3703 Prime Numbers and their Distribution
Based on the 2013 autumn problem classes by Mr C Daw.

(Part 2 of 2).

Excercise.

Show that:

$$\sum_{d^2 | n} \mu(d) = \begin{cases} 1 & \text{if } n \text{ is square free} \\ 0 & \text{otherwise}. \end{cases}$$

Define Mobus:

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-1) & \text{if } n = p_1 \dots p_r \text{ (distinct primes)} \\ 0 & \text{otherwise}. \end{cases}$$

Assume $f(n) = \sum_{d^2 | n} \mu(d)$ is multiplicative.

$$f(p^k) = \sum_{d^2 | p^k} \mu(d)$$

$$= \underbrace{\mu(1)}_{1} + \underbrace{\mu(p)}_{-1} + \underbrace{\dots + \mu(p^{[\frac{k}{2}]})}_{0}$$

$$= \begin{cases} 1 & \text{if } k \leq 1 \\ 0 & \text{if } k > 1 \end{cases}$$

Show that f is mult:

$f(ab) = f(a) f(b)$ if $(a, b) = 1$.

$$\{d : d^2 | ab\} \longleftrightarrow \{(u,v) : u^2 | a,\ v^2 | b\}.$$

$$f(ab) = \sum_{d^2|ab} \mu(d)$$

$$= \sum_{u^2|a,\, v^2|b} \mu(uv)$$

$$= \left(\sum_{v^2|a} \mu(u)\right)\left(\sum_{v^2|b} \mu(v)\right)$$

$$= f(a)f(b)$$

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Show that $\sum_p \frac{1}{p\log p}$ converages.

Merten's

Define $a(n) = \begin{cases} \frac{\log n}{n} & \text{if } n \text{ is prime} \\ 0 & \text{otherwise.} \end{cases}$

$$A(x) = \sum_{n \leq x} a(n) = \sum_{p \leq x} \frac{\log p}{p}$$

$$\underset{\text{mertens}}{=} \log(x) + O(1)$$

$$= \log(x) + r(x)$$

where $|r(x)| \leq C_0$, say, for all $x > 1$

Now $\sum_{p \leq x} \frac{1}{p\log p} = \sum_{2 \leq n \leq x} \frac{a(n)}{\log^2 n}$.

use summation by parts:

$$= \frac{a(2)}{\log^2 2} + \sum_{2 < n \leq x} \frac{a(n)}{\log^2 n}$$

$$= \cancel{\frac{a(2)}{\log^2 2}} + \frac{A(x)}{\log^2 x} - \cancel{\frac{a(2)}{\log^2 2}} + \int_2^x \frac{2A(t)}{t\log^3 t}\,dt$$

$$\frac{d}{dt}\left(\frac{1}{\log^3 t}\right) = \frac{-2}{t\log^3 t}$$

Recall, $A(x) = \log(x) + r(x)$

$$= \frac{1}{\log x} + \frac{r(x)}{\log^2 x} + \int_2^x \frac{2}{t\log^2 t}\,dt$$

$$- \int_2^x \frac{2r(x)}{t\log^3 t}\,dt.$$

Take abs. value:

$$\left| \frac{1}{\log x} + \frac{r(x)}{\log^2 x} + \int_2^x \frac{2}{t\log^2 t}\,dt\right.$$

$$\left. + \int_2^x \frac{2r(x)}{t\log^3 t}\,dt \right|$$

$$\frac{d}{dt}\left(\frac{-1}{\log t}\right) = \frac{1}{t\log^2 t}$$

Use triangle ineq.

$$\leq \frac{1}{\log x} + \frac{c_0}{\log^2 x} + 2\overbrace{\left(\frac{1}{\log 2} - \frac{1}{\log x}\right)}$$

$$+ 2c_0\underbrace{\left(\frac{1}{\log^2 2} - \frac{1}{\log^2 x}\right)} \to \frac{2}{\log x} + \frac{2c_0}{\log^2 2}$$

as $x \to \infty$

$$\frac{d}{dt}\left(-\frac{1}{\log t}\right) = \frac{1}{t\log^3 t}$$

$\Rightarrow$ Series conv. $\square$

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$$\sum_{n\le X} \frac{\phi(n)}{n} = cX + O(\log X)$$

for some const $c>0$ and all $X>0$.

<u>Thm 1R</u>: $\frac{\phi(n)}{n} = \sum_{d|n} \frac{\mu(d)}{d}$.

$$\sum_{n\le X} \frac{\phi(n)}{n} = \sum_{n\le X}\sum_{d|n} \frac{\mu(d)}{d}.$$

$$= \sum_{d\le X} \sum_{\substack{n\le X \\ d|n}} \frac{\mu(d)}{d} = \sum_{d\le X} \frac{\mu(d)}{d} \sum_{\substack{n\le X \\ d|n}} 1$$

$$= \sum_{d\le X} \frac{\mu(d)}{d} \left[\frac{X}{d}\right]$$

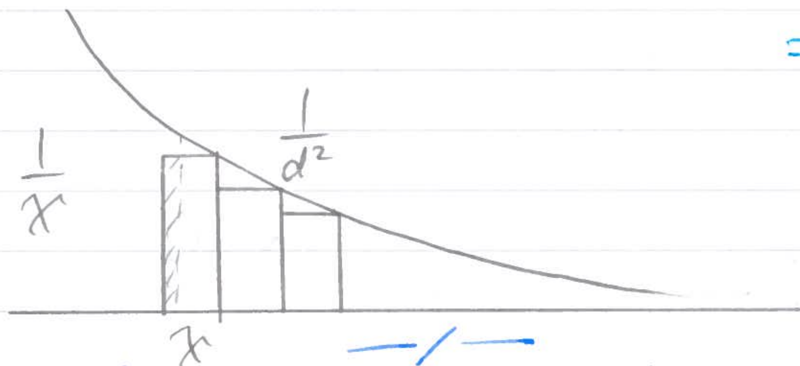
$$= X \sum_{d\le X} \frac{\mu(d)}{d^2} + O\underbrace{\left(\sum_{d\le X} \frac{\mu(d)}{d}\right)}_{O(\log X)}$$

taking $\left[\frac{X}{d}\right] = \frac{X}{d} + O(1)$.

$$\sum_{d\le x}\frac{\mu(d)}{d^2} = \underbrace{\sum_{d=1}^{\infty}\frac{\mu(d)}{d^2}}_{\frac{6}{\pi^2}\ \text{it's a const.}} - \overbrace{\underbrace{\sum_{d>x}\frac{\mu(d)}{d^2}}_{\text{use integral approximation}}}^{O\left(\frac{1}{x}\right)}$$

$$\left|\sum_{d>x}\frac{\mu(d)}{d^2}\right| \le \frac{1}{x} + \int_x^{\infty}\frac{1}{t^2}\,dt$$

$$= \frac{2}{x}.$$



Let χ be a Dirichlet char mod q, $\chi = \chi_0$.

Show that

$$\left|\sum_{n\le x}\chi(n)\right| \le q \qquad (x>0)$$

First show $\sum_{a \bmod q}\chi(a) = 0$.

<u>Sol</u>: $\chi \ne \chi_0 \Rightarrow \exists\, b\ (b,q)=1$ st $\chi(b)\ne 1$.

$$\chi(b)\sum_{a \bmod q}\chi(a) = \sum_{a \bmod q}\chi(b)\chi(a)$$

$$= \sum_{a \bmod q}\chi(ba) = \sum_{a \bmod q}\chi(a)$$

$$\underbrace{[1-\chi(b)]}_{\neq 0}\ \underbrace{\sum_{a \bmod q}\chi(a)}_{\Rightarrow\ =0} = 0.$$

So $\sum_{n \leq kq} \chi(n) = 0.$

$k \in \mathbb{N}.$

Choose $k \in \mathbb{N}$ st:

$$kq \leq x < (k+1)q.$$

So $\sum\limits_{n \leq x} \chi(n) = \sum\limits_{kq < n \leq x} \chi(n)$

So abs value of at most q terms with abs value 1 or 0 is at most q

<u>13/12/12</u>

$\chi = \chi_0$ Dirichlet char. mod q.

$|\sum_{n \leq x} \chi(n)| \leq q$.

$L(s, \chi)$. $\sigma > 0$.

Use Dirichlet's hyperbola method to show:

$$\sum_{n \leq x} \frac{f(n)}{\sqrt{n}} = 2\sqrt{x}\, L(1, \chi) + O(1)$$

$f(n) := \sum_{d|n} \chi(d)$, χ Dirichlet char mod q.

<u>Sol</u>:

Let $a(d) = \dfrac{\chi(a)}{\sqrt{d}}$

$b(e) = \dfrac{1}{\sqrt{e}}$

$Y = \sqrt{X}$

D.H.M: $* = \sum_{d \leq \sqrt{x}} \dfrac{\chi(d)}{\sqrt{d}} B\left(\dfrac{X}{d}\right)$

$+ \sum_{d \leq \sqrt{x}} \dfrac{1}{\sqrt{d}} A\left(\dfrac{X}{d}\right) - A(\sqrt{X}) B(\sqrt{X})$

A, B partial sums of a, b.

Hint: $B(x) = \sum_{n \le x} \frac{1}{\sqrt{n}} = 2\sqrt{x} + B + O\left(\frac{1}{\sqrt{x}}\right)$

$$\underbrace{A(x) = \sum_{n \le x} \frac{\chi(n)}{\sqrt{n}}} = L\left(\tfrac{1}{2}, \chi\right) + O\left(\frac{1}{\sqrt{x}}\right)$$

$$L\left(\tfrac{1}{2}, \chi\right) = \sum_{n=1}^{\infty} \frac{\chi(n)}{\sqrt{n}}$$

note this is conv. by summation by parts.

i.e. $$\frac{\sum_{n \le x} \chi(n)}{\sqrt{x}} + \frac{1}{2}\int_1^x \frac{\left|\sum_{n \le t} \chi(n)\right|}{t^{3/2}}\, dt$$

bounded

0

conv

0 as $x \to \infty$

$$\sum_{n > x} \frac{\chi(n)}{\sqrt{n}} = \frac{-\sum_{n \le x} \chi(n)}{\sqrt{x}} + \frac{1}{2}\int_x^{\infty} \frac{\sum_{n \le t} \chi(n)}{t^{3/2}}\, dt$$

$O\left(\frac{1}{\sqrt{x}}\right)$

$O(1)$

$O\left(\frac{1}{\sqrt{x}}\right)$

$$O\left(\sum_{d \le \sqrt{x}} \frac{1}{\sqrt{d}} \cdot \frac{\sqrt{d}}{\sqrt{x}}\right) = O(1)$$

D.H.M $* = \sum_{d \le \sqrt{x}} \frac{\chi(d)}{\sqrt{d}} B\left(\frac{x}{d}\right) + \left(\sum_{d \le \sqrt{x}} \frac{1}{\sqrt{d}}\right)\left(A\left(\frac{x}{d}\right) - A(\sqrt{x})\right)$

$$B\left(\frac{x}{d}\right) = \left\{\frac{2\sqrt{x}}{\sqrt{d}} + B + O\left(\frac{\sqrt{d}}{\sqrt{x}}\right)\right\}$$

$$\sum_{d \le \sqrt{x}} \frac{1}{\sqrt{d}} = B(\sqrt{x})$$

$A(x/d) - A(\sqrt{x})$

$= \left[L(\tfrac{1}{2}, \chi) + O(\sqrt{d}/\sqrt{x})\right]$

$\quad - \left[L(\tfrac{1}{2}, \chi) + O(x^{-1/4})\right]$

$= O(\sqrt{d}/\sqrt{x})$ $\quad O\left(\frac{1}{\sqrt{x}}\right)$

So $* = 2\sqrt{x}\, L(1, \chi) - 2\sqrt{x} \sum_{d > \sqrt{x}} \frac{\chi(d)}{d}$

$\underbrace{O(1)}\ -\cancel{2\sqrt{x}}\ \frac{\sum_{d \le \sqrt{x}} \chi(d)}{\cancel{\sqrt{x}}} + 2\sqrt{x} \int_{x}^{\infty} \underbrace{\frac{\sum_{n \le t} \chi(d)}{t^2}}_{O\left(\frac{1}{t}\right)}\, dt$

integral

$+ B \overbrace{\sum_{d \le \sqrt{x}} \frac{\chi(d)}{\sqrt{x}}}^{O(1)}$

$BL(1/2, \chi) + O(1/\sqrt{x})$

const

$+ \underbrace{O\left(\sum_{d \le x} \frac{\chi(d)}{\sqrt{x}}\right)}_{O(1)} + O(1)$

$\square$

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$\nu(d)$ = no of distinct prime factors of d.

Let $n \in \mathbb{N}$, $n > 1$. Show that

$$\sum_{\substack{d|n \\ \nu(d) \geq m}} \mu(d) = \begin{cases} \geq 0 & \text{if } m \text{ even} \\ \leq 0 & \text{if } m \text{ odd.} \end{cases}$$

You may use:

$$\sum_{j=0}^{m} (-1)^j \binom{\nu(n)}{j} = (-1)^m \binom{\nu(n)-1}{m}$$

$n = p_1^{u_1} \cdots p_{\nu(n)}^{u_{\nu(n)}}$ $\qquad j \leq m$

$$\sum_{j=0}^{m} (-1)^j \binom{\nu(n)}{j} = (-1)^m \binom{\nu(n)-1}{m}$$

$\geq$ if m even.

$\leq$ if m odd.

$\square$

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$d|n \quad d = p_1^{r_1} \cdots p_{\nu(n)}^{r_{\nu(n)}} \qquad 0 \leq r_i \leq 1$

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<u>Landau's Theorem</u>:

Let $a_n \geq 0$. Let $F(s) = \sum_{n=1}^{\infty} a_n / n^s$. Show that σ_0 is a singular point of $F(s)$

$F(s)$ has no analytic cont.

analytic at σ_0

Since $a_n \geq 0$ $\sigma_0 = \sigma_2 = \sigma_1$

i.e. we have uniform converage in $\sigma > \sigma_0 + \delta$ for any $\delta > 0$.

Therefore, $F(s)$ is holomorphic in $\sigma > \sigma_0$ and we can differentiate termwise.

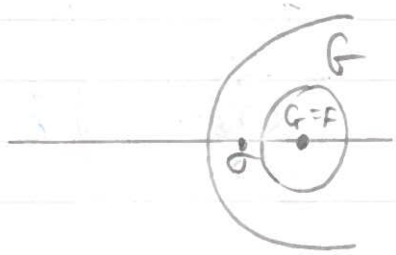
$$F^{(k)}(s) = \sum \frac{a_n (\log n)^k (-1)^k}{n^s}$$

If $F(s)$ is not singular at σ_0, there exists

$$D = \{ s : |s - \tau| < \delta \}$$

$\tau > \sigma_0$ $\delta > 0$ st $|\sigma_0 - \tau| < \delta$.

and a holomorphic function



$$G : D \to \mathbb{C}.$$

st.

$$G(s) = F(s) \quad \text{in } D \cap \{ \sigma > \sigma_0 \}.$$

So

$$\sum_{k=0}^{\infty} (-1)^k \frac{F^{(k)}(\tau)}{k!} (\tau - s)^k$$

converages abs in D.

By the termwise derivative:

$$= \sum_{k=0}^{\infty} \frac{(\tau - s)^k}{k!} \sum_{n=1}^{\infty} \frac{a_n (\log n)^k}{n^{\tau}}$$

abs conv in D.

If $s = \sigma$, $\tau - \delta < \sigma < \tau$ then: this series (conv.) consists of non-neg terms so I can interchange summation to get:

$$\sum_{n=1}^{\infty} \frac{a_n}{n^{\tau}} \underbrace{\sum_{k=0}^{\infty} \frac{\overbrace{(\tau - \sigma)^k (\log n)^k}^{(\log n^{\tau-\sigma})}}{k!}}_{n^{\tau - \sigma}}$$

$$\parallel$$

$$\sum_{n=1}^{\infty} \frac{a_n}{n^{\sigma}} < \infty$$

But: $\tau - \delta < \sigma_0 < \tau$ #

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