7112 Geometry and Groups Notes

Based on the 2017 spring lectures by Dr J Evans

MATH 7112

09-01-17

Office Hours
Mon 1-2
Thurs 3-4

Hw: 5 Qs:
Q1 : 2 marks
Q2 : 5 marks
Q3-5 : 3 marks
12 or more silver star
15 or more gold star

8 Hw sheets in total

Symmetry

Definition

An isometry of Euclidean space is a bijection $T : \mathbb{R}^n \mapsto \mathbb{R}^n$ s.t. $|T(x) - T(y)| = |x - y|$.

Isometries are distance-preserving bijections.
(eg. rotation, reflection, translation...)

$\mathrm{Isom}(\mathbb{R}^n) = \{ T : \mathbb{R}^n \mapsto \mathbb{R}^n : T \text{ is an isometry} \}$

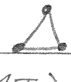
$\mathrm{Isom}(\mathbb{R}^n)$ forms a group under composition of maps.
ie.
- composition $T_1 \circ T_2(x) = T_1(T_2(x))$ is an isometry if T_1 & T_2 are.
- associativity ie. $T_1 \circ (T_2 \circ T_3) = (T_1 \circ T_2) \circ T_3$
- identity. $\exists I \in \mathrm{Isom}(\mathbb{R}^n)$ with $I(x) = x$ such that $T \circ I = I \circ T = T$
- inverses: $\forall T \in \mathrm{Isom}(\mathbb{R}^n)$ $\exists T^{-1} \in \mathrm{Isom}(\mathbb{R}^n)$ s.t. $T \circ T^{-1} = T^{-1} \circ T = I$

ie. undoing a distance preserving map still preserves distances.

Def

If $P \subseteq \mathbb{R}^n$, then $\mathrm{Sym}(P) := \{ T \in \mathrm{Isom}(\mathbb{R}^n) : TP = P \}$ is the symmetry group of P.

Example

P = Equilateral triangle in $\mathbb{R}^2$ 

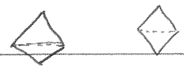
3 reflections, 3 rotations $(\underset{\text{identity}}{0}, \frac{2\pi}{3}, \frac{4\pi}{3})$ D_6

P = Regular hexagon 

6 reflections, 6 rotations D_{12}

dihedral groups

Regular p-gon $\longrightarrow D_{2p}$

Π : regular tetrahedron 

12 rotations 12 reflections $\Rightarrow$ 24 in total (will cover $\mathrm{Sym}^+ P$)

S_4 , $|S_4| = 4! = 24$

$$\left.\boxed{\begin{array}{l} \mathrm{Sym}\,\Pi \cong S_4 \\ \mathrm{Sym}^+\Pi \cong A_4 \end{array}}\right\}$$ Lemma Π is a regular tetrahedron.

Proof

Symmetries permute the four vertices so get a map

$A : \mathrm{Sym}\,\Pi \mapsto S_4$

$T \mapsto (v \mapsto Tv)$

i.e. $A(T)v = Tv$, $A(T) \in S_4$

Recall that a homomorphism $f : G \mapsto H$ is a map satisfying

- $f(gh) = f(g)f(h)$
- $f(1) = 1$

i.e. a homomorphism is a structure-preserving map of groups.

An isomorphism is a bijective homomorphism.

We're trying to prove that $\mathrm{Sym}\,\Pi \overset{\text{isomorphism}}{\cong} S_4$

We already have a map $A : \mathrm{Sym}\,\Pi \mapsto S_4$, we need to prove that it's a bijective homomorphism.

a). homomorphism , b). surjective , c). injective.

a). A is a homomorphism.

Proof: Need to show $A(1) = 1$

This is just saying $A(1)v = v$, i.e. $A(1)v = v$ i.e. $1(v) = v$ ✓

NTP : $A(T_1 T_2) = A(T_1)A(T_2)$

$$\begin{aligned} A(T_1 \circ T_2)v &= (T_1 \circ T_2)(v) = T_1(T_2(v)) \\ &= A(T_1)(T_2(v)) = A(T_1)(A(T_2)v) \\ &= (A(T_1) \circ A(T_2))v \quad ✓ \end{aligned}$$

$\square$

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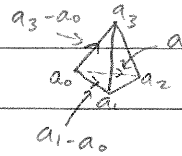
c). Lemma: (Next week)

An isometry of $\mathbb{R}^3$ is determined completely by its action on the vertices of a tetrahedron.

More precisely, if $T_1, T_2 \in \mathrm{Isom}(\mathbb{R}^n)$ & $a_0, \dots, a_n \in \mathbb{R}^n$ are such that

- $T_1 a_k = T_2 a_k \quad \forall k$
- $a_1 - a_0, \dots, a_n - a_0$ is a basis of $\mathbb{R}^n$

then $T_1 = T_2$.



} Tetrahedron case

This implies A is injective

Proof: we need $\ker A = \{1\}$

$\ker A = \{g \in \mathrm{Sym}\,\Pi : A(g) = 1\}$

Claim (Algebra 4)

If $A: G \mapsto H$ has $\ker A = \{1\}$ then A injective & conversely.

Proof:

If $g \in \ker A$ then $A(g) = 1$

But $A(1) = 1$ so unless $g = 1$, A is not injective.

Conversely, if $\ker A = \{1\}$, suppose $A(g_1) = A(g_2)$.

Then $A(g_1 g_2^{-1}) = 1 \Rightarrow g_1 g_2^{-1} \in \ker A$

$\Rightarrow g_1 g_2^{-1} = 1 \Rightarrow g_1 = g_2$. $\square$

So to prove A injective, we need to prove $\ker A = \{1\}$.

$\ker A = ?$ $\quad A: \mathrm{Sym}\,\Pi \mapsto S_4$

So $\ker A = \{$symmetries that fix all vertices$\}$

($1 \in S_4$ is the permutation that just fixes all vertices).

The Lemma we stated (without proof) says that if T fixes all vertices then it acts the same way as the identity $\therefore$ equals the identity. $\ker A = \{1\}$. $\square$

b). $A: \mathrm{Sym}\,\Pi \mapsto S_4$ is surjective

To transpose v & w, reflect in the plane equidistant from them.

To permute $(1\,2\,3)$ or $(1\,3\,2)$ etc, fix 4 and rotate

)

Fact: (Algebra 2?)

Transpositions generate S_4

(i.e. any permutation can be written as $t_1 t_2 \dots t_k$ with t_i transpositions e.g. $(1 2 3) = (1 3)(1 2)$).

Fact: If $t_1, \dots, t_n$ generate the group H, $F: G \mapsto H$ is a homomorphism, and $g_1, \dots, g_n$ are preimages of $t_1, \dots, t_n$ then F is surjective. i.e. $F(g_i) = t_i$.

Proof of Fact:

$\forall h \in H$ we can write h as $h = t_{i_1}^{\pm 1} \dots t_{i_m}^{\pm 1}$ ← can use inverses

$$\Rightarrow h = F(g_{i_1})^{\pm 1} \dots F(g_{i_m})^{\pm 1}$$

$$= F(g_{i_1}^{\pm 1} \dots g_{i_m}^{\pm 1}) \quad \text{as } F \text{ is a homomorphism.}$$

$$\in \text{Image}(F)$$

$\Rightarrow$ anything is in the image $\Rightarrow F$ surjective □

$\therefore$ Since A hits all transpositions & transpositions generate S_4 we see that A is surjective. □

$\Rightarrow A: \text{Sym}\,\Pi \mapsto S_4$ is an isomorphism. □.

Definition

Let G be a group, and X be a set. A group action of G on X is a homomorphism $G \xrightarrow{A} \text{Perm}(X)$

So in our previous example, we had a group action of $\text{Sym}\,\Pi$ on $X = \{\text{vertices of } \Pi\}$

So for every $g \in G$ we get a permutation $A(g): X \mapsto X$.

This is the "action" of g on X.

e.g. $\text{Sym}\,P$ acts on P, i.e. symmetries of P permute the points of P.

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eg. if P is a polytope (polygon/hedron/...) then $\text{sym}\,P$ acts on the set of $\begin{cases} \text{vertices of } P \\ \text{edges of } P \\ \text{faces of } P. \end{cases}$

As in the previous example, this map A sends a group we don't understand well ($\text{Sym}\,P$) to a group we do understand ($\text{Perm}\,X$).

Def

If G acts on a set X and $x \in X$, then

$$\text{Orb}(x) = \{y \in X : y = gx \text{ for some } g \in G\}$$

? abuse of notation: $A(g) : X \mapsto X$, we will write $g : X \mapsto X$.

$$\begin{tikzcd} & g_2 x \\ x \arrow[ur] \arrow[dr] & \\ & g_1 x \arrow[uu, "g_2 g_1^{-1}"'] \end{tikzcd} \Rightarrow g_2 g_1^{-1}(g_1 x) = g_2 x$$

$\text{Orb}(x)$ = "set of points you can map x to using the action of G"

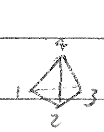
Def

$$\text{Stab}(x) = \{g \in G : gx = x\}$$
$$= \text{"set of } g \in G \text{ fixing } x\text{"}$$

$\text{Stab}(x) \leq G$ (stabliser)

Example

$X = \{\text{Vertices of } \Pi\}$, $G = \text{Sym}\,\Pi$

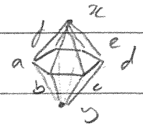


$\text{Orb}(x) = X$ "transitive"

$\text{Stab}(x) = S_3 \leq S_4$ (permutations of 1, 2, 3 keeping 4 fixed)

Example: hexagonal bipyramid $= P$

Sym P acts on $X = \{\text{vertices of } P\}$



$Orb(x) = \{x, y\}$

$a, b, c, d, e, f \notin Orb(x)$ (different # of incoming edges)

$Orb(a) = \{a, b, c, d, e, f\}$
$= Orb(b) = Orb(c) = \ldots$

Theorem: Orbit-Stabiliser Theorem

If G acts on a set X and $x \in X$, then there is a bijection between: $G/Stab(x)$ and $Orb(x)$.

$\overset{\shortparallel}{\text{cosets of } Stab(x) \leq G.}$

In particular $|Orb(x)| = \dfrac{|G|}{|Stab(x)|}$

$\iff |G| = |Orb(x)| \cdot |Stab(x)|$

For the hexagonal bipyramid

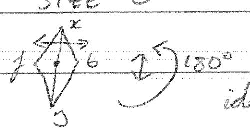
$Orb(x) = \{x, y\}$ so $|Orb| = 2$

$Stab(x) =$ symmetries of hexagon $\cong D_{12}$

$\Rightarrow |Sym\ P| = 2 \times 12 = 24$

Similarly: $Orb(a) = \{a, \ldots, f\}$ size 6

$|Stab(a)| = 4$



So $|Sym\ P| = 24$

Icosahedron $= \mathbb{I}$ (20 faces, triangles)

Sym $\mathbb{I}$ acts on $\{\text{faces}\}^{=X}$ transitively, i.e. $Orb(x) = X$

$Stab(x) = S_3$ (symmetries of triangular face)

$|Sym\ \mathbb{I}| = 20 \times 6 = 120$

$Sym\ \mathbb{I} \cong A_5 \times C_2$

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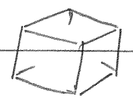
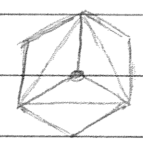
Orbit-Stabilizer Theorem

Let G be a group and suppose G acts on a set X.
Let $x \in X$. Then $\exists$ bijection $F: G/\underbrace{Stab(x)}_{\{g \in G : gx = x\}} \longmapsto \underbrace{Orb(x)}_{\{y \in X : y = gx \text{ for some } g \in G\}}$

Example

Cube 

$G = Sym(C)$
$X_1 = \{\text{vertices of } C\}$
$X_2 = \{\text{faces of } C\}$
$X_3 = \{\text{edges of } C\}$

$Orb(\text{vertex } x) = X_1$ (transitive) 
$Stab(x) = \{\text{Symmetries of a triangle}\}$
$= D_6 \cong S_6$

"Vertex figure of x is a triangle and symmetries in $Stab(x)$ preserve the vertex figure."

$|G| = |Orb(x)||Stab(x)|$
$= |X_1||D_6| = 6 \times 8 = 48$

$F \in X_2$
$Orb(F) = X_2$
$Stab(F) = \{\text{Symmetries of a square}\} = D_8$

So $|G| = |Orb(F)||Stab(F)|$
$= |D_8||X_2| = 8 \times 6 = 48.$

Example

Cuboctahedron $= P$ (root system of $S_3(4)$)
Faces: $\left\{\begin{array}{l}\text{square} - 6 \\ \text{triangle} - 8\end{array}\right\} = X$ "cube with corners cut off"

$G = Sym(P)$
F = square face
$Orb(F) = \{\text{square faces}\} \subseteq X$ $\quad Stab(F) = \{\text{symmetries of a square}\} = D_8$

To prove $Stab(F) = D_8$, observe that every $g \in Stab(F)$ is a symmetry of F which is square, so we get a homomorphism $Stab(F) \to D_8$

To check it's an isomorphism $(Stab(F) \to D_8)$ we need to show injectivity and surjectivity.

Inj. Suppose $g \in Stab(F)$ goes to $1 \in D_8$
i.e. g fixes the vertices of F.
Moreover, any symmetry of a bounded solid fixes the centre of mass, in particular O.

Recall: Lemma:
If $T \in Isom(\mathbb{R}^n)$ satisfies $Ta_0 = a_0, \dots, Ta_n = a_n$ for $a_0, \dots, a_n \in \mathbb{R}^n$ s.t. $a_1 - a_0, \dots, a_n - a_0$ form a basis then $T = 1$.

In our case use $a_0 = O$; $\{a_1, a_2, a_3\}$ form a basis for $\mathbb{R}^3$
$\Rightarrow$ if $g \in Stab\,F$ and g maps to 1 in D_8 then $g = 1$.

Surj.
Given a symmetry g of the square face F. We want to find a symmetry of P which stabilises the face F and acts as the symmetry g on F.

A symmetry of f is either a rotation (about the centre) or a reflection.
If it's a rotation, use the same rotation of P around the axis through the centre of F.
If it's a reflection in a line L, use the reflection of P in a plane of symmetry containing the line L & O.

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Proof of Orbit-Stabiliser Theorem

Let $F: G/\mathrm{Stab}(x) \longrightarrow \mathrm{Orb}(x)$ be the map $F(\underbrace{g\,\mathrm{Stab}(x)}_{\{gh \,:\, h \in \mathrm{Stab}(x)\}}) \rightarrow gx \in \mathrm{Orb}(x)$

NTS : injectivity, surjectivity and if it is well-defined.

Well-defined

If $g_1\mathrm{Stab}(x) = g_2\mathrm{Stab}(x)$ then $\{g_1h : h \in \mathrm{Stab}(x)\} = \{g_2h : h \in \mathrm{Stab}(x)\}$

$\Rightarrow g_1 \cdot 1 = g_2 \cdot h$ for some $h \in \mathrm{Stab}(x)$

$\Rightarrow g_1x = g_2hx \qquad h \in \mathrm{Stab}(x) \Rightarrow hx = x$

$\Rightarrow F(g_1\mathrm{Stab}(x)) = F(g_2\mathrm{Stab}(x))$ ✓

Injectivity

If $g_1x = g_2x$ we want to prove $g_1\mathrm{Stab}(x) = g_2\mathrm{Stab}(x)$.

$g_1x = g_2x \Rightarrow g_1^{-1}g_2x = x$

$\Rightarrow g_1^{-1}g_2 \in \mathrm{Stab}(x)$

$\Rightarrow g_1\mathrm{Stab}(x) = g_2\mathrm{Stab}(x)$. ✓

Surjectivity

$\mathrm{Orb}(x) = \{y : y = gx\}$

So given $y = gx \in \mathrm{Orb}(x)$ note that

$y = F(g\mathrm{Stab}(x)) \in \mathrm{Im}(F)$ ✓

$\square$

Rotations vs Reflections

We will see (next week) that any $T \in \mathrm{Isom}(\mathbb{R}^n)$ has the form $T(x) = Ax + b$ where $b \in \mathbb{R}^n$ (is a translation),

$A \in O(n) = \{A : A^TA = 1\}$ (A is $n \times n$ matrix).

Notice $|A^TA| = \det 1 = 1 = |A^T||A| = |A|^2 \Rightarrow \det A = \pm 1$.

Definition: $SO(n) = \{A \in O(n) : \det A = 1\}$ (Special orthogonal matrices)

Rotations!

Definition: $\mathrm{Isom}^+(\mathbb{R}^n) = \{T \in \mathrm{Isom}(\mathbb{R}^n) : Tx = Ax + b,\ A \in SO(n)\}$

[orientation preserving, i.e. right handed basis $\rightarrow$ right handed basis]

$\mathrm{Sym}^+(P) = \mathrm{Sym}\, P \cap \mathrm{Isom}^+(\mathbb{R}^n)$

eg. $\mathrm{Sym}^+(\text{triangle}) = C_3$ rotational symmetry.

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Last time, we defined

$Sym^+(P) = Sym(P) \cap Isom^+(\mathbb{R}^n)$

Rotational symmetries of P.

$Isom^+(\mathbb{R}^n) = \{T : \mathbb{R}^n \to \mathbb{R}^n : T \text{ isometry}, Tx = Ax + b, A \in SO(n), b \in \mathbb{R}^n\}$

Goal: Compute $Sym^+ C$ ← cube

$Sym^+ \mathbb{D}$ ← dodecahedron

(as groups, not just the size!)

What is the size of $Sym^+ C$?

Let $Sym^+ C$ act on $X = \{\text{vertices of } C\}$ (8 vertices)

$Stab(x) = \{3 \text{ rotational symmetries}\}$

$\Rightarrow$ (Orbit - Stabiliser theorem) $|Sym^+ C| = 3 \times 8 = 24$.

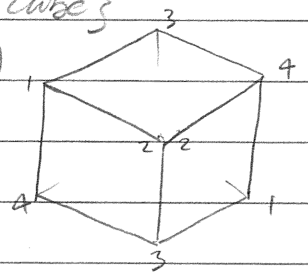
Lemma

$Sym^+ C \cong S_4$

Proof

Let $X = \{\text{diagonals inside the cube}\}$

(i.e. pairs of opposite vertices)



$Sym^+ C$ sends pairs of opposite vertices to pairs of opposite vertices $\therefore$ it acts on X

i.e. we have a homomorphism

$Sym^+ C \xrightarrow{A} Perm(X) = S_4$

We will prove that this (map, A) is surjective.

As both groups have size 24 this will imply that A is an isomorphism.

Recall: if we prove that all transpositions are in the image of A then we get that A is surjective (because transpositions generate S_4).

$(1\ 2) = 180°$ rot. around axis through edges $\vec{12}$.
By doing the same with other edges we get all transpositions.
transpositions $= \binom{4}{2} = 6 =$ # pairs of opposite edges. □

dodecahedron

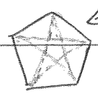
$Sym^+(\mathbb{D})$ (12 pentgon faces)
What is the size of $Sym^+\mathbb{D}$?
Let $Sym^+(\mathbb{D})$ act on $X = \{faces\}$, $|X| = 12$
$Stab(F) = \{$ 5 rotations of pentagon $\}$
face

so $|Sym^+\mathbb{D}| = 60$
$|S_5| = 120$, $|A_5| = 60$

even no. of transpositions

Lemma
even permutations of 5 objects
$Sym^+\mathbb{D} \cong A_5$.

Proof
We will find a set of 5 mystery objects acted upon by $Sym^+\mathbb{D}$ & show that the action homomorphism $Sym^+\mathbb{D} \mapsto S_5$ hits all even permutations.
Since $|Sym^+\mathbb{D}| = 60 = |A_5|$, this will prove the Lemma.


face has 5 diagonals
$\Rightarrow$ total $= 60$ diagonals

Claim:
The set $D = \{diagonals\}$ can be partitioned into 5 sets of 12, each of which forms a cube.

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Let $X = \{\text{inscribed cubes}\}$

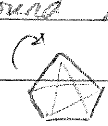
We get an action $\text{Sym}^+ \mathbb{D}$ on X

$\therefore \text{Sym}^+ \mathbb{D} \xrightarrow{\text{homo}} S_5$

We will show that the image is the subgroup A_5

$$\left.\begin{array}{l}(12)(34) \in A_5 \\ (123) \in A_5 \\ (12345) \in A_5\end{array}\right\} \text{All elements of } A_5 \text{ have one of these three cycle types}$$

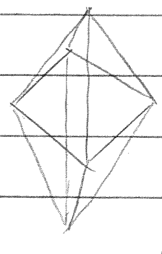
$(12345) \rightarrow \frac{2\pi}{5}$ rotation around midpoint of a face



$(123) \rightarrow \frac{2\pi}{3}$ rotation around a vertex

$(12)(34) \rightarrow \pi$ rotation about an axis through the mid points of an edge.

$\text{Sym}^+(\mathbb{O})$ octahedron $\qquad \text{Sym}^+(\mathbb{I})$ icosahedron



8 triangular faces

$$\left.\begin{array}{l}X = \{\text{faces}\} \\ |\text{Stab}(F)| = 3\end{array}\right\} \Rightarrow 3 \times 8 = 24 \text{ symmetries.}$$

triangle

Lemma

$\text{Sym}^+(\mathbb{O}) \cong \text{Sym}^+(C)$

$\text{Sym}^+(\mathbb{I}) \cong \text{Sym}^+(\mathbb{D})$

Proof

Given any polytope, P, there is a dual polytope $P^\vee$ obtained by putting a vertex at the centre of each face.

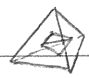
"P dual"

$\mathbb{O}^{\vee} = C \;, \quad C^{\vee} = \mathbb{O}$

$\#\,\text{faces}(P) = \#\,\text{vertices}(P^{\vee})$
& vice versa $\quad (P^{\vee})^{\vee} = P$ (up to scale)

Similarly $\begin{cases} \mathbb{I}^{\vee} = \mathbb{D} \\ \mathbb{D}^{\vee} = \mathbb{I} \end{cases}$

In general, any symmetry of P includes a symmetry of $P^{\vee}$ so we get a homomorphism $\operatorname{Sym} P \mapsto \operatorname{Sym} P^{\vee}$
and as $(P^{\vee})^{\vee} = P$ we get $\operatorname{Sym} P^{\vee} \mapsto \operatorname{Sym} P$
which is an inverse for this homomorphism
$\Rightarrow \operatorname{Sym}(P) \cong \operatorname{Sym}(P^{\vee})$. $\square$

$\mathbb{T}^{\vee} = \mathbb{T} \rightarrow$ 

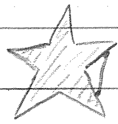
Convex polytopes in $\mathbb{R}^n$
If $x, y \in \mathbb{R}^n$ then
$tx + (1-t)y \quad t \in [0,1]$ parameterises all points on the straight line segment $\overline{xy}$

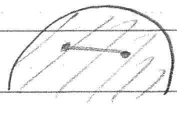
Def
If $X \subseteq \mathbb{R}^n$ is a finite set, then
$\operatorname{Conv}(X) = \{ \sum_{x \in X} t_x x : \sum_{x \in X} t_x = 1, t_x \geq 0 \} \leftarrow$ Convex hull of X

e.g. $\operatorname{Conv}(\{x, y\}) = \{ t_x x + t_y y : t_x + t_y = 1 \}$
$t_x = t \;, \quad t_y = 1 - t$

Def
A set $P \subseteq \mathbb{R}^n$ is called convex if $\forall\, x, y \in P$ the line segment $\overline{xy}$ is contained in P.

[Concave = complement is convex]


not convex!


Convex

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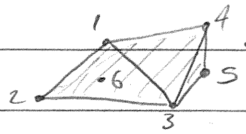
A convex polytope is the convex hull of a finite set in $\mathbb{R}^n$

Claim (exercise)
$\mathrm{Conv}(X)$ is convex, X is a finite set.

In fact $\mathrm{Conv}(X)$ is the smallest convex set containing all the points of X



$\leftarrow \mathrm{Conv}(\{1,2,3,4,5\})$
$= \mathrm{Conv}(\{1,2,3,4,5,6\})$

Definition
A vertex of a convex polytope $\mathrm{Conv}\, X$ is a point $x \in X$ s.t. $\mathrm{Conv}(X \setminus \{x\}) \neq \mathrm{Conv}(X)$

Exercise
If $V \subseteq X$ is the set of vertices then $\mathrm{Conv}\, V = \mathrm{Conv}\, X$.

Definition
Assume $X = \{\text{vertices}\}$, $P = \mathrm{Conv}(X)$.
$\mathrm{Int}(P) = \left\{ \sum_{x \in X} t_x x : \sum_{x \in X} t_x = 1,\ t_x > 0 \right\}$ $\leftarrow$ interior of P.

Definition
The boundary ∂P is $P \setminus \mathrm{int}(P)$

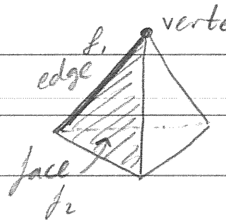
e.g. $X =$ , $P = \mathrm{Conv}(X) =$ 
$\mathrm{Int}\, P =$ , $\partial P =$ 

The boundary ∂P is stratified by k-dimensional faces ($\dim P = n$)

	codimension = n - dimension
0 - dim. faces = vertices	n
1 - dim. faces = edges	$n-1$
2 - dim. faces = faces	$n-2$
$\vdots$	
$n-3$ dim faces = peaks	3
$n-2$ dim faces = ridges	2
$n-1$ dim faces = facets	1

Facets of tetrahedron = triangular faces
" " triangle = edges
" " interval = endpoints (vertices)

Def
A flag in a polytope is a sequence
$f = (f_0 \subseteq f_1 \subseteq \dots \subseteq f_{n-1})$
of faces f_i with $\dim f_i = i$
(ie. vertex in an edge in a face ... in a facet.)



$f_0 \subseteq f_1 \subseteq f_2$

flags in T (tetrahedron) = $\underset{\#\text{ vertices}}{4} \times \underset{\#\text{ edges containing the fixed vertex}}{3} \times \underset{\#\text{ faces containing the fixed edge}}{2} = 24$

Def
A convex polytope, P, is called regular if $\operatorname{Sym} P$ acts transitively on flags.

ie. any vertex can be mapped to any other,
any edge containing it can be mapped to any other,
$\vdots$

$\Rightarrow$ as symmetrical as possible.

Platonic solid = reg. polyhedron in 3D

Theorem

Regular convex polytopes are classified by their Schläfli symbol, defined recursively as follows:

- Schläfli symbol of p-gon $\subseteq \mathbb{R}^2$ is $\{p\}$.
- Schläfli symbol of a Platonic solid with q copies of $\{p\}$ meeting at each vertex is $\{p, q\}$.

If we have p_{n-1} copies of $\{p_1, \dots, p_{n-2}\}$ at each peak then the Schläfli symbol is $\{p_1, \dots, p_{n-1}\}$.

Scläfli symbol for $T = \{3, 3\}$

" " " a cube $= \{4, 3\}$

" " " an octahedron $= \{3, 4\}$

" " " a dodecahedron $= \{5, 3\}$

" " " an icosahedron $= \{3, 5\}$

Duality : $\{p, q\} \to \{q, p\}$

"tesseract" $\leftrightarrow$ 4-d cube $= P$

facets are 3-d cubes

3 of these around each edge

$\Rightarrow \{4, 3, 3\}$

Action of Sym P on 8 facets, 48 symmetries in stab $\Rightarrow$ # symmetries $= 384$

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Schläfli symbol

$\{p, q\}$ — q faces meeting at each vertex

faces are p-gons

Theorem

$\mathbb{T}, \mathbb{C}, \mathbb{O}, \mathbb{D}, \mathbb{I}$ is a complete list of regular convex polytopes in 3D. (Platonic solids)

$\mathbb{T} : \{3, 3\}$
$\mathbb{C} : \{4, 3\}$
$\mathbb{O} : \{3, 4\}$
$\mathbb{D} : \{5, 3\}$
$\mathbb{I} : \{3, 5\}$

Sketch Proof

First suppose $p = 3$

Internal angle of $\triangle$ is $\pi/3$. Total angle is then $\frac{q\pi}{3}$.

If $q \leq 6$ then the total angle $\leq 2\pi$.

If the total angle $> 2\pi$ then the shape would not be convex.

$\Rightarrow q \leq 6$

If $q = 6$, we get a flat hexagon. By regularity of the polytope, all the vertices look like this

$\Rightarrow$ polytope is flat i.e. contained in a plane ※ $\Rightarrow q \leq 5$.

Internal angle for a p-gon is $\pi(1 - 2/p)$, so total angle is $q\pi(1 - 2/p) < 2\pi$ (for convexity).

$\Rightarrow 1 - 2/p < 2/q$

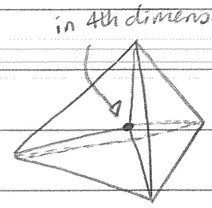
$\Rightarrow \frac{1}{2} < 1/p + 1/q$

In fact this means $\{p, q\} \in \{\text{the above list}\}$ $(p \geq 3,\ q \geq 3)$

4D regular convex polytopes

Schläfli Symbol	Name	# of facets	facets
$\{3, 3, 3\}$	4-simplex / 5-cell	5	Π
$\{3, 3, 4\}$	4-orthoplex / 16-cell	16	Π
$\{3, 3, 5\}$	600-cell	600	Π
$\{3, 4, 3\}$	24-cell	24	Ø
$\{4, 3, 3\}$	4-cube / tesseract / 8-cell	8	C
$\{5, 3, 3\}$	120-cell	120	$\mathbb{D}$

$\{3, 3, 3\}$



every face of Π gives a facet of the 4-simplex + 1 facet from original Π.

n-simplex is constructed similarly

Fact: In dim 5 & above there are precisely 3 regular polytopes

1-simplex

2-simplex 

3-simplex 

4-simplex

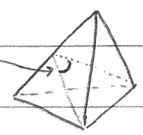
n-simplex $\{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} : x_i \geq 0 \ \& \ \sum x_i = 1\}$

For sheet 1 Q5, the facets are platonic solids.

There are q facets meeting along each edge.

In this example (→ 4-simplex) there are 3 tetrahedra meeting along each edge.

We need (in $\{p, q, r\}$), $r \times$ dihedral angle < 2



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Chapter 2 - Isometries of Euclidean Space

Def

An isometry of $\mathbb{R}^n$ is a bijection $T : \mathbb{R}^n \mapsto \mathbb{R}^n$ s.t. $|Tx - Ty| = |x - y| \quad \forall x, y \in \mathbb{R}^n$.

e.g. $Tx = x + b$ (translation by $b \in \mathbb{R}^n$) is an isometry.

e.g. $Tx = Ax$ ($A \in O(n)$) is an isometry.

Proof

$A \in O(n) \Rightarrow A^T A = 1$

$|x|^2 = x^T x \quad (*)$ $\quad \left[x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad x^T = (x_1, x_2, \dots, x_n) \right]$

$= x_1{}^2 + x_2{}^2 + \dots + x_n{}^2$

$$\begin{aligned}
|Ax - Ay|^2 &= |A(x-y)|^2 \\
&= [A(x-y)]^T A(x-y) && \text{using } (*) \\
&= (x-y)^T A^T A(x-y) \\
&= (x-y)^T (x-y) && \text{as } A^T A = 1 \text{ by } A \in O(n) \\
&= |x-y|^2 && \text{using } (*)
\end{aligned}$$

$\Rightarrow A$ is an isometry. $\square$

Lemma

The following are equivalent:

ⓐ The columns of A form an orthonormal basis (each element of basis has size 1 and $e_i \cdot e_j = \delta_{ij}$)

ⓑ $A^T A = 1$

ⓒ $(Ax) \cdot (Ay) = x \cdot y \quad \forall x, y \in \mathbb{R}^n$

ⓓ $|Ax|^2 = x^2 \quad \forall x \in \mathbb{R}^n$

$$\left[x \cdot y = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = (x_1, \dots, x_n) \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \right]$$

a ⇔ b

$$\begin{aligned}
(A^T A)_{ij} &= e_i{}^T A^T A e_j \\
&= (Ae_i) \cdot (Ae_j) \qquad Ae_i = i^{th} \text{ column of } A.
\end{aligned}$$

$\Rightarrow (A^T A)_{ij} =$ dot product of i & j th columns of A

$\Rightarrow \delta_{ij} \Leftrightarrow$ columns orthonormal

$c \Rightarrow d$

set $x = y$

$b \Rightarrow c$

$Ax \cdot Ay = x^T A^T A y = x^T y$ if $A^T A = I$

$c \Rightarrow a$

WTS: $Ax \cdot Ay = x \cdot y$

$\Rightarrow$ columns of A are orthonormal

Columns of A are Ae_i

So $Ae_i \cdot Ae_j = e_i \cdot e_j = \delta_{ij}$

as e_i are orthonormal

$d \Rightarrow c$

If $Ax \cdot Ax = x \cdot x \ \forall x$

Set $x = u+v \Rightarrow [A(u+v)] \cdot [A(u+v)] = (u+v) \cdot (u+v)$

$\Rightarrow \cancel{Au \cdot Au} + \cancel{Av \cdot Av} + 2Au \cdot Av = \cancel{u \cdot u} + \cancel{v \cdot v} + 2u \cdot v$

$\Rightarrow Au \cdot Av = u \cdot v$

$\square$

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Isometries of $\mathbb{R}^n$

We talked, last time about orthogonal matrices

$O(n) = \{A : A^T A = 1\}$

Example

2-d rotation

If $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Then the map $x \mapsto Ax$ is an isometry of $\mathbb{R}^2$ called a rotation by angle θ around the origin.

$A^T A = Id$ so A is orthogonal $\Rightarrow A \in O(2)$.

$\det(A) = 1 \Rightarrow A \in \underbrace{SO(2)}_{\text{rotations}}$.

Example

3-d rotation.

The matrix $\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \in SO(3)$

performs a rotation by θ around the z-axis

Rotating by θ around the y axis:

$$\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

Theorem

If $A \in SO(3)$ then A has an eigenvector with eigenvalue 1 (the axis of rotation).

If $A \neq 1$ then its eigenvector is unique and the restriction of A to the orthogonal complement of the axis is an element of $SO(2)$.

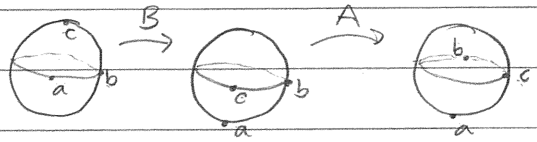
i.e. every $A \in SO(3)$ is a rotation around some axis.
In this example $A = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$ the eigenvector is $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$
(i.e. the y axis)
and we can see that $A|_{x-z\text{ plane}} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in SO(2)$

Example
(3-d). $\frac{\pi}{2}$ rotation around z-axis : $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = A$

$\frac{\pi}{2}$ rotation around y-axis : $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = B$

$$AB = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y \\ z \\ -x \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$



$\Rightarrow x = -y \;,\; z = y \quad \Rightarrow \begin{pmatrix} -z \\ z \\ z \end{pmatrix}$ $\quad \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector with eigenvalue 1.

Proof (of Thm)
If v is an eigenvector with eigenvalue 1 then $Av = v$
$\Leftrightarrow (A - 1)v = 0$
$\Leftrightarrow A - 1$ is not invertible
$\Leftrightarrow \det(A-1) = 0$

$\det(A-1) = \det((A-1)^T) = \det(A^T - 1)$
$= \det(A^{-1} - 1) \Leftarrow A \in O(3)$, so $A^T A = 1$
$= \det(A^{-1}(1-A))$
$= \det(A^{-1})\det(1-A)$ as $\det(\cdot)$ is a homomorphism
$= \det(1-A)$ as $A \in SO(3) \Rightarrow \det A = 1 = \det A^{-1}$
$= \det(-(A-1))$
$= \det(-1)\det(A-1)$
$= -\det(A-1)$ (as we're in 3-d)
$\Rightarrow \det(A-1) = 0$

So we see that if $A \in SO(3)$ then $Av = v$ for some v.
Let $W = \langle v \rangle^{\perp}$ (orthogonal complement of the line $\langle v \rangle$)
First, observe that W is preserved by the action of A.

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ie. $w \in W \Rightarrow Aw \in W$ because $w \in W$ iff $w \cdot v = 0$ iff $Aw \cdot Av = 0$ ($A \in O(3)$) iff $Aw \cdot v = 0$ ($Av = v$) iff $Aw \in W$.
Moreover, $A|_W$ (the restriction of A to W) is again orthogonal, and W is 2-dimensional, so $A|_W \in O(2)$.
Pick a basis v, w_1, w_2 ($w_i \in W$).
With respect to this basis, A is block diagonal

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{A|_W} \\ 0 & & \end{pmatrix}$$

$$\Rightarrow \underset{1}{\underset{\parallel}{\det A}} = \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{A|_W} \\ 0 & & \end{pmatrix}$$

$$= 1 \times \det(A|_W)$$

$$\Rightarrow \det(A|_W) = 1 \quad \Rightarrow A|_W \in SO(2) \quad \square.$$

So every $A \in SO(3)$ is really (geometrically) a rotation about some axis.

Example: Reflections.

$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ performs a reflection in $\mathbb{R}^2$ along the y axis.

$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ performs a reflection in $\mathbb{R}^3$ in the y-z plane.

In general for any hyperplane $H \subseteq \mathbb{R}^n$ we get a reflection r_H ($H \cong \mathbb{R}^{n-1}$).

The equation $x \cdot \underset{\text{normal vector}}{\nu} = \underset{\text{number}}{c}$ defines a hyperplane
$H = \{x \in \mathbb{R}^n : x \cdot \nu = c\}$ [ν is a unit vector in $\mathbb{R}^n$, $c \in \mathbb{R}$]
e.g. y-z plane is $\{x : x \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0\}$

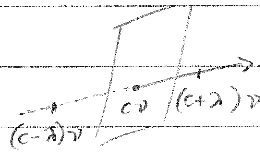
A reflection in H should:
- fix the points in H
- a point on the normal line to H should map to its mirror image.

$r_H(x) = x - 2((x \cdot \nu) - c)\nu$

If $x \in H$ then $x \cdot \nu = c$ so $r_H(x) = x$

$x \cdot \nu = c$ and $\nu \cdot \nu = 1 \Rightarrow (c\nu) \cdot \nu = c \Rightarrow c\nu \in H$

$$\begin{aligned} r_H((c+\lambda)\nu) &= (c+\lambda)\nu - 2\big(((c+\lambda)\nu)\cdot\nu - c\big)\nu \\ &= (c+\lambda)\nu - 2(c+\lambda-c)\nu \\ &= (c+\lambda)\nu - 2\lambda\nu \\ &= (c-\lambda)\nu \end{aligned}$$



Observe:

$H \ni 0 \iff c = 0$

In this case, $r_H(x) = x - 2(x\cdot\nu)\nu$ which is linear in x.

So $r_H(x) = Ax$ for some matrix A.

What matrix is this?

$A = 1 - M$ for some M where $Mx = 2(x\cdot\nu)\nu = 2\nu\nu^T x$

So $Ax = (1 - 2\nu\nu^T)x$ where $\nu\nu^T = \begin{pmatrix} \nu_1 \\ \vdots \\ \nu_n \end{pmatrix}(\nu_1 \dots \nu_n) = \begin{pmatrix} \nu_1\nu_1 & \nu_1\nu_2 & \dots & \nu_1\nu_n \\ \nu_2\nu_1 & & & \vdots \\ \vdots & & & \\ \nu_n\nu_1 & \dots & & \nu_n\nu_n \end{pmatrix}$

In the example with $\nu = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $\nu\nu^T = \begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}$

$\Rightarrow A = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - 2\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix} = \begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix}$

Theorem

If $T \in \mathrm{Isom}\,\mathbb{R}^n$, then $Tx = Ax + b$ for some $A \in O(n)$, $b \in \mathbb{R}^n$.

Lemma

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an isometry with the following property: there exists a basis $e_1, \dots, e_n$ of $\mathbb{R}^n$ s.t. $T(0) = 0$ and $T(e_i) = e_i \ \forall i$. Then $T = 1$.

Proof

Here $T(x)$ is written Tx

Let $x \in \mathbb{R}^n$ be a point. We have $|x| = |x - 0| = |Tx - T0|$ ③ (because $T \in \mathrm{Isom}\,\mathbb{R}^n$) $\Rightarrow |x| = |Tx|$ so T preserves norms of vectors.

Moreover: ① $|x - e_i|^2 = x\cdot x - 2(x\cdot e_i) + e_i\cdot e_i$

$= |x|^2 + |e_i|^2 - 2(x\cdot e_i)$

② $|Tx - Te_i|^2 = |Tx - e_i|^2$

$= |Tx|^2 + |e_i|^2 - 2(Tx)\cdot e_i$

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① = ② because T is an isometry ($T \in \mathrm{Isom}\,\mathbb{R}^n$)

③ $\Rightarrow |x| = |Tx|$

So overall we get

$$|x|^2 + |e_i|^2 - 2x \cdot e_i = |x|^2 + |e_i|^2 - 2Tx \cdot e_i$$

$\Rightarrow \quad x \cdot e_i = (Tx) \cdot e_i \quad \Rightarrow$ Components of Tx w.r.t. e_i are the same as for $x \Rightarrow Tx = x$. $\square$

Lemma:

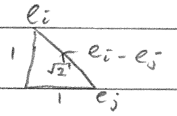
If T is an isometry of $\mathbb{R}^n$ fixing 0, and $e_1, \dots, e_n$ is an orthonormal basis for $\mathbb{R}^n$, then $Te_1, \dots, Te_n$ is an orthonormal basis for $\mathbb{R}^n$.

Proof

Let $f_i = T(e_i)$

$$|f_i| = |f_i - 0| = |Te_i - 0| = |e_i - 0| \qquad (T \in \mathrm{Isom}\,\mathbb{R}^n,\ T(0) = 0)$$
$$= |e_i| = 1$$

$$f_i \cdot f_j = \tfrac{1}{2}\left(|f_i|^2 + |f_j|^2 - |f_i - f_j|^2\right)$$
$$= \tfrac{1}{2}\left(|Te_i|^2 + |Te_j|^2 - |Te_i - Te_j|^2\right)$$
$$= \tfrac{1}{2}\left(1 + 1 - |e_i - e_j|^2\right)$$
$$= \tfrac{1}{2}(1 + 1 - 2) = 0$$



$\Rightarrow |e_i - e_j|^2 = 2$

$\square$

Lemma

If $\{e_i\}_{i=1}^n$ and $\{f_i\}_{i=1}^n$ are orthonormal bases for $\mathbb{R}^n$ then $\exists!$ matrix $A \in O(n)$ s.t. $f_i = Ae_i$.

Proof

Let A be the change of basis matrix so that $Ae_i = f_i$. Need to show $A_i \in O(n)$

$$\delta_{ij} = f_i \cdot f_j = (Ae_i) \cdot (Ae_j)$$
$$= (Ae_i)^T (Ae_j)$$
$$= e_i^T A^T A e_j = (A^T A)_{ij}$$

$\Rightarrow A^T A = I \Rightarrow A \in O(n)$. $\square$

Note: we can rephrase the lemma as follows:
$O(n)$ acts on orthonormal bases transitively and with stabiliser $= \{1\}$ (freely).

Proof of Theorem
[Any $T \in \mathrm{Isom}(\mathbb{R}^n)$ has the form $Tx = Ax + b$]

Define $S(x) = T(x) - T(0)$
(i.e. we set $b = T(0)$ and $S = t_{-b} \overset{\text{composition}}{\circ} T$ where $t_{-b}(x) = x - b$).
As t_{-b} is an isometry, S is still an isometry, but now $S(0) = T(0) - T(0) = 0$.
So we need to show that S is an orthogonal transformation.
By the lemma, if e_i is an orthonormal basis of $\mathbb{R}^n$ then $f_i = S(e_i)$ is another orthonormal basis.
By the other lemma $\exists!\, A \in O(n)$ s.t.
$$Ae_i = f_i \;\; \forall i.$$
Therefore $S(0) = 0$ and $S(e_i) = f_i = A(e_i)$
$\Rightarrow U = A^{-1} \circ S \in \mathrm{Isom}\,\mathbb{R}^n$ satisfies $U(0) = 0$ and $U(e_i) = e_i$.
By the first lemma, we see $U = 1$.
$\Rightarrow 1 = U = A^{-1} \circ S \Rightarrow Sx = Ax \;\; \forall x$
$\Rightarrow Tx = S(x) + T(0) = Ax + b$ $\square$

Theorem
An arbitrary isometry $T \in \mathrm{Isom}\,\mathbb{R}^n$ can be written as a product of reflections (we can actually write down a list of reflections of length $\leq n+1$ to multiply together to get a particular isometry).

Example
e.g. $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = A$, $Ax = -x$, $A \in \mathrm{Isom}(\mathbb{R}^3)$

It's not a reflection: it has no eigenvectors of eigenvalue 1, also not a rotation (same reason).

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$r_{H_1} \circ r_{H_2} \in \mathrm{Isom}^+ \mathbb{R}^3$, but $\det A = -1$ so A is not a product of 2 reflections.

$$\begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \end{pmatrix} = \underset{r_{yz\,plane}}{\begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}} \times \underset{r_{xz\,plane}}{\begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix}} \times \underset{r_{xy\,plane}}{\begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}}$$

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Theorem

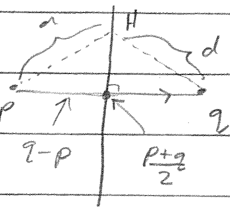
$\mathrm{Isom}\,\mathbb{R}^n$ is generated by reflections.

The "wordlength" of $\mathrm{Isom}\,\mathbb{R}^n$ w.r.t. this generating set is $\leq n+1$,
i.e. $\forall g\ \exists$ "word" $R_1 R_2 \dots R_k = g$ where R_i are reflections & the word length of g is the minimal number of letters needed to write g.

To prove this, we need a lemma.

Lemma

Given $p, q \in \mathbb{R}^n$ $(p \neq q)$ let H be the hyperplane of points equidistant from p & q.
Then $r_H(p) = q$, $r_H(q) = p$



Proof

Recall that if $H = \{x \mid x \cdot n = c\}$ then $r_H(x) = x - 2(x \cdot n - c)n$.

Claim

$\frac{p+q}{2} \in H$ & $n = \frac{p-q}{|p-q|}$

so $x = \frac{p+q}{2} \in H \Rightarrow \frac{p+q}{2} \cdot \frac{p-q}{|p-q|} = c$

$$r_H(x) = x - 2\left(x \cdot \frac{p-q}{|p-q|} - \frac{p+q}{2} \cdot \frac{p-q}{|p-q|}\right) \frac{p-q}{|p-q|}$$

$$\Rightarrow r_H(p) = p - 2\left(p - \frac{p+q}{2}\right) \cdot \left(\frac{p-q}{|p-q|}\right) \frac{p-q}{|p-q|}$$

$$= p - 2\left(\frac{p-q}{2} \cdot \frac{p-q}{|p-q|}\right) \frac{p-q}{|p-q|}$$

$$= p - (p-q) = q \quad \square$$

Proof of Theorem

Let $T \in \operatorname{Isom} \mathbb{R}^n$

Let H_0 be the plane of equidistance from 0 and $T(0)$.

$r_{H_0}(T(0)) = 0$ by the lemma. Set $T_0 = r_{H_0} \circ T$.

Suppose that $T_m \in \operatorname{Isom} \mathbb{R}^n$ s.t. $T_m e_k = e_k \ \forall k \leq m$ and $T_m(0) = 0$ (e.g. for $m = 0$ we have T_0)

Then let H_{m+1} be the plane of equidistance from e_{m+1} and T_{m+1}.

By the lemma, $r_{H_{m+1}}(T_m e_{m+1}) = e_{m+1}$.

Moreover, $r_{H_{m+1}} e_k = e_k$ $(k \leq m)$ because $e_k \in H_{m+1}$.

To see this, note that $|e_{m+1} - e_k| = |T_m e_{m+1} - T_m e_k|$

(as $T_m \in \operatorname{Isom} \mathbb{R}^n$) $= |T_m e_{m+1} - e_k|$

Now set $T_{m+1} = r_{H_{m+1}} \circ T_m$.

This now satisfies $T_{m+1} e_k = e_k$ $k \leq m+1$.

So T_n is such that $T_n(0) = 0$ and $T_n e_k = e_k \ \forall k$.

So $T_n = 1$.

But $T_n = r_{H_n} \circ r_{H_{n-1}} \circ \dots \circ r_{H_0} \circ T = 1$

$\Rightarrow T = r_{H_0} \circ r_{H_1} \circ \dots \circ r_{H_n}$ (which is $n+1$ reflections). $\square$

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Example

$Tx = -x$ in $\mathbb{R}^2$ $\quad T = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

Here $T(0) = 0$ so we do nothing, i.e. $T_0 = T$.

Next look at $Te_1 = -e_1$, so set $H_1 = y$-axis.

$T_1 = r_{H_1} \circ T$, $r_{H_1} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\Rightarrow T_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Next, $T_1 e_2 = -e_2$, so use r_{H_2} = reflection $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in x-axis.

$$T_2 = r_{H_2} \circ r_{H_1} \circ T$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

One more theorem about generation:

Theorem:

The group $SO(3)$ is generated by the elements

$R(z, \theta) \quad (\theta \in [0, 2\pi))$

$R(y, \pi/2)$

where $R(u, \theta)$ means (anticlockwise) rotation by θ around positive axis u.

Lemma:

If $A \in SO(3)$ and $Au = u'$ for axes u, u',
then $R(u', \theta) = A R(u, \theta) A^{-1}$.

$(u' \leftarrow u \longleftarrow u \leftarrow u')$

Proof:

A is just a change of coordinates sending u to u'.
so $AR(u,\theta)A^{-1}$ is just $R(u,\theta)$ viewed in a different coordinate system where u looks like u'. $\square$

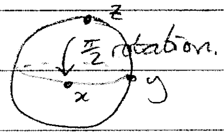
N.B. $R(u, \theta) \in \mathrm{Stab}(u)$ for the action of $SO(3)$ on unit vectors/axis.

Q1 Sheet 1 $\Rightarrow$ if $gu = u'$ $Stab(u) = g\,Stab(u')\,g^{-1}$ & this lemma is an example of that.

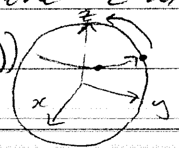
<u>Proof of Theorem:</u>

1). lemma above.

2). $R(x, \theta) = R(y, \pi/2)R(z, \theta)R(y, -\pi/2)$ (using the lemma).



3). Let u be any axis. By rotating about the z-axis we can put u in the yz plane. $(R(z, \theta_1))$
Then rotate u' around the x axis until it coincides with the z axis. $(R(x, \theta_2))$



$\therefore$ If $A = R(x, \theta_2)R(z, \theta_1)$ then $Au = z$.
Note A is a product of rotations from the generating set.

4). Finally, we see that since $Au = z$,
$$R(u, \theta) = A\,R(z, \theta)\,A^{-1} \quad \text{by the lemma.}$$
Now the RHS is a product of generators so the LHS is too. Since any element of $SO(3)$ is of the form $R(u, \theta)$ (some u, θ) we can generate all elements of $SO(3)$ from $R(y, \pi/2)$ and $R(z, \theta)$. $\square$

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Quaternions and Rotations

Def

Hamilton

The quaternion algebra, $\mathbb{H}$, is the $\mathbb{R}$-algebra with generators i, j, k subject to the relation $i^2 = j^2 = k^2$.

s.t. $\left\{\begin{array}{l} ij = -ji = k \\ jk = -kj = i \\ ki = -ik = j. \end{array}\right\} (*)$

In other words, a quaternion q is an expression of the form $q = t + ix + jy + kz$, $t, x, y, z \in \mathbb{R}$ and multiplication is $\mathbb{R}$-linear subject to the relations $(*)$

e.g.
$$\begin{aligned}(i+j)(j-2k) &= ij + jj - 2ik - 2jk \\ &= k - 1 + 2j - 2i \\ &= -1 - 2i + 2j + k\end{aligned}$$

Lemma

Quaternion multiplication is associative.

ie. $q_1(q_2 q_3) = (q_1 q_2) q_3$

Remark: This is non-trivial (but proof is tedious)

Def

If $q = t + ix + jy + kz$ then we define:

- $Re(q) = t$, $Im(q) = ix + jy + kz$
- $\bar{q} = t - ix - jy - kz$ (quaternionic conjugate)
- $|q|^2 = q\bar{q} = t^2 + x^2 + y^2 + z^2$

Lemma

a). $t^2 + x^2 + y^2 + z^2 = q\bar{q}$

b). $q\bar{q} = \bar{q}q$

c). If $q \neq 0$ then $q^{-1} = \frac{1}{|q|^2}\bar{q}$ and $qq^{-1} = q^{-1}q = 1$

d). If $q_1 = t_1 + ix_1 + jy_1 + kz_1$, $q_2 = t_2 + ix_2 + jy_2 + kz_2$, then
$Re(\bar{q}_1 q_2) = t_1 t_2 + x_1 x_2 + y_1 y_2 + z_1 z_2$

e). $\overline{q_1 q_2} = \bar{q}_2 \bar{q}_1$

f). $|q_1 q_2| = |q_1||q_2|$

Proof

a). $(t + ix + jy + kz)(t - (ix + jy + kz))$

$= t^2 - (ix + jy + kz)^2$

$= t^2 - (i^2x^2 + j^2y^2 + k^2z^2 + \cancel{xyij} + \cancel{xyji} + \cancel{ikxz} + \cancel{kixz} + \cancel{jkyz} + \cancel{kjyz})$

$= t^2 + x^2 + y^2 + z^2$ ✓

b). $q\bar{q} = t^2 + x^2 + y^2 + z^2$

$\bar{q}q \underset{\text{by (a)}}{=} t^2 + (-x)^2 + (-y)^2 + (-z)^2 = q\bar{q}$ ✓

c). $q \frac{1}{|q|^2} \bar{q} = \frac{q\bar{q}}{|q|^2} = 1$ so $q^{-1} = \frac{1}{|q|^2}\bar{q}$ is an inverse. ✓

as defined above

d). $$\left.\begin{aligned} q_1 \bar{q}_2 = {} & t_1t_2 - x_1x_2 - y_1y_2 - z_1z_2 \\ & + i(x_1t_2 + x_2t_1 + y_1z_2 - y_2z_1) \\ & + j(y_1t_2 + y_2t_1 + x_2z_1 - x_1z_2) \\ & + k(z_1t_2 + z_2t_1 + x_1y_2 - x_2y_1) \end{aligned}\right\} A$$

$\Rightarrow Re(q_1\bar{q}_2) = t_1t_2 + x_1x_2 + y_1y_2 + z_1z_2$ ✓

e). $\overline{q_1 q_2}$ is given by eqn. A with $i \mapsto -i,\ j \mapsto -j,\ k \mapsto -k$

$\bar{q}_2\bar{q}_1$ is given by eqn A with $\begin{cases} x \mapsto -x \\ y \mapsto -y \\ z \mapsto -z \end{cases}$ & $1 \mapsto 2$

(this switches the signs of the i, j, k terms). ✓

f). Now $\overline{q_1q_2} = \bar{q}_2\bar{q}_1$ so $|q_1q_2|^2 = q_1q_2\overline{q_1q_2}$

$= q_1q_2\bar{q}_2\bar{q}_1$

$= |q_1|^2|q_2|^2$ ✓

□

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What does this have to do with $SO(3)$?

Def

Let $G = \{q \in \mathbb{H} : |q| = 1\}$

be the group of unit quaternions.

- associative ✓
- $1 \in G$ is the identity ✓
- q^{-1} exists and is equal to $\bar{q}$, $|\bar{q}| = |q|$ ✓
- $q_1, q_2 \in G \Rightarrow |q_1 q_2| = |q_1||q_2| = 1$ so $q_1 q_2 \in G$ ✓

So its a group.

Note that topologically, G is the set of points in $\mathbb{H} \cong \mathbb{R}^4$

$S^3 = \{(x, y, z, t) \mid x^2 + y^2 + z^2 + t^2 = 1\} \subseteq \mathbb{R}^4$

This is a 3-dimensional sphere in 4-dimensional space (otherwise known as $SU(2)$).

There is an action of G on $\mathbb{H}$ defined as follows:

[Recall: A group action of G on a set X is a homomorphism $A : G \mapsto Perm(X)$ $Perm(X) = \{\text{permutations on } X\}$]

$A : G \mapsto Perm(\mathbb{H})$

$A(g)$ is a map $\mathbb{H} \mapsto \mathbb{H}$ and its the map $x \mapsto gxg^{-1}$

i.e. for each $g \in G$ we get a transformation

$A(g) : \mathbb{H} \mapsto \mathbb{H}$ which sends x to gxg^{-1}.

Lemma

This conjugation action of G on $\mathbb{H}$ satisfies:

1). It's linear, i.e. $x \mapsto gxg^{-1}$ is linear over $\mathbb{R}$.

2). $|gxg^{-1}| = |x|$ so actually this action is by orthogonal transformations of $\mathbb{H}$.

3). The subspace $Im\,\mathbb{H}$ is preserved by this action, i.e. if $x \in Im\,\mathbb{H}$ then $gxg^{-1} \in Im\,\mathbb{H}$.

Proof

1). $g(\lambda x + \mu y) g^{-1} = \lambda g x g^{-1} + \mu g y g^{-1} \quad \forall \lambda, \mu \in \mathbb{R}$

follows by $\mathbb{R}$-linearity of quaternion multiplication.

2). $|g x g^{-1}| = |g||x||g^{-1}| = |x|$

3). $\operatorname{Im} \mathbb{H} = \{ q \in \mathbb{H} : \bar{q} = -q \}$

So if $x \in \operatorname{Im}\mathbb{H}$ then
$$\begin{aligned} \overline{g x g^{-1}} &= \overline{g^{-1}}\, \bar{x}\, \bar{g} \\ &= g \bar{x} g^{-1} \qquad \left[\begin{matrix} \text{as } \bar{g} = g^{-1}, \\ \text{as } g\bar{g} = 1 \end{matrix}\right] \\ &= -g x g^{-1} \end{aligned}$$

$\square$

Example

$g = i$ what rotation of $\mathbb{R}^3 = \operatorname{Im}\mathbb{H}$ does this quaternion perform?

Let $q = ix + jy + kx$ be in $\operatorname{Im}\mathbb{H}$ $(i^{-1} = -i)$

$g q g^{-1} = -i(ix + iy + kz) i = ix - jy - kz$

using $-iii = i$, $-iji = -j$, $-iki = -k$.

This is the rotation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ which is a

π-rotation around the x-axis.

e.g. $-j(ix + jy + kz) j = -ix + jy - kz$

(π rotation around the y axis)

Example

Let $g = \cos\theta + i\sin\theta$, $g^{-1} = \cos\theta - i\sin\theta$

$$\begin{aligned} &(\cos\theta + i\sin\theta)(ix + jy + kz)(\cos\theta - i\sin\theta) \\ &= ix + (j\cos\theta + k\sin\theta) y (\cos\theta - i\sin\theta) + z(k\cos\theta - j\sin\theta)(\cos\theta - i\sin\theta) \\ &= ix + y(j\cos^2\theta + k\sin\theta\cos\theta + k\sin\theta\cos\theta - j\sin^2\theta) \\ &\qquad + z(k\cos^2\theta - j\sin\theta\cos\theta - j\sin\theta\cos\theta - k\sin^2\theta) \\ &= ix + y(j\cos 2\theta + k\sin 2\theta) + z(-j\sin 2\theta + k\cos 2\theta) \end{aligned}$$

So $A(g)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y\cos 2\theta - z\sin 2\theta \\ y\sin 2\theta + z\cos 2\theta \end{pmatrix}$

This is a rotation by 2θ around the x-axis.

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Lemma

Let q be a unit quaternion not equal to ± 1.
Then $\exists!\ u \in \operatorname{Im} \mathbb{H}$ with $|u| = 1$
and $\theta \in [0, 2\pi)$ s.t. $q = \cos\theta + u\sin\theta$.

Proof

Let $r = \operatorname{Re}(q)$ then $|q|^2 = r^2 + |\operatorname{Im} q|^2 = 1$
$\Rightarrow -1 \leq r \leq 1$
$\Rightarrow \exists!\ \theta \in [0, 2\pi)$ s.t. $(\cos\theta, \sin\theta) = (r, |\operatorname{Im} q|)$
Now let $u = \dfrac{\operatorname{Im}(q)}{\sin\theta} = \dfrac{\operatorname{Im}(q)}{|\operatorname{Im}(q)|}$. This is a unit quaternion.
So $q = \cos\theta + \operatorname{Im}(q) = \cos\theta + u\sin\theta$.
(If $q = \pm 1$ then $\theta = 0$ or π so we cannot divide by $\sin\theta$). $\square$

We will see that if $g = \cos\theta + u\sin\theta$ $(|u| = 1)$
then $A(g)$ (the rotation of $\operatorname{Im} \mathbb{H}$ associated to g)
is a rotation by 2θ around u.

Lemma 1

Suppose that $u_1 = ix_1 + jy_1 + kz_1$, $u_2 = ix_2 + jy_2 + kz_2$
are unit imaginary quaternions.
Then $u_1 u_2 = -u_1 \cdot u_1 + u_1 \times u_2$
where we think of u_1, u_2 as vectors in $\mathbb{R}^3$,
- $\cdot$ denotes dot product
- $\times$ denotes cross product.

Proof

Exercise on Sheet 3 $\square$

ie. the imaginary part is $i(y_1 z_2 - y_2 z_1) + j(z_1 x_2 - x_1 z_2) + k(x_1 y_2 - x_2 y_1)$

Lemma 2

Let u, v be orthogonal unit imaginary quaternions and
let $w = uv$ (quaternion product). Then u, v, w form an orthonormal right-handed

basis of $\mathbb{R}^3$. Any such basis satisfies

$$u^2 = v^2 = w^2 = -1, \quad uv = -vu = w$$
$$vw = -wv = u$$
$$wu = -uw = v$$

Proof

From lemma 1, $w = u \times v$ so u, v, w form a R.H. orthonormal basis by Methods 1.

By lemma: $u^2 = -1$ $\quad$ [$u \cdot u = 1$, $u \times u = 0$, $uv = -u \cdot v + u \times v$
$v^2 = -1$ $\quad$ $= w = -vu$ etc.]
$w^2 = -1$

Theorem

If $g = \cos\theta + u\sin\theta$ then $g(ix + jy + kz)g^{-1}$ is obtained by rotating $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ around u by angle 2θ.

Proof

Let u be the vector given by $g = \cos\theta + u\sin\theta$.
Let v be a vector in $\mathbb{R}^3$ orthogonal to it & unit length.
Let $w = uv$. Lemma 2 $\Rightarrow$ u, v, w is an orthonormal basis of $\mathbb{R}^3$ & $u^2 = v^2 = w^2 = -1$, $uv = -vu = w$ etc.
Write $a \in \operatorname{Im}\mathbb{H}$ in terms of the basis u, v, w.

$a = \alpha u + \beta v + \gamma w$.

Now $gag^{-1} = (\cos\theta + u\sin\theta)(\alpha u + \beta v + \gamma w)(\cos\theta - u\sin\theta)$

$(\cos\theta + u\sin\theta)u(\cos\theta - u\sin\theta) = u$

similar to earlier example $\rightarrow$ $(\cos\theta + u\sin\theta)v(\cos\theta - u\sin\theta) = v(\cos^2\theta - \sin^2\theta) + w\, 2\sin\theta\cos\theta$

as u, v, w obey same relations as i, j, k. $\quad$ $(\cos\theta + u\sin\theta)w(\cos\theta - u\sin\theta) = -v\, 2\sin\theta\cos\theta + w(\cos^2\theta - \sin^2\theta)$.

So $a \mapsto gag^{-1}$ has the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\theta & -\sin 2\theta \\ 0 & \sin 2\theta & \cos 2\theta \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

with respect to the basis u, v, w.

$\therefore$ it's a rotation as claimed. $\square$

Recall that A (our group action on $\mathrm{Im}\,\mathbb{H}$) is a group homomorphism $G \mapsto \mathrm{Perm}(\mathrm{Im}\,\mathbb{H})$.
We've now seen that A lands in the subgroup $SO(3) \subseteq \mathrm{Perm}(\mathrm{Im}\,\mathbb{H})$.

i.e. we can think of A as a homomorphism
$$G \mapsto SO(3)$$
unit quaternion $\mapsto$ corresponding rotation.

We know this because we just proved that every $g \in G$ has the form $\cos\theta + u\sin\theta$ & anything of this form acts as a rotation.

In fact $A : G \mapsto SO(3)$ is surjective as any element of $SO(3)$ is a rotation by some angle ϕ around an axis u so we pick $g = \cos\frac{\phi}{2} + u\sin\frac{\phi}{2}$ to get a preimage, so every rotation is given by some quaternion.

In fact A is 2-to-1 (and not injective)

Proof

It suffices to check that $\ker A$ has size 2.
($A^{-1}(h)$ is a coset of $\ker A$ so every preimage has the same size).

$\ker A = A^{-1}(1) = \{1, -1\}$ $\quad [g = -1 \Rightarrow gxg^{-1} = --x = x]$

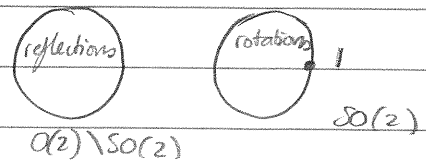
Any $\cos\theta + u\sin\theta$ acts as a nontrivial rotation if $\theta \neq 0, \pi$.

We know that 1 and -1 act as the identity rotation $\therefore$ live in the kernel.
Nothing else lives in $\ker A$ as any other quaternion rotates by nontrivial angle mod 2π.

$\square$

Every rotation is given by 2 different quaternions.
This "double-cover" of $SO(3)$ by G is responsible for spin in Quantum Field Theory.

Picture of $O(2)$:



Two disconected components of $O(2)$.
This is similar for all higher dimensions.

The group of unit quaternions, G, is the 3-sphere which has <u>one</u> connected component.
The action of G on $\mathbb{R}^3$ is a homomorphism $A: G \mapsto SO(3)$ so sends $1 \in G$ to $1 \in SO(3)$. Since G and $SO(3)$ are connected and A is continuous, $A(G) \in SO(3)$

[Chapter 4?]

<u>Spherical geometry</u>

0). What is a sphere?
1). What is a "straight line" on the sphere?
2). What is the distance between points on the sphere?

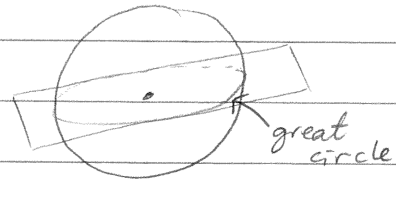
<u>Def</u>

The unit n-sphere S^n is the set $\{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$
The sphere is the boundary of the ball $\{x \in \mathbb{R}^{n+1} \mid |x| \leq 1\}$

<u>Def</u>

A <u>great circle</u> on S^n is the intersection of a 2-plane (2D) in $\mathbb{R}^{n+1}$ through O with S^n.

e.g. the equator is a great circle.
(use xy-plane)



Lemma

Through any two (non-antipodal) points there passes a unique great circle.

[antipodal means directly opposite eg. $\begin{pmatrix}x\\y\\z\end{pmatrix}$ and $\begin{pmatrix}-x\\-y\\-z\end{pmatrix}$]

Proof

Let p & q be the points, thought of as vectors in $\mathbb{R}^{n+1}$.
p & q are linearly independent $\Leftrightarrow$ not-antipodal.
Let π be the plane spanned by p & q.
The great circle we need is $\pi \cap S^n$. $\square$

If p & q are antipodal (like North & south poles) then there are infinitely great circles containing p & q.

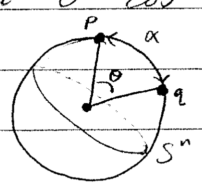
Def

A spherical line is just a segment of a great circle.

Distance on S^n

Lemma

Let α be a spherical line which subtends an angle of θ radians. Then the length of α is θ.
If p & q are the endpoints of α then $\theta = \cos^{-1}(p \cdot q)$



Proof

The angle in radians is defined so as to make this true.
The dot product $p \cdot q = |p||q|\cos\theta = \cos\theta$
as $|p||q| = 1$. $\square$

$\cos^{-1}$ is a multivalued function, so when we write $\cos^{-1}(p \cdot q)$ we just mean the smallest non-negative value in the range $[0, \pi]$

Def
$d(p,q)$ is defined to be this value of $\cos^{-1}(p\cdot q)$ in $[0,\pi]$

[$d(p,q)$ is the shortest path between p & q]

Lemma
p & q are antipodal iff $d(p,q)=\pi$

Theorem
If $p, q \in S^n$, consider all continuous paths $\gamma:[0,1] \mapsto S^n$.
Then if $l(\gamma)$ denotes the length of γ in the ambient R^n we have $l(\gamma) \geq d(p,q) = \cos^{-1}(p\cdot q)$ and equality holds $\Leftrightarrow$ γ is a parameterisation of a spherical line.
(i.e. a map $[0,1] \mapsto S^n$ whose image is a spherical line)
i.e. spherical lines minimise length of paths on S^n.

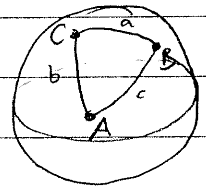
In general, given a space with a distance function $d(p,q)$, the "shortest paths" are called "geodesics"
(proof of thm next week).

Remark
To define "continuous" γ you could use the standard definition of continuous but replace $|p-q|$ with $d(p,q)$.
We could also ask for $\gamma(t) = (\gamma_1(t), \dots, \gamma_{n+1}(t))$ to be st. γ_i is continuous.

Spherical trigonometry on S^2

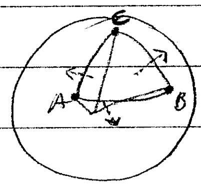
Def.
A spherical triangle is a triple of spherical lines of length $< \pi$ connecting 3 points $A, B, C \in S^2$.

Sides a, b, c opposite A, B, C, cut out by planes π_a, π_b, π_c



Lemma
The normals to π_a, π_b, π_c (pointing out of the triangle) are

$$\begin{cases} n_c = -\dfrac{1}{\sin c} A \times B \\ n_b = -\dfrac{1}{\sin b} C \times A \\ n_a = -\dfrac{1}{\sin a} B \times C \end{cases}$$



Proof
n_c is orthogonal to π_c which is spanned by A, B.
To get the correct direction we use $-A \times B$. To get the correct length $|n_c| = 1$ we have to divide by

$|A \times B| = |A||B| = \sin\theta = \sin c$ ← angle between A & B

So $n_c = -\dfrac{1}{\sin c} A \times B$.

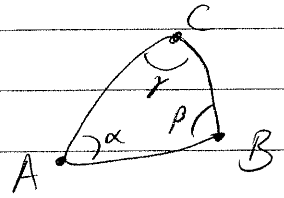
Similar for n_a and n_b. □

Lemma

$n_b \cdot n_c = -\cos(\alpha)$

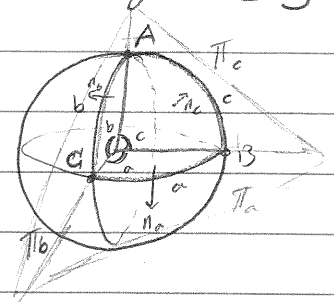
$n_c \cdot n_a = -\cos(\beta)$

$n_a \cdot n_b = -\cos(\gamma)$



where α, β, γ are the internal angles of the triangle.

Spherical Trigonometry



A, B, C unit vectors

$\cos a = B \cdot C$

$\cos b = C \cdot A$

$\cos c = A \cdot B$

The spherical line $\begin{cases} a & \text{is cut out by a plane } \Pi_a \\ c & \text{" " " " " " } \Pi_c \\ b & \text{" " " " " " } \Pi_c \end{cases}$

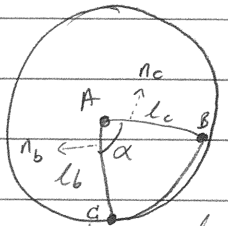
Last time we saw a formula for normals to Π_a, Π_b, Π_c:

$$n_a = -\frac{1}{\sin a} B \times C$$

$$n_b = -\frac{1}{\sin b} C \times A$$

$$n_c = -\frac{1}{\sin c} A \times B$$

From above: (ie. in the plane $T_A \leftrightarrow$ plane $\rightarrow$) $\nwarrow T_A$ tangent plane to sphere at A.



straight lines

$l_c = \Pi_c \cap T_A$

$l_b = \Pi_b \cap T_A$

We see two lines l_b and l_c intersecting at A at angle we call α.

see picture for reference

Lemma

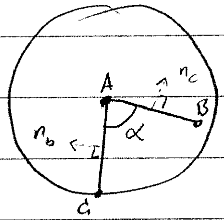
$n_a \cdot n_b = -\cos\gamma$

$n_b \cdot n_c = -\cos\alpha$

$n_c \cdot n_a = -\cos\beta$

where α, β, γ are the internal angles at A, B, C.

Proof:



From above:
we see that the angle between n_b & n_c is
$\frac{\pi}{2} + \alpha + \frac{\pi}{2} = \alpha + \pi$

so $n_b \cdot n_c = \cos(\alpha + \pi)$ (n_b, n_c unit vectors)
$= \cos\alpha \cos\pi - \sin\alpha \sin\pi$
$= -\cos\alpha$
$\square$

Theorem (Spherical cosine rule):
$\sin a \sin b \cos\gamma = \cos c - \cos a \cos b$

Remark:
If a, b, c small $\sin a \approx a$, $\cos a \approx 1 + \frac{a^2}{2}$
and spherical cosine rule reduces to Euclidean cosine rule (approximately). ← Exercise.

Proof
$-\cos\gamma = n_a \cdot n_b$ $\qquad n_a = -\frac{1}{\sin a} B \times C$, $n_b = -\frac{1}{\sin b} C \times A$
$= \frac{1}{\sin a \sin b} (B \times C) \cdot (C \times A)$

$x \cdot (y \times z) = y \cdot (z \times x)$ ①
$x \times (y \times z) = (x \cdot z) y - (x \cdot y) z$ ②

$\Rightarrow -\cos\gamma \sin a \sin b = (B \times C) \cdot (C \times A)$
$= (C \times A) \cdot (B \times C)$
$= B \cdot (C \times (C \times A))$ ①
$= B \cdot ((C \cdot A) C - (C \cdot C) A)$ ②
$= (B \cdot C)(C \cdot A) - (B \cdot A)(C \cdot C)$

$\Rightarrow \sin a \sin b \cos\gamma = A \cdot B - (B \cdot C)(C \cdot A)$ as $C \cdot C = 1$
$= \cos c - \cos a \cos b$
$\square$

Remark
We can therefore determine $\cos\gamma$ from the sidelengths a, b, c and therefore determine $\gamma \in [0, \pi]$ as $\cos$ is

injective in this interval.
On Hw4 Q5 there's a second cosine rule for determining lengths in terms of angles (has no analogue in Euclidean geometry).

Corollary (Spherical Pythagoras)
If $\gamma = \frac{\pi}{2}$ then $\cos c = \cos a \cos b$.

Exercise: this reduces to $a^2+b^2=c^2$ when a, b, c are small.

Theorem (Spherical sine rule)

$$\frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta} = \frac{\sin c}{\sin \gamma}$$

Proof:
We will show that $\sin a \sin b \sin \gamma$ equals $B\cdot(C\times A)$.
This means that

$$\sin a \sin b \sin \gamma = B\cdot(C\times A) = C\cdot(A\times B) = \sin a \sin c \sin \gamma$$

so $\frac{\sin c}{\sin \gamma} = \frac{\sin b}{\sin \beta}$ (i.e. $\sin a \sin b \sin \gamma = \sin b \sin c \sin \alpha = \sin c \sin a \sin \beta$ by cyclic symmetry $\Rightarrow$ sine rule).

C is contained in the planes Π_b and Π_a
$\therefore$ it is normal to both n_a & n_b so $n_a \times n_b$ is proportional to C.

$$|n_a \times n_b| = |n_a||n_b|\,|\sin(\pi+\gamma)|$$
$$= \sin\gamma$$

$$n_a \times n_b = C \sin\gamma$$

But $n_a = -\frac{1}{\sin a} B\times C$, $n_b = -\frac{1}{\sin b} C\times A$

so $$n_a \times n_b = \frac{(B\times C)\times(C\times A)}{\sin a \sin b}$$

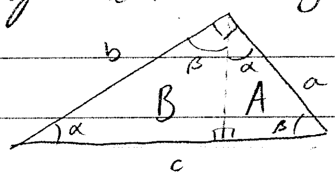
So $C \sin\gamma = \dfrac{(B \times C) \times (C \times A)}{\sin a \sin b}$

$$\Rightarrow C \sin a \sin b \sin\gamma = (B \times C) \times (C \times A)$$
$$= [(B \times C) \cdot A]C - [(B \times C) \cdot C]A \quad \text{triple product formula}$$
$$= [(B \times C) \cdot A]C \quad \text{as } (B \times C) \perp \text{ to } C$$

$$\Rightarrow \sin a \sin b \sin\gamma = A \cdot (B \times C)$$
$$= B \cdot (C \times A) = C \cdot (A \times B) \quad \text{by cyclic symmetry.}$$
$\square$

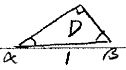
Digression

Proof of Euclidean Pythagoras: $a^2 + b^2 = c^2$



whole triangle: C

Determine angles in A & B as they must add up to π.
So they're similar, ie. rescalings of another triangle D,
with the same angles.



area $(D) = K$

rescale D by $a/b/c$ to get $A/B/C$

area$(A) = a^2K$, area$(B) = b^2K$, area$(C) = c^2K$

$$A + B = C \Rightarrow K(a^2 + b^2) = Kc^2.$$
$\square$

The triangle inequality in spherical geometry...

Theorem

If $x, y, z \in S^2$ are points on the sphere then
$d(x,y) \leq d(x,z) + d(z,y)$
with equality iff z lies on the shortest spherical line segment between x & y.

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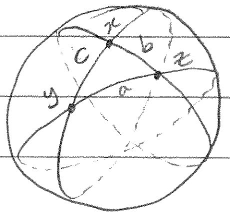
Proof

If two of the points coincide then it's easy, so assume the points are distinct.



They define a triangle with side lengths

$a = d(y,z)$

$b = d(x,z)$

$c = d(x,y)$

Let's assume for now $a, b, c < \pi$

Spherical cosine rule $\Rightarrow$

$$\cos c = \cos a \cos b + \sin a \sin b \cos \gamma$$
$$\geq \cos a \cos b - \sin a \sin b \quad = \cos(a+b)$$
$$\Rightarrow c \leq a+b \quad [\text{as } \cos \text{ is monotonic decreasing on } [0,\pi]]$$
$$\Rightarrow d(x,y) \leq d(x,z) + d(y,z)$$

Equality holds iff $\gamma = \pi$ which happens iff z is on the spherical line segment between x & y.

If one of the lengths equals π, say $d(x,y) = \pi$, then x and y are antipodal (directly opposite), so z automatically lives on a meridian curve from x to y and $d(x,z) + d(z,y) = d(x,y)$. □

Remark

In MATH 7102 (Analysis 4) you will see the definition of a metric space.

This is a set X with a "distance function"

$d: X \times X \to \mathbb{R}$ s.t. $d(x,y) \geq 0$ (equality $\Leftrightarrow x=y$)

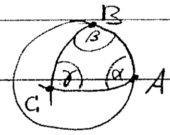
$d(x,y) \leq d(x,z) + d(z,y) \quad \forall x,y,z.$

So we've proved that (S^2, d) is a metric space.

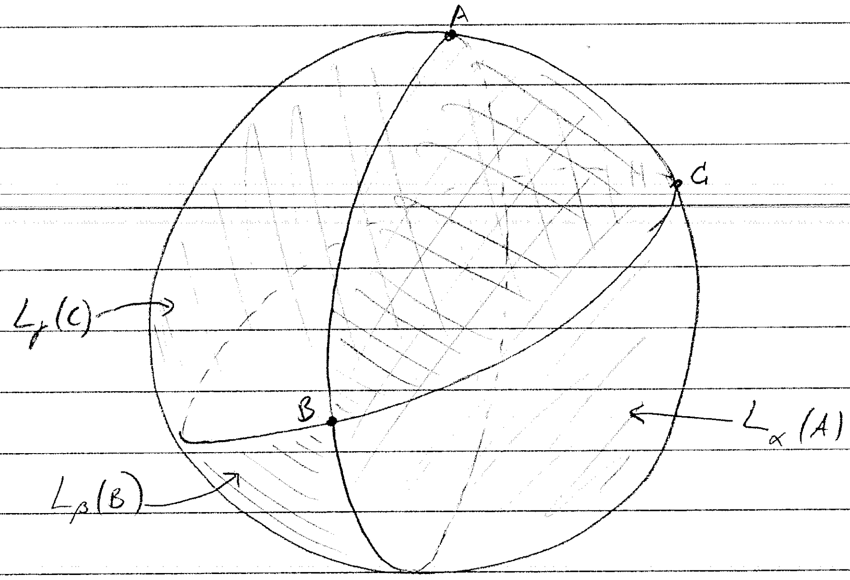
Theorem

$\alpha + \beta + \gamma = \pi + \text{area of } (\Delta)$

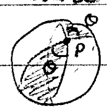
for a spherical triangle on the unit sphere.



(This is a very special case of the Gauss-Bonnet Theorem. See Differential Geometry in Year 3)



Let $L_\theta(p)$ denote the "double lune" at the point p subtending an angle θ



$\text{area}(L_\theta(p)) = \frac{\theta}{\pi} \cdot 4\pi = 4\theta$

$L_\alpha(A) + L_\beta(B) + L_\gamma(C) = 4\pi + 2\,\text{area}({}_B\overset{A}{\Delta}_C) + 2\,\text{area}({}_B\hat{\Delta}_C)$

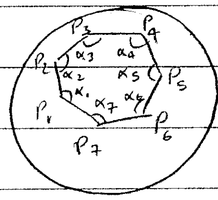
as we overcounted the area of the sphere twice on ${}_B\overset{A}{\Delta}_C$ and twice on the antipodal ${}_B\hat{\Delta}_C$

$\Rightarrow 4\alpha + 4\beta + 4\gamma = 4\pi + 4\,\text{area}({}_B\overset{A}{\Delta}_C)$

$\square$

Corollary

Let P be a spherical n-gon with vertices $p_1, \dots, p_n$ and internal angles $\alpha_1, \dots, \alpha_n$. If P is convex then $\text{area}(P) = \sum \alpha_i - (n-2)\pi$.



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Proof

Subdivide P into triangles and apply previous theorem:

$$\text{area}(P) = \sum \text{areas of triangles} = \sum_{\text{triangles}} \left(\sum \text{internal angles in } \Delta_i - \pi \right)$$

$$\Rightarrow \text{area}(P) = \sum \alpha_i - \sum_{\text{triangles}} \pi$$

$$= \sum \alpha_i - \pi(\text{number of triangles})$$

$$\# \text{ of triangles} = n-2$$

$$\Rightarrow \text{area}(P) = \sum \alpha_i - (n-2)\pi$$

(note: $\sum_{\text{triangles}} \sum \text{internal angles} = \sum \alpha_i$)

$\square$

Theorem (Euler's formula)

If we divide the sphere S^2 into convex spherical polygons so that there are:

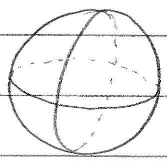
$F = \#$ polygons

$E = \#$ edges

$V = \#$ vertices

then $V - E + F = 2$

Examples

① 

"spherical octahedron"

$V=6$, $E=12$, $F=8$

$V-E+F = 2$ ✓

② dodecahedron on the sphere.

$V=20$, $F=12$, $E=30$

$V-E+F=2$ ✓

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Isometries of the sphere

Write $\mathrm{Isom}(S^2)$ for the group of isometries of S^2;
that is $\mathrm{Isom}(S^2) = \{t : S^2 \mapsto S^2 \mid t \text{ is a bijection}, d(tx, ty) = d(x,y)\}$
where $d(x,y)$ is the spherical distance between x & y,
$d(x,y) = \cos^{-1}(x \cdot y)$.

Theorem

$\mathrm{Isom}(S^2) = O(3)$

Proof

Let $t : S^2 \mapsto S^2$ be an isometry.
We will construct an isometry $T : \mathbb{R}^3 \mapsto \mathbb{R}^3$
s.t. $T(0) = 0$ & $T|_{S^2} : S^2 \mapsto S^2$ agrees with t.

$T|_{S^2} \Leftrightarrow$ restricting T to the sphere

$$T(x) = \begin{cases} 0 & \text{if } x = 0 \\ |x|\, t\left(\frac{x}{|x|}\right) & \text{if } x \neq 0 \end{cases}$$

eg. if t were $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ & $x = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$, $Tx = 2\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\frac{1}{2}x$.

Claim: T is an isometry of $\mathbb{R}^3$
(assume $x, y \neq 0$, other argument similar)

$$|Tx - Ty|^2 = \left| |x|\, t\hat{x} - |y|\, t\hat{y} \right|^2 \qquad \hat{x} = \frac{x}{|x|}, \quad \hat{y} = \frac{y}{|y|}$$

$$= |x|^2 |t\hat{x}|^2 + |y|^2 |t\hat{y}|^2 - 2|x||y|\, t\hat{x} \cdot t\hat{y}$$

$|\hat{x}| = 1$ so $t\hat{x} \in S^2$ so $|t\hat{x}| = |t\hat{y}| = 1$

$$\begin{aligned} \Rightarrow |Tx - Ty|^2 &= |x|^2 + |y|^2 - 2|x||y|\, t\hat{x} \cdot t\hat{y} \\ &= |x|^2 + |y|^2 - 2|x||y| \cos(d(t\hat{x}, t\hat{y})) \\ &= |x|^2 + |y|^2 - 2|x||y| \cos(d(\hat{x}, \hat{y})) \quad \text{as } t \in \mathrm{Isom}(S^2) \\ &= |x|^2 + |y|^2 - 2|x||y|(\hat{x} \cdot \hat{y}) \\ &= |x|^2 + |y|^2 - 2x \cdot y \\ &= |x - y|^2 \end{aligned}$$

$\square$ end of claim.

So by the classification of isometries of $\mathbb{R}^3$,

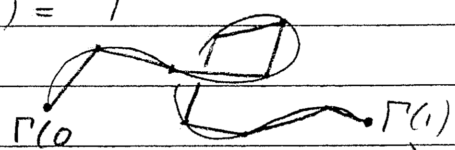
$Tx = Ax + b$ for some $A \in O(3)$, $b \in \mathbb{R}^3$.
Since $T0 = 0$, $A0 + b = 0 \Rightarrow b = 0$
$\Rightarrow Tx = Ax$ for some $A \in O(3)$
$\Rightarrow \mathrm{Isom}(S^2) = O(3)$.

$\square$

Geodesics on S^2

Theorem
The spherical line segment from p to q minimises length amongst all continuous paths on S^2 from p to q.

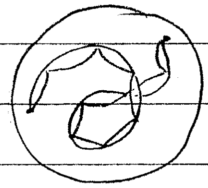
Def
Let $\Gamma : [0,1] \mapsto \mathbb{R}^3$ be a continuous map.
Take a dissection $(0 = t_0 < t_1 < \dots < t_n = 1) = T$
& let $s_T = \sum_{i=0}^{n-1} |\Gamma(t_i) - \Gamma(t_{i+1})|$.



(s_T is the length of the piecewise straight line approximation to Γ)

$l(\Gamma) = \sup_T (s_T)$ is defined to be the length of the path.
Note: $l(\Gamma)$ could be ∞.

Def
If $\Gamma(0,1) \subseteq S^2$, given a dissection T of $[0,1]$,
define $s_T' = \sum_{i=0}^{n-1} d(\Gamma(t_i), \Gamma(t_{i+1}))$.



$l'(\Gamma) = \sup_T (s_T')$

Lemma
$l(\Gamma) = l'(\Gamma)$ whenever $\Gamma([0,1]) \subseteq S^2$.

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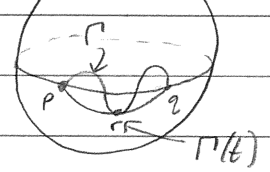
Given the lemma, here's how to prove the theorem: we want to show that a path Γ from p to q on S^2 is strictly longer than the spherical line from p to q if it's not equal to the spherical line.

Proof (of Thm) (assuming lemma)

Suppose Γ is a path on S^2 with $\Gamma(0) = p$, $\Gamma(1) = q$ and not equal to the spherical line segment.

If $\Gamma \neq$ spherical line then $\exists\, r = \Gamma(t)$ s.t. $r \notin$ spherical line $p \to q$.

Let $T = (0 = t_0 < t < t_2 = 1)$



$S_T' = \underbrace{d(p,r) + d(r,q) > d(p,q)}$ } triangle inequality

So $l'(\Gamma) = \sup_T S_T' > d(p,q)$

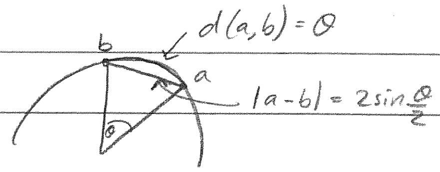
But by the lemma $l(\Gamma) = l'(\Gamma) > d(p,q)$

$\square$

Proof (of lemma: $l(\Gamma) = l'(\Gamma)$.)

$l'(\Gamma) \geq l(\Gamma)$:

We know that $d(a,b) \geq |a-b|$



$$\frac{2\sin\frac{\theta}{2}}{\theta} \leq 1$$

$\Rightarrow S_T \leq S_T'$

$\Rightarrow \sup S_T \leq \sup S_T' \Rightarrow l(\Gamma) \leq l'(\Gamma)$

$l'(\Gamma) \leq l(\Gamma)$:

$$\frac{2\sin\frac{\theta}{2}}{\theta} \to 1 \text{ as } \theta \to 0 \quad \text{(L'Hopital's rule)}$$

$$\Rightarrow \forall \varepsilon \quad \exists \delta \text{ s.t. } 1-\varepsilon < \frac{2\sin\frac{\theta}{2}}{\theta} \leq 1 \quad \forall\, 0 \leq \theta < \delta$$

Let $\delta' = 2\sin\frac{\delta}{2}$, this means

$$\theta(1-\varepsilon) < 2\sin\frac{\theta}{2} \leq \theta$$

whenever $2\sin\frac{\theta}{2} < \delta'$ (as $2\sin\frac{\theta}{2} < \delta' \Rightarrow \theta < \delta$).

$\Rightarrow$ For a sufficiently fine dissection,

$$S_T'(1-\varepsilon) < S_T \leq S_T'$$

(subtle point here)

$\therefore \sup S_T'(1-\varepsilon) \leq \sup S_T$

$\Rightarrow l'(\Gamma)(1-3) \leq l(\Gamma)$

as $\varepsilon \to 0 \Rightarrow l'(\Gamma) \leq l(\Gamma)$

$\Rightarrow l(\Gamma) = l'(\Gamma)$ $\square$

Remark

In this proof, we deduced

$$S_T'(1-\varepsilon) < S_T \quad ①$$

from $\theta(1-\varepsilon) < 2\sin\frac{\theta}{2}$ ② as follows

② holds if $2\sin\frac{\theta}{2} < \delta'$

so we get ① if we see that for all sufficiently fine dissections $0 = t_0 < t < t_1 < \dots < t_n = 1$, $|\Gamma(t_i) - \Gamma(t_{i+1})| < \delta'$

This holds because Γ is a continuous function on $[0,1]$ and hence uniformly continuous, so when the dissection is very fine we get uniform bounds on the distance $|\Gamma(t_i) - \Gamma(t_{i+1})|$.

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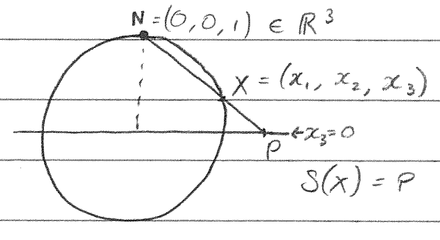
5: Möbius geometry

5.1 Stereographic projection

We define a map $S : S^2 \mapsto \mathbb{C} \cup \{\infty\}$ in the following way: $S(N) = \infty$

$S : S^2 \setminus \{N\} \mapsto \mathbb{C}$

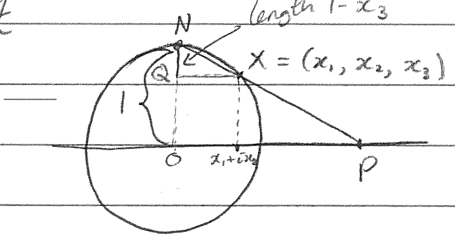


Lemma

$$S(x_1, x_2, x_3) = \frac{x_1 + ix_2}{1 - x_3}$$

As $X \to N$, $S(X)$ goes very far away in $\mathbb{C}$ towards ∞.

Proof



The triangles OPN & QXN are similar.
Sides ON and QN differ by a factor of $1 - x_n$, so the sides OP and QX also differ by this factor.
The vertical projection of X to $(x_3 = 0)$-plane is $x_1 + ix_2$ & we rescale by $\frac{1}{1-x_3}$ to get P.
So $P = \dfrac{x_1 + ix_2}{1 - x_3}$. $\square$

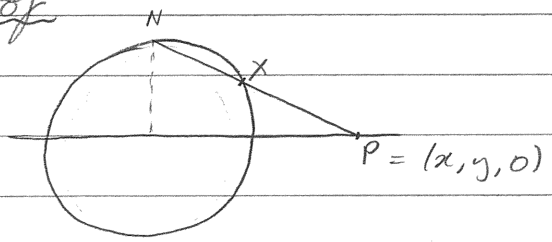
Lemma

There is an inverse map $\pi : \mathbb{C} \cup \{\infty\} \to S^2$ s.t. $S \circ \pi = id_{\mathbb{C} \cup \{\infty\}}$ & $\pi \circ S = id_{S^2}$

$z = x + iy$

$$\pi(z) = \left(\frac{2x}{|z|^2+1}, \frac{2y}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1} \right), \quad \pi(\infty) = N = (0,0,1)$$

Proof



The line NP is parameterised by

$(1-t)N + tP = (tx, ty, 1-t)$

What value of t gives the point X?

$X \in S^2 \Rightarrow (tx)^2 + (ty)^2 + (1-t)^2 = 1$

$\Rightarrow t^2x^2 + t^2y^2 + t^2 - 2t + 1 = 1$

$\Rightarrow$ at X, t satisfies $t(x^2+y^2+1) = 2$

$\Rightarrow t = \dfrac{2}{1+|z|^2}$

So $X = (tx + ty + 1-t)$

$= \left(\dfrac{2x}{1+|z|^2}, \dfrac{2y}{1+|z|^2}, 1 - \dfrac{2}{1+|z|^2}\right)$

$= \left(\dfrac{2x}{1+|z|^2}, \dfrac{2y}{1+|z|^2}, \dfrac{|z|^2-1}{1+|z|^2}\right)$

Examples

$S(0,0,-1) = 0 \in \mathbb{C}$

$S(\text{equator}) = \text{unit circle in } \mathbb{C}$

$\Rightarrow S(x_1, x_2, 0) = \left(\dfrac{2x_1}{2}, \dfrac{2x_2}{2}, \dfrac{1-1}{2}\right) \quad [x_1^2 + x_2^2 = 1]$

$= (x_1, x_2, 0)$.

$S(\text{lower hemisphere})$ = inside the unit disc.

$S(\text{upper hemisphere})$ = outside the unit disc.

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Remark

The stereographic projection realises S^2 as the Riemann sphere.

'Riemann' means that this is a Riemann surface which is a space in which you can do $\mathbb{C}$-analysis.

To see this we will cover S^2 with two complex 'charts':

1). $S^2 \overset{S}{=} \mathbb{C} \cup \{\infty\}$ gives a copy of $\mathbb{C} \subseteq S^2$ s.t. $S^2 \setminus \mathbb{C} = \{N\}$ (coordinate z).

2). $S^2 \setminus \{\text{south pole}\}$ is another copy of $\mathbb{C}$, let's say it has coordinate w related to the z-coordinate by $\underbrace{w = \frac{1}{z}, \quad z = \frac{1}{w}}$

changes of coordinates defined on the overlap of the two charts.

This coodinate change is holomorphic, so transforms holomorphic maps in one chart into holomorphic maps in the other chart, so we can do $\mathbb{C}$-analysis independently in each chart & the answer in each chart is the same as they're related by holomorphic changes of coordinates.

5.2 Möbius maps

Def

A Möbius map is a map $\mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ of the form $T(z) = \dfrac{az+b}{cz+d}$, with $ad - bc \neq 0$

"fractional linear transformation"

Write $\mathcal{M}$ for the group of Möbius maps and $GL(2, \mathbb{C})$ for the group $\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{C},\ ad - bc \neq 0 \right\}$

of 2-by-2 invertible $\mathbb{C}$-matrices.

Lemma

The map $GL(2,\mathbb{C}) \longrightarrow \mathcal{M}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \left(z \longmapsto \dfrac{az+b}{cz+d} \right)$ is a homomorphism

Proof

Exercise.

kernel of this homomorphism is the subgroup of $GL(2,\mathbb{C})$ s.t. $\dfrac{az+b}{cz+d} = z \quad \forall z$

claim

$\left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right\} = \text{kernel} \qquad (\lambda \neq 0)$

Proof

Pick $z = 0$:

$$\frac{a \times 0 + b}{c \times 0 + d} = 0 = \frac{b}{d} \Rightarrow b = 0$$

Pick $z = \infty$:

$$\frac{a\infty}{c\infty + d} = \frac{a}{c + \frac{d}{\infty}} = \frac{a}{c} = \infty \Rightarrow c = 0$$

So $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{kernel} \Rightarrow b = c = 0$

Pick $z = 1$:

$$\frac{a \times 1}{d} = 1 \Rightarrow a = d$$

$\Rightarrow \text{kernel} = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \neq 0 \right\} \quad \square$

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Our homomorphism $GL(2, \mathbb{C}) \xrightarrow{\varphi} M$ is surjective: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a preimage for the

Möbius map $z \mapsto \dfrac{az+b}{cz+d}$

$\therefore \operatorname{Im}\varphi = M \cong GL(2, \mathbb{C}) / \ker\varphi$
(First isomorphism thm)

Def $\quad$ projective
$PGL(2, \mathbb{C})$ is the quotient group
$GL(2, \mathbb{C}) / \{\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} : \lambda \neq 0\}$.

So we've proved $M \cong PGL(2, \mathbb{C})$

Examples

1). $z \mapsto z+b$, translations of $\mathbb{C}$ are Möbius transformations.
(sends $\infty \to \infty$, so fixes ∞).

2). $z \mapsto e^{i\theta}z$, rotation around $0 \in M$
(fixes ∞ and 0.

3). $z \mapsto \frac{1}{z}$, reciprocation $\in M$
(swaps 0 and ∞, fixes ± 1 : $z = \frac{1}{z} \Rightarrow z^2 = 1$)

$GL(2, \mathbb{C})$

general $\to$ linear $\quad$ general here means almost every element is invertible.

Theorem

1). $\mathcal{M}$ is generated by the set of all translations, homotheties ($z \mapsto \lambda z$, $\lambda \in \mathbb{C} \setminus \{0\}$), reciprocation. [Hw5, Q2a]
[If $\lambda = re^{i\theta}$ then the λ-homothety is a rescaling by r followed by a rotation by θ.]

2). Möbius maps send (straight lines and circles) to (straight lines and circles).

3). Möbius maps are conformal, i.e. they preserve angles. [Hw5, Q4]

Proof

1, 3 on hw sheet 5.

To prove 2, it suffices (by part 1) to prove it for the generators.

a). Translations: the translation of a circle is a circle, the translation of a straight line is a straight line. ✓

b). Homotheties: the rescaling / rotation
{ of a circle is a circle
{ of a straight line is a straight line. ✓

c). Reciprocation.

Case 1

Let C be a circle not passing through 0.
Parameterise C by $a + be^{i\theta}$ as θ varies.
We don't want $|a| = b$ (otherwise $a + be^{i(\arg a + \pi)} = 0$)
Rotate WLOG to make $a \in \mathbb{R}_{>0}$.
We want to see that $\frac{1}{a+be^{i\theta}} = C'$ parameterises another circle.
If C' were another circle then its centre would be halfway between the points $\frac{1}{a+b}$ and $\frac{1}{a-b}$.
So we compute $r(\theta) = \left| \frac{1}{a+be^{i\theta}} - \frac{1}{2}\left(\frac{1}{a+b} + \frac{1}{a-b}\right) \right|$ ← this should be the radius of C'

and we want to see that C' is a circle i.e. radius is constant, i.e. $r(\theta)$ independent of θ.

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$$r(\theta) = \left| \frac{1}{a+be^{i\theta}} - \frac{a}{a^2-b^2} \right|$$
$$= \left| \frac{a^2-b^2-(a^2+abe^{i\theta})}{(a^2-b^2)(a+be^{i\theta})} \right|$$
$$= \frac{|-b^2-abe^{i\theta}|}{|a^2-b^2||a+be^{i\theta}|}$$
$$= \frac{b|b+ae^{i\theta}|}{|a^2-b^2||a+be^{i\theta}|}$$

$$|b+ae^{i\theta}| = \sqrt{(b+a\cos\theta)^2 + a^2\sin^2\theta}$$
$$= \sqrt{b^2 + a^2\cancel{\cos^2\theta} + 2ab\cos\theta + a^2\cancel{\sin^2\theta}}$$
$$|a+be^{i\theta}| = \sqrt{(a+b\cos\theta)^2 + b^2\sin^2\theta}$$
$$= \sqrt{a^2 + b^2\cancel{\cos^2\theta} + 2ab\cos\theta + b^2\cancel{\sin^2\theta}}$$

so these factors cancel

and $r(\theta) = \dfrac{|b|}{|a^2-b^2|}$

So we've seen that if $C = \{a+be^{i\theta}\}$

then the image of C under reciprocation is

$$\left\{ z \in \mathbb{C} : \left| z - \frac{1}{2}\left(\frac{1}{a+b} + \frac{1}{a-b}\right) \right| = \frac{b}{|a^2-b^2|} \right\}$$

which is a circle.

<u>Case 2</u>

C is a circle passing through 0.

Rotate so that C is centred on $\mathbb{R}_{>0}$ at a point a. Then C is parameterised by $a+ae^{i\theta}$ (passes through 0 at $\theta=\pi$).

Now, the image of C under reciprocation is parameterised by

$$\frac{1}{a(1+e^{i\theta})} = \frac{1}{a} \frac{(1+e^{-i\theta})}{(1+e^{i\theta})(1+e^{-i\theta})}$$
$$= \frac{1}{a} \cdot \frac{1+e^{-i\theta}}{2+2\cos\theta}$$
$$= \frac{1}{2a}\left(\frac{1+\cos\theta}{1+\cos\theta} - \frac{i\sin\theta}{1+\cos\theta} \right)$$

$$= \frac{1}{2a}\left(1 - \frac{i\sin\theta}{\cos\theta}\right)$$

This has constant real part and as θ varies from $-\pi$ to π the imaginary part covers the whole imaginary direction. (straight line)

Case 3

If L is a straight line, rotate WLOG to make it vertical, then it is the reciprocal image of a circle $\Rightarrow$ reciprocal of the line is precisely this circle.

$\square$

Theorem

Given 3 distinct points z_0, z_1, z_∞ in $\mathbb{C} \cup \{\infty\}$

$$\exists! \, \tau \in \mathcal{M} \text{ s.t. } \begin{cases} \tau z_0 = 0 \\ \tau z_1 = 1 \\ \tau z_\infty = \infty. \end{cases}$$

In other words, the group $\mathcal{M}$ acts on the set X of triples of distinct points on $\mathbb{C} \cup \{\infty\}$, and, moreover, this action is transitive

The stabiliser of $(0,1,\infty)$ is the set of τ s.t. $\tau 0 = 0$, $\tau 1 = 1$, $\tau \infty = \infty$ $\Rightarrow$ $\tau = \mathrm{Id}$ by this theorem.

Orbit-stabiliser Thm $\Rightarrow$

$$\exists \text{ bijection } \underbrace{\mathcal{M}/_{\mathrm{Stab}(0,1,\infty)}}_{\substack{\| \\ \mathcal{M} \text{ as } \mathrm{Stab}(0,1,\infty) = \{1\}}} \to \underbrace{\mathrm{Orb}(0,1,\infty)}_{\substack{\| \\ \{\text{triples of distinct points on } \mathbb{C} \cup \{\infty\}\}}}$$

Since we need 3 complex numbers to specify 3 distinct points, this means that $\mathcal{M}$ is 3-complex dimensional or 6-real dimensional.

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Proof of theorem

$$\tau z = \frac{z - z_0}{z - z_\infty} \cdot \frac{z_1 - z_\infty}{z_1 - z_0}$$

sends $z_0 \mapsto 0$

$z_1 \mapsto 1$

$z_\infty \mapsto \infty$

(Fun exercise : the condition $ad - bc \neq 0 \Leftrightarrow z_0 \neq z_1 \neq z_\infty$)

To see uniqueness:

If τ and τ' satisfy the condition

$\tau z_0 = \tau' z_0 = 0$

$\tau z_1 = \tau' z_1 = 1$

$\tau z_\infty = \tau z_\infty = \infty$

Then $\begin{cases} \tau' \circ \tau^{-1}(0) = 0 \\ \tau' \circ \tau^{-1}(1) = 1 \\ \tau' \circ \tau^{-1}(\infty) = \infty \end{cases}$, Let $\sigma = \tau' \circ \tau^{-1}$

So we need to show that if $\sigma(0) = 0$, $\sigma(1) = 1$, $\sigma(\infty) = \infty$

Then $\sigma = Id$ (this will show that $\tau' \circ \tau^{-1} = Id \Rightarrow \tau' = \tau$)

If $\sigma(z) = \dfrac{az + b}{cz + d}$ then $\sigma(0) = \dfrac{b}{d} = 0 \Rightarrow b = 0$

$\sigma(\infty) = \dfrac{a}{c} = \infty \Rightarrow c = 0$

$\sigma(1) = \dfrac{a}{d} = 1 \Rightarrow a = d$

so $\sigma(z) = \dfrac{az}{a} = z \Rightarrow \sigma = Id$ $\square$

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The Möbius group $\mathcal{M}$ acts on the sphere S^2 in the following way

$$A : \mathcal{M} \longmapsto \text{Maps}(S^2, S^2)$$

$$[A(g)](x) = \pi(g(S(x)))$$

i.e. we stereographically project x to $\mathbb{C} \cup \{\infty\}$, apply the Möbius transformation g, then project back to the sphere (recall: inverse of S is π).

Example

$g(z) = e^{i\theta} z$ (a rotation by θ in the plane) performs a rotation by θ around x_3-axis

Check

$$x = (x_1, x_2, x_3), \quad S(x) = \frac{x_1 + i x_2}{1 - x_3}, \quad \pi(x+iy) = \left(\frac{2x}{1+|z|^2}, \frac{2y}{1+|z|^2}, \frac{|z|^2-1}{|z|^2+1}\right)$$

$$g(S(x)) = e^{i\theta}\left(\frac{x_1 + i x_2}{1 - x_3}\right)$$

$$\text{So } A(g)(x) = \pi\left(e^{i\theta}\left[\frac{x_1 + i x_2}{1 - x_3}\right]\right)$$

$$= \pi\left(\frac{x_1\cos\theta - x_2\sin\theta + i(x_1\sin\theta + x_2\cos\theta)}{1 - x_3}\right)$$

$$= \left(\frac{2(x_1\cos\theta - x_2\sin\theta)}{(1-x_3)\left(1 + \frac{x_1^2+x_2^2}{1-x_3^2}\right)}, \frac{2(x_2\cos\theta + x_1\sin\theta)}{(1-x_3)\left(1 + \frac{x_1^2+x_2^2}{1-x_3^2}\right)}, \frac{\frac{x_1^2+x_2^2}{1+x_3^2} - 1}{1 + \frac{x_1^2+x_2^2}{1+x_3^2}}\right)$$

$$x_1^2 + x_2^2 = 1 - x_3^2 \Rightarrow \frac{x_1^2+x_2^2}{(1-x_3)^2} = \frac{1+x_3}{1-x_3}$$

$$\Rightarrow (1-x_3)\left(1 + \frac{x_1^2+x_2^2}{1+x_3^2}\right) = 1 - x_3 + 1 + x_3 = 2$$

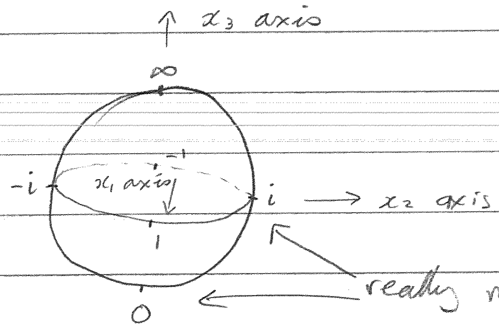
$$\text{So } A(g)(x) = (x_1\cos\theta - x_2\sin\theta, \ x_1\sin\theta + x_2\cos\theta, \ x_3)$$

So this is a rotation by θ about the x_3 axis.

We know that a Möbius map is determined by where it sends $0, 1, \infty$. This rotation fixes North & South poles, and sends points on equator to points on equator rotated by θ.
g (the Möbius map) fixes 0 & ∞ and sends 1 to $e^{i\theta}$.

Example

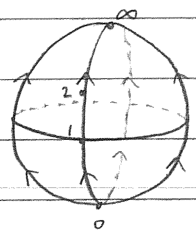
$$g(z) = \frac{z-1}{z+1}$$



$g(0) = -1$
$g(1) = 0$
$g(\infty) = 1$
$\Rightarrow$ rotation by $\frac{\pi}{2}$ about x_2 axis

Example

$g(z) = 2z$
$0, \infty$ fixed



not a rotation!
a 'squish' upwards!

Which Möbius maps correspond to rotations of S^2?
If $g(z) = \frac{az+b}{cz+d}$ then there's a nice necessary and sufficient condition for $A(g)$ to be a rotation in terms of a, b, c, d.

Def

The subgroup $SU(2) \subseteq GL(2, \mathbb{C})$ ← invertible 2×2 matrices (complex)
is the special unitary group of matrices A s.t. $A^\dagger = A^{-1}$.
Any $A \in SU(2)$ has $\det A = 1$ and is of the form $\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$ with $|a|^2 + |b|^2 = 1$.
We have a homomorphism $GL(2, \mathbb{C}) \mapsto \mathcal{M}$ $\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left(z \mapsto \frac{az+b}{cz+d}\right)\right]$
which exhibited $\mathcal{M} \cong PGL(2, \mathbb{C})$.

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Theorem

Let $PSU(2)$ denote the image of $SU(2)$ under this homomorphism inside M. Then $g \in M$ has $A(g)$ a rotation $\iff g \in PSU(2)$,

i.e. $A(g)$ is a rotation $\iff g(z) = \dfrac{az+b}{-\bar{b}z+\bar{a}}$

for some a, b with $|a|^2 + |b|^2 = 1$.

Example

$g(z) = e^{i\theta} z$

Looks like $g(z) = \dfrac{az+b}{cz+d}$ where $a = e^{i\theta}$, $b = 0$, $c = 0$, $d = 1 \neq \bar{a}$.

But also $g(z) = \dfrac{e^{i\theta/2} + 0}{0z + e^{i\theta/2}}$, $a = e^{i\theta/2} = \bar{d}$

A Möbius map doesn't uniquely determine a, b, c, d, only up to scale.

Example

$$g(z) = \frac{z-1}{z+1}$$

$$= \frac{\frac{1}{\sqrt{2}} z - \frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}} z + \frac{1}{\sqrt{2}}} \qquad \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \in SU(2)$$

Proof of Thm.

1). The Möbius map $g(z) = e^{i\theta} z$ performs a θ-rotation around x_3-axis.

2). The Möbius map $g(z) = \dfrac{z-1}{z+1}$ performs a $\frac{\pi}{2}$ rotation around x_2-axis.

3). These examples generate $SO(3)$ (rotations of S^2).

So I can get any rotation as $A(g)$ for some $g \in PSU(2)$

So this shows that $SO(3) \subseteq$ Image of $A|_{PSU(2)}$.

Want to see that $SO(3) =$ Image of $A|_{PSU(2)}$.
Take $g \in PSU(2)$, i.e. $g(z) = \dfrac{az+b}{-\bar{b}z+\bar{a}}$ s.t. $|a|^2+|b|^2=1$

Case 1

If $g(0)=0$ then $\dfrac{a0+b}{-\bar{b}0+\bar{a}} = \dfrac{b}{\bar{a}} = 0 \Rightarrow b=0$

$\Rightarrow |a|^2 = 1 \Rightarrow a = e^{i\phi}$ so $g(z) = e^{i2\phi}z$

So in this case $A(g)$ is a rotation (by 2ϕ around x_3-axis).

Case 2

If $g(0) = z \neq 0$ then there is a rotation which sends $\pi(z)$ back to $\pi(0)$. Call this rotation R.
By first part of proof $\exists h \in PSU(2)$ s.t. $R = A(h)$.
Then $hg(0)=0$, so by case 1 $A(hg)$ is a rotation.

$$\therefore A(g) = \underbrace{\underbrace{A(h^{-1})}_{\text{rotation}}\underbrace{A(hg)}_{\text{rotation}}}_{\text{rotation}}$$

$\Rightarrow A(g)$ is a rotation $\forall g \in PSU(2)$

$\Rightarrow A: PSU(2) \longmapsto SO(3)$ is an isomorphism $\square$

$\qquad \|$

$\{g \in M : g(z) = \dfrac{az+b}{-\bar{b}z+\bar{a}},\ |a|^2+|b|^2=1\}$

[This is another point of view on the the action of $SU(2)$ by rotations on S^2 which we saw whilst studying quaternions: $(\cos\phi + u\sin\phi)$ acts by rotating by 2ϕ around u]

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Recall:

Given a Möbius map $T: \mathbb{C} \cup \{\infty\} \mapsto \mathbb{C} \cup \{\infty\}$ we get a map $S^2 \to S^2$ as follows

$$\begin{array}{ccc} S^2 & \dashrightarrow & S^2 \\ s\downarrow & & \downarrow s \ \uparrow \pi \\ \mathbb{C} \cup \{\infty\} & \xrightarrow[T]{} & \mathbb{C} \cup \{\infty\} \end{array}$$

possible change of notation

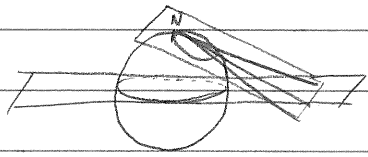
$$\left[\begin{array}{l} S = \sigma \\ \pi = \sigma^{-1} \end{array}\right]$$

Lemma

If C is a circle on S^2 passing through the north pole (N) then $S(C)$ is a straight line and conversely if $\gamma \subseteq \mathbb{C}$ is a straight line then $\pi(\gamma)$ is a circle on S^2 passing through N.

Proof

If C is a circle on S^2 then it is contained in a plane. In fact it's the intersection of a plane P with the sphere: $C = P \cap S^2$



Since C contains N, P also contains N.

$\Rightarrow$ P contains the straight line connecting N and any given point on C.

These are the lines along which we project to define stereographic projection.

$\therefore$ since $S(x)$ = intersection between the line Nx and the plane $\{z=0\}$ this means that $S(C) = P \cap \{z=0\}$ i.e. the image of C under projection is the where P intersects the complex plane.

Conversely, if $\gamma \subseteq \mathbb{C}$ is a straight line, then γ and N are contained in a unique plane, say Q.

Now $Q \cap S^2$ is a circle, and by definition it's $\pi(\gamma)$.
(S^2 is given by a quadratic equation, plane by linear eqn, so the intersection is given by a quatratic eqn again.)
$\square$

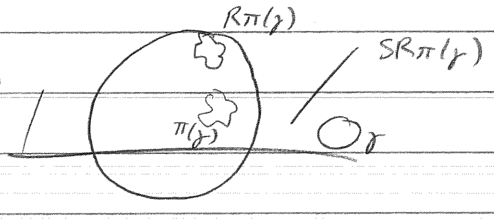
Theorem

A curve $\gamma \subseteq \mathbb{C} \cup \{\infty\}$ is a circle or straight line iff $\pi(\gamma) \subseteq S^2$ is a circle respectively not containing N or containing N.

Proof

If γ is a straight line, we just saw that $\pi(\gamma)$ is a circle containing N.
If γ is a circle, let $C = \pi(\gamma)$
$\exists$ a rotation R which moves C to another curve containing N.
Last time, we showed that for any rotation R of S^2
$\exists$ a Möbius map $T : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ such that

$$\begin{tikzcd} S^2 \arrow[r, "R"] \arrow[d, "S"'] & S^2 \arrow[d, "S"] \\ \mathbb{C} \cup \{\infty\} \arrow[r, "T"'] & \mathbb{C} \cup \{\infty\} \end{tikzcd} \qquad R = \pi \circ T \circ S$$



This means $S(R(\pi(\gamma))) = \cancel{S}(\cancel{\pi}(T(\cancel{S}(\cancel{\pi}(\gamma)))))$
$= T(\gamma)$ as $S \circ \pi = \mathrm{Id}$

So $R(\pi(\gamma))$ projects to $T(\gamma)$. Since γ is a circle, $T(\gamma)$ is a circle / straight line. Since $R(\pi(\gamma)) \ni N$ $T(\gamma)$ must be a straight line.
By the lemma, this means that $R(\pi(\gamma))$ is a circle. Since R is a rotation, $\pi(\gamma)$ is a circle, as required
$\square$

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We've just seen that the Möbius group acts on circles on S^2.

Theorem

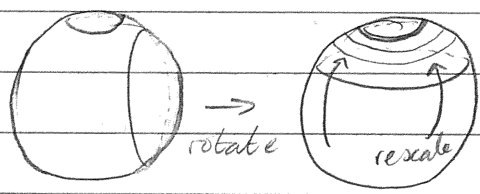
This action is transitive.

i.e. $\forall$ circles C, C' $\exists T \in M$ s.t. $TC = C'$.

Proof 1

Use a rotation (which can be done using a Möbius transformation) to make C and C' both centred at N.

Now use a homothety to rescale C until it has the same spherical radius as C'.



Proof 2

Three points on S^2 determine a unique 2-plane in $\mathbb{R}^3$ which then intersects S^2 in a circle.

If p_1, p_2, p_3 are points on S^2 & q_1, q_2, q_3 are three other points then $\exists! \, T \in M$ s.t. $Tp_1 = q_1$, $Tp_2 = q_2$, $Tp_3 = q_3$.

Given circles C & C', pick p_1, p_2, p_3 on C & q_1, q_2, q_3 on C', $\exists! \, T \in M$ s.t. $Tp_i = q_i$, $i = 1, 2, 3$ and T is a Möbius map, so TC is a circle.

But TC contains q_1, q_2, q_3 by construction.

Since C' is the unique circle containing q_1, q_2, q_3 we know that $TC = C'$.

Any complex differentiable map $\mathbb{C} \xrightarrow{f} \mathbb{C}$ is angle preserving.
This is because $f(z+\varepsilon) = f(z) + f'(z)\varepsilon + O(\varepsilon^2)$
and $v \mapsto av$ (v: vector in $\mathbb{C}$, a: number $\neq 0$ in $\mathbb{C}$) is angle preserving as $v \mapsto av$ is just rescaling v by $|a|$ and rotating by $\arg(a)$.
So if we take $a = f'(z)$, if this is non zero, then the angles are preserved by f.

Theorem

Stereographic projection preserves angles

Proof

Suppose I have two great circles on S^2 meeting at an angle α.
Rotate until they intersect at N.
The projections of these circles now are straight lines through the origin (as both pass through the south pole) meeting at angle $\alpha \Rightarrow S$ is angle preserving.

$\square$

Remark:

If $T \in M$ it gives us a transformation of the sphere by combining it with S & π.
Since all of these preserve angles, the corresponding transformation of S^2 is also angle preserving.

Addendum

Given 3 points in the plane, $\exists!$ circle through those 3 points.
Through 2 points there's a 1-parameter "pencil" of circles, so we need that third point to "rigidify" the situation & pick out a particular circle.

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Cross-ratios

Everything we've seen so far indicates that Möbius transformations are "flexible", they don't preserve distances, they act transitively on triples of points.

However, Möbius transformations do not act transitively on 4-tuples of points.

Q: Given 4 points $z_1, \dots, z_4$ and 4 points $w_1, \dots, w_4$ when is there a Möbius map T s.t. $Tz_i = w_i$ $\forall 1 \le i \le 4$?

A: Given 4 points $z_1, \dots, z_4$ define the cross-ratio

$$[z_1, z_2; z_3, z_4] = \frac{z_1 - z_3}{z_2 - z_3} \cdot \frac{z_2 - z_4}{z_1 - z_4}$$

Thm

$\exists T \in \mathcal{M}$ with $Tz_i = w_i$ $\forall 1 \le i \le 4$ iff

$[z_1, z_2; z_3, z_4] = [w_1, w_2; w_3, w_4]$.

If we think of

$$\frac{z_1 - z_3}{z_2 - z_3} \quad \frac{z_2 - z_4}{z_1 - z_4}$$

as a Möbius transformation τ where the variable is z_3

i.e. $\tau(z) = \dfrac{z_1 - z}{z_2 - z} \cdot \dfrac{z_2 - z_4}{z_1 - z_4}$

then $\tau(z_1) = 0$, $\tau(z_4) = 1$, $\tau(z_2) = \infty$.

So the cross-ratio $[z_1, z_2; z_3, z_4]$ is $\tau(z_3)$ where τ is the unique Möbius map sending

$$\begin{cases} z_1 \to 0 \\ z_2 \to \infty \\ z_4 \to 1 \end{cases}$$

In particular, $[0, \infty; z, 1] = z$

Proof of Thm

Given $z_1, \dots, z_4$ & $\omega_1, \dots, \omega_4$ quadruples of distinct points, if $[z_1, z_2; z_3, z_4] = [\omega_1, \omega_2; \omega_3, \omega_4]$ this means there are unique Möbius maps τ_z, τ_w s.t.

$$\tau_z(z_1) = 0, \quad \tau_z(z_2) = \infty, \quad \tau_z(z_4) = 1, \quad \tau_z(z_3) = [z_1, z_2; z_3, z_4]$$

by definition of the cross-ration.

Also, $\tau_w(\omega_1) = 0$, $\tau_w(\omega_2) = \infty$, $\tau_w(z_4) = 1$, $\tau_w(\omega_3) = [\omega_1, \omega_2; \omega_3, \omega_4]$.

Since the cross-ratios of z's and ω's coincide, this means that $\tau_w(\omega_3) = \tau_z(z_3)$

so
$$\tau_w^{-1} \circ \tau_z(z_1) = \omega_1$$
$$\tau_w^{-1} \circ \tau_z(z_2) = \omega_2$$
$$\tau_w^{-1} \circ \tau_z(z_3) = \omega_3$$
$$\tau_w^{-1} \circ \tau_z(z_4) = \omega_4$$

So $\tau_w^{-1} \circ \tau_z$ sends z_i to ω_i.

Conversely, if $\exists\, \sigma$ s.t. $\sigma(z_i) = \omega_i$ for $i = 1, \dots, 4$.

Let τ be the ! Möbius map sending ω_1 to 0, ω_2 to ∞, ω_4 to 1.

By definition, ω_3 goes to $[\omega_1, \omega_2; \omega_3, \omega_4]$, so under $\tau \circ \sigma$ we send

$$z_1 \to \omega_1 \to 0$$
$$z_2 \to \omega_2 \to \infty$$
$$z_4 \to \omega_4 \to 1$$
$$z_3 \to \omega_3 \to [\omega_1, \omega_2; \omega_3, \omega_4]$$

But given a Möbius map sending $z_1 \to 0$, $z_2 \to \infty$, $z_4 \to 1$, we know $z_3 \to [z_1, z_2; z_3, z_4]$.

However here, z_3 is mapping to $[\omega_1, \omega_2; \omega_3, \omega_4]$ under such a map, so $[z_1, z_2; z_3, z_4] = [\omega_1, \omega_2; \omega_3, \omega_4]$.

$\square$

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Exercise

$z_1 = \infty \qquad w_1 = \infty \qquad [z_1, z_2; z_3, z_4] = \dfrac{z_1 - z_3}{z_2 - z_3} \cdot \dfrac{z_2 - z_4}{z_1 - z_4}$

$z_2 = 0 \qquad w_2 = i$

$z_3 = 2 \qquad w_3 = 0$

$z_4 = 1 \qquad w_4 = -i$

Is there a Möbius map T st. $Tz_i = w_i$?

$$[\infty, 0; 2, 1] = \frac{\infty - 2}{0 - 2} \cdot \frac{0 - 1}{\infty - 1}$$

$$= \frac{1}{2} \cdot \frac{\infty}{\infty} = \frac{1}{2} \qquad \left(\frac{\infty}{\infty} = 1\right)$$

$$[\infty; i; 0, -i] = \frac{\infty - 0}{i - 0} \cdot \frac{i - (-i)}{\infty - (-i)}$$

$$= \frac{2i}{i} \cdot \frac{\infty}{\infty} = 2$$

So no Möbius map between them as they have different cross-ratios.

Theorem

The cross-ratio is invarient under Möbius transformations, i.e. if $\tau \in M$ then

$$[\tau z_1, \tau z_2; \tau z_3, \tau z_4] = [z_1, z_2; z_3, z_4]$$

Proof!

Recall from hw sheet 5 that M is generated by $h_\lambda z = \lambda z$, $t_b z = z + b$, $z \mapsto \frac{1}{z}$.

It suffices to prove the theorem for h_λ, t_b and reciprocation.

h_λ introduces a factor of λ on the top and on the bottom, they cancel so cross-ratio is unchanged.

The cross-ratio is a product of differences, eg. $z_1 - z_3$, so $z_1 \mapsto z_1 + b$, $z_3 \mapsto z_3 + b \Rightarrow z_1 - z_3 \mapsto z_1 - z_3$

& the cross ratio is unchanged.

For reciprocation:

$$\frac{\frac{1}{z_1}-\frac{1}{z_3}}{\frac{1}{z_2}-\frac{1}{z_3}} \cdot \frac{\frac{1}{z_2}-\frac{1}{z_4}}{\frac{1}{z_1}-\frac{1}{z_4}} = \left[\frac{1}{z_1}, \frac{1}{z_2}; \frac{1}{z_3}, \frac{1}{z_4}\right]$$

$$= \frac{(z_3-z_1)(z_4-z_2)/\cancel{z_1 z_2 z_3 z_4}}{(z_3-z_2)(z_4-z_1)/\cancel{z_1 z_2 z_3 z_4}}$$

$$= \frac{z_1-z_3}{z_2-z_3} \cdot \frac{z_2-z_4}{z_1-z_4} = [z_1, z_2; z_3, z_4]$$

$\square$

<u>Proof 2</u>

Let σ be the unique Möbius map with

$\sigma(z_1)=0$, $\sigma(z_2)=\infty$, $\sigma(z_4)=1$, $\sigma(z_3)=[z_1, z_2; z_3, z_4]$

$\sigma \circ \tau^{-1}$ sends τz_1 to 0

τz_2 to ∞

τz_4 to 1

τz_3 to $[z_1, z_2; z_3, z_4]$

But by definition if $\tau z_1 \to 0$, $\tau z_2 \to \infty$, $\tau z_4 \to 1$,

$\tau z_3 \to [\tau z_1, \tau z_2, \tau z_3, \tau z_4]$

But since $\tau z_3 \to [z_1, z_2; z_3, z_4]$

this means $[z_1, z_2; z_3, z_4] = [\tau z_1, \tau z_2; \tau z_3, \tau z_4]$

$\square$

We call the cross-ratio an <u>invariant</u> as it is unchanged under Möbius maps.

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1). Given 4 points $z_1, z_2, z_3, z_4 \in \mathbb{C} \cup \{\infty\}$ we defined their cross-ratio

$$[z_1, z_2; z_3, z_4] = \frac{z_1 - z_3}{z_2 - z_3} \cdot \frac{z_2 - z_4}{z_1 - z_4}$$

2). We saw that $\exists!$ Möbius map τ with $\tau z_i = w_i$, $1 \le i \le 4$ iff $[z_1, z_2; z_3, z_4] = [w_1, w_2; w_3, w_4]$

3). We saw that $[\sigma z_1, \sigma z_2; \sigma z_3, \sigma z_4] = [z_1, z_2; z_3, z_4]$ $\forall \sigma \in M$

What if we ask for existence of $\tau \in M$ s.t. $\tau z_i = w_{s(i)}$ for some permutation s?

This would require computing the 24 cross-ratios $[w_{s(1)}, w_{s(2)}; w_{s(3)}, w_{s(4)}]$.

Using invarience of the cross ratio under Möbius maps we can reduce the number of computations by a factor of 6.

e.g. W.L.O.G. $z_1 = 0$, $z_2 = \infty$, $z_4 = 1$

$$[0, \infty; z_3, 1] = z_3$$

Given any permutation s of $0, 1, \infty$ $\exists!$ Möbius map, τ_s, which does this permutation.

Then
$$[s(0), s(\infty); z_3, s(1)] = [\tau_s(0), \tau_s(\infty); z_3, \tau_s(1)]$$
$$= [0, \infty; \tau_s^{-1}(z_3), 1] = \tau_s^{-1}(z_3)$$

So we can find 6 of the 24 permuted cross-ratios given one of them, just by applying Möbius maps sending $\{0, 1, \infty\} \longmapsto \{0, 1, \infty\}$ in some order.

Lemma

The Möbius maps preserving the set $\{0,1,\infty\}$ (but possibly changing the order) are

$T_1 = $ Identity $\qquad T_2(z) = \frac{1}{z}$ $\qquad T_3(z) = \frac{1}{1-z}$

$\qquad\qquad\qquad\qquad 0 \leftrightarrow \infty \qquad\qquad (0,1,\infty)$

$T_4(z) = 1 - z \qquad T_5(z) = 1 - \frac{1}{z} = \frac{z-1}{z} \qquad T_6(z) = \frac{1}{1-\frac{1}{z}} = \frac{z}{z-1}$

$\quad 0 \leftrightarrow 1 \qquad\qquad (0,\infty,1) \qquad\qquad 1 \leftrightarrow \infty$

Example

Last time we saw

$[\infty, 0 ; 2, 1] = \frac{1}{2}$

$[\infty, i ; 0, -i] = 2$

We see that using T_2 we get $[T_2\infty, T_2 0 ; 2, T_2 1]$

$= T_2^{-1}[0, \infty ; 2, 1] = 2$

$\Rightarrow \exists$ a Möbius map τ with $\tau(0) = \infty$, $\tau(\infty) = i$, $\tau(2) = 0$, $\tau(1) = -i$

The set of 6 Möbius maps in the lemma form a subgroup of $\mathcal{M}$. In fact it is the stabiliser of $\{0,1,\infty\}$ under the action of the Möbius group on unordered triples of points.

It is isomorphic to the group $S_3 = D_6$ of permutations of 3 objects.

Theorem

4 points in $\mathbb{C} \cup \{\infty\}$ lie on a circle or straight line iff their cross-ratio is real

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In the next part of the course, we will be studying "hyperbolic geometry" which is a strange, non-Euclidean geometry defined on the UPPER-HALF PLANE.

Def
The upper half plane $\mathbb{H}$ is the subset $\{x+iy \in \mathbb{C} : y > 0\}$

Theorem
The subgroup of $T \subset \mathcal{M}$ s.t. $T\mathbb{H} = \mathbb{H}$ is precisely the set of $T(z) = \frac{az+b}{cz+d}$ where $a,b,c,d \in \mathbb{R}$, $ad-bc=1$.

[This is called $PSL(2,\mathbb{R})$]

Projective special $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ real coefficients

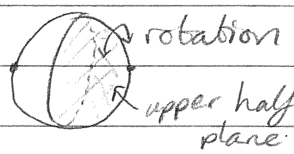
$\Rightarrow ad-bc=1$

i.e. Möbius maps rather than matrices.

Examples
Translations by real numbers preserve $\mathbb{H}$

$z \mapsto -\frac{1}{z}$ sends $i \to i$ & preserves $\mathbb{H}$

$z \mapsto \lambda z$ $\lambda \in \mathbb{R}$, $\lambda > 0$ sends $\mathbb{H} \mapsto \mathbb{H}$



Proof of Thm
If $T\mathbb{H} = \mathbb{H}$ then T preserves the boundary $\mathbb{R} \cup \{\infty\}$ of $\mathbb{H}$ i.e. $z \in \mathbb{R} \cup \{\infty\}$ then $Tz \in \mathbb{R} \cup \{\infty\}$.

Suppose $Tz = \frac{az+b}{cz+d}$

Consider $T(0) = \frac{b}{d} \in \mathbb{R} \cup \{\infty\}$, $T(1) = \frac{a+b}{c+d} =: m \in \mathbb{R} \cup \{\infty\}$

$T(\infty) = \frac{a}{c} =: n \in \mathbb{R} \cup \{\infty\}$

We need to show that, after rescaling, $a,b,c,d \in \mathbb{R}$

Assume $d=0$.

Since $ad-bc \neq 0$ we see that $c \neq 0$.

So $\frac{a}{c} \in \mathbb{R}$

Rescale so that $c=1$. Then we get $a \in \mathbb{R}$

Then $\frac{a+b}{c+d} \in \mathbb{R} \cup \{\infty\}$ tells us that $a+b \in \mathbb{R} \Rightarrow b \in \mathbb{R}$

$\Rightarrow$ after rescaling, $a,b,c,d \in \mathbb{R}$.

If $d \neq 0$, we can rescale to make $d=1$

Then $T(0) = \frac{b}{d} \in \mathbb{R} \Rightarrow b \in \mathbb{R}$

Case 1 $c=0$

$\Rightarrow T(1) = \frac{a+b}{d} \in \mathbb{R} \Rightarrow a+b \in \mathbb{R} \Rightarrow a \in \mathbb{R}$ as $b \in \mathbb{R}$.

Case 2 $c \neq 0$

$T(\infty) = \frac{a}{c} \overset{=n}{\in} \mathbb{R}$, $T(1) = \frac{a+b}{c+1} \overset{=m}{\in} \mathbb{R} \cup \{\infty\}$

So $b \in \mathbb{R}$ & $\frac{a+b}{c+1} = m \in \mathbb{R} \cup \{\infty\}$ $a = nc$, $n \in \mathbb{R}$

$\Rightarrow \frac{nc+b}{c+1} = m \Rightarrow nc+b = mc+m$

$\Rightarrow (n-m)c = m-b \Rightarrow c = \frac{m-b}{n-m} \in \mathbb{R} \Rightarrow a = nc \in \mathbb{R}$.

We still need to control $ad-bc=1$

We will show that it has to be positive

Then, rescaling a,b,c,d by $\lambda \in \mathbb{R}$ we can rescale $ad-bc$ to get $\lambda^2(ad-bc)=1$

$\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im} > 0\}$

Ex on Hw sheet N ($N>6$): $a,b,c,d \in \mathbb{R} \Rightarrow \mathrm{Im}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{(ad-bc)\mathrm{Im}(z)}{|cz+d|^2}$

So if $\mathrm{Im}(z) > 0$ and $\mathrm{Im}\left(\frac{az+b}{cz+d}\right) > 0$

$\Longleftrightarrow ad-bc > 0$ $\square$

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$\mathbb{H}$ the upper half-plane
$\{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$

We saw that if $T(z) = \dfrac{az+b}{cz+d}$ is in the group

$PSL(2, \mathbb{R})$ ($\Leftrightarrow$ $a,b,c,d \in \mathbb{R}$ & $ad - bc = 1$)
then $T\mathbb{H} = \mathbb{H}$, and conversely.

Recall that $\operatorname{Im}\left(\dfrac{az+b}{cz+d}\right) = \dfrac{(ad-bc)\operatorname{Im}(z)}{|cz+d|^2}$ (if $a,b,c,d \in \mathbb{R}$)

So we see that if $a,b,c,d \in \mathbb{R}$, complex numbers with $\operatorname{Im}(z) > 0$ are sent to complex numbers with positive imaginary part iff $ad - bc > 0$.
So if $ad - bc > 0$, we can rescale all coefficients by $\lambda \in \mathbb{R}$ to get $(\lambda a)(\lambda d) - (\lambda b)(\lambda c) = \lambda^2 (ad - bc)$ and taking $\lambda = 1/\sqrt{ad-bc}$ we get $\det = 1$.

Chapter 6. Hyperbolic Geometry

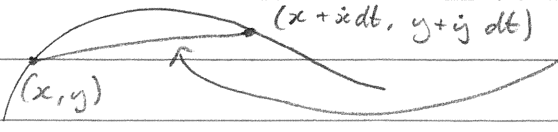
Def

Let $\mathbb{H}$ be the upper half-plane & let $\gamma(t) = x(t) + iy(t)$ be a path (parameterised by $t \in [0,1]$) in $\mathbb{H}$ i.e. $y(t) > 0$ $\forall t$.
The hyperbolic length of γ is defined to be

$$L(\gamma) := \int_0^1 \frac{\sqrt{\dot{x}^2 + \dot{y}^2}}{y} \, dt$$

Let's compare this formula with the formula for the length of a curve γ in $\mathbb{R}^2$.

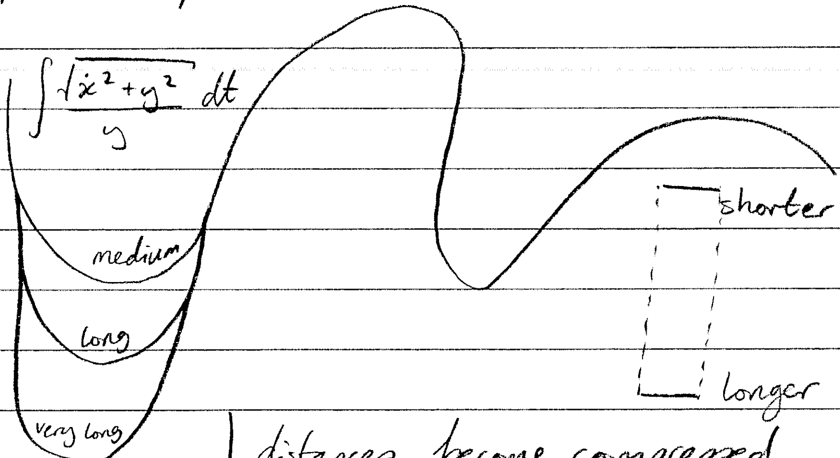
Length of curve in $\mathbb{R}^2 = \displaystyle\int_0^1 \sqrt{\dot{x}^2 + \dot{y}^2} \, dt$



length $= \sqrt{(x - x - \dot{x}dt)^2 + (y - y - \dot{y}dt)^2}$
$= \sqrt{\dot{x}^2 + \dot{y}^2} \, dt$

Total length of all segments $= \displaystyle\int \sqrt{\dot{x}^2 + \dot{y}^2} \, dt$ (limit of very fine segments)

Hyperbolic space:

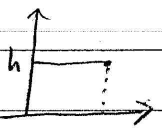


distances become compressed towards $y=0$ ← can fit more in!

INFINITY!

You can see in the Escher handout that hyperbolic geometry fits in a lot near the $y=0$ line: all of the lizards in the picture are the same (hyperbolic) size.

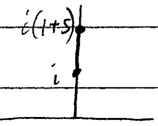
What is the hyperbolic length of $\gamma(t) = t + ih$, $t \in [0,1]$



$$L(\gamma) = \int_0^1 \frac{\sqrt{\dot{x}^2+\dot{y}^2}}{y}\,dt \qquad \dot{x}=1,\ \dot{y}=0$$

$$= \int_0^1 \frac{\sqrt{1}}{h}\,dt = \frac{1}{h} \to \infty \text{ as } h \to 0$$

What is the hyperbolic length of $\gamma(t) = i(1+st)$ ($\overset{\in\mathbb{R}}{s}$ const >0)



$$L(\gamma) = \int_0^1 \frac{\sqrt{\dot{x}^2+\dot{y}^2}}{y}\,dt \qquad x=0,\ y=1+st$$
$$\dot{x}=0,\ \dot{y}=s$$

$$= \int_0^1 \frac{\sqrt{0+s^2}}{1+st}\,dt$$

$$= \int_0^1 \frac{s}{1+st}\,dt = \log(1+st)\Big|_0^1$$

$$= \log(1+s) - \log(1) = \log(1+s)$$

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So the hyperbolic length of the line segment along the imaginary axis from i to ix is $\log x$.

Def

The hyperbolic distance between $p, q \in \mathbb{H}$ is defined to be the infimum over all piecewise smooth paths γ with $\gamma(0) = p$, $\gamma(1) = q$.

This will turn out to be the length of the unique shortest path connecting them (hyperbolic geodesic / hyperbolic line).

Shortest w.r.t. hyperbolic length.

Isometries of $\mathbb{H}$

Theorem

The Möbius maps $T \in PSL(2,\mathbb{R})$ (which we know preserve $\mathbb{H}$) preserve the hyperbolic lengths of curves, i.e. if $\gamma \in \mathbb{H}$ then $L(\gamma) = L(T\gamma)$.
i.e. $PSL(2,\mathbb{R}) \subseteq Isom(\mathbb{H})$

Proof

We will check that if T is one of
- translation $Tz = z + b$ $(b \in \mathbb{R})$
- homothety $Tz = \lambda z$ $(\lambda \in \mathbb{R}, \lambda > 0)$
- reciprocation $Tz = -1/z$

then $L(T\gamma) = L(\gamma)$

For translations let $\gamma(t) = x(t) + iy(t)$ be our curve.
Then $T(\gamma(t)) = (x(t) + b) + iy(t) = u(t) + iv(t)$
$\Rightarrow \dot{u} = \dot{x}$, $\dot{v} = \dot{y}$

$$\Rightarrow L(T\gamma) = \int \frac{\sqrt{\dot{u}^2 + \dot{v}^2}}{v}\, dt$$

$$= \int \frac{\sqrt{\dot{x}^2 + \dot{y}^2}}{y}\, dt \quad = L(\gamma) \checkmark$$

For homotheties $Tz = \lambda z$, $T(\gamma) = \lambda x(t) + i \lambda y(t)$

$$\Rightarrow L(T\gamma) = \int \frac{\sqrt{\lambda^2 \dot{x}^2 + \lambda^2 \dot{y}^2}}{\lambda y}\, dt$$

$$= \int \frac{\lambda \sqrt{\dot{x}^2 + \dot{y}^2}}{\lambda y}\, dt$$

$$= \int \frac{\sqrt{\dot{x}^2 + \dot{y}^2}}{y}\, dt \quad = L(\gamma) \checkmark$$

For reciprocation $Tz = -1/z$, $\gamma(t) = x(t) + iy(t)$.

We can write $L(\gamma) = \int \frac{|\dot{\gamma}|}{\operatorname{Im}(\gamma)}\, dt$

$$\frac{d}{dt}\left(-\frac{1}{\gamma}\right) = \frac{\dot{\gamma}}{\gamma^2}$$

so $\left|\frac{d}{dt}\left(-\frac{1}{\gamma}\right)\right| = \frac{|\dot{\gamma}|}{|\gamma|^2}$

$$\operatorname{Im}\left(\frac{-1}{\gamma}\right) = \operatorname{Im}\left(\frac{-\bar{\gamma}}{\gamma\bar{\gamma}}\right) = \operatorname{Im}\left(\frac{-\bar{\gamma}}{|\gamma|^2}\right) = \frac{\operatorname{Im}\gamma}{|\gamma|^2}$$

So $L(T\gamma) = \int \frac{|\dot{\gamma}|/|\gamma|^2}{\operatorname{Im}\gamma/|\gamma|^2}\, dt$

$$= \int \frac{|\dot{\gamma}|}{\operatorname{Im}\gamma}\, dt \quad = L(\gamma) \checkmark$$

Since these generate $PSL(2,\mathbb{R})$, we get the theorem. $\square$

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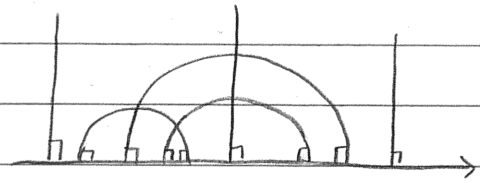
Hyperbolic lines

Def

A hyperbolic line is a circle or straight line in $\mathbb{H}$ which intersects $\mathbb{R}$ at right angles.



- vertical half-lines
- circles centred on $\mathbb{R}$

Lemma

$PSL(2,\mathbb{R})$ preserves this set, i.e. if γ is a hyperbolic line then so is $T(\gamma)$ $\forall T \in PSL(2,\mathbb{R})$

Proof

1). Möbius maps send circles and straight lines to circles and straight lines

2). Möbius maps are conformal (preserve angles)

3). $PSL(2,\mathbb{R})$ preserves $\mathbb{R}$

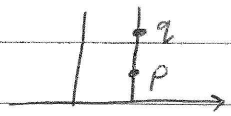
so the class of hyperbolic lines is preserved. $\square$

Lemma

Given two distinct points, p & $q \in \mathbb{H}$, there exists a unique hyperbolic line through both p & q.

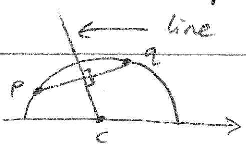
Proof

Case 1: $Re(p) = Re(q)$



Here we have the unique vertical half-line segment connecting p and q.

Case 2: $Re(p) \neq Re(q)$



← line equidistant from p and q

c = unique real point equidistant from p and q.

c is unique.
So there is a unique circle centred at c passing through both p & q. $\square$

<u>Lemma</u>

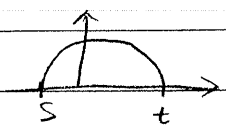
$PSL(2, \mathbb{R})$ acts transitively on hyperbolic lines.
i.e. any two hyperbolic lines are related by at least one of these Möbius maps.
Moreover $PSL(2, \mathbb{R})$ acts transitively on pairs (l, p) where l is a hyperbolic line & p is a point on l.

<u>Proof</u>

We will prove that $\forall$ hyperbolic line γ $\exists T \in PSL(2, \mathbb{R})$ s.t. $T\gamma$ is the imaginary axis.

<u>Case 1</u> : vertical line, $\gamma = \{z : Re(z) = c\}$
Let $Tz = z - c$
Then $T\gamma$ is the imaginary axis $\{z : Re(z) = c\}$

<u>Case 2</u>: γ is a circle intersecting $\mathbb{R}$ at two points, say s & t



Let $Tz = \dfrac{z-t}{z-s}$

Then $T(t) = 0$, $T(s) = \infty$
So $T\gamma$ is the vertical line at 0, i.e. imaginary axis.
$ad - bc = -s + t = t - s > 0$ (here $t > s$)
so this is in $PSL(2, \mathbb{R})$ (after rescaling by $\frac{1}{\sqrt{t-s}}$)
and $T\gamma$ = im. axis.

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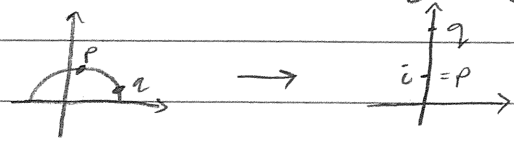
Moreover, if we are given a point p on γ it goes to a point iq on imaginary axis. Using a homothety $h_{1/q}$ we map $iq \to i$ so $h_{1/q} \circ T$ sends γ to the imaginary axis and p to i. □

__Theorem__

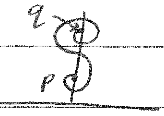
Let γ be a piecewise smooth path in $\mathbb{H}$ between p & q i.e. $\gamma(0) = p$, $\gamma(1) = q$. Let δ be the unique hyperbolic line segment between p & q. Then $L(\gamma) \geq L(\delta)$ with equality iff γ is a monotone parameterisation of δ (i.e. $\gamma(t) \in \delta \; \forall t$).

__Proof__

WLOG, using the action of $PSL(2,\mathbb{R})$ by isometries, we can move δ to be the imaginary axis we can move p to be the point i & then q is some point on imaginary axis.



Let γ be the curve in the statement of the theorem, $\gamma(t) = x(t) + iy(t)$



$$L(\gamma) = \int_0^1 \frac{\sqrt{\dot{x}^2 + \dot{y}^2}}{y} \, dt$$

$$\geq \int_0^1 \frac{\sqrt{\dot{y}^2}}{y} \, dt \qquad \text{as } \dot{x}^2 \geq 0$$

equality iff $\dot{y} \geq 0$.

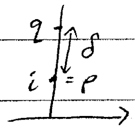
$$= \int_0^1 \left| \frac{\dot{y}}{y} \right| dt \geq \log(y(1)) - \log(y(0))$$

$$= \log(\operatorname{Im} q)$$

$[y(0) = 1 \quad (\operatorname{Im}(p)) \; , \; y(1) = \operatorname{Im}(q)]$

But we saw earlier that

$$L(\delta) = \log(\operatorname{Im} q)$$



So we get $L(\gamma) \geq L(\delta)$

means curve moves upwards always.

If we have equality then $\dot{x} \equiv 0$ so $x(t) = 0$, and $\dot{y} \geq 0$, and we get that image of $\gamma \subseteq \delta$. $\square$

<u>Corollary</u>

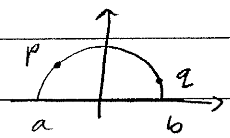
Let p, q be points in $\mathbb{H}$

Let γ be the unique hyperbolic line segment between them.

Let a & b be the points where γ intersects $\mathbb{R} \cup \{\infty\}$.

Then the hyperbolic length

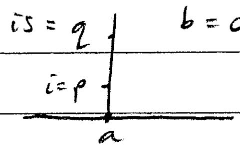
$L(\gamma)$ is $\log[a, b; q, p]$



<u>Proof</u>

WLOG we can act using $PSL(2, \mathbb{R})$ and assume that γ = imaginary axis and $p = i$.

(This is because $PSL(2, \mathbb{R})$ acts by isometries & doesn't change cross-ratios)



$Tz = -iz$

$$[a, b; p, q] = [\overset{0}{0}, \overset{\infty}{\infty}; \overset{-i(is)=s}{is}, \overset{1}{i}]$$

$$= \frac{\cancel{i}(s)}{\cancel{\infty} - is} \cdot \frac{\cancel{\infty} - i}{\cancel{i}} = s = \operatorname{Im} q$$

We already know that $L(\gamma) = \log(\operatorname{Im} q)$

$= \log(s)$

$= \log[0, \infty; is, i]$

$= \log[a, b; p, q]$ $\square$

Escher

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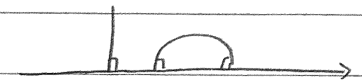
Last time we introduced hyperbolic length of a path in the upper half plane $\gamma\,[0,1] \to \mathbb{H}$, $\gamma(t) = x(t) + iy(t)$

$$L(\gamma) = \int_0^1 \frac{\sqrt{\dot{x}^2 + \dot{y}^2}}{y}\,dt$$

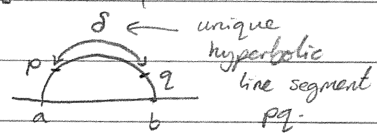
We've introduced "hyperbolic lines" →



and showed that the shortest path between p & $q \in \mathbb{H}$ is the (unique) hyperbolic line segment.

We showed that $PSL(2,\mathbb{R})$ of Möbius maps preserving $\mathbb{H}$ act by isometries of hyperbolic length and that $PSL(2,\mathbb{R})$ acts transitively on hyperbolic lines.

Finally we defined hyperbolic distance $d(p,q)$ to be $\inf\limits_{\gamma \text{ from } p \text{ to } q} L(\gamma)$ and showed this to be equal to $\log[a,b;q,p] = L(\delta)$

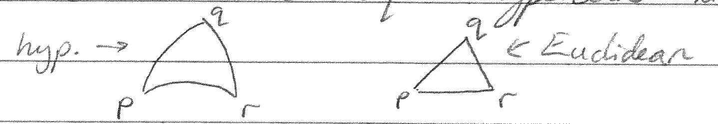


Today

- Triangle inequality for hyperbolic geometry
- Isometries of the hyperbolic upper half-plane send hyperbolic lines to hyperbolic lines.
- Gauss-Bonnet formula for hyperbolic triangles
  $\text{area}(\Delta) = \pi - \alpha - \beta - \gamma$

Triangle inequality

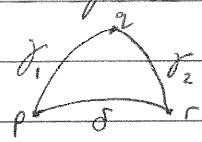
If p, q, r are points in $\mathbb{H}$ then
$d(p,r) \leq d(p,q) + d(q,r)$ with equality iff q lies on the unique hyperbolic line segment pr.



Proof

We saw last time that if γ is a piecewise smooth curve $\gamma(0) = p$, $\gamma(1) = r$ then $L(\gamma) \geq L(\delta)$ where δ is the hyperbolic line segment with equality iff $\gamma = \delta$.

Take γ to be the concatenation of γ_1 & γ_2 where



γ_1, γ_2 are hyperbolic lines $p \to q$, $q \to r$.

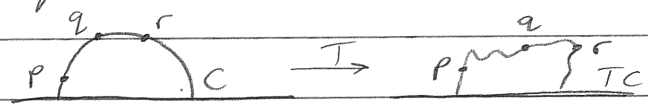
So we see $d(p,r) = L(\delta) \leq L(\gamma) = d(p,q) + d(q,r)$.

If there is equality, then $\gamma = \delta$ so γ_1 & γ_2 are subsets of δ and since $q \in \gamma_1$ we see $q \in \delta$. $\square$

Corollary

If T is an isometry of H and C is a hyp. line then TC is also a hyp. line.

Proof



Pick $p, q, r \in C$ with q between p & r.

We know $d(p,r) = d(p,q) + d(q,r)$

$\Rightarrow d(Tp, Tr) = d(Tp, Tq) + d(Tq, Tr)$ because distance is preserved by isometries.

The triangle inequality $\Rightarrow$ Tq lies on the hyp. line segment between Tp & Tr.

This is for all q in the segment pr.

So segment pr must map to the segment $TpTr$.

This is true $\forall p, r \in C$ so TC is a hyperbolic line (not a wiggly line as in the picture). $\square$

i.e. every point on the segment pr is sent to a point on the line segment T_pT_r so hyperbolic line segments are mapped to hyperbolic line segments.

Hyperbolic Gauss-Bonnet

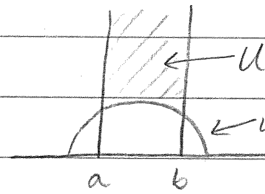
Def

If $\mathcal{U} \subseteq \mathbb{H}$, we define the area of $\mathcal{U}$ to be

$$\iint_{\mathcal{U}} \frac{dx\,dy}{y^2}$$

[dx, dy — Euclidean area element $\frac{dx}{y}$, $\frac{dy}{y}$ — Hyperbolic area element]

Example



unit circle $C \Leftrightarrow y = \sqrt{1-x^2}$

What is the hyperbolic area of $\mathcal{U}$?

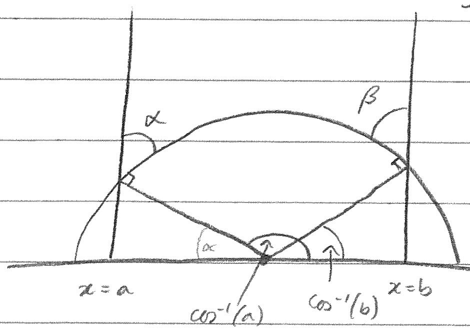
$$\int_a^b\int_{\sqrt{1-x^2}}^{\infty} \frac{dx\,dy}{y^2} = \text{area}(\mathcal{U}) = \int_a^b \left[\frac{-1}{y}\right]_{\sqrt{1-x^2}}^{\infty} dx$$

$$= \int_a^b 0 - \frac{-1}{\sqrt{1-x^2}}\,dx$$

$$= \int_a^b \frac{dx}{\sqrt{1-x^2}} \qquad x = \cos\theta, \quad dx = -\sin\theta\, d\theta$$

$$= \int_{\cos^{-1}(a)}^{\cos^{-1}(b)} \frac{-\sin\theta\, d\theta}{\sin\theta}$$

$$= \int_{\cos^{-1}(b)}^{\cos^{-1}(b)} d\theta = \cos^{-1}(b) - \cos^{-1}(a)$$



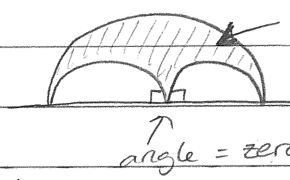
If we write α, β for the internal angles of $\mathcal{U}$ then

$$\cos^{-1}(a) = \pi - \alpha$$

$$\cos^{-1}(b) = \beta$$

$$\Rightarrow \text{area}(\mathcal{U}) = \pi - \alpha - \beta.$$

If is a hyperbolic triangle with a vertex at ∞, we call this an "ideal vertex"



This is a hyp. triangle with all 3 vertices on the boundary $\mathbb{R} \cup \{\infty\}$ of $\mathbb{H}$. "ideal triangle".

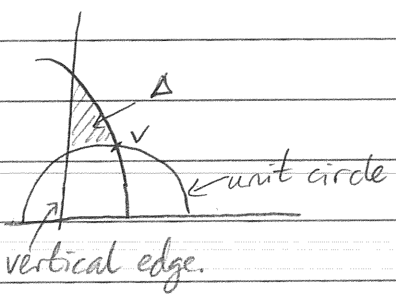
The internal angle at an ideal vertex is zero.
If you have two semicircles meeting $\mathbb{R}$ at right angles, the angle between them is zero.

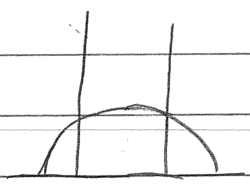
<u>Theorem</u>

If Δ is a hyp. triangle with internal angles α, β, γ then $\text{area}(\Delta) = \pi - \alpha - \beta - \gamma$ (in particular $\leq \pi$)

<u>Proof</u>

Use an isometry $T \in PSL(2, \mathbb{R})$ to move one of the edges to become vertical. Let C be one of the other semicircular edges. Translate & rescale to make C the unit circle.





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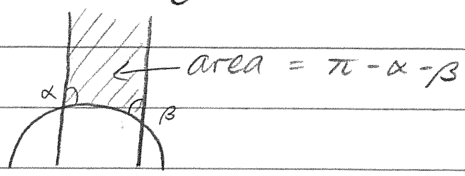
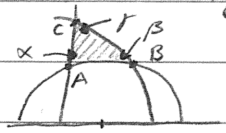
Recall that last time we stated that

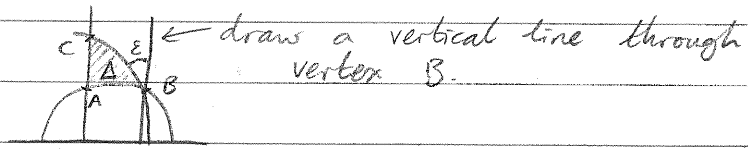
(*) $\text{area}(\Delta) = \pi - \alpha - \beta - \gamma$ for a hyperbolic triangle Δ with α, β, γ the internal angles

We actually showed

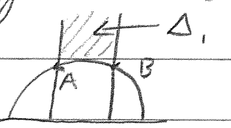


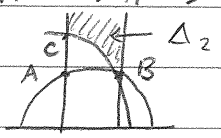
To complete the proof of (*) we use a Möbius transformation to make one of the sides of Δ vertical, to make another one equal to the unit circle, so W.L.O.G we only need to consider triangles of the form:





Observe that our triangle Δ is the set theoretic difference between Δ_1 & Δ_2, where





i.e. $\text{area}\,\Delta = \text{area}\,\Delta_1 - \text{area}\,\Delta_2$

$\text{area}(\Delta_1) = \pi - \alpha - (\beta + \varepsilon)$

$\text{area}(\Delta_2) = \pi - \varepsilon - (\pi - \gamma)$

as the internal angles of $\begin{cases} \Delta_1 \text{ are } \alpha \text{ \& } \beta + \varepsilon \\ \Delta_2 \text{ are } \varepsilon \text{ \& } \pi - \gamma \end{cases}$

so $\text{area}\,\Delta = \pi - \alpha - \beta - \cancel{\varepsilon} - (\cancel{\pi} - \cancel{\varepsilon} - \cancel{\pi} + \gamma)$

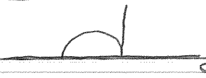
$= \pi - \alpha - \beta - \gamma$ $\square$

So hyperbolic triangles have $(\alpha + \beta + \gamma) < \pi$ and $\text{area}(\Delta) < \pi$

In Euclidean geometry, all lines are either
parallel (so only meet at ∞)
or intersect only once.

In spherical geometry, all spherical lines intersect.

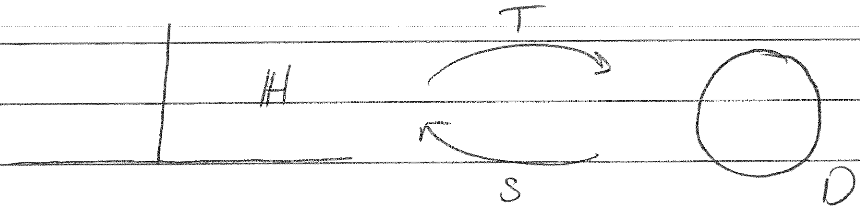
In hyperbolic geometry, lines do one of the following:

-  intersects in $\mathbb{H}$
-  "parallel lines"
  "infinity in $\mathbb{H}$" i.e. at the boundary.
-  "ultraparallel lines" (don't ever intersect at the boundary of $\mathbb{H}$)

Hyperbolic disc / Poincaré disc model

Consider $S(z) = \dfrac{i(1+z)}{1-z}$ & $T(z) = \dfrac{z-i}{z+i}$

Let $D = \{z : |z| < 1\} \subseteq \mathbb{C}$ be the unit disc



Claim

$SD = \mathbb{H}$ & $T\mathbb{H} = D$

In fact $S = T^{-1}$.

Proof

$$T(S(z)) = \frac{S(z) - i}{S(z) + i}$$

$$= \frac{i\left(\frac{1+z}{1-z}\right) - i}{i\left(\frac{1+z}{1-z}\right) + i}$$

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$$T(S(z)) = \frac{\cancel{1}+z - (\cancel{1}-z)}{1+\cancel{z}+(1-\cancel{z})}$$

$$= \frac{2z}{2} = z$$

So $T = S^{-1}$

So its sufficient to show that $SD = \mathbb{H}$

$(\Rightarrow \underbrace{TSD}_{=D} = T\mathbb{H})$

Let's compute $S(1)$, $S(i)$, $S(-i)$.

$S(1) = \infty$

$S(i) = -1$

$S(-i) = 1$

The unique circle through $i, -i, 1$ maps to the unique circle / straight line through $1, -1, \infty$, namely to $\mathbb{R}$.

So the boundary of D maps to $\mathbb{R}$.

We also see that $S(0) = i$, so the interior of the disc maps to the upper half plane.

$\square$

Def

If $\gamma : [0,1] \longmapsto D$ is a path in the interior of the unit disc then we define the hyperbolic length of γ to be $L(S(\gamma))$

now a curve in $\mathbb{H}$

hyperbolic length in $\mathbb{H}$

Lemma

If γ is a path in the disc then its hyperbolic length is $\displaystyle\int_0^1 \frac{2|\dot{\gamma}|}{1-|\gamma|^2}\, dt$

ie. if $\gamma(t) = x(t) + iy(t)$ then $|\dot{\gamma}| = \sqrt{\dot{x}^2 + \dot{y}^2}$

Proof

We have $L(S(\gamma)) = \int_0^1 \frac{|\frac{d}{dt} S(\gamma)|}{\mathrm{Im}(S(\gamma))} \, dt$

where $S(\gamma) = \frac{i(1+\gamma)}{1-\gamma}$

$$\frac{d}{dt} S(\gamma) = \frac{(1-\gamma)i\dot{\gamma} - i(1+\gamma)(-\dot{\gamma})}{(1-\gamma)^2}$$

$$= \frac{i\dot{\gamma}(2 - \gamma + \gamma)}{(1-\gamma)^2} = \frac{2i\dot{\gamma}}{(1-\gamma)^2}$$

$$\mathrm{Im}(S(\gamma)) = \mathrm{Im}\left(\frac{i(1+\gamma)}{1-\gamma}\right)$$

$$= \mathrm{Im}\left(\frac{i(1+\gamma)(1-\bar{\gamma})}{(1-\gamma)(1-\bar{\gamma})}\right)$$

$$= \frac{1}{|1-\gamma|^2} \mathrm{Im}\Big(i\big(\underbrace{1 + \underbrace{\gamma - \bar{\gamma}}_{\text{imaginary}} - |\gamma|^2}_{\text{real}}\big)\Big)$$

$$= \frac{1-|\gamma|^2}{|1-\gamma|^2}$$

So the integrand is $\frac{\frac{d}{dt}(S(\gamma))}{\mathrm{Im}(S(\gamma))} = \left(\frac{2|\dot{\gamma}|}{\cancel{|1-\gamma|^2}}\right) \Big/ \left(\frac{1-|\gamma|^2}{\cancel{|1-\gamma|^2}}\right)$

$$= \frac{2|\dot{\gamma}|}{1-|\gamma|^2}$$

$\square$

Exercise

If we take the path $\gamma(t) = rt$

$$L(\gamma) = \int \frac{2|\dot{\gamma}|}{1-|\gamma|^2} \, dt = 2\tanh^{-1}(r)$$

$\Rightarrow$ the point $\tanh(a/2) \in D$ lives a distance a from 0 (measured in hyperbolic geometry).

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$\Rightarrow$ for example, a circle of radius r in Euclidean geometry is a hyperbolic circle of radius $2\tanh^{-1}(r)$.

What do isometries in $PSL(2,\mathbb{R})$ look like when they act on the disc?

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{g \in PSL(2,\mathbb{R})} & \mathbb{H} \\ T\downarrow\uparrow S & & T\downarrow\uparrow S \\ D & \xrightarrow[T\circ g\circ S]{} & D \end{array}$$

So every $g \in PSL(2,\mathbb{R})$ acts on D by the Möbius map $T\circ g\circ S$

Lemma

These Möbius maps (which then act by hyperbolic isometries of D) are precisely those of the form $z \longmapsto \dfrac{mz+n}{\bar{n}z+\bar{m}}$ where $m, n \in \mathbb{C}$

with $|m|^2 - |n|^2 = 1$.

Examples

1). $gz = z+b$, $b \in \mathbb{R}$ $\quad g \in PSL(2,\mathbb{R})$

acts as translation by b in horizontal direction. What is $T\circ g\circ S$?

$$T(g(S(z))) = T\left(g\left(\frac{i(1+z)}{1-z}\right)\right) \qquad \left[Tx = \frac{x-i}{x+i}\right]$$

$$= T\left(\frac{i(1+z)}{1-z} + b\right) = \frac{\frac{i(1+z)}{1-z} + b - i}{\frac{i(1+z)}{1-z} + b + i}$$

$$= \frac{i(1+z) + (b-i)(1-z)}{i(1+z) + (b+i)(1-z)} = \frac{(2i-b)z + b}{-bz + (2i+b)}$$

$$\Rightarrow T(g(S(z))) = \frac{-(2+bi)z + ib}{-ibz + (bi-2)}$$

$n = ib/2$

$m = (-2-bi)/2$

$$|m|^2 - |n|^2 = \frac{4+b^2-b^2}{4} = \frac{4}{1} = 1$$

$$\Rightarrow T(g(S(z))) = \frac{-\frac{(2+bi)}{2}z + ib/2}{-\frac{ib}{2}z + \frac{(bi-2)}{2}}$$

How does this look as a transformation of D?

$$z \longmapsto \frac{-(2+ib)z + ib}{-ibz + bi - 2}$$

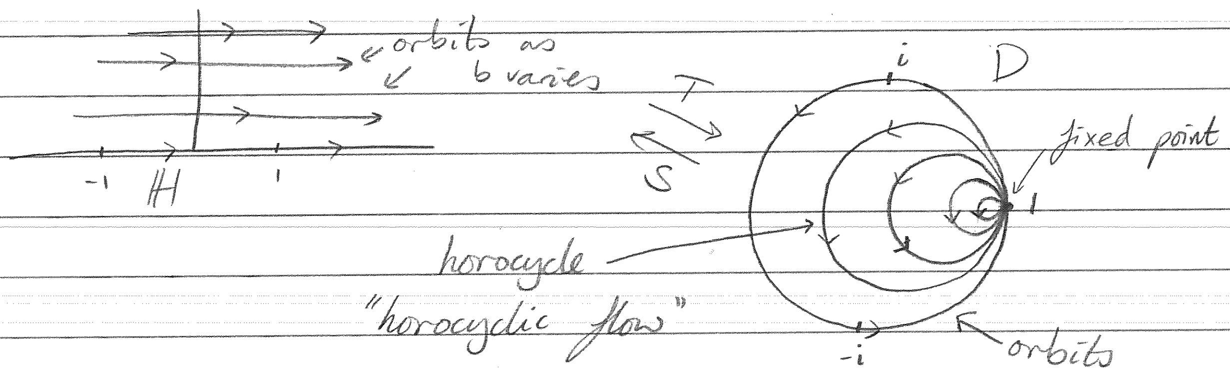
Fixed points:

$$-(\cancel{2}+ib)z + ib = -ibz^2 + (bi-\cancel{2})z$$

$$\Rightarrow ibz^2 - 2ibz + ib = 0$$

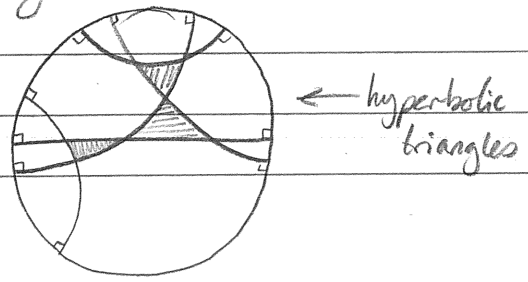
$$\Rightarrow z^2 - 2z + 1 = 0$$

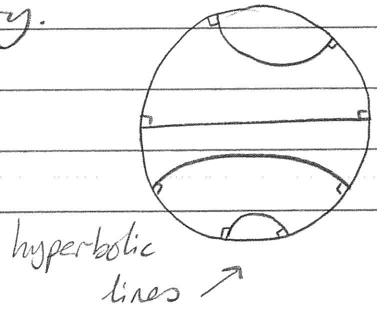
$$\Rightarrow (z-1)^2 = 0 \quad \Rightarrow z = 1$$



<u>Hyperbolic cosine rule</u>

Hyperbolic lines in D are straight lines or circles orthogonal to the boundary.





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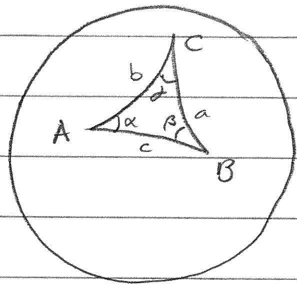
Theorem

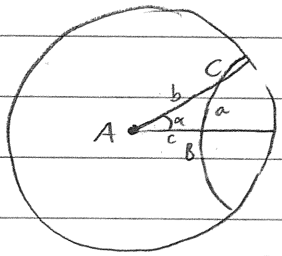
If ABC is a hyperbolic triangle with edges a, b, c and angles α, β, γ, then

$$\cosh a = \cosh b \cosh c - \cos\alpha \sinh b \sinh c.$$

Proof



Translate so that A is at the origin.



Now two of the edges are straight lines.
Rotate to make B lie on the positive real axis.

Claim:

If $B = r$ and $C = se^{i\alpha}$ then $r = \tanh(\frac{c}{2})$, $s = \tanh(\frac{b}{2})$.

This follows immediately from the exercise / example earlier in the lecture.

Moreover, if I use the Möbius map

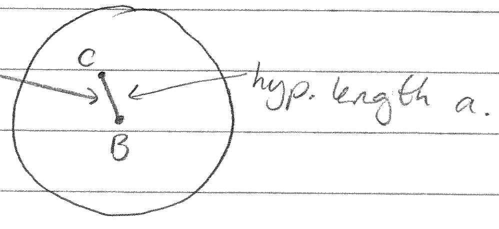
$$z \longmapsto \frac{-z + r}{rz - 1}$$

$C = se^{i\alpha}$, $B = r = \tanh(\frac{c}{2})$
$s = \tanh(\frac{b}{2})$

this is an isometry $\left[\frac{mz+n}{\bar{n}z + \bar{m}} \quad (m = -1,\ n = r)\right]$

sending r to 0. Since $B = r$, this maps B to 0, so now BC becomes a straight line segment of hyperbolic length a and C maps to the point

$$\frac{r - se^{i\alpha}}{rse^{i\alpha} - 1}.$$



So $\tanh(\frac{a}{2}) = \left|\frac{r - se^{i\alpha}}{re^{i\alpha} - 1}\right|$

We know that

$$\cosh a = \frac{1+\tan^2(\frac{a}{2})}{1-\tan^2(\frac{a}{2})} = \frac{1+\left|\frac{r-se^{i\alpha}}{rse^{i\alpha}-1}\right|^2}{1-\left|\frac{r-se^{i\alpha}}{rse^{i\alpha}-1}\right|^2}$$

$$= \frac{|rse^{i\alpha}-1|^2+|r-se^{i\alpha}|^2}{|rse^{i\alpha}-1|^2-|r-se^{i\alpha}|^2}$$

<u>Remark</u>: $\tanh(\frac{x}{2})$ substitutions hint that we're really doing a proof that looks nicer after stereographic projection.

$$|rse^{i\alpha}-1|^2 = (rs\cos\alpha-1)^2+(rs\sin\alpha)^2$$
$$= r^2s^2\cos^2\alpha - 2rs\cos\alpha + 1 + r^2s^2\sin^2\alpha$$
$$= r^2s^2 - 2rs\cos\alpha + 1$$

$$|r-se^{i\alpha}|^2 = (r-s\cos\alpha)^2+(-s\sin\alpha)^2$$
$$= r^2 - 2rs\cos\alpha + s^2\cos^2\alpha + s^2\sin^2\alpha$$
$$= r^2+s^2-2rs\cos\alpha$$

So $$\cosh a = \frac{r^2s^2-2rs\cos\alpha+1+r^2+s^2-2rs\cos\alpha}{r^2s^2-\cancel{2rs\cos\alpha}+1-r^2-s^2+\cancel{2rs\cos\alpha}}$$
$$= \frac{(r^2+1)(s^2+1)-4rs\cos\alpha}{(r^2-1)(s^2-1)}$$
$$= \frac{r^2+1}{r^2-1}\,\frac{s^2+1}{s^2-1} - \frac{2r}{r^2-1}\,\frac{2s}{s^2-1}\cos\alpha \qquad (+)$$

[Aim: $\cosh a = \cosh b\cosh c - \sinh b\sinh c\cos\alpha$]

Recall that $r=\tanh\frac{c}{2}$, $s=\tanh\frac{b}{2}$

$$\Rightarrow \frac{r^2+1}{r^2-1} = \frac{\tanh^2(\frac{c}{2})+1}{\tanh^2(\frac{c}{2})-1} = -\cosh c, \quad \frac{s^2+1}{s^2-1} = -\cosh b,$$

$$\frac{2r}{r^2-1} = \frac{2\tanh(\frac{c}{2})}{\tanh^2(\frac{c}{2})-1} = \sinh c, \quad \frac{2s}{s^2-1} = \sinh b$$

So $(+) \Rightarrow \cosh a = \cosh b\cosh c - \sinh b\sinh c\cos\alpha$. □

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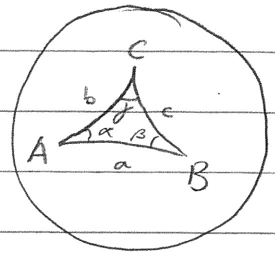


$\cosh a = \cosh b \cosh c - \cos\alpha \sinh b \sinh c$

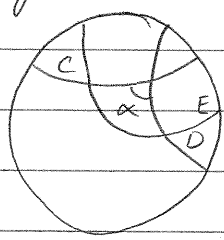
↑ Cosine rule

(Sine rule on Hw sheet 8)

<u>Lemma</u>

If $T \in \text{Isom}\,\mathbb{H}$ then T preserves angles, i.e. if C, D are hyperbolic lines meeting at angle α then TC, TD also meet at an angle α.

<u>Proof</u>

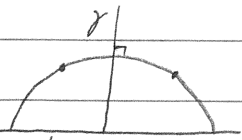


Pick a third line E to make a hyp. triangle. We saw previously that isometries send hyperbolic lines to hyperbolic lines.

So this triangle CDE gets sent to some triangle. Since T is an isometry, it preserves side lengths of this triangle. By the cosine rule, it $\therefore$ preserves angles (angles determined by side lengths).

<u>Hyperbolic reflections</u>

Fix a (hyp) line γ in $\mathbb{H}$. We will construct an isometry R_γ s.t. $R_\gamma(p) = p \;\; \forall p \in \gamma$.



<u>Claim</u>

$\forall p \in \mathbb{H} \; \exists!$ hyp. line containing $p \perp \gamma$

<u>Proof</u>

First, if γ = imaginary axis, this claim is true.

To see this, note that the hyp. lines meeting γ at right angles are the circles centred at 0. Precisely one of thes (radius $|p|$) passes through p.

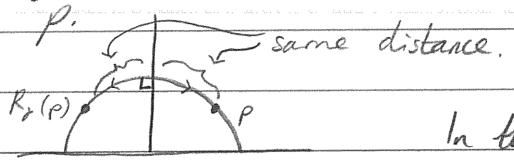
If γ is not the imaginary axis, we can use some Möbius map T such that $T\gamma$ is δ, the imaginary axis.

Then by previous case, $\exists!$ hyp. line C with $Tp \in C$ & $C \perp T\gamma$. So $T^{-1}C$ is the unique hyp. line with $p \in T^{-1}C$ and $T^{-1}C \perp \gamma$

$\square$

Def

Let γ be a hyp. line, $p \in \mathbb{H}$ a point, and let C be the unique hyp. line through p, $C \perp \gamma$. Define $R_\gamma(p)$ to be the unique point on C living the same hyp. distance from γ as p but not equal to p.



In the case $\gamma = i\mathbb{R}$, $R_\gamma(z) = -\bar{z}$

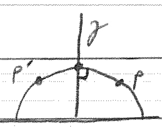
In general, if $\gamma = T(i\mathbb{R})$ for some $T \in PSL(2,\mathbb{R})$ then $R_\gamma = T \circ R_{i\mathbb{R}} \circ T^{-1}$

Lemma

Let γ be a hyp. line. If $T \in \text{Isom}\,\mathbb{H}$ st. $Tq = q$ $\forall q \in \gamma$ then $T = id$ or R_γ.

Proof

Let T be an isometry st. $Tq = q$ $\forall q \in \gamma$.



Let $p \notin \gamma$ be a point

Claim: Either $Tp = p$ or $Tp = R_\gamma(p) = p'$

Proof of claim:

T sends the unique hyp. line C containing p & $\perp \gamma$ to a hyp. line C' containing Tp $\perp$ to γ.

However $Tu = u$ where $u = C \cap \gamma$, so $u \in C$ & $u \in C'$.

So $C = C'$, the unique hyp. line through u, $\perp$ to γ

so $TC = C$.

$p \in C$ therefore gets sent to another point on C

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and $d(p,u) = d(Tp,u)$.
But there are precisely 2 points on C a distance $d(p,u)$ from u, namely p & p'. $\square$

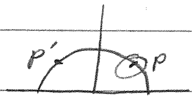
Let $\begin{cases} U = \{p \in \mathbb{H} \setminus \gamma : Tp = p\} \\ U' = \{p \in \mathbb{H} \setminus \gamma : Tp = p' = R_\gamma(p)\} \end{cases}$

Claim: Both of these are open sets.
We can't write $\mathbb{H}$ as a disjoint union of open sets because it's connected!
$\Rightarrow U = \mathbb{H}$ or $U' = \mathbb{H}$

Note: Technically I need to apply this argument to the two connected halves of $\mathbb{H} \setminus \gamma$ separately.
Then we need a separate argument to rule out the possibility that one half is reflected and the other half is fixed. This can be ruled out because

$$d(p,p) = 0 \quad \text{but} \quad d(p, R_\gamma(p)) \neq 0.$$

Why is $U = \{p \in \mathbb{H} \setminus \gamma : Tp = p\}$ open?
If $p \in U$ then $B_\varepsilon(p) \subseteq U$ for some ε.



Pick $0 < \varepsilon < \dfrac{d(p,p')}{3}$

Let $r \in B_\varepsilon(p)$
Suppose $Tr = r'$. We know $Tp = p$

$$3\varepsilon < d(p,p') \leq \underbrace{d(p,r')}_{\substack{= d(Tp,Tr) \\ = d(p,r) < \varepsilon}} + d(r',p')$$
$$< \varepsilon + d(r',p') < 2\varepsilon$$

$\times$

$\Rightarrow Tr = r \Rightarrow r \in U$
$\Rightarrow B_\varepsilon(p) \subseteq U$
$\Rightarrow U$ is open.

$\square$

Theorem

The isometry group Isom $\mathbb{H}$ is

$PSL(2,\mathbb{R}) \cup R_\delta \, PSL(2,\mathbb{R})$

where δ = imaginary axis.

i.e. every isometry is a composition of a Möbius map and possibly a reflection.

Proof

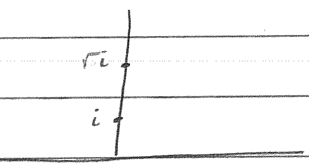
Let $T \in \text{Isom}\,\mathbb{H}$

T sends the imaginary axis δ to some hyp. line γ. Pick $S \in PSL(2,\mathbb{R})$ s.t. $S\gamma = \delta$.

Now $S \circ T$ sends δ to δ (as a set).

Moreover, if $p = Ti$ we can also assume $Sp = i$

so $S \circ T(i) = i$



Given a point ri on δ

Since $S \circ T$ is an isometry

$$d(S \circ T(ri), i) = d(ri, i) = |\log r| = d(\tfrac{i}{r}, i)$$

So $S \circ T(ri) = \begin{cases} ri \\ \frac{i}{r} \end{cases}$

If it's ri then $S \circ T$ fixes δ pointwise.

In this case $S \circ T = \begin{cases} 1 \\ R_\delta \end{cases}$

$$\Rightarrow T = \begin{cases} S^{-1} \\ S^{-1} \circ R_\delta \end{cases}$$

In the other case, if $S \circ T = i/r$ then

$R(z) = -\frac{1}{z} \in PSL(2,\mathbb{R})$

$R(\frac{i}{r}) = ir$ so $R \circ S \circ T = \begin{cases} 1 \\ R_\delta \end{cases} \Rightarrow T = \begin{cases} S^{-1} \circ R^{-1} \\ R_\delta \circ S^{-1} \circ R \end{cases}$

$\square$

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Last time:

Any isometry of hyperbolic space is in the group generated by $PSL(2,\mathbb{R})$ & reflections in hyperbolic lines.

Today

We will study the isometries in $PSL(2,\mathbb{R})$ in more depth, in particular we'll classify them into 3 types:

Theorem:

Any non-identity element $A \in PSL(2,\mathbb{R})$ is one of

Hyperbolic	Parabolic	Elliptic
Has two fixed points on $\mathbb{R} \cup \{\infty\}$.	Has one fixed point which is in $\mathbb{R} \cup \{\infty\}$.	Has two complex conjugate fixed points $z, \bar{z}$ with $z \in \mathbb{H}$, $\bar{z} \in -\mathbb{H}$.
$\lvert Tr(A) \rvert > 2$	$\lvert Tr(A) \rvert = 2$	$\lvert Tr(A) \rvert < 2$
A is conjugate to $\begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}$, i.e. to a rescaling.	A is conjugate to $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ i.e. $z \mapsto z+b$.	A is conjugate to $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, i.e. rotation.

$\left[A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ Tr A = a+d \right]$

Proof

We know that the fixed point condition:

$$\frac{az+b}{cz+d} = z$$

is a quadratic eqn in z:

$$az+b = cz^2 + dz$$

i.e. $cz^2 + (d-a)z - b = 0$

So we have ≤ 2 fixed points. Moreover, since $a,b,c,d \in \mathbb{R}$, these fixed points are either real or else complex conjugates.

If $c = 0$ (not a quadratic!) then the eqn. is just

$az + b = z$ i.e. $-b = (a-1)z$

So $z = \dfrac{-b}{a-1}$. In this case ∞ is fixed: $a\infty + b = \infty$.

So if $c=0$ we have either

- one fixed point at ∞ and $a=1$ (parabolic)
- one fixed point at ∞ and another at $\frac{-b}{a-1} \in \mathbb{R}$ (hyperbolic)

When $c=0$, $A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ & $ad=1$

so $A = \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix}$, $\mathrm{Tr}\,A = a + \frac{1}{a}$

The function $|a + \frac{1}{a}|$ has an absolute minimum at $a=1$. Differentiate: $1 - \frac{1}{a^2} = 0 \Rightarrow a = \pm 1$ where the function takes the value 2 (or -2). So $|\mathrm{Tr}\,A| \geq 2$ with equality iff $a=1$. So in parabolic case $|\mathrm{Tr}\,A| = 2$, in hyperbolic case $|\mathrm{Tr}\,A| > 2$.

Moreover in this $c=0$ case, the matrix A is either $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ $(a=1)$ as claimed in the theorem.

or $\begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix}$ $(a \neq 1)$

and this second matrix is conjugate to $\begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}$ (Jordan Normal Form of $\begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix}$ is $\begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}$ as the eigenvalues are distinct).

When $c \neq 0$, the fixed point equation is quadratic with real coefficients, so fixed points are either real or come in conjugate pairs.

Case 1: 2 real fixed points (Hyperbolic)
Pick a T in $PSL(2, \mathbb{R})$ st. $Tp = 0$, $Tq = \infty$.
So w.l.o.g. $p=0$, $q=\infty$. In this case

$$\frac{a0+b}{c0+d} = 0 \quad \& \quad \frac{a\infty + b}{c\infty + d} = \infty$$

$$\Rightarrow b = 0 \qquad\qquad \Rightarrow c = 0$$

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So we have reduced to the previous case ($c=0$).

<u>Case 2:</u> 1 real fixed point (multiplicity 2 solution to the quadratic) at p.

Pick $T \in PSL(2,\mathbb{R})$ s.t. $Tp = \infty$,

conjugating A by T: if $Ap = p$, then $TAT^{-1}(Tp) = Tp$

so w.l.o.g. ∞ is a fixed point $\Rightarrow \frac{a\infty + b}{c\infty + d} = \infty$

$\Rightarrow c = 0$ so we're back in the $c=0$ case.

<u>Case 3:</u> 2 complex conjugate fixed points $p \in \mathbb{H}$, $\bar{p} \in -\mathbb{H}$

Need to prove that $|\mathrm{Tr}(A)| < 2$

and that A is conjugate to a rotation matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

We'll prove that if A is conjugate to $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, then its trace is $2\cos\theta$ which has absolute value < 2.

Pick $T \in PSL(2,\mathbb{R})$ s.t. $Tp = i$.

Then (after conjugating) $\frac{ai+b}{ci+d} = i$

$\Rightarrow ai + b = -c + di \Rightarrow b = -c$ & $a = d$

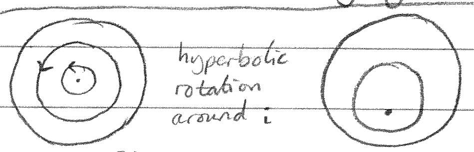
$\Rightarrow A = \begin{pmatrix} a & -c \\ c & a \end{pmatrix}$ with $a^2 + c^2 = 1$

$\therefore \exists\, \theta$ s.t. $a = \cos\theta$, $c = \sin\theta$

and then we see that A was conjugate (via T) to

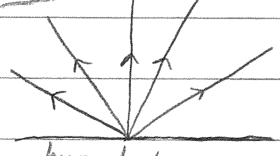
$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

□

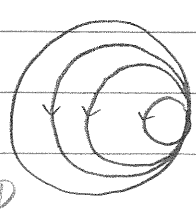


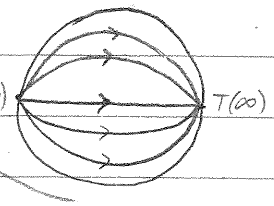
i.e. an <u>elliptic</u> isometry.

<u>Hyperbolic</u> case A conjugate to $\begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}$ $z \mapsto a^2 z$

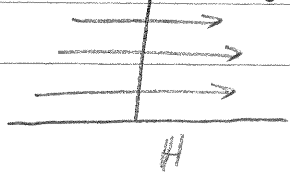


hyperbolic isom. of $\mathbb{H}$





Parabolic isometry



Hyperboloid model

In special relativity light always moves with the same speed, c (equal to 1 for this lecture).

Define the "spacetime interval" between two points in spacetime as follows:

If the points are

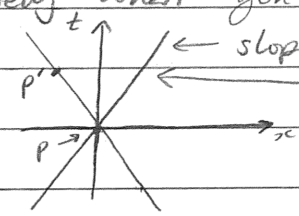
$p = (t, x, y, z)$ and $p' = (t', x', y', z')$

then the "interval" between them is

$$s(p, p') = -c^2(t-t')^2 + (x-x')^2 + (y-y')^2 + (z-z')^2$$

When is $s(p, p') = 0$?

Precisely when you can connect p & p' with a light beam.



$s(p, p') = 0$ means the distance p to p'
$= c \times$ (time difference between p & p')

Because we can test whether $s(p,p') = 0$, this contition should look the same in all frames of reference.

i.e. a change of frame could be anything that preserves $s(p, p')$

e.g. $A = \begin{array}{cc} t & x \\ \end{array}$
$$A = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}$$

Claim:

A preserves spacetime interval.

Proof

w.l.o.g. $p' = 0$ (ignore y & z)

$$Ap = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} t\cosh\theta + x\sinh\theta \\ t\sinh\theta + x\cosh\theta \end{pmatrix}$$

$$s(0, Ap) = -(t\cosh\theta + x\sinh\theta)^2 + (t\sinh\theta + x\cosh\theta)^2$$
$$= -t^2\cosh^2\theta - x^2\sinh^2\theta - \cancel{2xt\cosh\theta\sinh\theta}$$
$$+ t^2\sinh^2\theta + x^2\cosh^2\theta + \cancel{2xt\cosh\theta\sinh\theta}$$
$$= x^2 - t^2 = s(0, p) \qquad \left(p = \begin{pmatrix} t \\ x \end{pmatrix}\right) \quad \square$$

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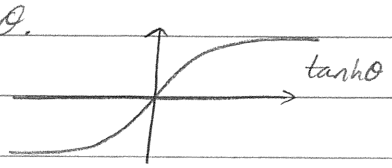
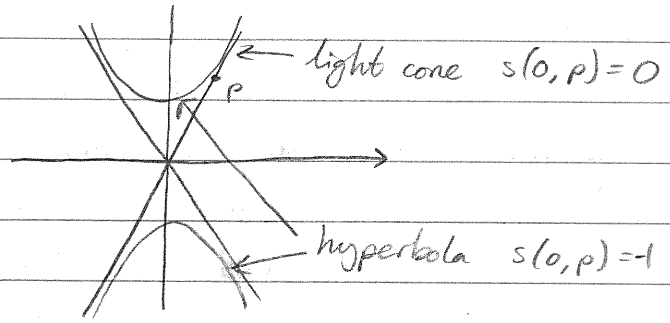
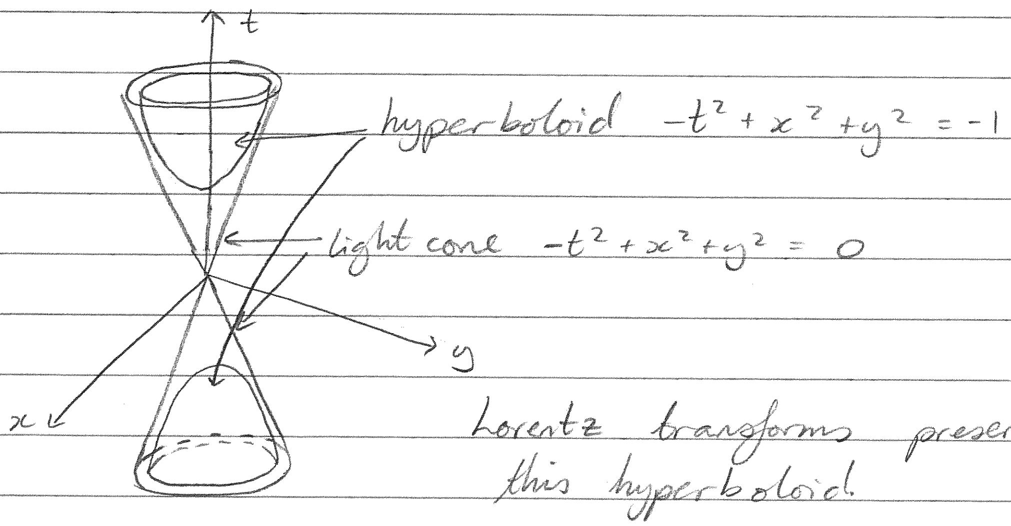
Physically, this A is called a "Lorentz boost" and is called a change of reference frame from a frame at rest to one that's moving at a speed $c\tanh\theta$.



Note: $\tanh\theta < 1$
(Can't use a Lorentz transformation to change a slower-than-light speed to a faster-than-light speed).

Def
A Lorentz transformation is a linear change of coordinates in space-time which preserves spacetime interval.





Lorentz transforms preserve this hyperboloid.

We equip the upper hyperboloid $Y = \{-t^2+x^2+y^2=-1,\ t>0\}$ with a geometry as follows: if γ is a curve in Y we define the length of γ to be

$$L(\gamma) = \int \sqrt{-\dot{t}^2 + \dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, d\tau$$

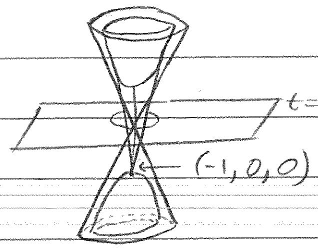
$$\gamma(\tau) = (t(\tau), x(\tau), y(\tau), z(\tau)).$$

Theorem

Y, equipped with this notion of length, is isometric to the hyperbolic disc.

Proof

Consider the stereographic projection:



$$F(t,x,y) = \frac{x+iy}{1+t}$$

Claim: If $p=(t,x,y) \in Y = \{-t^2+x^2+y^2=-1 : t>0\}$ then $F(p)$ lies in the unit disc in $\mathbb{C}$.

Proof

$$\left|\frac{x+iy}{1+t}\right|^2 = \frac{x^2+y^2}{(1+t)^2} = \frac{t^2-1}{(1+t)^2} = \frac{(1+t)(t-1)}{(1+t)(1+t)}$$

$$= \frac{t-1}{t+1} < 1 \qquad \square$$

If $\delta(\tau)$ is a curve in the hyperbolic disc then we proved

$$L(\delta) = \int \frac{2|\dot{\delta}|}{1-|\delta|^2}\, d\tau$$

Let $\gamma(\tau) = (t(\tau), x(\tau), y(\tau))$ be a curve on Y

$$L(\gamma) = \int \sqrt{-\dot{t}^2+\dot{x}^2+\dot{y}^2}\, d\tau$$ where dot denotetes $\frac{d}{d\tau}$

Let $u = \frac{x}{1+t}$, $v = \frac{y}{1+t}$.

These are the components of $\delta = F(\gamma)$

Compute $\dot{\delta}$ and $1-|\delta|^2$ & check that

$$\frac{2|\dot{\delta}|}{1+|\delta|^2} = \sqrt{-\dot{t}^2+\dot{x}^2+\dot{y}^2}$$

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This will show that length measured using spacetime interval on Y is the same as length in the hyperbolic disc.

$$\delta(\tau) = (u(\tau), v(\tau)) = \left(\frac{x(\tau)}{1+t(\tau)}, \frac{y(\tau)}{1+t(\tau)} \right)$$

$$\dot{\delta}(\tau) = \left(\frac{\dot{x}}{1+t} - \frac{x\dot{t}}{(1+t)^2}, \frac{\dot{y}}{1+t} - \frac{y\dot{t}}{(1+t)^2} \right)$$

$$|\dot{\delta}(\tau)|^2 = \frac{\dot{x}^2}{(1+t)^2} + \frac{x^2\dot{t}^2}{(1+t)^4} - \frac{2x\dot{x}\dot{t}}{(1+t)^3} + \frac{\dot{y}^2}{(1+t)^2} + \frac{y^2\dot{t}^2}{(1+t)^4} - \frac{2y\dot{y}\dot{t}}{(1+t)^3}$$

$$= \frac{\dot{x}^2+\dot{y}^2}{(1+t)^2} + \frac{(x^2+y^2)\dot{t}^2}{(1+t)^4} - \frac{2(x\dot{x}+y\dot{y})\dot{t}}{(1+t)^3}$$

$$= \frac{\dot{x}^2+\dot{y}^2}{(1+t)^2} + \frac{(t^2-1)\dot{t}^2}{(1+t)^4} - \frac{t\dot{t}^2}{(1+t)^3}$$

since $\gamma \subseteq Y$, $-t^2 + x^2 + y^2 = -1$

$\Rightarrow -t\dot{t} + x\dot{x} + y\dot{y} = 0$

$$\text{so } |\dot{\delta}|^2 = \frac{\dot{x}^2+\dot{y}^2}{(1+t)^2} + \left(\frac{t-1}{(1+t)^3} - \frac{2t}{(1+t)^3} \right)\dot{t}^2$$

$$= \frac{\dot{x}^2+\dot{y}^2-\dot{t}^2}{(1+t)^2}$$

$$1-|\delta|^2 = 1-u^2-v^2 = 1 - \frac{x^2}{(1+t)^2} - \frac{y^2}{(1+t)^2}$$

$$= \frac{(1+t)^2 - x^2 - y^2}{(1+t)^2}$$

$$= \frac{1+t^2-x^2-y^2+2t}{(1+t)^2} = \frac{2+2t}{(1+t)^2} = \frac{2}{1+t}$$

$$\Rightarrow \underbrace{\frac{2|\dot{\delta}|}{1-|\delta|^2}}_{\text{integrand in hyperbolic length of } \delta} = 2\frac{\sqrt{\dot{x}^2+\dot{y}^2-\dot{t}^2}}{1+t} \Big/ \frac{2}{1+t} = \underbrace{\sqrt{\dot{x}^2+\dot{y}^2-\dot{t}^2}}_{\text{integrand in spacetime interval}}$$

The geometry of the spacetime interval is called Minkowski spacetime & we see that hyperbolic geometry occurs naturally on a hyperboloid in Minkowski spacetime.

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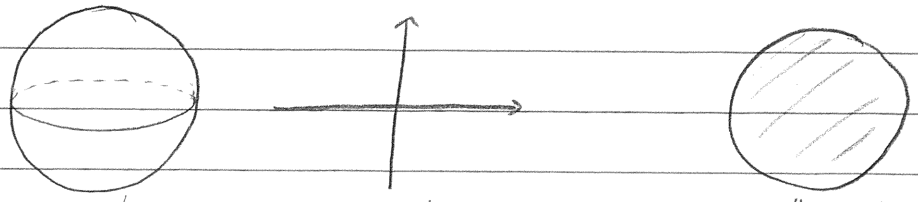
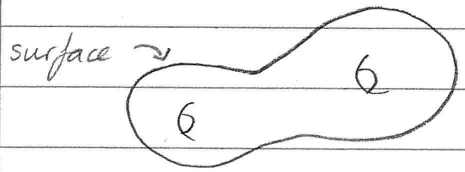
Non examinable.

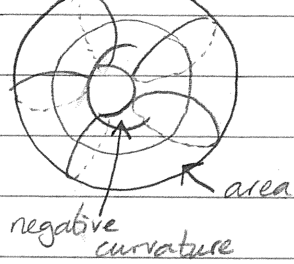


These geometries are the only 2-d geometries where the isometry group acts transitively.
One way of studying geometry / topology is to start with some random geometry and "flow" it to make it "nicer".



In fact every 2-d surface admits a geometry locally modelled on one of these three

Example
The torus



$S^1 \times S^1$

$\{(x, y) : x \in [0, 2\pi), y \in [0, 2\pi)\}$

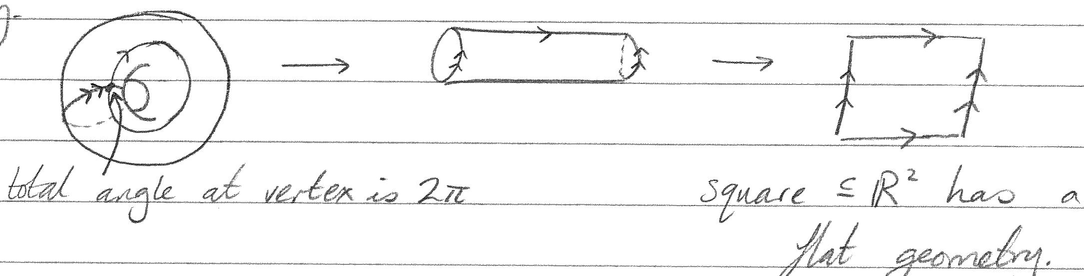
$x = x + 2\pi, \quad y = y + 2\pi$

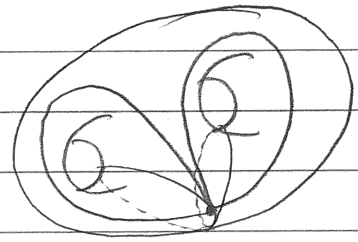
positively curved

$+$ flat

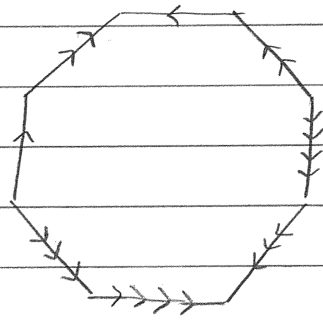
negatively curved

In fact, the torus can be given a flat geometry globally.

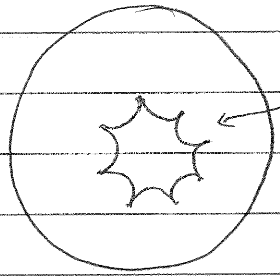




surface with genus 2.

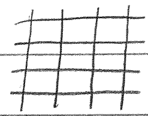


Sum of the angles should be $> 2\pi$ if the octagon were Euclidean.
But, we can find a hyperbolic octagon with internal angles $\pi/4$.



You can tile $\mathbb{H}$ with disjoint images of this octagon under certain Möbius maps.

You can tile $\mathbb{R}^2$ with translates of the square.



The group $\mathbb{Z}^2$ of pairs of integers acts by translating horizontally and vertically by an integer amount.
The orbit of the square under this $\mathbb{Z}^2$ action is the set of all tiles in the tiling.
Put another way, given any $(p,q) \in \mathbb{R}^2$
$\exists (a,b) \in \mathbb{Z}^2$ s.t. $(p-a, q-b) \in$ square.
i.e. each orbit of $\mathbb{Z}^2$ action on $\mathbb{R}^2$ contains a unique point in the square $[0,1) \times [0,1)$.

This tells us that the torus parameterises the orbits of this $\mathbb{Z}^2$ action.
i.e. there's a bijection between orbits of $\mathbb{Z}^2$ and points in the torus

$$\underbrace{\mathbb{R}^2/\mathbb{Z}^2}_{\text{space of orbits}} = \underbrace{T^2}_{\text{torus.}}$$

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In the octagonal case we have a group

$\Gamma \subseteq PSL(2, \mathbb{R})$

$\mathbb{H} / \Gamma =$ 

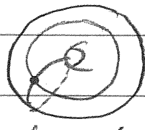
i.e. the genus 2 surface parameterises as the space of orbits.

<u>Theorem</u> (Uniformisation theorem)

For any surface Σ with genus >1

$\exists \, \Gamma \subseteq PSL(2, \mathbb{R})$ s.t. $\mathbb{H} / \Gamma = \Sigma$

"fundamental group" of Σ

In the case of T^2 

the group $\mathbb{Z}^2$ corresponds to loops that wrap $\begin{cases} m \text{ times around one } S^1 \\ n \text{ times around the other.} \end{cases}$

For higher genus, Γ is non-abelian.

(For more info. see Topology & Groups course)

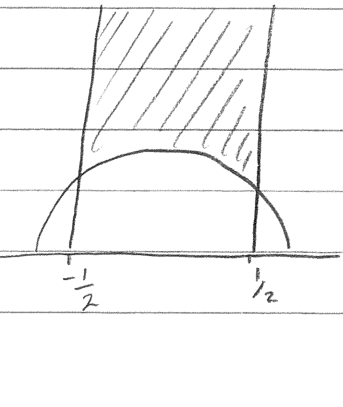
<u>Example</u>

For T^2 we looked at $\mathbb{Z}^2 \subseteq \mathbb{R}^2$.

We will now look at $PSL(2, \mathbb{Z}) \subseteq PSL(2, \mathbb{R})$

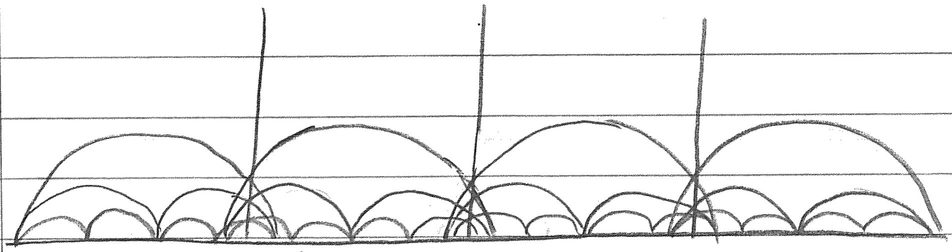
the group of Möbius maps with integer coefficients and det $=1$

e.g. $Sz = -\frac{1}{z}$ $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $Tz = z+1$ $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$



<u>Claim</u>: Any $z \in \mathbb{H}$ can be moved into $D = \{ z \in \mathbb{H} : |z| > 1, Re(z) \in [-\frac{1}{2}, \frac{1}{2}] \}$ by an element of $PSL(2, \mathbb{Z})$

fundamental domain.



This tiling looks like modular tiling.

Proof

Let $\frac{az+b}{cz+d}$ be a Möbius map in $PSL(2, \mathbb{Z})$

We saw that $\operatorname{Im}\left(\frac{az+b}{cz+d}\right) = \frac{\operatorname{Im} z}{|cz+d|^2}$ (note $ad-bc=1$ here)

We want to end up in D, so we need $\operatorname{Im}\left(\frac{az+b}{cz+d}\right)$ to

be as big as it could be.

$$\operatorname{Im} z > 0 \text{ so } |cz+d|^2 = |cx+ciy+d|^2$$
$$= c^2x^2 + d^2 + 2cxd + c^2y^2$$
$$> \min(c^2y^2, d^2)$$
$$\geq \min(y^2, 1) \text{ with equality iff } c = \pm 1$$

$$\Rightarrow \operatorname{Im}\left(\frac{az+b}{cz+d}\right) = \frac{\operatorname{Im} z}{|cz+d|^2} \leq \frac{\operatorname{Im} z}{\min(y^2, 1)}$$

For sufficiently large c or d it becomes much smaller, so in fact there's a finite number of pairs (c, d) s.t. $\operatorname{Im}\left(\frac{az+b}{cz+d}\right) >$ any given lower bound.

Pick an $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ s.t. $\operatorname{Im}\left(\frac{az+b}{cz+d}\right)$ is maximal amongst

$$\{\operatorname{Im}(gz) : g \in PSL(2, \mathbb{Z})\}$$

Once you have translated $\frac{az+b}{cz+d}$ until it lies in the region $\{\operatorname{Re}(z) \in [-\frac{1}{2}, \frac{1}{2}]\}$ it must also satisfy $|z| > 1$.

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Proof

This translated point has same (maximal) imaginary point.

If $|z| < 1$, we could apply $Sz = -\frac{1}{z}$

but $\operatorname{Im}(Sz) = \operatorname{Im}\left(-\frac{1}{z}\right)$

$$= \frac{\operatorname{Im} z}{|z|^2}$$

If $|z| < 1$ then $\operatorname{Im}(Sz) > \operatorname{Im} z$

This contradicts the fact that z maximises the imaginary part of its orbit under $PSL(2, \mathbb{Z})$.

$\square$