3703 Prime Numbers and their Distribution

Based on the 2013 autumn lectures by Mr C Daw.

(Part 1 of 2)

1/10/13

www.homepages.ucl.ac.uk/~zcahc32.

William Chen: "Distrubution of primes numbers"

—/—

Def: An arithmetic function is a function $f: \mathbb{N} \to \mathbb{C}$. where $\mathbb{N} = \{1, 2, 3, \ldots\}$.

Def: An arithmetic function $f: \mathbb{N} \to \mathbb{C}$ is multiplicative if $f(mn) = f(m)f(n)$ when $(m, n) = 1$ (coprime).

Eg: $U: \mathbb{N} \to \mathbb{C}$

$$U(n) = 1 \text{ for all } n \in \mathbb{N}.$$

is a multiplicative function.

Thm (1A): If $f: \mathbb{N} \to \mathbb{C}$ is a multiplicative arithmetic functions then $g: \mathbb{N} \to \mathbb{C}$, define by $g(n) = \sum_{m|n} f(m)$ is multiplicative.

N.B: $\sum_{m|n}$ = "sum over all divisors of m".

Proof (1A): Let $a, b \in \mathbb{N}$ st $(a, b) = 1$

We have a bijection;

$\{m \in \mathbb{N} : m | ab\} \Longleftrightarrow \{(u, v) \in \mathbb{N}^2 \text{ st } u|a, v|b, u, v \text{ coprime}\}$

since $a = p_1^{r_1} \dots p_n^{r_n}$, $b = q_1^{s_1} \dots q_t^{s_t}$. $p_i \neq q_j$ for all i,j (distinct primes p_i, q_j)

so $m = p_1^{m_1} \dots p_n^{m_n} q_1^{m_1'} \dots q_t^{m_t'}$ $\quad 0 \leq m_i \leq r_i$
$0 \leq m_i' \leq s_i$

so $m \longmapsto (p_1^{m_1} \dots p_n^{m_n}, q_1^{m_1'} \dots q_t^{m_t'})$

$uv \longleftarrow (u, v)$

$$g(ab) \overset{def}{=} \sum_{m|ab} f(m) = \sum_{u|a} \sum_{v|b} f(uv)$$

$$= \sum_{u|a} \sum_{v|b} f(u) f(v)$$

$$= \Big(\sum_{u|a} f(u)\Big)\Big(\sum_{v|b} f(v)\Big)$$

$$= g(a) g(b)$$

Def: Divisor function $d : \mathbb{N} \to \mathbb{C}$ defined by $d(n) = \sum_{m|n} 1$ for all $n \in \mathbb{N}$

i.e. we count positive divisors of n.

Def: $\sigma : \mathbb{N} \to \mathbb{C}$ defined by

$$\sigma(n) = \sum_{m|n} m \quad \text{for all } n \in \mathbb{N}$$

i.e we sum positive divisors of n.

Dirichlet's primes in progressions: ← L-function

$(a, q) = 1 \Rightarrow$ there exist infinitely many primes congruent to a mod q.

Prime number theorem: ← complex analysis

$$\lim_{x \to \infty} \frac{\sum_{p \le x} 1 . \log x}{x} = 1$$

04/10/13

d, σ

Thm (1B): Suppose $n \in \mathbb{N}$, $n = p_1^{u_1} \dots p_r^{u_r}$ p_i distinct
Then $d(n) = (1 - u_1) \dots (1 + u_r)$

$$\sigma(n) = \frac{p_1^{u_1+1} - 1}{p_1 - 1} \dots \frac{p_r^{u_r+1} - 1}{p_r - 1}$$

Proof: Every divisor $m | n$ is of the form

$$m = p_1^{v_1} \dots p_r^{v_r} \quad 0 \leq v_i \leq u_i$$

m is a choice of the r-tuple $(u_1, \dots, u_r)$.

i.e. $d(n) = (1 + u_1) \dots (1 + u_r)$

On the other hand,

$$\sigma(n) = \sum_{v_r=0}^{u_r} \dots \sum_{v_1=0}^{u_1} p_1^{v_1} \dots p_r^{v_r}$$

$$= \underbrace{\left(\sum_{v_r=0}^{u_r} p_r^{v_r} \right)}_{\text{geom. series}} \dots \left(\sum_{v_1=0}^{u_1} p_1^{v_1} \right)$$

$$= \frac{p_r^{u_r+1} - 1}{p_r - 1} \dots \frac{p_1^{u_1+1} - 1}{p_1 - 1} \quad \square.$$

Cor (1C) : $d, \sigma : \mathbb{N} \to \mathbb{C}$ are multiplcative.

$\square$.

Interlude (perfect no.s)

Def: A perfect no is an $n \in \mathbb{N}$ st $\sigma(n) = 2n$
i.e. n is the sum of its proper divisors.

Eg: $6 = 1+2+3$, $28 = 1+2+4+7+14$.

No one know if an odd prefect no exists.

We have a classificatcoin of the even perf. no.s.

Thm (1D) (Euler - Euclid)
Let $m \in \mathbb{N}$ st $2^m - 1$ is prime, then $2^{m-1}(2^m - 1)$ is an even perfect no. Furthermore, all even prefect no.s arise this way.

NB: Primes of the $2^m - 1$ are called Mersenne primes.

Proof (1D): Let $n = 2^{m-1}(2^m - 1)$ st $2^m - 1$ is prime.

Clearly $(2^m, 2^m - 1) = 1$

So $\sigma(n) \overset{\text{mult}}{=} \underbrace{\sigma(2^{m-1})}_{\text{prime -power}} \underbrace{\sigma(2^m - 1)}_{\text{prime}}$

$$= \frac{2^m - 1}{2 - 1} 2^m$$

$$= (2^m - 1) 2^{m-1} 2$$

$$= 2n$$

i.e. n is perf. + even. $(m \geq 2)$

Let $n \in \mathbb{N}$, perfect + even;

So $n = 2^{m-1} u$, $m > 1$, u is odd.

$$2^m \cdot u = 2\, 2^{m-1} u = 2n \overset{\text{perf}}{=} \sigma(n) \overset{\text{mult}}{=} \sigma(2^{m-1}) \sigma(u)$$

$$= (2^m - 1) \sigma(u)$$

$$\sigma(u) = \frac{2^m u}{2^m - 1} = u + \frac{u}{2^m - 1}$$

$u, \sigma(u)$ integers and $u < \sigma(u)$.

$\Rightarrow \frac{u}{2^m - 1} \in \mathbb{N}$ and a divisor of u.

$m > 1$ $\quad 2^m - 1 > 1$ so $\frac{u}{2^m - 1} \neq u$.

$\sigma(u)$ is the sum of two distinct divisors of u.

But $\sigma(u)$ (by def) is the sum of all divisors of u.

So $\frac{u}{2^m-1} = 1$, $u = 2^m - 1$ prime.

$n = \underline{2^{m-1}(2^m-1)}$ $\square$

We like to study $d(n), \sigma(n)$ asymptotically

i.e. as $n \to \infty$

Def: Let $f, g : \mathbb{R} \to \mathbb{C}$ then $f(x) = O(g(x))$ if the exist $M, N \in \mathbb{R}$ st $|f(x)| \leq M|g(x)|$ for all $x > N$.

Eg: $6x^4 + 3x^2 + 1 = f(x)$

$$x^4 = g(x)$$

$$|6x^4 + 3x^2 + 1| \leq 7|x^4| \qquad x > N.$$

$$x^4 \geq 3x^2 + 1.$$

Eg: $12x^8 + 4x^3 = f(x)$

$$g(x) = x^8.$$

I will also write $f(x) \ll g(x)$

Eg: $x \gg \log x$

Eg : $x \gg 1000 . \log x$

If n is prime $d(n) = 2$.

On the other hand $d(n)$ can be larger than any power of $\log n$.

Thm (IE) : For any $c > 0$.

$$d(n) \ll (\log n)^c \text{ does not hold.}$$

i.e. for all M, N there exists $n > N$ st $d(n) > M (\log n)^c$

Proof : We look at integers divisible by "many" primes. Fix $c > 0$. Let $l \in \mathbb{N} \cup \{0\}$ st $l \leq c \leq l+1$. Let p_j denote the jth prime in ascending order.

take logs

Consider $n = (p_1 \cdots p_{l+1})^m$ for some $m \in \mathbb{N}$ to choose

$$\text{So } d(n) = (m+1)^{l+1} > \left(\frac{\log n}{\log p_1 \cdots p_{l+1}} \right)^{l+1}$$

$$> K(c) (\log n)^{l+1}$$

where $K(c) = \left(\frac{1}{\log p_1 \cdots p_{l+1}} \right)^{l+1}$ – depends only on c.

Now, fix $M, N > 0$.

$$d(n) > K(c)(\log n)^{c+1} = K(c)(\log n)^{\overbrace{c+1-c}^{>0}}(\log n)^{c}$$

Choose n st $n > N$ and

$$K(c)(\underbrace{\log n}_{\text{increasing}})^{c+1-c} > M$$

$\square$

—/—

On the other hand:

Thm (IF): For any $\varepsilon > 0$ $d(n) \ll_{\varepsilon} n^{\varepsilon}$

M,N depend on ε

Proof: If $n \in \mathbb{N}$, $n > 1$ let $p_1^{u_1} \cdots p_r^{u_r}$ p_i distinct so:

$$\frac{d(n)}{n^{\varepsilon}} = \frac{1+u_1}{p_1^{\varepsilon u_1}} \cdots \frac{1+u_r}{p_r^{\varepsilon u_r}}$$

We may assume $\varepsilon < 1$

If $2 \le p_i < 2$ then

$$p_j^{\varepsilon u_j} \geq 2^{\varepsilon u_j} = e^{\varepsilon u_i \log 2} > 1 + \varepsilon u_j \log 2$$
$$> (1+u_j)\underbrace{\varepsilon \log 2}_{<1}$$

$$\frac{1+u_j}{p_j^{\varepsilon u_j}} < \frac{1}{\varepsilon \log 2}.$$

On the other hand if $p_j \geq 2^{1/\varepsilon}$, then $p_j^{\varepsilon} \geq 2$.

$$\Rightarrow \frac{1+u_j}{p_j^{\varepsilon u_j}} \leq \frac{1+u_j}{2^{u_j}} \leq 1$$

So, $\frac{d(n)}{n^{\varepsilon}} < \prod_{p<2^{1/\varepsilon}} \frac{1}{\varepsilon \log 2} \longleftrightarrow$ const depending only on ε.

$\square$

So $d(n) = 2$ infinitely often.

but fluctuates a lot.

To say something more precise we take an average i.e

$$\frac{1}{X} \sum_{n \leq X} d(n) \quad , X \in \mathbb{R}_{>0}$$

Thm (1G) (Dirichlet)

$$\sum_{n \leq X} d(n) = X \log X + (2\gamma - 1) X + O(X^{1/2})$$

(γ is Euler's const $:= \lim_{Y \to \infty} \left(\sum_{n \leq Y} 1/n - \log Y \right)$

$= 0.57721\ldots$)

$\uparrow$ is it irrational?

We need:

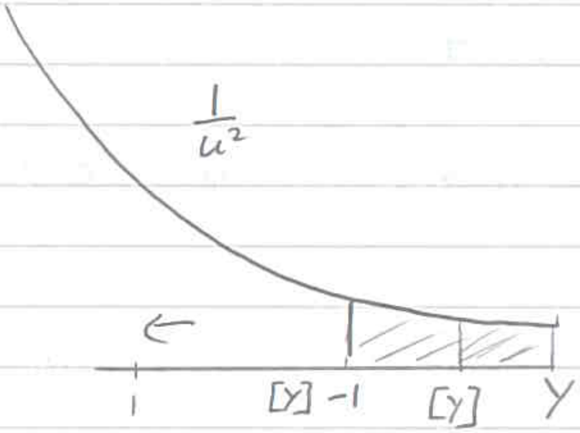
Thm (IH): $\sum_{n\le Y} 1/n = \log Y + \gamma + O(1/Y)$

Proof: $\sum_{n\le Y} 1/n = \sum_{n\le Y}\left(\frac{1}{Y} + \int_n^Y \frac{1}{u^2}\,du\right)$

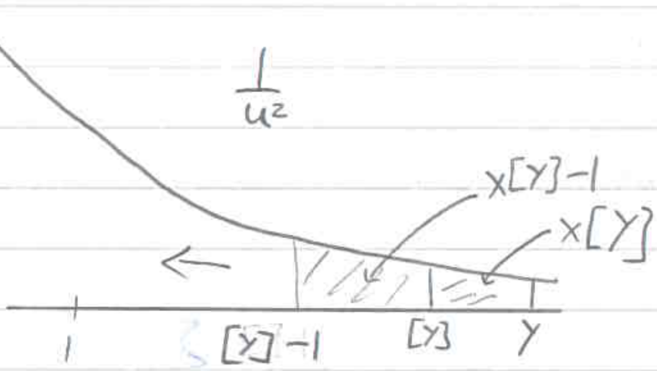
$$= \frac{\overset{\equiv \lfloor Y \rfloor}{[Y]}}{Y} + \sum_{n\le Y}\int_n^Y \frac{1}{u^2}\,du. \quad ①$$

$$= \frac{[Y]}{Y} + \int_1^Y \frac{1}{u^2}\sum_{n\le u} 1\,du.$$

①



②



Later this is called "summation by parts"

$$= \frac{[Y]}{Y} + \int_1^Y \frac{[u]}{u^2}\,du.$$

$$= \frac{[Y]}{Y} + \int_1^Y \frac{1}{u}\,du - \int_1^Y \frac{u-[u]}{u^2}\,du.$$

$$= \overbrace{1 + \Phi(1/Y)} + \log Y - \int_1^\infty \frac{u-[u]}{u^2}\,du + \int_Y^\infty \frac{u-[u]}{u^2}\,du.$$

Note: $\frac{[Y]}{Y} - \frac{Y}{Y} = \left|\frac{[Y]-Y}{Y}\right| < \frac{1}{Y}$

*

$$= \log Y + \left(1 - \int_1^\infty \frac{u-[u]}{u^2}\,du\right) + \Phi\left(\frac{1}{Y}\right)$$

$$* \left|\int_Y^\infty \frac{u-[u]}{u^2}\,du\right| \leq \int_Y^\infty \frac{|u-[u]|}{u^2}\,du$$

$$\leq \int_Y^\infty \frac{1}{u^2}\,du = \frac{1}{Y}.$$

Ex: show $\gamma = 1 - \int_1^\infty \frac{u-[u]}{u^2}\,du$ $\square$.

Proof (1G)

$$\sum_{n\leq x} d(n) = \sum_{n\leq x}\sum_{m|n} 1 = \sum_{m\leq x}\sum_{\substack{n\leq x\\ m|n}} 1 \qquad (n=rm)$$

$$= \sum_{m\leq x}\sum_{\substack{r\\ rm\leq x}} 1 = \sum_{\substack{rm\leq x\\ r,m}} 1$$

$$= \sum_{r\le x^{1/2}}\sum_{m\le \frac{x}{r}} 1 + \sum_{m\le x^{1/2}}\sum_{r\le \frac{x}{m}} 1 - \underbrace{\sum_{r\le x^{1/2}}\sum_{m\le x^{1/2}} 1}_{=\left(\underbrace{\sum_{r\le x^{1/2}} 1}_{=[x^{1/2}]}\right)\left(\sum_{m\le x^{1/2}} 1\right)}$$

This will later become a special case of the Dirichlet hyperbola method.

$$= 2\sum_{r\le x^{1/2}}\left[\frac{x}{r}\right] - [x^{1/2}]^2$$

$$= 2\sum_{r\le x^{1/2}}\frac{x}{r} + O(\underline{x^{1/2}}) - (x^{1/2} + O(1))^2$$

$$= 2x\underbrace{\left(\log x^{1/2} + \gamma + O\left(\frac{1}{x^{1/2}}\right)\right)}_{\text{Thm (1H)}} + O(x^{1/2}) - x$$

$$= x\log x - (2\gamma - 1)x + O(x^{1/2})$$

□.

$$f(x)O(g(x)) = O(f(x)g(x))$$

7/10/13.

We will study $\sigma(n)$ as $n \to \infty$ $\left(\sigma(n) = \sum_{m|n} m\right)$

For all $n \in \mathbb{N}$ $1|n$, $n|n$ so $\sigma(1) = 1$ and $\sigma(n) > n$ if $n > 1$

Thm (1F)

But, $\sigma(n) \leq n \cdot d(n) \ll_{\varepsilon} n^{1+\varepsilon}$. But we can do slightly better.

Thm (1J) $\sigma(n) \ll n \log n$

Proof $\sigma(n) = \sum_{m|n} \frac{n}{m} \leq n \sum_{m \leq n} \frac{1}{m} \ll n \log n$

Thm (1H)

$$\left\{\frac{n}{m} : m|n\right\} \longleftrightarrow \{m : m|n\}$$

Thm (1K): $\sum_{n \leq x} \sigma(n) = \frac{\pi^2}{12} x^2 + O(x \log x)$

Proof:
$$\begin{aligned}\sum_{n \leq x} \sigma(n) &= \sum_{n \leq x} \sum_{m|n} \frac{n}{m} \\ &= \sum_{m \leq x} \sum_{\substack{n \leq x \\ m|n}} \frac{n}{m} \\ &= \sum_{m \leq x} \sum_{rm \leq x} r \\ &= \sum_{m \leq x} \sum_{r \leq \frac{x}{m}} r\end{aligned}$$

$$= \sum_{m \le x} \frac{1}{2}\left[\frac{x}{m}\right]\left(\left[\frac{x}{m}\right]+1\right)$$

$$= \frac{1}{2}\sum_{m\le x}\left(\frac{x}{m} + O(1)\right)^2$$

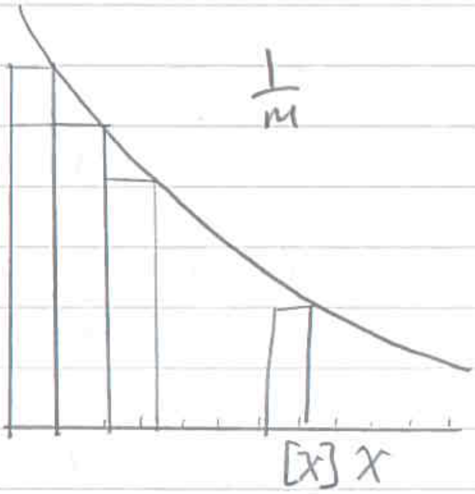
$$= \frac{1}{2}\sum_{m\le x}\frac{x^2}{m^2} + O\left(\sum_{m\le x}\frac{x}{m}\right)^{*}$$

$$+ O\left(\sum_{m\le x} 1\right)^{**}$$

$$= \frac{x^2}{2}\sum_{m=1}^{\infty}\frac{1}{m^2} + O\left(x^2\sum_{m>x}\frac{1}{m^2}\right) + O(x\log x)$$

since $\sum_{m\le x}\frac{1}{m} \le 1 + \int_1^x \frac{1}{t}\,dt = 1 + \log x$.

$$= O(\log x)$$



$* \; O\left(x\sum_{m\le x}\frac{1}{m}\right) = O(x\log x)$

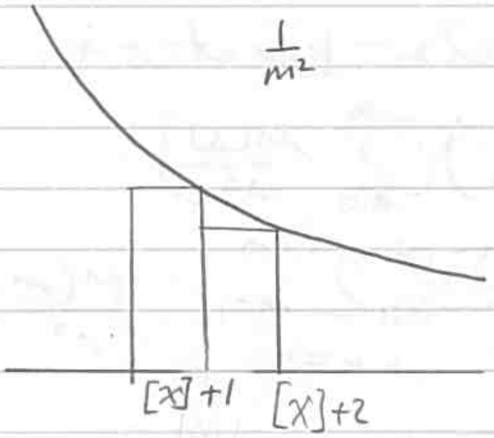
$** \; O([x])$

$$= \frac{X^2}{2}\sum_{m=1}^{\infty}\frac{1}{m^2} + O\left(X^2\sum_{m>X}\frac{1}{m^2}\right) + O(X\log X)$$

Claim : $\frac{X^2}{2}\cdot\frac{\pi^2}{6} + O(X\log X)$

To prove the claim, I need to show

$$X^2\sum_{m>X}\frac{1}{m^2} \ll X\log X$$



But $\sum_{m>X}\frac{1}{m^2} \rightarrow O\left(\frac{1}{X}\right)$

$$< \frac{1}{X} + \underbrace{\int_X^{\infty}\frac{1}{t^2}\,dt}_{\frac{1}{X}}$$

so $X^2\sum_{m>X}\frac{1}{m^2} \ll X$

$$= \frac{X^2}{2}\cdot\frac{\pi^2}{6} + O(X\log X)$$

$\square$

Def: Möbius function $\mu : \mathbb{N} \to \mathbb{C}$.

$$\mu(n) = \begin{cases} 1 & \text{if } n=1 \\ (-1)^r & \text{if } n = p_1 \cdots p_r \text{ distinct} \\ 0 & \text{otherwise.} \end{cases}$$

Thm (1L) : μ is multiplicative (Exercise)

μ is defined so that :

$$\left(\sum_{n=1}^{\infty} \frac{1}{n^s}\right)\left(\sum_{n=1}^{\infty} \frac{\mu(s)}{n^s}\right) = 1$$

i.e. $\displaystyle\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}$

$\zeta(s)$: Riemann zeta-function.

Heuristically, $\displaystyle\left(\sum_{n=1}^{\infty} \frac{1}{n^s}\right)\left(\sum_{n=1}^{\infty} \frac{\mu(s)}{n^s}\right)$

$$= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \sum_{\substack{m=1 \\ km=n}}^{\infty} \frac{\mu(m)}{n^s}.$$

$$= \sum_{n=1}^{\infty} \left(\sum_{m|n} \mu(m)\right) \cdot \frac{1}{n^s}.$$

is to be $\underline{1}$ then I'm claiming

<u>Thm</u> (1M) : $\displaystyle\sum_{m|n} \mu(m) = \begin{cases} 1 & \text{if } m=1 \\ 0 & \text{otherwise}. \end{cases}$

<u>Proof</u> : Put $f : \mathbb{N} \to \mathbb{C}$ and

$$f(n) = \sum_{m|n} \mu(m)$$

By thm 1A + 1L $\Rightarrow$ f is multiplicative

If $n=1$, trivial.

So, if $n = p_1^{u_1} \dots p_r^{u_r}$

$$f(n) = f(p_1^{u_1}) \dots f(p_r^{u_r})$$

So, I can assume $n = p^r$

$$f(p^r) = \sum_{m|p^r} \mu(m)$$

$$= \mu(1) + \mu(p) + \mu(p^2) + \dots + \mu(p^r) = 0$$

$$= 1 - 1. \qquad \square$$

Thm (IN) (Möbius inverson)
Let $f, g : \mathbb{N} \to \mathbb{C}$ and $g(n) = \sum_{m|n} f(m)$
then

$$f(n) = \sum_{m|n} \mu(m)\, g\left(\frac{n}{m}\right) \underset{\text{obv}}{=} \sum_{k|n} \mu\left(\frac{n}{k}\right) g(k)$$

Proof: $\displaystyle\sum_{m|n} \mu(m)\, g\left(\frac{n}{m}\right) \overset{\text{def}}{=} \sum_{m|n} \mu(m) \left(\sum_{k|\frac{n}{m}} f(k) \right)$

$$= \sum_{\substack{k,n \\ km|n}} \mu(m) f(k) = \sum_{k|n} f(k) \underbrace{\left(\sum_{m|\frac{n}{k}} \mu(m) \right)}_{\substack{0 \text{ unless} \\ \frac{n}{k} = 1 \iff n = k}}$$

$$= f(n)$$

11/10/13

N.B: (Change of summation)

$\{(m,k) : m|n,\ k|\frac{n}{m}\}$ $\quad n=am \quad a=rk$

$\qquad\qquad\qquad\qquad\qquad\qquad\quad n=rkm$

$\updownarrow$ bijection

$\{(m,k) : mk|n\}$

$\updownarrow$ bijection

$\{(k,m) : k|n,\ m|\frac{n}{k}\}$

Thm (IF): If $f,g : \mathbb{N} \to \mathbb{C}$. $f(n) = \sum_{m|n} \mu(\frac{n}{m}) g(m)$ for all n. then $g(n) = \sum_{m|n} f(m) \overset{\text{obv}}{=} \sum_{k|n} f(\frac{n}{k})$ for all n.

Proof: $\sum_{k|n} f(\frac{n}{k}) \overset{\text{def}}{=} \sum_{k|n} \sum_{m|\frac{n}{k}} \mu(\frac{n}{km}) g(m)$

$$= \sum_{m|n} g(m) \sum_{k|\frac{n}{m}} \mu(\frac{n}{km})$$

$$= \sum_{m|n} g(m) \underbrace{\left(\sum_{k|\frac{m}{n}} \mu(k)\right)}_{\substack{0 \text{ if } n\neq m \\ 1 \text{ if } n=m}}$$

$$= g(n)$$

□

Def (Euler function):

$\phi : \mathbb{N} \to \mathbb{C}$. $\phi(n) = \#\{m \in \{1,\dots,n\} : (m,n) = 1\}$

Thm (1Q): $\sum_{m|n} \phi(m) = n$.

Proof: For $m|n$. let $B_m = \{k \in \{1, \ldots, n\} : (k, n) = m\}$

So $\{1, \ldots, n\} = \coprod_{m|n} B_m$ $d(n)$ disjoint sets

If $k \in B_m$; $k = mk'$

and $(mk', n) = m \iff (k', \frac{n}{m}) = 1$

$k \in \{1, \ldots, n\} \iff k' \in \{1, \ldots, \frac{n}{m}\}$.

So, if $B_m' = \{k' \in \{1, \ldots, \frac{n}{m}\} : (k', \frac{n}{m}) = 1\}$

then $\# B_m = \# B_m'$

$$k \longmapsto k'$$

but $\# B_m' = \phi(\frac{n}{m})$

So $n = \# \{1, \ldots, m\}$

$$= \sum_{m|n} \# B_m$$

$$= \sum_{m|n} \# B_m'$$

$$= \sum_{m|n} \phi(\tfrac{n}{m})$$

$$\ldots = \sum_{m|n} \phi(m)$$

□

<u>Thm</u> (1R): $\phi(n) = \sum_{m|n} \mu(m) \frac{n}{m}$

$$\overset{\text{obv}}{=} n \sum_{m|n} \frac{\mu(m)}{m}$$

Note: $n = \sum_{m|n} \phi(m)$ "n": $\underset{m \mapsto m}{\mathbb{N} \to \mathbb{C}}$ f $\overset{\text{möbus}}{\Rightarrow} \sum_{m|n} \mu(m) f(\frac{n}{m}) = \frac{n}{m}$

<u>Thm</u> (1S) $\phi(n)$ is multiplicative.

<u>Proof</u>: (1A : if $f(n)$ is mult $\Rightarrow g(n) = \sum_{m|n} f(n)$ is mult)

$\underbrace{\frac{\mu(m)}{m}}_{\text{mult.}}$, μ is mult, "m" is mult.

define $f: \mathbb{N} \to \mathbb{C}$, $f(n) = \frac{\mu(n)}{n}$

but $\phi(n) = \underbrace{n \underbrace{\sum_{m|n} f(m)}_{\text{mult.}}}_{\text{mult}}$ (THM 1A)

□

Thm (1T): Suppose $n \in \mathbb{N}$ $n \geq 1$, $n = p_1^{u_1} \dots p_r^{u_r}$, p_i distinct. Then

$$\phi(n) = n \prod_{j=1}^{r} \left(1 - \frac{1}{p_j} \right) \overset{\text{obv.}}{=} \prod_{j=1}^{r} p_i^{u_j - 1} (p_j - 1)$$

Proof: For all primes p and $u \in \mathbb{N}$,

$$\frac{\phi(p^u)}{p^u} \underset{\text{THM 1R}}{=} \sum_{m \mid p^u} \frac{\mu(m)}{m} = 1 + \frac{\mu(p)}{p} = 1 - \frac{1}{p}.$$

The result follows since ϕ is mult. $\square$.
(THM 1S).

As before, we study $\phi(n)$ as $n \to \infty$

$\phi(1) = 1$. $\phi(n) < n$ if $n > 1$

Heuristically, $\sigma(n)$ and $\phi(n)$ should be "inversely related".

-If n has many factors, $\sigma(n)$ should be "large" with respect to n. But then many of $1, \dots, n$ cannot be coprime to n, so $\phi(n)$ should be small - and vice versa.

Thm (1U): $\frac{1}{2} \leq \frac{\sigma(n)\phi(n)}{n^2} \leq 1$ for all $n \in \mathbb{N}$.

Proof: If $n=1$, trivial.

If $n>1$, $n = p_1^{u_1} \dots p_r^{u_r}$ p_i distinct.

$$\sigma(n) \underset{\text{Thm 1B}}{=} \prod_{j=1}^{r} \frac{p_j^{u_j+1}-1}{p_j-1}$$

$$= n \prod_{j=1}^{r} \underbrace{\frac{1-p_j^{-u_j-1}}{1-p_j^{-1}}}_{\substack{n \cdot \text{large multiple} \\ u_j \to \infty}}$$

$$\phi(n) = n \prod_{j=1}^{r} \underbrace{\left(1-\frac{1}{p_j}\right)}_{<1}$$

$$\frac{\sigma(n)\phi(n)}{n^2} = \prod_{j=1}^{r}\left(1-p_j^{-u_j-1}\right) < 1$$

So the upper bound is known.

But $\prod_{j=1}^{r}\left(1-p_j^{-u_j-1}\right) \geq \prod_{p|n}\left(1-p^{-2}\right)$

$$\geq \prod_{m=2}^{n}\left(1-\frac{1}{m^2}\right)^{*}$$

Now $\left(1-\frac{1}{m^2}\right) = \frac{m^2-1}{m} = \frac{(m-1)(m+2)}{m^2}$

$$* \frac{\cancel{(2-1)(2+1)}}{\cancel{2}\cdot 2} \cdot \frac{\cancel{(3-1)(3+1)}}{\cancel{3\cdot 3}} \cdots \frac{\cancel{(n-1)}(n+1)}{\cancel{n}\cdot n}$$

$$= \frac{n+1}{2n} = \frac{1+\frac{1}{n}}{2} > \frac{1}{2} \quad \square .$$

<u>Thm</u> (IV) : $\phi(n) \gg \frac{n}{\log n}$

<u>Proof</u>: $\sigma(n) \overset{\text{Thm IJ}}{\ll} n \log n$, but

$$\phi(n) \underset{\text{Thm IU}}{>} \frac{n^2}{2\sigma(n)} \gg \frac{n}{\log n} \qquad \square$$

Now we average ϕ.

<u>Thm</u> (IW) : $\sum_{n \le x} \phi(n) = \frac{3}{\pi^2} x^2 + O(x \log x)$

note similarity with $\sigma(n)$.

<u>Proof</u>: $\sum_{n\le x} \phi(n) \underset{\text{Thm IR}}{=} \sum_{n \le x} \sum_{m \mid n} \mu(m) \frac{n}{m}$

$$= \sum_{m \le x} \mu(m) \sum_{\substack{n \le x \\ m \mid n}} \frac{n}{m} = \sum_{m \le x} \mu(m) \sum_{r \le \frac{x}{m}} r$$

$$= \sum_{m \le x} \mu(m) \frac{1}{2} \left[\frac{x}{m}\right] \left(\left[\frac{x}{m}\right] + 1 \right)$$

$$= \frac{1}{2}\sum_{m\le x} \mu(m)\left(\frac{x}{m} + O(1)\right)^2$$

$$= \frac{x^2}{2}\sum_{m\le x}\frac{\mu(m)}{m^2} + O\left(x\sum_{m\le x}\frac{1}{m}\right)$$

$$+ O\left(\sum_{m\le x} 1\right)$$

$$= \frac{x^2}{2}\sum_{m=1}^{\infty}\frac{\mu(m)}{m^2} + O\left(x^2\sum_{m>x}\frac{1}{m^2}\right)$$

$$+ \underbrace{O(x\log x)}_{\text{Proof THM 1K}}$$

$$= \frac{x^2}{2}\sum_{m=1}^{\infty}\frac{\mu(m)}{m^2} + \underbrace{O(x\log x)}_{\text{Proof of THM 1K}}$$

So, we need $\sum_{m=1}^{\infty}\frac{\mu(m)}{m^2} = \frac{6}{\pi^2}$

i.e. $\underbrace{\left(\sum_{m=1}^{\infty}\frac{1}{m^2}\right)}_{\frac{\pi^2}{6}}\left(\sum_{m=1}^{\infty}\frac{\mu(m)}{m^2}\right) = 1$

$$\left(\sum_{m=1}^{\infty}\frac{1}{m^2}\right)\left(\sum_{k=1}^{\infty}\frac{\mu(k)}{k^2}\right) \overset{\text{expect}}{=} \sum_{n=1}^{\infty}\frac{1}{n^2}\underbrace{\sum_{k\mid n}\mu(k)}_{\substack{0 \text{ if } n>1\\ 1 \text{ if } n=1}} = 1$$

So need...

<u>THM</u>: Let $A=\sum_{n=1}^{\infty} a_n$, $B=\sum_{n=1}^{\infty} b_n$ and assume that $\sum_{n=1}^{\infty} |a_n|$ is conv. Let

$$c_n = \sum_{m|n} a_m b_{\frac{n}{m}}$$

and $C_N = \sum_{n=1}^{N} c_n$. Then $\{C_n\}$ converges to AB as $N \to \infty$.

This proof is wrong

Proof: $$C_N \overset{def}{=} \sum_{n=1}^{N} \sum_{m|n} a_m b_{\frac{n}{m}} = \sum_{m=1}^{N} a_m \sum_{\substack{n=1 \\ m|n}}^{N} b_{\frac{n}{m}}$$

$$= \sum_{m=1}^{N} a_m \sum_{k=1}^{[\frac{N}{m}]} b_k = \sum_{m=1}^{N} a_{[\frac{N}{m}]} \sum_{k=1}^{m} b_k$$

$$= \sum_{m=1}^{N} a_{[\frac{N}{m}]} (B_m - B) + A_N B.$$

$(B_m = \sum_{k=1}^{m} b_k)$ $(A_N = \sum_{n=1}^{N} a_n)$

<u>Fix $\varepsilon > 0$</u>: $\exists N_0$ st if $N \geq N_0$

$$\Rightarrow |B_N - B| < \frac{\varepsilon}{3\sum_{n=1}^{\infty} |a_n|}$$

by convergence of $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} |a_n|$
$\exists L$ st if $N \geq L$ $|A_n - A| < \frac{\varepsilon}{3} |B|^{-1}$ by convergence of $\sum_{n=1}^{\infty} a_n$.

$\exists M$ st. if $m \geq M$. $|a_m| < \dfrac{\varepsilon}{3N_0 \max\limits_{N \leq N_0} |B_N - B|}$

because $|a_m| \to 0$ as $m \to 0$ by convergence of $\sum_{n=1}^{\infty} |a_n|$.

Put $N \geq \max\{L, N_0 M\}$.

$$|C_N - AB| = \left|\sum_{m=1}^{N} a_{[\frac{N}{m}]}(B_m - B) + (A_N - A)B\right|$$

$$\leq \underbrace{\sum_{m=1}^{N_0} |a_{[\frac{N}{m}]}||B_m - B|}_{} + \sum_{m=N_0+1}^{N} |a_{[\frac{N}{m}]}|\underbrace{|\overset{\leq \frac{\varepsilon}{3\sum_{n=1}^{\infty}|a_n|}}{B_m - B}|}_{m \geq N_0} + |A_N - A|\underbrace{|B|}_{N \geq L}$$

$$\left[\frac{N}{m}\right] \geq \left[\frac{N}{N_0}\right] \geq \left[\frac{N_0 M}{N_0}\right] = M$$

so

$< \frac{\varepsilon}{3}$

so $< \frac{\varepsilon}{2|B|}|B| = \frac{\varepsilon}{3}$

$< \varepsilon$

$\square$

(Proof not on the exam.)

14/10/10.

$$\underbrace{\sum a_n \quad \sum b_n}_{\text{Converges}} \qquad c_n = \sum_{m|n} a_m b_{\frac{n}{m}}$$

$$\sum_{n=1}^{\infty} c_n$$

Exercise: Fixed the proof.
Find a counter-example.

Thm: $\sum a_n$, $\sum b_n$ abs. conv. series.

$$c_n = \sum_{mk=n} a_m b_k$$

then $(\sum_{n=1}^{\infty} a_n)(\sum_{n=1}^{\infty} b_n) = \sum_{n=1}^{\infty} c_n$.

Proof: So:

$$\left(\lim_{N\to\infty} \sum_{n=1}^{N} |a_n|\right)\left(\lim_{N\to\infty} \sum_{n=1}^{N} |b_n|\right)$$
$$= \lim_{N\to\infty}\left(\sum_{m=1}^{N}\sum_{k=1}^{N} |a_m b_k|\right) \text{ exist.}$$

Claim: If $\sum \alpha_n$ is abs convergent $\sum_{n=1}^{\infty} \alpha_n = S'$ then $\sum_{n=1}^{\infty} \alpha_{\sigma(n)} = S'$ for any bijcetion $\sigma : \mathbb{N} \xrightarrow{\sim} \mathbb{N}$.

Proof: Fix $\varepsilon > 0$: $\exists N_1$ st $\sum_{n=N}^{\infty} |\alpha_n| < \varepsilon/2$ for all $N \geq N_1$ $\exists N_2 \geq N_1$ st $\{1, \dots, N_1\} \subseteq \{\sigma(1), \dots, \sigma(N_2)\}$

If $N \geqslant N_2$.

$$|\sum_{n=1}^{N} \alpha_{\sigma(s)} - S| \leq |\sum_{n=1}^{N_1} \alpha_n - S| + \sum_{\substack{\sigma(n) > N_1 \\ n \leq N}} |\alpha_{\sigma(n)}|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \square.$$

$$\sum_{n=1}^{\infty} a_n \sum_{n=1}^{\infty} b_n = (a_1 b_1) + (a_2 b_1 + a_2 b_2 + a_1 b_2) + (a_3 b_1 + a_3 b_2 + a_3 b_3 + a_2 b_3 + a_1 b_3 + \dots$$ exist

Choose $\sigma : \mathbb{N} \xrightarrow{\sim} \mathbb{N}$ so that we sum up:

$\lim_{N \to \infty} \sum_{n=1}^{N} \sum_{mk=n} a_m b_k$ i.e $(a_1 b_1) + (a_1 b_2 + a_2 b_1) + (a_1 b_3 + a_3 b_1) + (a_1 b_4 + a_2 b_2 + a_4 b_1) + \dots$ exist.

Then $(\sum_{n=1}^{\infty} a_n)(\sum_{n=1}^{\infty} b_n) = \sum_{n=1}^{\infty} \sum_{mk=n} a_m b_k$ by the claim

(not on exam) $\square$.

Now, we'll formalise our treatment of arithemtic functions.

Let $\mathcal{A}$ denote the class of all arthmetic functions.

Let $\mathcal{M}$ denote the class of all multiplicative arithmetic functions.

Def: Let $f, g \in \mathcal{A}$. Define $f * g : \mathbb{N} \to \mathbb{C}$ by $(f*g)(n) = \sum_{m|n} f(m) g(\frac{n}{m})$ for all $n \in \mathbb{N}$. We call this the Dirichlet convolution of f and g.

Ex: For all $f, g, h \in \mathcal{A}$, $f*g = g*f$ and $(f*g)*h = f*(g*h)$.

Def: Let $I : \mathbb{N} \to \mathbb{C}$ defined by

$$I(n) = \begin{cases} 1 & \text{if } n=1 \\ 0 & \text{otherwise.} \end{cases}$$

Ex: $f * I = I * f = f$.

There is not always $g \in \mathcal{A}$ st $f*g = I$ e.g. $f(n) = 0$ for all $n \in \mathbb{N}$.

Thm (1X') Let $f \in \mathcal{A}$ the following are equivalent:

i) $f(1) \neq 0$.

ii) $\exists ! \, g \in A$ st $g*f = I$.

$\exists !$ = "there exist a unique".

Proof: (ii) $\Rightarrow$ $1 = I(1) = (f*g)(1)$

$$= f(1)g(1)$$

$$\Rightarrow f(1) \neq 0.$$

Suppose i). Define $g \in \mathcal{A}$.

$$g(1) = \frac{1}{f(1)}, \quad g(n) = -\frac{1}{f(1)} \sum_{\substack{m|n \\ n>1}} f(m) g\left(\tfrac{n}{m}\right) \quad (n>1)$$

If $n=1$ $(f*g)(n) = f(1)g(1) = 1 \; (= I(1))$

If $n>1$ $(f*g)(n) \overset{def.}{=} \sum_{m|n} f(m) g\left(\tfrac{n}{m}\right)$

so $$\frac{(f*g)(n) - \overset{(m=1)}{f(1)g(n)}}{-f(1)} = g(n)$$

i.e. $-g(1)(f*g)(n) + g(n) = g(n)$

i.e. $-g(1)(f*g)(n) = 0.$

$\Rightarrow (f*g)(n) = 0.$

$$= I(n)$$

Ex: Inverse is unique

So:

$$\mathcal{A}^{`} = \{ f \in \mathcal{A} : f(1) \neq 0 \}.$$

is an abelian group.

Thm (14): $M^{`} = \{ f \in \mathcal{M} : f(1) = 1 \}$ from an ablian group.

Note, if $f \in \mathcal{M}$ is not identically zero,

$$f(1) f(n) = f(n) \text{ so } f(1) = 1.$$

Proof: Note $I \in \mathcal{M}$

Let $f, g \in \mathcal{M}^{'}$ and $(n, m) = 1$.

$$(f * g)(nm) \overset{def.}{=} \sum_{d | nm} f(d) g\left(\frac{nm}{d}\right)$$

$$= \sum_{d_1 | n} \sum_{d_2 | m} f(d_1 d_2) g\left(\frac{nm}{d_1 d_2}\right)$$

$$= \left(\sum_{d_1 | n} f(d_1) g\left(\frac{n}{d_1}\right)\right)\left(\sum_{d_2 | n} f(d_2) g\left(\frac{m}{d_2}\right)\right)$$

$$= (f * g)(n) \ (f * g)(m)$$

So $(f * g) \in \mathcal{M}$ and $(f * g)(1) \overset{def}{=} f(1) g(1)$ $= 1 \cdot 1 = 1$ so $(f * g) \in \mathcal{M}^{'}$

Given $f \in \mathcal{M}'$, we need $g \in \mathcal{M}'$ st $f * g = I$.

Now, f has an inverse $h \in \mathcal{A}'$.

Define $g \in \mathcal{M}'$ by $g(1) = 1$ and $g(p^k) = h(p^k)$ for all primes p, $k \in \mathbb{N}$.

and

$$g(n) = \prod_{p^k \| n} g(p^k)$$ where $\|$ = "largest power of p dividing".

$*$ closed in $\mathcal{M}$.

$\Rightarrow (f * g) \in \mathcal{M}'$ $f * g \in \mathcal{M}'$

$$\text{So } (f*g)(n) = \prod_{p^k \| n} (f*g)(p^k)$$

$$= \prod_{p^k \| n} (f*g)(p^k)$$

$$= \prod_{p^k \| n} I(p^k) = I(n) \qquad \square.$$

18/10/13

Thm (1Z) : (Note $u : \mathbb{C} \to \mathbb{R}$, $u(n) = 1 \ \forall n$)

i) $M * u = I$

ii) if $f \in \mathcal{A}$ and $g = f * u$ then $f = \mu * g$ (Möbius inverson).

iii) If $g \in \mathcal{A}$, $f = \mu * g$ then $g = f * u$.

Proof:

i) Same as the proof of Thm (1M)

ii) $\mu * g = \mu * (u * f) = (\mu * u) * f = I * f = f$.

iii)
$$\begin{aligned} f * u = (\mu * g) * u &= (g * \mu) * u \\ &= g * (\mu * u) \\ &= g * I \\ &= g \end{aligned}$$

□

—/—

Summation by parts (imporant topic)

Recall: in the proof of Thm 1H we wrote

$$\sum_{n \le y} \frac{1}{n} = \frac{[y]}{y} + \int_n^y \frac{1}{u^2} \left(\sum_{n \le u} n \right) du$$

This is a special case of:

Thm: Let $a : \mathbb{N} \to \mathbb{C}$ be an arithmetic functcoi. Denote the partial sums of a by

$$A(x) = \sum_{r \le x} a(r)$$

Let $0 \le y < x$; $f : \mathbb{R} \to \mathbb{C}$ be a continuously differentiable on $[y, x]$. Then

$$\sum_{y<r\leq x} a(r)f(r) = A(x)f(x) - A(y)f(y) - \int_y^x A(t)f'(t)\,dt.$$

Proof: Let $n, m \in \mathbb{N} \cup \{0\}$ st $n \leq x < n+1$, $m \leq y < m+1$

So $\sum_{y<r\leq x} a(r)f(r) = \sum_{r=m+1}^{n} a(r)f(r)$ $*$

Now $a(r) = A(r) - A(r-1)$, so

$* = [A(m+1) - A(m)]f(m-1) + \dots + [A(n) - A(n-1)] \cdot f(n) = -A(m)f(m+1)$

$+ \sum_{r=m+1}^{n-1} A(r)[f(r) - f(r+1)]$

$+ A(n)f(n)$

(which is called the "discrete version" of summation by parts)

Now, $\sum_{r=m+1}^{n-1} A(r)[f(r) - f(r+1)]$

$= -\sum_{r=m+1}^{n-1} A(r) \int_r^{r+1} f'(t)\,dt = **$

For $r \leq t < r+1$, we have $A(t) = A(r)$, so

$$** = -\sum_{r=m+n}^{n-1} \int_r^{r+1} A(t) f'(t)\, dt$$

→ Continuous almost everywhere

value at $A(r+1)$ doesn't affect integrae

$$= -\left\{ \int_{m+1}^{m+2} A(t) f'(t)\, dt + \dots + \int_{n-1}^{n} A(t) f'(t)\, dt \right\} = -\int_{m+1}^{n} A(t) f'(t)\, dt \quad ③$$

For $n \leq t \leq x < n+1$, $A(t) = A(n)$ $(= A(x))$

So $A(x)f(x) - A(n)f(n) = A(n)[f(x) - f(n)]$

$$= \int_n^x A(t) f'(t)\, dt \Rightarrow A(n)f(n) = A(x)f(x) - \int_n^x A(t) f'(t)\, dt \quad ①$$

For $y \leq t < m+1$, $A(t) = A(m)$

So $A(m)f(m+1) - A(y)f(y) = A(m)[f(m+1) - f(y)]$

$$= \int_y^{m+1} A(t) f'(t)\, dt \Rightarrow -A(m)f(m+1)$$

$$= -A(y)f(y) - \int_y^{m+1} A(t) f'(t)\, dt \quad ②$$

Now we substitute ①, ②, ③ into the "discrete version" to get.

$$\sum_{y\le r\le x} a(r)f(r) = -A(y)f(y) - \int_y^{m+1} A(t)f'(t)\,dt$$

$$- \int_{m+1}^{n} A(t)f'(t)\,dt + A(x)f(x) - \int_n^x A(t)f'(t)\,dt$$

As required. □.

In Th(1.1) let $a = u$, $f(x) = 1/x$, $[y, x] -$ $[1, x]$ then

$$\sum_{n\le x} a(n)f(n) = a(1)f(1) + \sum_{1<n\le x} a(n)f(n)$$

$$= \sum_{1<n\le x} \frac{1}{n}$$

$$= 1 + \underbrace{\frac{[x]}{x}}_{A(x)f(x)} \underbrace{-1}_{A(y)f(y)'} + \int^x \underbrace{\frac{1}{t^2}}_{f'(t)} \Big(\underbrace{\sum_{n\le t} 1}_{A(t)}\Big)\,dt$$

$$= \frac{[x]}{x} + \int_1^x \frac{1}{t^2}\Big(\sum_{n\le t} 1\Big)\,dt \qquad \square.$$

Dirchlet's Hyperbolic method.

In the proof of Thm (1G) we convinced ourselves of the following result:

$$\sum_{n \leq x} d(n) = \sum_{r \leq x^{1/2}} \sum_{m \leq \frac{x}{m}} 1 + \sum_{m \leq x^{\frac{1}{2}}} \sum_{r \leq \frac{x}{m}} 1 - \sum_{r \leq x^{\frac{1}{2}}} \sum_{m \leq x^{\frac{1}{2}}} 1$$

(where $n = rm$).

We can generalise this as follows:

Theorem: If $1 < y < x$, $a, b : \mathbb{N} \to \mathbb{C}$ with

$$A(x) = \sum_{n \leq x} a(n), \quad B(x) = \sum_{n \leq x} b(x)$$

then $\sum_{n \leq x} (a * b)(n) = \sum_{r \leq y} a(r) B(\frac{x}{r})$

$+ \sum_{m \leq \frac{x}{y}} b(m) A(\frac{x}{m}) - A(y) B(\frac{x}{y})$

Proof:
Claim:

$$\sum_{n \leq x} (a * b)(n) = \sum_{n \leq x} \sum_{rm = n} a(r) b(m)$$

$$= \sum_{rm \leq x} a(r) b(m) = \sum_{r \leq x} \sum_{m \leq \frac{x}{r}} b(m)$$

$$= \sum_{r \leq x} a(r) B(\tfrac{x}{r})$$

So, $\sum_{n\le x} (a*b)(n) = \sum_{rm\le x} a(r)b(m)$

Let S_1 be the sum of all terms with $r\le y$
S_2 be the sum of all terms with $m\le x/y$.

Note that every term belongs to at least one of these sums so,

$$S_1 = \sum_{r\le y} a(r) \sum_{m\le \frac{x}{r}} b(m) = \sum_{r\le y} a(r) B(\tfrac{x}{r})$$

$$S_2 = \sum_{m\le y} b(m) \sum_{r\le \frac{x}{m}} a(r) = \sum b(m) A(\tfrac{x}{m})$$

But S_1+S_2 contains those terms satisfying $r\le y$ and $m\le x/y$ twice, and their sum is:

$$A(y)\, B\left(\frac{x}{y}\right)$$

<u>Ex</u>: Find a, b, y in Thm(1G).

We know that there are infinitely many primes. The following is slightly stronger.

Thm: $\sum_{p \text{ prime}} \frac{1}{p}$ is divergent.

Proof: For $x \geq 2$, let $P_x = \prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{-1}$

So $\log P_x = -\sum_{p \leq x} \log(1 - 1/p)$

$$= S_1 + S_2$$

where $S_1 = \sum_{p \leq x} \frac{1}{p}$ and $S_2 = \sum_{p \leq x} \sum_{n=2}^{\infty} \frac{1}{np^n}$

To see this recall $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots$

$$-\sum_{p \leq x} \log\left(1 - \frac{1}{p}\right) = \sum_{p \leq x} \frac{1}{p} + \frac{1}{2p^2} + \ldots$$

Now, $0 \leq \sum_{n=2}^{\infty} \frac{1}{p^n} \leq \sum_{n=2}^{\infty} \frac{1}{p^n} = \frac{1}{p^2} \sum_{n=0}^{\infty} \frac{1}{p^n}$

$$= \frac{1}{p^2}\left(\frac{1}{1 - \frac{1}{p}}\right) = \frac{1}{p(p-1)}$$

So $0 \leq S_2 \leq \sum_{p} \frac{1}{p(p-1)} \leq \sum_{n=2}^{\infty} \frac{1}{n(n-1)}$

$$= \sum_{n=2}^{\infty} \left(\frac{-1}{n} + \frac{1}{n-1} \right) = \lim_{N \to \infty} \sum_{n=2}^{N} \left(\frac{-1}{n} + \frac{1}{n+1} \right)$$

$$= \lim_{N \to \infty} 1 - \frac{1}{N} = 1$$

Put $P_x = \underbrace{\prod_{p \le x} \sum_{n=0}^{\infty} \frac{1}{p^n}}_{\underbrace{\prod_{p \le x} 1 + \frac{1}{p} + \frac{1}{p^2} + \dots}_{\text{reciprocals of all products of primes powers for primes } p \le x.}} \geq \sum_{n \le x} \frac{1}{n} \longrightarrow \infty$

as $x \to \infty$

So $S_1 \to \infty$ as $x \to \infty$

$\square$.

—/—

Suppose $x = 10$, $n = 6 = 2 \cdot 3$.

$2, 3 \le 10$.

$$\prod_{p \le 10} \sum_{n=0}^{\infty} \frac{1}{p^n} = \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots\right)\left(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots\right)$$

$$\cdot \left(\frac{1}{\text{powers of } 5} \right) \leq \left(\frac{1}{\text{powers of } 7} \right)$$

—/—

__Def__: For any real no. $X \geq 2$, put:

$$\pi(X) = \sum_{p \leq X} 1 .$$

Later we will prove $\pi(x) \sim \frac{x}{\log x}$

i.e. $\lim_{x \to \infty} \frac{\pi(x) . \log x}{x} = 1$ (P.N.T)

prime number theorem.

For now we will concentrate on the weaker result of Chebyshev: there exist positive absolute constants c_1, c_2 st:

$$\frac{c_1 x}{\log x} = \leq \pi(x) \leq \frac{c_2 x}{\log x}$$

__Def__: Von Mangoldt function $\Lambda : \mathbb{N} \to \mathbb{C}$

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \ \ p \text{ prime } \ k \in \mathbb{N}. \\ 0 & \text{otherwise}. \end{cases}$$

__Thm__ (2B) $\underset{(\Lambda * U)(n)}{\underset{\| \text{def}}{\sum \Lambda(m)}} = \log n$

Proof: If $n=1$, trivial

If $n>1$, $n = p_1^{u_1} \cdots p_r^{u_r}$, p_i distinct.

$\Lambda(m)$ for $m|n$, is only non-zero for $m = p_i^{v_i}$ $1 \leq v_i \leq u_i$

So
$$\sum_{m|n} \Lambda(m) = \sum_{i=1}^{r} \sum_{v_i=1}^{u_i} \log p_i$$

$$= \sum_{i=1}^{r} u_i \log p_i$$

$$= \sum_{i=1}^{r} \log p_i^{u_i} = \log p_1^{u_1} \cdots p_r^{u_r} = \log n$$

We will need an "average" of Λ (to prove Chebyshev)

Thm (2C') $\displaystyle\sum_{m \leq x} \Lambda(m) \left[\frac{x}{m}\right]$

$$= x \log x - x + O(\log x)$$

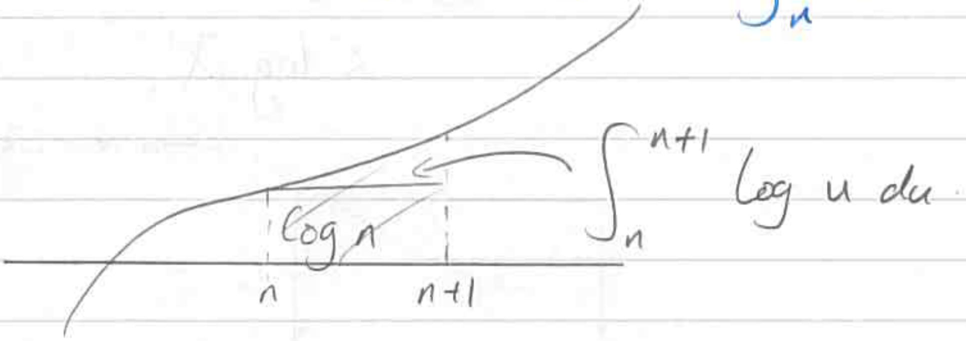
Proof:
$$\sum_{n \leq x} \log n \overset{2B}{=} \sum_{n \leq x} \sum_{m|n} \Lambda(m)$$

$$= \sum_{m \leq x} \Lambda(m) \sum_{\substack{n \leq x \\ m|n}} 1$$

$$= \sum_{m \leq x} \Lambda(m) \left[\frac{x}{m}\right]$$

So we must show result for $\sum_{n \le x} \log n$.

But $\log n$ is increasing so $\log n \le \int_n^{n+1} \log u \, du$.



So $\sum_{n \le x} \log n - \log(x+1) \le \sum_{n \le x} \int_n^{n+1} \log u \, du - \log(x+1)$

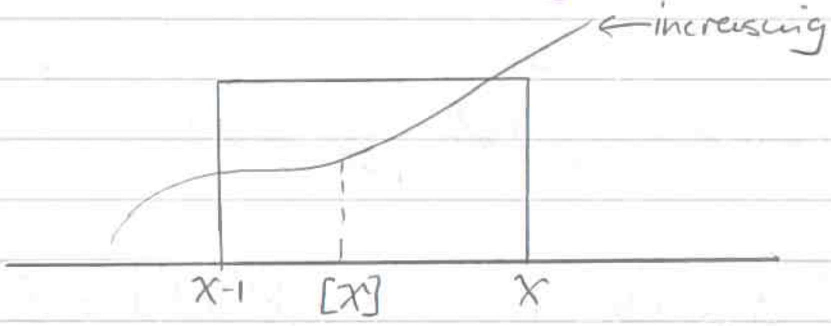
$$= \underbrace{\int_1^{[x]} \log u \, du}_{\le \int_1^{x+1} \log u \, du} - \underbrace{\log(x+1)}_{\ge \int_x^{x+1} \log u \, du}$$

$$\le \int_1^{x} \log u \, du$$

But $\log n \ge \int_{n-1}^{n} \log u \, du$ so

$$\sum_{n \le x} \log n \ge \sum_{2 \le n \le x} \int_{n-1}^{n} \log u \, du = \int_1^{[x]} \log u \, du.$$

$$= \int_1^x \log u \, du - \underbrace{\int_{[x]}^x \log u \, du}_{\leq \log x}.$$



Now $\int_1^x \log u \, du = x \log x - x + 1$

(by parts with $\log u = \log u \cdot 1$)

□

21/10/13.

Thm (1C) : $\sum_{m \leq x} \Lambda(m) \left[\frac{x}{m} \right]$

$$= x \log x - x + O(\log x)$$

We have :

$$x \log x - x + 1 - \log x \leq \sum_{n \leq x} \log n$$

$$\leq \log(x+1) + x \log x - x + 1 .$$

$$1 - \log x \leq \sum_{n \leq x} \log n - (x \log x - x)$$

$$\leq \log(x+1) + 1$$

i.e $\sum_{n \leq x} \log n - (x \log x - x) \ll \log x$

□

Now, to prove Chebyshev's theorem, we'll need the following :

Thm (2D): There exist absolute positive c_3, c_4 st

$$\sum_{m \leq x} \Lambda(m) \geq \frac{1}{2} x \log 2 \quad \text{if } x \geq c_3 \; (1)$$

$$\sum_{\frac{x}{2} < m \leq x} \Lambda(m) \leq c_4 x \quad \text{if } x \geq 0 \quad (2)$$

Proof: If $m \in \mathbb{N}$ and $x/2 < m \leq x$. then

$$\left[\frac{x}{2m}\right] = 0.$$

So. $\sum_{m \leq x} \Lambda(m)\left(\left[\frac{x}{m}\right] - 2\left[\frac{x}{2m}\right]\right) =$

$$\sum_{m \leq x} \Lambda(m)\left[\frac{x}{m}\right] - 2\sum_{m \leq \frac{x}{2}} \Lambda(m)\left[\frac{x}{2m}\right]$$

$$\overset{\text{THM(2C)}}{=} \left[x\log x - x + O(\log x)\right]$$

$$-2\left[\underbrace{\frac{x}{2}\log\frac{x}{2}}_{\log x - \log 2} - \frac{x}{2} + O(\log x)\right]$$

$$= x\log 2 + O(\log x)$$

So, there exists a pos. abs. const c_5 st

$$\frac{1}{2}x\log 2 < \sum_{m \leq x} \Lambda(m)\left(\left[\frac{x}{m}\right] - 2\left[\frac{x}{2m}\right]\right) < c_5 x$$

provided $x \geq c_3$, for some pos c_3.

Consider the funcion $\underbrace{[\alpha]}_{\leq \alpha} - 2\underbrace{[\tfrac{\alpha}{2}]}_{> \frac{\alpha}{2} - 1} < 2$

But $[\alpha] - 2\left[\frac{\alpha}{2}\right]$ is an integer.

So $0 \leq [\alpha] - 2\left[\frac{\alpha}{2}\right] \leq 1$

$$\sum_{m \leq x} \Lambda(m) \geq \sum_{m \leq x} \left(\left[\frac{x}{m}\right] - 2\left[\frac{x}{2m}\right] \right) \Lambda(m)$$

$$> \frac{1}{2} x \log 2 \quad \text{for } x \geq c_3$$

$$\Rightarrow (1) \quad \square$$

If, however, $x/2 < m \leq x$

$\Rightarrow \left[\frac{x}{m}\right] = 1$ and $\left[\frac{x}{2m}\right] = 0$.

So, (assuming $x \geq c_3$)

$$\sum_{\frac{x}{2} < m \leq x} \Lambda(m) = \sum_{\frac{x}{2} < m \leq x} \Lambda(m) \left(\left[\frac{x}{m}\right] - 2\left[\frac{x}{2m}\right] \right)$$

$$\leq \sum_{m \leq x} \Lambda(m) \left(\left[\frac{x}{m}\right] - 2\left[\frac{x}{2m}\right] \right)$$

$$\leq c_5 x.$$

For $x < c_3$,

$\sum_{\frac{x}{2} < m \leq x} \Lambda(m)$ take finitely many values, which we bound by a const. times x.

We obtain constants $d_1, \dots, d_n$

Let $c_4 = \max\{d_1, \dots, d_n, c_5\} \Rightarrow (2)$

$\square$.

<u>Th</u> (2E) (Chebyshev):
There exist positive absolute constants c_1 and c_2 st for real $x \geq 2$.

$$\frac{c_1 x}{\log x} \leq \pi(x) < \frac{c_2 x}{\log x}$$

<u>Proof</u>: (Lower bound):

$$\sum_{m \leq x} \Lambda(m) = \sum_{\substack{p^n \leq x \\ p, n}} \log p = \sum_{p \leq x} \log p \sum_{1 \leq n \leq \left[\frac{\log x}{\log p}\right]} 1$$

$$= \sum_{p \leq x} \log p \left[\frac{\log x}{\log p}\right] \overset{*}{\leq} \pi(x) \log x .$$

$$* \left[\frac{\log x}{\log p}\right] = \frac{\log x}{\Delta} \quad \Delta \geq \log p$$

$$\overset{\text{Thm }(2D)(1)}{\Longrightarrow} \pi(x) \geq \left.\frac{\frac{1}{2} x \operatorname{Log} 2}{\operatorname{Log} x}\right\} \text{ if } x \geq c_3.$$

$$\left(\tfrac{1}{2}\operatorname{Log} 2\right)\cdot\left(\operatorname{Log} x\right)^{-1}$$

For $2 \leq x < c_3$.

$$\pi(x) > \text{const}([x]) . \frac{x}{\operatorname{Log} x}$$

since $\pi(2) = 1$ so, we obtain finitely many pos. consts $\{d_1, \dots, d_n\}$.

Let $c_1 = \min\{d_1, \dots, d_n, \frac{1}{3}\log 2\}$ ∎.

(Upper bound): By Thm 2D(2)

$$\sum_{\frac{x}{2^{j+1}} < p \leq \frac{x}{2^j}} \operatorname{Log} p \leq c_4 \frac{x}{2^j} \quad \text{for all } j \in \mathbb{N} \text{ and } x \geq 0.$$

Since $x/2^j \geq 0$, summing over just primes rather all powers of primes.

Suppose $x \geq 2$. Let $k \in \mathbb{N}$ st $2^k \leq x \leq 2^{k+1}$
Then:

$$\sum_{x^{1/2} < p \leq x} \operatorname{Log} p \leq \sum_{j=0}^{k} \sum_{\frac{x}{2^{j+1}} < p \leq \frac{x}{2^j}} \operatorname{Log} p$$

since if $x^{\frac{1}{2}} \leq 2^{k+1}$, $x^{1/2} = \frac{x}{x^{1/2}} \geq \frac{x}{2^{k+1}}$

Using $\sum_{\frac{x}{2^{j+1}}<p\leq\frac{x}{2^j}} \log p \leq c_4 \frac{x}{2^j}$

$$\leq \sum_{j=0}^{k} c_4 x \cdot 2^{-j} = c_4 x \underbrace{\sum_{j=0}^{k} 2^{-j}}_{2} < 2c_4 x$$

So $\sum_{x^{1/2}<p<x} 1 < \sum_{x^{1/2}<p<x} \frac{\log p}{\log x^{1/2}} < \frac{1}{\log x^{1/2}} \cdot 2c_4 \cdot x.$

$$= \frac{4c_4 x}{\log x}$$

Hence, $\pi(x) < x^{1/2} + \frac{4c_4 x}{\log x} < \frac{c_2 x}{\log x}$

for some abs. pos const c_2. $\square$

25/10/13.

The following are "improvement" on previous result.

Thm(2F) (Mertens). (Comes up in exam)

i) $\sum_{m \leq x} \frac{\Lambda(m)}{m} = \log x + O(1)$

ii) $\sum_{p \leq x} \frac{\log p}{p} = \log x + O(1)$

iii) $\sum_{p \leq x} \frac{1}{p} = \log\log(x) + O(1)$

Proof:
Thm (2C):

$$\sum_{m \leq x} \Lambda(m) \left[\frac{x}{m}\right] = x\log x - x + O(\log x)$$

Put $\left[\frac{x}{m}\right] = \frac{x}{m} + O(1)$

$$\sum_{m \leq x} \Lambda(m) \left[\frac{x}{m}\right] = x \sum \frac{\Lambda(m)}{m} + O\left(\sum_{m \leq x} \Lambda(m)\right)$$

$$\sum_{m \leq x} \Lambda(m) \leq \sum_{j=0}^{\infty} \underbrace{\sum_{\frac{x}{2^{j+1}} < m \leq \frac{x}{2^j}} \Lambda(m)}_{\leq c_4 \frac{x}{2^j}} \leq 2c_4 x$$

Thm 2D(2)

So : $$x \sum_{m \leq x} \frac{\Lambda(m)}{m} = x \log x + O(x)$$

divide by x

$\Rightarrow$ (i)

Note, $$\sum_{m \leq x} \frac{\Lambda(m)}{m} = \sum_{\substack{p, k \\ p^k \leq x}} \frac{\log p}{p^k}$$

$$= \sum_{p \leq x} \frac{\log p}{p}$$

$$+ \sum_{p \leq x} \log p \sum_{2 \leq k \leq \frac{\log x}{\log p}} \frac{1}{p^k}$$

$$\sum_{p \leq x} \log p \sum_{2 \leq k \leq \frac{\log x}{\log p}} \frac{1}{p^k} \leq \sum_{p \leq x} \log p \sum_{k=2}^{\infty} \frac{1}{p^k}$$

geom series

$$= \sum_{p \leq x} \frac{\log p}{p(p-1)}$$

$$\leq \sum_{n=2}^{\infty} \frac{\log n}{n(n-1)} = O(1)$$

—/—

$\sum \frac{1}{n^{1+\varepsilon}}$ for $\varepsilon > 0$ is conv

$\log n = O(n^{\varepsilon})$ for any $\varepsilon > 0$

—/—

For any real $x \geq 2$ put

$$T(x) = \sum_{p \leq x} \frac{\log p}{p}.$$ so by (ii)

$$|T(x) - \log x| < C_6, \text{ for } x \geq 2$$ (C_6 a pos const)

(Holds for $x > N$ (N is a positive integer) so by increasing, we can achieve the above)

$$\sum_{p \leq x} \frac{1}{p} = \frac{1}{2} + \sum_{2 < n \leq x} a(n) \quad ,$$

where $a(n) = \begin{cases} \frac{1}{n} & \text{if } n \text{ is prime} \\ 0 & \text{otherwise} \end{cases}$

$a \in \mathcal{A}$.

$$= \frac{1}{2} + \sum_{2 < n \leq x} \underbrace{\log n . a(n)}_{\text{arithmetic fn}} . \underbrace{\frac{1}{\log n}}_{\text{"f"}}$$

sum. by. parts.

$$= \cancel{\frac{1}{2} +} \frac{T(x)}{\log x} - \cancel{\frac{T(2)}{\log 2}}$$

$$+ \int_2^x \frac{T(y)}{y \log^2 y} \, dy.$$

Note $\frac{d}{dy}\left(\frac{1}{\log y}\right)$

$$= -\frac{1}{(\log y)^2} \frac{1}{y}$$

$$= \frac{T(x) - \log x}{\log x} + 1 + \int_2^x \frac{T(y) - \log y}{y \log^2 y} \, dy$$

$$+ \overbrace{\int_y^x \frac{1}{y \log y} \, dy}^{\log \log x - \log \log 2} .$$

Note: $\frac{d}{dy}(\log \log y) = \frac{1}{y \log y}$

So:

$$\left| \sum_{p \leq x} \frac{1}{p} - \log \log x \right| \overset{\substack{\text{triangle} \\ \text{ineq}}}{<} \frac{C_6}{\log x} + 1 + \int_2^x \frac{C_6}{y \log^2 y} + \log \log 2$$

$$= \frac{C_6}{\log x} + 1 + \frac{C_6}{\log x} - \frac{C_6}{\log 2} + \log \log 2 = O(1)$$

□

—/—

Dirichlet series:

Def: A Dirichlet series is a series of the form

$$F(s) = \sum_{n=1}^{\infty} f(n) \cdot n^{-s}$$

where $s \in \mathbb{C}$ and $f : \mathbb{N} \to \mathbb{C}$.

It is a functcoi on $\mathbb{C}$; where it converages.

I will also assume $s = \sigma + it$, $\sigma, t \in \mathbb{R}$.

Thm (3A): Suppose $F(s)$ is a Dirichlet series, convergent for some $s \in \mathbb{C}$. There exist unique real numbers $-\infty \leqslant \sigma_0 \leqslant \sigma_1 \leqslant \sigma_2 < \infty$ st:

i) $F(s)$ converages for all s with $\sigma > \sigma_0$. For every $\varepsilon > 0$.
$F(s)$ diverages for some $s \in \mathbb{C}$ with $\sigma_0 - \varepsilon < \sigma \leqslant \sigma_0$.

ii) For every $\eta > 0$, $F(s)$ converages uniformly on the set $\{s \in \mathbb{C} : \sigma > \sigma_1 + \eta\}$ and does not converage uniformly on the set $\{s \in \mathbb{C} : \sigma > \sigma_1 - \eta\}$.

iii) $F(s)$ converages absolutely for every $s \in \mathbb{C}$ $\sigma > \sigma_2$. For every $\varepsilon > 0$, $F(s)$ does converage abs. for some $s \in \mathbb{C}$ with $\sigma_2 - \varepsilon < \sigma \leq \sigma_2$

Proof: Suppose that $F(s)$ converages for $s = s^* = \sigma^* + it^*$

So $f(n) n^{-s^*} \to 0$ as $n \to \infty$ so.

$$f(n) n^{-s^*} = O(1) \quad \text{i.e. } f(n) = O(n^{\sigma^*})$$

Note $|n^{it}| = |e^{\log n^{it}}| = |e^{it\log n}| = 1$

So, for all $s \in \mathbb{C}$, with $\sigma > \sigma^* + 1$,

$$|f(n) n^{-s}| = |f(n) n^{-\sigma}| = O(n^{\sigma^* - \sigma})$$

$$(\sigma^* - \sigma < 1)$$

So by the comparsion test $F(s)$ converages (absolutely) for all $s \in \mathbb{C}$, $\sigma > \sigma^* + 1$.

Now, let $\sigma_0 = \inf\{u \in \mathbb{R} : F(s)$ conv. for all $s \in \mathbb{C}$, st $\sigma > u\}$.

Note $\sigma^* + 1 \in$ ↗

and $\sigma_2 = \inf\{u \in \mathbb{R} : F(s)$ conv abs for all $s \in \mathbb{C}$ st $\sigma > u\}$.

So $\sigma_0 \leqslant \sigma_2 \Rightarrow (i), (ii)$

(Suppose $F(s)$ diverages for $\sigma > \sigma_0 = \inf\{\dots\}$.
But for all $\varepsilon > 0$, if $\sigma > \sigma_0 + \varepsilon$, $F(\varepsilon)$ converages.
So choose $\varepsilon < \sigma - \sigma_0 \Rightarrow \sigma_0 > \sigma + \varepsilon$ ※)

contradiction

For (ii), choose $\delta > 0$ and $\varepsilon > 0$, $\exists N \in \mathbb{N}$ st $\sum_{n=N+1}^{\infty} |f(n)| n^{-\sigma_2 - \delta} < \varepsilon$ $(\sigma_2 + \delta > \sigma_2)$.

Hence;

$$\sup\left\{ \left| \sum_{n=1}^{N} f(n)n^{-s} - \sum_{n=1}^{\infty} f(n)n^{-s} \right| : \sigma \geq \sigma_0 + \delta \right\}$$
$$\leq \sum_{n=N+1}^{\infty} |f(n)| n^{-\sigma_2 - \delta} < \varepsilon .$$

So, $F(s)$ converages uniformly on the set $\{ s \in \mathbb{C} : \sigma \geq \sigma_2 + \delta \}$.

Let $\sigma_1 := \inf \{ u \in \mathbb{R} : F(s)$ conv. uniformly on $\{ s \in \mathbb{C} : \sigma \geq u \} \}$.

Clearly, $\sigma_0 \leq \sigma_1 \leq \sigma_2 + \delta$.

But $\delta > 0$ was arbitrary $\Rightarrow \sigma_0 \leq \sigma_1 \leq \sigma_2$.

(If not $\sigma_1 > \sigma_2$: pick $0 < \delta < \sigma_1 - \sigma_2 \Rightarrow$ #)

Thm (3B) : For every $s \in \mathbb{C}$ with $\sigma > \sigma_1$, $F(s)$ may be differentiated term by term:

$$F'(s) = - \sum_{n=1}^{\infty} f(n) \log n . n^{-s}$$

Note: $n^{-s} = e^{\log n^{-s}} = e^{-s \log n}$

$$\frac{d}{ds}(n^{-s}) = - \log n \, e^{-s\log n} = - \log n \, n^{-s} .$$

Proof: Follows from uniform convergence $\square$

Thm (3C) : Suppose $F(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$ and $G(s) = \sum_{n=1}^{\infty} g(n)n^{-s}$, where $f, g \in \mathcal{A}$, $s \in \mathbb{C}$. Suppose further, there exists $\sigma_3 \in \mathbb{R}$ st $F(s) = G(s)$ for all s with $\sigma \geq \sigma_3$. Then $f(n) = g(n)$ for all $n \in \mathbb{N}$.

—/—

An example of Dirichlet series:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

—/—

Rather we will prove:

Thm (3D) : Suppose $F(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$, where $f \in \mathcal{A}$, $s \in \mathbb{C}$. Suppose that $F(s) = 0$ for all $\sigma \geq \sigma_3 \in \mathbb{R}$. Then $f(n) = 0$ for all $n \in \mathbb{N}$.

28/10/13.

THM (3D): Suppose $F(s) = \sum_{n=1}^{\infty} f(n) n^{-s}$ and $f \in \mathcal{A}$, $s \in \mathbb{C}$. Suppose $\exists\, \sigma_3 \in \mathbb{R}$ st $F(s) = 0$ for all $\sigma > \sigma_3$. Then $f(n) = 0$ for all $n \in \mathbb{N}$.

Proof: $F(s)$ converages at $s = \sigma_3$ so $f(n) = \mathcal{O}(n^{\sigma_3})$ for all $n \in \mathbb{N}$. Let $\sigma \geq \sigma_3 + 2$. Then.

$$(1) \quad \sum_{n=1}^{\infty} f(n) n^{-\sigma} = \mathcal{O}\Big(\sum_{n=1}^{\infty} n^{\sigma_3 - \sigma}\Big)$$

and $y^{\sigma_3 - \sigma}$ is a decreasing function so

$$(2) \quad \sum_{n=N}^{\infty} n^{\sigma_3 - \sigma} = N^{\sigma_3 - \sigma} + \sum_{n=N+1}^{\infty} n^{\sigma_3 - \sigma}$$

$$\left[\frac{y^{\sigma_3 - \sigma + 1}}{\sigma_3 - \sigma + 1}\right] \qquad \leq N^{\sigma_3 - \sigma} + \int_N^{\infty} y^{\sigma_3 - \sigma}\, dy$$

$$= N^{\sigma_3 - \sigma} + \mathcal{O}(N^{\sigma_3 - \sigma + 1})$$

So (3) $\sum_{n=N}^{\infty} f(n) n^{-\sigma} = \mathcal{O}(N^{\sigma_3 - \sigma + 1})$

So, put $N = 2$ since $\sigma \geq \sigma_3 + 2 \geq \sigma_3$

$$0 = F(\sigma)$$

$$= f(1) + \sum_{n=2}^{\infty} f(n) n^{-\sigma}$$

$$= f(1) + \mathcal{O}(2^{\sigma_3 - \sigma + 1})$$

$$\to f(1) \text{ as } \sigma \to \infty.$$

Hence, $f(1) = 0$. Now suppose $f(1) = \dots = f(M-1) = 0$.

Now put $N = M+1$.

$$0 = F(\sigma) = f(M)M^{-\sigma} + \sum_{n=M+1}^{\infty} \int f(n)n^{-\sigma}$$

$$= f(M)M^{-\sigma} + O\left(\left(M+1\right)^{\sigma_3 - \sigma + 1}\right)$$

$$\Rightarrow 0 = f(M) + O\left(\left(M+1\right)^{\sigma_3 + 1}\left(\frac{M}{M+1}\right)^{\sigma}\right)$$

$$\to f(M) \text{ as } \sigma \to \infty$$

Hence $f(M) = 0$. Result follows by induction

$\square$.

Multiplication of Dirichlet series (IMPORANT)

Thm (3E): Suppose $j = 1, 2, 3$.

$$F_j(s) = \sum_{n=1}^{\infty} f_j(s)n^{-s}$$

$f_j \in \mathcal{A}$, $s \in \mathbb{C}$. Suppose that $f_3 = f_1 * f_2$. Then $F_1(s) F_2(s) = F_3(s)$ for $\sigma > \max\{\sigma_2^{(1)}, \sigma_2^{(2)}\}$, where for $j = 1, 2$. $F_j(s)$ conv. abs. for all $s \in \mathbb{C}$ st $\sigma > \sigma_2^{(j)}$

Proof:

$$\sum_{n=1}^{N} f_3(n)\, n^{-s}$$

$$= \sum_{n=1}^{N} \sum_{xy=n} f_1(x) f_2(x) (xy)^{-s}$$

$$= \sum_{\substack{x,y \\ xy \le N}} f_1(x) x^{-s} f_2(y) y^{-s}$$

So $\sum_{n=1}^{N} f_3(n) n^{-s} - \sum_{x \le \sqrt{N}} f_1(x) x^{-s} \sum_{y \le \sqrt{N}} f_2(y) y^{-s}$

$$= \sum_{\sqrt{N} < x \le N} f_1(x) x^{-s} \sum_{y < \frac{N}{x}} f_2(y) y^{-s}$$

$$+ \sum_{x \le \sqrt{N}} f_1(x) x^{-s} \sum_{\sqrt{N} < y \le \frac{N}{x}} f_2(y) y^{-s} .$$

So:

$$\left| \sum_{n=1}^{\infty} f_3(n) n^{-s} - \sum_{x \le \sqrt{N}} f_1(x) x^{-s} \sum_{y \le \sqrt{N}} f_2(y) y^{-s} \right|$$

$$< \left(\sum_{x > \sqrt{N}} |f_1(x)| x^{-\sigma} \right) \left(\sum_{y=1}^{\infty} |f_2(y)| y^{-\sigma} \right)$$

$$+ \left(\sum_{x=1}^{\infty} |f_1(x)| x^{-\sigma} \right) \left(\sum_{y > \sqrt{N}} |f_2(y)| y^{-\sigma} \right) \quad \ast$$

So, if $\sigma > \max\{ \sigma_2^{(1)}, \sigma_2^{(2)} \}$ then $\sum_{x > \sqrt{N}} |f_1(x)| x^{-\sigma}$ and $\sum_{y \ge \sqrt{N}} |f_2(y)| y^{-\sigma} \to 0$ as

$N \to \infty$ and $\sum_{y=1}^{\infty} |f_2(y)| y^{-\sigma}$, $\sum_{x=1}^{\infty} |f_1(x)| x^{-\sigma}$ are finite.

So $* \to 0$ as $N \to \infty$

But $\sum_{x \leq \sqrt{N}} f_1(x) x^{-s}$, $\sum_{y \leq \sqrt{N}} f_2(y) y^{-s}$

$$\to \overset{= F_1(s)}{\sum_{x=1}^{\infty} f_1(x) x^{-s}}, \overset{= F_2(s)}{\sum_{y=1}^{\infty} f_2(y) y^{-s}}$$

as $N \to \infty$. $\square$

Remark : Theorem 3E generalises to a product of $k+1$ Dirichlet series i.e suppose $F_j(s) = \sum_{n=1}^{\infty} f_j(n) n^{-s}$ $j = 1, \dots, k+1$ where $f_j \in \mathcal{A}$, $s \in \mathbb{C}$. Suppose that

$$f_{k+1}(n) = \sum_{\substack{x_1, \dots, x_k \\ x_1 \dots x_k = n}} f_1(x_1) \dots f_k(x_k)$$

Then $F_{k+1}(s) = F_1(s) \dots F_k(s)$, provided $\sigma > \max\{\sigma_2^{(1)}, \dots, \sigma_2^{(k)k}\}$ where $F_j(s)$ conv. abs. for all $s \in \mathbb{C}$ st $\sigma > \sigma_2^{(j)}$

Thm (3F) : Suppose that $f \in \mathcal{M}$. For all $s \in \mathbb{C}$ st $\sigma > \sigma_2$

$$F(s) = \sum_{n=1}^{\infty} f(n) n^{-s}$$

$$= \prod_p \left(\sum_{n=0}^{\infty} f(p^n) p^{-ns} \right)$$

Proof: By the remark, if p_j is the j^{th} prime increasing order,

$$\prod_{j=1}^{k}\left(\sum_{k=0}^{\infty} f(p_j^h)\, p_j^{-hs}\right)$$

$$= \sum_{n=1}^{\infty}\left(\sum_{\substack{h_1,\dots,h_k \\ p_1^{h_1}\cdots p_k^{h_k}=n}} f(p_1^{h_1})\dots f(p_k^{h_k})\right) n^{-s}.$$

Since we can define k arithmetic fns f_j:

$$f_j(n) = \begin{cases} n & \text{if } n = p_j^h \\ 0 & \text{otherwise.} \end{cases}$$

So $\sum_{h=0}^{\infty} f(p_j^h) f(p_j^{hs}) = \sum_{n=1}^{\infty} f_j(n) n^{-s}$

So $\prod_{j=0}^{k}\left(\sum_{h=0}^{\infty} f(p_j^h)\, p_j^{-hs}\right)$

$$\underset{\text{(Remark)}}{=} \sum_{n=1}^{\infty}\left(\sum_{x_1\cdots x_k=n} f_1(x_1)\dots f_k(x_k)\right) n^{-s}$$

$$= \sum_{n=1}^{\infty} \sum_{\substack{h_1,\dots,h_k \\ p_1^{h_1}\cdots p_k^{h_k}=n}} f(p_1^{h_1}) \dots f(p_k^{h_k}) n^{-s} \dots$$

to be continued.

1/11/13

No lectures for 11th Nov.

Thm (3F): Suppose $f \in \mathcal{M}$. For all $s \in \mathbb{C}$, st. $\sigma > \sigma_2$.

$$F(s) = \prod_p \left(\sum_{h=0}^{\infty} f(p^h)\, p^{-hs} \right)$$

Order the primes: p_j.

$$\prod_{j=1}^{k} \left(\sum_{h=0}^{\infty} f(p^h)\, p^{-hs} \right)$$

$$= \sum_{n=1}^{\infty} \overbrace{\left(\sum_{\substack{h_1,\dots,h_k \\ p_1^{h_1}\cdots p_k^{h_k} = n}} f(p_1^{h_1}) \dots f(p_k^{h_k}) \right)}^{\text{Contains at most one term by unique factorisation}} n^{-s}$$

Hence, $\prod_{j=1}^{k} \left(\sum_{h=0}^{\infty} f(p_j^h)\, p_j^{-hs} \right)$

$$= \sum_{n=1}^{\infty} \theta_k(n) f(n)\, n^{-s}.$$

where

$$\theta_k(n) = \begin{cases} 1 & \text{if all prime divisors of } n \text{ belong to } \{p_1, \dots, p_k\} \\ 0 & \text{otherwise} \end{cases}$$

So

$$\prod_{j=1}^{k}\left(\sum_{h=0}^{\infty} f(p_j^h)\, p_j^{-hs}\right) - \sum_{n=1}^{\infty} f(n)\, n^{-s}$$

$$= \sum_{n=1}^{\infty}\left[\theta_k(n) - 1\right] f(n)\, n^{-s}$$

$$= O\left(\textstyle\sum_{n=k+1}^{\infty} |f(n)|\, n^{-\sigma}\right) \to 0 \text{ as } k \to \infty$$

because if $n \in \{1, \dots, k\}$, then its prime divisors must belong to $\{p_1, \dots, p_k\}$. □.

Def: An arithmetic function f is called completely / totally / strongly multiplicative if $f(mn) = f(m)f(n)$ for all $n, m \in \mathbb{N}$.

Thm (3G): Suppose $f \in \mathcal{A}$ is totally multiplicative and not identically zero. For every $s \in \mathbb{C}$ st $\sigma > \sigma_2$.

$$F(s) = \prod_{p}\left(1 - f(p)\, p^{-s}\right)^{-1} \quad \text{(Euler-Product)}$$

Proof: $\sum_{h=0}^{\infty} f(p^h)\, p^{-hs}$ is convergent by comparison with $\sum_{n=1}^{\infty} |f(n)|\, n^{-\sigma}$.

Also $\sum_{h=0}^{\infty} f(p^h)\, p^{-hs} \underset{\text{total mult}}{=} \sum_{h=0}^{\infty} \left(f(p)\, p^{-s}\right)^h$ geometric series.

And $f(1) = 1 \Rightarrow \quad = \left(1 - f(p)\, p^{-s}\right)^{-1}$ Apply THM 3F

□

Primes in arithmetic progesscon:

Objective: Show if $q \in \mathbb{N}$, $a \in \mathbb{Z}$ st $(a, q) = 1$ then there exist infinitely many primes $p \equiv a \mod q$.

Note: $(a, q) = 1$ is necessary

$\rightarrow$ Suppose $n \equiv a \mod q$

So $n = a + bq$ $b \in \mathbb{Z}$

$(a, b) \mid n$. So if $(a, q) > 1$ and n is prime, then $(a, q) = n$ i.e. there is at most one prime $p \equiv a \mod q$.

EG: We can prove certain cases of Dirichlet's theorem using elementary methods. In general, the theorem is hard.

(i). There are infinitely many primes $p \equiv -1 \mod 4$.

Suppose not, List such primes: $p_1, \ldots, p_r$

The $4p_1 \ldots p_r - 1 \equiv -1 \mod 4$ so it must have a prime factor $p \equiv -1 \mod 4$ and p cannot be one of the p_i $i = 1, \ldots, r$. To see the latter, $p \neq 2$.

So $p \equiv 1$ or $-1 \mod 4$. But if all prime divisors are congruent to 1 mod 4 – Contradiction.

(ii) Suppose are infinitely many primes $p \equiv 1 \mod 4$
Suppose not, List all such primes $p_1, \dots, p_r$

Consider a prime $p \mid 4(p_1 \dots p_r)^2 + 1$. $p \neq 2$.

i.e. $4(p_1 \dots p_r)^2 + 1 \equiv 0 \mod p$.

$$\text{so } -1 \equiv \underbrace{4(p_1 \dots p_r)^2}_{\text{a square}} \mod p$$

so -1 is a "quadratic residue mod p".

So $p \equiv 1 \mod 4$ but $p \neq p_i$ $i = 1, \dots, r$.
For $p \neq 2$
Put
$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \equiv x^2 \text{ mod. } p \quad x \in \mathbb{Z}_{\neq 0} \\ -1 & \text{if not} \\ 0 & \text{if } p \mid a \end{cases}$$

$\{n \in \mathbb{Z} : n \neq 0\}$

<u>Claim</u>: $\left(\frac{a}{p}\right) \equiv a^{\frac{(p-1)}{2}} \mod p$.

<u>Proof</u>: if $p \mid a$ $\left(\frac{a}{p}\right) \overset{def}{=} 0 \equiv a^{\frac{(p-2)}{2}} \pmod{p}$

If $p \nmid a$. Let $g \in \mathbb{Z}$ be a primitive p^{th} root of unity mod p.

—/—

Def: Primitive root of 1 mod p

$g^{p-1} \equiv 1 \bmod p$. but $g^{\alpha} \not\equiv 1 \bmod p$. $0 < \alpha < p-1$

—/—

Then $a \equiv g^r \bmod p$ $0 \leq r \leq p-2$.

r is even iff a is a quadratic residue mod p.

So $a^{\frac{p-1}{2}} \equiv (g^{\frac{p-1}{2}})^r \bmod p$.

Claim $h = g^{\frac{p-1}{2}} \equiv -1 \bmod p$

Proof: Note: $h^2 = g^{p-1} \equiv 1 \bmod p$

So $p \mid h^2 - 1 = (h+1)(h-1)$

So $p \mid (h+1)$ or $p \mid (h-1)$

So $h \equiv 1$ or $-1 \bmod p$.

But $h \not\equiv 1 \bmod p$ because g is primitive.

$\square$

So $a^{\frac{p-1}{2}} \equiv (-1)^r \bmod p$ (r even iff a is quad res mod p)

$\square$

So $\left(\frac{-1}{p}\right) \equiv (-1)^{\frac{p-1}{2}} \mod 2$.

So $\left(\frac{-1}{p}\right) = 1$ iff $\frac{p-1}{2}$ is even.

iff $p \equiv 1 \mod 4$. $\square$

Dirichlet's idea. $a \in \mathbb{Z}$, $q \in \mathbb{N}$.
To show, if $(a,q) = 1$; then

$$\sum_{p \equiv a \bmod q} \frac{1}{p} \text{ is divergent}$$

For technical reasons, it turns out to be easier to show:

$$\sum_{p \equiv a \bmod q} \frac{\log p}{p^\sigma} \to \infty \quad \text{as } \sigma \to 1^+ \text{ (from above)}$$

We illustrate this idea in the case of primes $p \equiv 1 \mod 4$.

First of all, we need a function that distinguishes between integers $n \equiv 1 \mod 4$ and the rest. Suppose n is odd. Then:

$$\frac{1+(-1)^{\frac{n-1}{2}}}{2} = \begin{cases} 1 \text{ if } n \equiv 1 \mod 4 \\ 0 \text{ if } n \equiv -1 \mod 4. \end{cases}$$

So

$$\sum_{p \equiv 1 \bmod 4} \frac{\log p}{p^\sigma} = \frac{1}{2} \sum_{p \text{ odd}} \frac{\log p}{p^\sigma} \left(1 + (-1)^{\frac{p-1}{2}}\right)$$

Now, $\sum_{p \text{ odd}} \frac{\log p}{p^\sigma} \xrightarrow{\text{Chap } 2} \infty$ as $\sigma \to 1^+$

So it suffices to show that the series

$\sum_{p \text{ odd}} \frac{(-1)^{\frac{p-1}{2}} \cdot \log p}{p^\sigma}$ converages as $\sigma \to 1^+$

The next idea is that, if we show that the contributcon from non-prime, natural numbers in:

$$(1) \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \frac{(-1)^{\frac{n-1}{2}} \Lambda(n)}{n^\sigma} \text{ is convergent}$$

it suffices to show that the series (1) converages as $\sigma \to 1^+$

Note that:

$$\chi(n) = \begin{cases} (-1)^{\frac{n-1}{2}} & n \text{ is odd} \\ 0 & n \text{ is even.} \end{cases}$$

is totally multiplicative, since nm is odd iff n and m are odd and

$$\chi((2a+1)(2b+1)) = (-1)^{\frac{(2a+1)(2b+1)-1}{2}}$$

$$= (-1)^{\overset{0}{\cancel{2ab}}+a+b}.$$

$$\chi(2a+1)\chi(2b+1) = (-1)^{\frac{2a}{2}}(-1)^{\frac{2b}{2}}$$

$$= (-1)^{a+b}$$

Write $L(\sigma) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{\sigma}}$.

Note that, for ever $n \in \mathbb{N}$,

$$\chi(n)\log n = \chi(n)\sum_{m|n}\Lambda(m)$$

$$\overset{\text{Totally mult}}{=} \sum_{m|n} \underbrace{\chi(m)\Lambda(m)\chi\left(\frac{n}{m}\right)}_{\chi\Lambda * \chi(n)}$$

Hence, for $\sigma > 1$,

$$L'(\sigma) \overset{\text{Thm 3B}}{=} -\sum_{n=1}^{\infty} \frac{\chi(n)\log n}{n^{\sigma}}$$

$$\overset{\text{Thm 3E}}{=} -\left(\sum_{n=1}^{\infty} \frac{\chi(n)\Lambda(n)}{n^{\sigma}}\right)\overbrace{\left(\sum_{n=1}^{\infty} \frac{\chi(n)}{n^{\sigma}}\right)}^{L(\sigma)}$$

Hence, $\sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \frac{(-1)^{\frac{n-1}{2}}\Lambda(n)}{n^{\sigma}} = \sum_{n=1} \frac{\chi(n)\Lambda(n)}{n^{\sigma}}$

$\ldots = -\frac{L'(\sigma)}{L(\sigma)}$

Recall: $L(\sigma) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^\sigma}$

Now as $\sigma \to 1^+$, we expect:

$L(\sigma) - L(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots > 0.$

$L'(\sigma) - L'(1) = \frac{\log 3}{3} - \frac{\log 5}{5}$

$+ \frac{\log 7}{7} \; - - -$

converages by the alternative series test.

i.e. we $-\frac{L'(\sigma)}{L(\sigma)}$ to converage as $\sigma \to 1^+$.

15/11/13

Dirichlet Characterics.

Dirichlet's crucial discovery was that for all $q \in \mathbb{N}$ there exists a family of $\phi(q)$ functoin $\chi : \mathbb{N} \to \mathbb{C}$ now referred to as "Dirichlet characters modulo q" which generalise χ from before and satisfy.

$$\frac{1}{\phi(q)} \sum_{\chi \bmod q} \frac{\chi(n)}{\chi(a)} = \begin{cases} 0 & \text{if } n \equiv a \bmod q \\ 1 & \text{if } n \not\equiv a \bmod q \end{cases}$$

"sum over Dirichlet characters modulo q"

$\#\{n \in \{1, \dots, q\} . (n,q) = 1\}$

Group Characters.

Let G be a finite abelian group of order h with identity "e".

So recall $G = \prod_{i=1}^{s} C_{h_i}$, $e = (1, \dots, 1)$

$$C_{h_i} \simeq \mathbb{Z}/h_i\mathbb{Z}$$

$$\shortparallel$$

$$\{1, x, x^2, x^3, \dots, x^{h_i - 1}\}$$

Def: A character on G is a non-zero, complex valued functoin χ on G such that $\chi(uv) = \chi(u)\chi(v)$ for all $u, v \in G$.

i.e. χ is a group homomorphism.

$$G \to \mathbb{C}^{\times} = \mathbb{C} \setminus \{0\}.$$

Remark (check)
(i) $\chi(e) = 1$
(ii) for all $u \in G$, $\chi(u)$ is an h^{th}-root of unity.
(iii) the number c of characters is finite.
(iv) the characters form a abelian group
e.g χ_1, χ_2. $\underbrace{\chi_1 \chi_2(u) := \chi_1(u)\chi_2(u)}_{\text{group operation on characters}}$

Remark: If $u \in G$, $u \neq e$, then there exist a character χ on G st $\chi(u) \neq 1$

Proof $G \cong \prod_{i=1}^{s} C_{h_i}$ so $h_1 \dots h_s = h$
write x_i for a generator of $C_{h_i} \cong G$.
so $u = x_1^{n_1} \dots x_s^{n_s}$ where n_i mod h_i is unique. since $u \neq e$, there exist $k \in \{1, \dots, s\}$ st $n_k \neq 0$ mod h_k.

Let $\chi(x_k) = e^{2\pi/h_k}$ and $\chi(x_i) = 1$ if $n \neq k$ so $\chi(u) = \chi(x_1^{n_1} \dots x_s^{n_s}) \overset{\text{homo}}{=} \chi(x_1)^{n_1} \dots \chi(x_s)^{n_s} = \chi(x_k)^{n_k} = e^{\frac{2\pi i n_k}{h_k}} \neq 1$

Let χ_0 denote the principle character on G i.e. $\chi_0(u) = 1$ for all $v \in G$.

Note: $\sum_{\chi}$ = "sum over all characters on G"

Thm (4B): Suppose G is a finite abelian group of order h, with identity e. Suppose χ_0 is the principal on G.

(i) For every χ on G

$$\sum_{u\in G} \chi(u) = \begin{cases} h & \text{if } \chi = \chi_0 \\ 0 & \text{if } \chi \neq \chi_0 \end{cases}$$

(ii) For every $u \in G$

$$\sum_{\chi} \chi(u) = \begin{cases} c & \text{if } u = e \\ 0 & \text{if } u \neq e \end{cases}$$

where c is the number of chars on G.

(iii) $c = h$

(iv). For every $u, v \in G$, we have:

$$\frac{1}{h}\sum_{\chi} \frac{\chi(u)}{\chi(v)} = \begin{cases} 1 & \text{if } u = v \\ 0 & \text{if } u \neq v \end{cases}$$

Proof: If $\chi = \chi_0$, done.
If $\chi \neq \chi_0$, so there exist $u_1 \in G$ st $\chi(u_1) \neq 1$.

so $\chi(u_1)\sum_{u\in G} \chi(u) = \sum_{u\in G} \chi(u_1)\chi(u)$

$$= \sum_{u\in G} \chi(u_1 u) \underset{\substack{\uparrow \\ \text{since } \{u_1 u : u\in G\} = \{u \in G\}}}{=} \sum_{u\in G} \chi(u)$$

$$\underbrace{[\chi(u_1)-1]}^{\neq 0}\sum_{u\in G}\chi(u)=0.$$

$$\Rightarrow \sum_{u\in G}\chi(u)=0$$

(ii) If $u=e$, done.

If $u\neq e$, there exist χ_1 st $\chi_1(u)\neq 1$ (by earlier remark).

So $\chi_1(u)=\sum_{\chi}\chi(u)$

$$=\sum_{\chi}\chi_1(u)\chi(u)$$

$$=\sum_{\chi}(\chi_1\chi)(u)$$

$$=\sum_{\chi}\chi(u)$$

$$\Rightarrow \underbrace{[\chi_1(u)-1]}_{\neq 0}\sum_{\chi}\chi(u)=0.$$

$$\Rightarrow \sum_{\chi}\chi(u)=0.$$

$$\text{(iii)}\quad h=\sum_{\chi}\overbrace{\sum_{u\in G}\chi(u)}=\begin{cases} h \text{ if } \chi=\chi_0 \\ 0 \text{ if } \chi\neq\chi_0\end{cases}$$

$$=\sum_{u\in G}\underbrace{\sum_{\chi}\chi(u)}_{=\begin{cases} c \text{ if } u=e \\ 0 \text{ if } u\neq e\end{cases}}=c$$

(iv) Note. $\frac{1}{h}\sum_{\chi} \frac{\chi(u)}{\chi(v)}$

$$= \frac{1}{h}\sum_{\chi} \chi(u)\chi(v^{-1})$$

$$= \frac{1}{h}\sum_{\chi} \chi(uv^{-1})$$

$$= \begin{cases} c/h & \text{if } uv^{-1} = e \\ 0 & \text{if } uv^{-1} \neq e \end{cases}.$$

and $c = h$.

Ex: Consider a ring $\mathbb{Z}/7\mathbb{Z}$ i.e. the residue classes modulo 7.

$$1 \equiv 8 \equiv 15.$$

Recall, addition and mult. commute with "mod 7".

$\mathbb{Z}/7\mathbb{Z}$ contains a mult. group of "units".

$$\left(\mathbb{Z}/7\mathbb{Z}\right)^* = \{\bar{n} \in \mathbb{Z}/7\mathbb{Z} : \exists \bar{m}, \overline{nm} = \bar{1}\}.$$

Because 7 is prime,

$$(\mathbb{Z}/7\mathbb{Z})^* = \{\bar{n} \in \mathbb{Z}/7\mathbb{Z} : \bar{n} \neq \bar{0}\}.$$

(Suppose not, let $\bar{n} = \bar{0}$ st $\overline{nm} \neq \bar{1}$ for all $\bar{m}$, $\exists\, \bar{m} = \bar{m}'$ st $\overline{nm} = \overline{nm'} \iff \bar{n}\,(\overline{m-m'}) = 0$ i.e. $7 \mid n(m-m') \Rightarrow 7 \mid n$ or $7 \mid m-m'$ contradiction).

In fact, $(\mathbb{Z}/7\mathbb{Z})^*$ is cyclic (order $\underset{\phi(7)}{\underset{\shortparallel}{6}}$)

$$\bar{3}^2 = \bar{2},\quad \bar{3}^3 = \bar{6},\quad \bar{3}^4 = \bar{4},$$

$$\bar{3}^5 = \bar{5},\quad \bar{3}^6 = \bar{1}$$

$$(\mathbb{Z}/7\mathbb{Z})^\times = \langle \bar{3} \rangle.$$

so, a character on $(\mathbb{Z}/7\mathbb{Z})^\times$ is determined by its value at $\bar{3}$ since $\chi(\bar{m}) = \chi(\bar{3}^r) = \chi(\bar{3})^r$.

Let $\omega = e^{\frac{\pi i}{3}}$ — primitive 6th root of unity in $\mathbb{C}^\times$.

There will be $|(\mathbb{Z}/7\mathbb{Z})^\times| = 6$ characters, $\chi(\bar{1}) = 1$

	$\bar{1}$	$\bar{3}$	$\bar{2}$	$\bar{6}$	$\bar{4}$	$\bar{5}$
χ_1	1	1	1	1	1	1
χ_2	1	ω	ω^2	ω^3	ω^4	ω^5
χ_3	1	ω^2	ω^4	1	ω^2	ω^4
χ_4	1	ω^3	1	ω^3	1	ω^3
χ_5	1	ω^4	ω^2	1	ω^4	ω^2
χ_6	1	ω^5	ω^4	ω^3	ω^2	ω

} Character table

Note that the group of chars is $\langle \chi_2 \rangle$.

Def: A character χ is real if $Im(\chi) \subseteq \mathbb{R}^{\times}$ i.e. $Im(\chi) \subseteq \{\pm 1\}$.

We now introduce Dirichlet Characters.

Let $q \in \mathbb{N}$. There are $\phi(q)$ residue classes $a \bmod q$ st $(a,q)=1$.

They form a group under multiplication – $(\mathbb{Z}/q\mathbb{Z})^{\times}$

Take a set of representatives. $\{a_1, \dots, a_{\phi(q)}\}$

Let $G = \{\bar{a}_1, \dots, \overline{a_{\phi(q)}}\} = (\mathbb{Z}/q\mathbb{Z})^{\times}$.

Define a character $\chi : G \to \mathbb{C}^{\times}$

For every $n \in \mathbb{N}$.

$$\chi(n) = \begin{cases} \chi(\bar{a}_j) & \text{if } n \equiv a_j \bmod q \quad j=1,\dots,\phi(q) \\ 0 & \text{otherwise } (n,q)>1 \end{cases}$$

Def: A function $\chi : \mathbb{N} \to \mathbb{C}$ of the above form is called a DIRICHLET CHARACTER MODULO q. It is always totally multiplicative.

There are $\phi(q)$ chars. mod q.

Def: The principal character χ_0 modulo q is defined by $\chi_0(n) = \begin{cases} 1 & \text{if } (n,q)=1 \\ 0 & \text{if } (n,q)>0. \end{cases}$

THM (4C): Let $q \in \mathbb{N}$, χ_0 the princ Dirichlet chara mod q.

(i) For any Dirichlet character χ mod q

$$\sum_{n \bmod q} \chi(n) = \begin{cases} \phi(q) & \text{if } \chi = \chi_0 \\ 0 & \text{if } \chi \neq \chi_0 \end{cases}$$

(ii) For any $n \in \mathbb{N}$,

$$\underset{\uparrow}{\sum_{\chi \bmod q}} \chi(n) = \begin{cases} \phi(q) & \text{if } n=1 \bmod q \\ 0 & \text{if } n \neq 1 \bmod q \end{cases}$$

"sum over Dirichlet chars mod q"

(iii) For any $a \in \mathbb{Z}$ st $(a,q)=1$ and $n \in \mathbb{N}$

$$\frac{1}{\phi(q)} \sum_{\chi \bmod q} \frac{\chi(n)}{\chi(a)} = \begin{cases} 1 & \text{if } n \equiv a \bmod q \\ 0 & \text{if } n \not\equiv a \bmod q \end{cases}$$

Proof: Exercise :- Just use THM 4B and the DEFs. We now

We now introduce the analogues of $L(\sigma)$ from before. Let $s = \sigma + it$, $\sigma, t \in \mathbb{R}$.

For $\sigma > 1$, let $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, and, more generally, $L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}$. where $q \in \mathbb{N}$ and χ is a Dirichlet char mod q.

Def: $L(s,\chi)$ is called DIRICHLET L-FUNCTION. Note, $L(s,\chi)$ is a Dirichlet series and converages absolutely for $\sigma > 1$ and uniformly for $\sigma > 1+\delta$ for any $\delta > 0$.

Since the coefficients are totally multiplicative. THM 3G $\Rightarrow$ for $\sigma > 1$

$$L(s,\chi) = \prod_p \left(1 - \chi(p)p^{-s}\right)^{-1}$$

THM (4D): Suppose $\sigma > 1$. Then $\zeta(s) \neq 0$. In fact, $L(s,\chi) \neq 0$ for any Dirichlet char χ mod q.

Proof: $\sigma > 1 \Rightarrow |\zeta(s)| = |\prod_p (1-p^{-s})^{-1}|$

$$\geqslant \prod_p (1-p^{-\sigma})^{-1} = \prod \frac{1-p^{-\sigma}}{\underbrace{1-p^{-2\sigma}}_{(1-p^{-\sigma})(1+p^{-\sigma})}}$$

$$= \frac{\zeta(2\sigma)}{\zeta(\sigma)} > 0$$

(σ real)

Also, $|L(s,\chi))| = |\prod_p (1 - \underset{\overset{\uparrow}{0 \text{ if } p|q}}{\chi}(p)p^{-s})^{-1}|$

$$\geqslant \prod_{p \nmid q} (1 + p^{-\sigma})^{-1}$$

$$\geqslant \prod_p (1 + p^{-\sigma})^{-1} > 0$$ as before

$\square$.

18/11/13

Thm (4E): Let χ_0 be the principal Dirichlet character modulo $q \in \mathbb{N}$. Then for $\sigma > 1$

$$L(s, \chi_0) = \prod_{p \mid q} (1 - p^{-s}) \zeta(s)$$

Proof : $L(s, \chi_0) = \prod_{p} (1 - \chi_0(p) p^{s})^{-1}$

$$= \prod_{p \nmid q} (1 - p^{-s})^{-1}$$

$$= \underbrace{\prod_{p} (1 - p^{-s})^{-1}}_{\zeta(s)} \prod_{p \mid q} (1 - p^{-s}) \qquad \square.$$

Thm (4F): Suppose $\sigma > 1$. Then:

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \Lambda(n) n^{-s}$$

In fact, for any Dirichlet character χ mod q,

$$-\frac{L'(s, \chi)}{L(s, \chi)} = \sum_{n=1}^{\infty} \chi(n) \Lambda(n) n^{-s}$$

Proof: Since $\sigma > 1$, Thm 3B

$$\Rightarrow L'(s, \chi) = -\sum_{n=1}^{\infty} \chi(n) (\log n) n^{-s}.$$

Recall $\chi(n)\log n = \chi(n)\sum_{m|n}\Lambda(m)$

$$= \sum_{m|n}\chi(m)\Lambda(m)\chi(\tfrac{n}{m})$$

$$= (\chi\Lambda * \chi)(n).$$

So THM 3E $\Rightarrow$

$$-L'(s,\chi) = \left(\sum_{n=1}^{\infty}\chi(n)\Lambda(n)n^{-s}\right)\cdot$$

$$\cdot\underbrace{\left(\sum_{n=1}^{\infty}\chi(n)n^{-s}\right)}_{L(s,\chi)}$$

$\square$

Thm (4G): If $\sigma > 1$, for any Dirichlet char χ mod q

$$\log L(s,\chi) = \sum_{p}\sum_{m=1}^{\infty} m^{-1}\chi(p^m)p^{-ms}$$

Proof: $\log L(s,\chi) = \log\prod_{p}(1-\chi(p)p^{-s})^{-1}$

$$= \sum_{p}\log(1-\chi(p)p^{-s})^{-1}$$

Note that the right hand-side converages uniformly for $\sigma > 1+\delta$ for any $\delta > 0$.

by Weierstass M-test:

($\{f_n\}$ seq. of complex functions on a set A and $\exists \{M_n\}$ seq of positive numbers st $|f_n(x)| \leq M_n$ for all $x \in A$ and $\sum_{n=1}^{\infty} M_n < \infty$, then $\sum_{n=1}^{\infty} f_n(x)$ converages uniformly on A).

$$|\mathrm{Log}(1 - \chi(p)p^{-s})|$$
$$\leq 2|\chi(p)p^{-s}| \leq 2p^{1-\delta}$$

For the first inequality:

$$(1)\ \mathrm{Log}(1 - \chi(p)p^{-s}) = \sum_{m=1}^{\infty} - \frac{\chi(p)^m}{m p^{ms}}$$

total multipli-cativity of χ

So:

$$|\mathrm{Log}(1 - \chi(p)p^{-s}|$$
$$= \left|\frac{\chi(p)}{p^s}\right| \left|\sum_{m=0}^{\infty} - \frac{\chi(p)^m}{m p^{ms}}\right|$$
$$\leq |\chi(p)p^{-s}| \sum_{m=0}^{\infty} \frac{1}{p^{\sigma m}}$$
$$\leq |\chi(p)p^{-s}| \sum_{m=0}^{\infty} \frac{1}{2^{\sigma m}}$$
$$\leq |\chi(p)p^{-s}| \sum_{m=0}^{\infty} \frac{1}{2^m} = 2|\chi(p)p^{-s}|$$

Applying (1), yields the result. □.

Analytic Continuation.

We would like to extend our definition of these functions to the half plane $\sigma > 0$ ($\zeta(s)$, $L(s,\chi)$ - functions on $\sigma > 1$)

Ex: Consider the function

$$f(s) = \sum_{n=0}^{\infty} s^n$$

- geometric series.

We know that it converages absolutely on the set $\{ s \in \mathbb{C} : |s| < 1 \}$ and uniformly in the set $\{ s \in \mathbb{C} : |s| < 1 - \delta \}$, for any $\delta > 0$. to $1/1-s$. So let $g(s) = 1/1-s$ defined on $\mathbb{C}$.

This is an analytic function on $\mathbb{C} \setminus \{1\}$ and $g(s) = f(s)$ in the set $\{ s \in \mathbb{C} : |s| < 1 \}$ and has a pole at $s = 1$.

So we say g is the analytic continuation of f to the complex plane with a pole at $s=1$.

Ex: Uniform convergence:

$$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$$

Suppose $\sum_{n=1}^{\infty} 1/n^{\alpha}$ conv. uniformly on $\{\alpha \in \mathbb{R} : \alpha > 1\}$.

Fix $\varepsilon > 0$, $\exists N$ st $\sum_{n=N-1}^{\infty} 1/n^{\alpha} < \varepsilon$ for all $\alpha > 1$.

But $\sum_{n=N-1}^{\infty} \frac{1}{n^{\alpha}} \geq \int_{N}^{\infty} \frac{1}{t^{\alpha}}\, dt$.

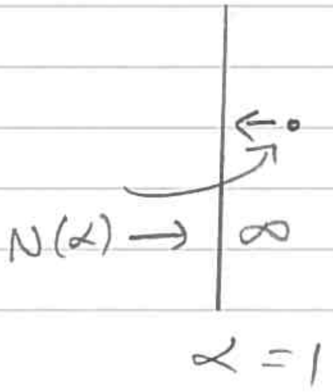
$$= \left[\frac{t^{-\alpha+1}}{-\alpha+1} \right]_{N}^{\infty}$$

$$= -\frac{N^{-\alpha-1}}{-\alpha+1}$$

$$= \frac{N^{1-\alpha}}{\alpha-1}$$

$$> (\alpha-1)^{-1}\frac{1}{N} \text{ for } 2 > \alpha > 1.$$

∞ as $\alpha \to 1^{+}$.



Thm (4H): $\zeta(s)$ admits an analytic continuation to the half plane $\sigma > 0$, and is analytic in the half-plane except for a simple pole at $s=1$ with residue 1.

Recall: Suppose f is a meromorphic function i.e. f is a holomorphic/analytic function except for a set of isolated points at which f has a Laurent series

e.g. if we call one such point s_0,

$$f(s) = \sum_{n=-\infty}^{\infty} \overset{\text{consts}}{a_n} (s-s_0)^n$$

in a punctured neighbourhood of s_0.

Then the residue at s_0 is the coefficient a_{-1}. The pole is simple if $a_n = 0$ for all $n < -1$.

Thm (4J): Let $q \in \mathbb{N}$, χ_0 the prin. Dirich. char. mod q. Then $L(s, \chi_0)$ admits an analytic continuation to $\sigma > 0$ and is analytic except for a simple pole at $s=1$ with residue

$$\frac{\phi(q)}{q}.$$

Thm (4K) : Let $q \in \mathbb{N}$, χ a non-prin Dirichlet char mod q. Then $L(s, \chi))$ admits an analytic cont. to $\sigma > 0$ and is analytic in this half-plane.

22/11/13.

Thm (4L): Suppose $a : \mathbb{N} \to \mathbb{C}$. $a(n) = \mathcal{O}(1)$
i.e. a is bounded

Put $A(x) = \sum_{n \leq x} a(n)$

and Define a Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} a(n) n^{-s} \text{ for } \sigma > 1$$

For all $x > 0$, $\sigma > 1$,

$$\sum_{n \leq x} a(n) n^{-s} = A(x) x^{-s} + s \int_1^x A(t) t^{-s-1}\, dt \quad (1)$$

Further, for $\sigma > 1$,

$$F(s) = s \int_1^{\infty} A(t) t^{-s-1}\, dt \quad (2)$$

Proof: (1) is immediate from "summation by parts".

(2) follows from (1). letting $x \to \infty$ observing that $A(x) = \mathcal{O}(x)$ and $\sigma > 1$.

We will require a technical result, stated without proof.

Thm (4M): Define a path $\Gamma \subseteq \mathbb{C}$ by $w(t) = u(t) + iv(t)$ where $u, v[0,1] \rightarrow \mathbb{R}$ are continuous differentiable.

Let $D \subseteq \mathbb{C}$ be a domain (open, connected subset).

For $s \in D$, let

$$F(s) = \int_\Gamma f(s,w)\, dw,$$

where $f(s,w)$ is continuous for every $s \in D$ and every $w \in \Gamma$, and analytic in D.

Then $F(s)$ is analytic in D. $\square$.

Proof (4H) So $F(s) = \zeta(s)$. In the notation of Thm 4L

$a(n) = 1$ for all $n \in \mathbb{N}$ so $A(x) = [x]$ for all $x > 0$.

so $\zeta(s) = s\int_1^\infty [t] t^{-s-1}\, dt$.

$= s\int_1^\infty t^{-s}\, dt - s\int_1^\infty \{t\} t^{-s-1}\, dt$;

where $\{t\} = t - [t]$.

$$= \underbrace{\frac{s}{s-1}}_{1+\frac{1}{s-1}} - \underbrace{s\int_1^\infty \{t\} t^{-s-1}\, dt}$$

To prove Thm 4H, we need to show this is an analytic function for $\sigma > 0$.

Put:

$$\cancel{s}\int_1^\infty \{t\} t^{-s-1}\, dt$$

$$= \sum_{n=1}^\infty F_n(s), \text{ where}$$

$$F_n(s) = \cancel{s}\int_n^{n+1} \{t\} t^{-s-1}\, dt$$

Therefore it remains to show that $F_n(s)$ is analytic and in $\mathbb{C}$ and $\sum_{n=1}^\infty F_n(s)$ converages uniformly to $F(s)$ on $\sigma > \delta$ for any $\delta > 0$.

<u>N.B</u>: If $F(s) = \sum_{n=1}^\infty F_n(s)$ converages uniformly for $\sigma > \delta$ for all $\delta > 0 \Rightarrow$ $F(s)$ is analytic for $\sigma > 0$.

For the first, put $t = n + r$

$$F_n(s) = \int_0^1 r(n+r)^{-s-1}\, dr$$

$$= \int_0^1 r e^{-(s+1)\log(n-r)} \, dr$$

$\Rightarrow$ the result follows from Thm 4M

For the second statement for $\sigma > \delta$,

$$|F_n(s)| = \left| \int_n^{n+1} \{t\} t^{-s-1} \, dt \right|$$

$$\leqslant n^{-\sigma-1} \leqslant n^{-\delta-1}$$

$\delta > 0$, so the result follows from the Weierstass M-test. $\square$

<u>Thm (4J)</u>: Suppose $\sigma > 1$, By Thm 4E

$$L(s, \chi_0) = \zeta(s) \prod_{p|q} \left(1 - \frac{1}{p^s} \right)$$

Clearly the second term of the RHS is analytic for $\sigma > 0$.

$\Rightarrow$ The RHS is analytic for $\sigma > 0$ except for a simple pole at $s = 1$.

At $s=1$, $\zeta(s)$ has a simple pole with residue 1, wheras.

$$\prod_{p|q} \left(1 - \frac{1}{p} \right) = \frac{\phi(q)}{q} \quad \text{by Thm 1T.}$$

$\square$

Proof (4K): Excercise (Imporant).

We are in a positcon to prove THM (4A)

Keep in mind the case 1 mod 4.

Thm (4N): Suppose that $\sigma > 1$. Then

$$\sum_{p \equiv a \bmod q} \frac{\log p}{p^{\sigma}} = \sum_{n \equiv a \bmod q} \frac{\Lambda(n)}{n^{\sigma}} + O(1)$$

Proof: Clearly the LHS does not exceed the RHS

$$\sum_{n \equiv a \bmod q} \frac{\Lambda(n)}{n^{\sigma}} - \sum_{p \equiv a \bmod q} \frac{\log p}{p^{\sigma}}$$

$$\leq \sum_{p} \sum_{m=2}^{\infty} \frac{\log p}{p^{m\sigma}}$$

$$\leq \sum_{p} \sum_{m=2}^{\infty} \frac{\log p}{p^{m}} = \sum_{p} \frac{\log p}{p(p-1)}$$

—/—

Recall $\sum_{m=2}^{\infty} \frac{1}{p^m} = \frac{1}{p^2} \sum_{m=0}^{\infty} \frac{1}{p^m}$

$$= \frac{1}{p^2} \cdot \frac{1}{1-\frac{1}{p}}$$

—/—

$$\ldots = \sum_{p} \frac{\log p}{p(p-1)}$$

$$\leq \sum_{n=2}^{\infty} \frac{\log n}{n(n-1)} = O(1) \quad \square .$$

Let $\sigma > 1$

$$\underbrace{\sum_{p \equiv a \bmod q} \frac{\log p}{p^{\sigma}}}_{\text{Does it (as } \sigma \to 1^{+}) \text{ diverage?}} \underset{\text{Thm 4N}}{=} \underbrace{\sum_{n \equiv a \bmod q} \frac{\Lambda(n)}{n^{\sigma}}}_{\text{We'll show this diverages}} + O(1)$$

$$\sum_{\substack{n \equiv a \\ \bmod q}} \frac{\Lambda(n)}{n^{\sigma}} = \sum_{n=1}^{\infty} \underbrace{\left(\frac{1}{\phi(q)} \sum_{\chi \bmod q} \frac{\chi(n)}{\chi(a)} \right)}_{\text{Thm 4C}} \frac{\Lambda(n)}{n^{\sigma}} + O(1)$$

$$= \begin{cases} 1 & \text{if } n \equiv a \bmod q \\ 0 & \text{if not} \end{cases}$$

$$\ldots = \frac{1}{\phi(q)} \sum_{\chi \bmod q} \frac{1}{\chi(a)} \sum_{n=1}^{\infty} \frac{\chi(n)\Lambda(n)}{n^{\sigma}} + O(1).$$

where the change of summation by abs conv.

$$\overset{\text{Thm 4F}}{=} -\frac{1}{\phi(q)} \sum_{\chi \bmod q} \frac{1}{\chi(a)} \frac{L'(\sigma,\chi)}{L(\sigma,\chi)} + O(1)$$

Suppose that (1)

$$\sum_{\substack{\chi \bmod q \\ \chi \neq \chi_0}} \frac{1}{\chi(a)} \frac{L'(\sigma,\chi)}{L(\sigma,\chi)} = O(1) \text{ as } \sigma \to 1^+$$

Then $$\sum_{p \equiv a \bmod q} \frac{\log p}{p^\sigma} = \frac{-1}{\phi(q)} \frac{L'(\sigma,\chi_0)}{L(\sigma,\chi_0)} + O(1) \text{ as } \sigma \to 1^+$$

$$= \frac{1}{\phi(q)} \frac{1}{\sigma - 1} + O(1) \text{ as } \sigma \to 1^+$$

because $\frac{L'(s,\chi_0)}{L(s,\chi_0)}$ has a simple pole at $s=1$ with residue -1

(Check this!)

and $\frac{1}{\phi(q)} \cdot \frac{1}{\sigma-1} \to \infty$ as $\sigma \to 1^+$

Thus, to prove Thm 4A, it remains to prove (1).

We will show, that for every non-principal Dirichlet char χ. $L(1,\chi) \neq 0$.

Thm (4P): Suppose $q \in \mathbb{N}$, χ a Dirichlet char mod q st χ is non-real. Then $L(1,\chi) \neq 0$.

<u>Proof</u>: For $\sigma > 1$, by Thm 4G.

$$\sum_{\chi \bmod q} \log L(\sigma,\chi) = \sum_{\chi \bmod q} \sum_{p} \sum_{m=1}^{\infty} \chi(p^m) m^{-1} p^{-m\sigma}$$

$$\overset{\text{abs conv}}{=} \sum_{p} \sum_{m=1}^{\infty} \underbrace{\left(\sum_{\chi \bmod q} \chi(p^m) \right)}_{\substack{\text{TM 4C (ii)} \\ = \begin{cases} \phi(q) & \text{if } p^m \equiv 1 \bmod q \\ 0 & \text{if not} \end{cases}}} m^{-1} p^{-m\sigma}$$

$$= \phi(q) \sum_{\substack{p \\ p^m \equiv 1 \bmod q}} \sum_{m=1}^{\infty} m^{-1} p^{-m\sigma} > 0.$$

$$\Rightarrow \left| \prod_{\chi \bmod q} L(\sigma,\chi) \right| \underset{(*)}{> 1} \text{ by exponentiation}$$

Suppose that χ_1 is a non-real Dirichlet char mod q st $L(1,\chi_1) = 0$.

$\overline{\chi_1} \neq \chi_1$ $(\overline{\chi_1}(n) = \overline{\chi_1(n)}$ for all $n \in \mathbb{N})$

$L(1,\overline{\chi_1}) = \overline{L(1,\chi_1)} = 0.$

In the above product $(*)$

$L(\sigma,\chi_0)$ has a simple pole at $\sigma = 1$; whereas $L(\sigma,\chi)$ is analytic for $\sigma > 0$,

χ non-prin. so $(*)$ has a zero at $\sigma = 1$, which is a contradiction.

Thm (4Q): Suppose that $q \in \mathbb{N}$, χ is a real non-principal Dirichlet char. mod q. Then $L(1, \chi) \neq 0$.

Proof: Suppose it is flase i.e. $\exists \chi$, real, non-prin st $L(1, \chi) = 0$.

Then: $F(s) = \underbrace{\zeta(s)}_{\text{simple pole}} \underbrace{L(s, \chi)}_{\text{zero at } s=1}$ is analytic $\sigma > 0$.

Note that for $\sigma > 1$, we have $F(s) = \sum_{n=1}^{\infty} f(n) n^{-s}$ where for every $n \in \mathbb{N}$.

$$f(n) = \sum_{m|n} \chi(m) \quad (\text{Thm } 3E)$$

Let $g : \mathbb{N} \to \mathbb{R}$ be the function

$$g(n) = \begin{cases} 1 & \text{if } n \text{ is a square} \\ 0 & \text{if not.} \end{cases}$$

First we will show than $f(n) \geq g(n)$ for all $n \in \mathbb{N}$.

- χ is totally mult $\Rightarrow$ f is mult (THM 1A)
Clearly g is mult.

So it suffices to check for $n = p^k$, p prime

$k \in \mathbb{N}$.

Note $\chi(n) \in \{-1, 0, 1\}$ because χ is real.

We have $f(p^k) = 1 + \chi(p) + \ldots + \chi(p)^k$

$$= \begin{cases} 1 & \text{if } \chi(p) = 0 \\ k+1 & \text{if } \chi(p) = 1 \\ 1 & \text{if } \chi(p) = -1, \ k \text{ is even} \\ 0 & \text{if } \chi(p) = -1, \ k \text{ is odd.} \end{cases}$$

So $f(p^k) \geqslant g(p^k) = \begin{cases} 1 & \text{if } k \text{ even} \\ 0 & \text{if } k \text{ odd.} \end{cases}$

25/11/13

χ - real, non-prime Dirichlet char.

$F(s) := \underbrace{\zeta(s)}_{\substack{\text{simple} \\ \text{pole at} \\ s=1}} \underbrace{L(s, \chi)}_{\substack{\text{zero} \\ \text{at } s=1}}$ analytic in $\sigma > 0$.

$F(s) = \sum_{n=1}^{\infty} f(n) n^{-s}$ for $\sigma > 1$,

$f(n) = \sum_{m|n} \chi(m)$

$g : \mathbb{N} \to \mathbb{R}$, $g(n) = \begin{cases} 1 \text{ if } n \text{ is a sq} \\ 0 \text{ if not}. \end{cases}$

$f(n) \geq g(n)$ for all $n \in \mathbb{N}$.

$0 < r < \frac{3}{2}$

$F(s)$ is analytic function for $\sigma > 0$ so we have a Taylor expansion.

$$F(2-r) = \sum_{\nu=0}^{\infty} \frac{F^{(\nu)}(2)}{\nu!} (-r)^{\nu}$$

Recall:

$$f(x) = \sum_{\nu=0}^{\infty} \frac{f^{(\nu)}(a)}{\nu!} (x-a)^{\nu}$$

in a neighbourhood of a.

So we choose $x = 2 - r$, $a = 2$.

By Thm 3B:

$$F^{(v)}(2) = \sum_{n=1}^{\infty} f(n)(-\log n)^{v} n^{-2}.$$

So, for every $v \in \mathbb{N} \cup \{0\}$.

$$\frac{F^{(v)}(2)}{v!}(-r)^{v} = \frac{r^{v}}{v!}\sum_{n=1}^{\infty} f(n)(\log n)^{v} n^{-2}.$$

and, since $f(n) \geq g(n)$,

$$\ldots \geq \frac{r^{v}}{v!}\sum_{n=1}^{\infty} g(n)(\log n)^{v} n^{-2}.$$

$$= \frac{r^{v}}{v!}\sum_{k=1}^{\infty} (\log k^{2})^{v}(k^{2})^{-2}.$$

$$= \frac{(2r)^{v}}{v!}\sum_{k=1}^{\infty} (\log k)^{v} k^{-4}.$$

$$= \frac{(-2r)^{v}}{v!}\underbrace{\sum_{k=1}^{\infty} (-\log k)^{v} k^{-4}}_{\zeta^{(v)}(4)}$$

By Thm 3B

It follows that, for $0 < r < 3/2$.

$$F(2-r) \geq \sum_{v=0}^{\infty} \frac{(-2r)^{v}}{v!}\zeta^{(v)}(4)$$

Taylor series of $\zeta(s)$
$a=4,\ x=4-2r$

$$\ldots = \zeta(4-2r)$$

Now, as $r \to 3/2$ from below.

$$F(2-r) \to \infty,$$

which is contradicting to our hypothesis that $F(s)$ was analytic for $\sigma > 0$.

$$L(1, \chi) \neq 0 \quad \square$$

$$\square$$

The Prime number theorem.

Let $\pi(x) = \sum_{p \leq x} 1$ for any $x > 0$.

Put $f(x) \sim g(x)$ as $x \to \infty$ stands for

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1.$$

Thm (5A) (PNT)

$$\pi(x) \sim \frac{x}{\log x} \quad \text{as } x \to \infty$$

Note, this does not imply $|\pi(x) - x/\log x| \to 0$ as $x \to \infty$ or is even particularly small.

ex: x^2+x , x^2

$x^2+x \sim x^2$ as $x \to \infty$

but $x^2+x \quad - x^2 = x$ as $x \to \infty$

Def: $\psi(x) = \sum_{n \leq x} \Lambda(n)$ for any $x > 0$ is called the Chebyshv Psi function.

Thm (SB): As $x \to \infty$, $\psi(x) \sim x$ iff $\pi(x) \sim x/\log x$.

Proof: Recall the proof of Chebyshev's theorem:

1) $\psi(x) = \sum_{n \leq x} \Lambda(x)$

$$= \sum_{\substack{p,k \\ p^k \leq x}} \log p.$$

$$= \sum_{p \leq x} \log p \sum_{1 \leq k \leq \frac{\log x}{\log p}} 1$$

$$= \sum_{p \leq x} \log p \left[\frac{\log x}{\log p} \right]$$

$$\leq \pi(x) \log x.$$

On the other hand, for any $\alpha \in (0,1)$

2) $\psi(x) \geq \sum_{p \leq x} \log p$

$$\geq \sum_{x^\alpha < p \leq x} \log p$$

$$\geq [\pi(x) - \pi(x^\alpha)] \cdot \log x^\alpha .$$

$$= \alpha [\pi(x) - \pi(x^\alpha)] \cdot \log x .$$

Combining (1) and (2):

$$\frac{\alpha \pi(x)}{x/\log x} - \frac{\alpha \pi(x^\alpha)}{x/\log x}$$

$$\overset{\text{divide (2) by } x}{\leq} \frac{\psi(x)}{x}$$

$$\overset{\text{divide (1) by } x}{\leq} \pi(x) \Big/ x/\log x$$

Since $\alpha < 1$

$$\pi(x^\alpha) \overset{\text{chebyshev's theorem}}{<} C_2 \frac{x^\alpha}{\log x^\alpha} = \frac{C_2 x^{\alpha - 1}}{\alpha} \cdot \frac{x}{\log x}$$

$$0 < \frac{\pi(x)}{x/\log x} < \frac{C_2 x^{\alpha-1}}{\alpha} \to 0 \text{ as } x \to \infty$$

So, suppose $\pi(x) \sim x/\log x$ as $x \to \infty$

$$\frac{\alpha \pi(x)}{x/\log x} - \frac{\alpha \pi(x^\alpha)}{x/\log x} \to \alpha \text{ as } x \to \infty$$

It follows from (3), that for any $\varepsilon \to 0$.

$$\alpha - \varepsilon \leq \frac{\psi(x)}{x} \leq 1 + \varepsilon \text{ for } x \geq C(\varepsilon), \text{ a const depending on } \varepsilon.$$

But $\alpha < 1$ was arbitrary, so

$$\frac{\psi(x)}{x} \to 1 \text{ as } x \to \infty.$$

Now, note (3) can be written.

$$(3') \quad \frac{\psi(x)}{x} \underset{(3)}{\leq} \frac{\pi(x)}{x/\log x} \leq \frac{1}{\alpha}\frac{\psi(x)}{x} + \frac{\pi(x^\alpha)}{x/\log x}$$

So, suppose $\psi(x) \sim x$ as $x \to \infty$

Then $\frac{1}{\alpha}\frac{\psi(x)}{x} + \frac{\pi(x^\alpha)}{x/\log x} \to \frac{1}{\alpha}$ as $x \to \infty$

So, for any $\varepsilon > 0$, by (3')

$$1 - \varepsilon \leq \frac{\pi(x)}{x/\log x} \leq \frac{1+\varepsilon}{x} \text{ for } x \geq C'(\varepsilon)$$

But $\alpha > 1$ was arbitrary, so

$$\frac{\pi(x)}{x/\log x} \to 1 \quad \text{as } x \to \infty.$$

$\square$.

So $\psi(x) \sim x$ as $x \to \infty$

$\Rightarrow$ PNT.

However, looking at $\psi(x)$ directly, we have difficult issue of convergence

We consider $\psi_1(x) = \int_0^x \psi(t)\, dt$ for $x > 0$

Thm (SC'): Suppose $\psi_1(x) \sim \frac{1}{2} x^2$ as $x \to \infty$
Then $\psi(x) \sim x$ as $x \to \infty$.

29/11/13

Thm (5C) : Suppose $\psi_1(x) \sim \frac{1}{2}x^2$ as $x \to \infty$ then $\psi(x) \sim x$ as $x \to \infty$

Proof : Suppose $0 < \alpha < 1 < \beta$. Since $\Lambda(n) \geq 0$ for all $n \in \mathbb{N}$, $\psi(x)$ is increasing. $\Rightarrow$

$$\psi(x) \leq \frac{1}{\beta x - x} \int_x^{\beta x} \psi(t)\, dt.$$

$$= \frac{\psi_1(\beta x) - \psi_1(x)}{\beta x - x} = \frac{\psi_1(\beta x) - \psi_1(x)}{(\beta - 1)x}$$

so $\frac{\psi(x)}{x} \leq \frac{\psi_1(\beta x) - \psi_1(x)}{(\beta - 1)x^2}$

On the other hand, for all $x > 0$.

$$\psi(x) \geq \frac{1}{x - \alpha x} \int_{\alpha x}^{x} \psi(t)\, dt = \frac{\psi_1(x) - \psi_1(\alpha x)}{(1 - \alpha)x}$$

so $\frac{\psi(x)}{x} \geq \frac{\psi_1(x) - \psi_1(\alpha x)}{(1 - \alpha)x^2}$

As $x \to \infty$,

by assumption

$$\frac{\psi_1(\beta x) - \psi_1(x)}{(\beta - 1)x^2} \sim \frac{1}{\beta - 1} \underbrace{\left(\frac{1}{2}\beta^2 - \frac{1}{2}\right)}_{\frac{1}{2}(\beta^2 - 1)}$$

$$= \frac{1}{2}(\beta + 1)$$

Also,

$$\frac{\psi_1(x) - \psi_1(\alpha x)}{(1-\alpha)x^2} \sim \frac{1}{1-\alpha}\left(\overbrace{\frac{1}{2} - \frac{1}{2}\alpha^2}^{\frac{1}{2}(1-\alpha^2)}\right)$$

$$= \frac{1}{2}(1+\alpha)$$

Since α, β are arbitrary,

$$\frac{\psi(x)}{x} \sim 1 \quad \text{as } x \to \infty \quad . \quad \square.$$

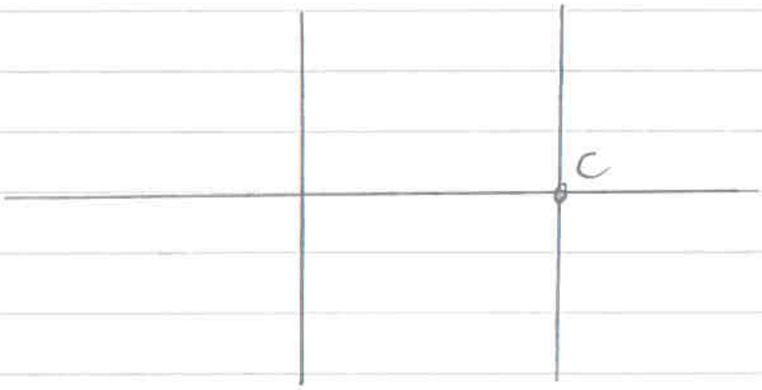
Therefore to prove the P.N.T, we're left to prove.

<u>Thm (5D)</u> : $\psi_1(x) \sim \frac{1}{2}x^2$ as $x \to \infty$

<u>Thm (5E)</u>: Suppose $x > 0$, $c > 1$. Then,

$$\psi_1(x) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)} \frac{\zeta'(s)}{\zeta(s)}\, ds.$$

where the contour is the line $\sigma = c$.



This is proved via:

Thm (SF) ← Comes up in exam

Suppose $Y>0$, $c>1$. Then

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{Y^s}{s(s+1)}\,ds = \begin{cases} 0 & \text{if } Y\leq 1 \\ 1-\frac{1}{Y} & \text{if } Y\geq 1 \end{cases}$$

if $Y=1$ $\Rightarrow 1-\frac{1}{1}=0$

Proof: Note that the integral is absolutely convergent since:

$$\left|\frac{Y^s}{s(s+1)}\right| \leq \frac{Y^s}{|t|^2}$$

$|s|\geq|t|$

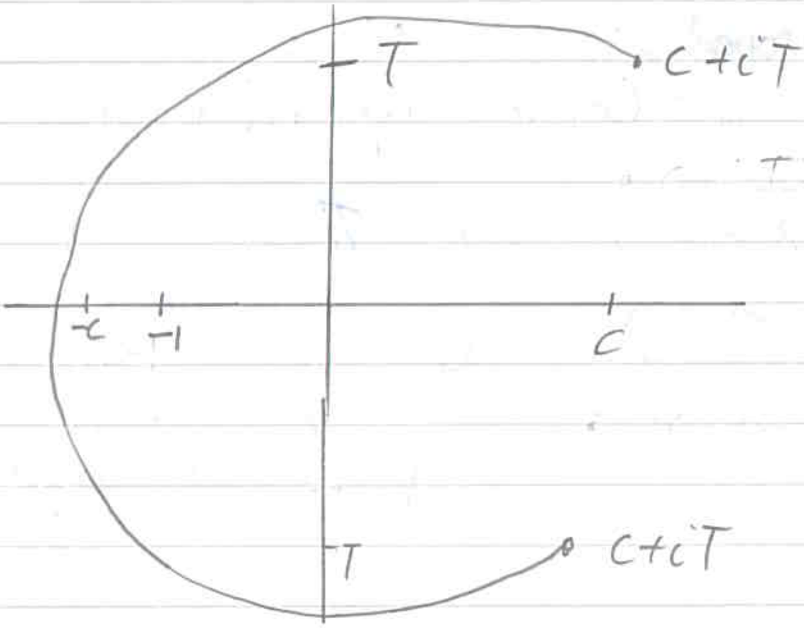
$|s+1|\geq|t|$

Let $T>1$ and write:

$$I_T := \frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{Y^s}{s(s+1)}\,ds.$$

Suppose, firstly, that $Y\geq 1$. Consider the circular arc $A^-(c,T)$ centred at $s=0$, passing from $c-iT$ to $c+iT$ on the left of the line $\sigma=c$, oriented in the clockwise direction.

i.e.



Let $J_T^- = \frac{1}{2\pi i} \int_{A^-(c,T)} \frac{y^s}{s(s+1)} ds$

On the $A^-(c,T)$

$|y^s| = y^\sigma \leqslant y^c$ since $y \geqslant 1$, $\sigma \leqslant c$.

Also, $|s| = R = (c^2+T^2)^{1/2}$, the radius of $A^-(c,T)$.

$|s+1| \geqslant |s| - 1 = R - 1$

So $|J_T^-| \leqslant \frac{1}{2\pi} \cdot \frac{y^c}{R(R-1)} \, 2\pi R$ ← circumfrence

$\leqslant \frac{y^c}{T-1} \to 0$ as $T \to \infty$ $(T>1)$

↑ $R \geqslant T$

Recall (Cauchy Residue Theorem)
If $U \subseteq \mathbb{C}$ is a simply connected subset $a_1, \dots, a_n \in \mathbb{C}$ st f is a holomorphic functon on $U \setminus \{a_1, \dots, a_n\}$ then if $\gamma \subseteq \mathbb{C}$ is a positively oriented rectifiable Jordan curve bounding the a_k (not meeting any) then:

$$\int_\gamma f(s)\,ds = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f, a_k)$$

where $\mathrm{Res}(f, a_k)$ is the residue of f at a_k.

positively oriented ↺

rectifiable → finite length

Jordan curve - continuous injective map of the circle into $\mathbb{C}$

Recall: Order of a pole of $f(z)$ at z_0 is the smallest $n \in \mathbb{N}$ st $(z-z_0)f(z)$ is holomorphic in a neighbourhood of z_0. The existence of n is the defnition of a pole (provided $f(z)$ is not holomorphic at z_0)

- If $f(z)$ has a pole of order n at z_0,

$$\mathrm{Res}(f, z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} \left((z-z_0)^n f(z) \right)$$

Ex: $f(s) = \dfrac{Y^s}{s(s+1)}$ $\quad \mathrm{Res}(f, 0) = \lim_{s \to 0} \dfrac{Y^s}{s+1} = 1$

$$\mathrm{Res}(f, -1) = \lim_{s \to -1} \frac{Y^s}{s} = -\frac{1}{Y}.$$

What about $\text{Res}\left(\frac{Y^s}{s^2}, 0\right)$

$$= \lim_{s \to 0} \frac{d}{ds}\left(\underbrace{Y^s}_{e^{s \log Y}}\right)$$

$$= \lim_{s \to 0} \log Y . Y^s = \log Y.$$

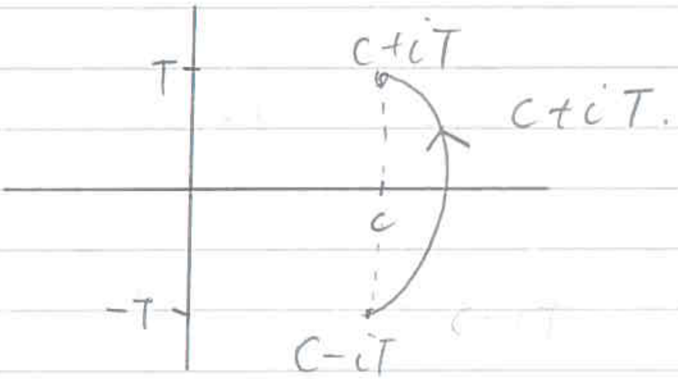
So $I_T = J_T^- + \text{Res}\left(\frac{Y^s}{s(s+1)}, 0\right)$

Cauchy $\quad + \text{Res}\left(\frac{Y^s}{s(s+1)}, -1\right)$

$$= J_+^- + 1 - \frac{1}{Y} \longrightarrow 1 - \frac{1}{Y} \text{ as } T \to \infty$$

Suppose that $Y \leq 1$.

Consider the circular arc $A^+(c,T)$. centred at $s=0$, passing from $c-iT$ to $c+iT$ on the right-hand side of the line $\sigma = c$, oriented anti-clockwise



Let $J_T^+ = \frac{1}{2\pi i} \int_{A^+(c,T)} \frac{y^s}{s(s+1)} \, ds$

$|y^s| = y^\sigma \leqslant y^c, \quad y \leq 1, \quad \sigma \geqslant c.$

Also $|s| = R, \quad |s+1| \geqslant R, \quad Re(s) = \sigma > 1$

It follows that:

$$|J_T^+| \leqslant \frac{1}{2\pi} \frac{y^c}{R^2} \cdot 2\pi R = \frac{y^c}{R} \leqslant \frac{y^c}{T} \to 0.$$

as $T \to \infty$

Since there are no singularities of $y^s/s(s+1)$ inside the contour $[c-iT, c+iT] \cup A^+(c,T)$.

$$I_T = J_T^+ \to 0 \text{ as } T \to \infty$$

□ I would learn this!

Proof (5E): For $x \geqslant 1$

$$\psi_1(x) = \int_0^x \psi(t) \, dt$$

$$= \int_1^x \psi(t) \, dt = \int_1^x \left(\sum_{n \leq t} \Lambda(n) \right) dt$$

where we have just used summation by parts

$$= \sum_{n\le X} (X-n)\,\Lambda(n)$$

$$f(x) = X - x \quad , \quad a(n) = \Lambda(n)$$

i.e. $\sum_{n\le X} (X-n)\,\Lambda(n)$

$$= \Big(\sum_{n\le X} \Lambda(n)\Big)\overbrace{(X-X)}^{0}$$

$$-\int_1^X \Big(\sum_{n\le t} \Lambda(t)\Big)(x-\underset{\uparrow\,(-1)}{t})'\,dt$$

$$\frac{\psi_1(x)}{X} = \sum_{n\le X}\left(1-\frac{n}{X}\right)\Lambda(n)$$

$$= \sum_{n\le X}\left(1-\frac{1}{\left(\frac{X}{n}\right)}\right)\Lambda(n)$$

$$\overset{\text{Thm 5F}}{=} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{2\pi i}\underbrace{\int_{c-i\infty}^{c+i\infty} \frac{\left(\frac{X}{n}\right)^s}{s(s-1)}\,ds^{(*)}}\ \text{for } c>1$$

$$= \begin{cases} 0 & \text{if } X/n \le 1 \\ 1-\frac{1}{(X/n)} & \text{if } X/n \ge 1 \end{cases}$$

Now,

$$\sum_{n=1}^{\infty} \int_{c-i\infty}^{c+i\infty} \left| \frac{\Lambda(n)\,(X/n)^s}{s(s+1)} \right| ds$$

$$\leq X^c \underbrace{\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^c}}_{\text{finite since } c>1} \underbrace{\int_{-\infty}^{\infty} \frac{dt}{c^2+t^2}}_{\text{finite: } \pi = \left[\tan^{-1}\frac{t}{c}\right]_{-\infty}^{\infty}} \qquad \Lambda(n) = O(\log n)$$

since $|s(s+1)| \geq |s|^2$ $(\sigma > 0)$

Therefore, we may change the order of summation + integration.

$$* \quad \frac{\psi_1(X)}{X} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{X^s}{s(s+1)} \underbrace{\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}}_{-\frac{\zeta'(s)}{\zeta'(s)}} ds$$

by Thm 4F

Recall Thm 4L, $\sigma > 1$, $X > 0$.

$$(1) \quad \sum_{n \leq X} n^{-s} = s \int_1^X [t]\, t^{-s-1}\, dt + [X]\, X^{-s}$$

$$= s \int_1^X t^{-s}\, dt - s \int_1^X \{t\}\, t^{-s-1}\, dt + X^{-s+1}$$

$$- \{X\}\, X^{-s}.$$

$$= \frac{s}{s-1} - \frac{s}{(s-1)x^{s-1}} - \int_1^x \frac{\{t\}}{t^{s+1}}\,dt$$

$$+ \frac{1}{x^{s-1}} - \frac{\{x\}}{x^s}$$

Letting $x \to \infty$.

(2) $\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{\{t\}}{t^{s+1}}\,dt$

The analytic continuation of $\zeta(s)$ to $\sigma > 0$

We are interested in $|\zeta(s)|$

Note that $\zeta(\sigma - it) = \overline{\zeta(\sigma + it)}$
so it suffices to study $t \geq 0$.

<u>Thm</u> (5G): For every $\sigma \geq 1$, $t \geq 2$

(i) $|\zeta(s)| = \Phi(\log t)$

(ii) $|\zeta'(s)| = \Phi(\log^2 t)$

Suppose further that $0 < \delta < 1$. Then for every $\sigma \geq \delta$, $t \geq 1$,

(iii) $|\zeta(s)| = \Phi_\delta(t^{1-\delta})$

Const. depends on δ.

Proof: For $\sigma > 0$, $t \geq 1$, $x \geq 1$, we have from (1), (2),

$$(3) \quad \zeta(s) - \sum_{n \leq x} \frac{1}{n^s}$$

$$= \frac{s}{(s-1)x^{s-1}} - \frac{1}{x^{s-1}} + \frac{\{x\}}{x^s} - s\int_x^\infty \frac{\{t\}}{t^{s+1}}\,dt$$

$$= \frac{1}{(s-1)x^{s-1}} + \frac{\{x\}}{x^s} - s\int_x^\infty \frac{\{t\}}{t^{s+1}}\,dt$$

$$(4) \quad |\zeta(s)| \overset{\text{triangle ineq}}{\leq}$$

$$\sum_{n \leq x} \frac{1}{n^\sigma} + \frac{1}{\underset{\substack{\uparrow \\ |s-1| \geq |t| = t}}{t} x^{\sigma-1}} + \frac{1}{x^\sigma} + |s|\int_x^\infty \frac{dt}{t^{\sigma+1}}$$

$$\leq \sum_{n \leq x} \frac{1}{n^\sigma} + \frac{1}{t x^{\sigma-1}} + \frac{1}{x^\sigma} + \left(1 + \frac{t}{\sigma}\right)\frac{1}{x^\sigma}$$

Since $\displaystyle\int_x^\infty \frac{dt}{t^{\sigma+1}} = \frac{1}{\sigma x^\sigma}$

$$\frac{|s|}{\sigma} = \sqrt{1 + \frac{t^2}{\sigma}} \leq 1 + \frac{t}{\sigma}$$

If $\sigma \geqslant 1$, $t \geqslant 1$, $X \geqslant 1$

$$|\zeta(s)| \leqslant \sum_{n \leqslant X} \frac{1}{n} + \frac{1}{t} + \frac{1}{X} + \frac{1+t}{X}$$

$$\leqslant \left(\log X + 1\right) + 3 + \frac{t}{X}$$

Choose $X = t \Rightarrow |\zeta(s)| \leqslant \log t + 5$

$$= O(\log t)$$

$$\Rightarrow (i) \quad (t \geqslant 2)$$

$\square$

2/12/13.

Thm (5G): For every $\sigma \geq 1, t \geq 2$,
i) $|\zeta(s)| = O(\log t)$
ii) $|\zeta'(s)| = O(\log^2 t)$
Suppose further that $0 < \delta < 1$. Then for every $\sigma \geq \delta,\ t \geq 1$
(iii) $|\zeta(s)| = O_\delta(t^{1-\delta})$

If $\sigma \geq \delta, t \geq 1, X \geq 1$ by (4)

$$|\zeta(s)| \leq \sum_{n \leq X} \frac{1}{n^\delta} + \frac{1}{t X^{\delta - 1}} + \left(2 + \frac{t}{\delta}\right)\frac{1}{X^\delta}$$

$$\leq \int_0^X \frac{dt}{t^\delta} + X^{1-\delta} + \frac{3t}{\delta X^\delta} \leftarrow 2 \leq \frac{2t}{\delta}$$

$$\leq \frac{X^{1-\delta}}{1-\delta} + X^{1-\delta} + \frac{3t}{\delta X^\delta} \quad \text{Choose } X = t.$$

$$(5)\quad |\zeta(s)| \leq t^{1-\delta}\Big(\underbrace{\frac{1}{1-\delta} + 1 + \frac{3}{\delta}}_{\text{const. depending on } \delta \Rightarrow \text{(iii)}}\Big)$$

For (ii) we could differentiate (3) wrt s and proceed similarly. Instead, suppose $s_0 = \sigma_0 + it_0$ st $\sigma_0 \geq 1$, $t_0 \geq 2$.

Let C be the circle centred at s_0 with radius $\rho < 1/2$.

<u>Recall</u> (Cauchy's integral formula)

$U \subseteq \mathbb{C}$ open subset

$f : U \to \mathbb{C}$ holomorphic function.

$D := \{ z : |z - z_0| \leq r \} \subseteq U$

$\partial D = \gamma$, $a \in Int\, D$

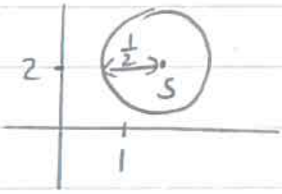
Then $f^{(n)}(a) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{n+1}} dz$.

i.e. $|\zeta'(s_0)| = \left| \frac{1}{2\pi i} \int_C \frac{\zeta(s)}{(s-s_0)^2} ds \right|$

$$\leq \frac{1}{2\pi i} \frac{M}{\rho^2} 2\pi\rho = \frac{M}{\rho} \quad (*)$$

where $M := \sup_{s \in C} |\zeta(s)|$

Note that, for every $s \in C$, $\sigma \geq \sigma_0 - \rho \geq 1 - \rho$ and $2t_0 > t \geq t_0 - \rho > 1$



$t < t_0 + 1/2$
$< t_0 + t_0$
$< 2t_0$

By (5), with $\delta = 1 - \rho > 1/2$, for every $s \in C$.

$$|\zeta(s)| \leq t^{\rho}\left(\frac{1}{\rho} + 1 + \frac{3}{1-\rho}\right)$$

$$\leq (2t_0)^{\rho}\left(\frac{1}{\rho} + \underbrace{1 + \frac{3}{1-\rho}}_{\leq 5/\rho}\right) \leq \frac{10t_0^{\rho}}{\rho}.$$

Now, $1-\rho > \frac{1}{2} > \rho$

i.e. $\frac{1}{1-\rho} < \frac{1}{\rho}$.

(*) i.e. $|\zeta'(s_0)| \leq \frac{10t_0^{\rho}}{\rho^2}$.

Let $\rho = (\log t_0 + 2)^{-1}$

Then $t_0^{\rho} = e^{\rho \log t_0} < e$

So $|\zeta'(s_0)| \leq 10e(\log t_0 + 2)^2$

$$= O(\log^2 t_0) \Rightarrow (ii) \quad \square$$

Thm (SH): $\zeta(s)$ has no zeros on the line $\sigma = 1$. Furthermore, there is a pos. const. A st for $\sigma \geq 1$

$$\frac{1}{\zeta(s)} = O((\log t)^A) \text{ as } t \to \infty$$

Proof:

$$3+4\cos\theta+\cos 2\theta = 2(1+\cos\theta)^2 \geq 0.$$

since $\cos 2\theta = 2\cos^2\theta - 1$.

On the other hand;

$$\log \zeta(s) \overset{\text{THM 4G}}{=} \sum_p \sum_{m=1}^{\infty} \frac{1}{mp^{ms}} \quad \text{for } \sigma > 1$$

So: $\log|\zeta(\sigma+it)| = \mathrm{Re}\left(\sum_{n=2}^{\infty} c_n n^{-\sigma-it}\right)$

where $c_n = \begin{cases} \frac{1}{m} & : n = p^m : p \text{ prime } m \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$

since: $\log z = \log|z| + i\arg(z)$.
(assuming principal value of logarithm)

and $\mathrm{Re}\left(\sum_{n=2}^{\infty} c_n n^{-\sigma-it}\right)$

$$= \sum_{n=2}^{\infty} c_n n^{-\sigma} \cos(t \log n).$$

since $n^{-it} = e^{(-it)\log n} = \underbrace{\cos(-t\log n)}_{=\cos(t\log n)} + i\underbrace{\sin(-t\log n)}_{\text{purely imaginary}}$

Thus, $\log|\zeta^3(\sigma)\zeta^4(\sigma+it)\zeta(\sigma+2it)|$

$$= 3\log|\zeta(\sigma)| + 4\log|\zeta(\sigma+it)| + \log|\zeta(\sigma+2it)|$$

$$= \sum_{n=2}^{\infty} c_n n^{-\sigma}\underbrace{(3+4\cos(t\log n) + \cos(2t\log n))}_{2(1+\cos(t\log n))^2} \geq 0$$

Thus for $\sigma > 1$, we have:

$$(1) \quad |(\sigma-1)\zeta(\sigma)|^3 \left|\frac{\zeta(\sigma+it)}{(\sigma-1)}\right|^4 |\zeta(\sigma+2it)|$$

$$\geq \frac{1}{\sigma-1}$$

Let $t>0$. Suppose that $s=1+it$ is a zero of $\zeta(s)$.

$\zeta(s)$ is analytic at the points $s=1+it$ and $s=1+2it$ and has a simple pole at $s=1 \Rightarrow$ LHS(1) must converage to a finite limit as $\sigma \to 1^+$. But this condradicts the fact that RHS' diverages at $\sigma \to 1^+$. Hence, $s=1+it$ is not a zero of $\zeta(s)$

6/12/13

Thm (SH): $\zeta(s)$ has no zeros on the line $\sigma = 1$ ✓

Furthermore,

$$\frac{1}{\zeta(s)} = O\left((\log t)^{A} \right) \quad \text{as } t \to \infty$$

where A is a pos abs const, for $\sigma \geq 1$

Proof: We may assume $1 \leq \sigma \leq 2$ since, for $\sigma \geqslant 2$,

$$\left| \frac{1}{\zeta(s)} \right| = \left| \prod_{p} \left(1 - p^{-s} \right) \right|$$

$$\leq \prod_{p} 1 - p^{-\sigma}$$

$$< \zeta(\sigma)$$

$$1 - p^{-2\sigma} < 1 \Leftrightarrow 1 + p^{-\sigma} < \left(1 - p^{-\sigma} \right)^{-1}$$

$$\leq \zeta(2)$$

Suppose $1 < \sigma \leq 2$ and $t \geqslant 2$. By (1)

$$(\sigma - 1)^3 \leq |(\sigma - 1)\zeta(\sigma)|^3 |\zeta(\sigma + it)|^4 \cdot |\zeta(\sigma + 2it)|$$

abs. pos. const

$$\leq A_1 \cdot |\zeta(\sigma+it)|^4 \cdot \underline{\log(2t)}$$

THM 5G(ii)

<u>Note</u> $(\sigma-1)\zeta(\sigma)$ is a real continuous function on $[0,1]$, so it abtains a max value

Since $\log 2t \leq 2\log t \quad (t \geq 2)$

$$(2) \quad |\zeta(\sigma+it)| \geq \frac{(\sigma-1)^{3/4}}{A_2(\log t)^{1/4}}$$

A_2 abs pos const.

<u>Note</u>: (2) holds when $\sigma=1$, by the first result.

Suppose $1<\eta<2$. If $1 \leq \sigma \leq \eta$ and $t \geq 2$

$$|\zeta(\sigma+it)-\zeta(\eta+it)| \geq |\zeta(\eta+it)| - |\zeta(\sigma+it)|$$

$$= \left| \int_\sigma^\eta \zeta'(x+it)\,dx \right|$$

$$\leq A_3(\eta-1)\log^2 t$$

THM 5G(ii)

length of integral.

Combine with. (2).

(3) $|\zeta(\sigma+it)| \geq |\zeta(\eta+it)| - A_3(\eta-1)\log^2 t$

$$\geq \frac{(\eta-1)^{3/4}}{A_2(\log t)^{1/4}} - A_3(\eta-1)\log^2 t.$$

On the other hand if $\eta \leq \sigma \leq 2$ and $t \geq 2$ in view of (2), (3) must still hold since.

$$\frac{(\sigma-1)^{3/4}}{A_2(\log t)^{1/4}} \geq \frac{(\eta-1)^{3/4}}{A_2(\log t)^{1/4}}$$

Therefore, (3) holds for $1 \leq \sigma \leq 2, t \geq 2$ and $1 < \eta < 2$.

Choose η so that $\frac{(\eta-1)^{3/4}}{A_2(\log t)^{1/4}}$

$$= 2A_3(\eta-1)\log^2 t.$$

i.e. $\eta \overset{*}{=} 1 + (2A_2A_1)^{-4}(\log t)^{-9}$,

where $t > t_0$ st $\eta < 2$ (def of t_0). Then.

$$|\zeta(\sigma+it)| \geq A_3(\eta-1)\log^2 t.$$

$$= A_4(\log t)^{-7} \quad \text{using } *$$

where A_4 is an abs. pos. const; $1 \leq \sigma \leq 2$, $t > t_0$.

$\square$

We are now ready to prove the P.N.T.

i.e. we would like to show THM (5D)

$$\psi_1(x) \sim \frac{1}{2}x^2, \quad \text{as } x \to \infty .$$

—/—

$$\psi(x) = \sum_{n \leq x} \Lambda(n)$$

$$\psi_1(x) = \int_0^x \psi(t)\, dt$$

$$\lim_{x \to \infty} \frac{\psi_1(x)}{\frac{1}{2}x^2} = 1$$

—/—

By THM (5E).

$$\frac{\psi_1(x)}{x^2} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} G(s)\, x^{s-1}\, ds .$$

where $c > 1$, $x > 0$, and

$$G(s) = -\frac{1}{s(s+1)} \frac{\zeta'(s)}{\zeta(s)}$$

By THM (4H), $G(s)$ is analytic for $\sigma \geq 1$, except at $s = 1$.

By THM 5G and 5H.

$$G(s) = O(|t|^{-2}(\log|t|)^2(\log|t|)^4)$$

$$|s(s+1)| \geq |t|^2$$

So $|G(s)| < |t|^{-3/2}$ for all $|t| > t_0$
(def of t_0)

Let $\varepsilon > 0$ be given. We consider a contour made up of the following line segments. $\alpha < 1$, $U > T$

$L_1 = [1-iU, 1-iT]$ "go from $1-iU$ to $1-iT$"

$L_2 = [1-iT, \alpha-iT]$

$L_3 = [\alpha-iT, \alpha+iT]$

$L_4 = [\alpha+iT, 1+iT]$

$L_5 = [1+iT, 1+iU]$

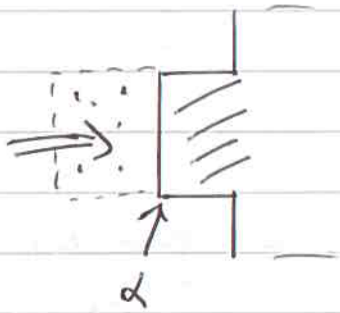
where $T := T(\varepsilon) > \max\{t_0, 2\}$, $\alpha := \alpha(T) := \alpha(\varepsilon)$
$\alpha \in (0,1)$.

and U choosen s.t.

(T) (i). $\int_T^\infty |G(1+it)|\, dt < \varepsilon$.

(recall $|G(s)| < |t|^{-3/2}$)

(α). (ii) The $[\alpha, 1] \times [-T, T]$ contains no zeros of $\zeta(s)$. This is possible since, THM SH implies $\zeta(s)$ has no zeros on the line $\sigma = 1$, so as an analytic function on $[\frac{1}{2}, 1) \times [-T, T]$ has only finitely many zeros in this region.

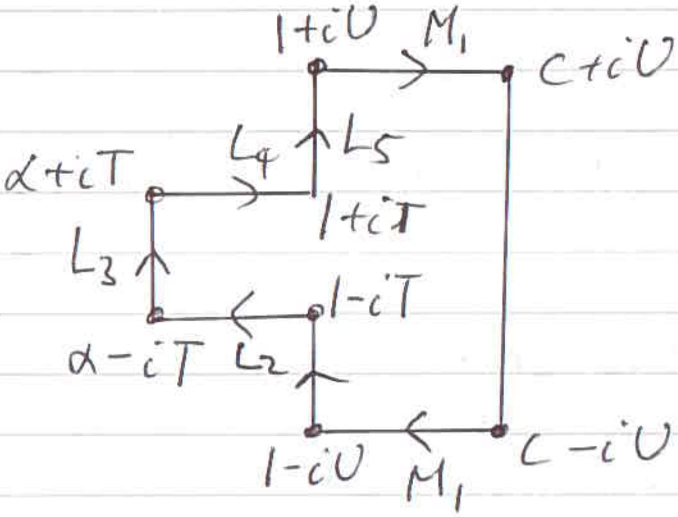


(iii) $U > T$.

Furthermore, define straight line segments.

$$M_1 = [c - iU, \ 1 - iU]$$

$$M_2 = [1 + iU, \ c + iU]$$



By Cauchy residue theorem

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} G(s)\, X^{s-1}\, ds$$

$$= \frac{1}{2\pi i}\sum_{j=1}^{2}\int_{M_j} G(s)\, X^{s-1}\, ds$$

$$+ \frac{1}{2\pi i}\sum_{j=1}^{5}\int_{L_j} G(s)\, X^{s-1}\, ds$$

$$+ \operatorname{Res}\left(G(s) X^{s-1},\ 1\right)$$

Note the pole 0, -1 are outside the contour.

We have $\operatorname{Res}\left(G(s)X^{s-1},\ 1\right)$

$$= \underbrace{\operatorname{Res}\left(\frac{\zeta'(s)}{\zeta(s)},\ 1\right)}_{-1}\ \underbrace{\lim_{s\to 1} -\frac{X^{s-1}}{s(s+1)}}_{-\frac{1}{2}} = \frac{1}{2}$$

Now. $\left|\int_{L_1} G(s)\, X^{s-1}\, ds\right|$

$$= \left|\int_{L_5} G(s)\, X^{s-1}\, ds\right| \le \int_T^{\infty} |G(1+it)|\, dt$$

$$< \varepsilon$$

On the other hand;

$$\left|\int_{L_2} G(s) X^{s-1} ds\right| = \left|\int_{L_4} G(s) X^{s-1} ds\right|$$

$$\leq M \int_{\alpha}^{1} X^{\sigma-1} d\sigma \leq \frac{M}{\log X}$$

where $M = \underset{L_2 \cup L_3 \cup L_4}{\mathrm{Sup}} |G(s)|$

$$M = M(\alpha, T) = M(\varepsilon)$$

and we use $\int_{\alpha}^{1} X^{\sigma-1} d\sigma$.

$$= \int_{\alpha}^{1} e^{(\sigma-1)\log X} d\sigma.$$

$$= \frac{1}{\log X}\left[\underbrace{e^{(\sigma-1)\log X}}_{X^{\sigma-1}}\right]_{\alpha}^{1}$$

$$= \frac{1}{\log X}\left[1 - \underbrace{X^{\alpha-1}}_{<1}\right] < \frac{1}{\log X}$$

Also, $\left|\int_{L_3} G(s) X^{s-1}\, ds\right|$

$$\leqslant 2TMX^{\alpha-1}$$

$|X^{s-1}|$

length of L_3

Since, $|G(s)| < |t|^{-3/2}$ for $|t| > t_0$

$$\left|\int_{M_1} G(s) X^{s-1}\, ds\right|$$

$$= \left|\int_{M_2} G(s) X^{s-1}\, ds\right|$$

$$\leqslant \int_1^c |u|^{-3/2} X^{\sigma-1}\, d\sigma.$$

$$\leqslant \frac{X^{c-1}}{\log X} \cdot |u|^{-3/2}.$$

where again

$$\int_1^c X^{\sigma-1}\, d\sigma = \int_1^c e^{(\sigma-1)\log X}\, d\sigma$$

$$= \frac{1}{\log X}\left[X^{\sigma-1}\right]_1^c = \frac{1}{\log X}\left[X^{c-1} - 1\right]$$

$$\leqslant \frac{x^{c-1}}{\log x} \quad \text{for } x > 1$$

Combining these estimites:

$$\left| \frac{1}{2\pi i} \int_{c-iu}^{c+iu} G(s)\, x^{s-1}\, ds - \frac{1}{2}\, ds \right|$$

$$\leqslant \frac{\varepsilon}{\pi} + \frac{M}{\pi \log x} + \frac{TM}{\pi x^{1-\alpha}} + \cancelto{0}{\frac{x^{c-1}|u|^{-3/2}}{\pi \log x}}$$

triangle inequality.

Letting $u \to \infty$

$$\left| \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} G(s)\, x^{s-1}\, ds - \frac{1}{2}\, ds \right|$$

$$\leqslant \frac{\varepsilon}{\pi} + \frac{M}{\pi \log x} + \frac{TM}{\pi x^{1-\alpha}}$$

Therefore,

$$\lim_{x \to \infty} \left| \frac{\psi_1(x)}{x^2} - \frac{1}{2} \right| \leqslant \frac{\varepsilon}{\pi} .$$

because L.H.S is independent of ϵ.

i.e. $\displaystyle \lim_{x \to \infty} \frac{\psi_1(x)}{\frac{1}{2} x^2} = 1$ □

—/—