7102 Analysis 4: Real Analysis Notes

Based on the 2013 spring lectures by Dr N Sidorova

MATH7102 — Real Analysis

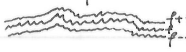
7 January 2013
Dr. Nadia SIDOROVA
CILT.

Overview of course: three parts. (chapters)

① Uniform convergence: sequences $\langle a_n \rangle$, $a_n \to a$ (numbers); sequence of functions $f_n \to f$ $\begin{cases} \text{pointwise conv. (weak)} \\ \text{uniform conv. (strong)} \\ \text{other types} \end{cases}$

In particular, we will discuss and prove that $\to$ $\exists$ a continuous but nowhere differentiable function.

Also, we ask if we can approximate any continuous function f by "nice" functions.



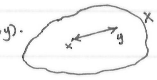
Given ε, can we find a polynomial that fits in the spanned tube?

Answer is positive: every continuous function on $[a,b]$ can be approximated by polynomials.

② Fourier series: $\{1, \cos(nx), \sin(nx)\}$. We will extend this set to more general orthogonal systems.

③ Metric spaces: consider a set X s.t. we can define "distance" as an abstract function between two points within.

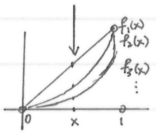
dist. $= d(x,y)$.



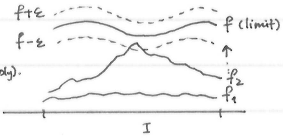
chapter 1.
UNIFORM CONVERGENCE.

$\boxed{\text{Definition}}$ Let $I \subset \mathbb{R}$, and let $\{f_n\}_{n=1}^{\infty}$ and f be real-valued functions on I. We say that <u>f_n converges to f pointwise</u> if $\forall x \in I$, $f_n(x) \to f(x)$. Equivalently, $\underline{\forall x \in I}$, $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $|f_n(x) - f(x)| < \varepsilon$. (we have an N for each value of x).

e.g. let $I = (0,1)$, $f_n(x) = x^n$. Then $\forall x \in I$, $f_n(x) = x^n \to 0$. $\Rightarrow$ f_n converges pointwise to the zero function.



$\boxed{\text{Definition}}$ Let $I \subset \mathbb{R}$, and let $\{f_n\}_{n=1}^{\infty}$ and f be real-valued functions on I. We say that <u>f_n converges to f uniformly</u> if $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $|f_n(x) - f(x)| < \varepsilon$ $\underline{\forall x \in I}$. (we need to find a universal N that would work for all $x \in I$ simultaneously).

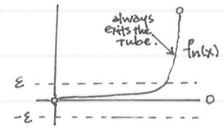


Note: We unpack this definition a bit, using a graphical representation. Draw a tube around f, using graphs $f+\varepsilon$, $f-\varepsilon$. Then if for some $N \in \mathbb{N}$, $n \geq N \Rightarrow f_n$ is completely in the tube, then it converges uniformly.

$\boxed{\text{Ex}}$ We have seen that if $I = (0,1)$, $f_n(x) = x^n$, then $\{f_n\}$ converges pointwise. Does it converge uniformly?

Soln. The only candidate for a limit, if it converges uniformly, is the same as above (i.e. 0 function)

Draw a tube from $-\varepsilon$ to ε as shown. Any f_n always exits the tube as $x \to 1$. i.e. f_n does not converge uniformly to 0, q.e.d.



$\boxed{\text{Theorem}}$ 1.1 If $f_n \to f$ uniformly, then $f_n \to f$ pointwise.

$\boxed{\text{Corollary}}$ If f_n does not converge pointwise, then it does not converge uniformly.

Proof — By contrapositive, q.e.d.

$\boxed{\text{Corollary}}$ Suppose $f_n \to f$ pointwise and $f_n \to g$ continuously, then $f = g$.

Proof — $f_n \to g$ continuously $\Rightarrow$ $f_n \to g$ pointwise. Then $f_n \to f, g$ pointwise. By uniqueness of limits, $f = g$, q.e.d.

Proof — If $f_n \to f$ uniformly, then $\forall \varepsilon > 0$ $\exists N$ (depending only on ε and not x) s.t. $\forall n > N$, $|f_n(x) - f(x)| < \varepsilon$ $\forall x \in I$ $\Rightarrow$ $f_n \to f$ pointwise, q.e.d.

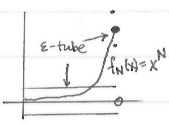
<u>Testing for pointwise and uniform convergence</u>

- Test pointwise convergence $\to$ if there is <u>no</u> pointwise convergence, then there is <u>no</u> uniform convergence.
- If $f_n \to f$ pointwise, test whether $f_n \to f$ uniformly.

We now move on to evaluate some examples for practice.

$\boxed{\text{Ex}}$ Determine whether the following functions converge pointwise? uniformly?

① $I = [0,1]$, $f_n(x) = x^n$



Soln. If $x = 1$, $f_n(1) \to 1$. If $x = 0$, $f_n(0) \to 0$. If $0 < x < 1$, $f_n(x) \to 0$. Then f_n converges pointwise, to f s.t. $f(x) = \begin{cases} 0, & x \in [0,1) \\ 1, & x = 1. \end{cases}$

10 January 2013
Dr. Nadia SIDOROVA
Roberts G06.

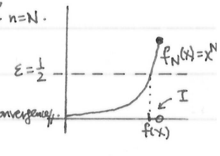
We then must determine if $f_n \to f$ uniformly? We claim it does not, and to do that we apply the negated definition of uniform convergence: f_n does not converge to f uniformly, i.e. $\boxed{\exists \varepsilon > 0 \text{ s.t. } \forall N \in \mathbb{N} \ \exists n \geq N \ \exists x \in I, \ |f_n(x) - f(x)| \geq \varepsilon}$

in most cases, one can take $n = N$.

i.e. we must find an ε-tube that is "bad enough". Choose $\varepsilon = \frac{1}{2}$, let $N \in \mathbb{N}$ be fixed.

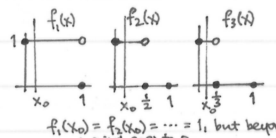
Choose $n = N$, then $f_n(x) = f_N(x) = x^N$. We see that x is any value where the function exits the $\frac{1}{2}$-tube.

Take left limit, $x^N = \frac{1}{2} \Rightarrow x = \frac{1}{2^{1/N}}$. Then for this x, $|f_n(x) - f(x)| = |f_N(x) - f(x)| = |\frac{1}{2} - 0| = \frac{1}{2} = \varepsilon \geq \varepsilon \Rightarrow$ no uniform convergence.



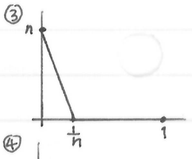
② $I = [0,1]$, $f_n(x) = \begin{cases} 1 & \text{for } x \in [0, \frac{1}{n}) \\ 0 & \text{for } x \in [\frac{1}{n}, 1]. \end{cases}$

Soln. If $x = 0$, $f_n(0) = 1 \to 1$. If $x = 1$, $f_n(1) = 0 \to 0$. For $x \in (0,1)$, $f_n(x) = \underbrace{\begin{cases} 1 & \text{if } x \in (0, \frac{1}{n}) \\ 0 & \text{if } x \in [\frac{1}{n}, 1) \end{cases}}^{\{1,1,1,\ldots,1,0,0\ldots\}} \to 0$ as $n \to \infty$. Thus, $f_n \to f$ pointwise, where $f(x) = \begin{cases} 1 & x = 0 \\ 0 & x \in (0,1]. \end{cases}$ Does $f_n \to f$ uniformly? No. Choose $\varepsilon = \frac{1}{2}$, $\forall N \in \mathbb{N}$. Choose $n = N$, and pick x s.t. $|f_n(x) - f(x)| = |f_N(x) - f(x)| \geq \frac{1}{2}$. Take $x = \frac{1}{2N}$. Then $|f_N(x) - f(x)| = 1 - 0 = 1 \geq \frac{1}{2} = \varepsilon$, q.e.d.



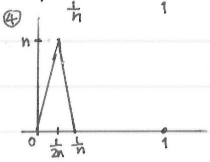
$f_1(x_0) = f_2(x_0) = \cdots = 1$, but beyond a point goes to 0.

③ $I=[0,1]$, $f_n(x)=\begin{cases} 0 & x\in[\frac{1}{n},1] \\ n-n^2x & x\in[0,\frac{1}{n}) \end{cases}$



Soln. Take $x=0$. Then $f_n(0)=n\to\infty$ (diverges) $\Rightarrow$ no pointwise convergence // $\Rightarrow$ no uniform convergence //.

④ $I=[0,1]$, $f_n(x)=\{$ (see graph marked ④)



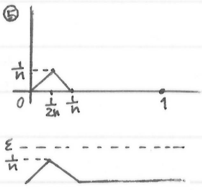
Soln. Take $x=0$, $f_n(0)=0\to 0$; $x=1$, $f_n(1)=0\to 0$. Then we take $x\in(0,1)$. Consider, for instance, x between $\frac{1}{3}$ and $\frac{1}{4}$.

Then $f_1(x)<f_2(x)<f_3(x)$, but $f_3(x)>f_4(x)>\cdots\to 0$. Hence, for any fixed point $x\in(0,1)$, $f_n(x)\to 0$ as $n\to\infty$.

Thus, $f_n(x)\to f(x)=0$; $f_n\to f$ pointwise //, $f_n\not\to f$ uniformly //. Choose $\varepsilon=100$. Let $N\in\mathbb{N}$, take $n=101+N$

and find x s.t. $|f_n(x)-f(x)|\geq 100$. Pick $x=\frac{1}{2n}$. Then $|f_n(x)-f(x)|=|n-0|=n=101+N\geq 100=\varepsilon$

⑤ $I=[0,1]$, $f_n(x)=\{$



Soln. As in Ex ④, $f_n\to f(x)=0$ pointwise //. However, we claim here that $f_n(x)\to f(x)=0$ uniformly.

Working with definitions, we seek N s.t. given $\varepsilon>0$, $\exists N$ s.t. $\forall n\geq N$ $\forall x\in I$ s.t. $|f_n(x)-f(x)|<\varepsilon$.

$\forall n\geq N$, $\forall x$, $|f_n(x)-f(x)|=|f_n(x)-0|=|f_n(x)|<\varepsilon$. We want $\frac{1}{n}<\varepsilon$, so set $\frac{1}{N}<\varepsilon$

$|f_n(x)-f(x)|=|f_n(x)|\leq\frac{1}{n}\leq\frac{1}{N}<\varepsilon$ // q.e.d.

⑥ $I=(0,\infty)$, $f_n(x)=x+\frac{1}{x+n}$.

Soln. As $n\to\infty$, $f_n(x)\to x+0=0$ $\forall x$. Hence pointwise, $f_n\to f(x)=x$ //. We then test uniform convergence. Evaluate $|f_n(x)-f(x)|$.

$|f_n(x)-f(x)|=|\frac{1}{x+n}|=\frac{1}{x+n}\leq\frac{1}{n}$ (since $x>0$). Is $\frac{1}{n}<\varepsilon$? We suspect that convergence is uniform.

$\forall\varepsilon>0$, choose N s.t. $\frac{1}{N}<\varepsilon$; let us say, $N=\lfloor\frac{1}{\varepsilon}\rfloor+2$. Then $\forall n\geq N$, $|f_n(x)-f(x)|=\frac{1}{x+n}\leq\frac{1}{n}\leq\frac{1}{N}<\varepsilon$ // q.e.d.

Theorem 1.2 (Uniform convergence preserves continuity)

Let $I\subset\mathbb{R}$, let $\{f_n\}_{n=1}^{\infty}$, f be real-valued functions on I. Suppose (i) all f_n are continuous and (ii) $f_n\to f$ uniformly.

Then f is continuous.

Note: Not true for pointwise convergence! e.g. $f_n(x)=x^n$ on $[0,1]$. f_n converges pointwise to discontinuous $f=\begin{cases}0 & x\in[0,1) \\ 1 & x=1.\end{cases}$

Proof – NTP: Let $x\in I$ and prove that f is continuous at x, i.e. $\forall\varepsilon>0$ $\exists\delta>0$ s.t. $y\in I$, $|y-x|<\delta\Rightarrow|f(y)-f(x)|<\varepsilon$. Let $\varepsilon>0$ be given.

- We know $f_n\to f$ uniformly $\Rightarrow$ $\exists N$ s.t. $\forall n\geq N$, $\forall z\in I$ $|f_n(z)-f(z)|<\varepsilon/3$. In particular, $\forall z\in I$, $|f_N(z)-f(z)|<\frac{\varepsilon}{3}$.
- By continuity of f_N at x, $\exists\delta>0$ s.t. $y\in I$, $|y-x|<\delta\Rightarrow|f_N(y)-f_N(x)|<\frac{\varepsilon}{3}$.

If $y\in I$ and $|y-x|<\delta$, then $|f(y)-f(x)|=|f(y)-f_N(y)+f_N(y)-f_N(x)+f_N(x)-f(x)|\leq|f(y)-f_N(y)|+|f_N(y)-f_N(x)|+|f_N(x)-f(x)|$

From our observation above, and since $x,y\in I$, then $|f(y)-f(x)|<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon\Rightarrow f$ is continuous // q.e.d.

Covers and subcovers.

Consider the open interval: (a,b). Let us examine a collection of open intervals. These can be denoted by $\{I_i\}_{i\in A}$, where A is an index set. This reflects the diversity:

(1) finite: $I_1, I_2, \ldots, I_m$; in which case we write $\{I_i\}_{i\in\{1,2,\ldots,m\}}$

(2) infinite: $I_1, I_2, \cdots$; in which case we index with natural numbers $\{I_i\}_{i\in\mathbb{N}}$

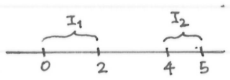
(3) infinite: $\{(x-1,x+1)\}_{x\in\mathbb{R}}$ for instance.

Definition A collection $\{I_i\}_{i\in A}$ of open intervals is a cover of a set $S\subset\mathbb{R}$ if $S\subset\bigcup_{i\in A}I_i$.

Let S be a set and $\{I_i\}_{i\in A}$ be a cover of S. A subcollection of $\{I_i\}_{i\in A}$ is called a subcover of S if this smaller collection is itself a cover of S.

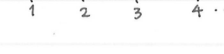
Ex ① Let $I_1=(0,2)$ and $I_2=(4,5)$. Consider the sets $S=$

(a) $(0,1)$, (b) $[0,\frac{1}{2})$, (c) $=\{1\}\cup(4,4.5)$. Which of these ~~are~~ is $\{I_1,I_2\}$ a covers of?



Soln. (a) and (c) //. $\{I_1,I_2\}$ is a cover of $S=(0,1)$ and $S=\{1\}\cup(4,4.5)$. However, $0\in S$ but $0\notin\{I_1,I_2\}$.

② Let $I_n=(n-\frac{1}{3},n+\frac{1}{3})$, $n\in\mathbb{N}$. Is $\{I_n\}_{n\in\mathbb{N}}$ a cover for the sets $\mathbb{N}$, $\mathbb{Z}$, $\{1\}\cup(3-\frac{1}{5},3+\frac{1}{5})$, $(3-\frac{1}{5},3+\frac{1}{5})$? If so, does it have a finite subcover?

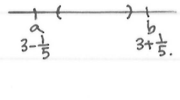


Soln.

Set, S.	Is $\{I_n\}_{n\in\mathbb{N}}$ a cover for S?	Does the cover $\{I_n\}_{n\in\mathbb{N}}$ have a finite subcover?
$\mathbb{N}$	yes	no.
$\mathbb{Z}$	no	–
$\{1\}\cup(3-\frac{1}{5},3+\frac{1}{5})$	yes	yes, we can take $\{I_1,I_3\}$
$(3-\frac{1}{5},3+\frac{1}{5})$	yes	yes, we can take I_3.

③ Consider the set $S=(3-\frac{1}{5},3+\frac{1}{5})$. In example ②, we had a cover of S which has a finite subcover. Construct a cover of S which has no finite subcover.

Soln. Take $I_n=(3-\frac{1}{5}+\frac{1}{n},3+\frac{1}{5}-\frac{1}{n})$. Then $\bigcup I_n=(3-\frac{1}{5},3+\frac{1}{5})$ ~~is a cover of~~ I_n is a cover of $(3-\frac{1}{5},3+\frac{1}{5})$.



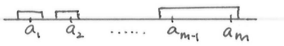
Then, this cover of $(3-\frac{1}{5},3+\frac{1}{5})$ has no finite subcover.

Question: Are there sets for which every cover has a finite subcover?

14 January 2013
Dr. Nadia SIDOROVA
CLT.

Definition A set $K \subset \mathbb{R}$ is <u>compact in $\mathbb{R}$</u> if every cover of K (by open intervals) has a finite subcover.

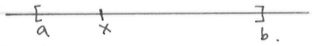
e.g. $S = \{a_1, \dots, a_m\}$ up to finite m is compact.



Theorem 13 (Heine-Borel)

Every closed interval $[a,b]$ is compact in $\mathbb{R}$.

Proof - NTP: every cover of $[a,b]$ has a finite subcover. Let $\{I_i\}_{i \in A}$ be some cover of $[a,b]$.



Let $B = \{x \in [a,b] : [a,x]$ has a finite subcover$\}$. Obviously, we claim $a \in B \Rightarrow B \neq \phi \Rightarrow$ has sup.

Denote $c = \sup B$. Suppose $c \neq b \Rightarrow c < b$. $\exists$ interval I_γ s.t. $c \in I_\gamma$, $I_\gamma \in [a,b]$. Since $c = \sup B$,

we can pick $x \in B$ and $x \in I_\gamma$ s.t. $x \leq c$ (by definition). Then by definition of B, the interval $[a,x]$ can be covered by finite collection $I_{\alpha_1}, \dots, I_{\alpha_m}$. Pick $\begin{cases} y \in I_\gamma \\ y > c \end{cases}$. Then the interval $[a,y]$ is covered by $I_{\alpha_1}, I_{\alpha_2}, \dots, I_{\alpha_m}, I_\gamma$.

This is a finite collection, so $y \in B$. But $y > c = \sup B \Rightarrow$ contradiction. Thus, $c = b$. We still must establish that $c = b \in B$.

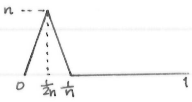
Pick an interval that covers the point b, labelling it newly I_γ. Then pick $x \in I_\gamma$, $x < b$. Then $x \in B \Rightarrow$ we need only finitely many intervals to cover $[a,x]$, say $I_{\beta_1}, \dots, I_{\beta_m}$. Then the interval $[a,b]$ is covered by $I_{\beta_1}, \dots, I_{\beta_m}, I_\gamma \Rightarrow b \in B$. $\therefore$

$\{x \in [a,b] : [a,x]$ has a finite subcover$\} = [a,b] \Rightarrow [a,b]$ is compact in $\mathbb{R}$ // q.e.d.

Recall that theorem 1.2 claimed that $\begin{cases} f_n \to f \text{ uniformly} \\ \text{all } f_n \text{ are continuous} \end{cases} \Rightarrow f$ is continuous. Can we make a claim about uniform convergence as well, if f is continuous?

No. For instance, recall the example where $f_n \to 0$ uniformly, shown in graph on right. All f_n are continuous, 0 is continuous; but no uniform convergence.

We can however claim this by imposing more constraints.

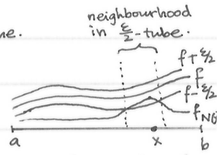


Theorem 14 (Dini's theorem)

Let $\{f_n\}_{n=1}^{\infty}$, f be real-valued functions on $[a,b]$ such that

(a) $f_n \to f$ pointwise, (b) all f_n are continuous, (c) f is continuous, and (d) $\forall x \in [a,b]$, $\{f_n(x)\}_{n=1}^{\infty}$ is monotone.

Then $f_n \to f$ uniformly.



At x, $|f_{N(x)}(x) - f(x)| < \frac{\varepsilon}{2}$.

Proof - We want $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ $\forall n \geq N$ $\forall x \in [a,b]$ $|f_n(x) - f(x)| < \varepsilon$. Let $\varepsilon > 0$, $x \in [a,b]$ fixed.

Since $f_n \to f$ pointwise, we have $f_n(x) \to f(x)$. $\Rightarrow$ $\exists N(x)$ $\forall n \geq N(x)$, $|f_n(x) - f(x)| < \frac{\varepsilon}{2}$.

In particular, $|f_{N(x)}(x) - f(x)| < \frac{\varepsilon}{2}$. Denote $g(y) = f_{N(x)}(y) - f(y)$ (difference of graph from f at $y = x$).

$f_{N(x)}$, f are continuous at $x \Rightarrow g$ is continuous at x. Given $\varepsilon > 0$, $\exists \delta(x) > 0$ s.t. $|y - x| < \delta(x), y \in [a,b] \Rightarrow |g(y) - g(x)| < \frac{\varepsilon}{2}$.

$\Rightarrow |f_{N(x)}(y) - f(y)| = |g(y)| = |g(y) - g(x) + g(x)| \leq |g(y) - g(x)| + |g(x)| = |g(y) - g(x)| + |f_{N(x)}(x) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

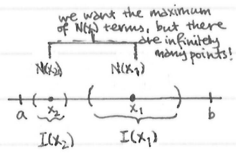
17 January 2013
Dr. Nadia SIDOROVA
CLT.

whenever $y \in [a,b]$ and $|y - x| < \delta(x)$. i.e. there is a neighbourhood of $f_{N(x)}(x)$ in ε-tube.

???

$\forall y \in [a,b]$, $f_n(y)$ is monotone $\Rightarrow$ $f_n(y) - f(y)$ is monotone and converges to $0 \Rightarrow |f_n(y) - f(y)|$ is decreasing.

$\therefore$ $|f_n(y) - f(y)| \leq |f_{N(x)}(y) - f(y)| < \varepsilon$ whenever $y \in [a,b]$, $|y - x| < \delta(x)$.



Denote $I(x) = (x - \delta(x), x + \delta(x))$. We have proven that $\forall x \in [a,b]$, $\forall y \in I(x)$, $\forall n \geq N(x) \Rightarrow |f_n(y) - f(y)| < \varepsilon$.

In this interval $I(x)$, $f_n \to f$ uniformly. But depending on x, we have different values for $N(x)$. We want to pick maximum of it, but there are infinitely many points $x \in [a,b]$. We work around this:

Let $\{I(x)\}_{x \in [a,b]}$ be a cover of $[a,b]$. By Heine-Borel theorem, $\exists$ finite subcover $I(x_1), \dots, I(x_m)$.

Choose $N = \max\{N(x_1), \dots, N(x_m)\}$. Take $n \geq N$. $x \in [a,b] \Rightarrow x \in I(x_i)$ for some $1 \leq i \leq m$. $n \geq N \geq N(x_i) \Rightarrow |f_n(x) - f(x)| < \varepsilon$ // q.e.d.

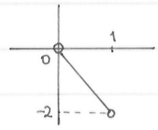
Ex Find an example to show that Dini's Theorem does not work on an open interval.

Soln. Let $(0,1)$ be the open interval I. Take $f_n(x) = x^n$, certainly, (a) $f_n \to 0$ pointwise, (b) all $f_n(x) = x^n$ are continuous, (c) $f = 0$ is continuous,

(d) $f_n(x)$ is monotone increasing. However, $f_n \not\to f$ uniformly. // q.e.d.

Definition Let $I \subset \mathbb{R}$ and $f: I \to \mathbb{R}$. Then the <u>supremum norm</u> of f is $\|f\|_{\sup} = \sup_{x \in I} |f(x)|$.

e.g. ① $I = \mathbb{R}$, $f(x) = \sin(x)$. $\|f\|_{\sup} = 1$. ② $I = (0,1)$, $f(x) = -2x$. $\|f\|_{\sup} = 2$.



Theorem 1.5 f_n converges to f uniformly on I $\Leftrightarrow$ $\|f_n - f\|_{sup} \to 0$ as $n \to \infty$.

Proof – $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. for $n > N$, $\forall x \in I$, $|f_n(x) - f(x)| \le \varepsilon$ $\Leftrightarrow$ $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $n > N$, $\forall x \in I$, $\|f_n(x) - f(x)\|_{sup} \le \varepsilon$ $\Leftrightarrow$ $\|f_n - f\|_{sup} \to 0$ // q.e.d.

e.g. ① $I = (0,1)$, $f_n(x) = x^n$. $f_n \to 0$ pointwise. $\|f_n - 0\|_{sup} = \|f_n\|_{sup} = 1 \not\to 0$. $\Rightarrow$ no uniform convergence.

② $I = \mathbb{R}$, $f_n(x) = \begin{cases} 1 & x \le n \\ 0 & x > 0 \end{cases}$. Then $f_n \to 1$ pointwise. $\|f_n - 1\|_{sup} = \left\| \begin{matrix} 0, & x \le n \\ -1, & x > 0 \end{matrix} \right\|_{sup} = 1 \not\to 0$ as $n \to \infty$. $\Rightarrow$ no uniform convergence.

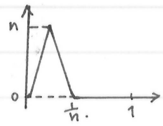
③ $I = \mathbb{R}$, $f_n(x) = \begin{cases} \frac{1}{n} & x \le n \\ 0 & x > 0 \end{cases}$. Then $f_n \to 0$ pointwise. $\|f_n - 0\|_{sup} = \|f_n\|_{sup} = \frac{1}{n} \to 0$ as $n \to \infty$. $\Rightarrow$ uniform convergence.

Recall that $\left.\begin{matrix} f_n \to f \text{ uniformly} \\ \text{all } f_n \text{ continuous} \end{matrix}\right\} \Rightarrow f$ is continuous. We make a similar statement concerning Riemann integrability.

Theorem 1.6 Let $\{f_n\}_{n=1}^{\infty}, f$ be real-valued functions on $[a,b]$, such that

- $f_n \to f$ uniformly, and
- all f_n are Riemann integrable,

then f is Riemann integrable and $\int_a^b f_n(x)\,dx \longrightarrow \int_a^b f(x)\,dx$.



Note: This theorem does not work if we replace uniform convergence by pointwise convergence. For example, consider the functions f_n on the right. $f_n \to 0$ pointwise but not uniformly. 0 is integrable, but $\int_0^1 f_n(x)\,dx = \frac{1}{2} \not\to 0 = \int_0^1 0\,dx$.

Also, sheet 2 Q6 contains an example where f is not integrable.

Recall: Let $P \in \mathbb{P}[a,b]$ be a partition of $[a,b]$. $a = t_0 < t_1 < \dots < t_n = b$. Define upper and lower Darboux sums: $U(f,P) = \sum_{i=1}^{n} \sup_{[t_{i-1},t_i]} f(x)\,(t_i - t_{i-1})$, $L(f,P) = \sum_{i=1}^{n} \inf_{[t_{i-1},t_i]} f(x)\,(t_i - t_{i-1})$. We say that f is Riemann integrable iff $\forall \varepsilon > 0$ $\exists P$ s.t. $U(f,P) - L(f,P) < \varepsilon$.

Proof – Let $\varepsilon > 0$. Since $f_n \to f$ uniformly, $\exists N$ s.t. $|f_N(x) - f(x)| < \frac{\varepsilon}{4(b-a)}$ $\forall x \in [a,b]$. Since f_N is Riemann integrable, there is a partition P s.t. $U(f_n, P) - L(f_n, P) < \frac{\varepsilon}{4}$. Then $|f_N(x) - f(x)| < \frac{\varepsilon}{4(b-a)}$ $\Rightarrow$ $f_N(x) - \frac{\varepsilon}{4(b-a)} < f(x) < f_N(x) + \frac{\varepsilon}{4(b-a)}$, $\forall x \in [a,b]$.

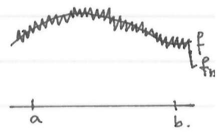
$U(f,P) = \sum_{i=1}^{n} \sup_{[t_{i-1},t_i]} f(x) \cdot (t_i - t_{i-1}) \le \sum_{i=1}^{n} \left(\sup_{[t_{i-1},t_i]} \left[f_N(x) + \frac{\varepsilon}{4(b-a)} \right] \right)(t_i - t_{i-1}) = U(f_n, P) + \frac{\varepsilon}{4(b-a)} \sum_{i=1}^{n} (t_i - t_{i-1}) = U(f_n, P) + \frac{\varepsilon}{4}$.

$L(f,P) = \sum_{i=1}^{n} \inf_{[t_{i-1},t_i]} f(x) \cdot (t_i - t_{i-1}) \ge L(f_N, P) - \frac{\varepsilon}{4}$. Thus, $U(f,P) - L(f,P) \le U(f_n,P) + \frac{\varepsilon}{4} - L(f_N,P) + \frac{\varepsilon}{4} = \frac{\varepsilon}{2} + [U(f,P) - L(f,P)] = \frac{3\varepsilon}{4} < \varepsilon$ // q.e.d.

$\left| \int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx \right| = \left| \int_a^b (f_n(x) - f(x))\,dx \right| \le \int_a^b |f_n(x) - f(x)|\,dx$ by triangle inequality of integrals. We estimate by supremum norm:

$\left| \int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx \right| \le \int_a^b |f_n(x) - f(x)|\,dx \le \int_a^b \|f_n(x) - f(x)\|_{sup}\,dx = \underbrace{\|f_n(x) - f(x)\|_{sup}}_{\text{(just a real number)}} \int_a^b dx = (b-a)\,\|f_n(x) - f(x)\|_{sup} \to 0$ as $f_n \to f$ uniformly

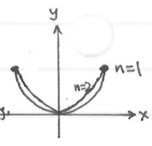
Hence, $\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx$ // q.e.d.



How about differentiability? Is there a condition similarly for differentiability? No.

e.g. ① $I = \mathbb{R}$, $f_n(x) = \frac{1}{n} \sin(n^2 x)$. Then $f_n \to 0$ pointwise and uniformly: $\|f_n - 0\|_{sup} = \frac{1}{n} \to 0$. $f_n'(x) = n \cos(n^2 x)$. At $x = 0$, $f_n'(0) = n$, which does not converge. i.e. f_n' does not converge, not even pointwise; although $f = 0$ certainly does.

② In the previous example, we have $f_n \to f$ uniformly, all f_n differentiable. Although f_n' did not converge, at least we had f differentiable.



Now, we construct $f_n \to f$ uniformly, all f_n differentiable; but f is not differentiable. Take $I = [-1,1]$, $f_n(x) = |x|^{1+\frac{1}{n}}$.

Clearly, $f_n \to |x|$ pointwise. Using Dini's theorem, $f_n \to |x|$ pointwise, all f_n are continuous, $|x|$ is continuous and $\{f_n\}$ is monotone increasing.

$\Rightarrow f_n \to f$ uniformly. We see that at $x = 0$, $\lim_{h \to 0} \frac{f_n(h) - f_n(0)}{h} = \lim_{h \to 0} \frac{|h|^{1+\frac{1}{n}}}{h} = \lim_{h \to 0} |h|^{\frac{1}{n}} \operatorname{sign}(h) = 0$, so $f_n(x)$ are differentiable, but $f(x) = |x|$ is not differentiable.

Series

A series is given by $\sum_{n=1}^{\infty} g_n$, where g_n are functions s.t. $g_n : I \to \mathbb{R}$. Let $f_n(x)$ be the partial sum of the series from 1 to n: $f_n(x) = \sum_{i=1}^{n} g_i(x)$.

This gives us a sequence of partial sums.

Definition A series $\sum_{n=1}^{\infty} g_n$ converges pointwise on I if f_n converges pointwise on I.

If $f_n \to f$, then we call f ~~the sum~~ of the series, and write $f = \sum_{n=1}^{\infty} g_n$.

The series $\sum_{n=1}^{\infty} g_n$ converges uniformly on I if f_n converges uniformly on I.

Ex The series $\sum_{n=0}^{\infty} x^n$ converges on $(-1,1)$, diverges otherwise $\Rightarrow$ converges pointwise on $(-1,1)$: Let $f_n(x) = \sum_{i=0}^{n} x^i = \frac{1 - x^{n+1}}{1 - x}$ for $x \in (-1,1)$.

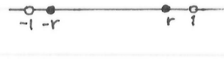
21 January 2013
Dr. Nadia SIDOROVA
CLT.

Does it converge uniformly on $(-1,1)$? on $[r,-r]$ where $0 < r < 1$?

Soln. We know that as $n \to \infty$, $f_n(x) \to \frac{1}{1-x}$ for all $x \in (-1,1)$. Then $|f_n(x) - f(x)| = \left| \frac{1 - x^{n+1}}{1-x} - \frac{1}{1-x} \right| = \frac{|x|^{n+1}}{1-x}$.

On $x \in (-1,1)$; $\|f_n - f\|_{sup} = \sup_{x \in (-1,1)} \frac{|x|^{n+1}}{1-x} = \infty$ (as $x \to 1$) $\not\to 0$ $\Rightarrow$ no uniform convergence on $(-1,1)$ //

On $x \in [-r,r]$; $|x|^{n+1} \le r^{n+1}$ and, $1 - x \ge 1 - r$. Then, $\|f_n - f\|_{sup} = \sup_{[-r,r]} \frac{|x|^{n+1}}{1-x} \le \sup_{[r,r]} \frac{r^{n+1}}{1-r} \to 0$.



Thus, uniform convergence on $[-r,r]$ //

Definition Let $I \subseteq \mathbb{R}$ and $\{f_n\}_{n=1}^{\infty}$ be real-valued functions on I. The sequence $\{f_n\}$ is called a <u>uniform Cauchy sequence</u> if

$\forall \varepsilon > 0, \ \exists N \in \mathbb{N} \ \ \forall n, m \geq N \ \ \|f_n - f_m\|_{sup} < \varepsilon.$

Theorem 1.7 (Central Principle of Uniform convergence).

A sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly on I $\iff$ $\{f_n\}_{n=1}^{\infty}$ is a uniform Cauchy sequence on I.

Proof $-$ ($\Rightarrow$). Suppose $f_n \to$ some f uniformly on I, i.e. given fixed $\varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. $\forall n > N$, $|f_n - f| < \frac{\varepsilon}{4}$ $\forall x \in I$. Then we have:

$|f_n(x) - f_m(x)| = |f_n(x) - f(x) + f(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f_m(x) - f(x)| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2} \ \ \forall n, m \geq N, \ x \in I.$

Thus, $\|f_n(x) - f_m(x)\|_{sup} \leq \frac{\varepsilon}{2} < \varepsilon \ \ \forall n, m \geq N. \Rightarrow \{f_n\}_{n=1}^{\infty}$ is a uniform Cauchy sequence // q.e.d.

($\Leftarrow$). Suppose $\{f_n\}_{n=1}^{\infty}$ is a uniform Cauchy sequence. Then $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ $\forall n, m \geq N$, $\sup_{x \in I} |f_n(x) - f_m(x)| < \varepsilon/2$. In particular, for n, m fixed, $|f_n(x) - f_m(x)| < \varepsilon/2$ for any $x \in I$. Then $\forall x \in I \Rightarrow \{f_n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence of numbers. Then $f_n(x)$ converges to some value $f(x)$.

Now, $|f_n(x) - f_m(x)| < \varepsilon \ \ \forall n, m \geq N$. Let $m \to \infty$, then $|f_n(x) - f(x)| \leq \lim_{m \to \infty} \varepsilon/2 < \varepsilon \ \ \forall n \geq N, x \in I. \Rightarrow f_n \to f$ uniformly on $x \in I$.
or equal to!

Theorem 1.8 (Weierstraß M-test).

Let $\sum_{n=1}^{\infty} g_n$ be a series of real-valued functions on $I \subseteq \mathbb{R}$. Suppose there is a sequence of numbers $\{M_n\}_{n=1}^{\infty} \in \mathbb{R}$ s.t.

- $|g_n(x)| \leq M_n \ \ \forall n \ \forall x \in I$, and
- $\sum_{n=1}^{\infty} M_n$ converges,

then $\sum_{n=1}^{\infty} g_n$ converges uniformly.

Note: This is similar to the comparison-test for series, except that $\forall x \in I$, g_n is being compared with a fixed constant M_n.

Proof $-$ $f_n(x) = \sum_{i=1}^{n} g_i(x)$. We want to prove that $\{f_n\}$ is a uniform Cauchy sequence (and thus by central Principle of Uniform converges, f_n converges uniformly).

Let $\varepsilon > 0$. $\sum_{n=1}^{\infty} M_n$ converges $\iff$ $S_n = \sum_{i=1}^{n} M_i$ converges $\iff$ $\{S_n\}$ is a Cauchy sequence $\iff$ $\exists N \in \mathbb{N}$, $\forall n, m \geq N$, $|S_n - S_m| < \frac{\varepsilon}{2}$.

We have, assuming $n > m$ WLOG, $|f_n(x) - f_m(x)| = \left|\sum_{i=1}^{n} g_i(x) - \sum_{i=1}^{m} g_i(x)\right| = \left|\sum_{i=m+1}^{n} g_i(x)\right| \leq \sum_{i=m+1}^{n} |g_i(x)| \leq \sum_{i=m+1}^{n} M_i = S_n - S_m < \frac{\varepsilon}{2} \ \ \forall x \in I.$

Then $\|f_n(x) - f_m(x)\|_{sup} \leq \frac{\varepsilon}{2} < \varepsilon \ \ \forall n, m \geq N \Rightarrow \{f_n\}_{n=1}^{\infty}$ is uniform Cauchy sequence $\Rightarrow$ by CPUC, $f_n(x)$ converges uniformly // q.e.d.

24 January 2013
Dr. N Sidorova
Roberts G06.

Theorem 1.9 If $\sum_{n=1}^{\infty} g_n$ converges uniformly, then $g_n \to 0$ uniformly.

Note: In particular, if $g_n \not\to 0$ uniformly, then $\sum_{n=1}^{\infty} g_n$ does not converge uniformly.

Proof $-$ $\sum_{n=1}^{\infty} g_n$ converges uniformly $\Rightarrow$ f_n converges uniformly, where $f_n = \sum_{i=1}^{n} g_i$. By CPUC, f_n is a uniform Cauchy sequence

$\Rightarrow \forall \varepsilon > 0 \ \exists N \ \forall n, m \geq N$, $\|f_m - f_n\|_{sup} < \varepsilon$. In particular, $\|f_{n+1} - f_n\|_{sup} < \varepsilon \Rightarrow \|g_{n+1}\|_{sup} < \varepsilon \Rightarrow g_{n+1} \to 0$ uniformly, $g_n \to 0$ uniformly // q.e.d.

Note: This theorem does not work in the opposite direction. For instance, $\sum_{n=1}^{\infty} \frac{x^n}{n}$ does not converge uniformly although $\frac{x^n}{n} \to 0$ uniformly.

Ex

① Does the series $\sum_{n=1}^{\infty} \frac{\sin(nx)}{2^n}$ converge uniformly?

Soln. $\frac{\sin(nx)}{2^n} \leq \frac{1}{2^n}$. $\sum_{n=1}^{\infty} \frac{1}{2^n} < \infty$, so by Weierstraß M-test, $\sum_{n=1}^{\infty} \frac{\sin(nx)}{2^n}$ converges uniformly //

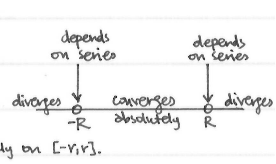
② Does $\sum_{n=1}^{\infty} (\sin x)^n$ on $x \in (0, \frac{\pi}{2})$ converge uniformly?

Soln. We check if $(\sin x)^n \to 0$. $\|\sin(x)^n\|_{sup} = 1 \not\to 0 \Rightarrow \sum_{n=1}^{\infty} (\sin x)^n$ does not converge uniformly //.

③ Does $\sum_{n=1}^{\infty} \frac{1}{n^2 + x}$ converge uniformly on $[0, \infty)$?

Soln. $\frac{1}{n^2 + x} \leq \frac{1}{n^2}$, and $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$, so by M-test, $\sum_{n=1}^{\infty} \frac{1}{n^2 + x}$ converges uniformly //.

Consider a power series $\sum_{n=0}^{\infty} a_n x^n$, with radius of convergence R. Then, series converges absolutely for $|x| < R$, diverges for $|x| > R$.



Recall that for $\sum_{n=0}^{\infty} x^n$, with $r = 1 - \varepsilon$, the series converges uniformly on $[-r, r]$ but does not converge uniformly on $(-1, 1)$.

Theorem 1.10 Let R be the radius of convergence of a power series $\sum_{n=1}^{\infty} a_n x^n$. Let $0 < r < R$. Then the series $\sum_{n=1}^{\infty} a_n x^n$ converges uniformly on $[-r, r]$.

Proof $-$ If $|x| \leq r$, then $|a_n x^n| \leq |a_n| r^n = |a_n r^n| \ \ \forall x \in [-r, r]$. Then $\sum_{n=1}^{\infty} |a_n r^n| < \infty$ since the series converges absolutely at the point r.

By the M-test, $\sum_{n=1}^{\infty} a_n x^n$ converges uniformly on $[-r, r]$ // q.e.d.

Recall that $\left.\begin{array}{l} f_n \to f \text{ uniformly} \\ \text{all } f_n \text{ are continuous} \end{array}\right\} \Rightarrow f$ is continuous. Same result holds for integrability.

Theorem 1.11 If all g_n are continuous on I, and $\sum_{n=1}^{\infty} g_n$ converges uniformly on I, then the function $\sum_{n=1}^{\infty} g_n$ is continuous on I.

Proof $-$ Denote by $f_n = g_1 + \cdots + g_n$, the n^{th} partial sum. All g_n are continuous $\Rightarrow$ f_n is continuous. f_n converges uniformly.

Then $f(x) = \lim_{n \to \infty} f_n(x)$ is continuous $\Rightarrow \sum_{n=1}^{\infty} g_n$ is continuous // q.e.d.

7102-05.

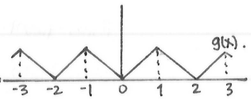
Theorem 1.12 If all g_n are Riemann integrable on $[a,b]$ and $\sum_{n=1}^{\infty} g_n$ converges uniformly on $[a,b]$, then $\sum_{n=1}^{\infty} g_n$ is Riemann integrable on $[a,b]$, and $\int_a^b \left(\sum_{n=1}^{\infty} g_n(x)\right) dx = \sum_{n=1}^{\infty} \int_a^b g_n(x)\, dx$.

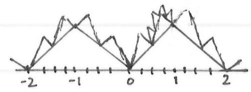
Proof – Denote $f_n = g_1 + \cdots + g_n$. Since all g_i are Riemann integrable, f_n is Riemann integrable. Since $\sum_{n=1}^{\infty} g_n$ converges uniformly, $\Rightarrow f_n$ converges uniformly to $f(x)$.

We claim that $\sum_{n=1}^{\infty} g_n$ is Riemann integrable. Then $\int_a^b \left(\sum_{n=1}^{\infty} g_n(x)\right) dx = \lim_{n\to\infty} \int_a^b f_n(x)\, dx = \lim_{n\to\infty} \int g_1 + \cdots + g_n \, dx = \lim_{n\to\infty} \sum_{i=1}^{n} \int_a^b g_i(x)\, dx = \sum_{i=1}^{\infty} \int_a^b g_i(x)\, dx$ // q.e.d.

Theorem 1.13 There is a continuous function on $\mathbb{R}$ which is nowhere differentiable.

Proof – We define $g(x) = |x|$ on $[-1,1]$ and extend it to $\mathbb{R}$ 2-periodically.





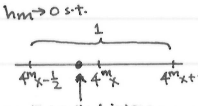
We can increase the jumps to "fill in the gaps".

Define $f(x) = \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n g(4^n x)$. The 4^n term scales the function in the x-direction, while the factor of $\left(\frac{3}{4}\right)^n$ ensures that terms of sequence converge to 0.

Claim 1: The series $\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n g(4^n x)$ converges uniformly on $\mathbb{R}$. $\left|\left(\frac{3}{4}\right)^n g(4^n x)\right| \le \left(\frac{3}{4}\right)^n \ \forall x\ \forall n$. Naturally $\sum \left(\frac{3}{4}\right)^n < \infty$, so series converges uniformly by M-test. $\Rightarrow f$ is well-defined.

Claim 2: f is a continuous function. Since each $\left(\frac{3}{4}\right)^n g(4^n x)$ is a product of continuous functions, it is continuous. Since series converges uniformly, f is continuous.

Claim 3: f is nowhere differentiable. Fix arbitrary $x \in \mathbb{R}$. $\frac{f(x+h) - f(x)}{h}$ is what we consider. We will construct a sequence $h_m \to 0$ s.t. $\left|\frac{f(x+h_m) - f(x)}{h_m}\right| \to \infty$. Choose $h_m = \begin{cases} \frac{1}{2} 4^{-m} & \text{if } \exists \text{ integer in } (4^m x - \frac{1}{2}, 4^m x) \text{ left half} \\ \frac{1}{2} 4^{m} & \text{if } \exists \text{ integer in } (4^m x, 4^m x + \frac{1}{2}] \text{ right half.} \end{cases}$ (either of them otherwise).



Then $\frac{f(x+h_m) - f(x)}{h_m} = \frac{\sum_n \left(\frac{3}{4}\right)^n g(4^n(x+h_m)) - \sum_n \left(\frac{3}{4}\right)^n g(4^n x)}{h_m} = \frac{1}{h_m} \sum_n \left(\frac{3}{4}\right)^n \left[g(4^n(x+h_m)) - g(4^n x)\right]$

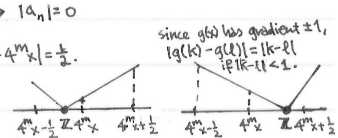
Fix m, then let $\left(\frac{3}{4}\right)^n \frac{g(4^n(x+h_m)) - g(4^n x)}{h_m} = a_n$ by definition.

Claim: $|a_n| \begin{cases} = 0 & n > m \quad (a) \\ = 3^n = 3^m & n = m \quad (b) \\ \le 3^n & n < m. \quad (c) \end{cases}$ For (a), $|g(4^n(x+h_m)) - g(4^n x)| = |g(4^n x \pm \frac{1}{2} 4^{n-m}) - g(4^n x)|$. Then $n - m \in \mathbb{N}$.

As such, $\frac{1}{2} 4^{n-m}$ is divisible by 2. By periodicity of g with period 2, $g(k+2) = g(k) \Rightarrow |a_n| = 0$

For (b), $|g(4^n(x+h_m)) - g(4^n x)| = |g(4^n x \pm \frac{1}{2}) - g(4^n x)| = |g(4^m x \pm \frac{1}{2}) - g(4^m x)| = |4^m x \pm \frac{1}{2} - 4^m x| = \frac{1}{2}$.

since $g(x)$ has gradient ± 1, $|g(k) - g(l)| = |k - l|$ if $|k - l| < 1$.



(or flipped vertically).

Then $|a_n| = \left(\frac{3}{4}\right)^m \cdot \frac{\frac{1}{2}}{\frac{1}{2} 4^{-m}} = 3^m$. We finally evaluate for (c):

For (c), $|g(4^n(x+h_m)) - g(4^n x)| = |g(4^n x \pm \frac{1}{2} 4^{n-m}) - g(4^n x)| \le |(4^n x \pm \frac{1}{2} 4^{n-m}) - (4^n x)| = \frac{1}{2} 4^{n-m}$.

This is a valid inequality: we want to explain why $|g(a) - g(b)| \le |a - b|$. By inspection, this is true if a, b are in same interval, but even more so if not.

$\therefore |a_n| \le \left(\frac{3}{4}\right)^n \frac{\frac{1}{2} 4^{n-m}}{\frac{1}{2} 4^{-m}} = \left(\frac{3}{4}\right)^n 4^n = 3^n$, and the inequality holds.

Having proven our subsidiary claim, we see that $\left|\frac{g(x+h_m) - g(x)}{h_m}\right| = \left|\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n \frac{g(4^n x + h_m) - g(4^n x)}{h_m}\right|$.

28 January 2013.
Dr. Nadia SIDOROVA
9LT.

Then $\left|\frac{f(x+h_m) - f(x)}{h_m}\right| = |a_1 + a_2 + \cdots + a_{m-1} + a_m| \ge |a_m| - |a_1 + \cdots + a_{m-1}|$ ($\because |a| = |a+b-b| \le |a+b| + |b|$).

and $|a_m| - |a_1 + \cdots + a_{m-1}| \ge |a_m| - |a_1| - |a_2| \cdots - |a_{m-1}| = 3^m - (|a_1| + |a_2| + \cdots + |a_{m-1}|) \le 3^m - (3^1 + 3^2 + \cdots + 3^{m-1}) = 3^m - \frac{3(3^{m-1} - 1)}{3-1} = \frac{3^m}{2} + \frac{3}{2} \to \infty$.

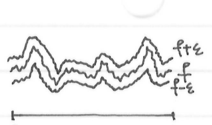
Hence, f is nowhere differentiable as $\left|\frac{f(x+h_m) - f(x)}{h_m}\right| \to \infty$, f is not differentiable at arbitrary $x \in \mathbb{R}$ // q.e.d.

Note: this theorem was first proved by Weierstraß, who instead used a cosine function rather than $|x|$, i.e. he chose $f(x) = \sum_{n=1}^{\infty} a^n \cos(b^n x)$ rather than $f(x) = \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n g(4^n x)$.

Approximation of Continuous Functions by Polynomials

defined on the closed interval $[a,b]$.

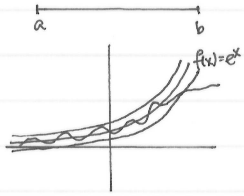
Consider a continuous function f. Take $\varepsilon > 0$, and construct the epsilon tube from $f - \varepsilon$ to $f + \varepsilon$.



Can we always find a polynomial lying within the ε-tube? Yes, but only on $[a,b]$.

For example, ① take $f(x) = \frac{1}{x}$ on $(0,1)$. Then we see that $f(x)$ is unbounded, and near $x = 0$, the ε-tube also goes to infinity.

However, polynomials cannot converge to infinity near $x = 0 \Rightarrow$ not true on $(0,1)$



② Not true on $\mathbb{R}$: take $f(x) = e^x$. As $x \to \infty$, polynomial growth cannot match exponential growth.

Definition We define $P_{nk}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, $n = 1, 2, 3, \ldots$, $0 \le k \le n$. These polynomials correspond to the binomial distribution $B(n, x)$.

Let us consider $Y_{n,x}$ to be a binomial random variable. Then $Y_{n,x} \approx nx$ for large n. Then $\frac{Y_{n,x}}{n} \approx x$, $f\left(\frac{Y_{n,x}}{n}\right) \approx f(x)$.

To get rid of the randomness, we use the mean/expectation value: $f(x) \approx E\left[f\left(\frac{Y_{n,x}}{n}\right)\right] = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) P(Y_{nk} = k) = \sum_{k=0}^{n} \underbrace{f\left(\frac{k}{n}\right) \cdot \binom{n}{k}}_{\text{numbers}} \underbrace{x^k (1-x)^{n-k}}_{\text{polynomials}}$.

We claim that this polynomial, $\boxed{B_n^f(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}}$, the Bernstein polynomial, gives an approximation to f.

Theorem 1.14 (Weierstraß's Approximation Theorem on $[0,1]$).

Let $f: [0,1] \to \mathbb{R}$ be a continuous function. Then $B_n^f \to f$ uniformly on $[0,1]$.

Lemma Let $x \in [0,1]$, then

(a) $\sum_{k=0}^{n} P_{nk}(x) = 1$, (b) $\sum_{k=0}^{n} k P_{nk}(x) = nx$, (expectation) (c) $\sum_{k=0}^{n} (k - nx)^2 P_{nk}(x) = nx(1-x)$. (variation)

Proof – (a)

(c) $k(k-1)\binom{n}{k} = k(k-1) \cdot \frac{n!}{k!(n-k)!} = \frac{(n-2)!}{(k-2)!(n-k)!} n \cdot (n-1) = \frac{(n-2)!}{(k-2)!((n-2)-(k-2))!} n \cdot (n-1) = \binom{n-2}{k-2} n(n-1)$. Then we have, by definition,

$\sum_{k=0}^{n} k(k-1) P_{nk}(x) = \sum_{k=0}^{n} k(k-1)\binom{n}{k} x^k (1-x)^{n-k} = \sum_{k=2}^{n} \binom{n-2}{k-2} n(n-1) x^k (1-x)^{n-k} = n(n-1)x^2 \sum_{k=2}^{n} \binom{n-2}{k-2} x^{k-2} (1-x)^{(n-2)-(k-2)} = n(n-1)x^2 \sum_{k=0}^{n-2} \binom{n-2}{k} x^k (1-x)^{(n-2)-k}$

Hence, since $\sum_{k=0}^{n-2}\binom{n-2}{k}x^k(1-x)^{(n-2)-k}=1$ from part (a), $\sum_{k=0}^{n}k(k-1)P_{nk}(x)=n(n-1)x^2$. Then we have

$\sum_{k=0}^{n}(k-nx)^2P_{nk}(x)=\sum_{k=0}^{n}(k^2-2knx+n^2x^2)P_{nk}(x)=\sum_{k=0}^{n}[k(k-1)+k-2knx+n^2x^2]P_{nk}(x)=\sum k(k-1)P_{nk}(x)+\sum kP_{nk}(x)-2nx\sum kP_{nk}(x)+n^2x^2\sum P_{nk}(x)$.

$=n(n-1)x^2+nx-2n^2x^2+n^2x^2=nx(1-x)$ // q.e.d.

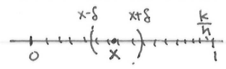
With that, we will prove Theorem 1.14.

31 January 2013.
Dr. Nadia SIDOROVA.
Roberts G06.

Proof — NTP: $\forall \varepsilon>0,\ \exists N\in\mathbb{N}$ s.t. $\forall n\geq N$, $|B_n^f(x)-f(x)|<\varepsilon$. Let $\varepsilon>0$. f is continuous on $[0,1]\Rightarrow f$ is uniformly continuous on $[0,1]\Rightarrow \exists\delta>0$ s.t. if $x,y\in[0,1]$ and $|x-y|<\delta$, then $|f(x)-f(y)|<\frac{\varepsilon}{2}$. Then

$|B_n^f(x)-f(x)|=\left|\sum_{k=0}^{n}f(\tfrac{k}{n})P_{nk}(x)-f(x)\underbrace{\sum_{k=0}^{n}P_{nk}(x)}_{1,\text{ by lemma}}\right|=\left|\sum_{k=0}^{n}(f(\tfrac{k}{n})-f(x))P_{nk}(x)\right|\overset{\Delta}{\leq}\sum_{k=0}^{n}|f(\tfrac{k}{n})-f(x)|P_{nk}(x)$

We can split this into two sums and evaluate them separately: $\sum_{k:|\frac{k}{n}-x|<\delta}|f(\tfrac{k}{n})-f(x)|P_{nk}(x)+\sum_{k:|\frac{k}{n}-x|\geq\delta}|f(\tfrac{k}{n})-f(x)|P_{nk}(x)$



For the first term, $|\frac{k}{n}-x|<\delta\Rightarrow|f(\frac{k}{n})-f(x)|<\frac{\varepsilon}{2}$ by uniform continuity. For the second sum,

$|f(\frac{k}{n})-f(x)|\leq|f(\frac{k}{n})|+|f(x)|\leq 2\|f\|_{sup}$. Hence, $|B_n^f(x)-f(x)|<\frac{\varepsilon}{2}\sum_{k:|\frac{k}{n}-x|<\delta}P_{nk}(x)+2\|f\|_{sup}\sum_{k:|\frac{k}{n}-x|\geq\delta}P_{nk}(x)$.

Surely, $\sum_{k:|\frac{k}{n}-x|<\delta}P_{nk}(x)\leq\sum_{k=0}^{n}P_{nk}(x)=1$, so $\frac{\varepsilon}{2}\sum_{k:|\frac{k}{n}-x|<\delta}P_{nk}(x)\leq\frac{\varepsilon}{2}$.

We see also that $|\frac{k}{n}-x|\geq\delta\Rightarrow\frac{(k-nx)^2}{n^2}\geq\delta^2\Rightarrow\frac{(k-nx)^2}{n^2\delta^2}\geq1$. Then $|B_n^f(x)-f(x)|<\frac{\varepsilon}{2}+2\|f\|_{sup}\sum_{k:|\frac{k}{n}-x|\geq\delta}1\cdot P_{nk}(x)$

i.e. $|B_n^f(x)-f(x)|<\frac{\varepsilon}{2}+2\|f\|_{sup}\frac{1}{n^2\delta^2}\sum_{k=0}^{n}(k-nx)^2P_{nk}(x)=\frac{\varepsilon}{2}+2\|f\|_{sup}\cdot\frac{1}{n^2\delta^2}\underbrace{nx(1-x)}_{\leq1}\leq\frac{\varepsilon}{2}+\frac{2\|f\|_{sup}}{n\delta^2}$.

Choose N s.t. $\frac{2\|f\|_{sup}}{N\delta^2}<\frac{\varepsilon}{2}$ (clearly possible ∵ f is bounded). Then for any $n\geq N$, $|B_n^f(x)-f(x)|<\frac{\varepsilon}{2}+\frac{2\|f\|_{sup}}{n\delta^2}<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$ // q.e.d.

We can expand this theorem to accomodate the general case for an arbitrary interval.

Theorem 1.15 (Weierstraß's Approximation Theorem on $[a,b]$)

Let $f:[a,b]\to\mathbb{R}$ be a continuous function. Then $\exists$ a sequence of polynomials $\{P_n\}_{n=1}^{\infty}$ s.t. $P_n\to f$ uniformly on $[a,b]$.

Proof — Use the above theorem as a base. We find a mapping g s.t. $[0,1]\xrightarrow{h}[a,b]\xrightarrow{f}\mathbb{R}$. Let h be the linear function $h(t)=a+(b-a)t$.

Then we define $g:[0,1]\to\mathbb{R}$, $g(t)=f(h(t))=f(x(t))$, which is continuous on $[0,1]$ as a composition of continuous functions.

Define $P_n(x)=B_n^g\,h^{-1}(x)=B_n^g(\frac{x-a}{b-a})$. We claim that $P_n:[a,b]\to\mathbb{R}$ converges to f uniformly.

$$\begin{tikzcd} {[0,1]} \arrow[r, shift left, "h"] \arrow[rr, bend right, "g \text{ or } B_n^g"'] & {[a,b]} \arrow[l, shift left, "h^{-1}"] \arrow[r, shift left, "f"] \arrow[r, shift right, "?"'] & \mathbb{R} \end{tikzcd}$$

$\|P_n-f\|_{sup}=\sup_{x\in[a,b]}|P_n(x)-f(x)|=\sup_{x\in[a,b]}|B_n^g(\frac{x-a}{b-a})-f(x)|=\sup_{t\in[0,1]}|B_n^g(t)-g(t)|=\|B_n^g-g\|_{sup}\to0$.

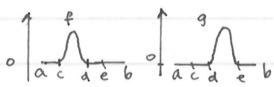
Chapter 2
FOURIER SERIES.

Consider the space $R[a,b]$, which denotes $R[a,b]=\{f:[a,b]\to\mathbb{R}$ s.t. f is Riemann integrable$\}$.

We also write $\langle f,g\rangle=\int_a^b f(x)g(x)\,dx$, the inner product of $f,g\in R[a,b]$. Then $\langle\cdot,\cdot\rangle$ satisfies all properties of an inner product from MATH2201, except positivity. $\exists f\not\equiv0$ but $\langle f,f\rangle=0$, since f need not be continuous e.g. $f(x)=\begin{cases}0, & x\in\mathbb{R}\setminus\{0\}\\ 1 & x=0\end{cases}$.

Definition Two functions $f,g\in R[a,b]$ are orthogonal if $\langle f,g\rangle\equiv\int_a^b f(x)g(x)\,dx=0$.

e.g. see the graphs of the two functions on right, which are orthogonal.



f,g are orthogonal.

Definition A sequence of Riemann-integrable functions $\{\varphi_n\}_{n=1}^{\infty}$ is an orthogonal system if $\langle\varphi_n,\varphi_m\rangle=0\ \forall n\neq m$.

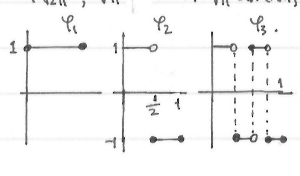
It is an orthonormal system if additionally, $\langle\varphi_n,\varphi_n\rangle=1\ \forall n$.

We call, for $f\in R[a,b]$, $\|f\|_2=\sqrt{\langle f,f\rangle}\equiv\sqrt{\int_a^b f(x)^2dx}$ the two-norm of f

Examples of systems:

① Trigonometric orthogonal system on $[-\pi,\pi]$: $\{1,\cos(nx),\sin(nx);\ n\in\mathbb{N}\}$. orthonormal system is $\{\frac{1}{\sqrt{2\pi}},\frac{1}{\sqrt{\pi}}\cos(nx),\frac{1}{\sqrt{\pi}}\sin(nx)\}$.

② On $[0,1]$. The sequence of functions as shown on the right.



Then $\{\varphi_n\}$ is an orthogonal system: $\langle\varphi_n,\varphi_n\rangle=\int_0^1\varphi_n^2(x)\,dx=\int_0^1 1\,dx=1$.

$\langle\varphi_m,\varphi_n\rangle=0$ if $n\neq m$: we try $\langle\varphi_2,\varphi_3\rangle=\int_0^1\varphi_2(x)\varphi_3(x)\,dx=0$ by drawing a picture of $\varphi_2(x)\varphi_3(x)$.

Definition Let $f\in R[a,b]$ and let $\{\varphi_n\}_{n=1}^{\infty}$ be an orthonormal system on $[a,b]$. Let $a_n=\langle f,\varphi_n\rangle\equiv\int_a^b f(x)\varphi_n(x)\,dx$ be the n^{th} Fourier coefficient of f w.r.t. $\{\varphi_n\}$.

Then $\sum_{n=1}^{\infty}a_n\varphi_n$ is the Fourier series of f w.r.t. $\{\varphi_n\}$.

Remarks: (1) $\sum_{n=1}^{\infty}a_n\varphi_n$ does not have to converge!

(2) Even if it converges, it is not always equal to f! (We cannot assume that this is a good representation for f).

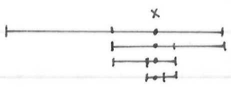
e.g. Take $[-\pi,\pi]$, $\{\frac{1}{\sqrt{\pi}}\cos(nx)\}_{n\in\mathbb{N}}$ be the orthonormal system. Take $f(x)=x$, $a_n=\langle f,\varphi_n\rangle=\frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi}\underset{\text{odd}}{x}\,\underset{\text{even}}{\cos(nx)}\,dx=0$.

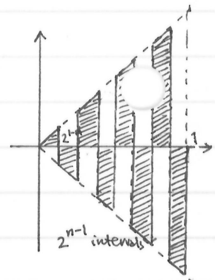
Fourier series is $\sum_{n=1}^{\infty}0\cdot\varphi_n=0\not\equiv x$.

e.g. Let $\{\varphi_n\}$ be as above. $f(x)=x$ on $[0,1]$. Find its Fourier series: $a_n=\int_0^1 x\,\varphi_n(x)\,dx$ are coefficients.

$a_n=\int_0^1 x\,\varphi_n(x)\,dx=-(2^{1-n})^2\cdot 2^{n-2}=-2^{2-2n+n-2}=-2^{n}$. FS: $-\sum 2^{-n}\varphi_n(x)$ converges. But then,

does the Fourier series converge to x? think of the binary representation of x:





The binary representation of $x=0.01\ldots$

if 0, $\varphi_2=1$ / if 1, $\varphi_2=-1$. if 0, $\varphi_3=1$ / if 1, $\varphi_3=-1$ ... etc.

$\boxed{\text{Theorem}}$ 2.1 (Least squares approximation).

Let $f\in\mathcal{R}[a,b]$ and let $\{\varphi_n\}$ be an orthonormal system on $[a,b]$. Denote by a_n the n^{th} Fourier coefficient of f w.r.t. $\{\varphi_n\}$.

Then $\|f-\sum_{i=1}^n a_i\varphi_i\|_2\le\|\sum_{i=1}^n c_i\varphi_i\|_2$ for any n and any sequence $\{c_i\}_{i=1}^n$. The equality holds $\Leftrightarrow$ $a_i=c_i$ $\forall\, i=1,2,\ldots,n$.

Proof – Find $\|f-\sum_{i=1}^n a_i\varphi_i\|_2^2=\langle f-\sum_{i=1}^n a_i\varphi_i,\ f-\sum_{j=1}^n a_j\varphi_j\rangle=\langle f,f\rangle-\sum_{j=1}^n a_j\underbrace{\langle f,\varphi_j\rangle}_{a_j}-\sum_{i=1}^n a_i\underbrace{\langle f,\varphi_i\rangle}_{a_i}+\sum_{i=1}^n\sum_{j=1}^n a_ia_j\underbrace{\langle\varphi_i,\varphi_j\rangle}_{\delta_{ij}}$

$\therefore\ \|f-\sum_{i=1}^n a_i\varphi_i\|_2^2=\langle f,f\rangle-\sum_{j=1}^n a_j^2-\sum_{i=1}^n a_i^2+\sum_{k=1}^n a_k^2=\langle f,f\rangle-\sum_{i=1}^n a_i^2.=\|f\|_2^2-\sum_{i=1}^n a_i^2$

$\|f-\sum_{i=1}^n c_i\varphi_i\|_2^2=\langle f-\sum_{i=1}^n c_i\varphi_i,\ f-\sum_{j=1}^n c_j\varphi_j\rangle=\langle f,f\rangle-\sum_{j=1}^n c_j\langle f,\varphi_j\rangle-\sum_{i=1}^n c_i\langle f,\varphi_i\rangle+\sum_{i=1}^n\sum_{j=1}^n c_ic_j\overbrace{\langle\varphi_i,\varphi_j\rangle}^{\delta_{ij}}$

$\therefore\ \|f-\sum_{i=1}^n c_i\varphi_i\|_2^2=\langle f,f\rangle-\sum_{j=1}^n a_jc_j-\sum_{i=1}^n a_ic_i+\sum_{i=1}^n c_i^2=\langle f,f\rangle-\sum_{i=1}^n a_i^2+\sum_{i=1}^n a_i^2-2\sum_{i=1}^n a_ic_i+\sum_{i=1}^n c_i^2$

$=\langle f,f\rangle-\sum_{i=1}^n a_i^2+\sum_{i=1}^n\left[a_i^2-2a_ic_i+c_i^2\right]=\|f-\sum_{i=1}^n a_i\varphi_i\|_2^2+\sum_{i=1}^n(a_i-c_i)^2\ge\|f-\sum_{i=1}^n a_i\varphi_i\|_2^2,$

with equality $\Leftrightarrow$ $a_i=c_i$ $\forall i$. Take square roots to get $\|f-\sum_{i=1}^n a_i\varphi_i\|_2\le\|f-\sum_{i=1}^n c_i\varphi_i\|_2$.

4 February 2013
Dr. Nadia SIDOROVA
UCL.

$\boxed{\text{Theorem}}$ 2.2 (Bessel's inequality)

$\sum_{n=1}^\infty a_n^2\le\|f\|_2^2$. In particular, $a_n\to 0$.

Proof – $\|f-\sum_{i=1}^n a_i\varphi_i\|_2^2=\|f\|_2^2-\sum_{i=1}^n a_i^2$ (see the proof of theorem 2.1). Thus $\sum_{i=1}^n a_i^2\le\|f\|_2^2$. $\Rightarrow$ $\sum_{i=1}^\infty a_i^2$ converges and $\sum_{i=1}^\infty a_i^2\le\|f\|_2^2$.

$\sum_{n=1}^\infty a_n^2<\infty\ \Rightarrow\ a_n^2\to 0\ \Leftrightarrow\ a_n\to 0.$

<u>Trigonometric Fourier series.</u>

$[-\pi,\pi]$, $\left\{\frac{1}{\sqrt{2\pi}},\ \frac{1}{\sqrt{\pi}}\cos(nx),\ \frac{1}{\sqrt{\pi}}\sin(nx)\right\}_{n\in\mathbb{N}}$. $a_n=\int_{-\pi}^{\pi}f(x)\frac{1}{\sqrt{\pi}}\cos(nx)\,dx$, $b_n=\int_{-\pi}^{\pi}f(x)\frac{1}{\sqrt{\pi}}\sin(nx)\,dx$, $a_0=\int_{-\pi}^{\pi}f(x)\frac{1}{\sqrt{2\pi}}\,dx$.

Fourier series is $\frac{a_0}{\sqrt{2\pi}}+\sum_{n=1}^\infty\frac{a_n}{\sqrt{\pi}}\cos(nx)+\sum_{n=1}^\infty\frac{b_n}{\sqrt{\pi}}\sin(nx)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)\,dt+\sum_{n=1}^\infty\frac{1}{\pi}\left(\int_{-\pi}^{\pi}f(t)\cos(nt)\,dt\right)\cos(nx)+\sum_{n=1}^\infty\frac{1}{\pi}\left(\int_{-\pi}^{\pi}f(t)\sin(nt)\,dt\right)\sin(nx)$.

Orthogonal system: $\{1,\cos(nx),\sin(nx)\}$. Take $a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos(nx)\,dx$, $b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin(nx)\,dx$. Then $\frac{a_0}{2}+\sum_{n=1}^\infty a_n\cos(nx)+b_n\sin(nx)$

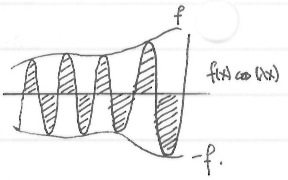
is the trigonometric Fourier series of f.

$\boxed{\text{Ex}}$ Find the trigonometric Fourier series of $f(x)=x$ on $[-\pi,\pi]$.

Soln. $a_n=0$, $b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}x\sin(nx)\,dx=\cdots=\frac{2(-1)^{n+1}}{n}$. Then Fourier series is $f(x)=\sum_{n=1}^\infty\frac{2}{n}(-1)^{n+1}\sin(nx)$.

$\boxed{\text{Theorem}}$ 2.3 (Riemann's Lemma)

Let $f\in\mathcal{R}[a,b]$. $\int_a^b f(x)\cos(\lambda x)\,dx\to 0$ as $\lambda\to\infty$. Likewise, $\int_a^b f(x)\sin(\lambda x)\,dx\to 0$.



Proof – (i) Let f be a simple step function, that is, $\exists$ partition $P=\{a=t_0<t_1<\cdots<t_{n-1}<t_n=b\}$ s.t. $f(x)=c_i$ if $x\in(t_{i-1},t_i)$.

Then $\left|\int_a^b f(x)\cos(\lambda x)\,dx\right|\le\sum_{i=1}^n\left|\int_{t_{i-1}}^{t_i}f(x)\cos(\lambda x)\,dx\right|=\sum_{i=1}^n\left|c_i\frac{\sin(\lambda x)}{\lambda}\Big|_{t_{i-1}}^{t_i}\right|=\sum_{i=1}^n c_i\frac{\sin(\lambda t_i)-\sin(\lambda t_{i-1})}{\lambda}$

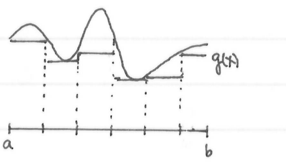
$\le\frac{1}{\lambda}\underbrace{\sum_{i=1}^n|c_i|\cdot 2}_{\text{const. number}}\longrightarrow 0$ as $\lambda\to\infty$. // q.e.d.

(ii) Then we generalise, let $f\in\mathcal{R}[a,b]\Rightarrow\forall\varepsilon>0$, $\exists\,\overset{\{a=t_0<t_1<\cdots<t_n=b\}}{P}\in\mathbb{P}[a,b]$ s.t. $U(f,P)-L(f,P)<\frac{\varepsilon}{2}$.

7 February 2013.
Dr. Nadia SIDOROVA.
Roberts G06.

Choose $g(x)=\inf_{t\in[t_{i-1},t_i]}f(t)$ $x\in(t_{i-1},t_i)$. Then by (i), $\int_a^b g(x)\cos(\lambda x)\,dx\to 0$. $\Rightarrow$ $\exists\,\lambda_0$, $\forall\lambda\ge\lambda_0$,

$\left|\int_a^b g(x)\cos(\lambda x)\,dx\right|<\frac{\varepsilon}{2}$. Then $\left|\int_a^b f(x)\cos(\lambda x)\,dx\right|\le\left|\int_a^b g(x)\cos(\lambda x)\,dx\right|+\left|\int_a^b(f(x)-g(x))\cos(\lambda x)\,dx\right|$.



$\Rightarrow\ \le\int_a^b|f(x)-g(x)|\,dx+\frac{\varepsilon}{2}.=\sum_{i=1}^n\int_{t_{i-1}}^{t_i}(f(x)-g(x))\,dx+\frac{\varepsilon}{2}.=\sum_{i=1}^n\int_{t_{i-1}}^{t_i}\left(f(x)-\inf_{t\in[t_i,t_{i-1}]}f(t)\right).dx+\frac{\varepsilon}{2}.$

Then on $[t_i,t_{i-1}]$, $f(x)\le\sup_{t\in[t_i,t_{i-1}]}f(t)$, so $\left|\int_a^b f(x)\cos(\lambda x)\,dx\right|\le\frac{\varepsilon}{2}+\sum_{i=1}^n\left(\sup_{t\in[t_{i-1},t_i]}f(t)-\inf_{t\in[t_{i-1},t_i]}f(t)\right)$

$=U(f,P)-L(f,P)+\frac{\varepsilon}{2}<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$. Then $\int_a^b f(x)\cos(\lambda x)\,dx\to 0$ // q.e.d.

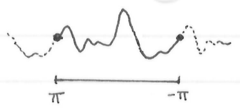
Statement about $\sin(\lambda x)$ follows in an analogous approach. // q.e.d.

$\boxed{\text{Definition}}$ We denote $S_n^f(x)$ by $S_n^f(x)=\frac{a_0}{2}+\sum_{k=1}^n a_k\cos(kx)+b_k\sin(kx)$. Then $S_n^f(x)$ is the <u>n^{th} partial sum of the trigonometric Fourier series</u> of f.

$\boxed{\text{Theorem}}$ 2.4 (Dirichlet's Theorem).

Let $f\in\mathcal{R}[-\pi,\pi]$, assume $f(-\pi)=f(\pi)$. Extend f to $\mathbb{R}$ 2π-periodically and denote the extended function by same letter.

Then $S_N^f(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)D_n(x-t)\,dt.\equiv\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x-t)D_n(t)\,dt$; where $D_n(t)=\begin{cases}\frac{\sin((n+\frac{1}{2})t)}{\sin(t/2)} & t\ne 2\pi m\\ 1+2n & t=2\pi m.\end{cases}$



and is known as the <u>Dirichlet kernel</u>.

Proof – $S_n^f(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k\cos(kx) + b_k\sin(kx) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt + \sum_{k=1}^{n}\left[\frac{1}{\pi}\left(\int_{-\pi}^{\pi} f(t)\cos(kt)\,dt\right)\cos(kx) + \frac{1}{\pi}\left(\int_{-\pi}^{\pi} f(t)\sin(kt)\,dt\right)\sin(kx)\right]$

$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\left[1 + 2\sum_{k=1}^{n}(\cos(kx)\cos(kt) + \sin(kx)\sin(kt))\right]dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\left[1+2\sum_{k=1}^{n}\cos(k(x-t))\right]dt.$

We look specifically at $\left(1+2\sum_{k=1}^{n}\cos(k\theta)\right)\sin\frac{\theta}{2} = \sin\frac{\theta}{2} + \sum_{k=1}^{n}\left(\sin\left[k\theta+\frac{\theta}{2}\right] - \sin\left[k\theta-\frac{\theta}{2}\right]\right) = \sin\frac{\theta}{2} + \sin\frac{3\theta}{2} - \sin\frac{\theta}{2} + \sin\frac{5\theta}{2} - \sin\frac{3\theta}{2} + \dots$

All terms cancel except $\sin(n\theta+\frac{\theta}{2})$. $\Rightarrow$ $1+2\sum_{k=1}^{n}\cos(k\theta) = \frac{\sin((n+\frac{1}{2})\theta)}{\sin(\theta/2)}$ for $\theta \neq 2\pi m$; $\theta = 1+2n$ otherwise.

Then $1+2\sum_{k=1}^{n}\cos(k\theta) = D_n(\theta)$, so $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\left[1+2\sum_{k=1}^{n}\cos(k(x-t))\right]dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)D_n(x-t)\,dt$ // q.e.d.

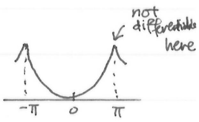
To get second formula, perform change of variables: $s = x-t$, then $S_n^f(x) = \frac{1}{2\pi}\int_{x+\pi}^{x-\pi} f(x-s)D_n(s)\,ds\ (-1) = \frac{1}{2\pi}\int_{x-\pi}^{x+\pi} f(x-s)D_n(s)\,ds.$

Since function f is 2π-periodic, $S_n^f(x) = \frac{1}{2\pi}\int_{x-\pi}^{x+\pi} f(x-s)D_n(s)\,ds = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)D_n(t)\,dt$ // q.e.d.

$\boxed{\text{Theorem}}$ 2.5 (Convergence of Trigonometric Fourier series: I)

Let $f \in \mathcal{R}[-\pi,\pi]$ s.t. $f(-\pi) = f(\pi)$, and extend it 2π-periodically to $\mathbb{R}$. Consider $x \in [-\pi,\pi]$. If f is differentiable at x, then $S_n^f(x) \to f(x)$ as $n \to \infty$.

Examples: ① $f(x) = x^2$. We don't know if Fourier series converges at $-\pi, \pi$.

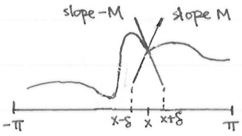


② $f(x) = |x|$. We do not know if Fourier series converges at 0.

$\boxed{\text{Theorem}}$ 2.6 (Convergence of Trigonometric Fourier series: II).

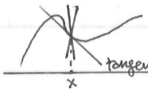
Let $f \in \mathcal{R}[-\pi,\pi]$ s.t. $f(-\pi) = f(\pi)$, and extend it 2π-periodically to $\mathbb{R}$. Let $x \in [-\pi,\pi]$.

Suppose $\exists M > 0$ and $\delta > 0$ s.t. $|f(x+t) - f(x)| \leq M|t|$ $\forall t \in (-\delta,\delta)$. Then $S_n^f(x) \to f(x)$ as $n \to \infty$.



Function does not go into the cone from $x-\delta$ to $x+\delta$.

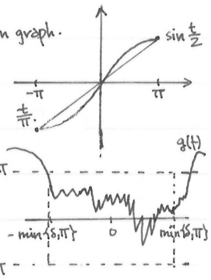
Example: If f is differentiable at x, then we note that $S_n^f(x) \to f(x)$ because it is just a restricted case of theorem 2.6.



Proof – Note that $\frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} 1 + 2\sum_{k=1}^{n}\cos(kt)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} dt = \frac{1}{2\pi}(2\pi) = 1$. Then we have:

$|S_n^f(x) - f(x)| = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)D_n(t)\,dt - f(x)\cdot\underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(t)\,dt}_{1}\right| = \frac{1}{2\pi}\left|\int_{-\pi}^{\pi}(f(x-t)-f(x))\cdot D_n(t)\,dt\right| = \frac{1}{2\pi}\left|\int_{-\pi}^{\pi}\frac{f(x-t)-f(x)}{\sin(t/2)}\sin((n+\frac{1}{2})t)\,dt\right|$

We can ignore the finitely many points where $t = 2\pi m$. We denote $g(t) = \frac{f(x-t)-f(x)}{\sin(t/2)}$ for fixed x. We cannot use Riemann's lemma since g may be non-integrable around 0. We examine $g(t)$: $|f(x-t) - f(x)| \leq M|t|$ if $t \in (-\delta,\delta)$. $|\sin\frac{t}{2}| \geq |\frac{t}{\pi}|$ from graph.

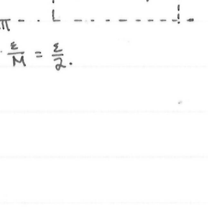


Then $|g(t)| = \frac{|f(x-t)-f(x)|}{|\sin(t/2)|} \leq \frac{M|t|}{|t/\pi|} = M\pi$ if $|t| < \min\{\delta,\pi\}$.

Let $\varepsilon > 0$. Choose $g_\varepsilon(t) = \begin{cases} g(t) & \text{if } |t| > \frac{\varepsilon}{2M} \\ 0 & \text{otherwise.}\end{cases}$ Then we have, from earlier,

$|S_n^f(x) - f(x)| = \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} g(t)\sin((n+\frac{1}{2})t)\,dt\right| \leq \frac{1}{2\pi}\left|\int_{-\pi}^{\pi}\underbrace{(g(t)-g_\varepsilon(t))}_{\substack{\text{rarely } \neq 0; \\ \text{in fact never, except on } [-\frac{\varepsilon}{2M},\frac{\varepsilon}{2M}].}}\sin(n+\frac{1}{2})t\,dt\right| + \frac{1}{2\pi}\left|\int_{-\pi}^{\pi}\underbrace{g_\varepsilon(t)}_{\substack{\text{Riemann integrable,} \\ \text{can evaluate by Riemann's lemma}}}\sin(n+\frac{1}{2})t\,dt\right|$

By Riemann's lemma, we have that:



Since g_ε is Riemann integrable, $\left|\int_{-\pi}^{\pi} g_\varepsilon(t)\sin(n+\frac{1}{2})t\,dt\right| \to 0$ i.e. $\exists N$ s.t. $\forall n \geq N$, $\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} g_\varepsilon(t)\sin(n+\frac{1}{2})t\,dt\right| < \frac{\varepsilon}{2}$.

Consider $\frac{1}{2\pi}\left|\int_{-\pi}^{\pi}(g(t)-g_\varepsilon(t))\sin(n+\frac{1}{2})t\,dt\right| \leq \frac{1}{2\pi}\int_{-\varepsilon/2M}^{\varepsilon/2M}|g(t)-0|\cdot 1\,dt \leq \frac{1}{2\pi}\int_{-\varepsilon/2M}^{\varepsilon/2M} g(t)\,dt \leq \frac{1}{2\pi}\int_{-\varepsilon/2M}^{\varepsilon/2M} M\pi\,dt = \frac{M}{2}\cdot\frac{\varepsilon}{M} = \frac{\varepsilon}{2}.$

Hence overall, $|S_n^f(x) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ $\forall n > N$ $\Rightarrow$ $S_n^f(x) \to f(x)$ as $n \to \infty$ // q.e.d.

Remark: this allows us to prove theorem 2.5 based on our groundwork done.

(Theorem 2.5) Proof – We want to show that f is differentiable at $x \Rightarrow \exists M, \delta$ s.t. $|f(x+t) - f(x)| \leq M|t|$ $\forall t \in (-\delta,\delta)$.

f differentiable at $x \Rightarrow \lim_{t\to 0}\frac{f(x+t)-f(x)}{t}$ exists. Take $\varepsilon = 1 \Rightarrow \exists\delta > 0$ s.t. $\left|\frac{f(x+t)-f(x)}{t} - f'(x)\right| < 1$ whenever $|t| < \delta$.

$\Rightarrow f'(x) - 1 \leq \frac{f(x+t)-f(x)}{t} \leq f'(x)+1$ for $|t| < \delta$. $f'(x)+1$ is a number, so $\exists M$ s.t. $\left|\frac{f(x+t)-f(x)}{t}\right| \leq M$ for $|t| < \delta$.

By Theorem 2.6, we get $S_n^f(x) \to f(x)$ for $n \to \infty$ // q.e.d.

However, these theorems clearly do not apply to all functions – there are examples, such as $f(x) = \sqrt{|x|}$. We cannot apply theorems 2.5 and 2.6 at $x = 0$.

Take note that this does not necessarily imply that Fourier series at $x = 0$ does not converge!

18 February 2013
Dr. Nadia SIDOROVA
ULT.

$\boxed{\text{Theorem}}$ 2.7 Let $f \in \mathcal{R}[-\pi,\pi]$ and $f(-\pi) = f(\pi)$. Suppose f'' exists and $f'' \in \mathcal{R}[-\pi,\pi]$. Then $S_n^f \to f$ uniformly on $[-\pi,\pi]$.

Proof – $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \left[\frac{1}{\pi}f(x)\frac{\sin(nx)}{n}\right]_{-\pi}^{\pi} - \frac{1}{n\pi}\int_{-\pi}^{\pi} f'(x)\sin(nx)\,dx = 0 - \frac{1}{n\pi}\Big[\underbrace{f'(x)(-\cos nx)}_{2\pi \text{ periodic}}\Big]_{-\pi}^{\pi} + \frac{1}{n^2\pi}\int_{-\pi}^{\pi} f''(x)(-\cos nx)\,dx = \frac{1}{n^2\pi}\int_{-\pi}^{\pi} f''\cos nx\,dx$

Then $|a_n| \leq \frac{1}{n^2\pi}\int_{-\pi}^{\pi}|f''(x)\cos(nx)|\,dx \leq \frac{1}{n^2\pi}\int_{-\pi}^{\pi}|f''(x)|\,dx = \frac{c}{n^2}$, setting $c = \frac{1}{\pi}\int_{-\pi}^{\pi}|f''(x)|\,dx$. Likewise, we have that $|b_n| \leq \frac{c}{n^2}$.

As such, we manipulate to get $|S_n^f(x) - f(x)| = \left|\sum_{k=n+1}^{\infty} a_k\cos(kx) + b_k\sin(kx)\right| \leq \sum_{k=n+1}^{\infty}(|a_k||\cos(kx)| + |b_k||\sin(kx)|) \leq \sum_{k=n+1}^{\infty}(|a_k| + |b_k|)$

Thus, $|S_n^f(x) - f(x)| \leq \sum_{k=n+1}^{\infty}(|a_k| + |b_k|) \leq \sum_{k=n+1}^{\infty}\frac{c}{k^2} + \frac{c}{k^2} = 2c\sum_{k=n+1}^{\infty}\frac{1}{k^2}$. Then $\|S_n^f(x) - f(x)\|_{\sup} \leq 2c\sum_{k=n+1}^{\infty}\frac{1}{k^2} \to 0$, since $\sum_{k=1}^{\infty}\frac{1}{k^2} < \infty$.

$\therefore$ $S_n^f \to f$ uniformly // q.e.d.

This marks the end of this chapter.

Chapter 3.

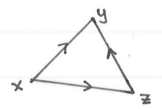
METRIC SPACES.

When we say $x_n \to x$, we mean that as the distance between x_n and x tends to 0, $|x_n - x| \to 0$. Also, $f_n \to f$ uniformly means as distance between f_n and f tends to 0, $\|f_n - f\|_{\sup} \to 0$.

Likewise, we can extend this concept further.

$\boxed{\text{Definition}}$ A pair (X, d) of a set X and a function $d: X \times X \to \mathbb{R}$ is called a <u>metric space</u> if:

(i) $d(x,y) \geq 0 \quad \forall x,y \in X$ and $d(x,y) = 0 \Rightarrow x = y$.

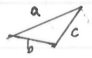


(ii) $d(x,y) = d(y,x) \quad \forall x,y \in X$.

(iii) $d(x,y) \leq d(x,z) + d(z,y) \quad \forall x,y,z \in X$ (triangle inequality).

d is called a <u>metric function</u> (or distance function).

e.g. ① $\mathbb{R}$, $d(x,y) = |x-y|$. ② $\mathbb{R}^n$, $d(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$ (Euclidean distance).



$a \leq b + c$ ($\mathbb{R}^2$).

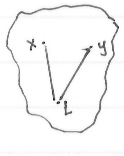
③ $C[a,b]$ i.e. continuous functions on $[a,b]$, $d(f,g) = \|f-g\|_{\sup}$.

④ Discrete space: any set X, $d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$. We check conditions for metric space. (iii): Let $x,y,z \in X$, NTP: $d(x,y) \leq d(x,z) + d(z,x)$

case 1: $x = y$, then $d(x,y) = 0 \leq d(x,z) + d(z,y)$. Case 2: $x \neq y$, then $d(x,y) = 1 \leq$ $1+1=2$ or $1+0 = 0+1 = 1$

⑤ "British railway metric". Let X be a set, specify $L \in X$. Let $f(x)$ be the distance from x to L.

Then $f(x) \geq 0$, $f(x) = 0 \Leftrightarrow x = L$. We define $d(x,y) = \begin{cases} f(x) + f(y) & x \neq y \\ 0 & x = y. \end{cases}$



$\boxed{\text{Definition}}$ A pair of a vector space V and a norm function $\|\cdot\|: V \to \mathbb{R}$ is called a <u>normed space</u> if:

(i) $\|x\| \geq 0 \quad \forall x \in V$, $\|x\| = 0 \Leftrightarrow x = 0$.

(ii) $\|\lambda x\| = |\lambda| \|x\| \quad \forall x \in V, \lambda \in \mathbb{R}$.

(iii) $\|x+y\| \leq \|x\| + \|y\| \quad \forall x,y \in V$.

$\boxed{\text{Theorem}}$ 3.1 Every normed space $(V, \|\cdot\|)$ can be made into a metric space (V, d) with $d(x,y) = \|x-y\|$.

Proof – (i) $d(x,y) = \|x-y\| \geq 0$, $d(x,y) = 0 \Rightarrow \|x-y\| = 0 \Rightarrow x = y$. (ii) $d(x,y) = \|x-y\| = \|-(y-x)\| = |-1| \|y-x\| = \|y-x\| = d(y,x)$.

(iii) $d(x,y) = \|x-y\| = \|x-z+z-y\| \leq \|x-z\| + \|z-y\| = d(x,z) + d(z,y)$ // q.e.d.

Remark: Not every metric space is a normed space: for instance, (X,d) discrete metric space: $d(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y. \end{cases}$

21 February 2013.
Dr. Nadia SIDOROVA
Roberts G06.

X is not necessarily a vector space, and even if it did, it would not apply.

e.g. ① $\mathbb{R}$, $\|x\| = |x|$ is a normed space with metric $d(x,y) = \|x-y\| = |x-y|$ corresponding to it.

② $\mathbb{R}^n$, with Euclidean norm (or 2-norm) $\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$ is a normed space. Likewise, we have 1-norm: $\|x\|_1 = \sum_{i=1}^{n} |x_i|$.

Of course, we can generalise this to the q-norm $(q \geq 1)$ $\|x\|_q = \left(\sum_{i=1}^{n} |x_i|^q\right)^{\frac{1}{q}}$. Proving this is a norm is difficult, particularly the triangle inequality.

$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$ is the infinity-norm.

The corresponding metrics are $d_2(x,y) = \|x-y\|_2 = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$, $d_1(x,y) = \|x-y\|_1 = \sum_{i=1}^{n} |x_i - y_i|$, $d_q(x,y) = \|x-y\|_q = \left(\sum_{i=1}^{n} |x_i - y_i|^q\right)^{\frac{1}{q}}$.

Also, $d_\infty(x,y) = \|x-y\|_\infty = \max_{1 \leq i \leq n} |x_i - y_i|$.

③ $C[a,b]$. Let $\|f\|_{\sup} = \sup_{x \in [a,b]} |f(x)|$ be the supremum norm. (corresponds to infinity norm in ②). 2-norm is $\|f\|_2 = \sqrt{\int_a^b f(x)^2 \, dx}$.

Also, we have q-norm $\|f\|_q = \left(\int_a^b |f(x)|^q\right)^{1/q}$, 1-norm $\|f\|_1 = \int_a^b |f(x)| \, dx$. Metrics are found analogously.

$\boxed{\text{Theorem}}$ 3.2 $\|\cdot\|_{\sup}$ and $\|\cdot\|_2$ are norms on $C[a,b]$.

Proof – (a) For $\|\cdot\|_{\sup}$: (i) $\|f\|_{\sup} \geq 0$ obvious. $\sup_{[a,b]} |f(x)| = 0 \Rightarrow f(x) = 0 \; \forall x \Leftrightarrow f = 0$. (ii) $\|\lambda f\|_{\sup} = \sup_{[a,b]} |\lambda f(x)| = |\lambda| \sup_{[a,b]} |f(x)| = |\lambda| \cdot \|f\|_{\sup}$.

(iii) $\|f+g\|_{\sup} = \sup_{[a,b]} |f(x) + g(x)| \leq \sup_{[a,b]} (|f(x)| + |g(x)|) \leq \sup_{[a,b]} |f(x)| + \sup_{[a,b]} |g(x)| = \|f\|_{\sup} + \|g\|_{\sup}$ // q.e.d.

(b) For $\|\cdot\|_2$: (i) $\|f\|_2 \geq 0 \because$ it is a square root. $\|f\|_2 = 0 \Leftrightarrow \int_a^b f(x)^2 \, dx = 0 \Leftrightarrow f \equiv 0$ (since f is continuous).

(ii) $\|\lambda f\|_2 = \sqrt{\int_a^b (\lambda f(x))^2 \, dx} = |\lambda| \sqrt{\int_a^b f(x)^2 \, dx} = |\lambda| \cdot \|f\|_2$.

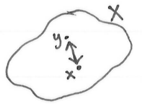
(iii) $\|f+g\|_2^2 = \langle f+g, f+g \rangle = \|f\|_2^2 + 2\langle f,g \rangle + \|g\|_2^2$. By Cauchy-Schwarz inequality, $|\langle f,g \rangle| \leq \|f\|_2 \|g\|_2$.

Then $\|f+g\|_2^2 \leq \|f\|_2^2 + 2|\langle f,g \rangle| + \|g\|_2^2 \leq \|f\|_2^2 + 2\|f\|_2\|g\|_2 + \|g\|_2^2 = (\|f\|_2 + \|g\|_2)^2$.

Hence, we see that $\|f+g\|_2^2 \leq (\|f\|_2 + \|g\|_2)^2$. Since both must have the square roots,

$\|f+g\|_2 \leq \|f\|_2 + \|g\|_2$ // q.e.d.

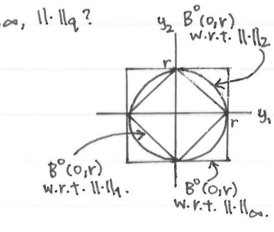
Definition Let (X,d) be a metric space. Let $x \in X$ and $r>0$. Then $B^\circ(x,r) = \{y \in X : d(y,x) < r\}$ is called the <u>open ball of radius r</u> with centre at x. Similarly, $B(x,r) = \{y \in X : d(y,x) \le r\}$ is the <u>closed ball of radius r</u> with centre at x.



e.g. ① $(\mathbb{R}, |\cdot|)$: $B^\circ(x,r) = \{y \in \mathbb{R} : |y-x| < r\} = (x-r, x+r)$. $B(x,r) = [x-r, x+r]$.

② $\mathbb{R}^2$. What are the open balls and closed balls $B^\circ(0,r)$, $B(0,r)$ with respect to the metrics $\|\cdot\|_2$, $\|\cdot\|_1$, $\|\cdot\|_\infty$, $\|\cdot\|_q$?

For $\|\cdot\|_2$,
$B^\circ(0,r) = \{y \in \mathbb{R}^2 : \|y\|_2 < r\}$ i.e. $y_1^2 + y_2^2 < r^2 \Rightarrow$ circle of radius r.
For $\|\cdot\|_\infty$,
$B^\circ(0,r) = \{y \in \mathbb{R}^2 : \|y\|_\infty < r\}$ i.e. $\max\{|y_1|, |y_2|\} < r$ i.e. $|y_1|, |y_2| < r$.
For $\|\cdot\|_1$,
$B^\circ(0,r) = \{y \in \mathbb{R} : \|y\|_1 < r\}$ i.e. $|y_1| + |y_2| < r$. $\Rightarrow$ $|y_2| < r - |y_1| \Rightarrow$ diamond (as shown)

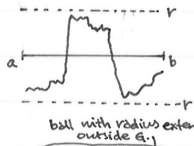


Heuristically, we think about $\|\cdot\|_q$, with $q = 3,4,5,\cdots$ we will get something in between the shape for $\|\cdot\|_2$ and $\|\cdot\|_\infty$. To get closed balls $B(0,r)$, we take $B^\circ(0,r) + \delta B^\circ(0,r)$, its boundary.

③. Discrete space: set X with $d(x,y) = \begin{cases} 1 & x \ne y \\ 0 & x = y \end{cases}$. $B^\circ(x,r) = \{y \in X : d(y,x) < r\} = \begin{cases} X, & r>1 \\ \{x\}, & r \le 1 \end{cases}$.

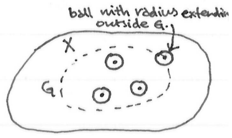
$B(x,r) = \{y \in X : d(y,x) \le r\} = \begin{cases} X, & r \ge 1 \\ \{x\}, & r < 1 \end{cases}$ (note the difference in inequality signs).

④ $C[a,b]$ with $\|\cdot\|_{sup}$. $B^\circ(\text{zero function}, r) = \{f : \|f\|_{sup} < r\} = \{$all functions remaining completely and strictly inside the r-tube around the zero function$\}$.



Definition Let (X,d) be a metric space. A set $G \subset X$ is called an <u>open set</u> if $\forall x \in G$, $\exists r > 0$ s.t. $B^\circ(x,r) \subset G$.

A set F is called a <u>closed set</u> if its complement $X \setminus F$ is open.

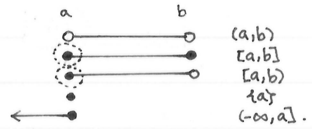


Remarks: There are sets which are neither open nor closed. More interestingly, there are sets which are open AND closed at the same time.

Hence, we cannot prove that a set is open by showing that it is not closed (and vice versa).

e.g. ① $(\mathbb{R}, |\cdot|)$: Then we examine the following sets:

	(a,b)	$[a,b]$	$[a,b)$	$\{a\}$	$(-\infty, a]$
Set is open?	✓	X	X	X	X
Set is closed?	X	✓	X	✓	✓



Then consider $\{\frac{1}{n} : n \in \mathbb{N}\}$. The set is made of discrete points, so it is not open. It is not closed either, as 0 in complement is a "bad" point.
since r is a fixed value.

However, while $\{0\} \cup \{\frac{1}{n}\}_{n \in \mathbb{N}}$ is also not open, it is closed.

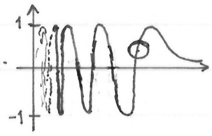
② Take $\mathbb{R}^2$ with $\|\cdot\|_2$. Consider $B^\circ(0,r)$. It is clearly an open set, but not a closed set. On the other hand, $B(0,r)$ is not an open set, but is a closed set.

$A = \{(x, \sin\frac{1}{x}) : x > 0\} \subset \mathbb{R}^2$.

The set is obviously not open. Take any point lying on the graph itself, a ball of any radius >0 contains points not within G.

Consider $\mathbb{R}^2 \setminus A$. Then at $x = 0$, the set is not open $\Rightarrow$ set A is not closed.

Note then that the set $F = \{(x, \sin\frac{1}{x}), x>0\} \cup \{(0,y) : -1 \le y \le 1\}$ is still not open, but is closed.

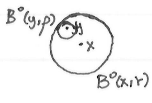


25 February 2013.
Dr. Nadia SIDOROVA.
UCL.

③ Let (X,d) be a discrete space, $d(x,y) = \begin{cases} 1 & x \ne y \\ 0 & x = y \end{cases}$. $B^\circ(x,r) = \begin{cases} \{x\}, & r \le 1 \\ X, & r > 1 \end{cases}$. Let $S \subset X$.

If S is open, $\forall x \in S$, take $r \le 1$. Then $B^\circ(x,r) = \{x\} \subset S \Rightarrow$ <u>any set</u> in a discrete metric space is open!

Likewise, any set in a discrete metric space is closed.



Theorem 3.3 (a) Every open ball $B^\circ(x,r)$ is an open set.

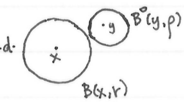
(b) Every closed ball $B(x,r)$ is a closed set.

Proof - (a). Let $y \in B^\circ(x,r) \Rightarrow d(x,y) < r$. Take $\rho = r - d(x,y) > 0$. Consider $B^\circ(y,\rho)$. We need to prove that $B^\circ(y,\rho) \subset B^\circ(x,r)$.

Pick $z \in B^\circ(y,\rho)$. Then $|z-y| < \rho$. Check $d(x,z) = |x-z| < |x-y| + |y-z| = d(x,y) + d(y,z) < d(x,y) + \rho = r \Rightarrow z \in B^\circ(x,r)$, q.e.d.

(b). Consider $B(x,r)$. Let $y \in X \setminus B(x,r) \Rightarrow d(x,y) > r$. Take $\rho = d(x,y) - r$, and consider $B^\circ(y,\rho)$. We want to prove that:

$B^\circ(y,\rho) \subset X \setminus B(x,r)$. Pick $z \in B^\circ(y,\rho)$, then $d(z,y) < \rho$. $d(x,z) \ge d(x,y) - d(y,z) > d(x,y) - \rho = r \Rightarrow z \in X \setminus B(x,r)$, q.e.d. $\Leftrightarrow z \notin B(x,r)$.



Theorem 3.4 Let (X,d) be a metric space. Then

(a) ϕ and X are both open and closed.

(b) Let $\{G_\alpha\}_{\alpha \in A}$ be a collection of open sets. Then $\bigcup_{\alpha \in A} G_\alpha$ is an open set.

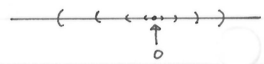
(c) Let $\{G_i\}_{i=1}^n$ (finite n) be a finite collection of open sets. Then $\bigcap_{i=1}^n G_i$ is an open set.

(d) Let $\{F_\alpha\}_{\alpha \in A}$ be a collection of closed sets. Then $\bigcap_{\alpha \in A} F_\alpha$ is a closed set.

(e) Let $\{F_i\}_{i=1}^n$ be a finite collection of ~~open~~ closed sets. Then $\bigcup_{i=1}^n F_i$ is a closed set.

Remark: What is wrong with (c) and (e) for infinite collections? We exhibit this through some examples.

e.g. For (c), consider $G_n = (-\frac{1}{n}, \frac{1}{n})$; which are open sets. Then $\bigcap_{n=1}^{\infty} = \{0\}$, which is not open. Also, for (e), consider closed sets $F_n = [-1+\frac{1}{n}, 1-\frac{1}{n}]$, $n \geq 2$. Then $\bigcup_{n=2}^{\infty} F_n = (-1,1)$, which is not closed.

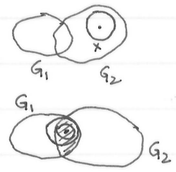


Proof - (a). X is open, since $\forall \alpha \in X$, $B^\circ(\alpha, r) \subset X$ for any r. ϕ is open, since there are no points for which we should check anything.

$\therefore$ Or, try the following: G is open means $\forall x \in G$, $\exists r$ $B^\circ(x,r) \subset G$. G is not open means $\exists x \in G$, $\forall r > 0$ $B^\circ(x,r) \not\subset G$. But $\nexists x \in \phi$ at all, so ϕ is not not open $\Rightarrow$ ϕ is open$_{//}$ q.e.d. Then for closure, we note the trivial observation that X, ϕ are merely complements of each other. Then X is closed $\Leftrightarrow$ ϕ is open, ϕ is closed $\Leftrightarrow$ X is open$_{//}$ q.e.d.

28 February 2013
Dr. Nadia Sidorova.
Roberts G06.

(b) Let $x \in \bigcup_{\alpha \in A} G_\alpha$. Then $x \in G_{\alpha_0}$ for some $\alpha_0 \in A$. Since G_α is open, $\exists r > 0$ s.t. $B^\circ(x,r) \subset G_{\alpha_0}$. However, since $\alpha_0 \in A$, $B^\circ(x,r) \subset G_{\alpha_0} \subset \bigcup_{\alpha \in A} G_\alpha \Rightarrow \bigcup_{\alpha \in A} G_\alpha$ is open$_{//}$ q.e.d.



(c) Let $x \in \bigcap_{i=1}^{n} G_i$. Then $x \in G_1$, $x \in G_2$, $\dots$, $x \in G_n$. Since all $\{G_i\}_{i=1}^{n}$ are open, for each $1 \leq i \leq n$, $\exists r_i > 0$ s.t. $B^\circ(x, r_i) \subset G_i$. Then let $r = \min_{1 \leq i \leq n} r_i$. Then $B^\circ(x,r) \subset B^\circ(x, r_i) \subset G_i$ $\forall i$. Since $B^\circ(x,r) \subset G_i$ for all G_i, $B^\circ(x,r) \subset \bigcap_{i=1}^{n} G_i \Rightarrow \bigcap_{i=1}^{n} G_i$ is open$_{//}$ q.e.d.

(d) All F_α are closed $\Rightarrow$ $X \setminus F_\alpha$ are open $\Rightarrow$ by part (b), $\bigcup_{\alpha \in A} (X \setminus F_\alpha)$ is open $\Rightarrow$ $X \setminus (\bigcap_{\alpha \in A} F_\alpha)$ is open $\Rightarrow$ $\bigcap_{\alpha \in A} F_\alpha$ is closed$_{//}$ q.e.d.

(e) All $\{F_i\}_{i=1}^{n}$ are closed $\Rightarrow$ all $X \setminus F_i$ are open $\Rightarrow$ by part (c), $\bigcap_{i=1}^{n} (X \setminus F_i)$ is open $\Rightarrow$ $X \setminus (\bigcup_{i=1}^{n} F_i)$ is open $\Rightarrow$ $\bigcup_{i=1}^{n} F_i$ is closed$_{//}$ q.e.d.

Definition Let (X,d) be a metric space. We say that a sequence of points in X converges to $x \in X$ (we write $x_n \to x$ as $n \to \infty$) if $d(x_n, x) \to 0$ as $n \to \infty$.
Equivalently, $x_n \to x$ $\Leftrightarrow$ $\forall \varepsilon > 0$, $\exists N$ $\forall n \geq N$, $d(x_n, x) < \varepsilon$. (or $x_n \in B^\circ(x, \varepsilon)$.)

e.g. ① $(\mathbb{R}, \|\cdot\|)$. Then $x_n \to x$ $\Leftrightarrow$ $|x_n - x| \to 0$ $\Leftrightarrow$ convergent sequences are the usual convergent sequences in $\mathbb{R}$.

② $((0,1), d)$; where $d(x,y) = |x-y|$. Let $x_n = \frac{1}{n}$. Then $\frac{1}{n} \to$ number strictly between 0 and 1? No $\Rightarrow$ $\{\frac{1}{n}\}$ does not converge.
However, $y_n = \frac{1}{2} + \frac{1}{n} \to \frac{1}{2}$ converges in this space.

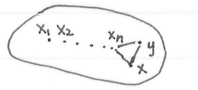
③ Discrete space, (X,d), then $x_n \to x$ $\Leftrightarrow$ $d(x_n, x) \to 0$.

x_i: x_1 x_2 x_3 x_4 $\cdots$
$d(x, x_i)$: $\underbrace{* \quad * \quad * \quad * \quad \cdots}_{\text{any values}}$ $0 \; 0 \; 0 \; 0 \; 0 \to 0$.

A sequence $\{x_n\}$ converges if it is eventually constant.

④ $(C[a,b], \|\cdot\|_{sup})$. Then $f_n \to f$ $\Leftrightarrow$ $d(f_n, f) \to 0$ i.e. $\|f_n - f\|_{sup} \to 0$. True $\Leftrightarrow$ $f_n \to f$ uniformly on $[a,b]$.

Theorem 3.5 If $x_n \to x$ and $x_n \to y$, then $x = y$.

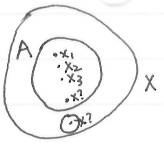


Proof – Assume $x \neq y$. Then $d(x,y) > 0$. $0 < d(x,y) \leq d(x, x_n) + d(x_n, y) \to 0 + 0 = 0$. $\Rightarrow 0 < 0$, which is a contradiction.
Hence, assumption is wrong $\Rightarrow$ $x = y_{//}$ q.e.d.

Recall some motivating examples: ① $(\mathbb{R}, |\cdot|)$. $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ was not closed. ② $(\mathbb{R}^2$, Euclidean $d)$. $(x,y) = (x, \sin\frac{1}{x})$.

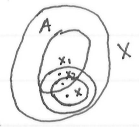
Theorem 3.6 Let (X,d) be a metric space and $A \subset X$. A is closed $\Leftrightarrow$ If sequence $\{x_n\}_{n \in \mathbb{N}}$ with $x_n \in A$ $\forall n$ converges, then it converges to point $x \in A$.

Proof - ($\Rightarrow$). Suppose A is closed, take $x_n \in A$ $\forall n$, $x_n \to$ some x. NTP: $x \in A$.



By contradiction. Suppose $x \notin A$, $x \in X \setminus A$. A is closed $\Rightarrow$ $X \setminus A$ is open. Then $\exists r > 0$ s.t. $B^\circ(x,r) \subset X \setminus A$
Then $x_n \to x$ $\Rightarrow$ $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $d(x_n, x) < r$ $\Rightarrow$ $x_n \in B^\circ(x,r) \subset X \setminus A$ $\Rightarrow$ $x_n \notin A$.
But $x_n \in A$ by definition $\Rightarrow$ contradiction.

($\Leftarrow$). Suppose RHS is true, but assume A is not closed. $\Rightarrow$ $X \setminus A$ is not open. (i.e. not every point can be surrounded by a ball)
$\Rightarrow$ at least one point cannot be surrounded by a ball. $\exists x \in X \setminus A$ s.t. $\forall r > 0$, $B^\circ(x,r) \cap A = \phi$.
In particular, for $r = \frac{1}{n}$, $B^\circ(x, \frac{1}{n}) \cap A \neq \phi$ $\forall n$. i.e. $\exists x_n \in B^\circ(x, \frac{1}{n}) \cap A$. Then, each $x_n \in A$.
Also, $d(x_n, n) < \frac{1}{n} \to 0$ i.e. $x_n \to x$. However, $x \notin A$ as $x \in X \setminus A$ $\Rightarrow$ contradicts RHS. Hence, $X \setminus A$ must be open
$\therefore$ A is closed$_{//}$ q.e.d.



Definition Let (X,d) be a metric space. A sequence $\{x_n\}_{n \in \mathbb{N}}$ of points in X is called a Cauchy sequence if $\forall \varepsilon > 0$ $\exists N$ $\forall n, m \geq N$, $d(x_n, x_m) < \varepsilon$.

e.g. ① $(\mathbb{R}, |\cdot|)$. Cauchy sequences are the "usual" Cauchy sequences in $\mathbb{R}$.

② $((0,1), d)$. $d(x,y) = |x-y|$. $x_n = \frac{1}{n}$. is Cauchy: $\forall \varepsilon > 0$ $\exists N$ $\forall n, m \geq N$, $|\frac{1}{n} - \frac{1}{m}| < \varepsilon$.

③ Discrete space, (X,d). Then x_n is Cauchy $\Leftrightarrow$ $\forall \varepsilon > 0$ $\exists N$ $\forall n, m \geq N$, $d(x_n, x_m) < \varepsilon$. $d(x_n, x_m) = \begin{cases} 1 & x_n \neq x_m \\ 0 & x_n = x_m \end{cases}$
i.e. $\forall n, m \geq N$, $x_n = x_m$. Hence $\{x_n\}$ is Cauchy $\Leftrightarrow$ $\{x_n\}$ is eventually constant.

④ $(C[a,b], \|\cdot\|_{sup})$. $\{f_n\}_{n \in \mathbb{N}}$ is Cauchy $\Leftrightarrow$ $\forall \varepsilon > 0$ $\exists N$ s.t. $\forall n, m \geq N$, $\|f_n - f_m\|_{sup} < \varepsilon$. Then $\{f_n\}_{n \in \mathbb{N}}$ is a uniform Cauchy sequence.

Remarks: For examples ①, ③, ④; convergence criteria are the same — but they differ for ②. We investigate this further.

7102-12

Lemma If x_n converges, then x_n is a Cauchy sequence. [Remark: Converse is not true!]

Proof - Suppose $x_n \to x$. $\Rightarrow \forall \varepsilon > 0, \ \exists N \ \forall n \geq N \ \ d(x_n, x) < \frac{\varepsilon}{2} \Rightarrow \forall n, m \geq N, \ \ d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ // q.e.d.

Definition A metric space is called <u>complete</u> if every Cauchy sequence in that metric space converges.

A complete normed space is called a <u>Banach space</u>.

e.g. ① $(\mathbb{R}, |\cdot|)$ is a complete space and Banach space.

② $((0,1), |\cdot|)$ is not complete (and of course not a Banach space).

③ the discrete space is complete, but it is not a Banach space because a discrete metric is not a norm.

④ $(C[a,b], \|\cdot\|_{sup})$ is complete by the CPUC. Since it is a vector space and $\|\cdot\|_{sup}$ is a norm, it is also a Banach space.

⑤ More complete spaces (Banach spaces, in fact): • $\mathbb{R}^n$ with $\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_\infty, \|\cdot\|_p$ for $p \geq 1$.

• $C[a,b]$ with $\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_{sup}$ or $\|\cdot\|_p, p \geq 1$.

⑥ the metric space $(\mathbb{Q}, |\cdot|)$ will not be complete. Consider $x_n \in \mathbb{Q}$, $x_n \to \sqrt{2}$ (or any irrational no.) as $n \to \infty$. x_n does not converge in $(\mathbb{Q}, |\cdot|)$.

However, x_n converges in $\mathbb{R} \Rightarrow x_n$ is Cauchy in $\mathbb{R} \Rightarrow x_n$ is Cauchy in $(\mathbb{Q}, |\cdot|)$. Thus, $(\mathbb{Q}, |\cdot|)$ is not complete.

Lemma Let (X, d) be a complete metric space. Then if $Y \subset X$, (Y, d) is complete $\Leftrightarrow$ Y is closed.

Proof - $(\Rightarrow)$ Suppose (Y, d) is complete. We want: Y is closed (use Theorem 3.6). Let $x_n \in Y$, then $x_n \to x \in X$. $\Rightarrow \{x_n\}$ is Cauchy in (X, d).

Thus, $\{x_n\}$ is Cauchy in (Y, d). Since (Y, d) is complete, $x_n \to x$ in $Y \Rightarrow x \in Y$ // q.e.d.

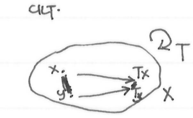
$(\Leftarrow)$. Suppose Y is closed. We want to show that (Y, d) is complete. Let $\{x_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence in (Y, d).

Then $\{x_n\}$ is a Cauchy sequence in (X, d). $\Rightarrow x_n \to x \in X$ since (X, d) is complete. As Y is closed, $x \in Y \Rightarrow x_n$ converges in (Y, d) // q.e.d.

4 March 2013.
Dr. Nadia SIDOROVA
CLT.

<u>Contraction mappings</u>:

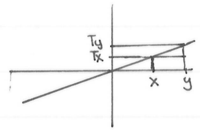
Definition Let (X, d) be a metric space. A mapping $T: X \to X$ is called a <u>contraction mapping</u> if $\exists c \in (0,1)$ s.t. $d(Tx, Ty) \leq c \cdot d(x,y) \ \forall x, y \in X$.



Note: We write Tx for $T(x)$, point on nomenclature.

e.g. ① $(\mathbb{R}, |\cdot|)$, $Tx = \frac{x}{2}$ is a contraction mapping. then $|Tx - Ty| = \frac{1}{2}|x-y| \leq \frac{1}{2}|x-y|$. $c = \frac{1}{2}$. $\Rightarrow T$ is a contraction with $c = \frac{1}{2}$.

② $(\mathbb{R}, |\cdot|)$, $T(x) = \sin\left(\frac{x}{3}\right)$. then $|T(x) - T(y)| = |\sin\frac{x}{3} - \sin\frac{y}{3}| \overset{MVT.}{=} |T'(\xi)||x-y| = |\frac{1}{3}\cos\frac{\xi}{3}||x-y| \leq \frac{1}{3}|x-y|$. $\leftarrow \xi \in (x,y)$



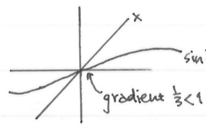
T is a contraction mapping with $c = \frac{1}{3}$.

More generally, for $(\mathbb{R}, |\cdot|)$, T is differentiable $\Rightarrow |T(x) - T(y)| \overset{MVT}{=} |T'(\xi)||x-y| \leq \|T'\|_{sup}|x-y|$. If $\|T'\|_{sup} < 1$ (strictly), then T is a contraction mapping.

Definition Let (X, d) be a metric space and $T: X \to X$. If $x \in X$ has the property $Tx = x$, then x is called a <u>fixed point</u> of T.

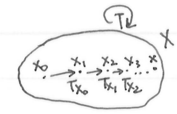
e.g. ① $(\mathbb{R}, |\cdot|)$, $Tx = \frac{x}{2}$. $x = 0$ is the unique fixed point.

② $(\mathbb{R}, |\cdot|)$, $Tx = \sin\left(\frac{x}{3}\right)$. Find $\sin\frac{x}{3} = x$. Then $x = 0$ is the unique fixed point.



Theorem 3.7 (Contraction Mapping Theorem).

Let (X, d) be a non-empty complete metric space, and let $T: X \to X$ be a contraction mapping. Then T has a unique fixed point.



Proof - Denote the contraction constant of T by c. Since $X \neq \phi$, we can pick some $x_0 \in X$. Define $\{x_n\}$ by $x_n = Tx_{n-1}$

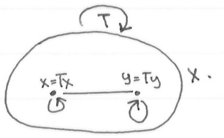
$\forall n \in \mathbb{N}$. Then $d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq c\, d(x_n, x_{n-1}) \leq c^2 d(x_{n-1}, x_{n-2})$ etc... $\leq c^n d(x_1, x_0)$.

WLOG, assume $m > n$. Then $d(x_m, x_n) \overset{\triangle}{\leq} d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n) \leq (c^{m-1} + c^{m-2} + \cdots + c^{n+1} + c^n)\, d(x_1, x_0)$.

i.e. $d(x_m, x_n) \leq \sum_{i=1}^{n-1} c^i\, d(x_1, x_0) \leq \sum_{i=1}^{\infty} c^i\, d(x_1, x_0) = \frac{c^n}{1-c} d(x_1, x_0) \to 0$ as $n \to \infty$ as $c \in (0,1)$.

Hence $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence $\Rightarrow$ since (X, d) is complete, $\{x_n\}_{n \in \mathbb{N}}$ converges to some $x \in X$.

$x_n \to x \Rightarrow x_{n+1} \to x$. Then $d(x_{n+1}, Tx) = d(Tx_n, Tx) \leq c\, d(x_n, x) \to 0 \Rightarrow x_{n+1} \to Tx$. Hence, $Tx = x$.



Suppose x, y are both fixed points, then $Tx = x, Ty = y$. $|Tx - Ty| = |x - y| = d(x,y) \leq c\, d(x,y) \Rightarrow c \geq 1$

However, T is a contraction mapping $\Rightarrow c \in (0,1) \Rightarrow$ contradiction $\Rightarrow$ only one fixed point // q.e.d.

Remark: Why are the assumptions of the CMT important?

① X must be complete: $X = (0, \infty)$ with $d(x,y) = |x-y|$. Take $T(x) = \frac{x}{2}$ is a contraction, but $x = Tx \Leftrightarrow x = \frac{x}{2} \Rightarrow$ no solutions in $(0,\infty)$. T has no fixed point.

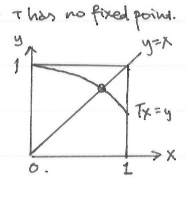
② We cannot replace $d(Tx, Ty) \leq c\, d(x,y)$ by $d(Tx, Ty) < d(x,y)$. The space $X = [1, \infty)$ with $d(x,y) = |x-y|$ is complete (since closed).

If $Tx = x + \frac{1}{x}$, $|Tx - Ty| \overset{MVT}{=} |1 - \frac{1}{\xi^2}||x-y| < |x-y|$. $Tx = x \Leftrightarrow x + \frac{1}{x} = x \Rightarrow \frac{1}{x} = 0 \Rightarrow$ no solution $\Rightarrow T$ has no fixed point.

7 March 2013.
Dr. Nadia SIDOROVA
CLT.

Applications of the CMT:

① Take $[0,1]$, $d(x,y) = |x-y|$. $Tx = \cos(x)$. T is a contraction mapping on $[0,1]$. We know that $\cos(x)$ is not a contraction mapping on $\mathbb{R}$.



For $Tx=x$ on $[0,1]$, there is a unique solution: set $x_0=0$, $\cos(x_{n-1})=x_n$. Then by contraction Mapping theorem, $x_n \to x$ (fixed point).

This gives us a method to technically compute x s.t. $Tx=x$.

② Consider differential equation $\{y'=f(x,y),\ y(x_0)=y_0\}$ ⊛. For instance, $\{y'=xy,\ y(0)=1.\}$ [Clearly: $y(x)=e^{x^2/2}$ in this (easy) case where f is a "nice" function].

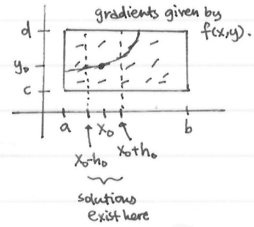
In general, we can integrate to get $\int_{x_0}^{x} y' = \int_{x_0}^{x} f(t,y(t))\,dt \Rightarrow \{y(x) = y_0 + \int_{x_0}^{x} f(t,y(t))\,dt.$ ⊛⊛ Assume $y(\cdot)$ is continuous [implicit since y differentiable]$\}$.

Hence, we have an equation $y=Ty$, where the mapping T is s.t. $(T(\varphi))(x) = \int_{x_0}^{x} f(t,\varphi(t))\,dt + y_0$. Hence, we are actually looking for a fixed point to mapping T.

One can show that under some reasonable assumptions, T is a contraction mapping (on some suitable space of functions). Then solving ⊛ is equivalent to solving ⊛⊛, which is equivalent to finding a fixed point of T. i.e. Take $\varphi_0(x)$ to be any function, $\varphi_1 = T\varphi_0$, $\varphi_2 = T\varphi_1$, $\varphi_3 = T\varphi_2$, ……

For instance, consider the system $\{y'=xy,\ y(0)=1\}$. Take $\varphi_0(x)=1$. $\varphi_1(x) = 1+\int_0^x t\cdot 1\,dt = 1+\frac{1}{2}x^2$. $\varphi_2(x) = 1+\int_0^x t(\varphi_1(t))\,dt = 1+\frac{x^2}{2}+\frac{x^4}{2\cdot 2^2}$

$\varphi_3(x) = 1+\int_0^x t\,\varphi_2(t)\,dx = 1+\frac{x^2}{2}+\frac{x^4}{8}+\frac{x^6}{6\cdot 8} = 1+\frac{x^2}{2}+\frac{(\frac{x^2}{2})^2}{2!}+\frac{(\frac{x^2}{2})^3}{3!}$. Continuing in this pattern, $\varphi_n(x) = \sum_{i=0}^{n} \frac{1}{i!}\left(\frac{x^2}{2}\right)^i$. As $n\to\infty$, $\varphi_n(x) \to e^{x^2/2}$ // q.e.d.



$\boxed{\text{Theorem}}$ 3.8 (Picard Theorem).

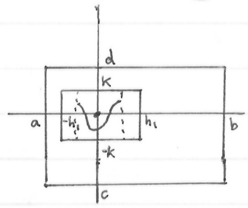
Suppose $f:[a,b]\times[c,d]\to\mathbb{R}$ is continuous, and $\frac{\partial f}{\partial y}$ exists and is continuous. Let $(x_0,y_0)\in(a,b)\times(c,d)$. Then $\exists h_0>0$ s.t. ⊛ $\begin{cases} y'=f(x,y) \\ y(x_0)=y_0 \end{cases}$ has a unique solution on $[x_0-h_0, x_0+h_0]$.

Proof-(existence) WLOG, assume that $a<0<b$, $c<0<d$ and $x_0=0$, $y_0=0$ (by translation of rectangle).

f is continuous on $[a,b]\times[c,d] \Rightarrow f$ is bounded *(rigourous proof next week) $\Rightarrow |f(x,y)|\le M$ for some M, $\forall x,y$.

$\frac{\partial f}{\partial y}$ is continuous on $[a,b]\times[c,d] \Rightarrow \frac{\partial f}{\partial y}$ is bounded $\Rightarrow |\frac{\partial f}{\partial y}(x,y)| \le \tilde{M}$ for some $\tilde{M}$, $\forall x,y$.

Then $\begin{cases} y'=f(x,y) \\ y(0)=0 \end{cases} \Leftrightarrow \begin{cases} y(x)=\int_0^x f(t,y(t))\,dt \\ y \text{ is continuous.} \end{cases}$ Choose h_1, k s.t. $[-h_1,h_1]\times[-k,k]\subset[a,b]\times[c,d]$.



Let $h\le h_1$. Consider $B(h,k) = \{\varphi:[-h,h]\to\mathbb{R},\ \varphi \text{ is continuous and } \|\varphi\|_{sup}\le k\}$.

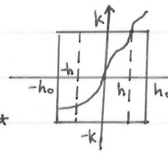
Consider mapping $(T\varphi)(x) = \int_0^x f(t,\varphi(t))\,dt$

- We will show that $T: B(h,k)\to B(h,k)$; $\|T\|_{sup} = \sup_{x\in[-h,h]} |\int_0^x f(t,\varphi(t))\,dt| \le M\cdot h \le \frac{k}{2} \le k$ (whenever $h<\frac{k}{2M}$)
- $B(h,k)$ with $\|\cdot\|_{sup}$ is complete, $B(h,k)\subset C[-h,h]$ (both with $\|\cdot\|_{sup}$). $C[-h,h]$ is complete. Then $B(h,k)$ is now complete since it is closed: if $\varphi_n\in B(h,k)$ s.t. $\varphi_n \longrightarrow \varphi$ uniformly, then $|\varphi(x)|\le k$ $\forall x \Rightarrow \varphi\in B(h,k)$.
- T is a contraction mapping on $B(h,k)$: let $\varphi,\psi\in B(h,k)$. Then $\|T\varphi - T\psi\|_{sup} = \sup_{x\in[-h,h]} |\int_0^x f(t,\varphi(t))\,dt - f(t,\psi(t))\,dt|$
- $\|T\varphi-T\psi\|_{sup} = \sup_{x\in[-h,h]} |\int_0^x (f(t,\varphi(t)) - f(t,\psi(t)))\,dt| \overset{MVT}{=} \sup_{x\in[-h,h]} |\int_0^x \frac{\partial f}{\partial y}(t,\xi)\,(\varphi(t)-\psi(t))\,dt| \le h\tilde{M}\,\|\varphi-\psi\|_{sup}$.

Set $h\tilde{M}<1 \Rightarrow h<\frac{1}{\tilde{M}}$, $h\le\frac{1}{2\tilde{M}}$. Choose $h_0 = \min\{h_1, \frac{k}{2M}, \frac{1}{2\tilde{M}}\}$. Then T is a contraction mapping on complete space $B(h_0,k)$

$\therefore$ By CMT, $T\varphi=\varphi$ has a unique solution $\Rightarrow$ ⊛ has a unique solution on $[-h_0,h_0]$ among functions which are bounded by k.

It remains to show that there is no solution on $[-h_0,h_0]$ with norm $>k$.

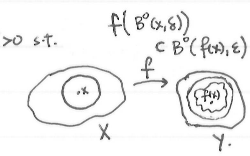


Suppose $\exists\varphi^*$ s.t. φ^* solves ⊛ on $[-h_0,h_0]$ and $\|\varphi^*\|_{sup}>k$. Let $h=\inf\{|t| : |h(t)|>k\}$.

Denote $\bar{\varphi}^*$ as the restriction of φ^* to $[-h,h]$. $\bar{\varphi}^*$ is a solution of ⊛ on $[-h,h] \Rightarrow \bar{\varphi}^*$ is a fixed point of T on $B(h,k)$. $\|\bar{\varphi}^*\|_{sup} = k$ (by definition) $= \|T\bar{\varphi}^*\|_{sup} \le \frac{k}{2} \Rightarrow$ contradiction // q.e.d.

<u>Continuity</u>

$\boxed{\text{Definition}}$ Let (X,d_X) and (Y,d_Y) be metric spaces, and $f:X\to Y$. Let $x\in X$. We say that f is <u>continuous</u> at x if $\forall\varepsilon>0$, $\exists\delta>0$ s.t. if $d_X(y,x)<\delta$ then $d_Y(f(y),f(x))<\varepsilon$.



Equivalently: $\forall\varepsilon>0$ $\exists\delta>0$ s.t. $y\in B^\circ(x,\delta) \Rightarrow f(y)\in B^\circ(f(x),\varepsilon)$; or $f(B^\circ(x,\delta))\subset B^\circ(f(x),\varepsilon)$.

We say that f is continuous if it is continuous at each $x\in X$.

e.g. ① $(\mathbb{R},|\cdot|)$: we get usual definition of continuity.

② Let X be a discrete space, Y be any metric space, f be any function from X to Y. Claim: any f is continuous.

Let $x\in X$, let $\varepsilon>0$. We want $\delta>0$ s.t. $f(B^\circ(x,\delta))\subset B^\circ(f(x),\varepsilon)$. $B^\circ(x,\delta)=\{x\}$ if $\delta\le1$, X if $\delta>1$.

Choose $\delta\le1$, say $\delta=\frac{1}{2}$. Then $f(B^\circ(x,\frac{1}{2})) = f(\{x\}) = \{f(x)\}\subset B^\circ(f(x),\varepsilon) \Rightarrow f$ is continuous at x. x is arbitrary $\Rightarrow f$ is continuous everywhere.

③ $F: C[0,1]\to C[0,1]$. Let $F(f)=f$ (identity map). Is F continuous as a function from $(C[0,1],\|\cdot\|_2)$ to $(C[0,1],\|\cdot\|_1)$?

11 March 2013
Dr. Nadia SIDOROVA
UCL.

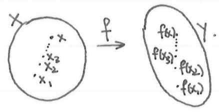
Yes: choose $\varepsilon>0$. We suppose $\|g-f\|_2<\delta$. Then $\|F(g)-F(f)\|_1 = \|g-f\|_1 = \int_0^1 |g(x)-f(x)|\,dx = \int 1\cdot|g(x)-f(x)|\,dx$.

Apply Cauchy-Schwarz inequality: $\int_0^1 1\cdot|g(x)-f(x)|\,dx \le \sqrt{\int_0^1 1^2dx}\sqrt{\int_0^1|g(x)-f(x)|^2dx} = \sqrt{\int_0^1|g(x)-f(x)|^2dx} = \|g-f\|_2 < \delta=\varepsilon$. Take $\delta=\varepsilon$, q.e.d//.

Is F continuous as a function from $(C[0,1],\|\cdot\|_{sup})$ to $(C[0,1],\|\cdot\|_2)$? Yes. Choose $\varepsilon>0$. Suppose $\|g-f\|_{sup}<\delta$. Then $\|F(g)-F(f)\|_2$ is

$\|F(g)-F(f)\|_2 = \|g-f\|_2 = \sqrt{\int_0^1|g(x)-f(x)|^2dx} \le \sqrt{\|g-f\|_{sup}^2\int_0^1dx} = \|g-f\|_{sup} < \delta=\varepsilon$.

Is F continuous as a function from $(C[0,1], \|\cdot\|_1)$ to $(C[0,1], \|\cdot\|_2)$? No! To prove this, we need a theorem:



Theorem 3.9 Heine condition / sequential Definition of continuity).

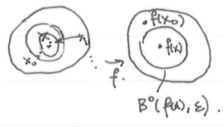
Let $f: X \to Y$ be a function between metric spaces (X, d_X) and (Y, d_Y). f is continuous at $x \in X \iff$ for any sequence $x_n \to x$ we have $f(x_n) \to f(x)$.

Proof - ($\Rightarrow$) suppose f is continuous at x. Pick arbitrary sequence $x_n \to x$. We want to show that $f(x_n) \to f(x)$. Let $\varepsilon > 0$. By continuity, $\exists \delta > 0$ s.t.

$f(B^\circ(x, \delta)) \subset B^\circ(f(x), \varepsilon)$. Since $x_n \to x$, $\exists N \in \mathbb{N}$ $\forall n > N$ s.t. $x_0 \in B^\circ(x, \delta)$. Hence, the two statements $\Rightarrow$ $f(x_n) \in B^\circ(f(x), \varepsilon)$

($\Leftarrow$) suppose RHS is true, but f is not distontinuous at x i.e. $\exists \varepsilon > 0$ $\forall \delta > 0$ $\exists$ $y \in B^\circ(x, \delta)$ s.t. $f(y) \notin B^\circ(f(x), \varepsilon)$.

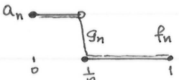
For any $n \in \mathbb{N}$, choose $\delta = \frac{1}{n}$ $\Rightarrow$ $\underbrace{\exists y_n \in B^\circ(x, \tfrac{1}{n})}_{\substack{d_X(y_n, x) < \frac{1}{n} \to 0 \\ y_n \to x}}$, s.t. $\underbrace{f(y_n) \notin B^\circ(f(x), \varepsilon)}_{\substack{d_Y(f(y_n), f(x)) \geq \varepsilon \\ f(y_n) \not\to f(x)}}$, $\Rightarrow$ contradiction // q.e.d.



e.g. (cont'd). $F(f) = f$, from $(C[0,1], \|\cdot\|_1)$ to $(C[0,1], \|\cdot\|_2)$. We will show that F is not continuous at the point $f \equiv 0$. It suffices to construct $f_n \to f$ w.r.t.

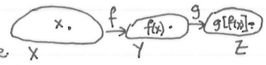
$\|\cdot\|_1$, $F(f_n) \not\to F(f)$: i.e. $\begin{cases} \|f_n\|_1 \to 0 \\ \|f_n\|_2 \not\to 0 \end{cases}$. set $f_n(x) = \begin{cases} a_n & 0 \leq x < \frac{1}{n} \\ 0 & \frac{1}{n} \leq x \leq 1 \end{cases}$. Then we have $\|f_n\|_1 = \int_0^1 |f_n(x)| dx = \frac{a_n}{n}$.

$\|f_n\|_2 = \sqrt{\int_0^1 f_n(x)^2 dx} = \sqrt{a_n^2 \cdot \frac{1}{n}} = \frac{a_n}{\sqrt{n}}$. then if $a_n = \sqrt{n}$, for example, $\|f_n\|_1 \to 0$ but $\|f_n\|_2 \not\to 0$.



However, f_n were not discontinuous, so for the correct proof, Take $f_n = g_n$ rather than just f_n, where g_n indicates f_n but with a connecting segment of small order.

④ suppose we have three metric spaces, X, Y, Z. If f, g are continuous everywhere, then $g \circ f$ is continuous everywhere.

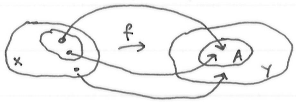


This follows naturally from our definitions. Let $x \in X$, $x_n \to x$, then $f(x_n) \to f(x)$ since f is continuous. Then $g(f(x_n)) \to g(f(x))$ since g is continuous.

Hence, $(g \circ f)$ is continuous at x.

14 March 2013.
Dr. Nadia SIDOROVA
Roberts G06.

Definition Let X, Y be sets and $f: X \to Y$. Let $A \subset Y$. The preimage of A under f is the set $f^{-1}(A) = \{x \in X : f(x) \in A\}$.

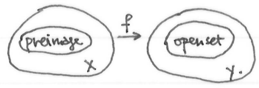


Theorem 3.10 Let (X, d_X) and (Y, d_Y) be metric spaces and let $f: X \to Y$. Then the following three statements are equivalent:

(i) f is continuous (at every point)

(ii) for every open set $G \subset Y$ holds $f^{-1}(G)$ is open (the preimage of every open set is open).

(iii) for every closed set $F \subset Y$ holds $f^{-1}(F)$ is closed (the preimage of every closed set is closed).



Comment: This is not true for images! You can have a continuous function f and an open set $A \subset X$ s.t. $f(A)$ is not open.

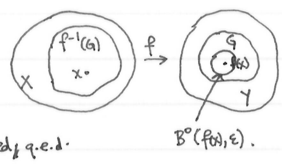
e.g. ① $f: X \to Y$, where X is the discrete space and Y is any metric space. This is continuous, since every set in discrete space is open/closed.

② composition of functions.

If f, g are both continuous, then $g \circ f$ is continuous.

Proof - (i)$\Rightarrow$(ii): Let $x \in f^{-1}(G) \Rightarrow f(x) \in G$. Since G is open, $\exists \varepsilon > 0$ s.t. $B^\circ(f(x), \varepsilon) \subset G$. Since f is continuous (at x),

$\exists \delta > 0$ s.t. $f(B^\circ(x, \delta)) \subset B^\circ(f(x), \varepsilon) \subset G$. $\Rightarrow B^\circ(x, \delta) \subset f^{-1}(G) \Rightarrow f^{-1}(G)$ is open // q.e.d.



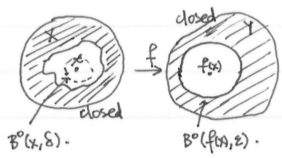
(ii)$\Rightarrow$(iii): Let $F \subset Y$ be a closed set. Then $Y \setminus F$ is open $\Rightarrow f^{-1}(Y \setminus F)$ is open $\Rightarrow X \setminus (f^{-1}(Y \setminus F))$ is closed $\Rightarrow f^{-1}(F)$ is closed // q.e.d.

(iii)$\Rightarrow$(i): Let $x \in X$, let $\varepsilon > 0$. $B^\circ(f(x), \varepsilon)$ is open since every open ball is open. $\Rightarrow Y \setminus B^\circ(f(x), \varepsilon)$ is closed.

By assumption, $\Rightarrow f^{-1}(Y \setminus B^\circ(f(x), \varepsilon))$ is closed and $x \notin f^{-1}(Y \setminus B^\circ(f(x), \varepsilon))$ since $f(x) \in B^\circ(f(x), \varepsilon)$.

$\Rightarrow X \setminus f^{-1}(Y \setminus B^\circ(f(x), \varepsilon)) = f^{-1}(B^\circ(f(x), \varepsilon))$ is open, and contains x. $\Rightarrow \exists \delta > 0$ s.t.

$B^\circ(x, \delta) \subset f^{-1}(B^\circ(f(x), \varepsilon)) \Rightarrow f(B^\circ(x, \delta)) \subset B^\circ(f(x), \varepsilon) \Rightarrow f$ is continuous at x $\forall x \in X$ // q.e.d.



Compactness.

Recall: We said that a set of open intervals $\{I_\alpha\}_{\alpha \in A}$ is a cover for a set S in $\mathbb{R}$, if $S \subset \bigcup_{\alpha \in A} I_\alpha$.

The set S is called compact if every cover of S has a finite subcover.

Definition Let (X, d) be a metric space. Let $S \subset X$. A collection of open sets $\{I_\alpha\}_{\alpha \in A}$ is a cover of S if $S \subset \bigcup_{\alpha \in A} I_\alpha$.

A finite subcollection $\{I_{\alpha_1}, \dots, I_{\alpha_m}\}$ is a finite subcover if $S \subset I_{\alpha_1} \cup \dots \cup I_{\alpha_m}$.

The set S is compact if every cover $\{I_\alpha\}_{\alpha \in A}$ of S has a finite subcover $\{I_{\alpha_1}, \dots, I_{\alpha_m}\}$ for some m.

The set S is called sequentially compact if for any sequence $\{x_n\}_{n=1}^\infty$ s.t. $x_n \in S$, $\exists$ a subsequence $\{x_{n_k}\}_{k=1}^\infty$ which converges, and $x_{n_k} \to x \in S$. (its limit is in S).

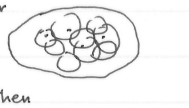
e.g. ① $(\mathbb{R}, |\cdot|)$, $S \subset \mathbb{R}$. (a) $S = [a,b]$ - compact (Heine-Borel Theorem), $[a,b]$ - sequentially compact (Bolzano-Weierstraß theorem).

(b) $S = \mathbb{R}$. Take intervals $\{(-n, n)\}_{n \in \mathbb{N}}$. Then this is a cover, but any finite subcollection does not cover $\mathbb{R} \Rightarrow$ no finite subcover $\Rightarrow$ not compact.

Take $x_n = n$. This is a sequence which has no convergent subsequence.

(iii) $S=(0,1)$. Consider the intervals $\{(\frac{1}{n}, 1-\frac{1}{n})\}_{n\in\mathbb{N}}$. This is a cover for $(0,1)$ without a finite subcover. $\Rightarrow$ not compact. For sequential compactness, pick any sequence. By Bolzano-Weierstraß Theorem, $\exists$ a convergent subsequence e.g. $x_n = \frac{1}{n} \to 0$, however $0 \notin (0,1)$. S is not sequentially compact.

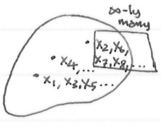
② Let (X,d) be a discrete space. We search for compact sets: let S be a finite set: $S=\{x_1,\dots,x_m\}$. Let $\{I_\alpha\}_{\alpha\in A}$ be a cover of S. $\Rightarrow$ Pick one I_α for each x_i, where $x_i \in I_i$. $\Rightarrow$ finite subcover $\Rightarrow$ every finite set is compact.



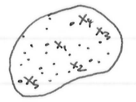
For an infinite set, $S=\{x_1,x_2,\dots\}$, we take the cover $\{\{x\}\}_{x\in S}$. Each set $\{x\}$ is open. (just the set of singletons). Then $\bigcup_{x\in S}\{x\} = S \Rightarrow$ this is a cover. However, it has no finite subcover, since every finite subcover would only cover finitely many points.

$\Rightarrow$ Every infinite set in the discrete space is not compact.

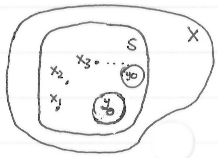
What about sequentially compact sets? Let S be a finite set, it is sequentially compact, since if $\{x_n\}$ is a sequence in S, it will only take finitely many values $\Rightarrow$ at least one value will be taken infinitely-many times. This is our convergent subsequence.



Now, let S be an infinite set. It is not sequentially compact: Take x_n to be a sequence in X with distinct values. $\Rightarrow$ each subsequence has distinct values $\Rightarrow$ not eventually constant $\Rightarrow$ does not converge.



$\boxed{\text{Theorem}}$ 3.11 A set S is compact $\Leftrightarrow$ S is sequentially compact.

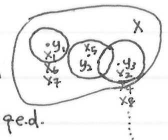


Note: We would only prove the forward direction, the reverse is harder and more time-consuming.

Also, the reverse implication holds in metric spaces, but not in the generalisation to topological spaces.

Proof – ($\Rightarrow$) Assume S is compact, but not sequentially compact. $\Rightarrow \exists \{x_n\}_{n=1}^{\infty}$ s.t. $x_n \in S$ $\forall n$ but x_n does not have a subsequence which converges to a point in S.

$\Rightarrow \forall y \in S$ $\exists r(y)>0$ s.t. $B^\circ(y, r(y))\setminus\{y\}$ does not contain any of the points $x_1, x_2, x_3, \dots$ Then $\{B^\circ(y, r(y))\}_{y\in S}$ is a cover for S. Since S is compact, this cover has a finite subcover: $B^\circ(y_1, r(y_1)), B^\circ(y_2, r(y_2)), \dots, B^\circ(y_m, r(y_m))$. The points $x_1, x_2, x_3, \dots$ can only belong to the set $\{y_1, \dots, y_m\}$. At least one of the values will be taken infinitely many times $\Rightarrow$ this produces a constant (hence convergent) subsequence with limit in S $\Rightarrow$ contradicts initial assumption $\Rightarrow$ S is sequentially compact // q.e.d.



$\boxed{\text{Definition}}$ Let $S \subset X$ be a set in a metric space, (X,d), $S \neq \emptyset$. The <u>diameter</u> of S is $\operatorname{diam}(S) = \sup_{x,y\in S} d(x,y)$.

e.g. If (X,d) is the discrete space, $S\neq\emptyset$, $S\subset X$, $\operatorname{diam}(S) = \begin{cases} 1 & \text{if } |S|\geq 2 \\ 0 & \text{if } |S|=1 \end{cases}$.

S is <u>bounded</u> if $\operatorname{diam}(S)<\infty$.

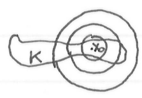
18 March 2013.
Dr. Nadia SIDOROVA
CLT.

$\boxed{\text{Theorem}}$ 3.12 Let (X,d) be a metric space and let $K\subset X$.

(i) If K is compact then K is bounded (in particular, if K is not bounded, it is not compact).

(ii) If K is compact then K is closed (in particular, if K is not closed, it is not compact.)

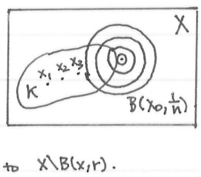
(iii) If K is closed and $K\subset L$ where L is compact, then K is compact.



Proof – (i) Pick $x_0\in K$. Consider $\{B^\circ(x_0,n)\}_{n\in\mathbb{N}}$. $\bigcup_{n\in\mathbb{N}} B^\circ(x_0,n) = X \Rightarrow \{B^\circ(x_0,n)\}_{n\in\mathbb{N}}$ is a cover for K.

K is compact $\Rightarrow \exists$ finite subcover $B^\circ(x_0,n_1),\dots, B^\circ(x_0,n_m)$. Take $R=\max\{n_1,\dots,n_m\} \Rightarrow K\subset B^\circ(x_0,R) \Rightarrow \forall x,y\in K$, $d(x,y)\leq d(x,x_0)+d(x_0,y)<2R$.

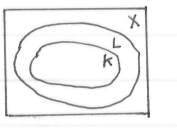
$\operatorname{diam}(K)\leq 2R<\infty \Rightarrow K$ is bounded // q.e.d.



(ii) Suppose K is not closed $\Rightarrow \exists\, x\in K$, $x_n\to x$ but $x\notin K$. Consider $\{X\setminus \overset{\text{closed}}{\overline{B}}(x,\frac{1}{n})\}_{n\in\mathbb{N}}$, which is open.

$\bigcup_{n\in\mathbb{N}} X\setminus \overline{B}(x,\frac{1}{n}) = X\setminus\{x\}\supset K \Rightarrow \{X\setminus \overline{B}(x,\frac{1}{n})\}_{n\in\mathbb{N}}$ is a cover of K. Since K is compact, there is a finite subcover $X\setminus \overline{B}(x,\frac{1}{n_1}),\dots, X\setminus \overline{B}(x,\frac{1}{n_m})$. Let $r=\min\{\frac{1}{n_1},\dots,\frac{1}{n_m}\}$. $K\subset X\setminus \overline{B}(x,r) \Rightarrow$ all x_n belong to $X\setminus \overline{B}(x,r)$.

$\Rightarrow d(x_n,x)>r$, hence $x_n\not\to x$, contradiction // q.e.d.



(iii) Let $\{I_\alpha\}_{\alpha\in A}$ be a cover of $K \Rightarrow \{\{I_\alpha\}_{\alpha\in A}, X\setminus K\}$ is a cover for L. Since L is compact, there is a finite subcover $I_{\alpha_1},\dots,I_{\alpha_m}$, $X\setminus K$ for $L \Rightarrow$ these also cover $K \Rightarrow I_{\alpha_1},\dots,I_{\alpha_m}$ covers K // q.e.d.

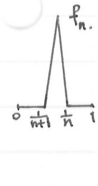
Comments: ① $\mathbb{R}^n$ with any norm (but not just metric) is compact $\Leftrightarrow$ it is closed + bounded (reverse direction without proof: do not use in exam).

It is not true in general that compact $\Leftrightarrow$ closed + bounded e.g. (X,d) – discrete space. S is an infinite set $\Rightarrow$ not compact, but S is closed (as every set is closed) and S is bounded, $\operatorname{diam}(S)=1$.

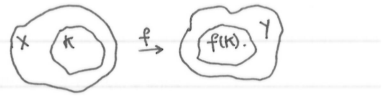
e.g. 2. Consider $(C[0,1], \|\cdot\|_{\sup})$. Let $S=\overline{B}(0,1)$ be the closed unit ball: • S is closed (every closed ball is closed) and bounded ($\|f-g\|_{\sup}\leq \|f\|_{\sup}+\|g\|_{\sup} \leq 2 \Rightarrow \operatorname{diam} S\leq 2$. However, S is not sequentially compact: $\|f_n - f_m\|_{\sup}=1 \Rightarrow$ no uniform Cauchy subsequence.

$\Rightarrow$ no convergent subsequence. $\Rightarrow$ S is not compact since it is not sequentially compact.

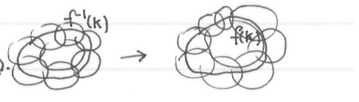
These demonstrate that we require normed spaces!



Theorem 3.13 Let (X, d_X) and (Y, d_Y) be two metric spaces, and let $f: X \to Y$ be a continuous function. Then, if $K \subset X$ is a compact set, then $f(K) \subset Y$ is a compact set.



Proof - Use definition. Let $\{I_\alpha\}_{\alpha \in A}$ be a cover of A. Consider $\{f^{-1}(I_\alpha)\}_{\alpha \in A}$. This is open, since I_α is open and f is continuous. This is a cover of K. Since K is compact, it has a finite subcover $f^{-1}(I_{\alpha_1}), \dots, f^{-1}(I_{\alpha_n})$. If we map these using f, then $I_{\alpha_1}, \dots, I_{\alpha_n}$ is a finite subcover for $f(K) \Rightarrow f(K)$ is compact.// q.e.d.



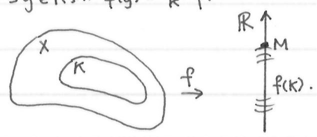
21 March 2013.
Dr. Nadia SIDOROVA
Darwin G06.

Recall from MATH1101, if $f: [a,b] \to \mathbb{R}$ is continuous, then f attains its maximum and minimum. This does not hold for open intervals. This holds, in general, because $[a,b]$ is compact.

Theorem 3.14 Suppose (X,d) is a metric space and $f: X \to \mathbb{R}$ is continuous. Let $K \subset X$ be a compact set. Then $\exists x \in K$ s.t. $f(x) = \sup_K f$, $\exists y \in K$ s.t. $f(y) = \inf_K f$.

Proof - Since K is compact and f is continuous $\Rightarrow f(K)$ is compact $\Rightarrow f(K)$ is closed and bounded.

Hence $M = \sup f(K) < \infty$. By definition, sup is least upper bound, $\exists a_n$ s.t. $a_n \in f(K)$, $a_n \to M \Rightarrow$ since set is closed, $M \in f(K)$. $\Rightarrow \exists x \in K$ s.t. $f(x) = M$ // q.e.d. Second part of proof is analogous // q.e.d.



Definition Let V be a vector space and $|\cdot|$, $\|\cdot\|$ be two norms on V. They are called <u>equivalent</u> if $\exists c, C > 0$ s.t. $c|x| \le \|x\| \le C|x| \ \forall x \in V$.

Remark: The definition is symmetric w.r.t. the two norms: $\|x\| \le C|x| \le \frac{C}{c}\|x\| \Rightarrow \frac{1}{C}\|x\| \le |x| \le \frac{1}{c}\|x\|$.

Comments:

① This is the same as saying $\forall r > 0 \ \exists \tilde{r} > 0$ s.t. $B^\circ_{\|\cdot\|}(x, \tilde{r}) \subset B^\circ_{|\cdot|}(x, r)$: let $r > 0$ be fixed, we choose $\tilde{r} = cr$.

Let $y \in B^\circ_{\|\cdot\|}(x, \tilde{r}) \Rightarrow \|y - x\| < \tilde{r} \Rightarrow |y - x| \le \frac{1}{c}\|y - x\| < \frac{\tilde{r}}{c} = r \Rightarrow y \in B^\circ_{|\cdot|}(x, r)$.

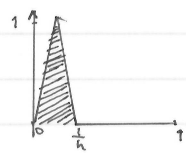
② The open sets w.r.t. $|\cdot|$ are identical to the open sets w.r.t. $\|\cdot\|$, $\because \forall x \in G$ open, $\exists r$ s.t. $x \in B^\circ_{\|\cdot\|}(x,r)$, but $\exists \tilde{r} > 0$ s.t. $B^\circ_{|\cdot|}(x, \tilde{r}) \subset B^\circ_{\|\cdot\|}(x, r)$.

e.g. ① $\mathbb{R}^n$: $\|x\|_\infty = \max_{1 \le i \le n} |x_i|$, $\|x\|_1 = |x_1| + \dots + |x_n|$. Show that $\|\cdot\|_\infty$ and $\|\cdot\|_1$ are equivalent. $\|x\|_1 \le n \cdot \|x\|_\infty \Rightarrow \|x\|_\infty \le \|x\|_1 \le n\|x\|_\infty$.

Hence, these norms are equivalent.

② Take $C[0,1]$: $\|f\|_{sup} = \sup_{[0,1]} |f(x)|$, $\|f\|_1 = \int_0^1 |f(x)|\,dx$. $\|f\|_1 \le \int_0^1 \|f\|_{sup}\,dx = \|f\|_{sup}$. However, there is no constant c s.t. we have $\|f\|_{sup} \le C\|f\|_1$. Suppose such a constant exists, define f_n as in the graph, $f_n \in C[0,1]$.

then $\|f_n\|_{sup} \le C\|f_n\|_1 \Rightarrow 1 \le c \cdot \frac{1}{2n} \to 0$ as $n \to \infty \Rightarrow$ contradiction



Theorem 3.15 $|\cdot|$ and $\|\cdot\|$ are equivalent $\iff$ $x_n \to x$ w.r.t. $|\cdot|$ and $\|\cdot\|$ simultaneously.

Proof - ($\Rightarrow$) Suppose $x_n \to x$ w.r.t. $|\cdot|$. Then $\|x_n - x\| \le C|x_n - x| \to 0$. i.e. $\|x_n - x\| \to 0$, $x_n \to x$ w.r.t. $\|\cdot\|$.

($\Leftarrow$). Suppose RHS is true but LHS is false, i.e. $|\cdot|$ and $\|\cdot\|$ are not equivalent $\Rightarrow$ one of the inequalities is not true. WLOG, state that $\nexists c$ constant s.t. $\|x\| \le C|x| \ \forall x$. Hence, $\forall n \ \exists x_n$ s.t. $\|x_n\| > n|x_n|$. Define $y_n = \frac{x_n}{\|x_n\|}$. Then $|y_n| = \left|\frac{x_n}{\|x_n\|}\right| = \frac{|x_n|}{\|x_n\|} < \frac{1}{n} \to 0 \Rightarrow y_n \to 0$ w.r.t. $|\cdot|$. Then $\|y_n\| = \left\|\frac{x_n}{\|x_n\|}\right\| = \frac{\|x_n\|}{\|x_n\|} = 1 \not\to 0. \Rightarrow y_n \not\to 0$ w.r.t. $\|\cdot\|$. This is a contradiction // q.e.d.

Exercise: Show that $|\cdot|$, $\|\cdot\|$ are equivalent $\iff$ they have the same collection of open sets.

Theorem 3.16 All norms on $\mathbb{R}^n$ are equivalent.

Comment: This is not true for metrics: e.g. $\mathbb{R}^d$, $d(x,y) = \begin{cases} 1 & x = y \\ 0 & x \ne y. \end{cases}$

Proof - We will prove that any norm $\|\cdot\|$ is equivalent to $\|\cdot\|_2$ $(\|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$. Let $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, e_n = \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}$. Let $x = x_1 e_1 + \dots + x_n e_n$.

Then $\|x\| = \|x_1 e_1 + \dots + x_n e_n\| \le \|x_1 e_1\| + \dots + \|x_n e_n\| = |x_1|\|e_1\| + \dots + |x_n|\|e_n\| \le \sqrt{x_1^2 + \dots + x_n^2}\sqrt{\|e_1\|^2 + \dots + \|e_n\|^2}$ by Cauchy-Schwarz inequality.

i.e. $\|x\| \le C \cdot \|x\|_2$ where $C = \sqrt{\|e_1\|^2 + \dots + \|e_n\|^2}$. For the other direction, take $f: \underset{\|\cdot\|_2}{\mathbb{R}^n} \to \mathbb{R}$ defined by $f(x) = \|x\|$.

Let $K = \{x \in \mathbb{R}^n : \|x\|_2 = 1\}$. Then K is bounded: $\|x - y\|_2 \le \|x\|_2 + \|y\|_2 = 1 + 1 = 2 \Rightarrow \mathrm{diam}(K) = 2$.

Also, K is closed: $K = B(0,1) \setminus B^\circ(0,1) = \underset{\text{closed}}{B(0,1)} \cap \underset{\text{closed}}{(\mathbb{R}^n \setminus B^\circ(0,1))} \Rightarrow K$ is closed. $\Rightarrow$ in $\mathbb{R}^n$, closed + bounded implies compact.

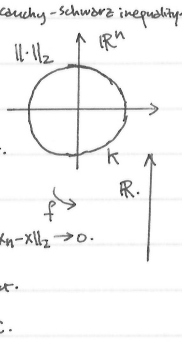
Hence, K is compact. Also, we claim f is continuous: take $x_n \to x$, NTP: $f(x_n) \to f(x)$. Then $|f(x_n) - f(x)| = |\|x_n\| - \|x\||$.

We use the fact that $\|\cdot\|$ is a norm to apply triangle inequality: $|\|x_n\| - \|x\|| = |\|x_n - x + x\| - \|x\|| \le |\|x_n - x\|| = \|x_n - x\| \le C\|x_n - x\|_2 \to 0$.

Thus, f is continuous. $\because f(x_n) \to f(x)$. Hence, by Theorem 3.14, f attains its infimum on K, as f is continuous and K is compact.

Denote $c = \inf_K f$. Then $\exists x_0 \in K$ s.t. $c = f(x_0) = \|x_0\| > 0$. Let $x \in \mathbb{R}^n$, $x \ne 0$. $\Rightarrow f\left(\frac{x}{\|x\|_2}\right) \ge c$ as $\frac{x}{\|x\|_2} \in K$. $\Rightarrow \left\|\frac{x}{\|x\|_2}\right\| \ge c$.

$\Rightarrow \frac{\|x\|}{\|x\|_2} \ge c \Rightarrow \|x\| \ge c\|x\|_2$. Hence, $c\|x\|_2 \le \|x\| \le C\|x\|_2 \Rightarrow \|\cdot\|$ and $\|\cdot\|_2$ are equivalent // q.e.d.



END OF SYLLABUS.

Some words on the exam:

Non-examinable content: equivalent norms, Picard's Theorem.

6 questions: 2 on uniform convergence, 1 on Fourier series, 3 on metric spaces. Similar to previous years.

Each question involves bookwork + problems to solve. Focus on homework and examples discussed in class.