7102 Analysis 4: Real Analysis Notes

Based on the 2017 spring lectures by Dr N Sidorova

MATH 7102

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Chapter 1

$\underbrace{a_n}_{\text{numbers}} \longrightarrow \underbrace{a}_{\text{number}}$

$\underbrace{f_n}_{\text{functions}} \longrightarrow \underbrace{f}_{\text{function}}$

- pointwise convergence (bad) (nice functions $\longrightarrow$ something horrid)
- uniform convergence (good) (preserves continuity, integrability, other nice properties)
- convergence w.r.t. $\|\cdot\|_1$ - norm
  $\|\cdot\|_2$ - norm
  $\|\cdot\|_\infty$ - norm
  $\|\cdot\|_q$ - norm

Highlights

- construct a continuous function which is not differentiable anywhere
- every continuous on $[a,b]$ can be approximated uniformly by polynomials

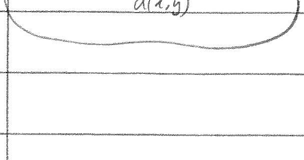
Chapter 2

Fourier Series

- justify the calculations from Methods 3
- replace $\{1, \sin(nx), \cos(nx)\}$ by orthonormal systems $\{\varphi_n(x)\}$ to generalise trig. F.S.

Chapter 3

Metric Spaces



$d(x,y) \leftarrow$ distance

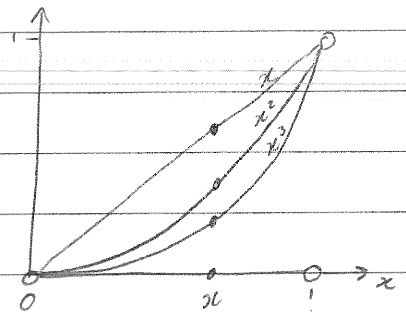
Chapter 1: Uniform Convergence

Def

Let $I \subset \mathbb{R}$ and let $\{f_n\}_{n=1}^{\infty}$ be ~~a sequence of~~ functions from I to $\mathbb{R}$. We say that $f_n \to f$ pointwise on I if $\forall x \in I$ $f_n(x) \to f(x)$ as $n \to \infty$.

Example

$I = (0,1)$, $f_n(x) = x^n$



$f_1(x) = x$

$f_2(x) = x^2$

$f_3(x) = x^3$

$f_n(x) = x^n \to 0 \quad \forall x \in (0,1)$

$f_n \longrightarrow f$ pointwise to $f(x) = 0$

Def

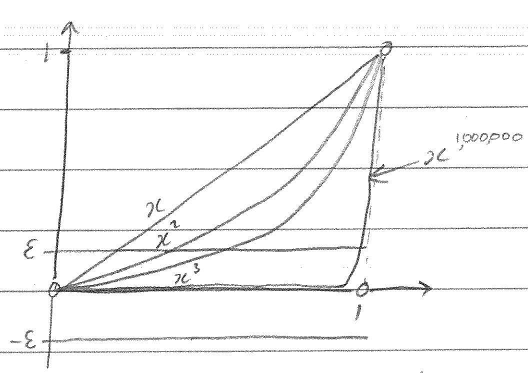
Let $I \subset \mathbb{R}$ and let $\{f_n\}_{n=1}^{\infty}$, f be functions from I to $\mathbb{R}$. We say that $f_n \to f$ uniformly on I if

$\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $\forall x \in I$, $|f_n(x) - f(x)| < \varepsilon$.

Example (as above)

$I = (0,1)$, $f_n(x) = x^n$, $f(x) \equiv 0$.

? : Does $f_n \to f$ uniformly?



Not a single f_n is in the ε-tube around the zero function.

$\Rightarrow f_n$ does not converge to the zero function uniformly!

No uniform convergence:

$\exists \varepsilon > 0$ s.t. $\forall N \ \exists n \geq N$ and $\exists x \in I$ s.t. $|f_n(x) - f(x)| \geq \varepsilon$

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Theorem 1.1

If $f_n \to f$ uniformly then $f_n \to f$ pointwise.

Proof

$f_n \to f$ pointwise $\iff$ $\forall x \in I$ $\forall \varepsilon > 0$, $\exists N$ s.t. $\forall n \geq N$ $|f_n(x) - f(x)| < \varepsilon$.

$f_n \to f$ uniformly $\iff$ $\forall \varepsilon > 0$ $\exists N$ s.t. $\forall n \geq N$ $\forall x \in I$ $|f_n(x) - f(x)| < \varepsilon$.

So uniform convergence $\Rightarrow$ pointwise convergence by definition. □

? : (i) Given $\{f_n\}_{n=1}^{\infty}$ (i) Does f_n converge pointwise?

(ii) If yes, what is the limit function?

(iii) Is the convergence uniform?

(i) fix x, see if $f_n(x)$ converges

(ii) the limit function from (i) will give you the limit function

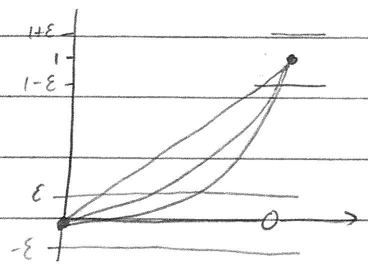
(iii) use the definition or its negation.

Examples (Three questions as above per example)

1). $I = [0,1]$, $f_n(x) = x^n$

$$\left.\begin{array}{ll} x \in (0,1) & f_n(x) = x^n \to 0 \\ x = 0 & f_n(0) = 0 \to 0 \\ x = 1 & f_n(1) = 1 \to 1 \end{array}\right\} f_n \text{ converges pointwise to } f(x) = \begin{cases} 0, & x \in [0,1) \\ 1, & x = 1. \end{cases}$$

Let's prove the convergence is not uniform (using the negation of uniform convergence).



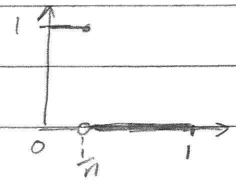
Let $\varepsilon = \frac{1}{2}$, let $N \in \mathbb{N}$.

Choose $n = N$ (works for most examples) and choose x to be the solution of $x^n = \varepsilon = \frac{1}{2}$.

Then $|f_n(x) - f(x)| = |x^n - 0| = \frac{1}{2} \geq \varepsilon$

2). $I = [0,1]$

$$f_n(x) = \begin{cases} 1 & , x \in [0, \frac{1}{n}] \\ 0 & , x \in (\frac{1}{n}, 1] \end{cases}$$



$x = 0$: $f_n(x) = 1 \to 1$

$x \in (0,1]$: $f_n(x) \longrightarrow 0$ as we have sequence $1, 1, 1, 1, \ldots, 1, 0, 0, \ldots$ forever $\to$

$\Rightarrow f_n$ converges pointwise to

$$f(x) = \begin{cases} 1 & , x = 0 \\ 0 & , x \neq 0. \end{cases}$$

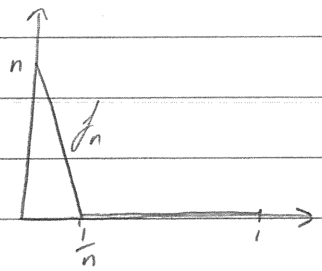
The convergence is not uniform.

Let $\varepsilon = \frac{1}{3}$, let $N \in \mathbb{N}$.

Choose $n = N$ and $x = 1/2n$ (or anything from $[0, \frac{1}{n}]$

$$|f_n(x) - f(x)| = |\underbrace{f_n(\tfrac{1}{2n})}_{=1} - 0| = 1 \geq \frac{1}{3} = \varepsilon$$

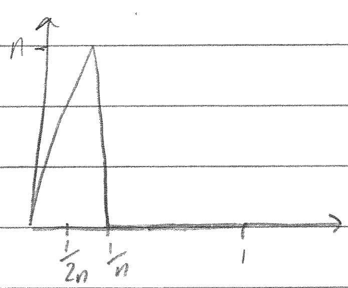
3). $I = [0,1]$ $\quad f_n(x) = \begin{cases} -n^2 x + n & , x \in [0, \frac{1}{n}) \\ 0 & , x \in [1/n, 1] \end{cases}$



$x = 0$ $\quad f_n(0) = n \to \infty$ doesn't converge

$\Rightarrow$ no pointwise convergence

4). $I = [0,1]$



$$f_n(x) \begin{cases} 2n^2 x & \text{if } x \in [0, \frac{1}{2n}] \\ -2n^2 x + 2n & \text{if } x \in (\frac{1}{2n}, \frac{1}{n}) \\ 0 & \text{if } x \in [\frac{1}{n}, 1] \end{cases}$$

$x = 0$, $f_n(0) = 0 \to 0$

$x \in (0,1]$, $f_n(x) \to 0$ as we have sequence $*, *, *, *, \ldots, *, 0, 0, 0, \ldots$

So f_n converges pointwise to the zero function.

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Q4 cont.

f_n doesn't converge to the zero function uniformly.
Choose $\varepsilon = \frac{1}{2}$, let $N \in \mathbb{N}$
choose $n = N$ and $x = 1/2n$
$|f_n(x) - f(x)| = |n - 0| = N \geq 1 \geq \frac{1}{2} = \varepsilon$

note: if the top of the point was not n but $\to 0$ then convergence would be uniform.

5). $I = (0, \infty)$, $f_n(x) = \dfrac{1}{x+n}$

$$\forall x \in (0,\infty) \quad f_n(x) = \frac{1}{\underset{\text{fixed}}{x} + \underset{\text{tends to } \infty}{n}} \longrightarrow 0$$

$\Rightarrow$ f_n converges pointwise to the zero function

f_n converges to the zero function uniformly since
$\forall \varepsilon > 0 \quad \exists N$ (namely, any $N > \frac{1}{\varepsilon}$) so that
$\forall n \geq N \quad \forall x \in (0,\infty)$, $|f_n(x) - f(x)| = \dfrac{1}{n+x} \leq \dfrac{1}{n} \leq \dfrac{1}{N} < \varepsilon$

6). $I = (0, \infty)$, $f_n(x) = \dfrac{xn}{1+x+n}$

$$\forall x \in (0,\infty) \quad f_n(x) = \frac{\overset{\text{fixed}}{x}\,\overset{\to\infty}{n}}{\underset{\text{fixed}}{1+x} + \underset{\to\infty}{n}} \longrightarrow x$$

$\Rightarrow$ f_n converges pointwise for $f(x) = x$

$$|f_n(x) - f(x)| = \left|\frac{nx}{1+x+n} - x\right| = \left|\frac{nx - x - x^2 - nx}{1+x+n}\right| = \left|\frac{x^2+x}{1+x+n}\right| \overset{?}{<} \varepsilon$$

Non uniform convergence.
Let $\varepsilon = 1/3$, let N be given.
Choose $n = N$ and $x = n$
Then $|f_n(x) - f(x)| = \left|\dfrac{x^2+x}{1+x+n}\right| \geq \dfrac{x^2}{1+x+n} \geq \dfrac{x^2}{3x} \geq \dfrac{x}{3} \geq \dfrac{1}{3} = \varepsilon.$

?: If all f_n are continuous, and $f_n \to f$ pointwise does this imply that f is continuous?

No: Example 1: $f_n(x) = x^n$ on $[0,1]$

$$f(x) = \begin{cases} 0 & x \neq 1 \\ 1 & x = 1 \end{cases} \quad \leftarrow \text{counterexample.}$$

Theorem 1.2

Let $\{f_n\}_{n=1}^{\infty}$, $f : [a,b] \mapsto R$.

If ⓐ all f_n are continuous on $[a,b]$

ⓑ $f_n \to f$ uniformly on $[a,b]$,

then f is continuous on $[a,b]$

Proof

Fix $x \in [a,b]$ and prove that f is continuous at x.

We need to show: $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. if $|y - x| < \delta$ then $|f(y) - f(x)| < \varepsilon$.

Let $\varepsilon > 0$ be fixed. Since $f_n \to f$ uniformly,

$\exists N \ \forall n \geq N \ \forall z \in [a,b] \ |f_n(z) - f(z)| < \frac{\varepsilon}{3}$.

In particular $\forall z \in [a,b] \ |f_N(z) - f(z)| < \varepsilon/3$.

Since f_N is continuous at $x \Rightarrow \exists \delta > 0$ s.t. if $|y - x| < \delta$, then $|f_N(y) - f_N(x)| < \varepsilon/3$.

Combining these gives:

$$\begin{aligned} |f(y) - f(x)| &= |f(y) - f_N(y) + f_N(y) - f_N(x) + f_N(x) - f(x)| \\ &\leq |f(y) - f_N(y)| + |f_N(y) - f_N(x)| + |f_N(x) - f(x)| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon \end{aligned}$$

□

Remark

$$\underset{\text{cont.}}{f_n} \to \underset{\text{discont.}}{f} \text{ pointwise} \quad \Rightarrow \text{the conv. is not uniform.}$$

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Compactness in $\mathbb{R}$

$(a,b) = \underset{a}{\circ}\!\!-\!\!-\!\!\underset{b}{\circ}$

Collections of $\overbrace{\text{open intervals}}$:

(a) finite collection: $I_1, I_2, \dots, I_n \; [= (a_1,b_1), (a_2,b_2), \dots, (a_n,b_n)]$

(b) countable collection: $I_1, I_2, \dots = \{I_n\}_{n\in\mathbb{N}}$

(c) arbitrary collection $\{I_\alpha\}_{\alpha\in A}$ (A is a set) $\leftarrow$ general set up

Example $\{(x-1, x+1)\}_{x\in\mathbb{R}}$

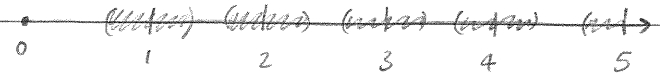
= the collection of all open intervals of length 2.

Def:

Let $S \subset \mathbb{R}$. A collection $\{I_\alpha\}_{\alpha\in A}$ is a cover of S if $S \subset \bigcup_{\alpha\in A} I_\alpha$.

Examples

$\{I_n\}_{n\in\mathbb{N}}$, where $I_n = (n-\frac{1}{3}, n+\frac{1}{3})$



For the following sets S, is $\{I_n\}_{n\in\mathbb{N}}$ a cover of S?

- $\mathbb{N}$ $\leftarrow$ yes
- $\mathbb{Z}$ $\leftarrow$ no
- $(1,2) \cup \{5\}$ $\leftarrow$ no
- $[1.9, 2.1]$ $\leftarrow$ yes

Def:

Let $S \subset \mathbb{R}$ and let $\{I_\alpha\}_{\alpha\in A}$ be a cover of S.
A finite subcollection $\{I_{\alpha_1}, \dots, I_{\alpha_n}\}$ is a finite subcover if $S \subset \bigcup_{i=1}^{n} I_{\alpha_i}$.

Examples

1). $I_n = (n - \frac{1}{3}, n + \frac{1}{3})$

(a) $S = \mathbb{N}$

take $\{I_2, I_4, I_{10}\}$

- this is not a finite subcover of $\mathbb{N}$.

• Is there any finite subcover? - no!

(b) $S = (3.9, 4.1)$

• Is there a finite subcover for this set? yes: I_4

or I_4 with any finite number of intervals, ie. $\{I_4, I_6, I_{100}\}$

2). $I_n = (-1 + \frac{1}{n}, 1 - \frac{1}{n})$, $n = 2, 3, \ldots$

(a) Is this a cover for $[-1, 1]$? no, $\bigcup_{n=2}^{\infty} (-1 + \frac{1}{n}, 1 - \frac{1}{n}) = (-1, 1)$

(b) • Is this a cover for $(-1, 1)$? yes → so $(-1, 1) \subset \bigcup_{n=2}^{\infty} (-1 + \frac{1}{n}, 1 - \frac{1}{n})$

• Is there a finite subcover? no

(c) $S = [-0.9, 0.9]$

• Is this a cover for S? yes

• Is there a finite subcover? yes, we need $n > 10$, we could choose I_{11}.

Def:

A set $S \subset \mathbb{R}$ is called compact in $\mathbb{R}$ if any cover of S has a finite subcover.

Examples: Is the set S a compact set?

1). $S = \{a\}$

Suppose $\{I_\alpha\}_{\alpha \in A}$ is a cover of $\{a\}$.

$\Rightarrow a \in$ some I_α. Pick $\{I_\alpha\}$ (collection consisting of one set)

this is a finite subcover, so S is compact.

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2). $S = \{a_1, \dots, a_n\}$

this is compact by the same argument.

3). $\mathbb{N}$ is not compact since $\{(n-\frac{1}{3}, n+\frac{1}{3})\}_{n\in\mathbb{N}}$ has no finite subcover.

4). $S = (3.9, 4.1)$

Previous example (1b) doesn't give an answer.

(2b) suggests that S is not compact:

Consider the cover $\{(3.9+\frac{1}{n}, 4.1-\frac{1}{n})\}_{n=11,12,\dots}$.

It has no finite subcover $\Rightarrow$ $(3.9, 4.1)$ is not a compact set.

5). Any unbounded set is not compact (homework).

4*). Any interval (a,b) is not compact.

Similarly $(a,b]$ is also not compact.

Theorem 1.3 (Heine - Borel Theorem)

Any closed interval $[a,b]$ is compact in $\mathbb{R}$.

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A set $S \subset \mathbb{R}$ is compact if every cover of S has a finite subcover. (finite sets, eg. $[a,b]$).

A set $S \subset \mathbb{R}$ is not compact if there is a cover that has no finite subcover. (eg. $\mathbb{N}$, (a,b), $(a,b]$, $[a,\infty)$ etc).

Theorem 1.3 (Heine - Borel)

$[a,b]$ is compact.

Proof

Let $\{I_\alpha\}_{\alpha \in A}$ be a cover of $[a,b]$.

$B = \{x \in [a,b] : [a,x] \text{ has a finite subcover}\}$

It is enough to show $b \in B$

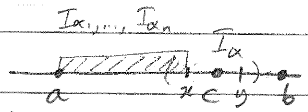
① $\sup B = b$

② $b \in B$

$B \neq \emptyset$ since $a \in B$

Let $c = \sup B$.

Claim 1 : $c = b$



Suppose $c \neq b \Rightarrow c < b$.

Let I_α be an interval, from the cover, covering c.

Since $c = \sup B$ there is $x \in B \cap I_\alpha$.

Since $x \in B$ there are finitely many $I_{\alpha_1}, \dots, I_{\alpha_n}$ covering $[a,x]$.

Let $y \in I_\alpha$ and $y > c$.

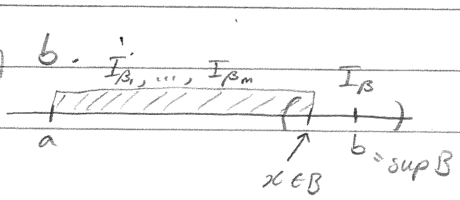
The interval $[a,y]$ is covered by $I_{\alpha_1}, \dots, I_{\alpha_n}, I_\alpha$

$\Rightarrow y \in B$ which contradicts $c = \sup B$ as $y > c$

$\Rightarrow c = b$.

Claim 2 : $b \in B$

Let I_β be an interval covering b.

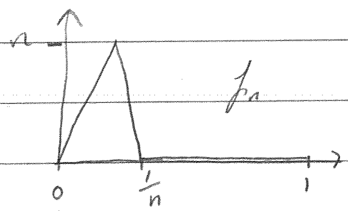


Let I_β be an interval covering b. Since $b = \sup B$,
$\exists x \in I_\beta \cap B$.
Since $x \in B$ there are finitely many intervals
$I_{\beta_1}, \dots, I_{\beta_m}$ covering $[a, x]$.
Hence $I_{\beta_1}, \dots, I_{\beta_m}, I_\beta$ covers $[a,b] \Rightarrow b \in B$. $\square$

We proved: $\left\{ \begin{array}{l} f_n \to f \text{ uniformly} \\ \text{all } f_n \text{ are continuous} \end{array} \right. \Rightarrow f$ is continuous.

? : $\left\{ \begin{array}{l} f_n \to f \text{ pointwise} \\ \text{all } f_n \text{ are continuous} \\ f \text{ is continuous} \end{array} \right. \; \cancel{\overset{?}{\Rightarrow}} \; f_n \to f$ uniformly
no!

Example



$\left\{ \begin{array}{l} f_n \to 0 \text{ pointwise} \\ \text{each } f_n \text{ is continuous} \\ \text{the zero function is continuous} \end{array} \right.$
but convergence is not uniform.

Theorem 1.4 (Dini)
Let $f_n, f : [a,b] \to \mathbb{R}$.
Suppose 1). $f_n \to f$ pointwise on $[a,b]$,
2). all f_n are continuous,
3). f is continuous,
4). $\forall x \in [a,b]$, $\{f_n(x)\}_{n=1}^{\infty}$ is monotone.
Then $f_n \to f$ uniformly on $[a,b]$

Remark
It is important in Dini's Thm that the domain is a closed interval.

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Example

$x \in [0,1)$, $f_n(x) = x^n$; $f(x) = 0$

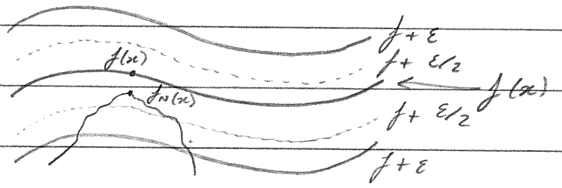
- $f_n \to 0$ pointwise
- f_n are continuous
- zero function is continuous
- x^n is decreasing to 0 $\forall x \in [0,1)$ as $n \to \infty$.

But $f_n \not\to 0$ uniformly as $x \in [0,1)$ $\leftarrow$ open at one side!

Proof:

We want to show: $\forall \varepsilon > 0 \ \exists N \ \forall n \geq N \ \forall x \in [a,b]$, $|f_n(x) - f(x)| < \varepsilon$.

Let $\varepsilon > 0$.

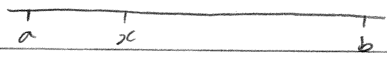


Step 1.

Fix $x \in [a,b]$, since $f_n(x) \to f(x)$

due to pointwise convergence

$\exists N(x)$ s.t. $|f_{N(x)}(x) - f(x)| < \varepsilon/2$ $(*)$



Step 2: Consider $g(y) = f_{N(x)}(y) - f(y)$

$g(y)$ is continuous by assumptions 2 and 3.

$\Rightarrow \exists \delta(x)$ s.t. if $y - x < \delta(x)$, $y \in [a,b]$, then $|g(y) - g(x)| < \frac{\varepsilon}{2}$ $(**)$

$\Rightarrow$ If $y \in (x - \delta x, x + \delta x) \cap [a,b]$ then

$$|f_{N(x)}(y) - f(y)| = |g(y)| \leq \underbrace{|g(x)|}_{(*)} + \underbrace{|g(y) - g(x)|}_{(**)} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

So $\exists \delta(x)$ s.t. $|f_{N(x)}(y) - f(y)| < \varepsilon$

$\forall y \in I(x) = (x - \delta(x), x + \delta(x))$.

Step 3

By 4, $|f_n(y) - f(y)| < \varepsilon$ $\forall y \in I(x)$ and $\forall n \geq N$ $(***)$

Step 4

$\{I(x)\}_{x \in [a,b]}$ is a cover of $[a,b]$.

Since $[a,b]$ is a compact set, there is a finite subcover

$I(x_1), \dots, I(x_m)$.

Choose $N = \max\{N(x_1), \dots, N(x_m)\}$

Let $n \geq N$ and $x \in [a,b]$.

Since $I(x_1), \dots, I(x_m)$ is a subcover, $x \in I(x_i)$ for some $1 \leq i \leq m$.

We know $n \geq N \geq N(x_i)$

$(***) \Rightarrow |f_n(x) - f(x)| < \varepsilon$. $\square$

__Exercise__

Look at the last example: $[0,1)$, $f_n(x) = x^n$, $f(x) = 0$; and see how the proof breaks down.

__Def__

Let $f: I \to R$

$\|f\|_{sup} = \sup_{x \in I} |f(x)|$ is the __supremum-norm__ of f.

__Examples__

(i) $I = \mathbb{R}$, $f(x) = 3\sin x$ $\|f\|_{sup} = 3$

(ii) $I = [0,1]$, $f(x) = -2x$ $\|f\|_{sup} = 2$

__Theorem 1.5__

$f_n \to f$ uniformly $\Leftrightarrow \|f_n - f\|_{sup} \to 0$.

__Proof__

$f_n \to f$ uniformly $\Leftrightarrow \forall \varepsilon > 0\ \exists N\ \forall n \geq N\ \forall x \in I$, $|f_n(x) - f(x)| < \varepsilon$

? $\|f_n - f\|_{sup} \to 0 \Leftrightarrow \forall \varepsilon > 0\ \exists N\ \forall n \geq N\ \forall x \in I$ $\sup_{x \in I} |f_n(x) - f(x)| < \varepsilon$

i.e. $|f_n(x) - f(x)| \leq \varepsilon$. equivalent

$\square$

__Examples__

1). $x \in [0,1)$, $f_n(x) = x^n$, $f_n \to 0$ pointwise but not uniformly

$\|f_n - f\|_{sup} = \sup_{x \in [0,1)} x^n = 1 \not\to 0$, so no uniform convergence

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2). $x \in [0,\infty)$, $f_n(x) = \begin{cases} 0, & 0 \le x \le n \\ 1, & x > n \end{cases}$

$f_n \to 0$ pointwise.

$\|f_n - 0\|_{sup} = \sup_{x\in[0,\infty)} f_n = 1 \not\to 0.$

$\Rightarrow$ convergence is not uniform.

Theorem 1.6

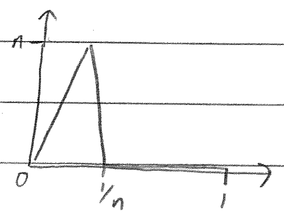
Let $f_n, f : [a,b] \mapsto \mathbb{R}$.

Suppose 1). $f_n \to f$ uniformly,
2). all f_n are Riemann-integrable on $[a,b]$.

Then f is Riemann-integrable and $\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx$

?: $\begin{cases} f_n \to f \text{ pointwise} \\ \text{all } f_n \text{ are Riemann integrable} \end{cases}$ $\overset{?}{\cancel{\Rightarrow}}$ no! $\begin{cases} f \text{ is integrable} \\ \int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx \end{cases}$

Example



$f_n \to 0$ pointwise & $f_n, 0$ are integrable,
but $\int_0^1 f_n(x)\,dx = \frac{1}{2} \not\to 0 = \int_0^1 0\,dx$

Hwk: Q5, limit function not integrable!
Q4, on $[0,\infty)$ the theorem is no longer true.

Proof

f is Riemann integrable $\Leftrightarrow$ $\forall \varepsilon > 0$ $\exists$ partition $P = \{a = t_0 < \dots < t_m = b\}$ s.t. $U(f,P) - L(f,P) < \varepsilon$.

Recall: $U(f,P) = \sum_{i=1}^m \sup_{x\in[t_{i-1},t_i]}(f(x))\,(t_i - t_{i-1})$, $L(f,P) = \sum_{i=1}^m \inf_{x\in[t_{i-1},t_i]}(f(x))(t_i - t_{i-1})$.

Let $\varepsilon > 0$.

Since $f_n \to f$ uniformly $\exists n$ s.t. $\|f_n - f\|_{sup} < \frac{\varepsilon}{4(b-a)}$ $(*)$

Since f_n is Riemann-integrable $\exists P$ st. $U(f_n, P) - L(f_n, P) < \varepsilon/2$

From $(*)$ $f_n(x) - \frac{\varepsilon}{4(b-a)} < f(x) < f_n(x) + \frac{\varepsilon}{4(b-a)}$

$$\sup_{x\in[t_{i-1},t_i]} f(x) \leq \sup_{x\in[t_{i-1},t_i]} f_n(x) + \frac{\varepsilon}{4(b-a)}.$$

$$\inf_{x\in[t_{i-1},t_i]} f(x) \geq \inf_{x\in[t_{i-1},t_i]} f_n(x) - \frac{\varepsilon}{4(b-a)}.$$

$$\begin{aligned} U(f,P) - L(f,P) &\leq \sum_{i=1}^{m} \sup_{x\in[t_{i-1},t_i]}(f_n(x))(t_i - t_{i-1}) + \frac{\varepsilon}{4(b-a)}\sum_{i=1}^{m}(t_i - t_{i-1}) \\ &\quad - \sum_{i=1}^{m} \inf_{x\in[t_{i-1},t_i]}(f_n(x))(t_i - t_{i-1}) + \frac{\varepsilon}{4(b-a)}\sum_{i=1}^{m}(t_i - t_{i-1}) \\ &= U(f_n,P) - L(f_n,P) + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\ &< \varepsilon/2 + \varepsilon/4 + \varepsilon/4 = \varepsilon \end{aligned}$$

[full proof next lecture]

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Theorem 1.6

$f_n, f : [a,b] \mapsto \mathbb{R}$

$\left.\begin{array}{l} f_n \to f \text{ uniformly} \\ \text{all } f_n \text{ are Riemann integrable} \end{array}\right\} \Rightarrow f$ is Riemann integrable and $\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx$

Proof

Aim: $\forall \varepsilon > 0 \ \exists P$ s.t. $U(f,P) - L(f,P) < \varepsilon$.

Let $\varepsilon > 0$.

① Since $f_n \to f$ uniformly choose f_n so that

$$\|f_n - f\|_{sup} < \frac{\varepsilon}{4(b-a)}$$

② Since f_n is integrable choose P so that

$$U(f_n,P) - L(f_n,P) < \varepsilon/2$$

③ Combine ① and ②:

$$U(f,P) - L(f,P) \leq U(f_n,P) - L(f_n,P) + \varepsilon/2 < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Let us prove $\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx$

$$\left|\int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx\right|$$

$$= \left|\int_a^b \big(f_n(x) - f(x)\big)\,dx\right|$$

$$\leq \int_a^b \underbrace{\left|f_n(x) - f(x)\right|}_{\leq \|f_n - f\|_{sup} \ \leftarrow \text{number}}\,dx$$

$$\leq \|f_n(x) - f(x)\|_{sup}(b-a) \to 0 \text{ since } f_n \to f \text{ uniformly.}$$

$\square$

$\left.\begin{array}{l} f_n \to f \text{ unif.} \\ \text{all } f_n \text{ are continuous} \\ \quad \text{or integrable} \end{array}\right\} \Rightarrow f$ is continuous or integrable with $\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx$

Does this work for differentiability? No.

Example

$\mathbb{R}$, $f_n(x) = \frac{1}{n}\sin(n^2 x)$

$$\|f_n - \underset{\substack{\uparrow \\ 0}}{f}\| = \frac{1}{n} \to 0$$

$\Rightarrow f_n$ converges to f uniformly

But $f_n'(x) = n\cos(n^2 x)$

- no uniform or pointwise convergence as $f_n'(0) \to \infty$.

Example

$[-1,1]$, $f_n(x) = |x|^{1+\frac{1}{n}}$ converges pointwise to $f(x) = |x|$

$f_n'(x)$ exists but $f(x)$ is not differentiable (at 0).
$f_n(x) \to f(x)$ uniformly by Dini's theorem, but
$f_n'(x) \not\to f'(x)$ as $f'(0)$ does not exist.

Def

Let $I \subset \mathbb{R}$ and $g_n : I \to \mathbb{R}$, $n \in \mathbb{N}$ (or $\mathbb{N}_0$).
Let $f : I \to \mathbb{R}$.
We say that the series $\sum_{n=1}^{\infty} g_n$ converges pointwise to f (f is called the sum of the series) if
$S_n \to f$ pointwise where
$S_n(x) = \sum_{i=1}^{n} g_i(x)$.

Def

We say that $\sum_{n=1}^{\infty} g_n$ converges to f uniformly if
$S_n \to f$ uniformly on I.

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Example

$\sum_{n=1}^{\infty} x^n$ (ie. $g_n(x) = x^n$) on $[-\frac{1}{2}, \frac{1}{2}]$ or $(-1, 1)$.

?: Does this series converge pointwise / uniformly?

$$S_n(x) = \sum_{i=1}^{n} x^i = x\left(\frac{x^n - 1}{x-1}\right) \xrightarrow{\text{pointwise}} \frac{x}{1-x} = f(x)$$

On both domains the series converges to $\frac{x}{1-x}$ pointwise

$$|S_n(x) - f(x)| = \left| x\left(\frac{x^n-1}{x-1}\right) - \frac{x}{x-1} \right|$$
$$= \frac{|x^n|}{1-x}$$

ⓐ on $[-\frac{1}{2}, \frac{1}{2}]$, $\|S_n - f\|_{sup} = \sup_{[-\frac{1}{2},\frac{1}{2}]} \frac{|x^n|}{1-x} \leq \frac{(\frac{1}{2})^n}{1/2} \to 0$

On this domain the convergence is uniform

ⓑ on $(-1, 1)$, $\|S_n - f\|_{sup} = \sup_{(-1,1)} \frac{|x^n|}{1-x} = \infty \not\to 0$

So on this domain, the convergence is <u>not</u> uniform.

ⓒ $[-r, r]$ where $r \in (0, 1)$
Convergence is uniform.

ⓓ What about $x = \pm 1$?
No pointwise convergence, so no uniform convergence, the series diverges.

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$g_n : I \mapsto \mathbb{R}$, $n = 1, 2, \ldots$

$\sum_{n=1}^{\infty} g_n$ converges pointwise / uniformly on I if
$f_n = \sum_{i=1}^{n} g_i \longrightarrow$ some f pointwise / uniformly on I.
f is called the sum of the series.

Example
$\sum_{n=1}^{\infty} x^n$ converges uniformly on $[0, r]$, $r \in (0,1)$ but only pointwise on $(0, 1)$.

Theorem 1.7
Let $g_n : [a,b] \mapsto \mathbb{R}$, $n \in \mathbb{N}$.
If ⓐ $\sum_{n=1}^{\infty} g_n$ converges uniformly
ⓑ all g_n are continuous
then the sum $f = \sum_{n=1}^{\infty} g_n$ is continuous.

Proof
Follows from the corresponding theorem for sequences.

Theorem 1.8
Let $g_n : [a,b] \mapsto \mathbb{R}$, $n \in \mathbb{N}$
If ⓐ $\sum_{n=1}^{\infty} g_n$ converges uniformly
ⓑ all g_n are Riemann-integrable on $[a,b]$
then the sum $f = \sum_{n=1}^{\infty} g_n$ is Riemann-integrable on $[a,b]$
and $\int_a^b f(x)\,dx = \sum_{n=1}^{\infty} \int_a^b g_n(x)\,dx$.

Proof
Follows from the corresponding theorem for sequences.

Def

Let $\{f_n\}_{n=1}^{\infty}$ be functions from I to $\mathbb{R}$.
It is called a uniform Cauchy sequence if
$\forall \varepsilon>0 \ \exists N\in\mathbb{N} \ \forall n,m\geq N \ \|f_n-f_m\|_{sup}<\varepsilon.$

Theorem 1.9 (Central Principle of Uniform Convergence = CPUC)
f_n converges uniformly (to some f)
$\iff (f_n)$ is a uniform Cauchy sequence.

Proof
$[\Rightarrow]$

$f_n\to f$ uniformly $\Rightarrow \forall\varepsilon>0 \ \exists N$ s.t. $\forall n\geq N, \forall x\in I$
$|f_n(x)-f(x)|<\varepsilon/4$
$\Rightarrow \forall n,m\geq N \ \forall x\in I \ |f_n(x)-f_m(x)|\leq|f_n(x)-f(x)|+|f_m(x)-f(x)|<\varepsilon/2$
so $\|f_n-f_m\|_{sup}\leq\varepsilon/2<\varepsilon.$

$[\Leftarrow]$
Let $\varepsilon>0$
(f_n) is uniform Cauchy
$\Rightarrow \exists N\in\mathbb{N} \ \forall n,m\geq N \ \|f_n-f_m\|_{sup}<\varepsilon$
Let $x\in I \Rightarrow |f_n(x)-f_m(x)|\leq\|f_n-f_m\|_{sup}<\varepsilon \ (\forall n,m\geq N)$
$\{f_n(x)\}_{n=1}^{\infty}$ a Cauchy sequence (of numbers)
$\Rightarrow$ it converges to some number $f(x)$
$\Rightarrow f_n\to f$ pointwise.
Let's prove $f_n\to f$ uniformly.
We know: $|f_n(x)-f_m(x)|<\varepsilon \ \forall n,m\geq N \ \forall x\in I$
let $m\to\infty$
$|f_n(x)-f(x)|\leq\varepsilon \ \forall n\geq N \ \forall x\in I.$ $\square$

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Theorem 1.10 (M-test)

Let $g_n : I \mapsto \mathbb{R}$, $n \in \mathbb{N}$, and let $(M_n)_{n=1}^{\infty}$ be a positive sequence (of numbers) such that

ⓐ $|g_n(x)| \leq M_n \ \forall x \in I \ \forall n \in \mathbb{N}$

ⓑ $\sum_{n=1}^{\infty} M_n < \infty$.

Then $\sum_{n=1}^{\infty} g_n$ converges uniformly.

Proof

Let $f_n(x) = \sum_{i=1}^{n} g_i(x)$ be the n-th partial sum.

$\sum_{n=1}^{\infty} g_n$ converges uniformly $\Leftrightarrow$ f_n converges uniformly

$\Leftrightarrow$ f_n is a uniform Cauchy sequence.

Let $\varepsilon > 0$

ⓑ $\Leftrightarrow \sum_{n=1}^{\infty} M_n \Leftrightarrow \left(\sum_{i=1}^{n} M_i\right)_{n=1}^{\infty}$ converges

$\Leftrightarrow \left(\sum_{i=1}^{n} M_i\right)_{n=1}^{\infty}$ is a Cauchy sequence.

$\Leftrightarrow \exists N \ \forall n > m \geq N \ \left|\sum_{i=1}^{n} M_i - \sum_{i=1}^{m} M_i\right| < \varepsilon$

$\Leftrightarrow \sum_{i=m+1}^{n} M_i < \varepsilon$

Now $|f_n(x) - f(x)| = \left|\sum_{i=1}^{n} g_i(x) - \sum_{i=1}^{m} g_i(x)\right|$

$= \left|\sum_{i=m+1}^{n} g_i(x)\right| \leq \sum_{i=m+1}^{n} |g_n(x)| \underset{ⓐ}{\leq} \sum_{i=m+1}^{n} M_i < \varepsilon$

$\Rightarrow \|f_n - f_m\|_{sup} \leq \varepsilon \Rightarrow \{f_n\}$ is a uniform Cauchy sequence. $\square$

Examples

Use the M-test to prove uniform convergence.

Use CPUC to disprove uniform convergence.

1). $\sum_{n=1}^{\infty} \frac{\sin(nx)}{2^n}$ $\quad \left|\frac{\sin nx}{2^n}\right| \leq \frac{1}{2^n}$ and $\sum_{n=1}^{\infty} \frac{1}{2^n} < \infty$

$\Rightarrow$ it converges by the M-test.

2). $\sum_{n=1}^{\infty} \frac{1}{n^2+x}$ on $[0, \infty)$ $\quad \left|\frac{1}{n^2+x}\right| \leq \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

$\Rightarrow$ it converges by the M-test.

3). $\sum_{n=1}^{\infty} x^n$ on $[0,1)$

We know it converges pointwise but not uniformly on $[0,1)$.

I want to show that the convergence is not uniform, i.e. $f_n(x) = \sum_{i=1}^{n} x^i$ doesn't converge uniformly $\underset{\text{CPUC}}{\iff}$ f_n is not uniform Cauchy.

So $\exists \varepsilon > 0 \ \forall N \ \exists n, m \geq N \ \|f_m - f_n\|_{sup} \geq \varepsilon$

Let $\varepsilon = 1/2$ $\forall N$ take $n = N+1$, $m = N$.

$\|f_n - f_m\|_{sup} = \|f_{N+1} - f_N\|_{sup} = \sup_{x \in [0,1)} |x^N| = 1 \geq \frac{1}{2} = \varepsilon$

* Hw hint: Q3:

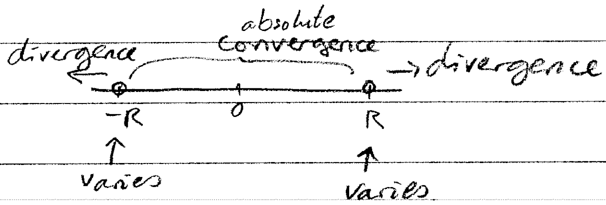
It is not enough to take $n, n+1$ but in Q1 it should be enough.

4). $\sum_{n=1}^{\infty} (\sin x)^n$ on $(0, \pi/2)$

$\|f_n - f_{n-1}\|_{sup} = \sup_{x \in (0, \pi/2)} |\sin(x)|^n = 1 \not\to 0 \underset{\text{CPUC}}{\Rightarrow}$ no uniform convergence

Power Series

$\sum_{n=0}^{\infty} a_n x^n$



We know by looking at $\sum_{n=0}^{\infty} x^n$ that we can't expect uniform convergence on $(-R, R)$.

Theorem 1.11

Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence R. For any $r \in (0, R)$ the power series converges on $[-r, r]$ uniformly.

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<u>Proof</u>:

$|a_n x^n| \leq |a_n r^n|$, $\forall x \in [-r, r]$ $\forall n$

and $\sum_{n=0}^{\infty} |a_n r^n| < \infty$

(follows from the absolute convergence of the power-series in r).

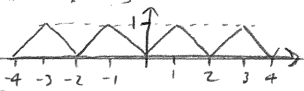
M-test $\Rightarrow \sum_{n=1} a_n x^n$ converges uniformly. $\square$

<u>Theorem 1.12</u>

There exists a continuous function on $\mathbb{R}$ which is nowhere differentiable.

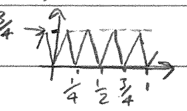
<u>Proof</u>

Let $\varphi(x) = |x|$ on $[-1, 1]$ and extended 2-periodically to $\mathbb{R}$.



$\varphi(x)$

Define $f(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \varphi(4^n x)$



$\left[\frac{3}{4}\varphi(4x)\right]$

$\left|\left(\frac{3}{4}\right)^n \varphi(4^n x)\right| \leq \left(\frac{3}{4}\right)^n \quad \forall x \in \mathbb{R}, \ \forall n \in \mathbb{R}.$

and $\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n < \infty \underset{\text{m-test}}{\Longrightarrow} \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \varphi(4^n x)$ converges uniformly.

$\Rightarrow f$ is well defined.

Since the convergence is uniform & each $\left(\frac{3}{4}\right)^n \varphi(4^n x)$ is continuous $\Rightarrow f$ is continuous.

Let's show that f is not differentiable anywhere.

Let $x \in \mathbb{R}$.

We have to show $\lim_{\delta \to 0} \frac{f(x+\delta) - f(x)}{\delta}$ doesn't exist.

We will construct a sequence (δ_m) s.t. $\delta_m \to 0$ and $\left|\frac{f(x+\delta_m) - f(x)}{\delta_m}\right| \to \infty$.

$$\delta_m = \begin{cases} +\ \frac{1}{2}4^{-m}, & \text{if there is no integer in } (4^m x, 4^m x + \frac{1}{2}) \\ -\ \frac{1}{2}4^{-m}, & \text{if the is no integer in } (4^m x - \frac{1}{2}, 4^m x) \end{cases}$$

[If $4^m x$ or $4^m x \pm \frac{1}{2}$ is an integer, we can take either sign]

$$\frac{f(x+\delta_m) - f(x)}{\delta_m} = \frac{\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \varphi(4^n(x+\delta_m)) - \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \varphi(4^n x)}{\delta_m}$$

$$= \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \underbrace{\frac{\varphi(4^n(x+\delta_m)) - \varphi(4^n x)}{\delta_m}}_{A_n \quad (m \text{ is fixed})}$$

Claim

(1) $A_n = 0 \quad \forall n > m$

(2) $|A_n| = 3^n \quad n = m$

(3) $|A_n| \leq 3^n \quad \forall n < m$

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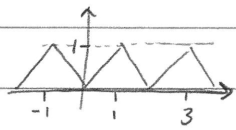
Theorem 1.12

$\exists$ a function $f: \mathbb{R} \mapsto \mathbb{R}$ which is everywhere continuous but nowhere differentiable.

Proof

$$f(x) = \sum_{n=0}^{\infty} \left(\tfrac{3}{4}\right)^n \varphi(4^n x), \text{ where } \varphi \longrightarrow$$



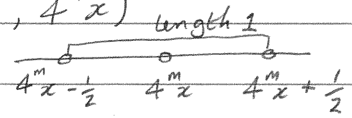
M-test $\Rightarrow$ unif. conv. $\Rightarrow$ f is well-defined and continuous.

Non-differentiability:

Let $x \in \mathbb{R}$

We will show that $\left|\dfrac{f(x+\delta_m) - f(x)}{\delta_m}\right| \to \infty$,

where $\delta_m = \begin{cases} +\frac{1}{2} 4^{-m} & \text{if no integer in } (4^m x, 4^m x + \frac{1}{2}) \\ -\frac{1}{2} 4^{-m} & \text{if no integer in } (4^m x - \frac{1}{2}, 4^m x) \end{cases}$



$$\frac{f(x+\delta_m) - f(x)}{\delta_m} = \sum_{n=0}^{\infty} \underbrace{\left(\frac{3}{4}\right)^n \frac{\varphi(4^n(x+\delta_m)) - \varphi(4^n x)}{\delta_m}}_{A_n \quad (m \text{ fixed})}$$

Claim: (a) $A_n = 0$ if $n > m$

(b) $|A_n| = 3^n$ if $n = m$

(c) $|A_n| \le 3^n$ if $n < m$

(a) $n > m$

$$\varphi(4^n(x+\delta_m)) - \varphi(4^n x) = \varphi(4^n x \pm \underbrace{\tfrac{1}{2} 4^{n-m}}_{\text{divisible by 2}}) - \varphi(4^n x)$$

$= 0$ since φ is 2-periodic.

(b) $n = m$

$$|\varphi(4^n(x+\delta_m)) - \varphi(4^n x)| = |\varphi(4^m x \pm \tfrac{1}{2}) - \varphi(4^m x)|$$

$= |\cancel{4^m x} \pm \frac{1}{2} - \cancel{4^m x}|$ by the choice of $\pm$ in δ_m

$= \frac{1}{2}$

so $|A_n| = \left(\frac{3}{4}\right)^m \cdot \frac{1}{2} \cdot 2 \cdot 4^m = 3^m = 3^n$.

ⓒ <u>$n < m$</u>

$$|\varphi(4^n(x+\delta_m)) - \varphi(4^n x)| = |\varphi(4^n x + 4^n \delta_m) - \varphi(4^n x)|$$
$$\leq |\cancel{4^n x} + 4^n \delta_m - \cancel{4^n x}|$$
$$= 4^n |\delta_m|$$

so $|A_n| \leqslant \left(\frac{3}{4}\right)^n \cdot \frac{4^n \cdot |\delta_m|}{|\delta_m|} = 3^n$

<u>Now</u> $\left|\frac{f(x+\delta_m) - f(x)}{\delta_m}\right| = |\underbrace{A_0 + \ldots + A_{m-1}} + \underbrace{A_m}|$

$\geqslant |A_m| - |A_0 + \ldots + A_{m-1}|$ by reverse triangle inequality.

$$\geqslant |A_m| - |A_0| - \ldots - |A_{m-1}|$$
$$\geqslant 3^m - 3^0 - \ldots - 3^{m-1}$$
$$= 3^m - \frac{3^m - 1}{3 - 1} \qquad (b) + (c)$$
$$= 3^m - \frac{3^m}{2} + \frac{1}{2} \longrightarrow \infty.$$

□

<u>Remark</u>

$$f(x) = \sum_{n=0}^{\infty} a^n \sin(b^n \pi x) \quad \text{for} \quad ab > 1 + \frac{3}{2}\pi$$

is also continuous but not differentiable.

(Example by Weierstrass)

It is the fluctuation of f as $n \to \infty$ that results in lack of differentiability.

<u>Theorem 1.13</u> (Weierstrass Approximation Theorem = WAT)

Let $f: [a, b] \mapsto \mathbb{R}$ be a continuous function.

Then there exists a sequence of polynomials $\{P_n\}_{n=1}^{\infty}$ such that $P_n \to f$ uniformly on $[a, b]$

Remarks:

Can we replace $[a,b]$ by something else?

Example: $f(x) = 1/x$

two domains: $(0,1]$ and $[1,\infty)$

Is there a sequence of polynomials (P_n) converging to f uniformly (on each domain)?

(a) $(0,1]$: No.

Suppose $P_n \to f$ uniformly

$$0 \leftarrow \|P_n - f\|_{sup} \geq \lim_{x\to 0} |P_n(x) - f(x)| = \infty$$

contradiction.

(b) $[1,\infty)$:

Suppose $P_n \to f$ uniformly

$$0 \leftarrow \|P_n - f\|_{sup} \geq \lim_{x\to\infty} |\underset{\substack{\downarrow \\ \infty}}{P_n(x)} - \underset{\substack{\downarrow \\ 0}}{\tfrac{1}{x}}| = \infty$$

if $P_n \neq$ const.

Hence $P_n = c_n$

$$0 \leftarrow \|P_n - f\|_{sup} \geq \lim_{x\to\infty} |c_n - \tfrac{1}{x}| = |c_n| \Rightarrow |c_n| \to 0.$$

$\Rightarrow P_n \to 0$ (enough to say "pointwise" but also uniformly)

Contradiction with $P_n \to \frac{1}{x}$ pointwise / unif.

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Theorem 1.15 (Weierstrass Approximation Theorem = WAT)
$f:[a,b] \mapsto \mathbb{R}$ continuous $\Rightarrow \exists$ a sequence of polynomials $\{p_n\}_{n=1}^{\infty}$ converging to f uniformly on $[a,b]$.

Def
$p_{nk}(x) = \binom{n}{k} x^k (1-x)^{n-k} \quad n \in \mathbb{N}, \ 0 \leq k \leq n$
For $f:[0,1] \mapsto \mathbb{R}$, $B_n^f(x) = \sum_{k=0}^{n} \underbrace{f(\tfrac{k}{n})}_{\text{number}} p_{nk}(x)$ are called Bernstein's polynomials.

Theorem 1.16 (WAT on $[0,1]$)
If $f:[0,1] \mapsto \mathbb{R}$ is continuous then $B_n^f \to f$ uniformly on $[0,1]$.

$x \in [0,1]$
Toss an unfair coin with $P(\text{head}) = x$, $P(\text{tail}) = 1-x$
$Y_{x,n}$ = number of heads after n tosses.
$\frac{Y_{x,n}}{n} \approx x \Rightarrow f\left(\frac{Y_{x,n}}{n}\right) \approx f(x)$

$$E f\left(\frac{Y_{x,n}}{n}\right) \approx f(x)$$

Observe that $E f\left(\frac{Y_{x,n}}{n}\right) = \sum_{k=0}^{n} f(\tfrac{k}{n}) \underbrace{P(Y_{x,n}=k)}_{\text{binomial distribution } \binom{n}{k} x^k (1-x)^{n-k}} = B_n^f(x)$

Proof (of the WAT on $[0,1]$)
Let $\varepsilon > 0$.
f is continuous on $[0,1] \Rightarrow f$ is uniformly continuous
$\Rightarrow \exists \delta > 0$ s.t. if $|x-y| < \delta$ $(x,y \in [0,1])$ then $|f(x)-f(y)| < \varepsilon/2$
$\forall x \in [0,1]$

$$|B_n^f(x) - f(x)| = \left|\sum_{k=0}^{n} f(\tfrac{k}{n}) p_{nk}(x) - f(x) \underbrace{\sum_{k=0}^{n} p_{nk}(x)}_{\text{as this } =1}\right|$$
by lemma following this proof!

$$= \left|\sum_{k=0}^{n} \left(f(\tfrac{k}{n}) - f(x)\right) p_{nk}(x)\right|$$

$$\leq \sum_{k=0}^{n} \left|f(\tfrac{k}{n}) - f(x)\right| p_{nk}(x) \quad \text{by } \Delta \text{ inequality.}$$

$$= \sum_{k:|\frac{k}{n}-x|<\delta} \underbrace{|f(\tfrac{k}{n}) - f(x)|}_{<\varepsilon/2} p_{nk}(x) + \sum_{k:|\frac{k}{n}-x|\geq\delta} \underbrace{|f(\tfrac{k}{n}) - f(x)|}_{<2\|f\|_{sup}} p_{nk}(x)$$

$$< \frac{\varepsilon}{2} \underbrace{\sum_{k=0}^{n} p_{nk}(x)}_{=1} + 2\|f\|_{sup} \sum_{k:|\frac{k}{n}-x|\geq\delta} 1 \cdot p_{nk}(x)$$

$$\left|\frac{k}{n} - x\right| \geq \delta \Leftrightarrow \frac{(k-nx)^2}{n^2\delta^2} \geq 1$$

$$\leq \frac{\varepsilon}{2} + 2\|f\|_{sup} \sum_{k:|\frac{k}{n}-x|\geq\delta} \frac{(k-nx)^2}{n^2\delta^2} p_{nk}(x)$$

$$\leq \frac{\varepsilon}{2} + \frac{2\|f\|_{sup}}{n^2\delta^2} \sum_{k=0}^{n} (k-nx)^2 p_{nk}(x)$$

$$= \frac{\varepsilon}{2} + \frac{2\|f\|_{sup}}{n^2\delta^2} (nx(1-x))$$

$$\leq \frac{\varepsilon}{2} + \frac{2\|f\|_{sup}}{\delta^2} \cdot \frac{1}{n}$$

Choose N so that $\frac{2\|f\|_{sup}}{\delta^2} \cdot \frac{1}{N} < \frac{\varepsilon}{2}$

then $\forall n \geq N \ \forall x \in [0,1] \ |B_n^f(x) - f(x)| < \frac{\varepsilon}{2} + \frac{2\|f\|_{sup}}{\delta^2} \frac{1}{n}$

$$\leq \frac{\varepsilon}{2} + \frac{2\|f\|_{sup}}{\delta^2} \cdot \frac{1}{N} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \square$$

Lemma

(1) $\sum_{k=0}^{n} p_{nk}(x) = 1$

(2) $\sum_{k=0}^{n} k \cdot p_{nk}(x) = nx$

(3) $\sum_{k=0}^{n} (k-nx)^2 p_{nk}(x) = nx(1-x)$.

Proof

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

(1) $\sum_{k=0}^{n} p_{nk}(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = (x+1-x)^n = 1$

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(2) $k\binom{n}{k} = k \cdot \dfrac{n!}{k!(n-k)!} = n \cdot \dfrac{(n-1)!}{(k-1)!((n-1)-(k-1))!} = n\binom{n-1}{k-1}, \quad k \geq 1$

$$\sum_{k=0}^{n} k p_{nk}(x) = \sum_{k=0}^{n} k \binom{n}{k} x^k (1-x)^{n-k} = \sum_{k=1}^{n} n \binom{n-1}{k-1} x^k (1-x)^{n-k}$$

$$= nx \sum_{k=1}^{n} \binom{n-1}{k-1} x^{k-1} (1-x)^{n-k}$$

$i = k-1$

$$= nx \sum_{i=0}^{n-1} \binom{n-1}{i} x^i (1-x)^{n-1-i}$$

$$= nx (x+1-x)^{n-1} = nx$$

(3) $k(k-1)\binom{n}{k} = k(k-1) \dfrac{n!}{k!\,(n-k)!}$

$$= n(n-1) \frac{(n-2)!}{(k-2)!\,((n-2)-(k-2))!}$$

$$\sum_{k=0}^{n} k(k-1)\, p_{nk}(x) = \sum_{k=0}^{n} k(k-1)\binom{n}{k} x^k (1-x)^{n-k}$$

$$= \sum_{k=2}^{n} n(n-1)\binom{n-2}{k-2} x^k (1-x)^{n-k}$$

$$= n(n-1)x^2 \sum_{k=2}^{n} \binom{n-2}{k-2} x^{k-2} (1-x)^{n-k}$$

$i = k-2$

$$= n(n-1)x^2 \sum_{i=0}^{n} \binom{n-2}{i} x^i (1-x)^{n-2-i}$$

$$= n(n-1)x^2 (x+1-x)^{n-2} = n(n-1)x^2$$

$$\sum_{k=0}^{n} (k-nx)^2 p_{nk}(x)$$

$$= \sum_{k=0}^{n} k(k-1) p_{nk}(x) + \sum_{k=0}^{n} k\, p_{nk}(x) - 2nx \sum_{k=0}^{n} k\, p_{nk}(x)$$

$$+ n^2 x^2 \sum_{k=0}^{n} p_{nk}(x)$$

$= n(n-1)x^2 + nx - 2n^2x^2 + n^2x^2$

$= n(n-1)x^2 + nx - n^2x^2$

$= nx(\cancel{nx} - x + 1 - \cancel{nx})$

$= nx(1-x)$

$\square$

<u>Proof</u> (WAT on $[a,b]$)

$f: [a,b] \mapsto \mathbb{R}$, define

$x: [0,1] \mapsto [a,b]$ where $x(t) = a + (b-a)t$, $t(x) = \dfrac{x-a}{b-a}$

(x is a bijection).

Define $g: [0,1] \mapsto \mathbb{R}$ by $g(t) = f(x(t))$.

WAT on $[0,1] \Rightarrow \exists \{p_n\}$ on $[0,1]$ s.t. $\|p_n - g\|_{sup} \to 0$.

Define $Q_n(x) = p_n(t(x))$.

Claim: $Q_n \to f$ uniformly on $[a,b]$.

$$\|Q_n - f\|_{sup} = \sup_{x \in [a,b]} |Q_n(x) - f(x)|$$

$$= \sup_{x \in [a,b]} |p_n(t(x)) - f(x)|$$

$$= \sup_{t \in [0,1]} |p_n(t) - \underbrace{f(x(t))}_{g(t)}|$$

$$= \|p_n - g\|_{sup} \to 0.$$

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Chapter 2 - Fourier Series

$R[a,b] = \{ f : [a,b] \to \mathbb{R} \text{ s.t. } f \text{ is Riemann-integrable} \}$

$\langle f, g \rangle = \int_a^b f(x)\, g(x)\, dx$

Def

A family (φ_n) of functions from $R[a,b]$ is an orthogonal system if $\langle \varphi_n, \varphi_m \rangle = \int_a^b \varphi_n(x)\varphi_m(x)\,dx = 0 \quad \forall n \neq m$.

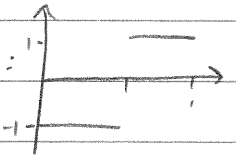
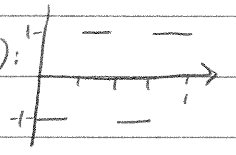
It is called an orthonormal system (o.n.s.) if it is orthogonal and in addition $\langle \varphi_n, \varphi_n \rangle = \int_a^b \varphi_n(x)^2\,dx = 1 \quad \forall n$.

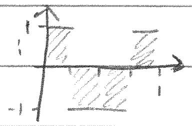
Examples

1). $\{1, \cos(x), \cos(2x), \cos(3x), \dots, \sin(x), \sin(2x), \sin(3x), \dots\}$ on $[-\pi, \pi]$.
This is an orthogonal system.

2). $\left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos(nx), \sin(nx) : n \in \mathbb{N} \right\}$ is orthonormal. (check = exercise) ↖ trigonometric system.

3). $\{1, x\}$ on $[-1, 1]$
$\int_{-1}^{1} 1 \cdot x\,dx = 0$ true
It is orthogonal.

4). $x \in [0,1]$ $\varphi_0(x) = 1$, $\varphi_1(x)$:  , $\varphi_2(x)$:  , etc...

$\{\varphi_i : i \in \mathbb{N}_0\}$ is orthonormal.
$\langle \varphi_n, \varphi_n \rangle = \int_0^1 1\,dx = 1$
$\langle \varphi_1, \varphi_2 \rangle = \int_0^1 \varphi_1(x)\varphi_2(x)\,dx = 0$: 
so $\langle \varphi_n, \varphi_m \rangle = 0$, $n \neq m$.

Def

For a Riemann-integrable function $f:[a,b]\mapsto \mathbb{R}$
and an o.n.s. $(\varphi_n)_{n=1}^{\infty}$ on $[a,b]$,

$$a_n = \langle f, \varphi_n \rangle \equiv \int_a^b f(x)\varphi_n(x)\,dx \ , \ n\in\mathbb{N},$$

are called Fourier coefficients of f w.r.t. (φ_n),
and $\sum_{n=1}^{\infty} a_n \varphi_n$ is called the Fourier series of f w.r.t. (φ_n).

Remark

$$f \rightsquigarrow \sum_{n=1}^{\infty} a_n \varphi_n$$

- We don't know if the series converges
- Even if it does, it doesn't have to converge to f.

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O.n.s

$\{\varphi_n\}_{n=1}^{\infty}$ on $[a,b]$: $\langle \varphi_n, \varphi_m \rangle = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases}$

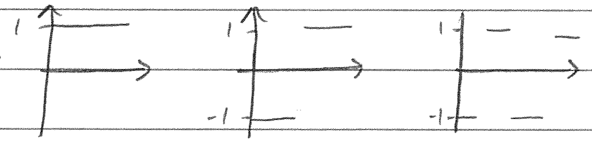
$f \in R[a,b]$, $\{\varphi_n\}$ o.n.s. $\Rightarrow$ $a_n = \langle f, \varphi_n \rangle$ - fourier coefficients
$\sum_{n=1}^{\infty} a_n \varphi_n$ - fourier series
of f w.r.t. $\{\varphi_n\}$

Recall: $\langle f, g \rangle = \int_a^b f(x) g(x)\, dx$

Examples

1). $\{1, x\}$ on $[-1, 1]$ (orthogonal), $\{\frac{1}{\sqrt{2}}, \frac{\sqrt{3}\,x}{\sqrt{2}}\}$ on $[-1, 1]$ (orthonormal)

2). $\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos(nx), \frac{1}{\sqrt{\pi}} \sin(nx)\}$

3). 

We don't know:
- if $\sum a_n \varphi_n$ converges
- if it converges to f.

Example

Let us compute the Fourier coefficients / Fourier series of $f(x) = x$ w.r.t. the systems above

1). $a_1 = \int_{-1}^{1} x \frac{1}{\sqrt{2}} = 0$, $a_2 = \int_{-1}^{1} x \cdot \frac{x\sqrt{3}}{\sqrt{2}}\, dx = \sqrt{\frac{2}{3}}$

Fourier series $= 0 \cdot \frac{1}{\sqrt{2}} + \sqrt{\frac{2}{3}} \cdot \sqrt{\frac{3}{2}}\, x = x$

Exercise: do the same for $f(x) = x^2$

$\rightarrow$ F.S. will be different from x^2 (as it has to be a polynomial of order 1).

2). $a_0 = \int_{-\pi}^{\pi} x \cdot \frac{1}{\sqrt{2\pi}}\, dx = 0$ (w.r.t. $\frac{1}{\sqrt{2\pi}}$)

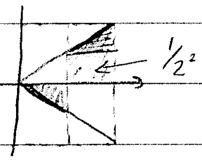
$a_n = \int_{-\pi}^{\pi} x \cdot \frac{1}{\sqrt{\pi}} \cos(nx)\, dx = 0$ (w.r.t. $\frac{1}{\sqrt{\pi}} \cos(nx)$)

$$b_n = \int_{-\pi}^{\pi} x \cdot \frac{1}{\sqrt{\pi}} \sin(nx)\,dx = \text{something...} \qquad (\text{w.r.t. } \tfrac{1}{\sqrt{\pi}} \sin(nx))$$

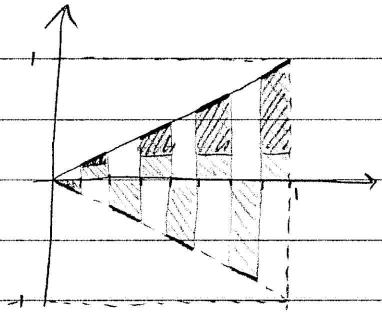
3). $$a_0 = \int_0^1 x \cdot 1 \, dx = \frac{1}{2}$$

$$a_1 = \int_0^1 x \cdot \varphi_1(x)\,dx = \int_0^{\frac{1}{2}} -x\,dx + \int_{\frac{1}{2}}^1 x\,dx = \left[-\frac{x^2}{2}\right]_0^{\frac{1}{2}} + \left[\frac{x^2}{2}\right]_{\frac{1}{2}}^1$$

$$= -\frac{1}{8} + \frac{1}{2} - \frac{1}{8} = \frac{1}{4}$$



$$a_n = \int_0^1 x \cdot \varphi_n(x)\,dx = 2^{n-1}(2^{-n})^2 = \frac{1}{2^{n+1}}$$



number of intervals $= 2^n$

each interval has length 2^{-n}

FS for $f(x) = x$ w.r.t. $\{\varphi_n\}$ is $\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \varphi_n$

Def

Let $f \in R[a,b]$, define its two-norm by

$$\|f\|_2 = \sqrt{\langle f, f \rangle} = \sqrt{\int_a^b f(x)^2\,dx}$$

Theorem 2.1 (Least-squares approximation)

Let $f \in R[a,b]$ and let $\{\varphi_n\}_{n=1}^{\infty}$ be an o.n.s.

Denote by $\{a_n\}_{n=1}^{\infty}$ the Fourier coefficients of f w.r.t. $\{\varphi_n\}_{n=1}^{\infty}$.

Then $$\left\| f - \sum_{i=1}^{n} a_i \varphi_i \right\|_2 \leq \left\| f - \sum_{i=1}^{n} c_i \varphi_i \right\|_2$$

equivalent.

for all n and all $c_1, \dots, c_n \in \mathbb{R}$, with the equality iff $c_1 = a_1, \dots, c_n = a_n$.

$$\int_a^b \left(f(x) - \sum_{i=1}^{n} a_i \varphi_i(x)\right)^2 dx \leq \int_a^b \left(f(x) - \sum_{i=1}^{n} c_i \varphi_i(x)\right)^2 dx$$

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Proof

$$\left\| f - \sum_{i=1}^n c_i \varphi_i \right\|_2^2 = \left\langle f - \sum_{i=1}^n c_i \varphi_i , f - \sum_{j=1}^n c_j \varphi_j \right\rangle$$

$$= \langle f, f \rangle - \left\langle f, \sum_{j=1}^n c_j \varphi_j \right\rangle - \left\langle \sum_{i=1}^n c_i \varphi_i , f \right\rangle + \left\langle \sum_{i=1}^n c_i \varphi_i , \sum_{j=1}^n c_j \varphi_j \right\rangle$$

$$= \langle f, f \rangle - \sum_{j=1}^n c_j \underbrace{\langle f, \varphi_j \rangle}_{a_j} - \sum_{i=1}^n c_i \underbrace{\langle \varphi_i , f \rangle}_{a_i} + \sum_{i=1}^n \sum_{j=1}^n c_i c_j \underbrace{\langle \varphi_i , \varphi_j \rangle}_{\delta_{ij}}$$

$$= \langle f, f \rangle - 2 \sum_{i=1}^n c_i a_i + \sum_{i=1}^n c_i^2 + \sum_{i=1}^n a_i^2 - \sum_{i=1}^n a_i^2$$

$$= \langle f, f \rangle - \sum_{i=1}^n a_i^2 + \sum_{i=1}^n (a_i - c_i)^2$$

So, $\left\| f - \sum_{i=1}^n c_i \varphi_i \right\|_2^2 = \langle f, f \rangle - \sum_{i=1}^n a_i^2 + \sum_{i=1}^n (a_i - c_i)^2$

In particular, $\left\| f - \sum_{i=1}^n a_i \varphi_i \right\|_2^2 = \underbrace{\langle f, f \rangle}_{\| f \|_2^2} - \sum_{i=1}^n a_i^2$

□

Theorem 2.2 (Bessel's Inequality)

Let $f \in R[a,b]$ and $\{\varphi_n\}_{n=1}^\infty$ be an o.n.s.
Denote by (a_n) the Fourier coefficients of f w.r.t. $\{\varphi_n\}$.
Then $\sum_{n=1}^\infty a_n^2 \leq \| f \|_2^2 = \int_a^b f(x)^2 \, dx$

In particular, $a_n \to 0$ as $n \to \infty$.

Proof

would have to write this fully in exam

As in the proof of Thm 2.1,

$$\sum_{i=1}^n a_i^2 = \| f \|_2^2 - \left\| f - \sum_{i=1}^n a_i \varphi_i \right\|_2^2$$

$$\leq \| f \|_2^2$$

$$\Rightarrow \sum_{i=1}^\infty a_i^2 \leq \| f \|_2^2$$

In particular, $a_n^2 \to 0$ and hence $a_n \to 0$. □

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Trigonometric Fourier Series

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos(nx), \frac{1}{\sqrt{\pi}} \sin(nx) \right\} \text{ on } [-\pi, \pi]$$

$$\hat{a}_0 = \int_{-\pi}^{\pi} f(x) \cdot \frac{1}{\sqrt{2\pi}} \, dx$$

$$\hat{a}_n = \int_{-\pi}^{\pi} f(x) \cdot \frac{1}{\sqrt{\pi}} \cos(nx) \, dx, \quad n \in \mathbb{N}$$

$$\hat{b}_n = \int_{-\pi}^{\pi} f(x) \cdot \frac{1}{\sqrt{\pi}} \sin(nx) \, dx, \quad n \in \mathbb{N}$$

$$\text{F.S.} : \hat{a}_0 \cdot \frac{1}{\sqrt{2\pi}} + \sum_{n=1}^{\infty} \left(\hat{a}_n \cdot \frac{1}{\sqrt{\pi}} \cos(nx) + \hat{b}_n \cdot \frac{1}{\sqrt{\pi}} \sin(nx) \right)$$

It is more convenient to consider $\{1, \cos(nx), \sin(nx)\}$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, dx, \quad n \in \mathbb{N}_0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \, dx, \quad n \in \mathbb{N}$$

$$\text{F.S.} : \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx) \right)$$

Exercise

Compute the F.S. for $f(x) = |x|$, $f(x) = x^2$, ...

WTP

- f cont $\Rightarrow$ F.S. doesn't conv. pointwise but converges almost everywhere
- f diff $\Rightarrow$ F.S. conv. to f pointwise.
- f twice diff $\Rightarrow$ F.S. conv. to f uniformly.

Theorem 2.3 (Riemann's Lemma)

If $f \in R[a,b]$ then $\int_a^b f(x)\cos(\lambda x)\,dx \xrightarrow{\lambda\to\infty} 0$ and $\int_a^b f(x)\sin(\lambda x)\,dx \xrightarrow{\lambda\to\infty} 0$.

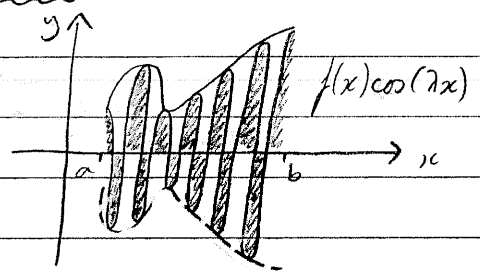
Remark

If $[a,b] = [-\pi,\pi]$ & $f \in R[-\pi,\pi]$

then $\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \to 0$ and $\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \to 0$.

(We already know that this is true from Bessel's inequality.)

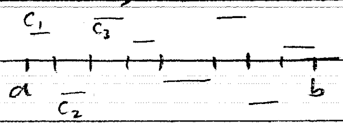
Remark



Proof

Step 1

Suppose f is a step function, that is, there is a partition $P: a = t_0 < \dots < t_n = b$ s.t. $f(x) = c_i \quad \forall i \in (t_{i-1}, t_i)$



$$\int_a^b f(x)\cos(\lambda x)\,dx = \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} f(x)\cos(\lambda x)\,dx$$

$$= \sum_{i=1}^{n} \frac{c_i \sin(\lambda x)}{\lambda}\Bigg|_{t_{i-1}}^{t_i}$$

$$= \frac{1}{\lambda}\underbrace{\sum_{i=1}^{n} c_i\big(\sin(\lambda t_i) - \sin(\lambda t_{i-1})\big)}_{\text{bounded by } 2\sum_{i=1}^{n}|c_i| \ \leftarrow \text{ constant.}} \to 0$$

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Step 2

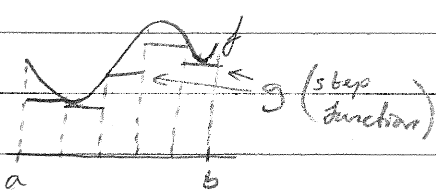
Now let $f \in R[a,b]$.

Let $\varepsilon > 0 \Rightarrow \exists P : a = t_0 < \dots < t_n = b$

such that $U(f,P) - L(f,P) < \frac{\varepsilon}{2}$

Define the step function

$$g(x) = \inf_{t \in [t_{i-1}, t_i]} f(t) \quad \text{if } x \in (t_{i-1}, t_i)$$



and any value at $t_0, t_1, \dots, t_n$

We know from Step 1 that $\int_a^b g(x)\cos(\lambda x)\,dx \to 0$

$\Rightarrow \exists \lambda_0$ s.t. $\forall \lambda \geq \lambda_0$,

$$\left| \int_a^b g(x)\cos(\lambda x)\,dx \right| < \frac{\varepsilon}{2}$$

Now $\left| \int_a^b f(x)\cos(\lambda x)\,dx \right|$

$$\leq \left| \int_a^b (f(x) - g(x))\cos(\lambda x)\,dx \right| + \overbrace{\left| \int_a^b g(x)\cos(\lambda x)\,dx \right|}^{< \varepsilon/2}$$

$$\leq \int_a^b (f(x) - g(x))\,dx + \frac{\varepsilon}{2}$$

f is always $\geq g$.

$$= \sum_{i=1}^n \int_{t_{i-1}}^{t_i} (f(x) - g(x))\,dx + \frac{\varepsilon}{2}$$

$$\leq \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \left(\sup_{[t_{i-1}, t_i]} f - \inf_{[t_{i-1}, t_i]} f \right) dx + \frac{\varepsilon}{2}$$

$$= \sum_{i=1}^n \sup_{[t_{i-1}, t_i]} f \cdot (t_i - t_{i-1}) - \sum_{i=1}^n \inf_{[t_{i-1}, t_i]} f \cdot (t_i - t_{i-1}) + \frac{\varepsilon}{2}$$

$$= U(f,P) - L(f,P) + \frac{\varepsilon}{2}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\square$

$\sin(\lambda x) \rightsquigarrow$ similar

$$S_n^f(x) = \frac{a_0}{2} + \sum_{m=1}^{n} a_m \cos(mx) + b_m \sin(mx)$$

<u>Theorem 2.4</u>

Let $f \in R[-\pi,\pi]$, assume that $f(-\pi) = f(\pi)$ and extend f 2π-periodically to R.

Then

$$S_n^f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, D_n(x-t)\, dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\, D_n(t)\, dt$$

where $D_n(t) = \begin{cases} \dfrac{\sin(n+\frac{1}{2})t}{\sin(t/2)}, & \text{if } t \neq 2\pi k \\ 2n+1, & \text{if } t = 2\pi k \end{cases}$

is called the Dirichlet Kernel.

<u>Proof</u>

$$S_n^f(x) = \frac{a_0}{2} + \sum_{m=1}^{n} a_m \cos(mx) + b_m \sin(mx)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, dt + \sum_{m=1}^{n} \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(mt)\cos(mx)\, dt$$

$$+ \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(mt)\sin(mx)\, dt.$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\left(1 + 2\sum_{m=1}^{n} \cos(mt)\cos(mx) + \sin(mt)\sin(mx)\right) dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\left(1 + 2\sum_{m=1}^{n} \cos(m(x-t))\right) dt$$

$\cos(m(x-t)) = \cos(m(t-x))$

It remains to show: $1 + 2\sum_{m=1}^{n} \cos(m\theta) = D_n(\theta) \quad \forall \theta.$

$$\sin\frac{\theta}{2}\left(1 + 2\sum_{m=1}^{n}\cos(m\theta)\right) = \sin\frac{\theta}{2} + \sum_{m=1}^{n} 2\sin\frac{\theta}{2}\cos m\theta$$

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$$= \sin\frac{\theta}{2} + \sum_{m=1}^{n} \sin\left(m\theta+\frac{\theta}{2}\right) - \sin\left(m\theta-\frac{\theta}{2}\right)$$

$$= \cancel{\sin\frac{\theta}{2}} \underbrace{- \cancel{\sin\frac{\theta}{2}} + \cancel{\sin\frac{3\theta}{2}}} \underbrace{- \cancel{\sin\frac{3\theta}{2}} + \cancel{\sin\frac{5\theta}{2}}} + \ldots$$

$$\underbrace{- \cancel{\sin\left(n\theta-\frac{\theta}{2}\right)} + \sin\left(n\theta+\frac{\theta}{2}\right)}$$

$$= \sin\left(\left(n+\tfrac{1}{2}\right)\theta\right)$$

$$\Rightarrow 1 + 2\sum_{m=1}^{n}\cos(m\theta) = \begin{cases} \dfrac{\sin\left(\left(n+\frac{1}{2}\right)\theta\right)}{\sin\left(\frac{\theta}{2}\right)}, & \theta \neq 2\pi k \\ 2n+1, & \theta = 2\pi k \end{cases}$$

$$= D_n$$

We have $S_n^f(x) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(t)\, D_n(x-t)\, dt \Big|_{s=x-t}$

$$= -\frac{1}{2\pi}\int_{x+\pi}^{x-\pi} f(x-s)\, D_n(s)\, ds$$

$$= \frac{1}{2\pi}\int_{x-\pi}^{x+\pi} f(x-s)\, D_n(s)\, ds$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-s)\, D_n(s)\, ds \quad \leftarrow \text{by } 2\pi\text{-periodicity of } f$$

$\square$

Theorem 2.5

Let $f \in R[-\pi,\pi]$, assume $f(-\pi) = f(\pi)$ and denote by f the 2π-periodic extension of f to $\mathbb{R}$. If f is differentiable at x then $S_n^f(x) \to f(x)$.

Theorem 2.6

Let $f \in R[-\pi, \pi]$, assume $f(-\pi) = f(\pi)$ and denote by f the 2π-periodic extension of f to $\mathbb{R}$.

Let $x \in \mathbb{R}$. If there exist $M > 0$ and $\delta > 0$ such that $\left| \frac{f(x+t) - f(x)}{t} \right| \leq M \quad \forall t \in (-\delta, \delta)$

than $S_n^f(x) \longrightarrow f(x)$.

Proof (of Thm 2.6)

$$|S_n^f(x) - f(x)| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) D_n(t)\,dt - f(x) \cdot \underbrace{\frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(t)\,dt}_{=1} \right|$$

Since $D_n(t) = 1 + 2 \sum_{m=1}^{n} \cos(mt)$

and $\int_{-\pi}^{\pi} D_n(t)\,dt = 2\pi + 0 = 2\pi$

So
$$|S_n^f(x) - f(x)| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(x-t) - f(x)) D_n(t)\,dt \right|$$

$$= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \underbrace{\frac{f(x-t) - f(x)}{\sin(\frac{t}{2})}}_{g(t)} \sin((n+\tfrac{1}{2})t)\,dt \right|$$

, x is fixed.

$$= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) \sin((n+\tfrac{1}{2})t)\,dt \right|$$

$$\leq \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} (g(t) - \hat{g}(t)) \sin((n+\tfrac{1}{2})t)\,dt \right| \qquad \hat{g}(t) = \begin{cases} 0, & t \in [\frac{-\varepsilon}{2M}, \frac{\varepsilon}{2M} \\ g(t) & \text{otherwise} \end{cases}$$

$$+ \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{g}(t) \sin((n+\tfrac{1}{2})t)\,dt \right|$$

By Riemann's Lemma $\exists N \quad \forall n \geq N \quad \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{g}(t) \sin((n+\tfrac{1}{2})t)\,dt \right| < \frac{\varepsilon}{2}$

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$$|f(x-t) - f(x)| \le M|t| \quad \forall\, |t| < \delta$$

$$\left|\sin \tfrac{t}{2}\right| \geq \frac{|t|}{\pi} \quad \forall\, |t| \le \pi$$

$$|g(t)| = \left|\frac{f(x-t) - f(x)}{\sin\left(\frac{t}{2}\right)}\right| \le \frac{M \cdot |t| \cdot \pi}{|t|} = M_\pi$$

$$\forall\, |t| < \min\{\delta, \pi\}$$

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(g(t) - \hat{g}(t)\right)\sin\left(\left(n+\tfrac{1}{2}\right)dt\right)dt\right| \le \frac{1}{2\pi}\int_{-\frac{\varepsilon}{2M}}^{\varepsilon/2M} M_\pi \, dt$$

$$= \frac{1}{2\pi} M_\pi \cdot \frac{\varepsilon}{M} = \frac{\varepsilon}{2}$$

$$\Rightarrow |S_n^f(x) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\square$

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Convergence of the F.S. to f

0). f continuous $\Rightarrow$ F.S. converges to f "almost everywhere"

1). f differentiable $\Rightarrow$ F.S. converges to f

2). f twice differentiable $\Rightarrow$ F.S. converges to f uniformly

Theorem 2.6

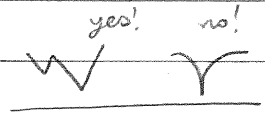
If f is differentiable at x then
$$S_n^f(x) \longrightarrow f(x)$$

Theorem 2.7

If $\exists\, M, \delta$ s.t. $\left|\dfrac{f(x+t)-f(x)}{t}\right| \leq M \;\; \forall\, 0<|t|<\delta$

then, $S_n^f(x) \to f(x)$.

(proved)



Proof of Thm 2.6

f is differentiable at x $\Rightarrow$ $\displaystyle\lim_{t\to 0}\left(\frac{f(x+t)-f(x)}{t}\right) = f'(x)$

Let $\varepsilon = 1$; $\exists\, \delta>0$ s.t. $0<|t|<\delta$

$$\Rightarrow \left|\frac{f(x+t)-f(x)}{t} - f'(x)\right| < 1$$

$$\Rightarrow \left|\frac{f(x+t)-f(x)}{t}\right| \leq \underbrace{|f'(x)| + 1}_{M}$$

$\Rightarrow S_n^f(x) \to f(x)$ by Thm 2.7. $\square$

Theorem 2.9 (Parceval's Theorem)

Let $f:[-\pi,\pi] \mapsto \mathbb{R}$ be s.t. $f(-\pi) = f(\pi)$ (f is 2π-periodic)

Assume f'' exists and is Riemann-integrable.

Then $S_n^f(x) \longrightarrow f$ uniformly on $[-\pi,\pi]$

and $\displaystyle\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2+b_n^2) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\,dx.$

Remark

Bessel's inequality: $\sum_{n=1}^{\infty} a_n^2 \leq \int_a^b f(x)^2 \, dx$

For "nice" orthonormal systems: $\sum_{n=1}^{\infty} a_n^2 = \int_a^b f(x)^2 \, dx$

trigonometric system is nice.

$\{1, \cos(nx), \sin(nx)\}$ vs $\left\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos(nx), \frac{1}{\sqrt{\pi}}\sin(nx)\right\}$

Proof

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \underbrace{\cos(nx)\,dx}_{\frac{d(\sin(nx))}{n}}$$

$$= \cancel{\frac{f(x)\sin(nx)}{n}\Big|_{-\pi}^{\pi}} - \frac{1}{\pi n}\int_{-\pi}^{\pi} f'(x) \underbrace{\sin(nx)\,dx}_{-\frac{d(\cos(nx))}{n}}$$

$$= \underbrace{\cancel{+\frac{1}{\pi n^2} f'(x)\cos(nx)\Big|_{-\pi}^{\pi}}}_{\text{as } f \text{ is } 2\pi\text{-periodic}} - \frac{1}{\pi n^2}\int_{-\pi}^{\pi} f''(x)\cos(nx)\,dx$$

So,

$$|a_n| = \frac{1}{\pi n^2}\left|\int_{-\pi}^{\pi} f''(x)\cos(nx)\,dx\right|$$

Similarly

$$|b_n| = \frac{1}{\pi n^2}\left|\int_{\pi}^{\pi} f''(x)\sin(nx)\,dx\right|$$

By Riemann's Lemma $\int_{-\pi}^{\pi} f''(x)\cos(nx)\,dx \to 0$,
$\int_{-\pi}^{\pi} f''(x)\sin(nx)\,dx \to 0$.

Let $\varepsilon > 0$, $\exists N \ \forall n \geq N$ $\left|\int_{-\pi}^{\pi} f''(x)\cos(nx)\,dx\right| < \varepsilon$,

$$\left|\int_{-\pi}^{\pi} f''(x)\sin(nx)\,dx\right| < \varepsilon.$$

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$$\left| S_n^f(x) - f(x) \right| \overset{\text{Thm 2.6}}{=} \left| \sum_{k=n+1}^{\infty} a_k \cos(kx) + b_k \sin(kx) \right|$$

$$\leq \sum_{k=n+1}^{\infty} |a_k| + |b_k|$$

$n \geq N$

$$\leq \sum_{k=n+1}^{\infty} \left(\frac{1}{\pi k^2} \varepsilon + \frac{1}{\pi k^2} \varepsilon \right) = \frac{2\varepsilon}{\pi} \underbrace{\sum_{k=n+1}^{\infty} \frac{1}{k^2}}_{\leq \text{const}}$$

$$< \frac{2c}{\pi} \cdot \varepsilon \qquad \forall x \in [-\pi, \pi]$$

$\Rightarrow S_n^f(x) \to f$ uniformly.

$\Rightarrow S_n^f(x) f \to f^2$ uniformly (since f is bounded:
$\| S_n^f \cdot f - f^2 \|_{sup} \leq \underbrace{\| S_n^f - f \|_{sup}}_{\to 0} \cdot \underbrace{\| f \|_{sup}}_{\text{const.}} \to 0$)

$$\Rightarrow \int_{-\pi}^{\pi} S_n^f(x) f(x) \, dx \to \int_{-\pi}^{\pi} f(x)^2 \, dx$$

$$\int_{-\pi}^{\pi} f(x)^2 \, dx = \lim_{n \to \infty} \int_{-\pi}^{\pi} f(x) S_n^f(x) \, dx$$

$$= \lim_{n \to \infty} \int_{-\pi}^{\pi} f(x) \left(\frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos(kx) + b_k \sin(kx) \right) dx$$

$$= \lim_{n \to \infty} \left[\frac{a_0}{2} \underbrace{\int_{-\pi}^{\pi} f(x) \, dx}_{\pi a_0} + \sum_{k=1}^{n} \left(a_k \underbrace{\int_{-\pi}^{\pi} f(x) \cos(kx) \, dx}_{\pi a_k} + b_k \underbrace{\int_{-\pi}^{\pi} f(x) \sin(kx) \, dx}_{\pi b_k} \right) \right]$$

$$= \pi \lim_{n \to \infty} \left(\frac{a_0^2}{2} + \sum_{k=1}^{n} a_k^2 + b_k^2 \right)$$

$$= \pi \left(\frac{a_0^2}{2} + \sum_{k=1}^{\infty} a_k^2 + b_k^2 \right)$$

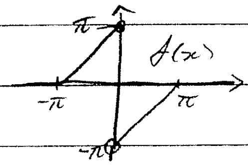
$\square$

Example

$f : [-\pi, \pi] \longmapsto R \quad , \quad f(-\pi) = f(\pi) \quad , \quad f \in R[-\pi, \pi]$

such that $S_n^f(x) \not\rightarrow f(x)$ at some x

$$f(x) = \begin{cases} x + \pi & , \quad x \in [-\pi, 0] \\ x - \pi & , \quad x \in (0, \pi] \end{cases}$$



$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{0} (x + \pi) \cos(nx) \, dx + \frac{1}{\pi} \int_{0}^{\pi} (x - \pi) \cos(nx)$$

$$= \cancel{\frac{1}{\pi} \int_{-\pi}^{\pi} x \cos(nx) \, dx} + \cancel{\int_{-\pi}^{0} \cos(nx) \, dx} - \cancel{\int_{0}^{\pi} \cos(nx) \, dx}$$

$$= 0$$

b_n can be computed...

$$F.S. = \sum_{n=1}^{\infty} b_n \sin(nx)$$

The F.S. is equal to zero at zero,
but $f(0) = \pi \neq 0$

So $S_n^f(0) \not\rightarrow f(0)$.

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Chapter 3 - Metric Spaces

Def

A metric space is (X, d), where X is a set and $d: X \times X \mapsto \mathbb{R}$ such that

1). $d(x,y) \geq 0 \quad \forall x,y \in X$ and $d(x,y) = 0 \Leftrightarrow x = y$

2). $d(x,y) = d(y,x) \quad \forall x,y \in X$

3). $d(x,y) \leq d(x,z) + d(z,y) \quad \forall x,y,z \in X$ [Δ inequality]

d is called the distance function or metric.

Examples

1). $\mathbb{R}$, $d(x,y) = |x-y|$ ($\mathbb{R}$ with the standard distance / metric)

2). $\mathbb{R}^n$, $d(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$

Δ inequality follows from Cauchy-Schwarz inequality.

3). Discrete metric space:

X - any set, $d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$

$\underset{0 \text{ or } 1}{d(x,y)} \overset{?}{\leq} \underset{0 \text{ or } 1}{d(x,z)} + \underset{0 \text{ or } 1}{d(z,y)}$

$x = z = y \Rightarrow 0 + 0 = 0 \Rightarrow \Delta$ inequality holds

4). $C[a,b]$ (cont. fns on $[a,b]$), $d(f,g) = \|f-g\|_{sup}$ ← problem class

Examples (non-examples!)

1). $\mathbb{R}$, $d(x,y) = x^2 + y^2$, $d(5,5) = 50 \neq 0$

2). $\mathbb{R}$, $d(x,y) = x^2 - y^2$, $d(1,2) = -3 < 0$ and $d(1,2) \neq d(2,1)$

3). $\mathbb{R}$, $d(x,y) = |x-y|^2$, $d(0,2) > d(0,1) + d(1,2)$

$2^2 > 1^2 + 1^2$

Non of these are metric spaces!

Metric spaces

Normal spaces

Inner product spaces

Def

A <u>normal space</u> is $(V, \|\cdot\|)$, where V is a real vector space and $\|\cdot\| : V \mapsto \mathbb{R}$ such that

1). $\|x\| \geqslant 0 \ \forall x$, $\|x\| = 0 \Leftrightarrow x = 0$

2). $\|\lambda x\| = |\lambda| \cdot \|x\| \ \forall x \in V, \ \lambda \in \mathbb{R}$

3). $\|x+y\| \leq \|x\| + \|y\| \ \forall x, y \in V$

$\|\cdot\|$ is called a norm.

Lemma

Let $(V, \|\cdot\|)$ be a normal space. Define $d(x,y) = \|x-y\| \ \forall x, y \in V$.
Then (V, d) is a metric space.

Proof

1). $d(x,y) = \|x-y\| \geqslant 0$

$d(x,y) = 0 \Leftrightarrow \|x-y\| = 0 \Leftrightarrow x-y=0 \Leftrightarrow x=y$

2). $d(x,y) = \|x-y\| = \|(-1)(y-x)\| = |-1| \cdot \|y-x\|$

$= 1 \cdot \|y-x\| = \|y-x\| = d(y,x)$.

3). $d(x,y) = \|x-y\| = \|(x-z)+(z-y)\|$

$\leq \|x-z\| + \|z-y\| = d(x,z) + d(z,y)$.

$\square$

Example

1). $\mathbb{R}$, $\|x\| = |x| \Rightarrow d(x,y) = \|x-y\| = |x-y| \leftarrow$ metric.

2). $\mathbb{R}^2$, $\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2} \leftarrow$ 2-norm

$\|x\|_1 = \sum_{i=1}^{\infty} |x_i| \leftarrow$ 1-norm

} see problem class.

non trivial $\rightarrow \|x\|_q = \left(\sum_{i=1}^{\infty} |x_i|^q\right)^{\frac{1}{q}} \leftarrow$ q-norm $\quad (q \in [1, \infty))$

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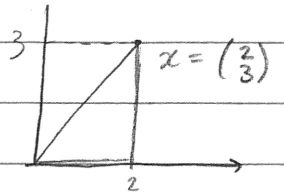
$$\|x\|_\infty = \max_{1 \le i \le n} |x_i| \quad \leftarrow \infty \text{ norm}$$

(in homework)



$\|x\|_1 = 5$

$\|x\|_2 = \sqrt{13}$

$\|x\|_\infty = 3$

3). Is there a normed space $(V, \|\cdot\|)$ s.t. $(V, d(x,y) = \|x-y\|)$ is a discrete metric space?

If yes: $1 = d(x,0) = \|x\|$

$1 = d(5x, 0) = \|5x\| = 5\|x\| = 5$

Contradiction!

4). $C[a,b]$, $\|f\|_2 = \sqrt{\int_a^b f(x)^2 \, dx} \quad \leftarrow$ 2-norm.

Problem class

homework $\rightarrow$ $\|f\|_1 = \int_a^b |f(x)| \, dx \quad \longleftarrow$ 1-norm.

not examinable $\rightarrow$ $\|f\|_q = \left(\int_a^b |f(x)|^q \, dx \right)^{\frac{1}{q}} \quad \leftarrow$ q-norm , $q \in [1, \infty)$

problem class $\rightarrow$ $\|f\|_{sup} = \sup_{x \in [a,b]} |f(x)| \quad \leftarrow$ supremum norm (∞-norm).

Remark

Let V be a vector space with an inner product $\langle \cdot, \cdot \rangle$.
Then $\|x\| = \sqrt{\langle x, x \rangle}$ is a norm.

(Exercise) (not examinable)

Theorem 3.1 (Cauchy-Schwarz inequality)

Let $f, g \in R[a,b]$. Then,

$$\left|\int_a^b f(x)g(x)\,dx\right| \leq \sqrt{\int_a^b f(x)^2\,dx}\sqrt{\int_a^b g(x)^2\,dx}$$

That is:

$|\langle f, g\rangle| \leq \|f\|_2 \cdot \|g\|_2$ (True for any inner product and the corresponding norm).

Proof

$\varphi(t) = \|tf - g\|_2^2 \geq 0 \quad \forall t \in \mathbb{R}$

$\varphi(t) = \langle tf - g, tf - g\rangle$

$\quad = t^2\langle f, f\rangle - 2t\langle f, g\rangle + \langle g, g\rangle$

$\mathcal{D} \leq 0, \quad \mathcal{D} = 4\langle f, g\rangle^2 - 4\|f\|_2^2 \cdot \|g\|_2^2$

$\Rightarrow \langle f, g\rangle^2 \leq \|f\|_2^2\|g\|_2^2$

$\square$

Def

Let (X, d) be a metric space.

$B^\circ(x, r) = \{y \in X : d(x,y) < r\}, \; x \in X, \; r > 0$

is the open ball around x of radius r.

(around $x \Leftrightarrow$ centre at x).

$B(x, r) = \{y \in X : d(x,y) \leq r\}, \; x \in X, \; r > 0$

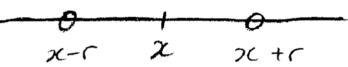
is the closed ball around x of radius r.

Examples

1). $\mathbb{R}$ with the standard distance

$B^\circ(x, r) = (x - r, x + r)$

$B(x, r) = [x - r, x + r]$



2). $\mathbb{R}^2 \; ((0,0), r)$ w.r.t $\|\cdot\|_2$

$\|\cdot\|_1$

$\|\cdot\|_\infty$

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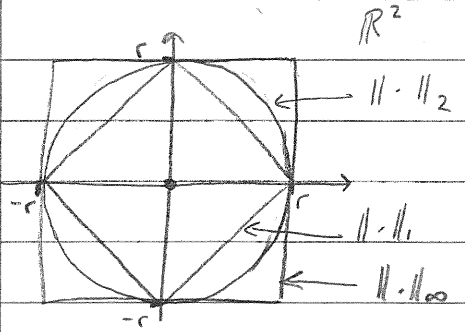


$\|y-0\|_1 < r$

$|y_1| + |y_2| < r$

$\|y-0\|_\infty < r$

$\max\{|y_1|, |y_2|\} < r$

3). Discrete space (X, d)

$$B^\circ(x,r) = \{y : d(x,y) < r\} = \begin{cases} X, & r > 1 \\ \{x\}, & r \le 1 \end{cases}$$

$$B(x,r) = \{y : d(x,y) \le r\} = \begin{cases} X, & r \geq 1 \\ \{x\}, & r < 1 \end{cases}$$

4). $C[a,b]$, $\|\cdot\|_{sup}$

$B^\circ(f, r) = \{g : \|g-f\|_{sup} < r\}$

= all continuous functions staying within the r-tube around f.

Def

Let (X, d) be a metric space.

A set $G \subset X$ is open if

$\forall x \in G$ $\exists r > 0$ s.t. $B^\circ(x,r) \subset G$.

A set $F \subset X$ is closed if $X \setminus F$ is open.

Examples

1). $\mathbb{R}$, standard distance.

ⓐ

	open?	closed?
(a,b)	✓	✗
(a,∞)	✓	✗
$[a,b]$	✗	✓
$(-\infty,b]$	✗	✓
$(a,b]$	✗	✗
$\{a\}$	✗	✓

ⓑ $\{\frac{1}{n} : n \in \mathbb{N}\}$

open? ✗

closed? ✗ (since the complement is not open because of 0).

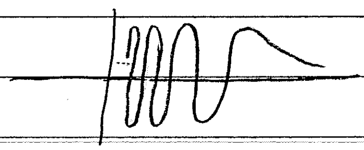
$\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$

open? ✗

closed? ✓

2). $\mathbb{R}^2$ with $\|\cdot\|_2$

$A = \{(x, \sin\frac{1}{x}) : x > 0\}$



open? ✗

closed? ✗

$A \cup \{(0,y) : |y| \leq 1\}$

open? ✗

closed? ✓

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3). Discrete space (X, d)

$A \subset X$ - open?

$\forall x \in A$ can I find $r > 0$ $B^{\circ}(x, r) \subset A$?

$\begin{cases} \{x\} & \text{if } r \leq 1 \\ X & \text{if } r > 1 \end{cases}$

$r = \frac{1}{2}$, $B^{\circ}(x, r) = \{x\}$ Yes!

Any set is open!

Any set is also closed!

Theorem 3.2

Any open ball is an open set.

Any closed ball is a closed set.

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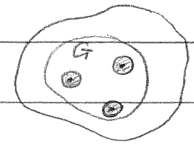
(X, d) - Metric space

$G \subset X$ is an <u>open set</u> if $\forall x \in G \ \exists r > 0$ s.t. $B^\circ(x,r) \subset G$

$F \subset X$ is a <u>closed set</u> if $X \setminus F$ is an open set.



Theorem 3.3

ⓐ $B^\circ(x,r)$ is open

ⓑ $B(x,r)$ is closed.

Proof

ⓐ Let $x \in X$, $r > 0$

Let us show that $B^\circ(x,r)$ is open.

Let $y \in B^\circ(x,r)$

$d(x,y) < r \Rightarrow \delta = r - d(x,y) > 0$

Let's show that $B^\circ(y,\delta) \subset B^\circ(x,r)$

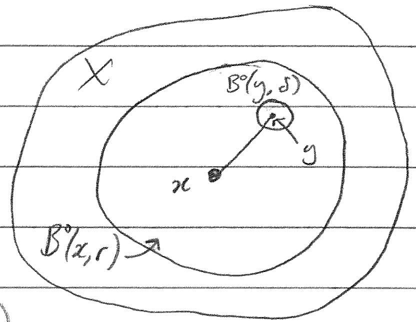
Let $z \in B^\circ(y,\delta)$, i.e. $d(y,z) < \delta$

$d(x,z) \leq d(x,y) + d(y,z)$ (Δ inequality)

$< d(x,y) + \delta = r$

$\Rightarrow d(x,z) < r$

$\Rightarrow z \in B^\circ(x,r)$.



ⓑ Let $x \in X$, $r > 0$

Let's show that $B(x,r)$ is closed

Let $y \in X \setminus B(x,r)$

$d(x,y) > r$

$\delta = d(x,y) - r > 0$

Let's show that $B^\circ(y,\delta) \subset X \setminus B(x,r)$

Let $z \in B^\circ(y,\delta)$ i.e. $d(y,z) < \delta$

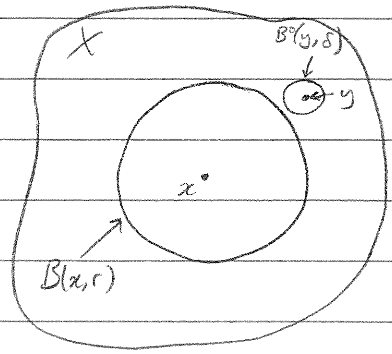
$d(x,y) \leq d(x,z) + d(z,y)$ (Δ inequality)

$d(x,z) \geq d(x,y) - d(z,y)$

$> d(x,y) - \delta = r$

$\Rightarrow d(x,z) > r$

$\Rightarrow z \in X \setminus B(x,r)$.



$\square$

Theorem 3.4

ⓐ $\phi, X \leftarrow$ whole space are open and closed

ⓑ If $\{G_\alpha\}_{\alpha \in A}$ is a collection of open sets then $\bigcup_{\alpha \in A} G_\alpha$ is open.

ⓒ If $\{G_i\}_{i=1}^{n}$ is a finite collection of open sets then $\bigcap_{i=1}^{n} G_i$ is open.

ⓓ If $\{F_\alpha\}_{\alpha \in A}$ is a collection of closed sets then $\bigcap_{\alpha \in A} F_\alpha$ is closed.

ⓔ If $\{F_i\}_{i=1}^{n}$ is a finite collection of closed sets then $\bigcup_{i=1}^{n} F_i$ is closed.

Counterexamples for (c) + (e)

(We cannot take infinite collections there)

ⓒ $G_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$

but $\bigcap_{n=1}^{\infty} G_n = \{0\}$ is not open.

ⓔ $F_n = \left[-1+\frac{1}{n}, 1-\frac{1}{n}\right]$

but $\bigcup_{n=1}^{\infty} F_n = (-1, 1)$ is not closed.

in general $\begin{cases} \text{not open} \neq \text{closed} \\ \text{not closed} \neq \text{open} \end{cases}$

Proof

ⓐ ϕ is open since it contains no point (for which the condition in the definition has to be checked)

$\Rightarrow X$ is closed since $X \setminus X = \phi$ is open.

X is open since $\forall x \in X \; \exists r =$ (anything) s.t. $B^\circ(x, r) \subset X$

$\Rightarrow \phi$ is closed since $X \setminus \phi = X$ is open.

ⓑ Let $x \in \bigcup_{\alpha \in A} G_\alpha \Rightarrow x \in G_\alpha$ for some $\alpha \in A$.

Since G_α is open $\exists r > 0$ s.t. $B^\circ(x, r) \subset G_\alpha \subset \bigcup_{\alpha \in A} G_\alpha$

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(c) Let $x \in \bigcap_{i=1}^{n} G_i \Rightarrow x \in G_i \ \forall G_i$

Since each G_i is open $\exists r_i > 0$ s.t.

$B^\circ(x, r_i) \subset G_i$

Choose $r = \min\{r_1, \dots, r_n\} > 0$

$B^\circ(x, r) \subset B^\circ(x, r_i) \subset G_i \ \forall i$

$\Rightarrow B^\circ(x, r) \subset \bigcap_{i=1}^{n} G_i$

(d) $X \setminus \bigcap_{\alpha \in A} F_\alpha = \bigcup_{\alpha \in A} (X \setminus F_\alpha)$

$\forall \alpha \in A$, F_α is closed $\Rightarrow X \setminus F_\alpha$ is open

$\Rightarrow \bigcup_{\alpha \in A} (X \setminus F_\alpha)$ is open by (b)

$\Rightarrow \bigcap_{\alpha \in A} F_\alpha$ is closed.

(e) $X \setminus \bigcup_{i=1}^{n} F_i = \bigcap_{i=1}^{n} (X \setminus F_i)$

For $1 \leq i \leq n$, F_i is closed $\Rightarrow X \setminus F_i$ is open

$\Rightarrow \bigcap_{i=1}^{n} (X \setminus F_i)$ is open by (c)

$\Rightarrow \bigcup_{i=1}^{n} F_i$ is closed.

$\square$

Def

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in a metric space (X, d). We say that $x_n \to x \in X$ ("x_n converges to x") if $\forall \varepsilon > 0 \ \exists N \in \mathbb{N}, \forall n \geq N, d(x_n, x) < \varepsilon \Leftrightarrow x_n \in B^\circ(x, \varepsilon)$.

Examples

1). $(\mathbb{R}, |\cdot|) \Rightarrow$ usual convergent sequences.

2). $(0,1)$, $d(x,y) = |x - y|$, $x_n = \frac{1}{n}$ doesn't converge.

3). Discrete space, $x_n \to x$. Take $\varepsilon = \frac{1}{2}$; $\exists N$ s.t. $\forall n \geq N$

$d(x_n, x) < \frac{1}{2} \Rightarrow d(x_n, x) = 0 \Rightarrow x_n = x$

Convergent series are of the form

$* * * \; * * * \; x \; x \; x \; x \; x \; x \ldots$

we call them "eventually constant"

4). $(C[a,b], \|\cdot\|_{sup})$ $f_n \to f$

$\Leftrightarrow \forall \varepsilon > 0 \;\; \exists N \;\; \forall n \geq N \;\; \|f_n - f\|_{sup} < \varepsilon$

$\Leftrightarrow$ f_n converges to f uniformly on $[a,b]$

Theorem 3.5

If $x_n \to x$ and $x_n \to y$, then $x = y$.

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$x_n \to x \iff \forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n \geq N \ d(x_n, x) < \varepsilon$
[equivalently $x_n \in B^\circ(x, \varepsilon)$]
$\iff d(x_n, x) \longrightarrow 0.$

Theorem 3.5

If $x_n \to x$ and $x_n \to y$ then $x = y$.

Proof

Suppose $x \neq y$. Then $d(x,y) > 0$

Let $\varepsilon = \dfrac{d(x,y)}{2} > 0$

Since $x_n \to x$ $\exists N_1, \ \forall n \geq N_1 \ d(x_n, x) < \dfrac{d(x,y)}{2}$

Since $x_n \to y$ $\exists N_2, \ \forall n \geq N_2 \ d(x_n, y) < \dfrac{d(x,y)}{2}$

Pick $n \geq \max\{N_1, N_2\}$

Then $d(x,y) \leq d(x, x_n) + d(x_n, y) < \dfrac{d(x,y)}{2} + \dfrac{d(x,y)}{2} = d(x,y)$

※.

$\square$

Theorem 3.6

Let (X, d) be a metric space and $A \subset X$.

A is closed $\iff$ for any sequence $x_n \in A$, $n \in \mathbb{N}$ which converges to some $x \in X$ we have $x \in A$.

Proof

[$\Rightarrow$] Suppose A is closed but the R.H.S. is false.

Then there is (x_n) s.t. $x_n \in A \ \forall n$,

$x_n \to x$ but $x \notin A$.

A is closed $\Rightarrow X \setminus A$ is open

$\Rightarrow \exists r > 0$ s.t. $B^\circ(x, r) \in X \setminus A$

However since $x_n \to x$ $\exists N, \ \forall n \geq N \ d(x_n, x) < r$,

i.e. $x_n \in B^\circ(x, r)$ i.e. $x_n \in X \setminus A$. ※

[$\Leftarrow$]

Suppose the R.H.S. is true but A is not closed.
i.e. $X \setminus A$ is not open.

$\Rightarrow \exists x \in X \setminus A$ s.t. $\forall n \in \mathbb{N}$

$B^\circ(x, \frac{1}{n}) \cap A \neq \emptyset$

Pick $x_n \in B^\circ(x, \frac{1}{n}) \cap A$

$$\begin{cases} \cdot\ x_n \in A \quad \forall n \\ \cdot\ d(x_n, x) < \frac{1}{n} \to 0 \Rightarrow x_n \to x \\ \cdot\ x \notin A \end{cases}$$

Contradicts the RHS.

$\square$

Def

A sequence (x_n) in a metric space (X, d) is a Cauchy sequence if
$\forall \varepsilon > 0 \ \exists N \in \mathbb{N}, \ \forall n, m \geq N \quad d(x_n, x_m) < \varepsilon$.

Examples

1). $\mathbb{R}$ with $|\cdot|$ - standard Cauchy sequences from Yr 1

2). $(0,1)$, $|\cdot|$, $x_n = \frac{1}{n}$ - Cauchy sequence

$\forall \varepsilon > 0 \ \exists N \ \forall n, m \geq N \quad |\frac{1}{n} - \frac{1}{m}| < \varepsilon$, true in the same way as in $\mathbb{R}$.

3). Discrete space

$(x_n) \quad \forall \varepsilon > 0 \quad \exists N \ \forall n, m \geq N \quad d(x_n, x_m) < \varepsilon$

$d(x_n, x_m) = 0 \Rightarrow x_n = x_m$

Cauchy $\Leftrightarrow$ eventually constant.

4). $C[a,b]$, $\|\cdot\|_{sup}$

$(f_n) \quad \forall \varepsilon > 0 \ \exists N \ \forall n, m \geq N \quad \|f_n - f_m\|_{sup} < \varepsilon$

Cauchy (in terms of our general def. of a Cauchy sequence)
$\Longleftrightarrow$ "uniform Cauchy" from Chapter 1.

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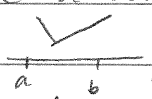
<u>Def</u>
A metric space is called <u>complete</u> if every Cauchy sequence in that space converges.
A normed space which is complete as a metric space is called a <u>Banach space</u>.

<u>Lemma</u>
If x_n converges then (x_n) is Cauchy.

<u>Proof</u>
Let $\varepsilon > 0$, since $x_n \to$ some x $\exists N, \forall n \geq N$
$d(x_n, x) < \frac{\varepsilon}{2}$.
Now $\forall n, m \geq N$ $d(x_n, x_m) \leq d(x_n, x) + d(x, x_m)$
$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

$\square$

<u>Examples</u>
1). $(\mathbb{R}, |\cdot|)$ - complete, Banach
2). $((0,1), |\cdot|)$ - not complete because of $x_n = \frac{1}{n}$.
3). discrete space - complete
(not a normed space so not a Banach space)
4). $(C[a,b], \|\cdot\|_{sup})$ - complete (by the CPUC), Banach.
5). $\mathbb{R}^n$, $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_q$, $\|\cdot\|_\infty$ complete (exercise)
6). $(\mathbb{Q}, |\cdot|)$ $q_n \to \sqrt{2}$ (in $\mathbb{R}$)
Cauchy but not convergent in $(\mathbb{Q}, |\cdot|)$
This is because $\mathbb{Q}$ is not closed in $\mathbb{R}$.
7). $P[a,b]$ (space of all polynomials)
$\|\cdot\|_{sup}$, not a complete space (so not a Banach space)
Let $f : [a,b] \mapsto \mathbb{R}$ be a continuous function but not a polynomial (eg. ).
We know from the WAT that there is a sequence of polynomials p_n converging to f uniformly.
- (p_n) is a uniform Cauchy sequence

$\Rightarrow \langle p_n \rangle$ is a Cauchy sequence in $(P[a,b], \|\cdot\|)$

- $\langle p_n \rangle$ does not converge in $(P[a,b], \|\cdot\|)$

These two bullet points show that

$(P[a,b], \|\cdot\|_{sup})$ is not complete / not Banach.

6+7 If • (X,d) is complete

• Y is <u>not</u> closed

then (Y,d) is <u>not</u> complete

(X,d) (Y,d)

If • (X,d) is complete

• Y <u>is</u> closed in X

then (Y,d) <u>is</u> complete.

Exercise.

<u>Def</u>

Let (X,d) be a metric space.

A mapping $T: X \mapsto X$ is called a <u>contraction mapping</u> if $\exists\, c \in (0,1)$ such that $d(T(x), T(y)) \leq c\, d(x,y) \quad \forall x,y \in X$.

<u>Examples</u>

1). $(\mathbb{R}, |\cdot|)$, $T(x) = \frac{1}{2}x$

$$|T(x) - T(y)| = \left|\frac{x}{2} - \frac{y}{2}\right| = \frac{1}{2}|x-y| \leq \frac{1}{2}|x-y| \quad \forall x,y \in X$$

2). $(\mathbb{R}, |\cdot|)$, $T(x) = \sin(\frac{x}{2})$

$$|T(x) - T(y)| = \left|\sin(\tfrac{x}{2}) - \sin(\tfrac{y}{2})\right|$$

$$\overset{MVT}{=} \left|\tfrac{1}{2}\cos(\tfrac{\xi}{2})\right| \cdot |x-y| \leq \tfrac{1}{2}|x-y|$$

3). $(\mathbb{R}, |\cdot|)$ $T: \mathbb{R} \mapsto \mathbb{R}$, differentiable and such that $\|T'(x)\|_{sup} < 1$ then T is a contraction mapping.

4). $(\mathbb{R}, |\cdot|)$ $T(x) = \sin x$ <u>not</u> a contraction mapping

$T'(x) = \cos x$; $\|T'\|_{sup} = 1$

$\Rightarrow |T(x) - T(y)| \leq c|x-y|$ is true iff $c \geq 1$

Suppose $\exists\, c \in (0,1)$ $|T(x) - T(y)| \leq c|x-y| \quad \forall x,y$.

$$\left|\frac{T(x) - T(y)}{x-y}\right| \leq c$$

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$y = 0,\ x \to 0$

$$\lim_{x \to 0} \left| \frac{T(x) - T(0)}{x} \right| \leq c$$

$$\Rightarrow \quad T'(0) \leq c$$

$$1 \leq c$$

$\times$ as $c \in (0,1)$

5). $([1,\infty), |\cdot|)$ $\quad T(x) = x + \frac{1}{x}$

$T'(x) = 1 - \frac{1}{x^2} < 1$ $\quad$ not a contraction mapping

but $\|T'\|_{sup} = 1$ $\quad$ Exercise.

Def

$x \in X$ is called a fixed point of T if $T(x) = x$.

Theorem 3.7 (Contraction Mapping Theorem)

Let (X, d) be a non-empty, complete metric space and let $T : X \mapsto X$ be a contraction mapping. Then T has a unique fixed point.

Proof

Since $X \neq \emptyset$ pick some $x_0 \in X$.

Let $x_n = T(x_{n-1})$, $n \in \mathbb{N}$.

$n > m$, $\displaystyle d(x_m, x_n) \leq \sum_{i=m}^{n-1} \underbrace{d(x_i, x_{i+1})}$

$$\left[\begin{aligned} & d(T(x_{i-1}), T(x_i)) \\ & \leq c\, d(x_{i-1}, x_i) \\ & \leq c^2 d(x_{i-2}, x_{i-1}) \leq c^i d(x_0, x_1) \end{aligned}\right]$$

$$\leq \sum_{i=m}^{n-1} c^i d(x_0, x_1)$$

$$\leq d(x_0, x_1) \sum_{m}^{n-1} c^i = \frac{d(x_0, x_1)\, c^m}{1 - c} \to 0$$

$\Rightarrow (x_n)$ is a Cauchy sequence

Since (X,d) is complete, x_n converges to some x

$x_n \to x \iff x_{n+1} \to x$

$$d(x_{n+1}, T(x)) = d(T(x_n), T(x)) \leq c\, d(x_n, x) \to 0$$

$\Rightarrow x_{n+1} \to T(x) \quad \Rightarrow T(x) = x \Rightarrow x$ is a fixed point.

Uniqueness:

Suppose $T(x) = x$ and $T(y) = y$, $x \neq y$

Then $\underbrace{d(T(x), T(y))}_{= d(x,y)} \leq c\, d(x,y)$

Since $d(x,y) \neq 0$ we get a contradiction.

$\square$

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$T : X \mapsto X$ is a contraction mapping
if $\exists c \in (0,1)$ s.t. $d(T(x), T(y)) \leq c\, d(x,y)$ $\forall x, y \in X$

Examples

$T : \mathbb{R} \mapsto \mathbb{R}$, differentiable,

$\|T'\|_{sup} < 1 \Rightarrow$ contraction

$\|T'\|_{sup} \geq 1 \Rightarrow \underbrace{\text{not a contraction}}_{\text{in homework}}$

$T(x) = \sin x$, i.e. $\|T'\|_{sup} = 1 \Rightarrow$ not a contraction

Theorem (Contraction Mapping Theorem)

$X \neq \emptyset$ complete, T contraction $\Rightarrow$ T has a unique fixed point i.e. $Tx = x$ has a unique solution.

Examples

1). $X = [1, \infty)$, $T(x) = x + \frac{1}{x}$

complete

If T is a contraction mapping $\overset{CMT}{\Rightarrow}$ $x + \frac{1}{x} = x$ has a unique solution.

$x + \frac{1}{x} = x$ has no solns. $\Rightarrow$ T is not a contraction mapping

Or: $T'(x) = 1 - \frac{1}{x^2}$ $\|T'\|_{sup} = 1 \Rightarrow$ T is not a contraction mapping.

MVT: $|T(x) - T(y)| = |T'(\xi)||x-y| < |x-y|$

The condition $d(T(x), T(y)) \leq c\, d(x,y)$ cannot be replaced by $d(T(x), T(y)) < d(x,y)$ in the CMT.

2). X needs to complete!

$X = (0, \infty)$

$\uparrow$ incomplete, $x_n = \frac{1}{n}$ Cauchy but not convergent.

$T(x) = \frac{x}{2}$ $\leftarrow$ contraction mapping

$\frac{x}{2} = x \iff x = 0 \notin (0, \infty)$

so no solution in our space.

3). $(*) \begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$ $y: \mathbb{R} \mapsto \mathbb{R}$, $f: \mathbb{R}^2 \mapsto \mathbb{R}$, $x_0, y_0 \in \mathbb{R}$

WTP (under some conditions) that the solution of $(*)$ exists and is unique

$(*) \iff (**)$ $\quad y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt$

Define $T(y) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt$

on "some" space of functions

- Make sure the space is complete
- Make sure T is a contraction $(T(y) = y)$

Def

Let (X, d_X) and (Y, d_Y) be two metric spaces and let $f: X \mapsto Y$.

We say that $\lim_{x \to a} f(x) = b$ $[a \in X, b \in Y]$

if $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. $0 < d_X(x, a) < \delta$ then $d_Y(f(x), b) < \varepsilon$.

f is continuous at $a \in X$ if $\lim_{x \to a} f(x) = f(a)$

i.e. $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. $d_X(x, a) < \delta \Rightarrow d_Y(f(x), f(a)) < \varepsilon$.

f is continuous if it is continuous at all $a \in X$.

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Examples

1). $(\mathbb{R}, |\cdot|) \leftarrow$ both (X, d_X), (Y, d_Y).
same definition as before.

2). $f: X \mapsto \mathbb{R}$
discrete ↑ ← with the standard distance

?: which functions f are continuous?

f is cont. at $a \Leftrightarrow \forall \varepsilon > 0 \ \exists \delta > 0$ s.t.

$$\underbrace{d(x,a)}_{0 \text{ or } 1} < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$$

$\forall \varepsilon > 0$ take $\delta = \frac{1}{2}$; $d(x,a) < \delta$

$\Rightarrow d(x,a) = 0 \Rightarrow x = a$

$\Rightarrow |f(x) - f(a)| = |f(a) - f(a)| = 0 < \varepsilon$.

Any function is continuous.

2*). $f: X \longmapsto Y$ ← continuous
discrete ↑ ↖ any

3). $F: C[a,b] \mapsto C[a,b]$
with $\|\cdot\|_2$ with $\|\cdot\|_1$

$F(f) = f$

Let $f \in C[a,b]$

Let $\varepsilon > 0$, choose $\delta = \varepsilon / \sqrt{b-a}$

Let $g \in C[a,b]$ be s.t. $\|g - f\|_2 < \delta$

Then $\|F(g) - F(f)\|_1 = \|g - f\|_1 = \int_a^b |g(x) - f(x)| \, dx$

$$\underset{\text{Cauchy Schwarz}}{\leq} \sqrt{\int_a^b |g(x) - f(x)|^2 dx} \cdot \sqrt{\int_a^b 1^2 dx}$$

$$= \|g - f\|_2 \cdot \sqrt{b-a}$$

$$< \delta \sqrt{b-a} = \varepsilon$$

4). $F(f) = f$ from $(C[a,b], \|\cdot\|_1)$ to $(C[a,b], \|\cdot\|_2)$ is discontinuous.

<u>Theorem 3.9</u>
f is continuous at $a \Leftrightarrow$ for any sequence $x_n \to a$ we have $f(x_n) \to f(a)$.

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$f : X \mapsto Y$ is continuous at $a \in X$
if $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $d_X(x, a) < \delta \Rightarrow d_Y(f(x), f(a)) < \varepsilon$

Examples
1). $f : \mathbb{R} \mapsto \mathbb{R}$, same as before.

2). $\underset{\text{discrete}}{f : X} \mapsto \underset{\text{any}}{Y}$ – all functions are continuous

[x 'close' to a in a discrete space $\Rightarrow x = a$
$\Rightarrow f(x) = f(a)$]

3). $\left.\begin{array}{l} F : (C[a,b], \|\cdot\|_2) \longrightarrow (C[a,b], \|\cdot\|_1) \\ F(f) = f \end{array}\right\}$ continuous

Theorem 3.9
f is continuous at $a \in X$
$\Leftrightarrow$ for any sequence $\langle x_n \rangle$ s.t. $x_n \to a$, $f(x_n) \to f(a)$

Proof
[$\Rightarrow$] Suppose f is continuous at a.
Let $x_n \to a$
We want to show: $f(x_n) \longrightarrow f(a)$,
i.e. WTS: $\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ s.t. $\forall n \geq N \ d_Y(f(x_n), f(a)) < \varepsilon$.
Let $\varepsilon > 0$; by continuity, $\exists \delta > 0$ s.t. $d_X(x, a) < \delta$
$\Rightarrow d_Y(f(x), f(a)) < \varepsilon$.
Since $x_n \longrightarrow a \ \exists N \ \forall n \geq N \ d_X(x_n, a) < \delta$. Hence
$d_Y(f(x_n), f(a)) < \varepsilon$.

[$\Leftarrow$] Suppose the R.H.S. holds, but f is not continuous at a.
This means $\exists \varepsilon > 0 \ \forall \delta$ (take $\frac{1}{n}$) there is x_n s.t.
$d_X(x_n, a) < \frac{1}{n}$ but $d_Y(f(x_n), f(a)) \geq \varepsilon$.
Observe that x_n contradicts the R.H.S. :

$d_X(x_n, a) < \frac{1}{n} \to 0 \Rightarrow x_n \to a$

$d_Y(f(x_n), f(a)) \geq \varepsilon \Rightarrow f(x_n) \not\to f(a)$

$\square$

Examples

4). $F : (C[a,b], \|\cdot\|_1) \longrightarrow (C[a,b], \|\cdot\|_2)$

$F(f) = f$

F is not continuous at any point!

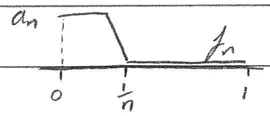
Let's show F is not continuous at $f \equiv 0$.

We will find a sequence $f_n \in C[a,b]$ s.t.

$\|f_n - 0\|_1 \to 0$ but $\|F(f_n) - F(0)\|_2 \not\to 0$.

Take $[a,b] = [0,1]$

not continuous



$$\|f_n - 0\|_1 = \int_0^1 |f_n(x)|\,dx = \frac{a_n}{n} \to 0$$

$$\|F(f_n) - F(0)\|_2 = \|f_n - 0\|_2 = \sqrt{\int_0^1 f(x)^2\,dx}$$

$$= \sqrt{\frac{a_n^2}{n}} = \frac{a_n}{\sqrt{n}} \not\to 0$$

[if $\alpha = \sqrt{n}$ or any $a_n = n^\alpha$ with $\frac{1}{2} \leq \alpha < 1$]

Theorem 3.10

If $f : (X, d_X) \mapsto (Y, d_Y)$ and $g : (Y, d_Y) \mapsto (Z, d_Z)$ are continuous, then $g \circ f$ is continuous

Proof

Let $x_n \to a \underset{f\text{ cont.}}{\Longrightarrow} f(x_n) \to f(a) \underset{g\text{ cont.}}{\Longrightarrow} g(f(x_n)) \to g(f(a))$

$\square$

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<u>Theorem 3.11</u>

$f : (X, d_X) \mapsto (Y, d_Y)$ is continuous (globally)

$\Leftrightarrow$ for any open set $G \subset Y$, $f^{-1}(G)$ is open in X.

<u>Proof</u>

[$\Rightarrow$]

Suppose f is continuous.

Let $G \subset Y$ be an open set

WTP: $f^{-1}(G)$ is open in X.

ie. $\forall a \in f^{-1}(G)$ $\exists \delta > 0$ s.t. $B^\circ(a, \delta) \subset f^{-1}(G)$.

Fix $a \in f^{-1}(G)$.

Since G is open, $\exists \varepsilon > 0$ s.t. $B^\circ(f(a), \varepsilon) \subset G$

Since f is continuous at a,

$\exists \delta > 0$ such that if $d_X(x, a) < \delta \Rightarrow d_Y(f(x), f(a)) < \varepsilon$

So $B^\circ(a, \delta) \in f^{-1}(B^\circ(f(a), \varepsilon)) \subset f^{-1}(G)$

$\Rightarrow f^{-1}(G)$ is open.

[$\Leftarrow$]

Suppose for any open set $G \subset Y$ we have $f^{-1}(G)$ is open. Let's show that f is continuous at every $a \in X$.

Let $\varepsilon > 0$; consider $B^\circ(f(a), \varepsilon)$ – it is open, hence $f^{-1}(B^\circ(f(a), \varepsilon))$ is open.

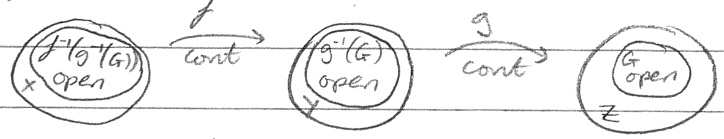
$\Rightarrow \exists \delta$ s.t. $B^\circ(a, \delta) \subset f^{-1}(B^\circ(f(a), \varepsilon))$

ie. $d_X(x, a) < \delta \Rightarrow d_Y(f(x), f(a)) < \varepsilon$.

$\square$

<u>Examples</u>

1).



$\Rightarrow g \circ f$ is continuous

2). $f : X \mapsto Y$ – all functions are continuous.

discrete open

Corollary

f is continuous $\Leftrightarrow$ for any closed set $F \subset Y$, $f^{-1}(F)$ is closed.

"Proof"

Opposite to Theorem: consider theorem with $Y \setminus F$ (open) so $f^{-1}(Y \setminus F)$ is open $\Leftrightarrow f$ cont.
so $f^{-1}(F)$ closed.

Remark

~~f continuous $\Leftrightarrow$ for any open set $G \subset X$, $f(G)$ is open~~

$f(x) = \sin x$
$f : \mathbb{R} \mapsto \mathbb{R}$ (with 1.1)
$G = \mathbb{R}$ is open
$f(G) = [-1, 1]$ is not open

Def

Let (X, d) be a metric space, and $A \subset X$.
The closure $\bar{A}$ is the smallest closed set containing A, that is, the intersection of all closed sets containing A.
The interior A° is the largest open set contained in A, that is, the union of all open sets contained in A.
The boundary ∂A of A is $\partial A = \bar{A} \setminus A^\circ$.

Remark

1). $A^\circ \subset A \subset \bar{A}$
2). A° is always open, $\bar{A}$ is always closed, ∂A is always closed.
3). If A is open, $A^\circ = A$, if A is closed, $\bar{A} = A$.

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Theorem 3.12

(a) $x \in \bar{A} \iff \exists x_n \in A$ s.t. $x_n \to x$.

(b) $x \in A^\circ \iff \exists r > 0$ s.t. $B^\circ(x,r) \subset A$

Proof

(a) [$\Rightarrow$]

Let $x \in \bar{A}$; $B^\circ(x, \frac{1}{n}) \cap A = \emptyset$

i.e. $A \subset X \setminus B^\circ(x, \frac{1}{n})$ (closed)

$\Rightarrow \bar{A} \subset X \setminus B^\circ(x, \frac{1}{n})$, but $x \in \bar{A}$

$\Rightarrow B^\circ(x, \frac{1}{n}) \cap A \neq \emptyset$, pick $x_n \in B^\circ(x, \frac{1}{n}) \cap A$

$x_n \in A$; $x_n \to x$ since $d(x_n, x) < \frac{1}{n} \to 0$

(a) [$\Leftarrow$]

$x_n \in A$, $x_n \to x$

$\Rightarrow x_n \in \bar{A}$, $\bar{A}$ is closed $\Rightarrow x \in \bar{A}$.

(b) [$\Rightarrow$]

$x \in A^\circ = \bigcup_\alpha G_\alpha$ ("union of all open sets contained in A")

$\Rightarrow x \in G_\alpha$ for some α

Since G_α is open $\exists r > 0$ s.t. $B^\circ(x,r) \subset G_\alpha \subset A^\circ$.

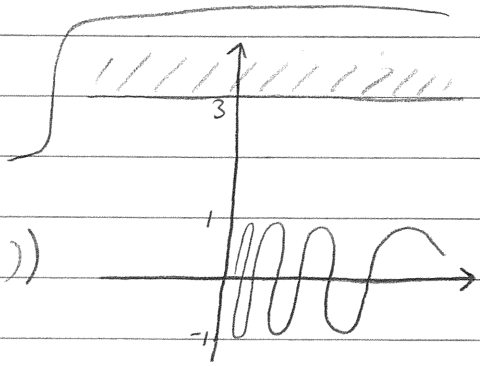
(b) [$\Leftarrow$]

$\exists r$ s.t. $\underbrace{B^\circ(x,r)}_{\text{open}} \subset A$

Since A° is the union of all open sets contained in A, $B^\circ(x,r) \subset A^\circ$

$\square$

Examples



1). $\mathbb{R}^2$, with the standard distance

$A = \{(x, \sin\frac{1}{x}) : x > 0\} \cup (\mathbb{R} \times [3, \infty))$

$A^\circ = \mathbb{R} \times (3, \infty)$

$\bar{A} = A \cup (\{0\} \times [-1, 1])$

$\partial A = \{(x, \sin\frac{1}{x}) : x > 0\} \cup (\{0\} \times [-1,1]) \cup (\mathbb{R} \times \{3\})$

?: $\overline{B^\circ(x,r)} = B(x,r)$ $\to$ true in normed space ③
$\searrow$ false in metric spaces ②

② Discrete space $= B^\circ(x,1) = \{x\}$ closed
$B^\circ(x,1) = \{x\}$ but $B(x,1) = X$ so they are not equal.

③ In a normed space, $\overline{B^\circ(x,r)} = B(x,r)$
$B^\circ(x,r) = \{y : \|y-x\| < r\}$
$B(x,r) = \{y : \|y-x\| \leq r\}$
We know $\overline{B^\circ(x,r)} \subset B(x,r)$ (since the closed ball is closed)
Let's take $y : \|y-x\| = r$
Construct a sequence of $\begin{cases} y_n \to y \\ y_n \in B^\circ(x,r) \end{cases}$
(this would mean $y \in \overline{B^\circ(x,r)}$ and so $\overline{B^\circ(x,r)} = B(x,r)$).

$$y_n = x + (y-x)(1-\tfrac{1}{n})$$
$$\|y_n - y\| = \|x + (y-x)(1-\tfrac{1}{n}) - y\| = \|y-x\| \cdot \tfrac{1}{n} = \tfrac{r}{n} \to 0$$
$$\|y_n - x\| = \|(y-x)(1-\tfrac{1}{n})\|$$
$$= \|y-x\|(1-\tfrac{1}{n}) = r(1-\tfrac{1}{n}) < r$$
$\Rightarrow y_n \in B^\circ(x,r)$. ✓

④ $(C[a,b], \|\cdot\|_{sup})$, $\underbrace{P[a,b]}_{\text{all polynomials}} \subset C[a,b]$

$P[a,b]^\circ = \emptyset$
$\overline{P[a,b]} = C[a,b]$
$\partial P[a,b] = C[a,b]$

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$\boxed{K \subset \mathbb{R} \text{ is } \underline{\text{compact in } \mathbb{R}} \text{ if every cover } \{I_\alpha\}_{\alpha \in A} \text{ of } K \text{ by open intervals has a finite subcover.}}$

Def

Let (X, d) be a metric space.
A set K is called compact if any cover $\{I_\alpha\}_{\alpha\in A}$ of K by open sets has a finite subcover.

$K \subset \bigcup_{\alpha \in A} I_\alpha$

Exercise

In $(\mathbb{R}, |\cdot|)$: "compact in $\mathbb{R}$" $\Leftrightarrow$ "compact"

trivial

Lemma: any open set in $\mathbb{R} = \bigcup_{\beta\in B} I_\beta$ where I_β are open intervals

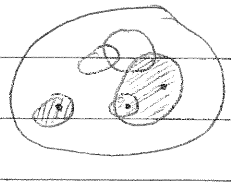
Def

Let (X, d) be a metric space.
A set $K \subset X$ is called sequentially compact if any sequence (x_n) s.t. $x_n \in K$ $\forall n$ has a subsequence converging to a point in K.

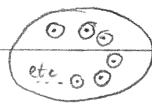
Examples

1). $(\mathbb{R}, |\cdot|)$
$[a, b]$ is compact by the Heine-Borel Thm.
$[a, b]$ is sequentially compact by Bolzano-Weierstrass.

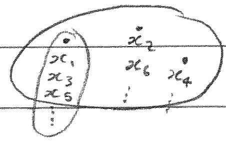
2). Discrete space.
(a) What are compact sets?
K finite $\Rightarrow$ compact
(any finite set in any metric space is compact)



K infinite $\Rightarrow$ not compact
(take the cover $\{\{x\}\}_{x\in K}$, it has no finite subcover)
K compact $\Leftrightarrow$ K finite.



(b) What are sequentially compact sets?



K - finite $\Rightarrow$ sequentially compact.

K - infinite $\Rightarrow$ not sequentially compact.

(x_n all distinct, any subsequence (x_{n_k}) still consists of distinct elements $\Rightarrow$ not eventually constant $\Rightarrow$ doesn't converge)

Theorem 3.13

K is compact $\Longleftrightarrow$ K is sequentially compact.

Remark

($\Leftarrow$) is hard to prove, no proof will be given.
We are not allowed to use this direction in homework / exam.

Remark

In topological spaces: compact $\overset{\Leftarrow}{\Rightarrow}$ sequentially compact

metric spaces: $d \rightsquigarrow$ open sets $\rightsquigarrow$ continuity

topological spaces: open sets $\rightsquigarrow$ continuity.

Proof

[$\Rightarrow$] (only this direction).

Suppose K is compact, but not sequentially compact.

$\exists (x_n)$ s.t. $x_n \in K$, $\forall n \in \mathbb{N}$,

but it has no subsequence converging to a point in K

Claim

$\forall y \in K \; \exists r(y) > 0$ s.t. $x_n \notin B^\circ(y, r(y)) \setminus \{y\}$

Proof of Claim

If $\exists x_{n_k} \in B^\circ(y, \frac{1}{k}) \setminus \{y\}$

then $d(x_{n_k}, y) < \frac{1}{k} \to 0$

$\Rightarrow x_{n_k} \to y$, contradiction. $\square$

Proof of Thm cont:

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$\{B^\circ(y, r(y))\}_{y \in K}$ ← cover of K

Since K is compact there is a finite subcover

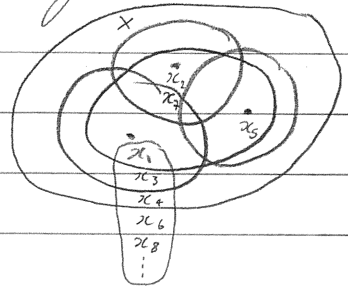
$B^\circ(y_1, r(y_1)), \dots, B^\circ(y_m, r(y_m))$

Now all x_n are equal to the values $y_1, \dots, y_m$

$\Rightarrow$ there is a subsequence $x_{n_k} = y_i$ for some $1 \leq i \leq m$ $\forall k$.

$\Rightarrow x_{n_k} \to y_i \in K$, contradiction.

So K is sequentially compact. $\square$



Def

A set A is called bounded if

$$\text{diam}(A) < \infty,$$

where $\text{diam}(A) = \sup_{x,y \in A} d(x, y)$.

Theorem 3.14

Let K be compact.

(a) K is closed and bounded

(b) if $L \subset K$ is closed then L is compact.

Remark

In $\mathbb{R}$ or $\mathbb{R}^n$ (with any norm),

compact $\Leftrightarrow$ closed + bounded

not allowed to use this!

It is not true in general.

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Theorem 3.14

Let K be a compact set in a metric space (X,d). Then

ⓐ K is closed and bounded

ⓑ if $L \subset K$ is closed then L is compact.

Recall

K compact = every cover has a finite subcover

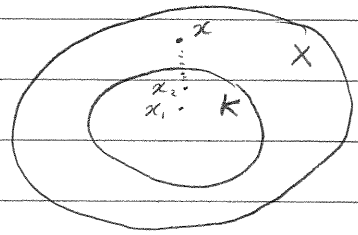
K seq. compat = every sequence has has a convergent subsequence (to a point in K)

compact $\Leftrightarrow$ seq. compact

[$\Rightarrow$ done, $\Leftarrow$ not done]

Proof

ⓐ K compact $\Rightarrow$ K closed.



Suppose K is compact but not closed.

$\Rightarrow \exists\, x_n \in K$ s.t. $x_n \to x \notin K$

$\underbrace{\{X \setminus B(x, \tfrac{1}{n})\}_{n\in\mathbb{N}}}_{\text{open sets covering } X\setminus\{x\}} \leftarrow$ cover for K

Since K is compact, it has a finite subcover

$X\setminus B(x,\frac{1}{n_1}), \dots, X\setminus B(x,\frac{1}{n_m})$.

That is, $K \subset \bigcup_{i=1}^{m} X \setminus B(x, \frac{1}{n_i})$

$$= X \setminus B\left(x, \frac{1}{\max\{n_1,\dots,n_m\}}\right)$$

Since $x_n \to x$ $\exists\, n$ s.t. $d(x_n, x) < \dfrac{1}{\max\{n_1,\dots,n_m\}}$

$$\Rightarrow \begin{cases} x_n \in B\left(x, \dfrac{1}{\max\{x_1,\dots,x_m\}}\right) \\ x_n \in K \end{cases}$$

$\times\!\!\!\times$ $\square$

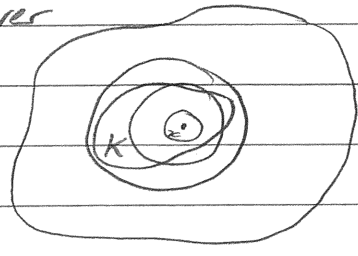
$\underline{K \text{ compact} \Rightarrow K \text{ bounded}}$

Pick $x \in K$

$\{B^\circ(x,n)\}_{n\in\mathbb{N}} \leftarrow$ cover for X and, in particular, for K.

K is compact so there is a finite subcover

$B^\circ(x,n_1), \dots, B^\circ(x,n_m)$



$\Rightarrow K \subset \bigcup_{i=1}^{m} B^\circ(x,n_i) = B^\circ(x, \max\{n_1,\dots,n_m\})$

$\Rightarrow \operatorname{diam}(K) \leq 2\max\{n_1,\dots,n_m\} < \infty$

since $d(\underset{K}{\underset{\cap}{y}}, \underset{K}{\underset{\cap}{z}}) \leq d(y,x) + d(x,z) \leq 2\max\{n_1,\dots,n_m\}$

(b) Let $\{I_\alpha\}_{\alpha\in A}$ be a cover for L.

$\left\{\{I_\alpha\}_{\alpha\in A}, \underbrace{X\setminus L}_{\text{open}}\right\}$ is a cover for K.

Since K is compact, there is a finite subcover of K

$I_{\alpha_1}, \dots, I_{\alpha_m}, X\setminus L$

Since $L \subset K$, $I_{\alpha_1}, \dots, I_{\alpha_m}, X\setminus L$ is a cover of L

$\Rightarrow I_{\alpha_1}, \dots, I_{\alpha_m}$ is a cover of L $\square$

Theorem 3.15

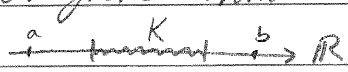
In $\mathbb{R}$ with the standard metric,

K is compact $\Leftrightarrow$ K is closed and bounded.

Proof

[$\Rightarrow$] follows from Thm 3.14.

[$\Leftarrow$]



Since K is bounded, $K \in [a,b]$ for some $a<b$

since $[a,b]$ is compact [Heine-Borel] and K is closed

$\Rightarrow K$ is compact (by Thm 3.14 b). $\square$

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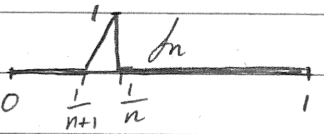
Example

$(C[0,1], \|\cdot\|_{sup})$

$B(0,1)$ - not compact

(but: bounded and closed).

We will show $B(0,1)$ is not seq. compact, which implies it is not compact.



$f_n \in B(0,1)$ since $\|f_n - 0\|_{sup} = 1 \leq 1$

$\|f_n - f_m\|_{sup} = 1, \ m \neq n$

For any subsequence $\|f_{n_k} - f_{n_m}\|_{sup} = 1 \ \forall k \neq m$

$\Rightarrow$ no subsequence is Cauchy

$\Rightarrow$ no subsequence converges

$\Rightarrow$ $B(0,1)$ is not seq. compact.

Theorem 3.16

Let (X, d_X), (Y, d_Y) be metric spaces, $f: X \mapsto Y$ be continuous, $K \subset X$ be compact. Then $f(K)$ is compact.

Proof

Let $\{I_\alpha\}_{\alpha \in A}$ be a cover of $f(K)$

$\{f^{-1}(I_\alpha)\}_{\alpha \in A}$ is a cover of K

(Open since f is continuous and I_α is open.)

Since K is compact, there is a finite subcover

$f^{-1}(I_{\alpha_1}), \dots, f^{-1}(I_{\alpha_m})$ of K

$\Rightarrow$ $\{I_{\alpha_1}, \dots, I_{\alpha_m}\}$ is a finite subcover of $f(K)$. $\square$

Theorem 3.17

Let $f : X \mapsto \mathbb{R}$ be continuous
(X: any metric; $\mathbb{R}$: standard metric)

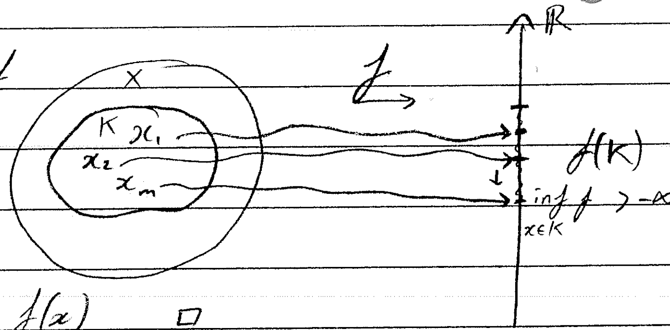
Let $K \subset X$ be compact. Then $\exists\, x_m \in K,\ x_M \in K$ such that $\inf\limits_{x \in K} f = f(x_m)$ and $\sup\limits_{x \in K} f = f(x_M)$.

Proof

We will only prove the inf. part.

By the previous theorem, $f(K)$ compact $\Rightarrow f(K)$ is closed and bounded. $\Rightarrow \inf\limits_{x \in K} f$ is finite.

$f(K)$ is closed, $f(x_n) \to \inf\limits_{x\in K} f$



$\Rightarrow \inf\limits_{x \in K} f \in f(K)$

$\Rightarrow \exists\, x_m \in K$ s.t. $f(x_m) = \inf\limits_{x \in K} f(x)$ $\square$

Def

Two norms, $\|\cdot\|$ and $|\cdot|$, on a vector space V are *equivalent* if $\exists\, c, C > 0$ such that

$$c|x| \le \|x\| \le C|x| \quad \forall x \in V.$$

Remark

$$\frac{1}{C}\|x\| < |x| < \frac{1}{c}\|x\| \qquad \text{(the definition is symmetric)}$$

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Examples

1). $\mathbb{R}^n$; $\|\cdot\|_1 \overset{\text{equivalent}}{\sim} \|\cdot\|_2$

$$\|x\|_1 = |x_1| + \dots + |x_n| \underset{\text{Cauchy Schwartz}}{\leq} \sqrt{1^2 + \dots + 1^2}\sqrt{x_1^2 + \dots x_n^2} = \sqrt{n}\|x\|_2$$

$$\|x\|_1^2 = (|x_1| + \dots + |x_n|)^2 \geq \|x\|_2^2$$

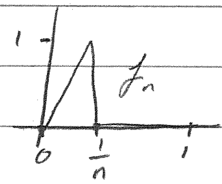
$$\underset{c}{\underset{\shortparallel}{1}} \cdot \|x\|_2 \leq \|x\|_1 \leq \underset{C}{\underset{\shortparallel}{\sqrt{n}}}\|x\|_2$$

2). $C[0,1]$, $\|\cdot\|_{sup} \not\sim \|\cdot\|_1$

Suppose $c\|f\|_1 \leq \|f\|_\infty \leq C\|f\|_1 \quad \forall f \in C[0,1]$

$\|f_n\|_\infty = 1 \quad \forall n$

$\|f_n\|_1 = \frac{1}{2} \cdot 1 \cdot \frac{1}{n} = \frac{1}{2n}$



So $\cancel{c \cdot \frac{1}{2n}} \leq 1 \leq \underbrace{C \cdot \frac{1}{2n}}_{\to 0}$ contradiction

Theorem 3.18

Two norms $|\cdot|$, $\|\cdot\|$ are equivalent

$\Leftrightarrow$ if $x_n \to x$ w.r.t. $|\cdot|$ then $x_n \to x$ w.r.t. $\|\cdot\|$ (and vice versa).

Proof

[$\Rightarrow$] Suppose $\exists c, C$ s.t. $c\|x\| \leq |x| \leq C\|x\| \quad \forall x$

If $x_n \to x$ w.r.t. $|\cdot|$ then $|x_n - x| \to 0$

then $\|x_n - x\| \leq \frac{1}{c}|x - x_n| \to 0$ then $x_n \to x$ w.r.t. $\|\cdot\|$.

[$\Leftarrow$] Suppose the R.H.S is true but $\|\cdot\| \not\sim |\cdot|$,

w.l.o.g. assume there is no C s.t. $|x| \leq C\|x\| \quad \forall x$.

This means $\forall n \quad \exists x_n$ s.t. $|x_n| > n\|x_n\|$.

$$y_n = \frac{x_n}{|x_n|}$$

• $\|y_n\| = \dfrac{\|x_n\|}{|x_n|} < \dfrac{1}{n} \to 0 \Rightarrow y_n \to 0$ w.r.t. $\|\cdot\|$

• $|y_n| = \dfrac{|x_n|}{|x_n|} = 1 \not\to 0 \Rightarrow y_n \not\to 0$ w.r.t. $|\cdot|$

Contradiction with the r.h.s.

$\square$

<u>Theorem</u>

Any norm $\|\cdot\|$ on $\mathbb{R}^n$ is equivalent to $\|\cdot\|_2$ (and in particular, any two norms on $\mathbb{R}^n$ are equivalent).

<u>Proof</u>

Let $e_1, \dots, e_n$ be the standard basis of $\mathbb{R}^n$.

$$\|x\| = \|x_1 e_1 + \dots + x_n e_n\| \underset{\substack{\uparrow \\ \text{property of norms}}}{\leq} |x_1| \cdot \|e_1\| + \dots + |x_n| \cdot \|e_n\|$$

$$\underset{\text{Cauchy-Schwart}}{\leq} \sqrt{x_1^2 + \dots + x_n^2} \underbrace{\sqrt{\|e_1\|^2 + \dots + \|e_n\|^2}}_{C_1}$$

$$\leq C_1 \cdot \|x\|_2$$

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$|\cdot| \sim \|\cdot\| \iff \exists c, C_1$ s.t. $c|x| \leq \|x\| \leq C_1|x| \quad \forall x.$

Theorem

$|\cdot| \sim \|\cdot\| \iff x_n \to x$ w.r.t. $|x|$ iff $x_n \to x$ w.r.t $\|x\|$.

Theorem

On $\mathbb{R}^n$ any norm $\|\cdot\|$ is equivalent to $\|\cdot\|_2$.
(In particular, any two norms are equivalent).

Proof

1). $\|x\| \leq C_1 \|x\|_2 \quad \forall x$ done.

2). Let's show $\exists c$ st. $\|x\| \geq c\|x\|_2 \quad \forall x \in \mathbb{R}^n$.

$S = \{x \in \mathbb{R}^n : \|x\|_2 = 1\}$

- S is bounded : $\|x-y\|_2 \leq \|x\|_2 + \|y\|_2 = 2$
- S is closed : $S = \underbrace{B_{\|\cdot\|_2}(0,1)}_{\text{closed}} \setminus \underbrace{B^\circ_{\|\cdot\|_2}(0,1)}_{\text{open}}$

not proven but left as exercise. hint: show $[a_1, b_1] \times \dots \times [a_n, b_n]$ is compact.

[In $\mathbb{R}^n$ closed & bounded $\Rightarrow$ compact]

$\Rightarrow S$ is compact

Define $f : S \mapsto \mathbb{R}$ (with standard metric)

$f(x) = \|x\|$

f is continuous : let $x_n \mapsto x \quad , x_n, x \in S$

$|f(x_n) - f(x)| = \big|\|x_n\| - \|x\|\big| \leq \|x_n - x\|$ by Δ inequality

$\leq C_1 \|x_n - x\|_2 \to 0$

So $\exists c = \inf_S f \quad , \quad c = f(\underset{S}{\underset{\cap}{x_0}}) = \|x_0\| > 0$ since $x_0 \neq 0$.

Let $x \in \mathbb{R}^n, x \neq 0$

$$\|x\| = \left\| \frac{x}{\|x\|_2} \cdot \|x\|_2 \right\| = \|x\|_2 \cdot f\left(\frac{x}{\|x\|_2}\right) \geq c \cdot \|x\|_2$$

$\square$

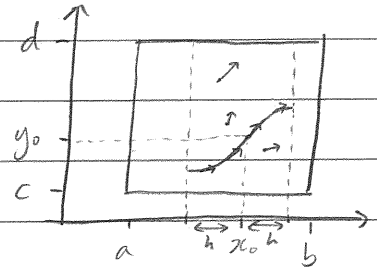
$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases} \quad (*)$$

Theorem (Picard)

Let $f : [a,b] \times [c,d] \mapsto \mathbb{R}$ be such that f and $\frac{\partial f}{\partial y}$ are continuous.

Let $(x_0, y_0) \in (a,b) \times (c,d)$

Then $\exists\, h > 0$ such that the solution of $(*)$ on $[x_0 - h,\ x_0 + h]$ exists and is unique.

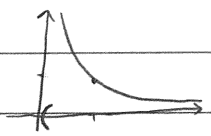


Example

$$\begin{cases} y' = -y^2 \\ y(1) = 1 \end{cases} \qquad f(x,y) = -y^2 \text{ defined on } \mathbb{R}^2$$

and is as nice as it gets!

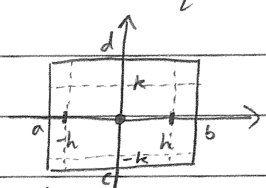
$y(x) = \frac{1}{x}$



Proof

W.l.o.g $x_0 = 0$, $y_0 = 0$

$(*)$ is equivalent to $\begin{cases} y(x) = \int_0^x f(t, y(t))\, dt \\ y \text{ is continuous} \end{cases} \quad (**)$



Let $h > 0$, $k > 0$

$B_{h,k} = \{ f \in C[-h, h] \text{ s.t. } \|f\|_{sup} \leq k \}$

with $\|\cdot\|_{sup}$

Assumptions

$h < b$, $h < |a|$

$k < d$, $k < |c|$.

$Mh \leq k/2$

$M'h \leq 1/2$

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Claim 1:

$B_{h,k}$ is a complete metric space.

We know that $C[-h,h]$ is complete,
hence we only need to check that $B_{h,k}$ is closed:

$$\left.\begin{array}{l} f_n \to f \\ \|f_n\|_{sup} \leq k \end{array}\right\} \Rightarrow \|f\|_{sup} \leq k$$

Define

$T: B_{h,k} \mapsto B_{h,k}$

by $(T(y))(x) = \int_0^x f(t, y(t))\,dt.$

Denote by M and M' bounds for $|f|$ and $\left|\frac{\partial f}{\partial y}\right|$ respectively.

$$\left|\int_0^x f(t,y(t))\,dt\right| \leq M \cdot |x| \leq Mh \leq k/2$$

$\Rightarrow T(y) \in B_{h,k} \Rightarrow T$ is well-defined.

T is a contraction mapping:

$$\|T(y) - T(\tilde{y})\|_{sup} = \sup_{x\in[-h,h]} \left|(T(y))(x) - (T(\tilde{y}))(x)\right|$$

$$= \sup_{x\in[-h,h]} \left|\int_0^x f(t,y(t))\,dt - \int_0^x f(t,\tilde{y}(t))\,dt\right|$$

$$\leq \sup_{x\in[-h,h]} \int_0^x \left|f(t,y(t)) - f(t,\tilde{y}(t))\right| dt$$

$$\leq \sup_{x\in[-h,h]} \int_0^x \left\|\frac{\partial f}{\partial y}\right\|_{sup} \cdot |y(t) - \tilde{y}(t)|\,dt$$

$$\leq M' \cdot h \cdot \|y - \tilde{y}\|_{sup} \quad (\text{since } |x| < h)$$

$$\leq \frac{1}{2}\|y - \tilde{y}\|_{sup}$$

By the contraction mapping, $\exists$ a unique fixed point $y \in B_{h,k} : y = Ty$.

MATH 7102

23-03-17

Theorem (Picard)

$f: [a,b] \times [c,d] \mapsto \mathbb{R}$

$(x_0, y_0) \in (a,b) \times (c,d)$

$f, \frac{\partial f}{\partial y}$ continuous

Then $\exists h > 0$ s.t. $\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$ has a unique solution on $[x_0 - h, x_0 + h]$

Proof

w.l.o.g $(x_0, y_0) = (0,0)$

Choose h, k: $h < a, |b|$, $k < c, |d|$

$Mh < k/2$

$M'h < 1/2$

- $B_{h,k} = \{ g : [-h, h] \mapsto \mathbb{R}$, continuous and $\|g\|_{sup} \leq k \}$

complete: $f_n \mapsto f$, should use $g_n \mapsto g$ instead [in proof write g instead of f]

- $T : B_{h,k} \mapsto B_{h,k}$ $\quad (T(y))(x) = y_0 + \int_0^x f(x, y(x))\, dx$

$y_0 = 0$ here

$T(y)$ is indeed in $B_{h,k}$: $\|T(y)\|_{sup} \leq \frac{k}{2}$

- T is a contraction.

CMT : $\exists$ a unique solution $T(y) = y$ i.e.

? $\begin{cases} y(x) = \int_0^x f(t, y(t))\, dt \\ y \text{ continuous} \end{cases}$ which is equivalent to $(*)$

Uniqueness

Suppose there is a solution φ s.t. $\|\varphi\|_{sup} > k$

Then $\exists h_0 > 0$ s.t. $\|\varphi\|_{sup} = k$ on $[-h, h]$

Then $T(\varphi) = \varphi$, $\cdot \ \|\varphi\|_{sup} = k$

$\cdot \ \|T(\varphi)\|_{sup} \leq \frac{k}{2}$ contradiction.

$\square$

Example

$$\begin{cases} y' = 2xy \\ y(0) = 1 \end{cases}$$

$$\frac{dy}{y} = 2x \Rightarrow \log y = x^2 + c$$

$$0 = 0^2 + c \Rightarrow c = 0$$

$$y(x) = e^{x^2}$$

$$y_0 \xrightarrow{T} y_1 \xrightarrow{T} y_2 \quad \cdots \quad y$$

$$y_0(x) = 1$$

$$y_1(x) = (T(y_0))(x) = 1 + \int_0^x 2t \cdot 1 \, dt = 1 + x^2$$

$$y_2(x) = (T(y_1))(x) = 1 + \int_0^x 2t(1+t^2)\, dt = 1 + x^2 + \frac{x^4}{2}$$

$$y_3(x) = \cdots\cdots = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{3!}$$

Exam info

Best 8 of 9 hw's will count.

4 out of 5 questions in exam

Each question: $\frac{1}{2}$ bookwork, $\frac{1}{4}$ easy unseen, $\frac{1}{4}$ hard unseen

3 past papers on moodle.