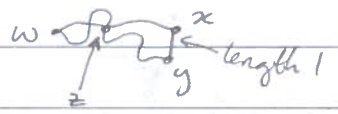
G is bipartite iff G contains no odd cycles.

<u>Proof</u>

($\Rightarrow$) last time

($\Leftarrow$) Suppose G is a graph with no odd cycles.
Can assume G is connected. Pick $w \in V$, put $w \in A$.
For any pair of vertices $x, y \in V(G)$ define
$d(x,y)$ = length of a shortest path from x to y.

$A = \{v \in V(G) \mid d(v,w) \text{ is even}\}$
$B = \{v \in V(G) \mid d(v,w) \text{ is odd}\}$
$\Rightarrow V(G) = A \dot\cup B$
Need to show there are no edges in A or in B.
By contradiction, suppose $\exists\ x, y \in A$ s.t. $xy \in E$



We know $d(w,x) = 2i$, $d(w,y) = 2j$
Let P_{wx} and P_{wy} be shortest w-x and w-y paths respectively.
Let z be the last common vertex of P_{wx} and P_{wy}.
Let $d(w,z) = d$.
Now consider the cycle formed by following P_{wx} from z to x, then the edge xy, then P_{wy} from y to z.
Since P_{wx} is a shortest w-x path, so the part of it from w to z is a shortest w-z path and the path from z to x is a shortest z-x path.
So the part of P_{wx} from z to x has length $2i-d$.
Similarly the part of P_{wy} from z to y has length $2j-d$.
Hence we have a cycle of length $2i+2j-2d+1$, odd. $\#$
$\square$

Graph colouring

If G is a graph and $k \in \mathbb{N}$ then a k-colouring is a map $c: V(G) \to [k]$ (call $c(v)$ the colour of v) such that $vw \in E(G)$ then $c(v) \neq c(w)$.

If G has a k-colouring then we say G is k-colourable and define the chromatic number of G by
$\chi(G) = \min\{k \in \mathbb{N} \mid G \text{ is } k\text{-colourable}\}$.

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If $|V(G)| = n$ then $\chi(G) \le n$.

$\chi(C_t) = \begin{cases} 2, & t \text{ even} \\ 3, & t \text{ odd} \end{cases}$ $\qquad \chi(K_t) = t$

$\chi(Q) = 2$ (Each vertex is a binary string of length n, two vertices connected if they differ in one place. Consider even / odd number of 1's).

If H is a subgraph of G then $\chi(H) \le \chi(G)$.

$\Delta(G) = \max d(v)$ for $v \in V$ (max degree of any vertex in G)

<u>Thm 2.8</u>

If G is a graph then $\chi(G) \le \Delta(G) + 1$

<u>Proof</u>

Let $k = \Delta(G) + 1$, and $V(G) = \{v_1, \dots, v_n\}$

Define $c : V(G) \to [k]$ as follows:

$c(v_1) = 1$.

If $v_1, \dots, v_{i-1}$ have all been coloured, consider v_i.

$F(v_i) = \{ \ell \in [k] \mid \exists \, v_j \in \Gamma(v_i) \; (j \le i-1) \text{ s.t. } c(v_j) = \ell \}$.

$|F(v_i)| \le d(v_i) \le \Delta(G) = k - 1$

So $C(v_i) = [k] \setminus F(v_i) \ne \emptyset$

$\Rightarrow c(v_i) = \min C(v_i)$ is well defined. $\square$

The <u>girth</u> of a graph G is the length of a shortest cycle in G, denoted $g(G)$.

(If G has no cycles, set $g(G) = \infty$.)

Thm 2.9 (Erdös 1959)
$\forall k, l \geq 3$, $\exists G$ with $\chi(G) \geq k$ and $g(G) \geq l$.

If $G = (V, E)$ we say $A \subseteq V$ is an *independent set* iff A contains no edges.


independent

$\alpha(G) = \max\{|A| \mid A \subseteq V(G) \text{ is independent}\}$ is the *independence number* of G.

Note in any colouring $c: V(G) \to [k]$, the *colour class* of any $i \in [k]$, $c^{-1}(i) = \{v : c(v) = i\}$, is an independent set.

Lemma 2.10
If $G = (V, E)$, $|V| = n$ then $\chi(G) \geq \lceil n/\alpha(G) \rceil$

smallest integer $\geq n/\alpha(G)$

Proof
Let $c: V(G) \to [k]$ be a colouring, with $k = \chi(G)$.
$V(G) = c^{-1}(1) \,\dot\cup\, c^{-1}(2) \,\dot\cup \dots \dot\cup\, c^{-1}(k)$
$n = \sum_{i=1}^{k} |c^{-1}(i)| \leq k\,\alpha(G)$ [since $c^{-1}(i)$ is an independent set]

$\Rightarrow \chi(G) \geq \lceil n/\alpha(G) \rceil$ since χ is an integer. $\square$

A *probability space* is a pair (Ω, P)
$P: \Omega \to [0,1]$, $\sum_{y \in \Omega} P(y) = 1$, if $A \subseteq \Omega$ then $P(A) = \sum_{a \in A} P(a)$.

E.g. $\Omega = \{1, 2, \dots, 6\}$, $P(i) = 1/6$ $\forall i \in \Omega$
$P(\text{odd}) = P(\{1, 3, 5\}) = 1/2$.

A *random variable* is $X : \Omega \to \mathbb{R}$

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e.g. $1_{\{1,3,5\}} = \begin{cases} 1 & , y \in \{1,3,5\} \\ 0 & , \text{otherwise} \end{cases}$

e.g. $X_{win} = \begin{cases} 100 & , i = 6 \\ -20 & , i \neq 6 \end{cases}$

The <u>expectation</u> of a random variable is its average value. If $O_X = \{X(y) \mid y \in \Omega\}$ is the set of values taken by X, then

$$\mathbb{E}[X] = \sum_{z \in O_X} z P(X = z).$$

<u>Lemma 2.11</u> (Linearity of expectation)

If $X_1, \dots, X_n$ are random variables on the same space, then $\mathbb{E}\left[\sum_{i=1}^{n} X\right] = \sum_{i=1}^{n} \mathbb{E}[X_i]$.

[Note: this lemma has nothing to do with independence of variables]

<u>Thm 2.12</u>

If $G = (V, E)$ has e edges then G contains a bipartite subgraph with $\geq \lceil e/2 \rceil$ edges.

<u>Proof</u>

Form a bipartition of $V = A \dot\cup B$ as follows.
For each vertex v toss a fair coin. If heads, put v in A, if tails put v in B.
All coin tosses are independent.
For each edge $uv \in E(G)$ define

$$X_{uv} = \begin{cases} 1 & , u, v \text{ in different classes} \\ 0 & , \text{otherwise} \end{cases}$$

Take H to be a bipartite subgraph with bipartition $A \dot\cup B$ by deleting edges inside either class.

$$|E(H)| = \sum_{uv \in E(G)} X_{uv}$$

So $\mathbb{E}[|E(H)|] = \sum_{uv \in E(G)} \mathbb{E}[X_{uv}]$
linearity of exp.

$\mathbb{E}[X_{uv}] = 1 \times P(X_{uv} = 1) = 1/2$ (by independence and fairness of coin toss).

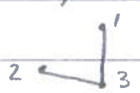
$\therefore$ expected # edges in H is $\frac{e}{2}$.
$\Rightarrow \exists$ a bipartite subgraph with $\geq \frac{e}{2}$ edges $\square$

Erdös-Renyi random graph model

$\Omega = \{ G \mid V(G) = [n], E(G) \subseteq \binom{[n]}{2} \}$ $\mathcal{G}(n, p)$
$n \in \mathbb{N}$, $p \in [0,1]$
p could be a function

For any $H \in \Omega$, $P(H)$ is the probability that the following random process generates H.
Start with E_n, for each $ij \in \binom{[n]}{2}$ flip a coin C_{ij}, $P(C_{ij} = \text{Heads}) = p$, $P(C_{ij} = \text{Tails}) = 1-p$.
If C_{ij} = Heads the insert edge ij, otherwise $ij \notin E$.
All coin tosses are independent.

e.g. $\mathcal{G}(3, 2/3)$



$C_{12} = T$
$C_{13} = H$
$C_{23} = H$

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Thm 2.9 (Erdős)

chromatic number min no. of colours needed to colour G — girth - length of shortest cycle

$\forall k, \ell \geq 3 \ \exists G$ with $\chi(G) \geq k$ and $g(G) \geq \ell$.

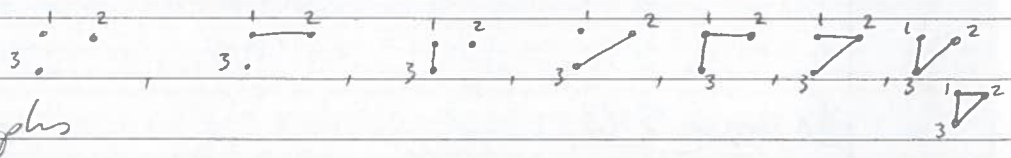
$\left(\chi(G) \geq \frac{|V(G)|}{\alpha(G)} \right)$ — $\alpha(G)$: independence number

$\mathcal{G}(n,p)$ - random graphs on $[n]$ with coin toss C_{ij} determining whether $ij \in E(G)$

$P(C_{ij} = \text{Heads}) = P(ij \text{ an edge}) = p$

Coin tosses independent.

$\mathcal{G}(3,p)$



8 graphs

$|\mathcal{G}(n,p)| = 2^{\binom{n}{2}}$

$P(G \in \mathcal{G}(3,p) \text{ has a triangle}) = p^3$ — 3 edges in

$P(G \in \mathcal{G}(3,p) \text{ is } \llcorner) = 3p^2(1-p)$ — 3 possibilities of 2 edges in, 1 out.

Let X_3 = # triangles in a graph $G \in \mathcal{G}(n,p)$

Suppose $Y_1, \ldots, Y_t$ are indicator variables for the existence of each possible triangle in G.

$X_3 = \sum_{i=1}^{t} Y_i \qquad \left(Y_i = \begin{cases} 1 & \text{triangle } T_i \text{ is in } G \\ 0 & \text{otherwise} \end{cases} \right)$

$\mathbb{E}[X_3] = \sum_{i=1}^{t} \mathbb{E}[Y_i] = \sum_{i=1}^{t} P(Y_i = 1) = tp^3$

A triangle is defined by an ordered triple of vertices v_1, v_2, v_3 : $v_3 \triangle^{v_1} v_2$. Sim. $v_2v_3v_1$, $v_3v_1v_2$, $v_1v_3v_2$, $v_3v_2v_1$, $v_2v_1v_3$.

So # ordered triples $= n(n-1)(n-2)$.

There are 6 ordered triples that give the same triangle. Hence # triangles $t = \binom{n}{3}$.

So $\mathbb{E}[X_3] = \binom{n}{3}p^3$

Lemma 2.13 (Markov's inequality)

If X is a non-negative random variable and $\lambda > 0$ then $P(X \geq \lambda) \leq \frac{\mathbb{E}[X]}{\lambda}$

Proved but not considered as part of the course

Proof

Let X take values in the set Ω_X.

$$\mathbb{E}[X] = \sum_{y \in \Omega_X} y P(X=y) \geq \sum_{y \geq \lambda} y P(X=y) \geq \lambda \sum_{y \geq \lambda} P(X=y) = \lambda P(X=y)$$

$\square$

Lemma 2.14

Let $G \in \mathcal{G}(n,p)$ and let $X_t = \#$ t-cycles in G.

For any possible t-cycle C, let Y_C be its indicator variable, i.e. $Y_C = \begin{cases} 1 & \text{if } C \subseteq G \\ 0 & \text{otherwise} \end{cases}$

Then $\mathbb{E}[X_t] = \sum_{C \text{ possible } t\text{-cycle}} \mathbb{E}[Y_C] = \sum_{C \text{ possible } t\text{-cycle}} P(Y_C = 1) = p^t \, \#\text{possible } t\text{-cycles}$

$= p^t \frac{n(n-1)\cdots(n-t+1)}{2t}$

linearity of $\mathbb{E}$

Proof

Any t-cycle is defined by an ordered t-tuple of distinct vertices $v_1 v_2 \dots v_t$.

There are $2t$ such t-tuples that give the same t-cycle:

$$\begin{array}{lcl} v_1 v_2 \cdots v_t & \text{and} & v_t v_{t-1} \cdots v_1 \\ v_2 v_3 \cdots v_t v_1 & & \vdots \\ \vdots & & \\ v_t v_1 \cdots v_{t-1} & & v_1 v_t v_{t-1} \cdots v_{t-2} \end{array}$$

so

$$\#\text{ possible } t\text{-cycles} = \frac{n(n-1)\cdots(n-t+1)}{2t}$$

$P(Y_C = 1) = p^t$.

$\square$

Suppose $A =$ "G contains more than $\frac{n}{2}$ short cycles"

$B =$ "G has independence number $\alpha(G) \geq n/2k$"

A cycle is short if its length is $\leq \ell$.

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Suppose $P(A) \leq \frac{1}{3}$ and $P(B) \leq \frac{1}{3}$
$\Rightarrow$ $P(\text{neither } A \text{ nor } B \text{ happens}) \geq \frac{1}{3}$

Suppose G is a graph with $\leq n/2$ short cycles and $\alpha(G) \leq n/2k$.

Form a new graph G' from G by removing a single vertex from each short cycle.
So $|V(G')| \geq n/2$, $g(G') > \ell$,

$$\chi(G') \geq \frac{|V(G')|}{\alpha(G')} \geq \frac{n/2}{n/2k} = k$$

$[\alpha(G') \leq \alpha(G)]$

So to prove Thm 2.9 need to prove $P(A) \leq 1/3$ and $P(B) \leq 1/3$.

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<u>Thm</u>
$\forall k, \ell \geq 3$ $\exists G$ with $\chi(G) \geq k$, $g(G) \geq \ell$.

<u>Lemma A</u>

$$\chi(G) \geq \frac{|V(G)|}{\alpha(G)}$$

<u>Lemma B</u>
Let $G \in \mathcal{G}(n,p)$ and $t \geq 3$
If $X_t = \#$ t-cycles in G then

$$\mathbb{E}[X_t] = \frac{n(n-1)\cdots(n-t+1)}{2t} p^t.$$

<u>Proof of Thm</u>
Let k, ℓ be given. Let n be sufficiently large so that $n \geq 36\ell^2$ and $\frac{n^{1/2\ell}}{8\log n} \geq 2k$.

We will call a cycle short if its length is at most l.
Define $\theta = \frac{1}{2l}$ and $p = \frac{1}{n^{1-\theta}}$.

Consider $G \in \mathcal{G}(n,p)$ a random graph.

A = "G contains $> \frac{n}{2}$ short cycles"
B = "$\alpha(G) > n/2k$".

We will show $P(A) \leq 1/3$, $P(B) \leq 1/3 \Rightarrow P(\text{neither } A \text{ nor } B) \geq 1/3$ (1)

Claim: if (1) holds then $\exists\, G'$ s.t. $\chi(G') > k$, $g(G') > l$.
Proof:
If (1) holds then there exists a graph G with $\leq \frac{n}{2}$ short cycles and $\alpha(G) \leq n/2k$.
Delete a vertex from each short cycle to give G'.
$g(G') > l$, $\alpha(G') \leq \frac{n}{2k} \Rightarrow \chi(G') \geq \frac{|V(G')|}{\alpha(G')} \geq \frac{n}{2} \cdot \frac{2k}{n} = k$

Let X = # short cycles in G
X_t = # t-cycles in G
Then $X = \sum_{t=3}^{l} X_t$, $\mathbb{E}[X] \underset{\substack{\uparrow \\ \text{linearity of expectation}}}{=} \sum_{t=3}^{l} \mathbb{E}[X_t]$

$$\Rightarrow \mathbb{E}[X] = \sum_{t=3}^{l} \frac{n(n-1)\dots(n-t+1)}{2t} p^t \leq \sum_{t=3}^{l} \frac{n^t}{2t n^{t-t\theta}} \leq \sum_{t=3}^{l} n^{t/2l} \leq l n^{1/2}$$

$\Rightarrow \mathbb{E}[X] \leq n/6$ since $n \geq 36 l^2$

Markov: $P(X \geq \lambda) \leq \frac{\mathbb{E}[X]}{\lambda}$

$\Rightarrow P(X \geq 3\mathbb{E}[X]) \leq 1/3$ hence $P(X \geq n/2) \leq 1/3$.

Consider $\alpha(G)$, let $s = \frac{4}{p} \log n + 1$
note $\frac{n}{2k} \geq \frac{8 n \log n}{n^{1/2l}} = \frac{8}{p} \log n \geq s$

$\left[P(C \cup D) \leq P(C) + P(D)\right]$

$P(B) = P(\alpha(G) \geq n/2k) \leq P(\alpha(G) \geq s)$

$\alpha(G) \geq s \iff \exists$ a set $T \subseteq V$ $|T| = s$ that is independent.

$P(\alpha(G) \geq s) = P(\exists T \in \binom{V}{s},\ T$ is independent$)$.

$$\leq \sum_{T \in \binom{V}{s}} P(T \text{ is independent}) = \binom{n}{s}(1-p)^{\binom{s}{2}}$$

$$\leq n^s e^{-ps(s-1)/2} = \left(n e^{-\frac{p(s-1)}{2}}\right)^s$$

$$= \left(n e^{-2\log n}\right)^s = n^{-s} \leq 1/3$$

for n large. $\square$

§3 Extremal Graph Theory

A Hamilton cycle in a graph G is a cycle in G containing all vertices in G.

The minimum degree of a graph G is $\delta(G) = \min_{v \in V(G)} d(v)$.

Thm 3.1 (Dirac 1952)

If G has $n \geq 3$ vertices and $\delta(G) \geq n/2$ then G contains a Hamilton cycle.

Thm 3.2 (Ore 1960)

If G has $n \geq 3$ vertices and $\forall u, v \in V(G)$ non-adjacent (ie $uv \notin E(G)$) $d(u) + d(v) \geq n$ then G contains a Hamilton cycle.

Ore's Thm $\Rightarrow$ Dirac

If $\delta(G) \geq n/2$ then $d(u) + d(v) \geq 2\frac{n}{2} = n$ $\forall u, v \in V$

$\therefore$ Ore's Thm $\Rightarrow G$ has a Hamilton cycle.

Proof (of Ore's Thm)

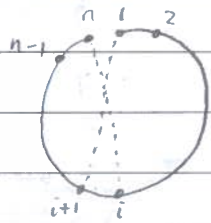
By contradiction.

Let G have $n \geq 3$ vertices s.t. $\forall\, u,v$ non-adjacent we have $d(u)+d(v) \geq n$, but G has no Hamilton cycle.

If there is a pair $uv \notin E(G)$ and adding uv to $E(G)$ does not create a Hamilton cycle, then do it. Repeat until adding any missing edge would create a Hamilton cycle.

Hence G now contains a Hamilton cycle with one edge missing.

wlog assume $V(G) = [n]$.



wlog $i(i+1) \in E(G)$ $i = 1, \dots, n-1$ and $1n \notin E(G)$.

$d(1) + d(n) \geq n$ ← hypothesis.

edge i to n

G has no Hamilton cycle $\Rightarrow$ $i(i+1)$ or in is missing from $E(G)$.

Possible edges involving vertices 1 and n:

1 2	n 1	$=1$ of these 2 edges
1 3	n 2	≤ 1 of these 2 edges
1 4	n 3	≤ 1 " " " "
⋮	⋮	⋮
1 (n-1)	n (n-2)	≤ 1 " " " "
~~1n~~	n (n-1)	$=1$ " " " "

So there are $\leq n-1$ edges.

# of edges present in 1st column is $d(1)$

# of edges present in 2nd column is $d(n)$

$\Rightarrow d(1) + d(n) \leq n-1$.

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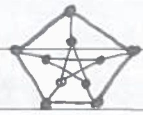
Peterson graph: 

§3.2 - Forbidden subgraphs

"extremal number of H"

Given a graph H, we define $\underset{(n\geq 1)}{ex(n,H)} = \max\{|E(G)| : \substack{|V(G)|=n \\ G \text{ is } H\text{-free}}\}$

i.e. $ex(n,H) = \max\{|E(G)| : G$ has n vertices and is H-free$\}$.

$ex(n,H) \leq \binom{n}{2}$

Lemma 3.3

If G and H are graphs and $\chi(H) > \chi(G)$ then G is H-free.

Proof

If H is a subgraph of G then any colouring of G gives a colouring of H $\Rightarrow \chi(H) \leq \chi(G)$.

$\square$

Thm 3.4 (Mantel 1903)

If $n \geq 1$ then $ex(n, K_3) = \lfloor \frac{n^2}{4} \rfloor$.

Proof

Example $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$  this is K_3-free (by lemma 3.3)

$\lfloor \frac{n}{2} \rfloor$ $\lceil \frac{n}{2} \rceil$

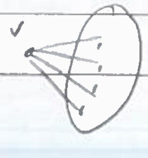
$\Rightarrow ex(n, K_3) \geq \lfloor \frac{n^2}{4} \rfloor$.

Now to show $ex(n, K_3) \leq \lfloor \frac{n^2}{4} \rfloor$

Let G be K_3-free with n vertices. Need $|E(G)| \leq \lfloor \frac{n^2}{4} \rfloor$.

Let $A \subseteq V(G)$ be a largest independent set

Consider A $V \setminus A$ with $|A| = a$.

 $\Gamma(v)$, $K_3 \overset{\text{free}}{\Rightarrow} \Gamma(v)$ is independent.

$d(v) = |\Gamma(v)| \leq a$ for $v \in V \setminus A$?)

Since A is independent, every edge in G meets $V \setminus A$.

Hence $\sum_{v \in V \setminus A} d(v) \geq |E(G)|$

So $|E(G)| \leq \sum_{v \in V \setminus A} d(v) \leq (n-a)a \leq \frac{n^2}{4}$ $\square$

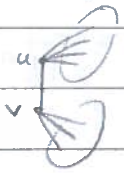
Cauchy-Schwarz inequality: for $a_i, b_i \geq 0$,

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \sum a_i^2 \sum b_i^2 .$$

<u>2nd Proof</u> (of upper bound)

Let G be K_3-free with n vertices.

Let $uv \in E(G)$



$\Rightarrow d(u) + d(v) \leq n$

$$\sum_{x \in V} (d(x))^2 = \sum_{uv \in E(G)} (d(u) + d(v)) \leq en \qquad (e = |E(G)|)$$

the degree of x occurs $d(x)$-times since x is part of $d(x)$-edges.

$a_x = \frac{1}{n}$ $x \in V$, $b_x = d(x)$, $x \in V$.

$$\sum_{x \in V} a_x = \frac{n}{n} = 1, \quad \sum_{x \in V} b_x = \sum_{x \in V} d(x) = 2e$$

So by Cauchy-Schwarz,

$$\left(\sum_{x \in V} \frac{d(x)}{n}\right)^2 \leq \frac{n}{n^2} \sum_{x \in V} (d(x))^2$$

$$\Rightarrow 4e^2 \leq n^2 e$$

$$\Rightarrow e \leq \frac{n^2}{4}$$ $\square$

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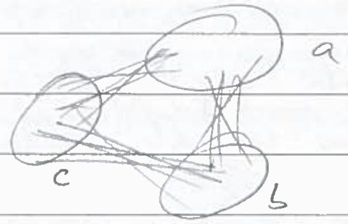
$H = K_4$, $ex(n, K_4) = ?$

$ab + ac + bc$

$a + b + c = n$



Take a, b, c as equal as possible
i.e. take $a = \lfloor \frac{n}{3} \rfloor$, $b = \lfloor \frac{n+1}{3} \rfloor$, $c = \lfloor \frac{n+2}{3} \rfloor$

[This gives $\underbrace{k, k, k}_{n=3k}$ or $\underbrace{k, k, k+1}_{n=3k+1}$ or $\underbrace{k, k+1, k+1}_{n=3k+2}$]

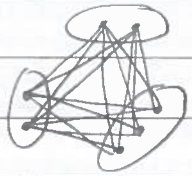
A graph G is k-partite iff $V(G) = V_1 \dot\cup V_2 \dot\cup \dots \dot\cup V_k$ and each V_i is an independent set.

It is complete k-partite iff all possible edges between different classes.

Let $n \geq r \geq 2$ be integers. Define the Turán graph $T_r(n)$ to be the complete r-partite graph order n (i.e. $|V(T_r(n))| = n$) with vertex classes as equal as possible in size.

Let $t_r(n)$ denote the number of edges in $T_r(n)$.

$T_3(7)$



so $t_3(7) = 2 \times 2 + 2 \times 3 + 2 \times 3 = 16$

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$ex(n, H) = \max\{|E(G)| : G \text{ is a } H\text{-free graph of order } n\}$

$T_r(n)$ is the complete r-partite graph of order n with vertex classes as equal as possible in size (Turán graph). $a\lfloor \frac{n}{r} \rfloor + (r-a)\lceil \frac{n}{r} \rceil = n$.

$|E(T_r(n))| = t_r(n)$.

E.g. $T_4(10)$ $\quad 2+2+3+3$

$t_4(10) = 4 + 9 + 4 \times 6$

Lemma 3.5

(a) Among all r-partite graphs of order n, $T_r(n)$ has the most edges, and

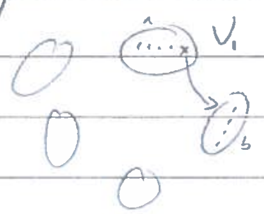
(b) $t_r(n) = t_r(n-r) + (r-1)(n-r) + \binom{r}{2}$.

Proof

(a) Let G be r-partite, order n, and have the most edges of any such graph.

Either G is $T_r(n)$ or there exist vertex classes V_1 and V_2 of G s.t. $|V_1| = a$, $|V_2| = b$ and $b \le a-2$.

Since G maximises the number of edges we can suppose G is complete r-partite.



Move a vertex from V_1 to V_2, while keeping the graph complete r-partite.

We lose $n-a$ edges and gain $n-(b+1)$ edges

$\Rightarrow$ gain ≥ 1 edge.

$\divideontimes$

Hence G was $T_r(n)$.

(b) $t_r(n) = t_r(n-r) + (r-1)(n-r) + \binom{r}{2}$ $\quad n \ge r$ $\quad \leftarrow$ WTS.

Choose a single vertex in each vertex class of $T_r(n)$ and colour this blue. Colour all other vertices red.

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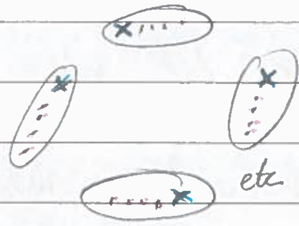
Edges in $T_r(n)$ are of 3-types: red-red, red-blue, blue-blue.

Red-red edges give a copy of $T_r(n-r)$, so have $t_r(n-r)$ edges red-red edges.



etc.

Blue-blue form K_r $\therefore$ $\binom{r}{2}$ such edges.

Each red vertex is joined to all blue vertices except the one in its class.

Since we have $n-r$ red edges, we must have $(n-r)(n-1)$ red-blue edges.

Hence $t_r(n) = t_r(n-r) + (n-r)(r-1) + \binom{r}{2}$.

$\square$

<u>Thm</u> 3.6 (Turán 1941).

If $2 \le r \le n$ and G is a K_{r+1}-free graph with n vertices and $ex(n, K_{r+1})$ edges, then G is $T_r(n)$.

<u>Proof</u>

(Induction on n).

True for $n=r$, $T_r(r) = K_r$.

Suppose $n \ge r+1$. Let G be K_{r+1}-free, n vertices and $ex(n, K_{r+1})$ edges. Then G must contain a copy K of K_r with vertices $\{v_1, v_2, \dots, v_r\}$.

Consider $G-K$. This is a K_{r+1}-free graph of order $n-r$ so by our hypothesis $|E(G-K)| \le t_r(n-r)$.

If $v \in V(G) \setminus \{v_1, \dots, v_r\}$ then v has $\le r-1$ neighbours in K (because G is K_{r+1}-free).

Finally K contains $\binom{r}{2}$ edges.

edges between K and $G-K$.

$$|E(G)| = |E(G-K)| + |E(K, G-K)| + |E(K)|$$
$$\le t_r(n-r) + \binom{r}{2} + (n-r)(r-1) = t_r(n).$$

But $|E(G)| = ex(n, K_{r+1}) \ge t_r(n)$ (since $T_r(n)$ is K_{r+1}-free)

Hence $|E(G)| = t_r(n)$ and all earlier inequalities are equalities.

Any complete graph is called a clique.

Hence every $v \in V(G) \setminus \{v_1, \dots, v_r\}$ has exactly $r-1$ neighbours in K.
For $1 \leq i \leq r$ let $W_i = \{v \in V(G) : vv_i \notin E(G)\}$

Claim: $W_1 \dot\cup W_2 \dot\cup \dots \dot\cup W_r$ is a partition of $V(G)$.
This follows, since if $v \in V(G) \setminus \{v_1, \dots, v_r\}$ then v has exactly $r-1$ neighbours in K.
$\therefore$ There is a unique $1 \leq i \leq r$ such that $v \in W_i$,
moreover $v_i \in W_i \ \forall\ 1 \leq i \leq r$.

Claim: W_i is an independent set.
If $v, w \in W_i$ and $vw \in E(G)$ then consider
$\{v, w, v_1, v_2, \dots, v_{i-1}, v_{i+1}, \dots, v_r\}$ this is a copy of K_{r+1}.
$\#$ $\Rightarrow$ claim holds.

Hence G is an r-partite graph of order n
but then Lemma ⓐ $\Rightarrow$ G is $T_r(n)$.
$\square$

Thm (Caro-Wei 1979)

independence number

If G is a graph of order n then $\alpha(G) \geq \sum_{v \in V(G)} \frac{1}{d(v)+1}$

[Recall $\sum_{v \in V} d(v) = 2e$]

Proof
Let $V(G) = [n]$ and let $\pi \in S_n$ be chosen uniformly at random.
For $i \in [n]$ let A_i = "$\pi(j) > \pi(i)\ \forall j \in \Gamma(i)$".
(i.e. vertex i comes before all of it neighbours after the permutation eg. $\cdot\ \cdot\ \underset{i}{\times}\ \cdot\ \times\ \times\ \cdot\ \times\ \cdot$).
Let $U = \{i \in [n] : A_i \text{ holds}\}$

Claim: U is an independent set.
If $u, v \in U$ and $uv \in E$ then we have a contradiction

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since $A_u \Rightarrow \pi(u) < \pi(v)$ but $A_v \Rightarrow \pi_v < \pi_u$.

$$\alpha(G) \geq \mathbb{E}[|U|] = \sum_{i \in [n]} P(A_i \text{ holds})$$

$|U| = \sum_{i \in [n]} X_i$ where $X_i = 1$, A_i holds; 0, otherwise

For A_i consider $d(i)+1$ vertices, since A_i is the event that one particular vertex (namely i) comes first in a set of $d(i)+1$ vertices under the permutation π. Since each vertex is equally likely to come first we have $P(A_i) = \frac{1}{d(i)+1}$
$\Rightarrow$ result.

$\square$

$ex(n, K_{r+1}) = t_r(n)$
For $H \neq K_r$ (any r) $ex(n, H) = ?$

$\frac{|E(G)|}{\binom{n}{2}}$ edge density?

If H is a graph then the Turán density of H is

$$\pi(H) = \lim_{n \to \infty} \left(\frac{ex(n,H)}{\binom{n}{2}} \right).$$

Lemma 3.8
(a) If H is a graph, then $\pi(H)$ is well defined.
(b) $\pi(K_{r+1}) = 1 - \frac{1}{r}$

Proof
(a) Let G have order n and $ex(n, H)$ edges s.t. G is H-free.

$$\underbrace{\sum_{v \in V(G)} |E(G-v)|}_{= (n-2)|E(G)| \; = (n-2)\, ex(n,H)} \leq n\, ex(n-1, H)$$

So $(n-2) ex(n, H) \leq n\, ex(n-1, H)$

$$\Rightarrow \quad \frac{ex(n,H)}{n(n-1)/2} \leq \frac{ex(n-1,H)}{n(n-1)/2}$$

$\Rightarrow$ the sequence $\left\{ \frac{ex(n,H)}{\binom{n}{2}} \right\}_{n=1}^{\infty}$ is decreasing

It is bounded below by zero $\therefore$ converges.

(b) $ex(n, K_{r+1}) = t_r(n)$

Each class in $T_r(n)$ has $< \frac{n+r}{r}$ vertices and $> \frac{n-r}{r}$ vertices.

So
$$\underbrace{\frac{\binom{r}{2}\left(\frac{n-r}{2}\right)^2}{\binom{n}{2}}}_{\to 1-\frac{1}{r}} < \frac{t_r(n)}{\binom{n}{2}} < \underbrace{\frac{\binom{r}{2}\left(\frac{n+r}{r}\right)^2}{\binom{n}{2}}}_{\to 1-\frac{1}{r}}$$

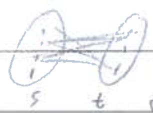
[n varies, r fixed]

Hence $\pi(K_{r+1}) = 1 - \frac{1}{r}$.

$\square$

$\pi(K_2) = 0$, $\pi(K_3) = \frac{1}{2}$, $\pi(K_4) = \frac{2}{3}$

We will show $\pi(H) = 1 - \frac{1}{\chi(H) - 1}$ (Erdős – Stone).

$ex(n, K_{s,t}) = O(n^{2-1/s})$  $K_{s,t}$ $\qquad \pi(K_{s,t}) = 0$

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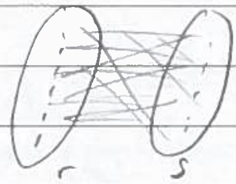
last time
$$ex(n, K_{r+1}) = t_r(n)$$
$$\pi(H) = \lim_{n\to\infty} \frac{ex(n,H)}{\binom{n}{2}}, \quad \pi(K_{r+1}) = 1 - \frac{1}{r}.$$

$ex(n, K_{r,s})$ $\qquad K_{r,s} =$  complete bipartite

Convexity

Let $a_1, \dots, a_n \geq 0$

$$\sum_{i=1}^n a_i = A \qquad \sum_{i=1}^n a_i^2 \geq n\left(\frac{A}{n}\right)^2 \quad \text{(i.e. all } a_i \text{ equal)}$$

Lemma (Convexity of Binomial Coefficients)

If $a_1, \dots, a_n \geq 0$, $k \geq 1$ and $\sum_{i=1}^n a_i = A$

then $\sum_{i=1}^n \binom{a_i}{k} \geq n \binom{A/n}{k}$.

[Jensen's Inequality]

Thm 3.9 (Kővári - Sós - Turán 1959)

If $r, s \geq 2$, n large, then

$$ex(n, K_{r,s}) \leq \frac{1}{2}(r-1)^{1/s} n^{2-1/s} + \frac{1}{2}(s-1)n.$$

In particular $ex(n, K_{r,s}) = O(n^{2-1/s})$.

[$g(n) = O(f(n))$ means $\exists M$ s.t. $|g(n)| \leq M|f(n)|$ $\forall n \geq n_0$]

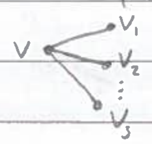
Proof

Let G be $K_{r,s}$-free of order n, with

$|E(G)| = ex(n, K_{r,s}) = e$ (here $e \neq 2.7\dots$, just a const!)

If $v \in V(G)$ and $S = \{v_1, \dots, v_s\} \in \binom{V(G)}{s}$ we say v covers S

iff $vv_i \in E(G)$, $1 \leq i \leq s$.

i.e.  $(K_{1,s})$

How many sets $S \in \binom{V(G)}{s}$ does v cover?

Exactly $\binom{d(v)}{s}$ sets S are covered by v.

Fix set $S \in \binom{V(G)}{s}$. How many vertices in $V(G)$ can cover S? At most $r-1$, since G is $K_{r,s}$-free.

$$\sum_{v \in V(G)} \binom{d(v)}{s}$$ counts pairs (v, S) where v covers the set S.

So $$\sum_{v \in V(G)} \binom{d(v)}{s} \leq (r-1)\binom{n}{s}$$

each s-set covered by $\leq r-1$ vertices. # possible s-sets

By Convexity of Binomial coefficients,
$$n\binom{\sum_{v \in V(G)} \frac{d(v)}{n}}{s} \leq \sum_{v \in V(G)} \binom{d(v)}{s} \leq (r-1)\binom{n}{s}.$$

$$\sum_{v \in V(G)} d(v) = 2e, \quad \text{let } e = n^{2-\alpha}.$$

$$\left[\binom{a}{b} = \frac{a(a-1)\cdots(a-b+1)}{b!}\right]$$

So $$n\binom{2n^{1-\alpha}}{s} \leq (r-1)\binom{n}{s}$$

In particular, $$n(2n^{1-\alpha} - s + 1)^s \leq (r-1)n^s$$

underestimation overestimation.

So $$(2n^{1-\alpha} - s + 1) \leq (r-1)^{1/s} n^{1-1/s}$$
$$\Rightarrow \quad 2n^{1-\alpha} \leq (r-1)^{1/s} n^{1-1/s} + s - 1$$
So $$e = n^{2-\alpha} \leq \frac{(r-1)^{1/s}}{2} n^{2-1/s} + \frac{(s-1)n}{2}.$$ $\square$

<u>Corollary</u> (Erdös 1946)

If $X \subseteq \mathbb{R}^2$, $|X| = n$ (i.e. n points in plane) then at most $\left(\frac{n^{3/2}}{\sqrt{2}} + \frac{n}{2}\right)$ pairs of points from X are at unit distance.

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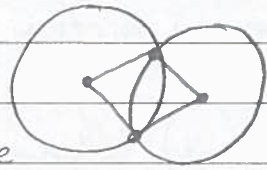
Proof

Construct a graph with $V(G) = X$ and $xy \in E(G)$ iff $|x-y| = 1$.



Since any two circles meet in ≤ 2 points, we have no copy of $K_{3,2}$ in G.

Hence $|E(G)| \leq \underset{\substack{r=3,\ s=2 \\ \text{in bound from thm.}}}{ex(n, K_{3,2})} \leq \dfrac{\sqrt{2}\, n^{3/2}}{2} + \dfrac{n}{2}$ $\square$

Thm 3.11 (Erdős - Stone 1946)

If $\chi(H) = r \geq 2$ then $\pi(H) = 1 - \frac{1}{r-1} \left(= \pi(K_r)\right)$.

Proof

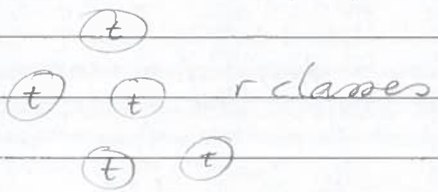
If $\chi(H) = r$, then $T_{r-1}(n)$ is H-free.

$\Rightarrow \pi(H) \geq \pi(K_r) = 1 - \frac{1}{r-1}$.

08-02-19 Need to prove $\pi(H) \leq 1 - \frac{1}{r-1}$.

Define $K_r(t)$ to be the complete r-partite graph with t vertices in each class.

i.e. $K_r(t) = T_r(rt)$



If $t \geq |V(H)|$ then there is a copy of H in $K_r(t)$, so $\pi(K_r(t)) \geq \pi(H)$

$\Rightarrow$ it is sufficient to prove $\pi(K_r(t)) \leq 1 - \frac{1}{r-1}$.

Lemma 3.12

Let $0 < c, \varepsilon < 1$ and $n > \overbrace{\frac{2}{\varepsilon}\left(1 + \frac{1}{c}\right)}^{①}$

If G is a graph order n with $\geq (c+\varepsilon)\binom{n}{2}$ edges then $\exists G' \subset G$ a subgraph of order $n' \geq \sqrt{\varepsilon}\, n$ and

min degree $\delta(G') \geq cn'$.

[Note if $n = |V(G)|$, $\delta(G) = cn$ then $e(G) \geq \frac{cn^2}{2}$.]

Proof (of 3.12)

Let $G_n = G$. If $\delta(G) \geq cn$, set $G' = G$.
Otherwise remove a vertex of degree $< cn$, call this graph G_{n-1}.
If $\delta(G_{n-1}) \geq c(n-1)$ then set $G' = G_{n-1}$, otherwise repeat.
Continue in this way to give
$G_n, G_{n-1}, \dots, G_k$ where G_k has k vertices and G_{k-1} is obtained from G_k by removing a vertex of minimum degree.

We claim this process stops at some $k \geq \sqrt{\varepsilon}\, n$.
Let $s = \lceil \sqrt{\varepsilon}\, n \rceil$.

Suppose we reach G_s.
Then $|E(G_s)| > |E(G)| - \sum_{k=s+1}^{n} ck \geq (c+\varepsilon)\binom{n}{2} - c\left(\binom{n+1}{2} - \binom{s+1}{2}\right)$

$$\left[\sum_{i=1}^{j} i = \frac{j(j+1)}{2} = \binom{j+1}{2} \right]$$

$$s = \lceil \sqrt{\varepsilon}\, n \rceil \quad \text{so} \quad \binom{s+1}{2} > \frac{s^2}{2} \geq \frac{\varepsilon n^2}{2} > n\left(1 + \frac{1}{c}\right) \text{ by } ①.$$

$$\text{So } |E(G_s)| > \varepsilon\binom{n}{2} - cn + c\binom{s+1}{2} \geq \varepsilon\binom{n}{2} - cn + (c+1)n$$

$$\text{ie. } |E(G_s)| > \varepsilon\binom{n}{2} + n$$

$$|E(G_s)| \leq \binom{s}{2}$$

$$\text{and } \binom{s}{2} \leq \binom{\sqrt{\varepsilon}\, n + 1}{2} = \frac{\sqrt{\varepsilon}\, n(\sqrt{\varepsilon}\, n + 1)}{2} = \frac{\varepsilon(n^2+n)}{2} < \varepsilon\binom{n}{2} + n$$

$$\Rightarrow \varepsilon\binom{n}{2} + n < |E(G_s)| < \varepsilon\binom{n}{2} + n \quad \text{※} \quad \square$$

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Thm 3.13

If $r \geq 2$, $t \geq 1$ and $0 < \varepsilon < 1/r$, then $\exists\, n_0(r,t,\varepsilon)$ an integer such that if G has $n \geq n_0(r,t\,\varepsilon)$ vertices and $\delta(G) \geq (1 - \frac{1}{r-1} + \varepsilon)n$, then G contains $K_r(t)$.

Proof (of Erdös-Stone) cont.

Need to show $\pi(K_r(t)) \leq 1 - \frac{1}{r-1}$.

If this fails, then $\exists\, \varepsilon > 0$ s.t. $\pi(K_r(t)) > 1 - \frac{1}{r-1} + 3\varepsilon$.

Let $n \geq \frac{n_0(r,t,\varepsilon)}{\sqrt{\varepsilon}}$ and let G have n vertices,

be $K_r(t)$ free, and at least $(1 - \frac{1}{r-1} + 2\varepsilon)\binom{n}{2}$ edges.
By Lemma 3.12 $\exists$ a subgraph G' of G such that G' has $n' \geq n_0(r,t,\varepsilon)$ vertices and $\delta(G') \geq (1 - \frac{1}{r-1} + \varepsilon)n'$.
But Thm 3.13 $\Rightarrow K_r(t) \subseteq G$.
$\divideontimes$
Hence the result follows. $\square$

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Proof (of 3.13)
Induction on r.
If $r = 2$, $K_2(t) = K_{t,t}$
Thm 3.9 $\Rightarrow ex(n, K_{t,t}) < \frac{1}{2}(t-1)^{1/t} n^{2-1/t} + \frac{1}{2}(t-1)n < tn^{2-1/t}$ $(*)$

Given $\varepsilon > 0$, $t \geq 1$ define $n_0(r,t,\varepsilon) = \left(\frac{2t}{\varepsilon}\right)^t$.

So if $n \geq n_0$, $n \geq \left(\frac{2t}{\varepsilon}\right)^t \Rightarrow \varepsilon \geq \frac{2t}{n^{1/t}}$

So if G has order $n \geq n_0$ and $\delta(G) \geq \varepsilon n$
then $|E(G)| = \frac{1}{2}\sum_{v \in V(G)} d(v) \geq \frac{1}{2}\varepsilon n^2 \geq tn^{2-1/t} \underset{\text{by}(*)}{>} ex(n, K_{t,t})$

So true for $r = 2$.

Now let $r \geq 3$ and suppose the result holds for $r-1$.
Let $t \geq 1$, $0 < \varepsilon < \frac{1}{r}$.

Let G have order n and $\delta(G) \geq (1 - \frac{1}{r-1} + \varepsilon)n$.

NTS: n sufficiently large $\Rightarrow$ G contains $K_r(t)$.

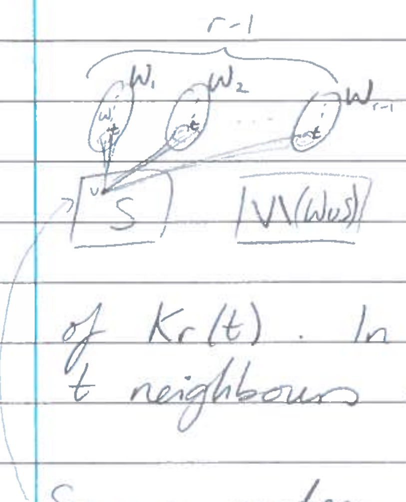
Let $w = \left\lceil \frac{2t}{\varepsilon} \right\rceil$, let $n \geq n_0(r-1, w, \varepsilon)$

Then $\delta(G) \geq (1 - \frac{1}{r-1} + \varepsilon)n > (1 - \frac{1}{r-1} + \varepsilon)n$
$\therefore$ we can use the $r-1$ case of the theorem.
So our inductive hypothesis $\Rightarrow$ G contains $K_{r-1}(w)$.

Let $W = W_1 \dot\cup W_2 \dot\cup \cdots \dot\cup W_{r-1}$ be the vertex classes of $K_{r-1}(w)$

Let $S = \{v \in V \setminus W : v \text{ has } \geq (r-2)w + t \text{ neighbours in } W\}$

$(r-2)w + t = |W| - (w-t)$



Note that S contains vertices which could potentially form an r^{th} vertex class to build a copy of $K_r(t)$. In particular, every $v \in S$ has at least t neighbours in each class W_i, $1 \leq i \leq r-1$.

Say a vertex $v \in S$ covers a copy H of $K_{r-1}(t)$ in W if v is adjacent to every vertex in H.

If G is $K_r(t)$-free then no copy of $K_{r-1}(t)$ in W is covered by more than $t-1$ vertices.

Also every vertex in S covers at least one copy of $K_{r-1}(t)$ in W.

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How many copies of $K_{r-1}(t)$ are there in W? $\binom{w}{t}^{r-1}$

So if G is $K_r(t)$-free then $|S| \leq \underbrace{(t-1)\binom{w}{t}^{r-1}}_{\text{const.}}$

<u>Claim:</u> For n sufficiently large, $|S| > (t-1)\binom{w}{t}^{r-1}$.

Hence $\exists\, n_0(r,t,\varepsilon)$ s.t. $n \geq n_0(r,t,\varepsilon)$
G contains a copy of $K_r(t)$.

We want to estimate the number of edges between W and $V \setminus W$, denoted by $e(W, V\setminus W)$.

$$\delta(G) \geq \left(1 - \frac{1}{r-1} + \varepsilon\right)n$$

$$\Rightarrow e(W, V\setminus W) = \left(\sum_{v \in W} d(v)\right) - 2\underbrace{e(W)}_{\#\text{edges inside } W}$$
$$\geq |W|\, n\left(1 - \frac{1}{r-1} + \varepsilon\right) - |W|^2$$

If $v \in V \setminus (W \cup S)$ then v has $< (r-2)w + t$ neighbours in W.

$$e(W, V\setminus W) < ((r-2)w + t)(n - |W| - |S|) + |W||S|$$
$$< n((r-2)w+t) + |S|(w-t) - |W|^2 + |W|(w-t)$$
$$[|W| = (r-1)w]$$

So $|W|n\left(1 - \frac{1}{r-1} + \varepsilon\right) - |W|^2 < n((r-2)w+t) + |S|(w-t) + |W|(w-t) - |W|^2$

So $|W|n\left(1 - \frac{1}{r-1} + \varepsilon\right) < n(|W| - w + t) + |S|(w-t) + |W|(w-t)$

So $n[(r-1)w\varepsilon - t] < |S|(w-t) + (r-1)w(w-t)$

$$\Rightarrow n\left(\frac{\varepsilon(r-1)w - t}{w-t}\right) - w(r-1) < |S|$$

$$\omega \geq \frac{2t}{\varepsilon} \Rightarrow |S| > tn\left(\frac{2(r-1)-1}{\omega - t}\right) - \omega(r-1)$$

$$\Rightarrow |S| > \frac{3tn}{\omega - t} - \omega(r-1) \to \infty \quad \text{as } n \to \infty$$

Hence the claim follows and the theorem is proved.

$\square$

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Stability

Thm 3.14 (Andrásfai - Erdős - Sós)

If G is K_3-free with n-vertices and $\delta(G) > \frac{2n}{5}$ then G is bipartite.

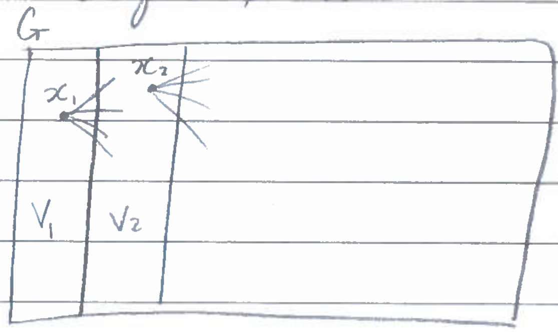
[no proof given]

Thm 3.15 (Füredi 2010)

If G is K_{r+1}-free with order n and at least $ex(n, K_{r+1}) - t$ edges, for some $t \geq 0$, then $\exists H \subseteq G$ such that H is r-partite and $|E(H)| \geq |E(G)| - t$

Proof

Let G be K_{r+1}-free, order n, $|E(G)| \geq ex(n, K_{r+1}) - t$.
Choose $x_1 \in V(G)$ of maximum degree, d_1.
Let $V_1 = V \setminus \Gamma(x)$.



Next choose $x_2 \in V \setminus V_1$ to have maximum degree in in $G[V \setminus V_1]$, set $V_2 = V \setminus (V_1 \cup \Gamma(x_2))$.
Let d_2 be the degree in $G[V \setminus V_1]$ of x_2.

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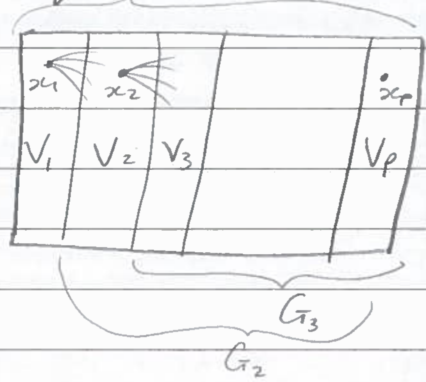
If $x_1, \ldots, x_j$ have been chosen, let x_{j+1} have maximum degree in $G[V \setminus (V_1 \cup \ldots \cup V_j)]$ and define $V_{j+1} = V \setminus (V_1 \cup \ldots \cup V_j \cup \Gamma(x_{j+1}))$.
Let d_{j+1} be the degree of x_{j+1}

If we construct $x_1, \ldots, x_p$ in this way then since they form a complete subgraph of G, we have $p \leq r$.

Let $G_i = G[V_i \cup V_{i+1} \cup \ldots \cup V_p]$.



Let $i(G) =$ # edges of G that are in a single vertex class.

Then
$$|E(G)| + i(G) = \sum_{i=1}^{p} \sum_{y \in V_i} d_{G_i}(y)$$
$$\leq \sum_{i=1}^{p} |V_i| \left(|V_{i+1}| + |V_{i+2}| + \ldots + |V_p| \right)$$
$$= E(\underbrace{K(V_1, V_2, \ldots, V_p)}_{\substack{\text{complete } p\text{-partite} \\ \text{graph with vertex classes} \\ V_1, \ldots, V_p}}) \leq t_p(n) \leq \underset{ex(n, K_{r+1})}{\underset{\shortparallel}{t_r(n)}}$$

$|E(G)| \geq ex(n, K_{r+1}) - t$
$\Rightarrow ex(n, K_{r+1}) - t + i(G) \leq ex(n, K_{r+1})$
$\Rightarrow i(G) \leq t$.

Hence by removing all internal edges from G we obtain a p-partite graph H with $\geq |E(G)| - t$ edges. $\square$

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Chapter 4 - Set Systems

Let $n \in \mathbb{N}$, $[n] = \{1, 2, \dots, n\}$

$\mathcal{P}([n]) = \{A \mid A \subseteq [n]\}$ ← power set ← non-uniform

If $0 \le k \le n$, $\binom{[n]}{k} = \{A \mid A \subseteq [n], |A| = k\}$

the "k-sets"

uniform

§4.1 - Chains and Antichains

A family of sets $\mathcal{A} \subseteq \mathcal{P}([n])$ is an antichain iff $\forall A, B \in \mathcal{A}$, $A \subseteq B \Rightarrow A = B$.

[Equivalently, $\mathcal{A}$ is an antichain iff $A, B \in \mathcal{A}$ and $A \neq B \Rightarrow A \not\subseteq B$ and $B \not\subseteq A$]

e.g. $\mathcal{A} = \{\emptyset\}$, $\mathcal{B} = \{1, 2, 3, \dots, n\}$ ← singleton sets

" $\binom{[n]}{0}$ " $\binom{[n]}{1}$

In general, $\binom{[n]}{k}$ is an antichain.

A family of sets $\mathcal{C} \subseteq \mathcal{P}([n])$ is a chain iff $\forall A, B \in \mathcal{C}$, $A \subseteq B$ or $B \subseteq A$.

e.g. $\mathcal{C} = \{\emptyset, 1, 12, \underbrace{123}_{\text{means } \{1\}, \{2\}, \{3\}}, \dots, [n]\}$ $\quad |\mathcal{C}| = n+1$.

$\{\emptyset, 1, 2\}$ is not an antichain or chain

Lemma 4.1

If $\mathcal{A}, \mathcal{C} \subseteq \mathcal{P}([n])$ and $\mathcal{A}$ is an antichain, $\mathcal{C}$ a chain, then $|\mathcal{A} \cap \mathcal{C}| \leq 1$.

Proof

(by contradiction)

If $|\mathcal{A} \cap \mathcal{C}| \geq 2$, let $A, B \in \mathcal{A} \cap \mathcal{C}$ s.t. $A \neq B$.

We know $A, B \in \mathcal{C} \Rightarrow A \subseteq B$ or $B \subseteq A$

but $A, B \in \mathcal{A} \Rightarrow A = B$. ※ $\square$

Lemma 4.2

If $\mathcal{C} \subseteq \mathcal{P}([n])$ is a chain then $|\mathcal{C}| \leq n+1$.

Proof

Partition $\mathcal{P}([n]) = \binom{[n]}{0} \dot\cup \binom{[n]}{1} \dot\cup \cdots \dot\cup \binom{[n]}{n}$

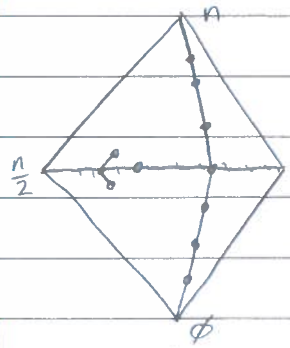
$$|\mathcal{C}| = |\mathcal{C} \cap \mathcal{P}([n])| = \sum_{k=0}^{n} \underbrace{\left|\mathcal{C} \cap \binom{[n]}{k}\right|}_{\leq 1 \text{ since } \mathcal{C} \text{ is a chain and } k\text{-sets are antichains}} \leq n+1.$$

$\square$

Theorem 4.3 (Sperner 1928)

If $\mathcal{A} \subseteq \mathcal{P}([n])$ is an antichain, then $|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$.

A chain $\mathcal{C} = \{C_1, \ldots, C_k\} \subseteq \mathcal{P}([n])$ is symmetric iff $|C_{i+1}| = |C_i| + 1$ $(1 \leq i \leq k-1)$, $C_1 \subset C_2 \subset \cdots \subset C_k$ and $|C_1| + |C_k| = n$.



If we partition $\mathcal{P}([n])$ into symmetric chains then we will have exactly $\binom{n}{\lfloor n/2 \rfloor}$ such chains.

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Lemma 4.4

$\forall n \in \mathbb{N}$, $\mathcal{P}([n])$ can be partitioned into symmetric chains.

Proof (of Thm 4.3 assuming Lemma 4.4)

Every symmetric chain in our partition meets $\binom{[n]}{\lfloor n/2 \rfloor}$ exactly once and so we have $\binom{n}{\lfloor n/2 \rfloor}$ chains.

Given an antichain $\mathcal{A} \subseteq \mathcal{P}([n])$, we know $\mathcal{A}$ meets any chain at most once, so $|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$. $\square$

Proof (of Lem 4.4)

(Induction on n)

$n=1$

$\mathcal{P}([n]) = \{\emptyset, 1\}$ is a symmetric chain.

Suppose $n \geq 2$ and the result holds for $n-1$.

Let $\mathcal{P}([n-1]) = \mathcal{C}_1 \,\dot\cup \dots \dot\cup\, \mathcal{C}_t$ be a partition into symmetric chains.

$$\mathcal{P}([n]) = \mathcal{P}([n-1]) \,\dot\cup\, \{A \cup \{n\} \mid A \in \mathcal{P}([n-1])\}.$$

Let $\mathcal{C}_i = \{C_1^i, C_2^i, \dots, C_{k_i}^i\}$.

Let $\mathcal{C}_i' = \{C_1^i \cup \{n\}, C_2^i \cup \{n\}, \dots, C_{k_i-1}^i \cup \{n\}\}$

This is a symmetric chain in $\mathcal{P}([n])$

since $|C_{j+1}^i \cup \{n\}| = |C_j^i \cup \{n\}| + 1$

and $|C_1^i \cup \{n\}| + |C_{k_i-1}^i \cup \{n\}| = |C_1^i| + |C_{k_i-1}^i| + 2$

$= n-1-1+2$

Let $\mathcal{C}_i'' = \{C_1^i, C_2^i, \dots, C_{k_i}^i, C_{k_i}^i \cup \{n\}\}$

This is a symmetric chain in $\mathcal{P}([n])$

since $|C_{j+1}^i| = |C_j^i| + 1$ for $1 \leq j \leq k_i - 1$,

$|C_{k_i}^i \cup \{n\}| = |C_{k_i}^i| + 1$ and $|C_1^i| + |C_{k_i}^i \cup \{n\}| = |C_1^i| + |C_{k_i}^i| + 1$

$= n-1+1 = n$

Result follows by induction. $\square$

[We are working with things very similar to posets]

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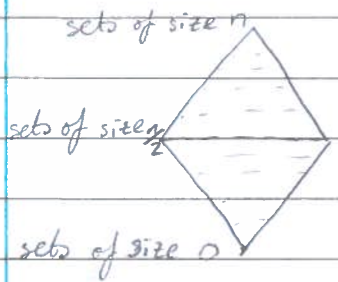
Thm 4.5 (LYM 1954)

If $\mathcal{A}$ is an antichain then $\displaystyle\sum_{A\in\mathcal{A}} \frac{1}{\binom{n}{|A|}} \le 1$.

$\mathcal{A} \subseteq \mathcal{P}([n])$, $0 \le k \le n$ $\quad \mathcal{A}_k = \mathcal{A} \cap \binom{[n]}{k}$

Let $a_k = |\mathcal{A}_k| = \left|\mathcal{A} \cap \binom{[n]}{k}\right|$

$$\sum_{A\in\mathcal{A}} \frac{1}{\binom{n}{|A|}} = \sum_{k=0}^{n} \underbrace{\frac{a_k}{\binom{n}{k}}}_{\text{the proportion of } k\text{-sets in } \mathcal{A}}$$



$\forall\, 0 \le k \le n$, $\binom{n}{k} \le \binom{n}{\lfloor n/2 \rfloor}$

Proof

Let $\mathcal{A} \subseteq \mathcal{P}([n])$ be an antichain.

Form a bipartite graph $G = (\mathcal{A}, S_n; E)$.

Here S_n = set of all permutations of $[n]$.

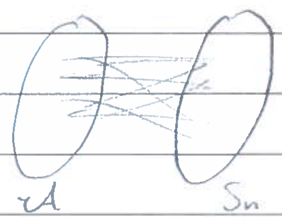
There is an edge $A\pi$, where $A \in \mathcal{A}$ and $\pi \in S_n$, iff the first $|A|$ elements of is the set A.

e.g. $A = 246$, $n = 6$

$\pi_1 = 412356$, $\pi_2 = 642135$

then $A\pi_2 \in E$ but $A\pi_1 \notin E$.

Since G is bipartite, $\displaystyle\sum_{A\in\mathcal{A}} d(A) = E = \sum_{\pi\in S_n} d(\pi)$



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Given $\pi \in S_n$, how large can $d(\pi)$ be?
$d(\pi) \leq 1$ since $\mathcal{A}$ is an antichain.

More precisely, if $d(\pi) \geq 2$ $\exists A, B \in \mathcal{A}$, $A \neq B$
s.t. A is the first $|A|$ elements of π and
B is the first $|B|$ elements of π
so wlog $|A| \leq |B| \Rightarrow A \subseteq B$ $\times$ $\Rightarrow d(\pi) \leq 1$.

Now let $A \in \mathcal{A}$, $|A| = k$. What is $d(A)$?

$A\pi$ is an edge iff $\pi = \underbrace{\overset{A}{\ldots\ldots}}_{k!}\,\underbrace{\overset{[n]\setminus A}{\ldots\ldots}}_{(n-k)!}$

so $d(A) = k!(n-k)! = |A|!(n-|A|)!$

$$\sum_{A \in \mathcal{A}} |A|!(n-|A|)! = |E| \leq \sum_{\pi \in S_n} 1 = n!$$

So $$\sum_{A \in \mathcal{A}} \frac{|A|!(n-|A|)!}{n!} \leq 1$$ $\square$

<u>Def^n</u>
If $\mathcal{A} \subseteq \mathcal{P}([n])$ then $\mathcal{A}$ is intersecting iff
$\forall A, B \in \mathcal{A}$, $A \cap B \neq \emptyset$.

$\mathcal{A} = \{A \subseteq [n] \mid 1 \in A\}$
$\mathcal{B} = \{A \subseteq [n] \mid A \cap [3] \geq 2\} = \underset{12\cancel{3}}{2^{n-3}} + \underset{1\cancel{2}3}{2^{n-3}} + \underset{\cancel{1}23}{2^{n-3}} + \underset{123}{2^{n-3}} = 2^{n-1}$

$A, B \in \mathcal{B}$ then $|A \cap B| \geq 2$ and $|B \cap [3]| \geq 2$
$\Rightarrow A \cap B \cap [3] \neq \emptyset$

$\mathcal{C}_k = \{C \subseteq [n] \mid |C \cap [2k-1]| \geq k\}$ $|\mathcal{C}_n| = 2^{n-1}$.

<u>Thm</u>
If $\mathcal{A} \subseteq \mathcal{P}([n])$ is intersecting then $|\mathcal{A}| \leq 2^{n-1}$.

Proof

If $|\mathcal{A}| > 2^{n-1}$ then $\exists A, A^c$ both in $\mathcal{A}$. #

$\square$

Thm 4.7 (Erdös - Ko - Rado)

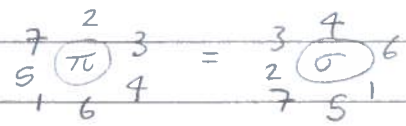
If $k \leq \frac{n}{2}$, $\mathcal{A} \subseteq \binom{[n]}{k}$ is intersecting, then $|\mathcal{A}| \leq \binom{n-1}{k-1}$

$$\binom{n-1}{k-1} = \#\left\{ A \in \binom{[n]}{k} \;\middle|\; 1 \in A \right\}.$$

Proof

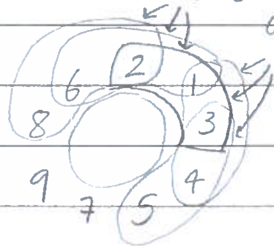
Let $n \geq 2k$, $\mathcal{A} \subseteq \binom{[n]}{k}$ intersecting.

Let $\mathcal{C}_n$ be the set of cyclic permutations of $[n]$, i.e. the set of permutations of $[n]$, where we consider $\pi = \sigma$ if we can get from π to σ by rotating where π and σ are both written around a circle. e.g.



$|\mathcal{C}_n| = (n-1)!$

$n = 9$
$k = 3$



$123 \in \mathcal{A}$

$$\frac{k}{n}\binom{n}{k} = \binom{n-1}{k-1}$$

$\leq 1 + \frac{1}{2} \cdot 2(k-1) = k$

A set A is an interval in a cyclic permutation C iff A appears as consecutive elements of C.

Lemma 4.8

If $C \in \mathcal{C}$ and $n \geq 2k$ and $\mathcal{I}$ is an intersecting family of intervals from $\mathcal{C}$ then $|\mathcal{I}| \leq k$.

Using this lemma construct a bipartite graph $G = (\mathcal{C}_n, \mathcal{A}; E)$

$C \in \mathcal{C}_n$ is joined to $\mathcal{A}$ by an edge iff A is an interval in C.

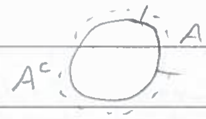
$$\sum_{C \in \mathcal{C}_n} d(C) = |E| = \sum_{A \in \mathcal{A}} d(A). \qquad (*)$$

Take $C \in \mathcal{C}_n$. $d(C) \leq ?$

$\Gamma(C)$ is an intersecting family of intervals in C.

So Lemma 4.8 $\Rightarrow |\Gamma(C)| \leq k$, $d(C) \leq k$.

If $A \in \mathcal{A}$, $d(A) = k!(n-k)!$

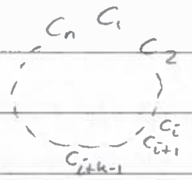


$$\sum_{A \in \mathcal{A}} k!(n-k)! = |E| \leq \sum_{C \in \mathcal{C}_n} k$$

so $|\mathcal{A}| k!(n-k)! \leq |\mathcal{C}_n| k = k(n-1)!$

$\Rightarrow |\mathcal{A}| \leq \binom{n-1}{k-1}$. $\square$

<u>Proof</u> (of Lemma 4.8)

Let $C =$ 

, suppose $\mathcal{I}$ is an intersecting family of intervals from C

$n \geq 2k$

wlog $\mathcal{I} \neq \emptyset$ so we suppose $I = \{c_i, c_{i+1}, \ldots, c_{i+k-1}\} \in \mathcal{I}$

For $j \in \mathbb{Z}$, define $I + j = \{c_{i+j}, \ldots, c_{i+j+k-1}\}$ where we interpret subscripts mod n.

ie. $I+j$ is I rotated j positions clockwise.

The only other intervals in C that meet I are

$I+1, I+2, \ldots, I+(k-1)$

$I-(k-1), I-(k-2), \ldots, I-1$.

Note $I+j$ and $I-(k-j)$ are disjoint.

So $\mathcal{I}$ contains at most $\frac{1}{2} \times 2(k-1)$ of these intervals

Hence $|\mathcal{I}| \leq k$. $\square$.

Compression

For $A \subseteq [n]$, $1 \leq i < j \leq n$ define the *ij th compression of A* to be

$$C_{ij}(A) = \begin{cases} (A \setminus \{j\}) \cup \{i\} & , j \in A,\ i \notin A \\ A & , \text{otherwise} \end{cases}$$

e.g. $C_{13}(234) = 124$, $C_{13}(123) = 123$, $C_{13}(456) = 456$.

Let $\mathcal{A} \subseteq \mathcal{P}([n])$

$C_{ij}(\mathcal{A}) = \{C_{ij}(A) : A \in \mathcal{A}\} \cup \{A \in \mathcal{A} : C_{ij}(A) \in \mathcal{A}\}$

e.g. $\mathcal{A} = \{146, 235, 124, 236\}$

$C_{12}(\mathcal{A}) = \{146, 135, 124, 136\} = \mathcal{A}'$

$C_{34}(\mathcal{A}') = \{146, 136, 135, 123\} = \mathcal{A}''$

$C_{23}(\mathcal{A}'') = \{146, 126, 125, 123\} = \mathcal{A}_3$

$C_{36}(\mathcal{A}_3) = \{134, 126, 123, 125\} = \mathcal{A}_4$

$C_{46}(\mathcal{A}_4) = \{134, 124, 123, 125\}$

$= \{123, 124, 125, 134\}$

Compressions

For $A \subseteq [n]$ and $1 \leq i < j \leq n$ define

$$C_{ij}(A) = \begin{cases} A \setminus \{j\} \cup \{i\} & \text{if } j \in A,\ i \notin A \\ A & \text{otherwise} \end{cases}$$

"Left i,j compression"

If $\mathcal{A} \subseteq \mathcal{P}([n])$ then $C_{ij}(\mathcal{A}) = \{C_{ij}(A) : A \in \mathcal{A}\} \cup \{A \in \mathcal{A} : C_{ij}(A) \in \mathcal{A}\}$

$\mathcal{A} = \{125,\ 134,\ 235,\ 135,\ 245\}$

$C_{25}(\mathcal{A}) = \{125,\ 134,\ 235,\ 123,\ 245\} =: \mathcal{A}_1$
$C_{45}(\mathcal{A}_1) = \{124,\ 134,\ 234,\ 123,\ 245\} =: \mathcal{A}_2$
$C_{34}(\mathcal{A}_2) = \{124,\ 134,\ 234,\ 123,\ 235\} =: \mathcal{A}_3$
$C_{13}(\mathcal{A}_3) = \{124,\ 134,\ 234,\ 123,\ 125\} =: \mathcal{A}_4$
So $\mathcal{A}_4 = \{123,\ 124,\ 125,\ 134,\ 234\}$

We say a family $\mathcal{A} \subseteq \mathcal{P}([n])$ is <u>left-compressed</u> if $C_{ij}(\mathcal{A}) = \mathcal{A}$ $\forall\, 1 \leq i < j \leq n$.

Lemma 4.4

If $\mathcal{A} \subseteq \binom{[n]}{k}$ and $1 \leq i < j \leq n$,

(i) $C_{ij}(\mathcal{A}) \subseteq \binom{[n]}{k}$.
(ii) $|C_{ij}(\mathcal{A})| = |\mathcal{A}|$.
(iii) If $\mathcal{A}$ is intersecting then so is $C_{ij}(\mathcal{A})$.
(iv) By repeatedly applying i,j compressions (for appropriate i,j) to $\mathcal{A}$, we will eventually (ie. in finite time) obtain a left compressed family.

Proof

(i) and (ii) follow from the definition.
(iii). Suppose $\mathcal{A}$ is intersecting but $C_{ij}(\mathcal{A})$ is not.
So $\exists\, A, B \in C_{ij}(\mathcal{A})$ s.t. $A \cap B = \emptyset$.

If $A, B \in \mathcal{A}$ then $A \cap B \neq \emptyset$
If $A, B \in C_{ij}(\mathcal{A}) \setminus \mathcal{A}$ then $i \in A \cap B$.
So wlog $A \in C_{ij}(\mathcal{A}) \setminus \mathcal{A}$ and $B \in \mathcal{A}$.

$A \in C_{ij}(\mathcal{A}) \setminus \mathcal{A} \Rightarrow i \in A,\ j \notin A$.
If $A \cap B = \emptyset$ then $\exists\, D \in \mathcal{A}$ s.t. $C_{ij}(D) = A$
Now $i \notin D$, $j \in D$ so $D \cap B \neq \emptyset \Rightarrow D \cap B = \{j\}$.
So $i \notin B$, $j \in B$.

Hence $E = B \setminus \{j\} \cup \{i\} \in \mathcal{A}$
But now $D \cap E = \emptyset$ ※
Hence $C_{ij}(\mathcal{A})$ is intersecting.

(iv) For $\mathcal{A} \subseteq \binom{[n]}{k}$ define $s(\mathcal{A}) = \sum_{A \in \mathcal{A}} \sum_{a \in A} a \in \mathbb{N}$

$s(\mathcal{A}) \leq nk|\mathcal{A}|$
Note if $C_{ij}(\mathcal{A}) \neq \mathcal{A}$ then $s(C_{ij}(\mathcal{A})) \leq s(\mathcal{A}) - 1$
(since this is a set $A \mapsto A \setminus \{j\} \cup \{i\}$)
Hence after at most $s(\mathcal{A})$ left compressions we obtain a left compressed family. □

Use compressions to prove EKR-thm (again):
If $n \geq 2k$ and $\mathcal{A} \subseteq \binom{[n]}{k}$ is intersecting, then $|\mathcal{A}| \leq \binom{n-1}{k-1}$

<u>Proof</u> (Induction on n)
$n = 2$ ✓
Note result is easy for $n = 2k$.
Since $A \in \mathcal{A} \Rightarrow [n] \setminus A \notin \mathcal{A}$
Hence for $n = 2k$, $|\mathcal{A}| \leq \frac{1}{2}\binom{2k}{k} = \binom{2k-1}{k-1}$ ✓

So suppose $n \geq 2k+1$ and result holds for smaller n. Let $\mathcal{A} \subseteq \binom{[n]}{k}$ be intersecting and left-compressed
(can assume this by previous Lemma)

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Let $\mathcal{B} = \{B \in \mathcal{A} : n \notin B\}$
$\mathcal{C} = \{C \setminus \{n\} : C \in \mathcal{A}\ n \in C\}$

Since $\mathcal{A} = \mathcal{B} \cup \{C \in \mathcal{A} : n \in C\}$
$|\mathcal{A}| = |\mathcal{B}| + |\mathcal{C}|$

$\mathcal{B} \subseteq \binom{[n-1]}{k}$ is intersecting

Inductive hypothesis $\Rightarrow$ $|\mathcal{B}| \leq \binom{n-2}{k-1}$

(note $n \geq 2k+1 \Rightarrow n-1 \geq 2k$)

$\mathcal{C} = \{C \setminus \{n\} : C \in \mathcal{A}, n \in C\} \subseteq \binom{[n-1]}{k-1}$
Claim: $\mathcal{C}$ is intersecting

If $\mathcal{C}$ is not intersecting, then $\exists\ C, D \in \mathcal{C}$ s.t. $C \cap D = \emptyset$.
Now let $E = C \cup \{n\} \in \mathcal{A}$
$F = D \cup \{n\} \in \mathcal{A}$
$|C \cup D| = 2k-2 \leq n-2$
So $\exists\ i \in [n-1] \setminus (C \cup D)$
Since $\mathcal{A}$ is left compressed, $G = \underbrace{E \setminus \{n\} \cup \{i\}}_{= C_{in}(E)} \in \mathcal{A}$
But $F \cap G \subseteq C \cap D = \emptyset$ ※
This proves the claim.

So $\mathcal{C} \subseteq \binom{[n-1]}{k}$ is intersecting.

Our inductive hypothesis $\Rightarrow |\mathcal{C}| \leq \binom{n-2}{k-2}$

So $|\mathcal{A}| = |\mathcal{B}| + |\mathcal{C}| \leq \binom{n-2}{k-1} + \binom{n-2}{k-2} = \binom{n-1}{k-1}$ □

Lemma 4.10 (The linear algebra bound)
If $v_1, \dots, v_t$ are linearly independent vectors in a vector space V, with $\dim(V) = d$, then $t \leq d$. $\square$

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Thm 4.11
Let $\mathcal{A} = \{A_1, \dots, A_m\} \subseteq \mathcal{P}([n])$ be a family of sets satisfying
(i) $|A_i|$ is odd $\forall i$
(ii) $|A_i \cap A_j|$ is even, $i \neq j$
Then $m \leq n$

Proof
Let $V = \mathbb{F}_2^{\,n}$.
For each $A_i \in \mathcal{A}$ define the incidence vector of A_i to be the vector v_i with jth entry
$$v_{ij} = \begin{cases} 1, & \text{if } j \in A_i \\ 0, & \text{otherwise.} \end{cases}$$
[e.g. $n = 5$, $A_2 = 125$ then $v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$]

So we want to show $|\mathcal{A}| = |\{v_1, \dots, v_m\}| \leq n$.

We do this by showing that $\{v_1, \dots, v_m\}$ are linearly independent (LI).

If $1 \leq i \neq j \leq m$, $v_i \cdot v_j = \sum_{k=1}^{n} v_{ik} v_{jk} = |A_i \cap A_j|$

$$v_i \cdot v_i = |A_i|$$

So $v_i \cdot v_j = 0$ $i \neq j$, $v_i \cdot v_i = 1$
(Recall in $\mathbb{F}_2$, even $= 0$, odd $= 1$).

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So $\{v_1, \dots, v_m\}$ is orthogonal $\therefore$ is LI.
Hence $|\mathcal{A}| = |\{v_1, \dots, v_m\}| \leq \dim V = \dim \mathbb{F}_2^n = n$

$\square$

Thm 4.12 (Fisher's Inequality 1940)
If $\mathcal{A} = \{A_1, \dots, A_m\} \subseteq \mathcal{P}([n])$ and $k \geq 1$ is an integer such that

Proof
$V = \mathbb{R}^n$, let v_i be the incidence vector of A_i. WTS: $\{v_1, \dots, v_m\}$ is LI over $\mathbb{R}$.

If $1 \leq i, j \leq m$, $v_i \cdot v_j = |A_i \cap A_j|$

so $v_i \cdot v_j = \begin{cases} |A_i| & , i = j \\ k & , i \neq j \end{cases}$ $(*)$

Suppose, for a contradiction that $\{v_1, \dots, v_m\}$ are linearly dependent.
So $\exists\ \lambda_1, \dots, \lambda_m \in \mathbb{R}$, not all zero, such that $\sum_{i=1}^{m} \lambda_i v_i = 0$.

$$\text{So } \underset{\in \mathbb{R}}{0} = 0 \cdot 0 = \left(\sum_{i=1}^{m} \lambda_i v_i \right) \cdot \left(\sum_{j=1}^{m} \lambda_j v_j \right)$$

$$= \sum_{i=1}^{m} \lambda_i^2 v_i \cdot v_i + \sum_{1 \leq i \neq j \leq m} \lambda_i \lambda_j v_i \cdot v_j$$

$$= \sum_{i=1}^{m} \lambda_i^2 |A_i| + \sum_{1 \leq i \neq j \leq m} \lambda_i \lambda_j k \quad \text{by } (*)$$

Note: $\forall\ 1 \leq i \leq m$, $|A_i| \geq k$ with equality at most once

$$0 = \sum_{i=1}^{m} \lambda_i^2 (|A_i| - k) + \sum_{i=1}^{m} \lambda_i^2 k + \sum_{1 \leq i \neq j \leq m} \lambda_i \lambda_j k$$

$$0 = \sum_{i=1}^{m} \lambda_i^2 (|A_i| - k) + k \left(\sum_{i=1}^{m} \lambda_i \right)^2$$

So $\sum_{i=1}^{m} \lambda_i^2 (|A_i| - k)$ (1) and $\sum_{i=1}^{m} \lambda_i = 0$ (2)

Since the $\lambda_1, \ldots, \lambda_m$ are not all zero,
(2) $\Rightarrow \exists\, a \neq b$ such that $\lambda_a \neq 0 \neq \lambda_b$.

But (1) $\Rightarrow \lambda_i^2(|A_i| - k) = 0 \quad 1 \leq i \leq m$
so $\lambda_a \neq 0$, $\lambda_b \neq 0 \Rightarrow |A_a| = k = |A_b|$
Impossible since then $|A_a \cap A_b| < k$ ※. $\square$

A family $\mathcal{A} \subseteq \mathcal{P}([n])$ is L-intersecting, $L \subseteq \{0, 1, \ldots, n\}$
iff $\forall A, B \in \mathcal{A}$, $A \neq B$, $|A \cap B| \in L$.

e.g. $L = \{k\}$ gives an L-intersecting family satisfying Fisher's Inequality.

Thm 4.13 (Ray-Chaudhuri - Wilson 1975)
If $\mathcal{A} \subseteq \mathcal{P}([n])$ is L-intersecting,
then $|\mathcal{A}| \leq \sum_{i=0}^{|L|} \binom{n}{i}$.

e.g. $L = \{0\}$, $\mathcal{A} = \{\emptyset, 1, 2, \ldots, n\}$
and $|\mathcal{A}| = \binom{n}{0} + \binom{n}{1}$.

$L = \{0, 1, s-1\}$ so $|L| = s$.
Let $\mathcal{A} = \{A \subseteq [n] \mid |A| \leq s\}$
$\mathcal{A}$ is L-intersecting and $|\mathcal{A}| = \sum_{i=0}^{|L|} \binom{n}{i}$

This example shows that the theorem is "best possible".

Proof (of thm 4.13)
Let $L = \{l_1, l_2, \ldots, l_s\}$ and $\mathcal{A} = \{A_1, \ldots, A_m\}$
with $|A_1| \leq |A_2| \leq \ldots \leq |A_m|$.
Let $v_i \in \mathbb{R}^n$ be the incidence vector of A_i.

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As before, $v_i \cdot v_j = |A_i \cap A_j|$.

For $1 \leq i \leq m$ define $p_i(x) \in \mathbb{R}[x_1, \dots, x_n]$ by $p_i(x) = \prod_{k : l_k < |A_i|} ((v_i \cdot x) - l_k)$ where $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

e.g. $\mathcal{A} = \{\underset{A_1}{1}, \underset{A_2}{12}, \underset{A_3}{123}, \underset{A_4}{124}\}$, $L = \{1, 2\}$, $n = 4$

$p_1(x) = 1$

$p_2(x) = (v_2 \cdot x - 1) = \left(\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} - 1 \right) = x_1 + x_2 - 1$

$p_3(x) = (v_3 \cdot x - 1)(v_3 \cdot x - 2)$
$= (x_1 + x_2 + x_3 - 1)(x_1 + x_2 + x_3 - 2)$

We have $p_i(v_j) = 0$, $1 \leq j < i \leq m$.

Since $v_i \cdot v_j = |A_i \cap A_j| \in L$ and $v_i \cdot v_j = l_k < |A_i|$, we have that the term $(v_i \cdot v_j - l_k) = 0$.

Note $p_i(v_i) \neq 0$, since $v_i \cdot v_i - l_k > 0$ for any k in the product.

($x_1^3 x_2^2$ has degree 5)

The degree of $p_i(x)$ is at most s, since it is the product of at most s linear terms.

[$x_1^3 x_2^4 x_3^5 = x_1 x_2 x_3$ if $x_1, x_2, x_3 \in \{0, 1\}$, $0^2 = 0$, $1^2 = 1$.]

Expand each $p_i(x)$ into terms of the form $C x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$.

We can replace any $\alpha_i \geq 2$ by 1 without changing the value of this polynomial on $\{0, 1\}^n$.

Let $q_i(x)$ be the resulting polynomial.

$\{q_1(x), \ldots, q_m(x)\}$

Let Q be a vector space over $\mathbb{R}$ of polynomials spanned by $\{q_i(x)\}_{i=1,\ldots,m}$.

Q is contained in the space M_s spanned by

$1, x_1, \ldots, x_n, x_1x_2, \ldots, x_{n-1}x_n, x_1x_2x_3, \ldots, x_{n-2}x_{n-1}x_n,$

$\ldots, \quad x_{i_1}x_{i_2}\ldots x_{i_s} \quad i_1 < i_2 < \ldots < i_s$

so $\dim M_s = \sum_{i=1}^{s} \binom{n}{i}$

So suppose for a contradiction that $\{q_1(x), \ldots, q_m(x)\}$ are linearly dependent.

$\exists \lambda_1, \ldots, \lambda_m$ not all zero such that

$\sum_{i=1}^{m} \lambda_i q_i = 0$

Let $k = \min\{j : \lambda_j \neq 0\}$

$0 = \lambda_k q_k(x) + \sum_{j=k+1}^{m} \lambda_j q_j(x)$

$0 = \lambda_k \underbrace{q_k(v_k)} + 0$

both $\neq 0$

$\#$ $\square$

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12-03-19

Chapter 5 - Ramsey Theory

"There is no such thing as total disorder"

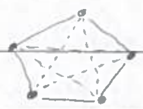
For $s, t \geq 2$, let $R(s,t)$ be the smallest $n \in \mathbb{N}$ such that whenever the edges of K_n are coloured red and blue there is always a red copy of K_s or a blue copy of K_t.

Prop^n 5.1

$R(3,3) = 6$

Proof

$R(3,3) > 5$ since



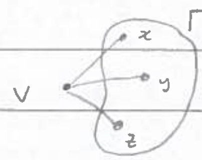
—— = blue
...... = red

$\leftarrow$ no monochromatic K_3.

$R(3,3) \leq 6$:

Suppose we have a red/blue-colouring of K_6, let $v \in V(K_6)$.

$d(v) = 5$ so wlog $\exists$ 3 blue edges containing v.



Now consider $\Gamma_B(v)$.

Either xy, xz, yz are all red $\Rightarrow$ red K_3,
or one of these is blue $\Rightarrow$ blue K_3.
So we always have a monochromatic K_3. $\square$

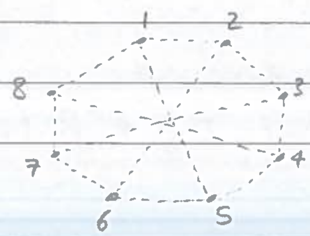
Prop^n 5.2

$R(3,4) = 9$

Proof

$R(3,4) > 8$

This has no red K_3 or blue K_4.



Red edges drawn, all other edges in the K_8 are blue (not drawn but still there!)

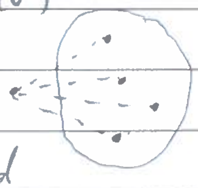
$R(3,4) \leq 9$:

Take a red / blue colouring of K_9.

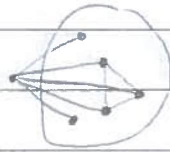
Let $v \in V(K_9)$. Define $\Gamma_R(v) = \{x : xv \text{ is red}\}$

$\Gamma_B(v) = \{y : yv \text{ is blue}\}$

$|\Gamma_R(v)| = d_R(v)$, $|\Gamma_B(v)| = d_B(v)$



Suppose $d_R(v) \geq 4$.
Either $\Gamma_R(v)$ contains a red edge $\Rightarrow$ red K_3
or all edges in $\Gamma_R(v)$ are blue $\Rightarrow$ blue K_4
$\Rightarrow$ wlog $d_R(v) \leq 3$.
So $d_B(v) \geq 5$ (since $d_R(v) + d_B(v) = d(v) = 8$)
If $\exists$ v with $d_B(v) \geq 6$ then since $R(3,3) = 6$ we know that $\Gamma_B(v)$ contains a red K_3 ✓
or a blue K_3 $\Rightarrow$ together with v gives a blue K_4.



So only possibility remaining is $d_B(v) = 5$ $\forall v \in V(K_9)$

$$2\#\text{blue edges} = \sum_{v \in V(K_9)} d_B(v) = 45 \quad \#$$

<u>Thm 5.3</u> (Ramsey)

If $s, t \geq 2$ then $R(s,t)$ is well-defined and satisfies $R(s,t) \leq \binom{s+t-2}{s-1}$

<u>Proof</u> (Induction on $s+t$)
$R(2,t) = t = \binom{t}{1}$, $R(s,2) = s = \binom{s}{s-1}$
So let $s, t > 2$ and suppose result holds for smaller s and t.
Let $n = R(s-1,t) + R(s,t-1)$ well-defined by inductive hypothesis.

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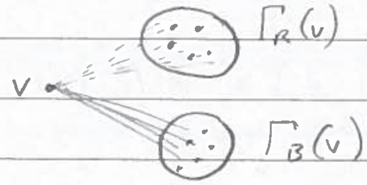
Take a red / blue colouring of the edges of K_n.
For $v \in V(K_n)$, $\Gamma_R(v) = \{x : xv \text{ red}\}$, $d_R(v) = |\Gamma_R(v)|$,
$\Gamma_B(v) = \{y : yv \text{ blue}\}$, $d_B(v) = |\Gamma_B(v)|$.

$$d_R(v) + d_B(v) = d(v) = n-1 = R(s-1,t) + R(s,t-1) - 1$$

Let $v \in V(K_n)$.



$$\underbrace{d_R(v)} + \overbrace{d_B(v)} \geq \underbrace{R(s-1,t)} + \overbrace{R(s,t-1)} - 1$$

Either $d_R(v) \geq R(s-1,t)$ ①
or $d_B(v) \geq R(s,t-1)$ ②
since if ① and ② both fail then
$d_R(v) + d_B(v) \leq R(s-1,t) - 1 + R(s,t-1) - 1$
※

In case ①, $\Gamma_R(v)$ contains a red K_{s-1} or a blue K_t, so with v we have a red K_s or a blue K_t.
In case ②, $\Gamma_B(v)$ contains a red K_s or a blue K_{t-1}, so with v we have a blue K_t.

Hence $R(s,t) \leq n = R(s-1,t) + R(s,t-1)$.
Also $R(s,t) \leq \binom{s-1+t-2}{s-2} + \binom{s+t-1-2}{s-1} = \binom{s+t-2}{s-1}$ □

helpful for how?

$R(s,s) \leq \binom{2s-2}{s-1}$, $R(s,s) \geq$?

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$R(s,t) = \min\{n \in \mathbb{N} \mid$ Any red / blue colouring of K_n has a red K_s or a blue $K_t\}$

$R(3,3) = 6$
$R(3,4) = 9$

$R(s,t) \leq R(s-1,t) + R(s,t-1) \leq \binom{s+t-2}{s-1}$

Prop^n 5.4

$R(4,4) = 18$

Proof

$R(4,4) \leq R(3,4) + R(4,3) = 18$

To see $R(4,4) > 17$ take $V(K_{17}) = \{0, \pm 1, \pm 2, \ldots, \pm 8\}$
and let xy be red iff $|x-y| \in \{1,2,4,8\}$ mod 17
otherwise xy is blue.
This does not contain a red K_4 or blue K_4
(not proved) $\square$

$R(5,5) = ?$

$43 \leq R(5,5) \leq 48$

Show $R(5,5) \leq 47 \rightarrow K_{47} - \binom{47}{2} = \frac{47 \cdot 46}{2}$
Naively need to check $2^{47 \times 23}$ $\leftarrow$ v. difficult to compute!

$R(s,s) \leq \binom{2s-2}{s-1} \approx \frac{4^s}{\sqrt{s}}$

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Thm 5.6 (Erdös 1947)
If $n > s > 1$ satisfy $\binom{n}{s} 2^{1-\binom{s}{2}} < 1$ $(*)$
then $R(s,s) > n$.

Proof
Let $n > s > 1$ satisfy $(*)$.
Consider a random red / blue colouring of the edges of K_n.
Each edge is coloured independently of all others and has $P(\text{edge is red}) = P(\text{edge is blue}) = 1/2$.

For an s-set $S \in \binom{[n]}{s}$ (where $V(K_n) = [n]$)

define $X_S = \begin{cases} 1 & \text{if } S \text{ is monochomatic} \\ 0 & \text{otherwise} \end{cases}$

If $X = \#$ monochromatic copies of K_s
then $X = \sum_{S \in \binom{[n]}{s}} X_S$

$E[X] = \sum_{S \in \binom{[n]}{s}} E[X_S]$

$E[X_S] = P(X_S = 1) = \overset{\text{red or blue}}{2} \times \frac{1}{2^{\binom{s}{2}}}$

So $E[X] = \binom{n}{s} 2^{1-\binom{s}{2}} \underset{(*)}{<} 1$

Since $X \geq 0$ is integer valued, $E[X] < 1 \Rightarrow$ there exists a red / blue colouring of K_n with no monochromatic K_s.

$\square$

Corollary 5.7
For $s \geq 2$, $R(s,s) \geq 2^{s/2}$

Proof
True for $s = 2, 3$.
Let $s \geq 4$, and let $n = \lfloor 2^{s/2} \rfloor$.
Need to check that $\binom{n}{s} 2^{1-\binom{s}{2}} < 1$.

$$\binom{n}{s} < \frac{n^s}{s!} \leq \frac{2^{s^2/2}}{2^s} \qquad \underset{\text{for } s \geq 4}{(s! > 2^s)}$$

$$\Rightarrow \binom{n}{s} 2^{1-\binom{s}{2}} < \frac{2^{\frac{s^2}{2}+1}}{2^{\frac{s^2}{2}-\frac{s}{2}+s}} = \frac{2}{2^{s/2}} < 1$$

$\square$

In particular, $(\sqrt{2})^s < R(s,s) < 4^s$

Thm (FLT)
There are no non-trivial integer solutions to $x^n + y^n = z^n$ for $n \geq 3$ integer.

Thm 5.9
For every $n \geq 1$, $\exists\, p_n$ such that for any prime $p > p_n$, the congruence
$x^n + y^n \equiv z^n \bmod p$ has a $\underbrace{\text{non-trivial}}_{\text{means not cong. to } 0 \bmod p.}$ solution.

Thm 5.10 (Schur's Thm)
For any $k \geq 1$, $\exists\, S(k) \geq 3$ such that for any k-colouring of the integers $\{1, 2, \ldots, S(k)\}$ $\exists\, u, v, w \in [S(k)]$ that are the same colour and satisfy $u + v + w$.

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For $k \in \mathbb{N}$ and $s_1, \ldots, s_k \geq 2$,

$R_k(s_1, \ldots, s_k) = \min\{n \in \mathbb{N} \mid$ Any k-colouring of edges of K_n with colours $c_1, \ldots, c_k$ contains a c_i-coloured K_{s_i} for some $1 \leq i \leq k\}$.

e.g. $R_3(2, 4, 6)$

$\Rightarrow$ asks for a c_1-coloured K_2 or c_2-coloured K_4 or c_3-coloured K_6.

Thm 5.12

$\forall k \in \mathbb{N}$, $s_1, \ldots, s_k \geq 2$, $R_k(s_1, \ldots, s_k)$ is well-defined.

Proof

* $k = 1$ ✓
* $k = 2$ $\leftarrow$ Ramsey's Thm.

Let $k \geq 3$. $n = R_{k-1}(s_1, \ldots, s_{k-2}, R(s_{k-1}, s_k))$.

Colour the edges of K_n with colours $c_1, \ldots, c_k$.

Pretend that colours c_{k-1} and c_k are indistinguishable.

So by def^n of $R_{k-1}(s_1, \ldots, s_{k-2}, R(s_{k-1}, s_k))$

we have a c_1-coloured K_{s_1}

or a c_2-coloured K_{s_2}

or ...

... or a c_{k-2}-coloured $K_{s_{k-2}}$

or a (c_{k-1}/c_k)-coloured $K_{R(s_{k-1}, s_k)}$.

In the last case, by def^n of $R(s_{k-1}, s_k)$,

$\exists$ a c_{k-1}-coloured $K_{s_{k-1}}$ or a c_k-coloured K_{s_k}. □

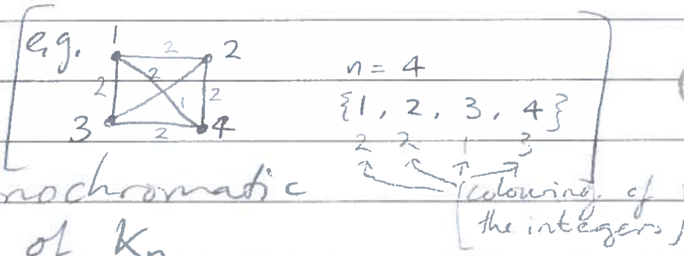
[Could also have argued with $n = R(R_{k-1}(s_1, \ldots, s_{k-1}), s_k)$]

Proof (of Schur's Thm)

Define $R_k(s) = R_k(\underbrace{s, s, \ldots, s}_{k})$

Given k, let $n = R_k(3)$.
Now suppose we have a k-colouring c of the integers $\{1, \ldots, n\}$. We need $u, v, w \in [n]$ s.t. $u+v=w$ and $c(u) = c(v) = c(w)$.

Define a k-colouring of K_n, $V(K_n) = [n]$
by $c'(ij) = c(|i-j|)$



Now $n = R_k(3) \Rightarrow \exists$ monochromatic triangle in the colouring of K_n
Say

, suppose $a > b > c$

$$\begin{cases} u = a-b \\ v = b-c \\ w = a-c \end{cases}$$

so $u + v = w$, and
$c(u) = c(a-b) = c(|a-b|) = c'(ab) = \gamma$
$c(v) = c(b-c) = c(|b-c|) = c'(bc) = \gamma$
$c(w) = c(a-c) = c(|a-c|) = c'(ac) = \gamma$.
Hence, we can take $S(k) = R_k(3)$. $\square$

Write $\mathbb{Z}_p^* =$ non-zero integers mod p
$= \{1, 2, \ldots, p-1\}$

$\mathbb{Z}_p^*$ is cyclic
i.e. $\exists$ g a generator of $\mathbb{Z}_p^*$ primitive root.
i.e. $\mathbb{Z}_p^* = \{g, g^2, \ldots, g^{p-1}\}$

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Proof (of Thm 5.9) [Always working mod p here]

Let $n \geq 1$ and let p be prime, $p > S(n)$ (given by Schur's thm)

Consider $\mathbb{Z}_p^*$ with generator g.

If $x \in \mathbb{Z}_p^*$, $\exists$ unique $1 \leq m \leq p-1$ such that $x = g^m \bmod p$.

Define an n-colouring of $\mathbb{Z}_p^*$ as follows.

Let $x = g^{nj+i}$, $0 \leq i \leq n-1$

Define $c(x) = i$ (since we can always write $m = nj+i$, i is the remainder $0 \leq i \leq n-1$).

By Schur's Thm, $\exists\, u, v, w \in \mathbb{Z}_p^*$ such that $u+v=w$ and $c(u) = c(v) = c(w) = i$.

So $u = g^{nj_u+i}$, $v = g^{nj_v+i}$, $w = g^{nj_w+i}$

$x = g^{j_u}$, $y = g^{j_v}$, $z = g^{j_w}$

$$x^n + y^n = g^{nj_u} + g^{nj_v} = g^{-i}\left(g^{nj_u+i} + g^{nj_v+i}\right)$$
$$= g^{-i}(u+v) = g^{-i}w = z^n \mod p$$

$\square$

19-03-19 Arithmetic Progressions (AP)

Thm (Green-Tao)

The primes contain A.P.s of all lengths.

An arithmetic progression is $a, a+d, a+2d, \ldots, a+(t-1)d$ this has length t.

<u>Thm</u> (Van der Waerden).
If $k \geq 1$, $t \geq 1$ then $\exists\ W(t,k)$ s.t. any k-colouring of $[W(t,k)]$ contains a monochromatic arithmetic progression (MAP) of length t.

Suppose we have k-coloured $\mathbb{N}$.
Then $P_1, \dots, P_r$ MAPs each of length t are said to be colour focused if $P_1, \dots, P_r$ are all MAPs of different colours with a common possible next term.

e.g.

P_1	1		6		11				$\boxed{16}$
P_2		4		8		12			16
P_3						13	14	15	16

In this case, if we only have 3 colours then $P_i \cup \{16\}$ is a MAP for some $1 \leq i \leq 3$.

<u>Proof</u> (of VdW thm) (Induction on t).

$W(1,k) = 1$, $W(2,k) = k+1$

Suppose $t \geq 3$ and $W(t-1,k)$ is well-defined for any k.
<u>Claim</u>: for any $1 \leq r \leq k$, $\exists\ n_r(t,k)$ such that any k-colouring of $[n_r(t,k)]$ contains either
(i) a MAP of length t
or (ii) r colour focused MAPs of length $t-1$.

Assume claim holds, now consider a k-colouring of $[n_k(t,k)]$,
either (i) holds $\Rightarrow$ this contains a MAP of length t
or (ii) holds $\Rightarrow$ we have $P_1, \dots, P_k$ colour focused MAPs of length $t-1$.

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Look at the common focus of $P_1, \dots, P_k$, say f.
Since these MAPs have colours $1, \dots, k$, one of them (say P_i) has the same colour as f, so $P_i \cup \{f\}$ is a MAP of length t.

$\Rightarrow$ Can set $W(t,k) = n_k(t,k)$

Result will follow by induction on t. $\square$

<u>Proof of claim</u>

For $r=1$, define $n_1(t,k) = W(t-1,k)$.
Now use induction on r.
Suppose $r \geq 2$ and $n_{r-1}(t,k)$ is well-defined.
Write $n = n_{r-1}(t,k)$.

We will show $n_r(t,k) = W(t-1, k^{2n})\, 2n$.

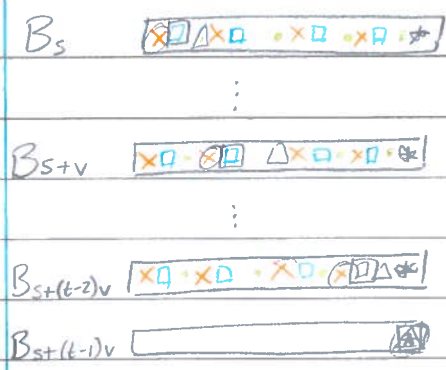
Take a k-colouring of $[n_k(t,k)] = [W(t-1,k^{2n})\, 2n]$.
Assume there is no MAP of length t.
NTS (ii) holds.

Consider this colouring as a colouring of blocks $B_1, B_2, \dots, B_{W(t-1,k^{2n})}$ each of length $2n$,

$B_i = [2n(i-1)+1, \; 2ni]$.

There are k^{2n} different ways to colour a block B_i

Looking at our colouring as a k^{2n} colouring of blocks, we see that since we have $W(t-1,k^{2n})$ blocks then by definition of $W(t-1,k^{2n})$ $\exists$ a MAP of blocks of length $t-1$.
Write this MAP as $B_s, B_{s+v}, \dots, B_{s+(t-2)v}$.



Each block $B_s, \dots, B_{s+(t-2)v}$ is coloured identically to the others.
Also B_i has size $2n$ where $n = n_{r-1}(t, k)$

$\therefore$ B_s contains $r-1$ colour focused MAPs of length $t-1$, say $P_1, \dots, P_{r-1}$ where $P_i = \{a_i, a_i + d_i, \dots, a_i + (t-2)d_i\}$.

Take $P_i' = \{a_i, a_i + (d_i + 2nv), \dots, a_i + (t-2)(d_i + 2nv)\}$, $1 \leq i \leq r-1$, all still colour focused MAPs of length $t-1$ with focus $f + (t-1)2nv$.

Define $P_r' = \{f, f + 2nv, \dots, f + (t-2)nv\}$ gives our r^{th} colour focused MAP. $\square$

22-03-19

__Thm 5.13__
Whenever $\binom{\mathbb{N}}{2} = K_{\mathbb{N}}$ is 2-coloured, $\exists$ $A \subseteq \mathbb{N}$ infinite such that $\binom{A}{2}$ is monochromatic

__Proof__
Take a red / blue colouring of the edges of $\binom{\mathbb{N}}{2}$. Set $a_1 = 1$.

$\exists$ $A_1 \subseteq \mathbb{N} \setminus \{1\}$ st. A_1 is infinite and all edges from a_1 to A_1 are the same colour, say c_1.

Repeat:
Set $a_2 = \min A_1$, then $\exists$ $A_2 \subset A_1 \setminus \{a_2\}$ infinite, with all edges from a_2 to A_2 being the same colour, say c_2.

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Continue:

Construct $\{a_k\}_{k=1}^{\infty} \subseteq \mathbb{N}$, $\{c_k\}_{k=1}^{\infty} \subseteq \{red, blue\}$

a_k is increasing, with the property that if $i < j$ then $a_i a_j$ has colour c_i (since $a_j \in A_i$ and all edges from a_i to A_i have colour c_i).

Take a subsequence of $\{c_k\}_{k=1}^{\infty}$ that is constant, say $\{c_{k_l}\}_{l=1}^{\infty}$.

Then $A = \{a_{k_l}\}_{l=1}^{\infty}$ gives an infinite set for which $\binom{A}{2}$ is monochromatic. $\square$