3802 History of Mathematics Notes

Based on the 2011 spring lectures by Mr S Rose

The Author has made every effort to copy down all the content on the board during lectures. The Author accepts no responsibility what so ever for mistakes on the notes or changes to the syllabus for the current year. The Author highly recommends that reader attends all lectures, making his/her own notes and to use this document as a reference only.

HISTORY OF MATHEMATICS

Steven Rose

Office hour: Mon 3-4 Room 806

Set text: History of Mathematics, Fauvel & Gray

Handouts
Lecture notes
Weekly essay 500-750 words every Thursday.

Exam 80% Best 4 essays (of 10) 20%

HISTORY OF MATHEMATICS

2010-11

Recommended Reading

The set anthology for the course is

J. Fauvel and J. Gray, (ed.), *The History of Mathematics – A Reader*, OU (1987).

Other useful source books are

D. J. Struik, (ed.), *A Source Book in Mathematics, 1200-1800*, Harvard University Press (1969)

Jacqueline Stedall, *Mathematics Emerging,* Oxford University Press (2008)

The standard histories are

V. Katz, *A History of Mathematics*, Addison Wesley Longman (1987)

C. Boyer and U. Merzbach, *A History of Mathematics*, Wiley (1989).

D. J. Struik, *A Concise History of Mathematics*, Dover (1987)

Other useful historical surveys include

T. Heath, *A History of Greek Mathematics*, OUP (1921)

S. Hollingdale, *Makers of Mathematics*, Penguin (1994)

C. H. Edwards, *The Historical Development of the Calculus*, Springer (1979)

E. Maor, *Trigonometric Delights*, Princeton UP (2002).

J. Stillwell, *Mathematics and its History*, Springer (2000)

Willian Dunham, *Euler- The Master of Us All*, The Mathematical Association of America (1999)

Further references will be provided on the individual coursework sheets.

HISTORY OF MATHEMATICS

Course Schedule (2010-11)

Week 1	Introduction Egyptian Mathematics Babylonian mathematics
Week 2	Early Greek Mathematics (*Thales, Pythagoras; Hippias,Archimedes,Pappus; Hippocrates, Menaechmus,Eratosthenes, Diocles; Hippocrates, Archimedes, Dinostratus*)
Week 3	Greek Philosophy of Mathematics (*Plato, Aristotle*) The Golden Age (*Euclid*)
Week 4	The Golden Age continued (*Archimedes, Apollonius*)
Week 5	Greek Astronomy (*Eudoxus, Aristarchus, Eratosthenes, Apollonius, Hipparchus, Ptolemy*) The Silver Age (*Diophantus, Pappus*) Transmission of Greek mathematics
Week 6	READING WEEK
Week 7	Indian Mathematics (**not for the exam**) Chinese Mathematics (**not for the exam**) Islamic Mathematics (*Al-Khwarizmi, Al-Uqlidisi, Al-Karaji, Al-Haytham, Khayyam, Al-Samawal*) Mathematics of Medieval Europe (*Hiyya, Gerson, Leonardo, Oresme*)
Week 8	Renaissance Algebra (*Cardano, Ferrari, Bombelli, Viète, Harriot*) Renaissance Trigonometry (*Regiomontanus, Viète, Mercator*) Renaissance Arithmetic (*Napier, Briggs*)
Week 9	Analytic Geometry (*Descartes, Fermat*) Pre-calculus (*Fermat, Roberval, Sluse; Cavalieri,Torricelli,Fermat, Roberval, Pascal, Barrow*)
Week 10	Calculus (*Newton, Leibniz*) Mechanics (*Galileo, Newton*)
Week 11	Analysis (*Euler*) Algebra (*Gauss, Galois*) Non-Euclidean geometry (*Lobachevsky, Bolyai*)

0. INTRODUCTION

We're trying to make sense of the past as with any history. We try to select significant things, i.e. things which (a) make an important mathematical contribution (so e.g. Principia)

(b) lead to an important advance (even incorrect maths can be helpful, e.g. non-Euclidean geometry)

(c) have a connection to social, political or economic contexts (e.g. Mercator projection)

We also want to establish facts, be it through:

(i) primary sources (treatises, letters, monuments etc)

(ii) secondary sources (commentaries, histories etc)

not always true, mind (Newton claimed he did Principia using calculus, not geometry — lies!)

We look at the causes of discoveries, and the previous work which inspired mathematicians. What was the social context at the time which caused people to work on these things? Although we can't really look at inspiration!

We examine the consequences and effects of discoveries, be they

(i) immediate (Napier's logarithms were taken up by Keppler in astronomy)

(ii) long term (number theory → cryptography)

Historians are also in a position to evaluate. Sometimes proofs turned out to be flawed; Euclid's work was looked at in the 19th C. and found to have some shortcomings.

I'll need to: - write short balanced essays on specific questions.

be familiar with the work and methods of mathematicians (won't have to solve problems or replicate proofs)

avoid anachronistic explanations, but modern notation is OK!

1. EGYPTIAN MATHEMATICS

There are two views:

Herodotus	5^{th} Cent. BC	practical
Aristotle	4^{th} Cent. BC	theoretical

Herodotus thought Egyptian maths came out of practical problems. Pyramids needed to be solved, taxes needed to be kept. The Nile flooded each year and it dissolved the physical boundaries of people's property. To set them back up, maths had to be set up (taxes $\propto$ land area).

Aristotle saw maths as the activity of a priestly, leisurely aristocratic class.

The truth is probably somewhere in between.

Sources: Rhind Mathematical Papyrus (now in British Museum) copied in 1600 BC from material dating back to 2000 BC. Has 84 solved problems and some helpful tables to go with it!

Moscow Papyrus (now, well still, in Moscow) has 25 solved problems.

Various monuments in Egypt with maths eiscribed on them.

Some secondary material too, namely what the Greeks said about the Egyptians.

Their script was (1) <u>hieroglyphics</u>:

Usually seen on monuments.

$\vert$	=	1
∩	=	10
?	=	100
[illegible]	=	1000

right-to-left tally system

e.g.

$|||\,||$ ∩∩∩∩ ∩∩∩ ?? = 275

Priests used <u>(2) hieratic</u>: e.g. 8 = $=$

Usually seen on papyrus.

used symbols for different numbers.

The Egyptians only used <u>unit fractions</u>. Noone really knows why,

e.g. $\frac{1}{5}$ $\frac{1}{7}$ $\frac{1}{23}$.

They even had tables to convert, say $\frac{2}{13}$, into $\frac{1}{8} + \frac{1}{52} + \frac{1}{104}$

The only exception is that they had a symbol for $\frac{2}{3}$ and $\frac{3}{4}$.

<u>Multiplication</u> was done using duplation (repeated doubling)

<u>Division</u> was done using repeated halving.

Example 1 "A quantity and its half make 16"
Done using 'false position' (ie. scaling)

Guess: 2.

$\rightarrow$ 1 $(\times 2)$ 2
$\rightarrow$ $\frac{1}{2}$ $(\times 2)$ 1
$\rightarrow$ $1\frac{1}{2}$ $(\times 2)$ 3

As many times as 3 must be multiplied to give 16, so must 2 be multiplied,
ie. answer = $2 \times \frac{16}{3}$.

What is $\frac{16}{3}$?

1 $(\times 3)$ 3 $\leftarrow$
2 $(\times 3)$ 6
4 $(\times 3)$ 12 $\leftarrow$
$\frac{2}{3}$ $(\times 3)$ 2
$\frac{1}{3}$ $(\times 3)$ 1 $\leftarrow$

$\Rightarrow \left(1 + 4 + \frac{1}{3}\right) \times 3 = 16$
$\rightarrow$ $5\frac{1}{3}$ is the scaling factor.

Scale by 2:
$1 \times 5\frac{1}{3} = 5\frac{1}{3}$
$2 \times 5\frac{1}{3} = 10\frac{2}{3}$.

$\rightarrow$ $10\frac{2}{3}$ is the answer.

Example 2: Divide 100 loaves amongst 10 men, 3 of whom get a double portion.

Technique is called these days 'ghosts'. You add 3 to 10, and then divide 100 by 13.

$$1 \times 13 = 13 \quad \leftarrow$$
$$2 \times 13 = 26$$
$$3 \times 13 = 39$$
$$6 \times 13 = 78 \quad \leftarrow$$
$$\tfrac{1}{3} \times 13 = 4\tfrac{1}{3}$$
$$\tfrac{2}{3} \times 13 = 8\tfrac{2}{3} \quad \leftarrow$$
$$\tfrac{1}{13} \times 13 = 1$$
$$\tfrac{1}{39} \times 13 = \tfrac{1}{3} \quad \leftarrow$$

$$\left(1 + 6 + \frac{2}{3} + \frac{1}{39}\right) \times 13 = 100$$

i.e. answer = $7 + \frac{2}{3} + \frac{1}{39}$.

Area of a circle: page 1 of Handout 1.

Example 2a: See P2 Handout 1, vol. of frustrum

Achievements of the Egyptians in maths:

practical, good for taxes etc, high influences on the Greeks!

Disadvantages:

- unwieldy tally nº system
- only liked unit fractions, requiring use of tallies
- Lack of general algorithms, every problem seems to be treated individually
- lack any concept of proof, they got the right answer and were happy with it
- failure to distinguish between what is exact and what is appropriate (e.g. area of circle is an approximation - were they aware of it?)

2. BABYLONIAN MATHEMATICS

Babylon is now Iraq. Their maths flourished between 2000 and 600 BC.

Their writing style is called <u>cuneiform</u> and they use wedges in wax etc. They use two signs:

𒁹 1

𒌋 10.

At first they had a tally system like the Egyptians but around 2000 they adopted a base 60 positional system.

e.g. 𒁹𒁹 𒌋𒌋𒌋𒌋𒁹𒁹𒁹 $= 2, 43 = 2 \times 60 + 43 = 163$

But ambiguity $\because$ there's no zero!

For example, the above might be $2 \times 60^2 + 43 \times 60 = 9780$

or $2 + 43 \times 60^{-1} = 2\frac{43}{60}$

Usually evident from the context though.

In later years, they used a $*$ to show a column was empty.

Why 60? Loads of divisors, making fractions easy.
They were able to express huge and tiny n°s very accurately.

Multiplication

They used tables, see p3 Handout 1.

Division

Multiplied by inverses and had tables of inverses.

e.g. $\frac{1}{1,0,45} = \frac{1}{1+\frac{45}{3600}}$

$= 59,15,33,20$ according to table
$= 0.987654321$!!!

Exponential tables

With these they could solve compound interest problems!

e.g. how long will it take to double your money at annual interest of 20% ? Ans = 3, 47, 13, 20 years
"decimal point"

Approximation to $\sqrt{2}$

See P8 Handout 1

Actual: 1.4142135 ..
Babylonian: 1.4142129

Pythagorean Triples (sides of R.A. triangles)

See P6 Handout 1, the Plimpton Tablet.

Square roots

Even if they knew $\sqrt{2}$, how did they physically derive it? Noone knows but it's thought they used a simple iterative system

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$$

BM 13901, Fauvel & Gray p31 HANDOUT 2

Problem 2 (SB p.31)

'I have subtracted the side of my square from the area : 14, 30 (870)

Write down 1, the coefficient

Break off half of 1 and multiply it by itself : 0,15 ($\frac{1}{4}$)

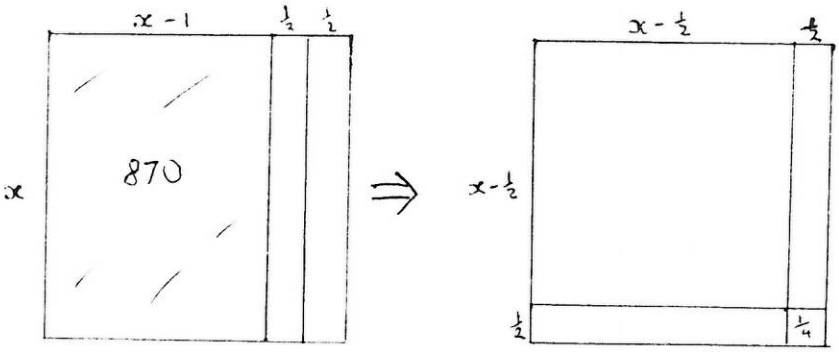
Add 0,15 to 14, 30 : 14, 30, 15 ($870\frac{1}{4}$)

This is the square of 29, 30 ($29\frac{1}{2}$)

Add 0, 30 to 29, 30 : 30 (30)

This is the side of the square.'

$$x^2 - x = 870$$
$$\left(x - \tfrac{1}{2}\right)^2 = 870\tfrac{1}{4}$$
$$x - \tfrac{1}{2} = 29\tfrac{1}{2}$$
$$x = 30$$



$$\left(x - \tfrac{1}{2}\right)^2 = 870 + \tfrac{1}{4}$$

Combinatoric formulae

They appeared to be familiar with:

$$(1+2+3+4+\cdots+n) = \tfrac{1}{2}n(n+1)$$
$$(1+2+4+8+\cdots+2^n) = 2^n+(2^n-1)$$
$$1^2+2^2+3^2+\cdots+n^2 = \left(\tfrac{1}{3}+\tfrac{2n}{3}\right)(1+2+\cdots+n)$$

Algebra

Were able to solve some linear, quadratic and even cubic equations. Their questions were usually asked in the form of practical situations and seemed to be aware of certain tricks, like transposition and 'multiplying through' to eliminate fractions. They also had grasped simple factorisation, and lastly, in some cases, they used substitutions.
And without algebraic notation!!

No negative coefficients suggests geometrical origins, though.

Completing the square

See Handout 2

Successes

(1) Positional system allowing representation of rational and irrational n°s to a high degree of precision

(2) Arithmetical techniques and inverses, and root extraction.

(3) Algebra

<u>Weaknesses</u>

(1) Absence of proof, get right answer and leave it at that.

(2) Absence of methodology, no formula that they seem to plug in to solve eqns, each one solved numerically. Must've had an idea in their minds, but not written down for us.

(3) Absence of zero causing ambiguity.

3. ANCIENT GREECE

Greece of course was centered in Athens, but there were many colonies that were very important in maths. For example, Euclid was a professor in Alexandria, Archimedes came from Syracuse in Sicily.

Greek numerals

See P10 Handout 1.

Two systems. Old one, <u>Attic</u>, essentially system of tallies again

$$45678 = \underbrace{MMMM}_{4\times 10000}\,\underbrace{\Gamma^{X}}_{5\times 1000}\,\underbrace{\Gamma^{H}H}_{5\times 100 + 100}\,\underbrace{\Gamma^{\Delta}\Delta\Delta}_{5\times 10 + 20}\,\underbrace{\Gamma III}_{5+3}$$

Replaced by <u>Ionian script</u>, based on classical Greek alphabet plus 3 additional archaic letters. Initially capitals used but later used little letters. Letters up to 1000, use a tick before letter for $\times 1000$, e.g.

$$,\alpha = 1000$$

$$,\alpha\tau\lambda\zeta = 1337$$

Used unit fractions. Put tick after letter for 'one-over', e.g. $1/34 = \lambda\delta'$

<u>Geometry</u> was the core thing, with number theory and algebraic geometry.

<u>Proof</u> – the Greeks took an interest in this!

Arguably geographic reasons for this. Greece was lots of city states, and there were lots of arguments about politics and even maths — you had to <u>prove</u> that what you were saying was true. (Maybe not!)

Three types of proof

(1) Inspection : would create a diagram and this would show the proof to be evident.

(2) Direct inference : If $A=B$, $A=C \Rightarrow B=C$.

(3) Reductio ad absurdum / proof by contradiction

<u>Influencial figures</u>

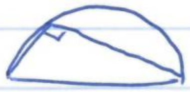
Thales 624–547 BC,

(a) credited with base angles of isosceles Δ are equal

(b) a circle is bisected by its ~~rad~~ diameter

(c) vertically opposite angles are equal

(d) angle in a semicircle $= 90^\circ$



no evidence of proofs but probably showed/gave logical arguments. <u>None</u> of his writings survive!

Pythagoras 572-497BC: Very mythical figure about whom little is known.
Founded a school of maths and philosophy, with a set of beliefs:

- transmigration of souls
- vegetarian

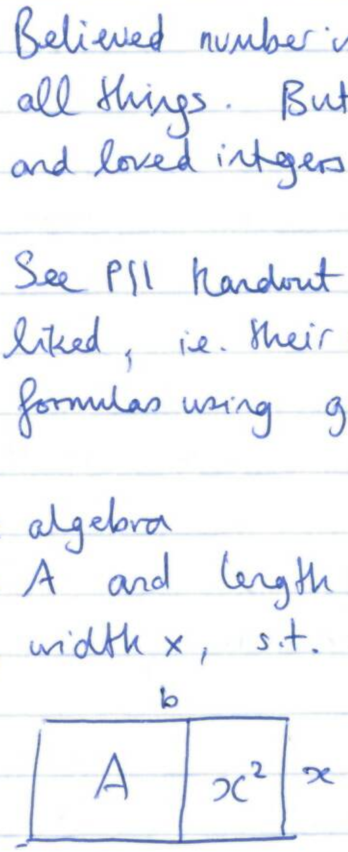
Unlikely that Pythag actually discovered his theorem, prob. one of his followers.

Discovered nº of regular solids: $\overset{6}{\text{cube}}$, $\overset{4}{\text{pyramid}}$, $\overset{12}{\text{dodecahedron}}$, to which they gave huge symbolic importance.

Believed number is the substance of all things. But disliked fractions and loved integers.

See P11 handout 1 for proofs they liked, ie. their combinatoric formulas using gnomons.

Also liked geometric algebra
e.g. Given area A and length b, construct a rectangle on b of width x, s.t. the rectangle exceeds A by x^2.



What happened to them?
THEY DISCOVERED IRRATIONAL NUMBERS !!

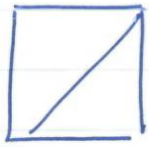
dun dun dun

They realised there exists things of "incommensurable magnitudes". Two magnitudes x,y are commensurable if there exists a unit s.t. each magnitude can be expressed as a whole nº of units,

ie. $x = a$, $y = b$ $(a, b \in \mathbb{N})$

$\Rightarrow \frac{x}{y} = \frac{a}{b} \in \mathbb{Q}$.

there is no unit s.t. the side and the diagonal can be expressed such!!



Plato (427–347 BC): A disciple of Socrates (470–399 BC), who was the most famous philosopher in Greece (all Soc's teachings were oral, no written remains).

When Greece lost to Sparta in a big war and Athens fell, people blamed Socrates because apparently he was not preparing ppl for war by asking big q^ns. They made Soc kill himself by hemlock!

Plato was disgusted, was going to become a politician but after Socrates' death he went very private & concentrated on studies. He founded the Academy in 386 BC, a school for statesmen. The statesmen bit kinda fell out but the Academy prospered.

Maths was a big part of the Academy — over the entrance said "Let noone who doesn't understand geometry enter here".

While at Academy, Plato produced his Dialogues, where characters argue about philosophical concerns. (one of the characters was Socrates!!)

Plato was not the only philosopher in town. Plato offered knowledge for its own sake, but his rivals <u>Sophists</u> offered knowledge for practical purposes. Plato was not amused.

Had an idea of <u>ideal forms</u>

Common nouns	vs	Proper nouns
little letter	vs	capital
dog	vs	Pippa

These are called universals

These are called individuals

In order for discourse to happen, we need to have <u>universals</u> in language. But is there such thing as <u>dog</u>? Where can we find <u>dog</u>? Do universals exist?

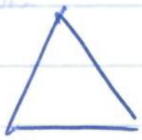
Plato belonged to the Realist or Essentialist school.
He believed that universals exist.

(the opposite view is held by Nominalists)

Plato thought that in each category there is an Ideal Form, ie. Pippa is a dog because she resembles the archetypal dog.

But where is this archetypal dog? Somewhere in the heavens.
What's this to do with maths?

GEOMETRY.

When a geometer draws  in the sand, he is actually interested in the <u>general</u> triangle, not this lame attempt!!

The highest aim of philosophy is to gain Insight into the Transcendental World.

Plato's influence: doubtful whether Plato offered any useful mathematics, but his influence was important

* maths was core of curriculum at Academy
* split into arithmetic
  geometry (plane + solid)
  astronomy
  music

  "quadrivium" in Medieval times, became centre of Medieval uni's.

* interest in 'ideal forms' engendered a bias towards pure maths (many followers despised applied maths!), exists today and has continued!

* Academy became important centre of research, especially in maths. ~~Most~~ Greatest scholars of Ancient Greece congregated there

Pupils

(a) Theatatus (417–369 BC) [not really a pupil, he was a learned collaborator]
- immensurable magnitudes
- surds e.g. $\sqrt{4+\sqrt{3}}$
- Euclid Elements X is mainly Th's work.
- discovered ^{8}octahedron, ^{20}icosahedron.

(b) Eudoxus (408–355 BC)
- greatest mathematician of 3^{rd} C. BC.

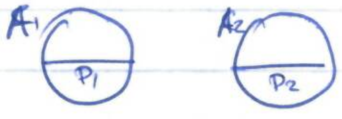
- ratio defⁿ: $a:b = c:d$ (rational and irrational numbers)

  iff given $m, n \in \mathbb{N}$

  $ma \lesseqgtr nb \implies mc \lesseqgtr nd$

  (10b of Euclid's Elements V).

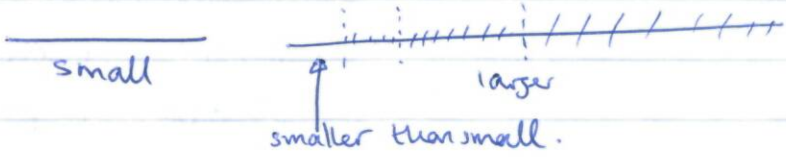
- credited by Archimedes as proving theorem that areas of circles are to one another as the squares of their diameters.



$$A_1 : A_2 = D_1^2 : D_2^2$$

this proof involves Eudoxus' method of exhaustion.

"Two unequal magnitudes being given, if from the greater there is subtracted a magnitude greater than its half, and from that which is left a magnitude greater than its half and so on, then there will be left some magnitude less than the lesser of the 2 original magnitudes" (given in Elements X).



- the first mathematical astronomers, trying to account for motion of heavenly bodies by introducing rotating spheres.

Aristotle (384–322BC) spoke with a Northern accent! The leading pupil of the academy - Plato called him the MIND!! He expected to become the director of the Academy after Pluto but he was overlooked so he did some marine biology.

He later came back to Athens and founded the Lycaeum. But then there was lots of anti-Macedonian feeling after death of Alexander the Great. Feeling threatened, Aristotle left!

He founded logic and zoology.

He made big contributions to politics, history, literary criticism, ethics, law, botany.

(1) Distinguished between
- definitions
- axioms (common to all sciences)
- postulates (specific to subjects)
- hypotheses

assumptions: axioms, postulates

→ Euclid.

(2) Insisted that def^ns do not assert existence, which must be demonstrated (e.g. I can define an equilateral △ with 3 equal sides, but does it exist, can I construct it?)

(3) Distinguished between geometrical truth and the diagram which illustrates it

(4) Distinguished between convergent and divergent series

(5) Distinguished between deductive sciences such as maths and empirical sciences such as physics and philosophy. logic, geometry

(6) Argued that not everything can be demonstrated - danger of infinite regress ("first have to create the universe")

(7) Distinguished between number (discrete) and magnitude (continuous), conflated by Pythagoreans.

THREE CLASSICAL PROBLEMS

1. Trisecting the angle
2. Doubling the cube
3. Squaring the circle

} construction problems

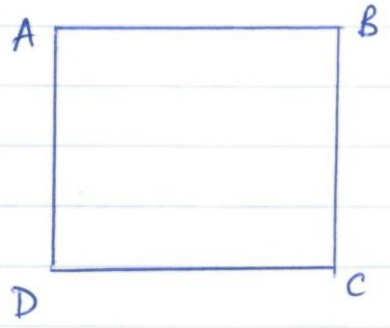
'cutting edge' of Greek mathematics

Types of sol^n: plane (compass and straight-edge)
solid (conics)
linear (transcendental curves)

In the 19th Cent. it was proved you can't do these 3 with plane methods alone.

Trisecting the angle

1. Hippias's trisectrix sol^n.

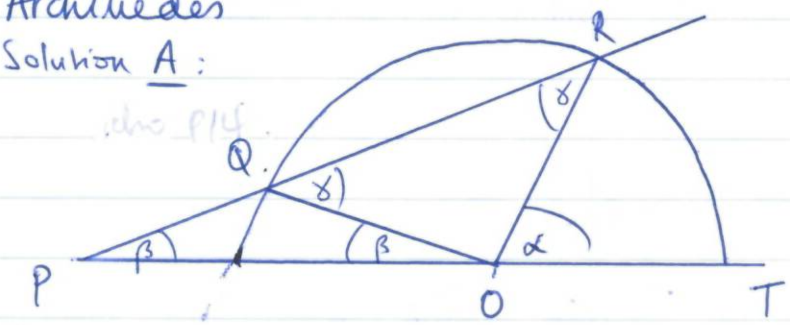


Imagine AB moves downward at const speed **and** AD rotates about D at const speed so they finish together

See Handout 1 P14.

Useless as a practical means!

2. Archimedes
Solution A:



Aim: trisect $R\hat{O}T = \alpha$

Draw semicircle centre O, radius OR.

Choose P s.t. $PQ = QO$ (can only use a ruler for this, not a straight-edge. This is called a NEUSIS)

Then $\underset{\beta}{O\hat{P}Q} = \frac{1}{3}\underset{\alpha}{R\hat{O}T}$

Proof: $\alpha = \beta + \gamma$ (external angle); $\gamma = 2\beta$ (ext. angle) $\Rightarrow \alpha = 3\beta$.

Solution B: this Spiral (see P14 again)

Doubling the cube

ie. construct a cube with double the volume
(ie. find the size of the side)

See P15 Handout 1.

Hippocrates discovered that the problem can be reduced (this is interesting - Greeks were familiar to reducing problems to things they were familiar with):

"mean proportionals"

Give a, b find x, y s.t

$$a : y = y : x = x : b \qquad \left(\underset{(1)}{\frac{a}{y}} = \underset{(2)}{\frac{y}{x}} = \underset{(3)}{\frac{x}{b}}\right)$$

$$\Rightarrow x^3 = ab^2$$

(1), (3) $\Rightarrow$ $xy = ab$ hyperbola
(2), (3) $\Rightarrow$ $y = \frac{x^2}{b}$ parabola

So for $a=2$, $b=1$, the problem is solved.

How to find mean proportionals?

1. Menaechmus (pupil of Plato)
   first recorded mention of conics, sol^n is given by pt of intersection between parabola and hyperbola.

new shapes!

Proved this using analysis and synthesis.

analysis: Assume sol^n found.
Deduce logical consequences till you arrive at something you know to be true.

synthesis : start at what you know to be true.
work backwards till you obtain the
desired sol^n

this is a <u>new</u> and <u>fruitful</u> way of solving
problems! we do it all the time.

<u>So</u>, (1) Hippocrates reduced it to mean proportionals
(2) Menaechmus reduced to parabola and hyperbola
(3) Archytas (428–347 BC)
intersection of circle, cone, torus (3D!!) gives mean prop's.

P16 (4) Eratosthenes (③BC mechanical device to find mean prop's).
(5) Diocles (②BC)

first device found
to solve a math problem!

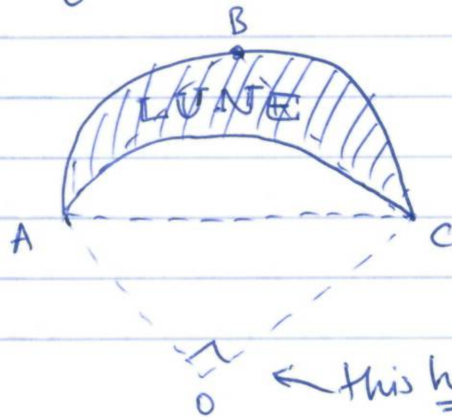
Plato wouldn't approve ☺

Squaring the circle

ie. draw a square equal in area to a given circle.

1. Hippocrates: he's not squaring the circle (nor did he think so)
but he is the first person recorded to find
the quadrature of a curved figure: a lune

See P17

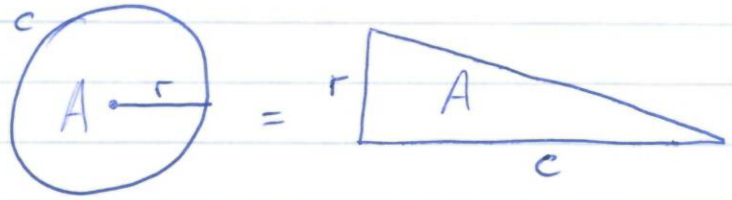


Area of lune =
area of ΔABC

← this <u>has</u> to be 90°.

You can actually only ~~st~~ do this with 5 different lunes!!

2. Archimedes : Area of circle = area of right-angled Δ whose sides are equal to radius and circumference



i.e. $A = \pi r^2$ in today's notation

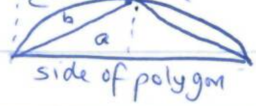
How does he prove it? Typically of Archimedes, reductio ad absurdum

discounts $A_C < A_T$ discounts $A_C > A_T$

But how did he get the right answer in the first place?!

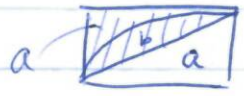
[See P18 Handout 1.]

Probably used Eudoxus' method of exhaustion

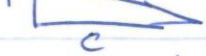


So if doubling the sides of the polygon cuts off more than half each time, we can do this.

if $a > b$ then we can use Eudoxus' method of exhaustion. Clearly so since



<u>Problem with this</u> : gives



↑ what is c !!

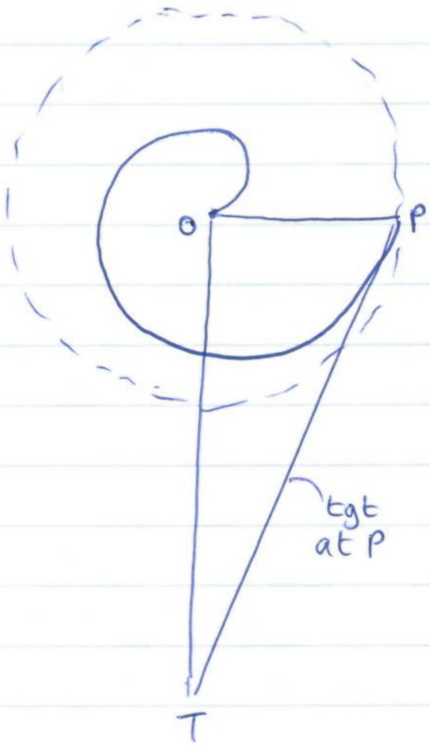
Reduced problem of <u>quadrature</u> to <u>rectification</u> !

↑ finding c.

How to solve rectification?

[P20 Handout 1]

Archimedes does it using his spiral ← $r = a\theta$



$C = OT$.

and area of circle = area of ΔOPT

3. Dinostratus quadratix

[P19 Handout 1]

Dinostratus (4th C. BC) found that Hippias' trisectrix can be adapted into a quadratix for squaring the circle.

Summary

The 3 classical problems:

(i) Prompted discovery of new curves (conics, cissoid, quadratix)

(ii) Inspired problem-solving techniques such as reduction of a problem, method of analysis + synthesis.

EUCLID

Very little is actually known of Euclid.

(1) Studied at Academy?
Taught at the Museum in Alexandria (c. 300 BC)

(2) Elements, Data, Optics are works he wrote
Many works did not survive, e.g. Conics
"treaties"

(3) The Elements is a compendium of intermediate mathematics, namely geometry, number theory and geometric algebra

There are 485 (synthetic) propositions in 13 books. All he gives us is the proof, we don't know how he found them.

'cutting edge'
But no advanced work on conics and no applied maths

(4) First 4 books of Elements are a distillation of much older material, going back to the Pythagoreans.
Later he draws on the work of Theatatus and Eudoxus for up-to-date stuff.

Proclus, writing 5th Cent AD, argued Euclid had 2 intentions
(i) Writing a manual for students
or (ii) To lay bare the mathematics of the cosmos !!
[e.g. 5 solids P24 Handout 1]

Elements

They begin with the { definitions, axioms, postulates } ← recall Aristotle required these before maths could begin

This is considered to be Euclid's own, original contribution.

[see P21 Handout]

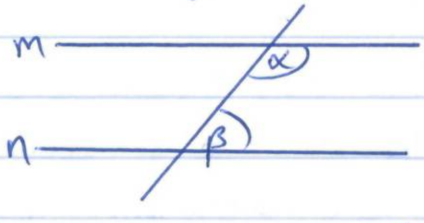
Some of these def^n's are a bit dodgy (e.g straight line) but modern mathematicians take things like this as given.

[P22] How do you know if something is parallel? Def^n 23 does not give an easy way of deciding if they're not "produced".

The postulates are quite interesting:

the first 3 underpin the construction of compass/straightedge. #1 interpreted as meaning only 1 straight line between 2 pts

postulate n° 5 is the Parallel Postulate:
given 2 lines and a transversal. If the transversal creates interior angles on both sides s.t. their sum is less than 2 rt angles, then the two lines will meet



$\alpha + \beta < 2 \times 90° \Rightarrow m, n$ meet

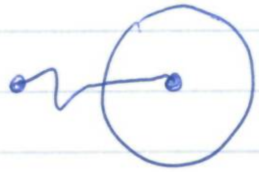
aka Playfair's axiom "

← only one line can be drawn through here which does not meet the first line"

Mathematicians were very unhappy with this as an assumption, and tried to prove it for 2000 years. In 1800s, Gauss and Lobachewsky and someone else realised you couldn't prove/disprove it, it just describes a type of geometry.

For many years, Elements' propositions were taken as absolute truth. Mathematicians now say the postulates are incomplete: you need more.

↳ e.g.



the line must intersect the circle.

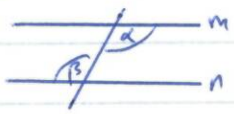
used in construction of equilateral triangle "principle of continuity".

Book I

- Constructions e.g. equil. Δ.
  they are existence thms – if you can't construct it, how can you say it exists?

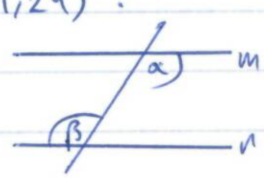
- Triangle theorems
- First 28 thms are 'neutral' theorems (no use of parallel postulate)
- Parallel postulate used for nº 29. (alt. angle thm)

- Last neutral thm $(\overset{\text{book}}{1},\overset{\text{nº}}{28})$ and $(1,27)$



$\alpha=\beta \Rightarrow m \parallel n$

- $(1,29)$:

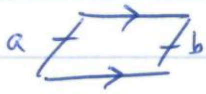


$m \parallel n \Rightarrow \alpha=\beta$

logically equivalent to parallel postulate

- He then uses the alternate angle thm to prove lots of stuff, e.g.

$\alpha + \beta + \gamma = 2 \times 90^\circ$

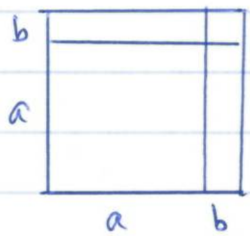


$a = b$

<u>Book II</u>

- Lots of geometric algebra from the Pythagoreans

e.g.

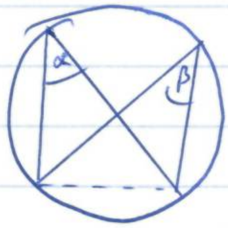


$(a+b)^2 = a^2 + 2ab + b^2$

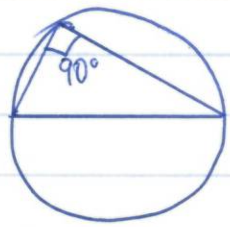
<u>Book III</u>

- Material on the circle

e.g.



$\alpha = \beta$



<u>Book IV</u>

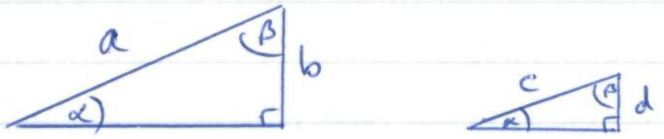
- Constructions of regular polygons
- needs to construct these in the plane to construct solids in 3D.

<u>Book V</u>

- Ratios
- material from Eudoxus

Book VI • Similar polygons

e.g. corresponding sides of similar triangles are proportional



$$a : b = c : d$$

[see P12 Handout 1]

- construction of mean proportional
- construction of square root.

Books VII, VIII, IX

- Number theory
- contain Euclidean algorithm
  - infinite nº of primes
  - sum of geometric progression

$$ar - a : S_n = a^{r} - a$$

i.e. $$\frac{a(r^n - 1)}{r-1} = S_n$$

Book X

- Surds e.g. $\sqrt{2+3\sqrt{5}}$
- longest book
- from work of Theatalus

e.g. $x + \sqrt{y} = a + \sqrt{b}$ for $\sqrt{y}, \sqrt{b} \notin \mathbb{Q}$

$$\Rightarrow \begin{cases} x = a \\ y = b \end{cases}.$$

Books XI, XII, XIII • Solid geometry culminating in construction of 5 regular solids – no others! [P12 H.1] [P24 H.1]

For over 2000 years, the material in the Elements was taken to be almost synonymous with geometry and the geometry of space as well (ie. there can be no other geometry!)

Only the Bible exceeds Elements in the nº of printed editions! It was a taught book until C20
↳ most successful textbook ever!!

ARCHIMEDES

Archimedes (287 – 212 BC) was a mathematician and physicist, making valuable contributions to
- geometry
- astronomy
- statics
- hydrostatics (first recorded use!)

He was an inventor and invented
- the screw pump.
- engines of war, magnifying glasses burning their ships the enemy's

He lived in Syracuse.

Eventually Syracuse fell to the Romans and Archimedes was killed by an ignorant Roman soldier.

Wrote a nº of treatises, most of which survived:

(1) Measurement of Circle

(a) area of circle = $\frac{1}{2} \cdot$ radius $\cdot$ circumference

$(= \pi r^2)$

(b) $3\frac{10}{71} < \pi < 3\frac{1}{7}$

"ratio of circumference : diameter"

Best approximation of the ancient world and as good as it got till the Renaissance.

Better than Egyptians 3
Babylonians 3.16

(2) Quadrature of parabola

Given a parabola, take 2 pts on it. The area of the parabola = $\frac{4}{3} \times$ area of triangle

Tools: (i) Eudoxus' method of exhaustion
(ii) understanding of 'symptoms' of a parabola
(iii) Euclid's area of triangle
(iv) His own summation formula [P30 H.1]

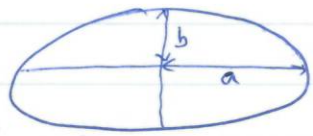
(3) Spirals

(a) Subtangent = circumference of first circle
in squaring the circle [P20 H.1]

(b) Area beneath the spiral = $\frac{1}{3}$ area of first circle
(a very stunning piece of what we'd call today integration)

(4) Conoids

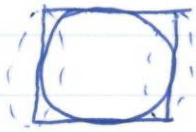
area of ellipse = πab



(5) On the sphere and the cylinder

vol. of sphere = $\frac{2}{3}$ vol. of circumscribed cylinder

(equiv. to $V = \frac{4}{3}\pi r^3$)

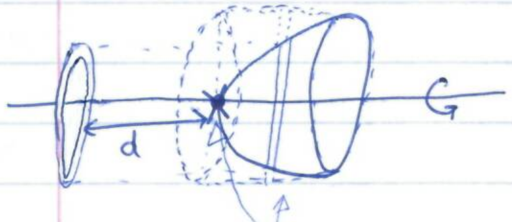


How Archimedes obtained his results was lost very very early in AD, and was only discovered by Hiburg in 1906. He heard that in Constantinople that there was a religious text that had been written over the top of Greek mathematics: the greatest mathematical find of all time.

(6) Method

Mechanical method, based on the law of the lever.
He would mentally 'weigh' it against a solid of known vol.

(i) Inscribe solid (e.g. paraboloid) within a solid of known centre of gravity (e.g cylinder)



(ii) Treat both solids as generated by rotating plane curves about a common horizontal axis.

(iii) Place a fulcrum at one extremity on the axis

(iv) Make an arbitrary vertical cut, of infinitissimal width, producing circular crosssections

(v) Use the known properties of the curves to show there exists a constant distance s.t. the circle in the parabola, if placed at that distance from the fulcrum, will balance the circle in the cylinder.

(vi) Assume that solids are composed of infinite n° of such slices

(vii) Whole parabola at distance d balances the whole cylinder <u>where it is</u>. It follows that vol. of cylinder = 2. vol of paraboloid

While Archimedes' work in applied mathematics (e.g. centre of gravity, hydrostatics) became influential in 16^{th} C, his geometry had comparatively little technical influence. His results (e.g. π) were admired but his methods were hard to replicate since they depend on knowing the right answer beforehand. In addition, his work only became readily available in the Middle Ages.

Euclid vs Archimedes: look at
- originality → Arch. wins → e.g. spirals
  (Euclid's work came from Platonites)
- range → Arch. wins: huge range
  Eucl. nº theory, geometry
- influence → Eucl. wins elements!
  → Arch loses arguably calculus may not have developed if not for quadrature

APOLLONIUS

with Euclid & Archimedes, the 3 figureheads of the Golden Age of Greek Mathematics.

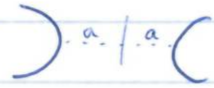
Apollonius (262-190BC) came from Perga a colony in Asia Minor (Turkey) thought to have studied in Alexandria and taught in the Museum. He is well known in ancient Greek world for his astronomy. Known for his CONICS.

Conics are slices of cone [P32 H.1]

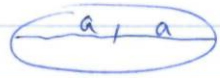
Apollonius derived the eqⁿs of these curves (but they didn't call it that of course). [P33].

Parabola: $y^2 = px$ (p parameter)

Hyperbola: $y^2 = x\left(p + \frac{p}{2a}x\right)$ where 2a = dist between 2 hyperbola



Ellipse $y^2 = x\left(p - \frac{p}{2a}x\right)$ where 2a = length of major axis



He wrote books, some based on Euclid, some his own work.

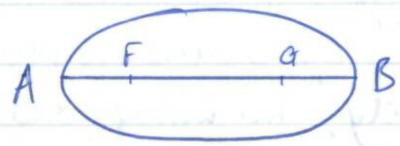
8 books → 4 Greek
3 Arabic
1 lost

Constructions: e.g. draw a tangent to a conic

Take $AE = ED$
AP is tgt.

e.g. construct a normal to a curve. [P34 H.1]

Focal properties of conics

e.g. 

$AF \cdot FB = \frac{1}{4} p AB$

Used in Newton 1800 years later in his work on planetary motion!

Didn't really invent coordinate geometry. His curves are not defined by equations, instead the curves produce certain symptoms or properties (the eq^n).

<u>Influenced</u>: (1) Analytic geometry (Descartes – pappus prob.)
(Fermat – drawing curves of eq^{n}s)
(2) Mechanics (Newton – elliptical orbits)

<u>Greek Astronomy</u>

[see P35 H.1] It was interest in the stars that brought about trigonometry (first spherical, later plane)

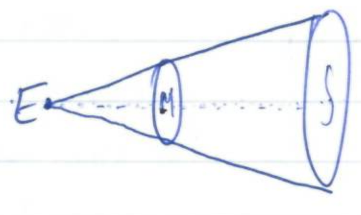
A geocentric concept, of course.

There are 6 phenomena that need to be accounted for, see handout 1.

Eudoxus came up with concentric spheres – 27 of them!!
In later years, ~~…~~ people believed in the spheres way up to the European Renaissance! (First mathematical astronomer, though).

Aristarchus, a contempory of Archimedes, was the first astronomer to propose a heliocentric system with the sun at the centre. Unfortunately, his writings and idea was lost. Taken up again by Copernicus in 16. Attempted to find angles between ~~sun~~ sun and moon (~ 87°) to find distance to moon vs distance to sun (~ 1/18, way off by a long way, ~ 1/400 !! Mistake is in measurement! 87° should be 89°50′).

Then he determined (he thought) the relative size of the sun and the moon. He noticed sun and moon ~~look~~ look same size.

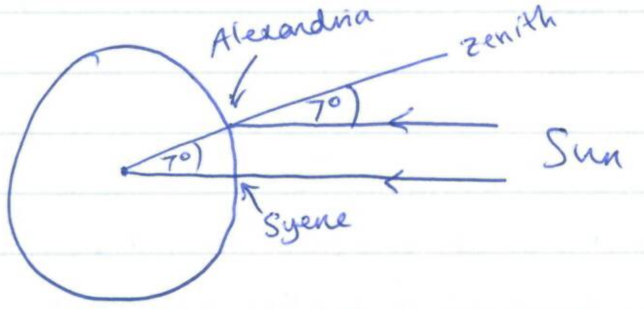


So he concluded (wrongly) that $\dfrac{\text{Radius earth}}{\text{Radius moon}} \approx \dfrac{60}{19}$

$\dfrac{\text{Radius sun}}{\text{Radius earth}} \approx \dfrac{43}{6}$

N°s wrong but mathematics impeccable!!

Eratosthenes is credited with the first accurate measurement of the circumference of the earth ~~but~~ by comparing the altitude of the sun in two places.



dist from Alex.
to Syene = 5000 stades

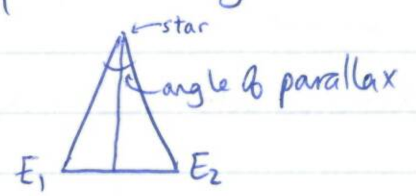
$7^\circ \times 50 = 350^\circ \approx 360^\circ$

$\times 50$ = 250,000 stades
$\approx$ 30,000 miles.
Not bad.

Apollonius proposed a better solution to Eudoxus' spheres model by coming up with ones that explain the inequality of the seasons.

He came up with epicycles and eccentric motion, and combinations thereof to explain the retrograding planets.

Hipparchus came up with bright idea of determining the moon's parallax, ~~e.g.~~



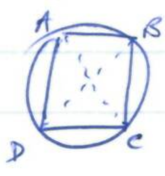
[see P39 H.1]

Ptolemy, the most important astronomer of antiquity.
He wrote the *Almagest*, the greater treatise, intensively studied right up to the Renaissance.

[see P40 H.1]

Constructed a table of chords (sines)
using his famous thm that within a cyclic quadrilateral,

$(AB \cdot CD) + (AD \cdot BC) = AC \cdot BD$



Like most astronomers, he found it very difficult to remove the idea of circular motion, for circles were so perfect! It was Keppler much much later who suggested the planets move in ~~circular~~ elliptical orbits.

SILVER AGE

There was a later burgeoning of maths again in AD years, centred in Alexandria.

Ptolemy was one (AD 100–178)

Diophantus (→ 3rd cent. AD?) was the first sight of algebra! He came up with a kind of algebraic notation,

e.g.

$$x^3 + 13x^2 - 5x + 2$$

$$K^Y \alpha \quad \Delta^Y \iota\gamma \quad \pitchfork \quad \varsigma \varepsilon \quad M^{\circ} \beta$$

K^Y	α	Δ^Y	$\iota\gamma$	$\pitchfork$	ς	ε	M°	β
Y unknown, K cubed	1	Y unknown, Δ squared	13	−	3 unknown	5	x° (unit)	2

big advance on the verbal algebra of the Babylonians

Solves a nº of determinate and indeterminate problems.

He's known for his work, 'Arithmetica'.

He was able to solve quadratic eqⁿs but ~~a~~ if given 2 solⁿs, he'd discard the smallest!

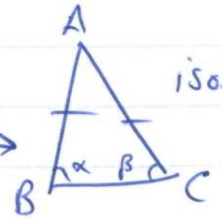
Presumably he'd be like "if you want more answers, work it out yourself".

Led to Diophantine eqⁿs — Fermat's Last Thm was scribbled in margin of Diophantus.

Pappus was the last of the great Greek mathematicians (early 4^{th} Cent. AD). His 'Collection' are 8 manuscripts written to rejuvinate mathematics. He provided this manual of older Greek mathematicians.

[See H1 P42] (last page)

But he also told us about these mathematician, so he's also a source.

He offers alternative proof of
(a) Elements Book I Thm 6 : that $\alpha = \beta$ → 

isos.

The original Euclid's thm is very long + laborious
(it was called 'the bridge of asses' - if you didn't understand it, give up mathematics!)

Also provided excellent description of analysis + synthesis.

Also proposed a generalised version of the problem first solved by Apollonius, called the Problem of Pappus.

Influenced on Descartes and Fermat's analytic geometry. and higher degree curves.

DECLINE OF GREEK MATHEMATICS

The Death of Hypatia, the daughter of Theon (AD 355-415)
↑
female mathematician

not a Christian. Encountered opposition of local patriarch. Cyril said she was a sorceress and incited a mob who killed her.

Christianity ~~local~~ Roman religion at time.

This essentially closed the book on Greek mathematics (in Alexandria, anyway).

<u>Reasons</u>: (1) Lack of support from Roman empire (Romans not interested in pure maths). As years progressed, knowledge of Greek (language of culture) died. That neglect of not promoting maths by the Romans took its toll.

(2) Destruction of library at Alexandria, some on purpose, some accidentally. Smashed up by 4th Cent AD. So much was lost. 1,000,000 manuscripts were at one point in the library. Most lost so very difficult for mathematical culture to flourish in those circumstances

(3) Christian prejudice against 'pagan' learning. A fundamentalist strain took over the Roman leaders: the whole truth is in the Bible. People otherwise are talking dangerous nonsense!

In AD 529, Emperor Justinian closed down all 'pagan' schools, including The Academy after 800 years.

This ushered in a period of decline that lasted about 1000 years in Europe. Elsewhere in Islamic/Chinese world, other stuff was happening...

NFE

CHINESE & INDIAN MATHEMATICS

Chinese Mathematics

this shape

(means the Gnomon)

(1) book 'Chou Pei', c. 300 BC
Right-angled triangle
manipulating fractions

(2) 'Nine chapters' (BC)
246 problems
taxation problems
geometrical areas
algebra
first recorded use of solving linear eq^{n}s using matrices!!
→ row echelon form!!!

(3) 'Sun Zi' (3rd cent. AD)
linear congruences
Chinese Remainder Thm. [See H3 P1]

(4) Jia Xian (11th cent. AD)
Pascal's triangle
Binomial coefficients to solve polynomial eq^{n}s
extracting nth roots.

Geared towards practical problems; did not dwell of proofs. The tradition of Chinese maths declined in the 17th when they made contact with the West.

(Of course, that's reversed recently ...)

Indian Mathematics

(1) Sulvasutras 800 BC
Pythagorean triples
Geometric algebra

(2) Number system (6^{th} C. AD)
Separate symbols for 1–9
Positional decimal system

decimal base – China
positional system – Babylon
zero – Cambodia

Later adopted by the Arabs
'Hindu-Arabic nos'.

(3) Trigonometry astrology / astronomy
some tables based on Hipparchus
Ptolemy

(4) Brahmagupta (b. AD 598)

$N \equiv 10 \pmod{137}$
$N \equiv 0 \pmod{60}$

Euclidean algorithm

area of cyclic quadrilateral

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

s = semi-perimeter

(5) Mahavira (9th C.) $\binom{n}{r} = \dfrac{n(n-1)\cdots(n-r+1)}{r!}$

(6) Bhaskara (1114-1185) $Dx^2 + 1 = y^2$ (Pell eq^n)

$\downarrow$

In case $D=61$

$\Rightarrow x = 22{,}615{,}390$

$y = 1{,}776{,}319{,}049$

(7) Infinite series (1300s, 1400s)

[See H3 P2]

Infinite series for arctan, $\underbrace{\sin, \cos}_{\text{usually credited to Newton!}}$

MATHEMATICS OF THE ISLAMIC WORLD

In the 9th Century in Baghdad, a group of Califf's people came to foster all types of knowledge and wisdom.

They had a library and encouraged all the best scholars to come there ("House of Wisdom"). It remained, for 200 years, the centre of culture.

One of the most important aspects of the mathematical side of their work was their <u>preservation and translation of Greek mathematics</u>. It's thanks to these guys that we have book of Apollonius' Conics, for example.

One of the most important scholars was
Thabit ibn Qurra (836-901) - translated Elements, Archimedes' sphere & cylinder, Apollonius' conics

They adopted the Hindu number system to form the Hindu-Arabic n° system

Al Kwarizmi (780-850) - 'On addition and subtraction' (oldest text of era)
zero, rules of number, extraction of square roots.
- a lot of work based on Brahmagupta

Al-Uqlidisi (not his real name, Arabic for 'The Euclidean'!!) - Book of Chapter (AD952)
- decimal fractions
- 19 halved → 9.5, 4.75 etc

3 They came up with algebra, based on systematic proof.

Al-Kwarizmi wrote the 'Condensed Book of Al-Jabr' (means 'transformation'), showed how to linear and 5 types of quadratic eqⁿ.

Used Babylonian geometrical methods (+ve coeffs only)

Completing the square, but proved his answer was correct.

Omar Khayyami (1048–1131) — Isfahan poetry
— philosophy
— 14 types of cubic solⁿs (again, geometric with +ve coeffs)

(can see influence of Greek here)

He shows you can construct and find the solⁿ to $x^3 + bx^2 + cx = d$. [See H3 P3]

All the algebra was done with words still!
Algebra as we know it really came with Descartes.

Menaechmus had done this kind of thing before but this is more complicated. Same idea, taken further.

Al-Karaji (d. 1019) — wrote a treatise called 'The Marvellous'
— developed a method for denoting x^n & $\frac{1}{x^n}$, $n > 3$
— moved algebra beyond geometry.
— identities, e.g. $\sqrt{A+B} = \sqrt{\frac{A+\sqrt{A^2-B^2}}{2}} + \sqrt{\frac{A-\sqrt{A^2-B^2}}{2}}$

Al-Samawal (1125–1174) Book of Calculation

algebraic division
e.g. $20x^2 + 30x \div 6x^2 + 12$

binomial coeffs for $(a+b)^n$ $n < 12$
using inductive argument.

HW hint Sheet 4: Consider Arch great
Ap great
Arch infl.
Ap infl.
Concl.

Great = range and originality

4 Combinatorics: Al Karaji ~~used~~ found
$$1^3 + 2^3 + \cdots + 10^3 = (1+2+\cdots+10)^2$$
using an early form of induction
(proper induction didn't come about till Jewish maths)

[from H3 P4]

Alhazen (965–1039), from Basra,
found
$$\sum_{k=1}^{n} k^2 = \frac{n^3}{3} + \frac{n^2}{2} + 6$$
$$\sum k^3 = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}$$
$$\sum k^4 = \left(\frac{n}{5} + \frac{1}{5}\right) n \left(n + \frac{1}{2}\right)\left[(n+1)n - \frac{1}{3}\right]$$
using inductive arguments

did work on optics and mirrors!

The pharoah summoned Alhazen to Egypt ∵ he thought Alhazen could create a contraption to stop the Nile flooding. Alhazen failed. Pharoah wanted to kill him. Alhazen feigned madness! (And escaped)

Ibn Al Barma (1256-1321) from Marakesh
used inductive arguments to show:

$$ {}^nC_r = \frac{n(n-1)\cdots(n-r+1)}{r!} $$

$$ {}^nP_r = \frac{n!}{(n-r)!} $$

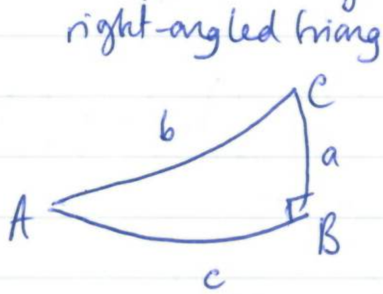
5 Trigonometry

Advancing Greek and Indian work on plane and spherical trigonometry.

Why? For astrology and to find Mecca! So there was an impetus to do this stuff.

Al-Birani (973-1055) from Uzbekistan
wrote the 'Exhaustive Treatise',
introduced cot, sec, cosec.
gave rules equivalent to $\tan^2\theta + 1 = \sec^2\theta$
$\cot^2\theta + 1 = \operatorname{cosec}^2\theta$

in spherical geometry, he proved for spherical right-angled triangle:



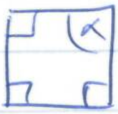
$$ \sin A = \frac{\sin a}{\sin b} $$

(by sin of length a, this means sin of angle subtended at centre of sphere where arc of great circle is a) gave precise instructions, on how, depending ~~of~~ on your latitude, you can find the dir^n of Mecca (quibla in Arabic).

6 <u>Geometry</u> Not as interested as Greeks but admired their work. Euclid was greatly admired. They tried obsessively to prove the parallel postulate. All the proofs were flawed.

Alhazen had one failed proof, in Alhazen's proof. The crutch of Alhazen's argument is:

parallel postulate $\Longleftrightarrow$ 4th angle of a quad with 3 rt angles is a rt angle.



At some point he introduced something equivalent to the p.p., making it circular. It's difficult to spot $\because$ the p.p. comes in many subtle guises! [in F&G].

The flaw: The assumption that the equidistant to a straight line is itself a straight line is logically equiv. to the p.p.

TRANSMISSION

or, How was Greek maths transmitted to Europe?

(1) Editors, commentators, translators of the Roman Empire.

(a) Rome: Seneca d. AD 68 } some references
Pliny the Elder AD 23–79 }

Boeterius 480–524 – an intellectual and philosopher. Translated Euclid into Latin.
↓
These translations were studied in Medieval times.

(b) Alexandria: Theon 4th Cent – prepared an edition of Euclid. He preserved most of the works. Our edition is thanks to him

Hypatia – edited Archimedes
Pappus – commentaries, info on lost material

(c) Athens: Proclus 5th cent – in his commentaries he quotes from Eudemus' lost 'History of Gk Maths'. (written 320 BC).

(d) Constantinople: Leon 9th cent. – collected and edited every treatise of Arch. he could find. Translated later into Latin in Middle Ages by Vatican.

(2) Arabic translation
of Euclid, Ptolemy,
Archimedes,
Apollonius — 9th, 10th Century.

most important translator
↓
Thabit ibn Qurra

(3) Contact between Europe and Islamic world (10th – 13th C.)

(a) Gerbert 946–1003 → Pope Sylvester (became?)

(b) Toledo — Gerard of Cremona — d. 1187

passed through Spain & Sicily
↓
both were split N/S between Islam/Chr.

Toledo, a city north of Madrid, fell in 1085, i.e. all the Muslims were expelled south and out of Spain. This was v. important ∵ all the European scholars flooded in to get their hands on this stuff. Loads of translation happened at this time.

His pupils set up Cathedral schools in Chatres, Utrecht. Acquired Latin translations of arabic math. treatises. All this before uni. set up.

Gerard of Cremona made over 70 translations!

(c) Vatican — William of Moerbeke 1215–81

(4) Medieval universities (12th C.)

embraced this form of education.
Had the first 9 books of Euclid in their MA degree.

(5) Fall of Constantinople (1453)

Used to be the centre of culture, last Roman past.
Fell to the Turks in 1453.

All the Greek scolars fled West and brought with them their Greek manuscripts. So now we have Greek manuscripts in Italy.

Best translator was Federigo Comandino, also a mathematician (so he corrected mistakes too!)

translated Euclid
Apollonius
Archimedes to Latin.

He also supplied notes.

So by the end of 16th Cent, Europeans scholars had everything we know about today, translated into Latin, (lang. of learned men), save the Archimedes' method.

<u>What was Lost?</u> Before Euclid wrote Elements, Hippocrates had written an axiomatic account of geometry.

Euclid's Conics, upon which Appollonius based his stuff.
Loci
Fallacies
Porisms

Archimedes' On Levers
On Centres of Gravity
Optics

Eudemus' History of Greek Mathematics (written 4th C.)

[H3 P6]

MEDIEVAL MATHEMATICS

Not noted for great mathematical ingenuity. More a time of translation.

But some interesting work was done:

(1) Jewish communities in Spain and southern France

(a) Abraham bar Hiyya (d. 1136)

[H3 P7]

First European mathematician to provide a table of chords. These kind of tables were well known to Indian and Arab mathematicians. To large degree of accuracy but $\pi = 22/7$.

(b) Abraham ibn Ezra (1090–1167)
Interested in astrology

This interested him in combinatorics — what combination of planets could be in conjunction?

$$^{n}C_{k} = \sum_{i=k-1}^{n-1} {}^{i}C_{k-1}.$$

e.g. $^{7}C_{4} = {}^{6}C_{3} + {}^{5}C_{3} + {}^{4}C_{3} + {}^{3}C_{3} = 35$

(c) Levi ben Gerson (1288–1344)
Mathematician (in Hebrew), astronomer, biblical commentator

Wrote 'Art of the Calculator', a book on combinatorics in 1321.

The fist to set out induction formally.

e.g. # permutations of n objects = n!

Given a perm abcde, add new element f.
But f can be placed in (n+1) pos^{n}s
so $P(n+1) = n!(n+1) = (n+1)!$

(first example of inductive proof)

(2) Leonardo del Pisa aka Fibonacci (1175-1250)

His father was a merchant who travelled extensively in North Africa. So he was brought up in Arab schools and learnt Arab mathematics. Became life's work to introduce Arab maths to Europe

In 'Book of Calculations' (1202) he introduced Arabic no. system to Europe, and gave 'rules of number'.

e.g. $83 \div 5\frac{2}{3}$: $(3\times5)+2=17$

$3\times83 = 249$

then $\frac{249}{17} = 14\frac{11}{17}$

Pretty simple stuff, but showed standard of learning in Europe if this was needed.

Most famous problem is the problem of the Rabbits! :

how many pairs of rabbits can be bred in one year if
(a) rabbits give birth to new pair each month
(b) new pairs breed in 2^{nd} month

After first month, there will be 2 pairs
second 3
third 5
fourth 8

$1, 2, 3, 5, 8, 13, \ldots$

(Answer = 377)

Was able to solve $x^3 + 2x^2 + 10x - 20 = 0$

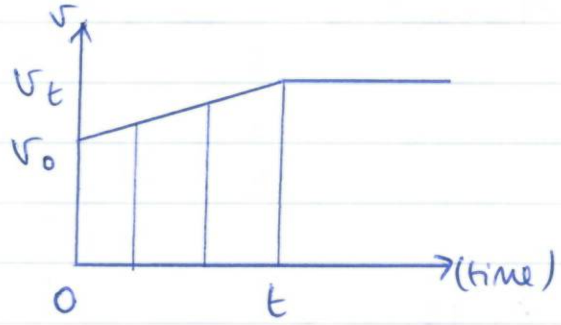
↳ gave $x = 1, 22, 7, 42, 33, 4, 40$ (in Baby. style!!!)
$\approx 1.368808107\ldots$
noone quite knows how he got this... ☹

(3) Oresme
A cleric and amateur mathematician
first graphical representation of moving body.

Verified Merton mean speed rule:
'a body uniformly accelerating from rest will traverse distance which it would have covered if in that time it had moved constantly at half its final speed'.

distance covered = area = $\frac{1}{2}(v_0 + v_t) \cdot t$

horizontal: time, space (longitude)
vertical: intensity (latitude)



(he invented the v-t graph!)

Also was first mathematician to sum an infinite series with an elegant geometric method.

[H3 P8]

Proved that harmonic series diverges:

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots$$
$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)$$
$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \not\to .$$

'the comparison test'.

So Medieval mathematics was pretty dire, and was made even worse by the Black Death in 1348, which swept Europe from Italy: people died before they could pass on info.

Bad times.

THE EUROPEAN RENAISSANCE

The revival of the learning of the arts and sciences in Europe in the 15^{th}, 16^{th} Cent.

Why?

(a) Fall of Constantinople (1453) brought Greek stuff here and stimulated debate and enquiry. Plato in particular came to be studied.

(b) Invention of printing press (in 1454) let maths be spread out, ie. dissemination of knowledge

(c) Development of money economy. need mathematicians to calculate interest and keep books.

(d) Voyages of discovery navigation requires plane and spherical trigonometry.

Renaissance algebra

Cast of remarkable characters in the cubic:

Scipione del Ferro (1465–1520), professor at Bologna Uni.
Fiore, assistant
Niccolò Tartaglia (1499–1557), itinerant mathematician
Gerolamo Cardano (1501–1576), physician, mathematician, gambler.

The task was to find an analytic solⁿ to the cubic. Generally thought impossible but del Ferro found it.

He didn't share it because at the time he told noone so that he would be employed to solve them!

On his deathbed he told Fiore, his assistant.

But later, Niccolò Tartaglia claimed to also have solved it. They set each other problems and Tartaglia won. His fame spread.

Later Cardano the historian wanted to publish a book so Tartaglia ~~could~~ ~~begged~~ was begged by him to release the info. Cardano agreed with Tartaglia that he wouldn't published it ∵ Tartaglia later wanted to publish his own book.

Later ~~Fa~~ Cardano found out del Ferro had ~~to~~ found an answer earlier and so Cardano decided his oath was no longer valid and he published the sol^n. Tartaglia was furious! They never reconciled! [H3 P9]

Cardano was a great character, btw.

Cardano published 'Ars Magna' in 1545.

↳ could solve 3 types of cubic

e.g. $x^3 + cx = d$ (in words)

Find u, v s.t. $u - v = d$ and $uv = \left(\frac{1}{3}c\right)^3$

multiply $u-v=d$ by u →

gives quadratic $u^2 - \left(\frac{c}{3}\right)^3 = du$

Solves to give $u = \sqrt{\left(\frac{d}{2}\right)^2 + \left(\frac{c}{3}\right)^3} + \frac{d}{2}$

$$v = \sqrt{\left(\frac{d}{2}\right)^2 + \left(\frac{c}{3}\right)^3} - \frac{d}{2}$$

then $x = u^{1/3} - v^{1/3}$.

Occasionally his formulas require taking the square root of a minus number. Cardano didn't understand how to deal with this.

Also he was often unsure why the formulas worked.

e.g. $x^3 + 6x = 20$.
By inspection, $x = 2$

The formula given gives

$$x = \sqrt[3]{\sqrt{108} + 10} - \sqrt[3]{\sqrt{108} - 10}$$

He couldn't explain how this was equal to 2.

Renaissance algebra

Cardano had for $x^3 + cx = d$

$$x = \sqrt[3]{\sqrt{\left(\frac{d}{2}\right)^2 + \left(\frac{c}{3}\right)^3} + \frac{d}{2}} - \sqrt[3]{\sqrt{\left(\frac{d}{2}\right)^2 + \left(\frac{c}{3}\right)^3} - \frac{d}{2}}$$

and for $x^3 = cx + d$

$$x = \sqrt[3]{\frac{d}{2} + \sqrt{\left(\frac{d}{2}\right)^2 - \left(\frac{c}{3}\right)^3}} + \sqrt[3]{\frac{d}{2} - \sqrt{\left(\frac{d}{2}\right)^2 - \left(\frac{c}{3}\right)^3}}$$

Quadratic: Divide 10 into 2 parts
whose product is 40.

$$x(10-x)=40$$
$$x^2-10x+40=0$$
$$x=5\pm\sqrt{-15}$$
$$(5+\sqrt{-15})(5-\sqrt{-15})=40.$$

complex roots?!?!

Problems: (1) Verbal sol^n
(2) Confusion over formula
(3) Complex roots?

"So refined so as to be useless!"
— Cardano

Enter:
Raffaello Bombelli (1526-1572), an engineer, but during his slack periods as an engineer he wrote the textbook 'Algebra'

his innovations: (1) Notation

x^3+6x^2-3 : $1^{(3}\ p\ 6^{(2}\ m\ 3^{(1}$

p = piu
m = meno

$\sqrt[3]{2+\sqrt{-121}}$: $R_c \lfloor 2\,p\ R_q\ 0\ m\ 121 \rfloor$

(2) Complex n.°s $\sqrt{-1}$ = 'pui di meno' (more of less)
'p di m'

$-\sqrt{-1}$ = 'meno di meno'
'm di m'.

$2+3i$: 2 p di m 3

he devised rules for adding, multiplying, simple complex n°s.

$$x^3 = 15x + 4$$

$$x = \underbrace{\overset{a+\sqrt{-b}}{\sqrt[3]{2+\sqrt{-121}}} + \overset{a-\sqrt{-b}}{\sqrt[3]{2-\sqrt{-121}}}}$$

has the clever idea to equate the reals and equate the imaginaries!

He wondered whether this made any sense at all! At first he thought it was crazy, but later he thought it was nice and useful.

He could use these to solve awkward quadratics /cubics

They were dealing with complex n°s to find real roots, didn't consider complex roots!

François Viète (1540-1603) was a privy counsellor, lawyer and cryptanalyst.

(a) Notation: $A^3 + CA = D$

A cubus + C plano in A aequatur D solido.

algebra had not yet broken free from geometry, this took Fermat + Descartes.

(b) He found an alternative soln to the cubic. He could reduce all cubics to manageable canonical forms by substitution. [H3 P11]

Quartic

(1) Ludovico Ferrari (1522-1565), Cardano's pupil

Solved things like $x^4 + 3 = 12x$

Completing the square by adding suitable terms to both sides [H3 P12]

to get $x = \sqrt{1\frac{1}{2}} \pm \sqrt{\sqrt{6} - 1\frac{1}{2}}$

have to solve a cubic on the way.

Gave no way of finding any complex roots; he stopped with these 2.

(2) Thomas Harriot (1560-1621), English cartographer, astronomer, physicist, linguist.

Greatest English mathematician before the advent of Newton!

He didn't really publish anything, instead he circulated papers among his friends. Also he worked for the Secret Service, so lots of his work (e.g navigation) may have been state secrets.

He noted that a cubic can be expressed as

$$a^3 - (b+c+d)a^2 - (bc+bd+cd)a - bcd = 0.$$

He also managed to extract all 4 roots (real and complex) of the quartic

'hypostatic' 'noetic'.

Reduces quartic to a cubic, and is then able to solve it.

He was the first mathematician to employ purely symbolic notation

$$25 = 6a - a^2 : \qquad 25 = +6a - aa$$

Renaissance Trigonometry

(1) Johann Müller (1436-1476) (aka Regiomontanus) from Königsberg.

He learnt Greek to translate Ptolemy's Almagest into Latin.

Known for 2 works:

(a) Epheremedes (1474)
position of sun/moon/planet. Table.

used by Columbus 1504, he threatened the Natives that if they didn't feed him, God would remove the moon (then he predicted an eclipse!)

→ a compendium

(6) <u>On Triangles</u> (1463) but published 70 years later

Very Euclidean-style set of results and proofs.
Defines sine as semi-chord of double the angle.

↳ part 1: plane trig e.g. sine rule
part 2: spherical trig e.g. sine rule
↓
most of this part is
plagiarised from ibn Afra in 12th C.
but some original stuff too.

[HB 814]

(2) <u>Joachim Rheticus</u> (1514–1574) German

Published 'On Sides and Angles of Triangles' in 1542, where he defined trig. ratios in terms of a right angled triangle for the first time, as today.

(3) <u>Bartholomew Pitiscus</u> (1561–1613) German

Coined the word 'trigonometry'
Published 'Book of Trigonometry', 1595
full of sine and tangent tables

(4) <u>François Viète</u> (1540–1603) French (duh)

Leading mathematician of the 16th C.

Published 'Canon Mathematicus' in 1579

↳ extensive tables of all six trig ratios
to the nearest minute.

Proved loads of identities, e.g.

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x+y}{2}\right)$$

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta.$$

Applications

- The first mathematician to apply trigonometry to algebra
  ↳ e.g. it was possible to solve cubics of certain form by comparing to identities for cubes of cos's. [H3 P15]

  Was also able to solve a polynomial eq^n of 45th degree!
  Used two trig identities: $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$
  $\sin 5\theta = \ldots$

  He didn't bother with the 22 negative sol^{n}s to the polynomial, only gave 23 +ve ones.

  He came up with beautiful infinite product formula for π:

$$\frac{2}{\pi} = \sqrt{\tfrac{1}{2}} \times \sqrt{\tfrac{1}{2}\left(1+\sqrt{\tfrac{1}{2}}\right)} \times \sqrt{\tfrac{1}{2}\left(1 \times \sqrt{\tfrac{1}{2}\left(1+\sqrt{\tfrac{1}{2}}\right)}\right)} \times \ldots$$

  First infinite product ~~~~ formula in maths!

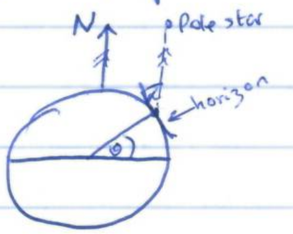
- Astronomy: Regiomontanus influenced → his book was carefully studied
  - Nicholas Copernicus (1473-1543) → first Europe[an] to give heliocentric view
  - Tycho Brahe (1546-1601)

- Geodesy: Gemma Frisius (1506-1555), a man invented triangulation.
  surveyed large areas thus.

• <u>Navigation</u> (a) Where am I ?
(b) How to I get to my destination

This was a period of voyaging - discovering India and America. These two questions arose!?.

(a) i. height of pole star = latitude



α = height of polaris = θ

You're so childish.

problems: daytime
clouds
~~north~~ southern hemisphere
polaris ≠ north exactly! A few degrees off.

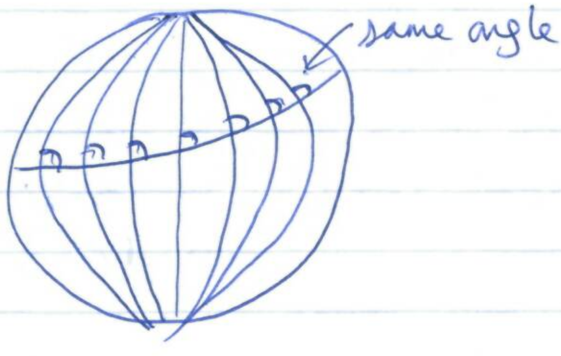
ii. longitude - compare a local clock to a clock set on GMT

British govt offered a £20k prize to anyone who could invent a chronometer. He won most of the prize. It was actually very big but in time it got smaller. Now in Greenwich!

John Harrison in 18th C.

(b) destination - course of constant bearing
magnetic compass

course of constant bearing cut all lines of longitude at a constant angle
'rhumb lines' or 'loxodrome'



Portuguese mathematician Pedro Nuñes (1502-1578)
all rhumb lines are spirals on surface on globe

Want a straight line on a map! [H3 P21]
Had to stick onto a rhumb line to get anywhere
∴ ~~we~~ can only create a course of constant bearing.
Great circle (like aeroplanes) would be better but
impossible with tech at the time.

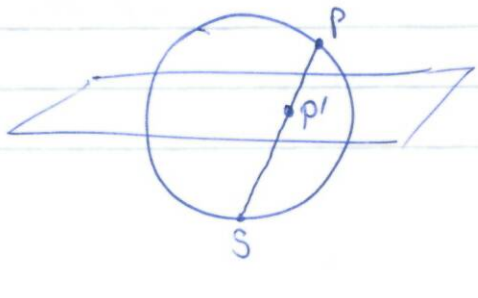
Dutch → Dutch for 'merchant'. Latinised →
Gerhard Kremer (1512-1594) gave the Mercator projection.
A big map! → a map <u>not</u> a projection!
→ In 1569, had 21 sections!
54" × 83".

The map distorted distances but preserved angles.
Lines of latitude and longitude are parallel

$$\Delta y = R\sec\theta\,\Delta\theta \qquad \theta = \text{latitude}$$

not Arch. spiral
$r = e^{-\theta}$

Thomas Harriot - rectification of logarithmic spiral
(which is the stereographic projection
of a rhumb line onto the equatorial
plane)

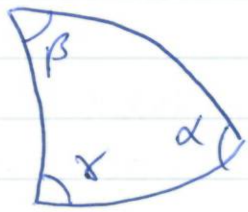


He came up with the first rectification in Mathematics: using a straightedge and compass, we can ~~us~~ find a straight line equal in length to the spiral.

[H3 P23]

He found that the area of a spherical triangle is
$$A = R^2(\alpha + \beta + \gamma - \pi).$$

[H3 P18]



Renaissance Arithmetic

(a) decimals

Simon Stevin (1548–1620), Dutchman published "De Thiende", better known in French as "La Disme",
gave $\pi = 3^{⓪} 1^{①} 4^{②} 1^{③} 6^{④}$
(not first use of decimals, of course, but helped ~~publicize~~ popularise them).
wanted decimal weights + measures.

(decimal point was invented later by Napier)

(b) logarithms

Sir John Napier (1550–1617), convinced that Pope was antichrist.

Spent 20 years in his castle creating a table of logarithms aim of logarithms is to reduce multiplication and division to addition and subtraction.

→ Astronomers had to make massive multiplications and divisions. This could reduce labour and errors. Continued to right up to when pocket calculators were invented.

Preceded by Michael Stifel (1487–1567).
Was convinced that the world would end on 18/10/15833.
So he gathered everyone then nothing happened! The church sent him away.

He noticed if you take powers of 2

0	1	2	3	4	5	← addition
1	2	4	8	16	32	← multiplication

If n°s are in geometrical progression, their exponents are in arithmetic progression.

'Admirable Canon of Logarithms' (1614)
published first in Latin, later English.

'Constructio' (1619), showed <u>how</u> he got it.

Presented in the 'Admirable Canon' is published a logarithms of sines

		log
$10^7(1-10^{-7})^0$	$= 10{,}000{,}000$	0
$10^7(1-10^{-7})^1$	$= 9{,}999{,}999$	1
$10^7(1-10^{-7})^2$	$= 9999998.0000001$	2

Multiplied by 10^7 to get rid of fractions.

Basis of Napier's logarithm is this:

If $n = 10^7(1-10^{-7})^r$
then $r = \underbrace{(N\log}_{\text{Naperian logs}})n$

Note it is a <u>decreasing</u> f^n (different from today).

And note the common ratio, $(1-10^{-7})$ is very close to 1, meaning the n°s were very closely spaces, so any sine you were looking for could be easily found.

[N3 P25]

Then Nlog of closely spaced n°s in a geometric progression from 10,000,000 to 0.

[N3 P29]

<u>Use</u>: $n_1 \times n_2$: Compute $N\log(n_1) = r_1$
Compute $N\log(n_2) = r_2$

Add $r_1 + r_2$

Look up n° corresponding to $r_1 + r_2$.
this is different from today → Divide this number by 10^7,

this is the antilogarithm, $n_1 \times n_2$.

Greatly simplifies astronomical calculations by Kepler etc.

Laplace said "logs had doubled the life of astronomers by halving their work!"

How does it work? [N3 P25].

He computed the first 100 logarithms
(can do this using simple subtraction, not hard)

He notices that $N\log(9,999,990) \simeq 100$

His second table, then has common ratio $\left(1-\frac{1}{10^5}\right)$.
He noted the last n° in the first table ≈ the second n° in the second table.

The third table is 21 rows, 69 cols.
The n°s he get are approximate but he calculates them better & later to form his radical table, then used that to form his 'Admirable Canon'.

Took him 20 years!

<u>Jobst Bürgi</u> Swiss came across same thing at same time, unknown to Napier. He used common ratio $\left(1+\frac{1}{10^4}\right)$, so n°s weren't so tightly spaced but logs were increasing.

<u>Henry Briggs</u> (1561-1631) had discussions with Napier. They thought it might be more convenient to use base 10 instead. But Napier was too old so he left it to Briggs to do. He brought out 'common Logarithms'

if $x = 10^y$
$y = \log_{10} x$

<u>Adrian Vlacq</u>, Dutchman, completed the tables. They served as the basis for math. computation up to the 1970s!

Natural logs: Discovered in 17^{th} C. by Belgian Gregory St Vincent in 1647.

Looked at hyperbola $(xy = 1)$ and noted that the area underneath it has logarithmic properties (no surprise! $y = \frac{1}{x}$ $\int y\,dx = \ln x$).

[H3 P30]

Newton calculated some natural logs in 1660s.

Euler defined in 1748 that
$$y = \log x \Leftrightarrow x = e^{y}.$$

Renaissance Astronomy

1. Copernicus (1473-1543), a polish monk.
Studied in Krakov then moved to Italy.
Came under influence of Renaissance.

Studied Ptolomy's Almanac and concluded it made no sense to have geocentric system. Hearing ancient Greek philosophers put forward a heliocentric system, he explored it

→ Heliocentric
7 concentric spheres → why spheres? He had no physics so assumed ∵ God would only make perfect things

[U3 P31] This system explained 'retrograde motion' as merely an illusion since the Earth rotates quickly.

2. Kepler (1571–1630), Imperial mathematician of Danish empire. Kepler decided Copernicus' theory was good (the heliocentric bit anyway).

[U3 P32]

On the basis of false mathematics and great intuition, he gets his three laws

K1: Planets move round in ellipse with sun at focus.

K2: Planets sweep out equal areas in equal times

K3: The (period of orbit)^2 varies with (mean dist. from Sun)^3.

No maths just tried different shapes on Mars!! Then briefly checked the rest

Newton later derived these mathematically.

3. Galileo He too agreed with Kepler and Copernicus, so he fell out with the Catholic Church. Kepler, being German, was at some distance from the Pope, but Galileo in Italy wasn't. He was arrested and threatened with torture if he didn't recant. So he recanted and was placed under house arrest but still smuggled expanded theory out in books.

'Two New Sciences' 1638

[U3 P33]

(i) Mean Speed Rule

(ii) Distance covered by falling body $\propto$ (time)^2

$\Rightarrow$ bodies fall as parabolas.

ANALYTIC GEOMETRY

(1) René Descartes (1596-1650)

Enjoyed living in obscurity in Holland. Became one of the most well-known intellectuals in Europe. Queen Christina of Sweden invited Descartes to give her maths lessons, but insisted they would be at dawn. So in the winter he would have to struggle in the morning snow. He caught a chill and died!

Better known (like Leibniz) as a philosopher than a mathematician. Belonged to the Rationalist side of opinion. (not empiricist)

Rules of deductive reasoning:

(a) accept nothing as true except self-evident
(b) subdivide complex subjects
(c) proceed from simple to complex
(d) revise everything

Disinclination to follow tradition meant he enlisted algebra in service of geometry to invent analytic geometry. Nevertheless influenced by Apollonius, Pappus and Viète.

Descartes' books look modern today because he invented many of the notation we use today, e.g

a, b, c	constants
x, y, z	variables
x^1, x^2, x^3	superscripts, treating them as objects rather than planes/solids

He expanded his idea in <u>Geometry</u>, an appendix to his <u>Discourse on Method</u> (1637)

He regarded himself as a geometer first and foremost.
He would apply algebra to solving geometry.
how?

(i) Assume the problem solved (analysis & synthesis - Pappus!)

(ii) Assign letters denoting the length of all the given lines, known and unknown

(iii) Establish relations between them, deriving as many eqⁿs as unknown

(iv) Use substitution to obtain one eqⁿ in one unknown (if poss.)

(v) Construct the root of this eqⁿ by tracing a curve and intersecting it with a straight line, a circle or another curve.

1 eqⁿ 1 unknown : <u>point</u> (determinat)
1 eqⁿ 2 unknowns : <u>locus</u> (indeterminent).

He solved Pappus problem for 4 lines
Used coordinates that weren't perpendicular!

He does most of the work and then leaves the rest to you. Fermat completed problems, Descartes CBA.

Also constructed normals to curves, in a very logical, beautiful, but laborious way!

What kind of curves was he interested in?
- Conics primarily although aware of more/higher powers

He distinguished (oddly) between
'geometrical curves' ← suitable for geometry.
continuous, algebraic.
Can trace w/ one cts motion
'mechanical curves' ← not suitable, e.g. quadratrix
thought you couldn't write
its eq^n down (wrong - Newton did).

But he showed that a hyperbola can be traced out by a mechanical device.
→ linkage
very fashionable in Victorian England.
Used in rail locomotives, turning piston movements into rotation.

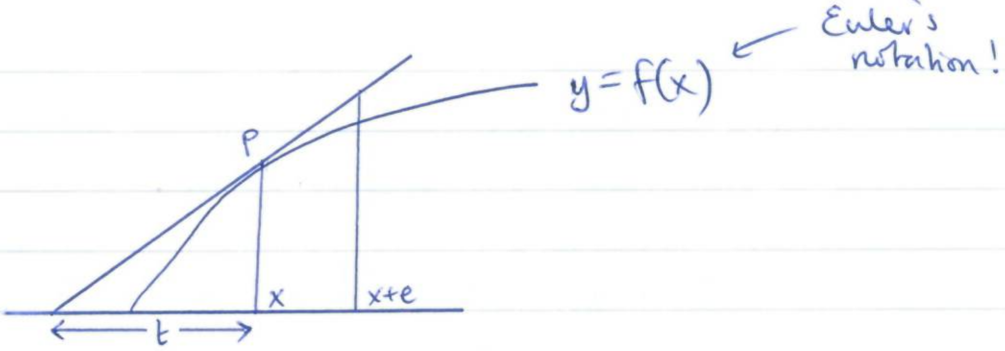
(2) Pierre de Fermat (1601-1675)

A lawyer, who studied maths briefly after finishing law school. A very gifted amateur mathematician.

Wrote 'Introduction to Plane and Solid Loci' (1633) which he didn't publish, he simply handed it out to friends.

(a) gives loci of algebraic eq^{n}s
(opposite to Descartes, but they're doing the same thing).
uses same notation as Viète.

(b) method of tangent construction



Aim: to construct a tangent to the curve at

$$\frac{f(x+e)}{f(x)} \simeq \frac{t+e}{t}$$

$$\Rightarrow t \text{ (subtangent) } \simeq \frac{e \cdot f(x)}{f(x+e)-f(x)}$$

This has lots of similarities with differential calculus.

CALCULUS

THE BEGINNINGS OF CALCULUS

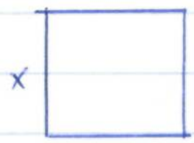
Calculus was developed to solve 4 types of problem:

(a) determination of min, max (projectiles)
(b) construction of tangent (optics)
(c) determination of instantaneous velocity (mechanics)
(d) quadrature and rectification of curves (astronomy)

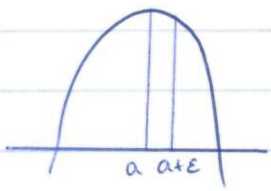
1. Fermat

(a) Tangent construction.
But he does not explicitly mention a limit and
[H4 P2] he does not derive the slope of the tangent

(b) Determination of maximum



semi-perimeter $= b$.
one side $= x$.
What is length of other side $(b-x)$ so as to maximise area?



Near a maximum, $f(a) \approx f(a+\varepsilon)$.

[H4 P1]

Wants to maximise $x(b-x)$.

equate the 2 values of x to be a and $a+e$

$$ba^2 - a^2 \approx b(a+e) - (a+e)^2 \quad = ba - a^2 + be - 2ae - e^2$$

Cancelling common terms and divided through by $e \neq 0$

$b \approx 2a + e$

'Suppressing' e, he obtains

$b = 2a$.

i.e. f^n reaches its max at $x = \frac{b}{2}$ as required.

This is naughty though – $\left.\begin{array}{l} e = 0 \\ e \neq 0 \end{array}\right\}$ at same time.

2. Gilles de Roberval (1602-1673)

A professor in France.

Kinematic method of drawing the tangent to a curve.

[H4 P3]

used vectors but not by name!

3. René de Sluse (1622-1685)

[H4 P5] Found a method in 1652 of finding the subtangent τ to a curve $f(x,y) = 0$.

INTEGRAL CALCULUS

(1) Bonaventura Cavalieri (1598-1647)

[H4 P6]

a disciple of Galileo
came up with idea of dividing lines into points
planes " strips
solids " planes.

Democritus' theory of indivisibles
but was not rigorous.

Was able to show in essence that $\int_0^a x^n\,dx = \frac{a^{n+1}}{n+1}$.

(2) Evangelista Torricelli (1608-1647)

a solid of infinite dimensions has finite volume! [~~H4~~ P7]
↳ he couldn't believe it and had to prove it in another way just to convince himself.

(3) Fermat (again)

[H4 P9]

If curve is $y = x^n$. $0 \le x \le a$.
Draw rectangle of sides a^n and a
divide by $n+1$.

and for hyperbola $y = px^{-k}$ $k \neq 1$. !! [H4 P10]

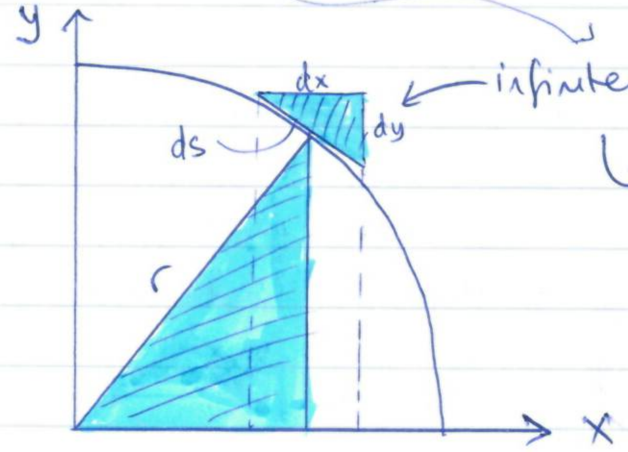
He can only do this for simple curves $y = x^2$
$y = x^{-2}$

because rectangles can be found ~~to~~ in a nice geometrical progression.

Wouldn't work for $y = x^3 + 2x^2 + 19x + \pi$.

(4) <u>Pascal</u> (1623–1662)

Use of differential triangle



infinitesimal triangle, hypotenuse along tgt.
↳ inspired Leibniz

similarity of shaded triangles

$$\Rightarrow \quad y\,ds = r\,dx$$

$$\int y\,ds = r\int dx$$

Area of hemisphere = $2\pi y\int ds = 2\pi r\int dx = 2\pi r x = 2\pi r^2$

} in Leibniz's notation

(5) <u>Van Heuraet</u> (1634–1660)

Rectification of curve [H4 P12]

(4) <u>Isaac Barrow</u> (1630–1677)

Barrow proved a theorem equivalent to the Fundamental Theorem of Calculus, without realising its importance!

[H4 P13]

Summing up the state of these matters by the late 1650s,

mathematicians could:

(1) Draw tangents to conics
(Fermat)

(2) Evaluate tangents to algebraic curves
(Sluse)

(3) Draw tangents to some transcendental curves → e.g. cycloid
(Roberval)

(4) Evaluate maxima/minima
(Fermat)

differentiation

(5) Obtain quadrature of curves $y = px^k$ $(k \neq 1)$
(Fermat)

(6) Obtain quadrature of some transcendental curves
(Roberval)

(7) Rectify simple curves
(Van Heuraet)

(8) Prove theorems equivalent to the FToC
(Barrow)

integration

But

(1) No concepts

(2) No notation

(3) No general algorithms for differentiation and integration (some, e.g. Sleus, but only for certain types of curves)

(4) No real understanding of the inverse relationship between finding tangents and finding quadrature (FToC).

NEWTON

Isaac Newton (1642–1727) didn't show much interest in maths till 16 when he was inspired by a teacher and made great strides.

Trinity College, Cambridge in 1662 as an undergraduate.

Influences: Descartes (plowed through his 'Geometry' in Latin, didn't like it much)
Wallis ('Arithmetica Infinitorum')
Viete.

Great plague in 1665 (not as bad as Black Death). Cambridge closed for a year and he was sent home. In his mother's house in Lincolnshire, he reflected and came up with many ideas on calculus and gravitation

In 1669, his professor Barrow resigned and Newton took his post. (Post today filled by Stephen Hawking).

In 1680's, he got bored with maths. Dabbled in alchemy, tried to draw a map of hell from Revelations.

Continued lecturing because he was paid to do so. Sometimes lectured to empty rooms! Not good lecturer.

1684, Christ. Wren had bet Hooke and Halley that they would not be able to tell him what the path of a planet orbiting under an inverse square law would be. They asked Newton, who already knew. Couldn't find papers so rewrote. Asked for more, he spent 3 yrs writing Principia.

Became later Warden then Master to the mint. Good job. Got to keep coins!

Became embroiled in controversy with Leibniz's followers as to who got there first.

(1) Binomial theorem
- Didn't invent (Chinese / Indian)
- Derived coefficients of binomial expansion for +ve, -ve and fractional indices [H4 P14]
- Never proved it as we do it today
- Knew about Pascal's Δ trick. he extended it backwards

(2) Reversion of series.

$$z = x - x^2 + x^3 - x^4 + \dots$$

$$\Rightarrow x = z + z^2 + z^3 + \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

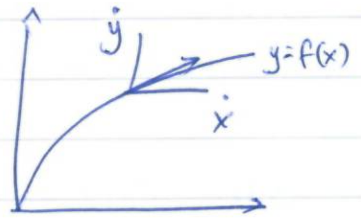
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4} + \dots$$

} preceded by Indians.

(3) <u>Theory of Fluxions</u>

Like Roberval, he had a very kinematic concept of curves

Slope of tangent $= \frac{\dot{y}}{\dot{x}} = \frac{dy/dt}{dx/dt} = \frac{dy}{dx}$ = instantaneous velocity



$\dot{x}$ = speed of fluent x over an infinitessimal time o during which it will travel $\dot{x}o$. Likewise y will travel $\dot{y}o$.

$= \frac{dx}{dt}dt = dx$

So in an equation $y = f(x)$,

$x + \dot{x}o$ can be substituted for x

$y + \dot{y}o$ can be substituted for y

$\Rightarrow$ differential eq^n

both approaching zero

(after refinement: $\frac{\dot{x}}{\dot{y}}$ is 'ratio of evanescent quantities')

so he did realise what he was doing.

[N4 P16] — chain rule!

(4) Anti-differentiation

no specific notation for integrals, he just sees it as opposite of ~~ant~~ diff^n.

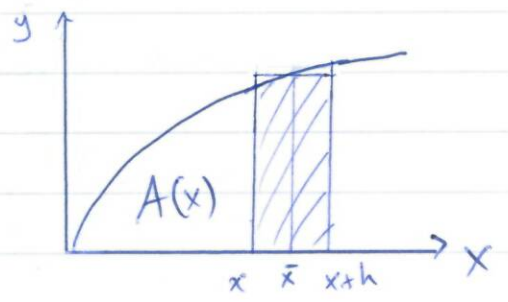
he uses example he gave before, but in reverse!

[N4 P17]

This algorithm is a bit like Sluse's method for finding tangents - very paint by numbers - quite crude.

But can do more complicated things, he uses power series! (uses his binomial thm). Unlike Leibniz, he was very happy to deal with infinite series (maybe ∵ Newton was a mathematical physicist).

(5) Fundamental Thm of Calculus



first use of Mean Value Thm for integrals!

Shaded area =
$A(x+h) - A(x) = f(\bar{x})\cdot h$

$h \to 0$, $f(\bar{x}) \to f(x)$.

$$\lim_{h\to 0} \frac{A(x+h) - A(x)}{h} = f(x)$$

$$\frac{d}{dx}(A(x)) = f(x)$$

ie. $\frac{d}{dx}\int f(x)\,dx = f(x)$

This proof is the one given in modern textbooks.

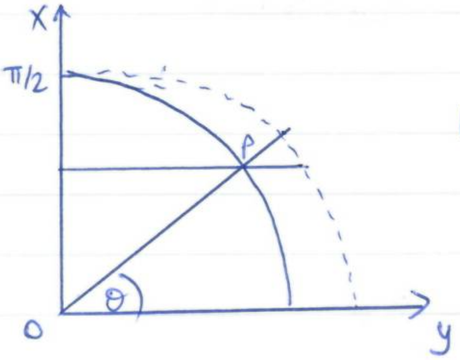
he gives a specific example of a curve and shows this [N4 P18]

He gives general algorithm for antidifferentiation which he is then able to apply to any equation.

(6) Standard integrals

Antidifferentiation, by substitution, by parts etc

Example: area under quadratrix (Descartes claimed this was impossible!)



Newton could also handle cissoid.

$$\left.\begin{array}{l} x = \frac{\pi}{2}(1-t) \\ \theta = \frac{\pi}{2}(1-t) \end{array}\right\} \; x = \theta = \tan^{-1}\left(\frac{x}{y}\right)$$

$$\Downarrow$$

$$\tan x = \frac{x}{y}$$

$$y = x \cot x$$

$$y = x\,\frac{\cos x}{\sin x}$$

$$= \frac{x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right)}{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} \dots\right)}$$

(using binomial thm) $= 1 - \frac{x^2}{3} \mp \frac{x^4}{45} - \dots$

$\Rightarrow$ Area under quadratrix

$$= \int y\,dx = \int_a^b \left(1 - \frac{x^2}{3} - \frac{x^4}{45} - \dots\right) dx$$

$$= \left[x - \frac{x^3}{9} - \frac{x^5}{225} - \dots\right]_a^b$$

LEIBNIZ (1646–1716)

More famous as a philosopher than a mathematician (like Descartes). Obtained doctorate at 20, having studied privately. Wangled diplomatic post, ended up in Paris, where he met Dutchman Christiaan Huyghens (1629–1695), who introduced him to advanced mathematics. Within a few years, he turned himself into Europe's greatest mathematician (bar maybe Newton). His later years were marred by controversy, his followers tried to claim he beat Newton to Calculus (prominantly the Bernoulli brothers). When George I of England came over from Germany (Hanover), he refused to take Leibniz so as not to annoy his new subjects. At Leibniz' funeral, there was only one mourner: his secretary.

He realised that partial sums and sequence of differences are inverses.

[H4 P24]

$$\left(d\int y = y \quad , \quad \int dy = y \right)$$

(1) Sums and differences

A prototype of the FToC

(2) Rectification using differential triangle

Perhaps unaware that others had preceded him [H4 P20]

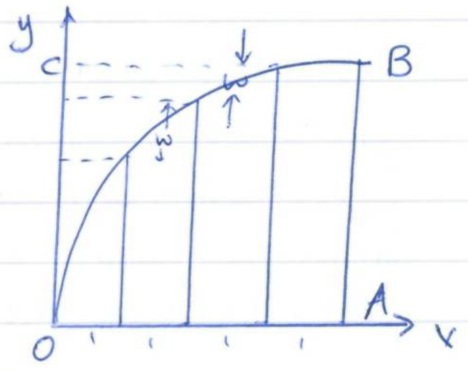
(3) Transmutation thm [H4 P21]

Evaluate the area under a given curve by equating it to the area beneath another curve which can be integrated more easily.

Was able to use this to prove an infinite expansion for $\frac{\pi}{4}$ (converges v. slowly though). Unfortunately he had been beaten by 300 yrs by an Indian mathematician!

(4) Integration

It started with a curve



area above curve = area ABCD − area below curve

omnes

$\rightarrow$ omn. xw = x. omn.w − omn.omn.w.1 — (1)

$\Rightarrow$ since w is variable, let w = 1

$\rightarrow$ omn x.1 = x.x − omn. x

$\rightarrow$ omn. x = $\frac{x^2}{2}$

Now let $w = x$.

$\Rightarrow$ omn. x^2 = x omn x − omn. omn. x

omn. x^2 = $x \dfrac{x^2}{2}$ − omn $\dfrac{x^2}{2}$

omn. x^2 = $\dfrac{x^3}{3}$

$\Rightarrow$ induction : omn $x^n = \dfrac{1}{n+1} x^{n+1}$

Later Leibniz wrote $\int$ for omn (∫umma, summa)

d for w (difference)

so (1) becomes

$$\int x\,dy = x\int y\,dy - \iint dy\,dx$$

i.e. $\int x\,dy = xy - \int y\,dx$ (parts !!)

(Oct/Nov 1675)

(5) Differentiation

Actually did this after diff^n !

Leibniz proved to himself the product rule → and quot. rule !

$d(xy) = (x+dx)(y+dy) - xy = x\,dy + y\,dy$

ignoring $dx\,dy$

→ More cavalier ~~with~~ than Newton.

(6) Differential eq^n s [N4 P28]

Also able to handle infinite series but didn't like it. Had very sophisticated understanding. [N4 P29]

NEWTON AND LEIBNIZ

(1) Developed general concepts
fluxions and fluents (N)
differences and sums (L)

(2) understood Inverse relationship between diff^n and $\int$^n

(3) Derived general rules of diff^n/$\int$^n.
applied to a variety of curves
(in contrast to their predecessors who ~~had~~ had an 'ad-hoc' method of difference approaches for different curves'

BUT

(1) Newton's integral (no notation) was simply a fluent to be derived from fluxion whereas Leibniz's integral ($\int$) was a sum of infinitesimal differences

(2) For ~~a~~ Newton, a second derivative was a fluxion of a f^n whereas Leibniz devised values and notation (dd) $\left(= \frac{d^2y}{dx^2} \text{ today}\right)$.
For Newton, derivative is rate of change
For Leibniz, it's small differences

(3) In general Newton solved particular problems by methods which he showed could be generalised whereas Leibniz emphasised general techniques which could be applied to specific problems

(4) Newton's concept while reflecting limiting processes was unwieldy while Leibniz's concept was cavalier but convenient, making calculus much easier to understand - it was *his* system which spread, not Newton's (e.g. $\dot{x}$ is only mechanics).

NEWTON'S *PRINCIPIA*

Published in 1687. Wrote in Latin (came from a (Latin) grammar school). 3 editions.

Newton had time to think about criticisms of previous editions.

English translation in 1729.

550 pages, 200 propositions – monumental work!

Structure

(1) Defns, Laws, corollaries [U4 P31]

(2) Book 1: orbit under a central force

(3) Book 2: motion in a resisting medium (fluids)

(4) Book 3: 'system of the World': gravitation

Influences: Euclid, Archimedes, Apollonius ← uses the style of.

Kepler, Galileo

Laws

('Laws of universe are written in mathematics'

$d \propto t^2$

Contemporary physicists

~~There is~~ No a

No analytic geometry – no x + y.

There is a myth that he invented the calculus, proved the results then converted ~~it~~ ~~it~~ into geometry for people to understand.

- Hardly any calculus as such but continual use of limiting processes

- Some quadratures that could only have been obtained by integration

Book I: Derivation of theorems justifying Kepler's 3 laws of planetary motion.

[H4 P33]

Kepler 2: A line joining the sun to the planet sweeps out equal areas in equal times. [H4 P34]

Kepler 1: Planets orbit in ellipses [H4 P35]
Doesn't actually prove this, he shows that if the body moves in an ellipse, the force must be inverse square.

Kepler 3: $\text{period}^2 \propto \text{Dist}^3$

Newton was aware that these only hold for an idealised one-body system, a situation 'such as hardly exists in the natural world'.

rushed

Attraction towards centre of sphere [H4 P38]

Book II: Motion in a resisting medium [P40 N4]

Body through air $\cong$ Still body, air moving!

↓

Wind tunnel started with Newton!

Book III: 'System of the World', brings together the more abstract results from Books I and II and applies it to the universe.

Before doing that, he lists rules he will follow (so he thought hard about it).

↳ Rule 1: Occam's razor
Rule 2: Homogeneity of the universe
Rule 3: Rock ~~~~ is always rock!
Rule 4: Induction. He doesn't believe you can prove things. You ~~can only~~ should believe in evidence until you can prove it wrong.

Moon test! Measured earth had taken place previously by French Picard
↳ 1° latitude = 69.1 miles on circumference
Circ. of earth = 69.1 × 360 mi

→ circumference of moon's orbit is
$69.1 \times 360 \times 60$
↑ dist from earth to moon in earth radii.

Know how fast moon travels around earth so can work out the speed of the moon's orbit is

200,308 ft/min

Newton wanted to measure the 'fall' of the moon

[H4 Last page]

Turns out in 1 min, the Earth falls 16ft towards the Earth.

He now considers that the distance ~~f~~ under falling travelled is

$$s = \frac{1}{2} a t^2$$

($32 ft/s^2 = 9.8 m/s^2$)

$$= \frac{1}{2} \times 32 \times 60^2$$

$$= 16 \times 60^2$$

~~(dist by object in 1~~ on surface of earth.

Dist fallen ~~on~~ in 1 min on surface at distance moon

$$= \frac{1}{2} \times 32 \times 60^2 \times \frac{1}{60^2}$$

60 earth rads away

assuming inverse square law

$= 16$ ft

he first got 13 as a student ∵ he had wrong value of circ. of earth.

The two results tally ⇒ there is one grav. force acting here and on the moon.

He ends by saying "I don't invent hypothesis" — i.e. he didn't know why.

He published a series of 'Queries' in 1704

'Do not bodies act upon light and bend it?'

'Are not gross bodies and light convertible?'

He gives no evidence. What an extraordinary idea!
In anticipation of Einstein's Gen. Theory of Relativity.
Second one is like $E=mc^2$!!

EULER

Leonard Euler (1707–1783) was Swiss, from Geneva.
Made contributions to loads of branches of mathematics.

The father of analysis — 1727–41 Russia
Number theory — 1741–1766 Germany
Differential geometry — 1766–83† Russia
Topology
Graph theory (founded!) — Leading mathematical physicist

One paper a month!
800 books, papers. 73 volumes!

Blind for many years up to his death. On the day he died he was calculating orbits of Uranus (recently ~~discovered~~ found) in his head.

(1) Notation: Devised/popularised:

$$e \qquad i = \sqrt{-1} \qquad lx = \log(x) \qquad fx = f(x)$$

(2) Analysis: his term! Published 'Analysis of infinitesimals' 1748, in which he investigated infinite processes.

Gave first defn. of a function:
if x denotes a variable then all quantities dependent on x are functions of x.

Treated sines, cosines, logarithms as functions rather than curves.

uses repeated use of binomial theorem

(a) series for e (his own invention)

[HS P3]

$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots$

$e = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n$

calculated to 23 dp's

(b) series for ln (not the first person to do this)
(preceded by Newton and a different Mercator)

$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$

also expanded log into complex n°s.

<u>(3) Complex Analysis:</u> Euler was the first mathematician after Bombelli to work with complex n°s. He derived many identities.

Takes DeMoivre's thm $\cos nz + i \sin nz = (\cos z + i \sin z)^n$ and manipulates.

[HS P5]

$\Rightarrow \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4} - \dots$ (Not new, Newton had done this, as had Indian mathematicians, but he was the first to use this method).

found (see handout)

$\cos x + i \sin x = e^{ix}$

let $x = \pi$: $e^{i\pi} + 1 = 0$. (not actually found explicitly in Euler's work but he uses it to find other things)

Found logarithms of negative n°s
$$\log(-x) = \log x + i\pi$$

(and much more).

(4) Classical Number theory

(a) Fermat conjectured that

$2^{2^k}+1$ is prime for $k=0,1,\ldots$

And $F_0, F_1, \ldots, F_4$ are prime.
But Euler found a counterexample:

$F_5 = 2^{2^5}+1 = 4,294,967,297$
$= 6,700,417 \times 641$. ← astonishing powers of computation

(As it happens, noone has found any more Fermat n°s that are prime).

(b) Fermat's Little Thm: proved. ← first proof was Euler's

$a^{p-1} \equiv 1 \pmod p$ for $\mathrm{hcf}(a,p)=1$

(c) Euler Totient function:

$\Phi(m)$ = n° integers less than and relatively prime to m

e.g. $\Phi(12) = 4$ $(\{1,5,7,11\})$

$$\Phi(m) = m\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_r}\right)$$

where p_i are distinct prime factors of m.

e.g. $\Phi(12) = 12(1-\frac{1}{2})(1-\frac{1}{3}) = 4$

and deduced $a^{\Phi(m)} \equiv 1 \pmod{m}$, $\mathrm{hcf}(a,m)=1$
a generalisation of Fermat's Little Thm.

(5) Analytical Number Theory

founded by Euler

(a) $\sum_{k=1}^{\infty} \frac{1}{k^2}$ known to be cvgt but not known to what to.

In 17~~88~~ 35, he solved it. This made his reputation. Such a hard problem

A cavalier but v. clever sol^n.

[U5 P11]

Also solved $\sum \frac{1}{k^4}, \sum \frac{1}{k^6}, \ldots, \sum \frac{1}{k^{26}}$. (even powers only)

In 1736, he asked what about $\sum \frac{1}{k^3}$?

Noone knows!!

(Used knowledge of infinite series to learn something about the integers)

(b) Gamma function. $\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt$

Showed

$$\Gamma(x+1) = x\Gamma(x).$$

(c) Euler's constant γ (he denoted it C)

$$\gamma = \lim_{n \to \infty} \left[\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) - \log(n+1) \right]$$

(d) Link between harmonic series and prime n^{o}s

$$\sum \frac{1}{n} = \prod \frac{1}{1-\frac{1}{p}}$$ (not good maths since they both diverge but the idea is this)

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots = \left(\frac{1}{1-\frac{1}{2}}\right)\left(\frac{1}{1-\frac{1}{3}}\right)\left(\frac{1}{1-\frac{1}{5}}\right)\cdots$$

Actually, the correct formulation is this by Kronecker:

zeta f^n! → $$\sum \frac{1}{n^s} = \prod \frac{1}{1-\frac{1}{p^s}} \qquad s > 1.$$

Euler's theorem can be interpreted as letting $s \to 1$

(6) Calculus

- Obtained derivatives of a no. of f^ns using binomial expansion and then ignoring any d^2's.

x^n, $\log x$, $\sin x$, $\sin^{-1} x$

Uses complex analysis sometimes

- Integration – wrote a book 'Institutiones Calculi Integralis'. First to look at $\iint$ for finding vols and first to use partial diff^n, using Jacobian

Euler invented the Jacobian, lol.

- Differential eq^ns: he found how to solve 1st order ODEs by multiplying by integrating factor.

For higher order linear DEs,
he found you had to solve Aux. Eq. first.
[HS P8]

• Calculus of variations [HS P9]

(7) <u>Graph Theory</u> (founded!)

Bridges of Königsberg [HS P15]

GAUSS

Carl Friedrich Gauss (1777-1855)

age 9: $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

1795 University of Göttingen
1801 Gained intl reputation by plotting the orbit of the asteroid Ceres
1807 Director of Observatory
Later Prof. of Astronomy. Didn't like teaching maths.

Contributed to number theory
algebra
differential geometry
potential theory
astronomy
numerical analysis
non-Euclidean geometry.

1. Constructability of a 17-gon

Found when 19. [HS P17]

2. Fundamental Thm of Algebra

His doctoral thesis. [HS P19]

Decided Euler's proof wasn't rigorous enough.
Offered 4 proofs in his time.

3. Quadratic Reciprocity

$0^2 = 0 \equiv 0 \pmod{11}$
$1^2 = 1 \equiv 1 \pmod{11}$
$2^2 = 4 \equiv 4 \pmod{11}$
$3^2 = 9 \equiv 9 \pmod{11}$
$4^2 = 16 \equiv 5 \pmod{11}$
$5^2 = 25 \equiv 3 \pmod{11}$
$6^2 = 36 \equiv 3 \pmod{11}$
$7^2 = 49 \equiv 5 \pmod{11}$
$8^2 = 64 \equiv 9 \pmod{11}$
$9^2 = 81 \equiv 4 \pmod{11}$
$10^2 = 100 \equiv 1 \pmod{11}$

$x^2 \equiv 5 \pmod{11}$ has 2 sol$^{\underline{n}}$s : $\{4, 7\}$

So 5 has to be a residue of 11
"5 R 11"

But $x^2 \equiv 7 \pmod{11}$ has no sol$^{\underline{n}}$
"7 N 11"

If p, q prime, $p R q$ what is $q R p$?
If $p N q$, what is $q N p$?

Gauss proved that for p, q prime,

$$q R p \Rightarrow p R q$$
$$q N p \Rightarrow p N q$$

(first stated by Euler, though)

unless both p, q are of the form $4n+3$, in which case

$$q R p \Rightarrow p N q$$
$$q N p \Rightarrow p R q$$

SOLUTIONS OF POLYNOMIAL EQUATIONS

In the 16th Cent, mathematicians found analytic sol^{n}s to the cubic and quartic, involving radicals.

<u>(1) Harriot</u> He understood that in the case of a cubic eq^n with b, c, d as roots, the eq^n is

$$x^3 - (b+c+d)x^2 - (bc+bd+cd)x - bcd = 0.$$

used this to solve quartic

<u>(2) Gerard</u> Generalised this into polynomials of degree n:
If the roots were x_i,

$$x^n - s_1 x^{n-1} + s_2 x^{n-2} - \dots + (-1)^n s_n = 0$$

where $s_1 = \sum x_i$

$s_2 = \sum_{i<j} x_i x_j$ etc.

<u>(3) Newton</u> Understood that all symmetric functions of roots can be expressed in terms of coefficients.

[HS P22]

Didn't prove it or state it as a thm, but obviously understood it.

<u>(4) Lagrange</u> actually Italian! carried out a systematic investigation into polyn. eq^{n}s. He looked <u>why</u> the sol^{n}s to quads and cubics worked. He came up with the resolvant

$$t = x_1 + \alpha x_2 + \alpha^2 x_3 \qquad (\alpha^3 = 1)$$

which can be exploited to solve cubic eq^{n}s.
Could solve 6-powers by solving quadratics in t, then solve cubics from there.

But no hope of solving the quintic by this method – a bit of a dead end.

But the idea of <u>permuting</u> the roots was very important and influenced Galois.

(5) Galois

A troubled young man, too clever for his own good. Career was cut short in a duel over a woman. He ended up with the pistol without a bullet in it.

<u>Innovations</u>: Galois explained the conditions under which a polynomial eq^n could be solved by <u>radicals</u>.

Corollary: the quintic (and eq^{n}s of higher degree) is insoluble, generally. Actually anticipated by Abel (Norwegian) but still.

Groups ('group' is his word!) and subgroups.

Field extensions ('field' not his word)

Ideas	Example
1 Given $P(x)=0$ (distinct roots) $\exists$ f^n V of the roots (the Galois resolvant) s.t. all roots can be expressed as rational f^{n}s of V.	$x^4+5x^2+6=(x^2-2)(x^2-3)=0$ roots $r_1=\sqrt{2}$ $r_2=-\sqrt{2}$ $r_3=\sqrt{3}$ $r_4=-\sqrt{3}$ Let $V=r_1+r_2=\sqrt{2}+\sqrt{3}$

$r_1 = f_1(V) = V^2 - \frac{1}{2V}$

$r_2 = f_2(V) = 1 - \frac{V^2}{2V}$

$r_3 = f_3(V) = V^2 + 1/2V$

$r_4 = f_4(V) = -(V^2+1)/2V$

---

2 Find the auxiliary eq^n (min. polynomial of which V is a root)

Square both sides of

$r_1 = \sqrt{2} = \frac{V^2-1}{2V}$

$\Rightarrow V^4 - 10V^2 + 1 = 0$

So V is a root of

$X^4 - 10X^2 + 1 = 0$

$V = V_1 = \sqrt{2} + \sqrt{3}$

$V_2 = \sqrt{2} - \sqrt{3}$

$V_3 = -\sqrt{2} + \sqrt{3}$

$V_4 = -\sqrt{2} - \sqrt{3}$

---

3 There exists a group of permutations of the roots s.t. (Galois gp)

(1) every f^n of the roots which can be expressed as a rational f^n of the coeffs is invariant under the permutations of the roots, and vice-versa.

$f_1(V_1)$	$f_2(V_1)$	$f_3(V_1)$	$f_4(V_1)$
$f_1(V_2)$	$f_2(V_2)$	$f_3(V_2)$	$f_4(V_2)$
$f_1(V_3)$	$f_2(V_3)$	$f_3(V_3)$	$f_4(V_3)$
$f_1(V_4)$	$f_2(V_4)$	$f_3(V_4)$	$f_4(V_4)$
r_1	r_2	r_3	r_4
r_1	r_2	r_4	r_3
r_2	r_1	r_3	r_4
r_2	r_1	r_3	r_4

[NS p25]

Take $r_1^2 + r_2^2 = 4$

Invariant under all 4 perms.

4 When one of the ~~a~~ roots of the auxiliary eq^n is adjoined, a new f^n of the roots which can be expressed in terms of the extended field may remain invariant under a normal subgroup.

Adjoint $V_4 = -\sqrt{2}-\sqrt{3}$.
New f^n $(r_2+r_4)^2 = (-\sqrt{2}-\sqrt{3})^2$.
Invariant only under

$r_1\ r_2\ r_3\ r_4$

$r_2\ r_1\ r_4\ r_3$

To each further extension, there corresponds a smaller subgroup (fundamental thm of Galois)

Adjoin $V_2 = \sqrt{2}-\sqrt{3}$
New f^n $r_1+r_4 = \sqrt{2}-\sqrt{3}$
only remains invariant under identity.

---

5 A polynomial is only solvable on radicals if its group can be successively decomposed leaving the identity permutation alone.

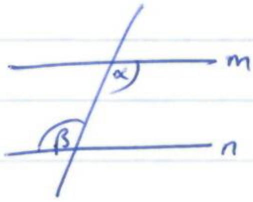
---

6 A solvable polynomial of prime degree p will have at most p(p−1) perms on its Galois gp. So a quintic would have 5×4 = 20 perms. But generally a poly. will have p! perms. Quintics would have 5! = 120 perms.
∗ ⇒ quintic can't be solved.

PARALLEL POSTULATE

The first 28 propositions of the first book of Elements require no use of the parallel postulate.

$\Longleftrightarrow$



alternate angle thm.

$\alpha + \beta < 2R \Rightarrow m, n$ meet

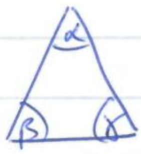
$m \parallel n \Rightarrow \alpha = \beta$

the rest of Euclidean geometry depends on the alternate angle theorem, e.g. angle sum of triangle = 2R.
pythagoras' thm
sides of similar Δs are proportional etc.

Why prove the parallel postulate?

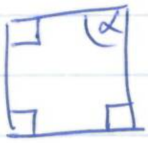
(1) lots ride on it
(2) p.p. less self-evident than other postulates
(3) its converse is a thm (1.27) $\alpha = \beta \Rightarrow m \parallel n$

alternate angle thm $\Longleftrightarrow$



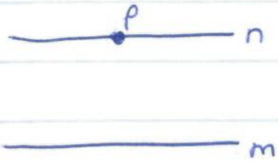
$\alpha + \beta + \gamma = 2R$

$\Longleftrightarrow$



$\alpha = R$

$\Longleftrightarrow$

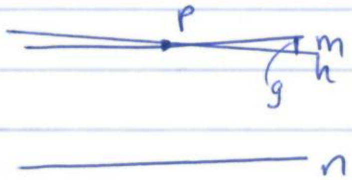


n is the unique straight line through P parallel to m

Playfair's axiom

Many attempts at the proof

(1) Proclus : right from the start he disagreed with Euclid. He did it by trying to prove Playfair's axiom.



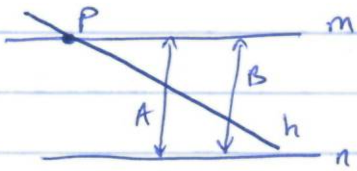
Eucl. I.27 says we can only draw 1 line through P // to n. Assume we have 2 lines through P.

Let h be the second line // to n.
Drop a ⊥ to m. Call its distance g.

As h moves to the right, g increases w/o limit.

⇒ eventually g will exceed distance between m and n,
⇒ h meets n, # to hypothesis.
⇒ it is not possible to find another line through P ~~with~~ which is parallel

$\boxed{\text{BUT}}$ there is the assumption that the distance between parallel lines remains constant or at any rate finite no matter how far the lines are extended



⟺ P.P.

ie assuming A ≡ B all the way.

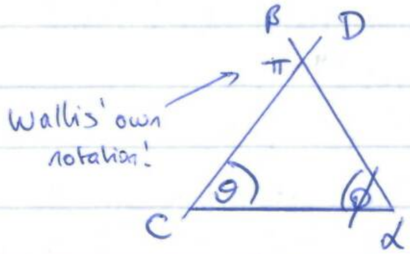
(2) Wallis Tries to prove Euclid's form of the p.p.

[HJ P27]

$\theta + \phi < 2R$.

Wants to prove CD meets AB.

Move AB parallel to itself till it cuts CD forming the triangle $\alpha C P$.



Construct ΔPCA similar to $\Delta \alpha C \pi$.

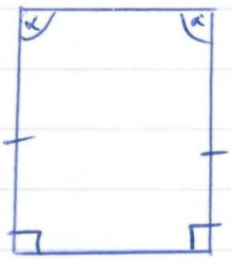
By similarity, CD and CP collinear
AB and AP collinear
$\Longrightarrow$ CD meet AB at P.

BUT Assumption in this excellent proof is this: given a triangle it is possible to draw a similar triangle of different size.

$\Longleftrightarrow$ P.P.

(3) Saccheri   Specialised in blindfold chess, could take on 3 people at once!

Saccheri quadrilateral:



$$? \begin{cases} \text{i} & \alpha = R & \text{hypothesis of rt. angle} \\ \text{ii} & \alpha > R & \text{hypothesis of obtuse angle} \\ \text{iii} & \alpha < R & \text{hypothesis of acute angle} \end{cases}$$

Tried to disprove (ii) and (iii).

He did successfully refute (ii) but his refutation of (iii) was suspect!

He did prove that if, in a single case, (i) or (ii) or (iii) is true, it is true in every case. Wonderful idea!

[M5 P28] for the suspect proof

BUT he's attempting to extend Euclid's geometry to infinity, and Euclid's geometry only works for finite geometry.

(4) Lambert

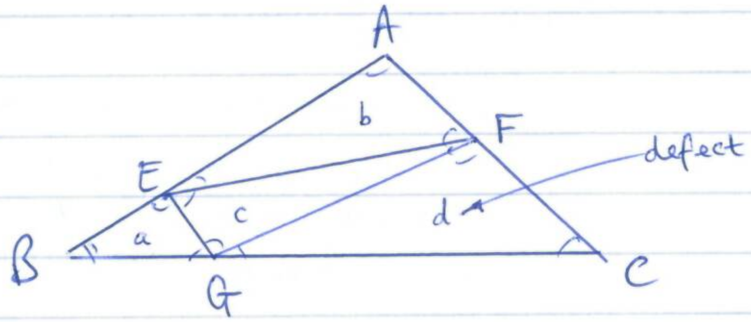
First mathematician to take an objective view of parallel lines. Simply wanted to explore, objectively, the implications of the right angle hypothesis
acute . . . .
obtuse . . . .

He refuted the obtuse angle hypothesis, like Saccheri.

But he was open minded to the hypothesis of the acute angle.

the amount the sum of angles falls short of 180°

He showed that on the acute angle hypothesis, the defect of a triangle increases with its size.



If we add all the angles in the triangles,
$(2R-a)+(2R-b)+(2R-c)+(2R-d) = 8R-(a+b+c+d)$

However, if we get rid of the angles at E, F, G, we have
$$2R-(a+b+c+d)$$

If we had a triangle of even bigger size, the defect increases ∵ we can fit more smaller triangles into it.

Implications : (a) $\exists$ an absolute standard of area and distance based on angular measure
defect $\propto$ area of Δ.

(b) $\exists$ a maximum triangle

(c) similarity does not exist
astronomy becomes an 'evil task'

serious trouble !!

By 1800 geometry is in crisis!
Obtuse angle hypothesis refuted
Acute angle hypothesis continues to resist refutation

Led to the discovery of non-Euclidean geometry, described by

Gauss → unwilling to reveal ideas
Bolyai (1802-1860) → noone listens
Lobachevsky (1796 - 1856) → publishes in Russian, German French, then noone listens.

Geometry based on acute angle hypothesis.
(later known as hyperbolic geometry)

It wasn't until Gauss' death that people looking through his papers saw that the great man had looked at it that people started taking an interest

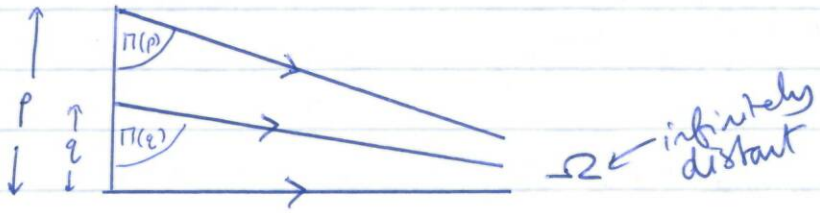
NON-EUCLIDEAN GEOMETRY

Lobachevsky and Bolyai proved some more neutral geometry theorems. In fact Bolyai proved that the p.p. is true in 3D regardless of whether it is or not in 2D.

[HS P33]

But there comes a point when we need to have some assumptions.

— boundary parallel, (separating intersecting and non-intersecting lines).



Π gets smaller as p increases

angle of parallelism $\Pi(p)$

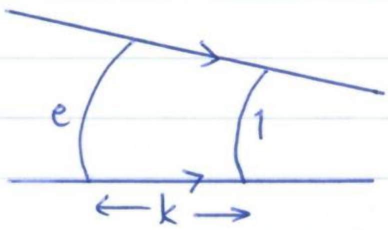
$\Pi(p) \to 0$ as $p \to \infty$

$\Pi(p) \to R$ as $p \to 0$

So we get tons of these boundary parallels all going to Ω.

We can draw a horocycle s.t. this line is a right angle to every parallel.

[HS P34]



$$\frac{\text{length of outer horocycle}}{\text{length of inner horocycle}} = e,$$

then distance between the horocycles is a linear constant k.

For the <u>horosphere</u>, a sphere of infinite radius, the geometry on its surface is Euclidean!! So we can use trigonometry on it.

Bolyai and Lob. both found

$$\tan\left[\tfrac{1}{2}\Pi(p)\right] = e^{-p/k}$$

k is the famous linear constant

$\Rightarrow \cot \Pi(p) = \sinh\left(\frac{p}{k}\right)$ etc.

They both discovered that spherical trigonometry is independent of the p.p.

<u>Area</u>: squares don't exist in non-E. geometry so how do you talk about it?

Turns out, if you have a triangle with defect α,

$$\text{Area} = k^2\alpha$$

π disappears! It is no longer the constant $\frac{C}{d}$.

$$C = 2\pi k \sinh(r/k)$$
$$4\pi k^2 \sinh^2(r/2k).$$

It's possible to <u>square the circle</u> in non-Eucl. geometry.
↑ well not a square, but a shape.

By the end of the 19th C., it was accepted there were many types of geometry.

It wasn't until Poincaré (1854-1912) who proved that you cannot prove or disprove the p.p.

The Exam

2 hours
½ hr per question.

Don't get into time trouble!!
Be strict.

(1) Answer the question,
the whole q^n
nothing but the q^n.

(2) Provide evidence
don't make bold assertions w/o proof

Q1 is context:
pick 1 from (a), (b), (c).

Indian, Chinese
are not for exam