7302 Analytical Dynamics Notes.

Based on the spring 2012 lectures by
Prof N R McDonald.

To my friends and colleagues that have supported me throughout my degree and to which I am humbly in their debts.

11/1/12

§ 0 Summation convention

The ith component of $\underline{a}$ is denoted a_i.
($i = 1,2,3$ in 3D)

The ij th component of matrix H is H_{ij}
Whenever an index i, j or k (etc.) is repeated in some term, a summation over 1,2,3 is understood.

Ex: $\underline{a} = a_1\underline{e}_1 + a_2\underline{e}_2 + a_3\underline{e}_3$ and
$\underline{b} = b_1\underline{e}_1 + b_2\underline{e}_2 + b_3\underline{e}_3$

$\underline{a}\cdot\underline{b} = a_1b_1 + a_2b_2 + a_3b_3$ (since $\underline{e}_1, \underline{e}_2, \underline{e}_3$ are orthonormal)
$= a_ib_i = a_jb_j$ etc.

Ex: $c_i = H_{ij}a_j$
$= H_{i1}a_1 + H_{i2}a_2 + H_{i3}a_3$
$= H_{ik}a_k$

$\underline{c} = H\underline{a}$

$$\begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ \cdot & \cdot \quad \cdot & \cdot \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$ (i.e row i of matrix H "dotted" with $\underline{a}$ to form $\underline{c} = H\underline{a}$)

$$\operatorname{div} \underline{u} = \nabla\cdot\underline{u} = \frac{\partial u_i}{\partial x_i} \left(= \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \right)$$

$$= \frac{\partial u_l}{\partial x_l} = \partial_l u_l = u_{i,i}$$

Note! - The expression $c_j a_j b_j$ has no meaning within this convention. An index may occur a maximum of 2 times!

Note that $a_j b_j + e_j$ is still legal.

An index which is not repeated is known as free index i.e it is not summed over.

Ex: $a_i H_{ij} J_{jk} C_\ell$ has 2 free indices k and ℓ.

Ex: $$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x_i \partial x_i} \quad (= \phi_{,ii})$$

$$= \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} + \frac{\partial^2 \phi}{\partial x_3^2}$$

Some special symbols:

i) delta symbol

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \quad , \delta_{12} = 0$$

$$\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33}$$

$$\delta_{ij}\delta_{ik} = \delta_{1j}\delta_{1k} + \delta_{2j}\delta_{2k} + \delta_{3j}\delta_{3k}$$

eg.: $j = 1$ and $k = 1 \Rightarrow 1$
$\quad$ " and $k = 2, 3 \Rightarrow 0$ $\Big| \Rightarrow \delta_{1k}$
$j = 2 \quad \delta_{2k}$
$j = 3 \quad \delta_{3k}$

$\delta_{ij}\delta_{ik} = \delta_{jk}$
$(\delta_{mj}\delta_{mk} = \delta_{jk}$ etc $)$

In fact, for any T_j we have $\delta_{ij}T_j = T_i$

"Substitution property of δ_{ij}".

$\delta_{ij}T_{iklm} = T_{jklm}$.

—/—

(ii) Permutation symbol

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } i,j,k \text{ is an even permutation of } 123 \\ -1 & \text{if } i,j,k \text{ is an odd perm. of } 123 \\ 0 & \text{if neither even or odd.} \end{cases}$$

i.e $\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{321} = +1$
$\varepsilon_{132} = \varepsilon_{321} = \varepsilon_{213} = -1$
(e.g $\varepsilon_{112} = \varepsilon_{222} = \varepsilon_{313} = 0$ etc)

Claim

$$(\underline{a} \times \underline{b}) = \varepsilon_{ijk} a_i b_j$$

16/1/12

Last lecture:

$a_{ij} b_{ik} c_{\alpha k}$

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

$$\delta_{ij} \Gamma_{i\alpha\beta} = \Gamma_{j\alpha\beta}$$

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{even} \\ -1 & \text{odd} \\ 0 & \text{neither.} \end{cases}$$

Claim: $\boxed{(\underline{a} \times \underline{b})_k = \varepsilon_{ijk} a_i b_j}$

Check: $\underline{a} \times \underline{b} = \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \underline{e}_1 (a_2 b_3 - a_3 b_2) - \underline{e}_2 (a_1 b_3 - a_3 b_1) + \underline{e}_3 (a_1 b_2 - b_1 a_2)$

$k=1$ $\quad (\underline{a} \times \underline{b})_1 = \varepsilon_{ij1} a_i b_j$

$$= \varepsilon_{231} a_2 b_3 + \varepsilon_{321} a_3 b_2$$

$$= a_2 b_3 - a_3 b_2$$

Similarly for $k = 2, 3$.

Recall: $\text{div}(\text{curl}\, \underline{u}) = 0$

$\underline{\nabla} = \frac{\partial}{\partial x_1} \underline{e}_1 + \frac{\partial}{\partial x_2} \underline{e}_2 + \frac{\partial}{\partial x_3} \underline{e}_3$

Note: $(\text{curl}\, \underline{u})_k = (\underset{\sim}{\nabla} \times \underline{u})_k$

$$= \varepsilon_{ijk} \frac{\partial}{\partial x_i} u_j$$

$$\operatorname{div}(\operatorname{curl}(\underline{u})) = \frac{\partial}{\partial x_k}(\operatorname{curl}\underline{u})_k$$

$$= \frac{\partial}{\partial x_k} \cdot \varepsilon_{ijk} \frac{\partial}{\partial x_i} u_j$$

$\operatorname{div}\underline{a}$
$= \frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial_3}$
$= \frac{\partial a_i}{\partial x_i} = \frac{\partial a_k}{\partial x_k}$

$$= \varepsilon_{ijk} \frac{\partial^2}{\partial x_k \partial x_i} u_j$$

$$= \varepsilon_{kji} \frac{\partial^2}{\partial x_i \partial x_k} u_j$$

$i \to k$
$k \to i$

$$= \varepsilon_{kji} \frac{\partial^2}{\partial x_k \partial x_i} u_j$$

(swap order of diff)

$$= -\varepsilon_{ijk} \frac{\partial^2 u_j}{\partial x_k \partial x_i}$$

$$= 0 \quad \text{(just like } y=-y\text{)}$$

$\varepsilon_{\overset{\frown}{123}} = \varepsilon_{231} = \varepsilon_{312} = 1$

$\varepsilon_{321} = \varepsilon_{132} = \varepsilon_{213} = -1$

Ex $\quad \operatorname{curl}(\operatorname{grad}\phi) = 0$.

Note: the following is useful

$$K_{ij} L_{jk} = M_{ik} \leftarrow ikth \text{ of } M = KL$$

$\begin{pmatrix}\text{ith row of}\\ \text{matrix } K\end{pmatrix} \cdot \begin{pmatrix}\text{kth column}\\ \text{of } L\end{pmatrix}$

i.e. $KL = M$

note also:

$$K_{ij}L_{jk} = L_{jk}K_{ij} = M_{ik}$$

What is $L_{kj}K_{ij} = L_{kj}(K)^T_{ji}$

$$= (LK^T)_{ki}$$

—/—

$$(A^T)_{jk}B_{ij} = (BA^T)_{ik}$$

—/—

1. Frames of Reference.

In order to describe a system it is essential to introduce a coordinate system to label the configurations of a system. This gives the degrees of freedom of the system.

Ex: A particle moving in space has 3 degrees of freedom eg 3 cartesian coods or 3 spherical polar coords etc.

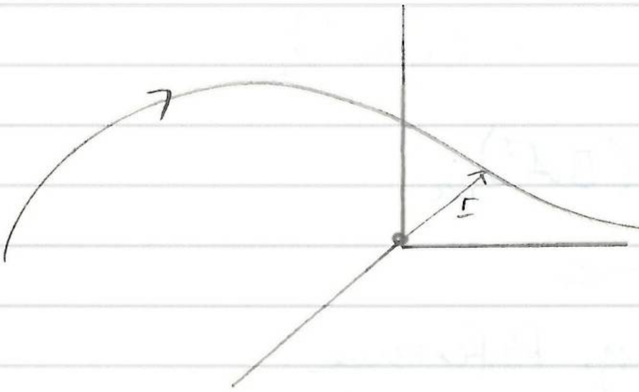
Ex: N particles has 3N degrees of freedom.

Ex: 2 particles connected by a rigid rod has 3 degrees of freedom: 3 for one of the particles + 2 for the other.

Ex: A rigid body (see later) has 6 degree of freedom.

Newton's law.

Def$\underline{n}$: velocity relative to frame of reference R is $\underline{u} = (\dot{r}_1, \dot{r}_2, \dot{r}_3) = \dot{\underline{r}}$ where $\underline{r}$ is the position vector of a particle from orgin of R.



acceleration relative to R is

$\underline{a} = \ddot{\underline{r}}$
$= (\ddot{r}_1, \ddot{r}_2, \ddot{r}_3)$

There exist special frames of reference called inertial frames s.t. Newton's law hold.

$$F_i = ma_i = m\ddot{r}_i \qquad i = 1, 2, 3.$$

where m is the (constant) mass of the particle and F_i is the ith component of the force acting on the particle.

A frame of refrence is repersented by an orgin O and $\underline{B} = (\underline{e}_1, \underline{e}_2, \underline{e}_3)$ a triad of unit vectors along coordinate axes.

Note that $\underline{e}_i \cdot \underline{e}_j = \delta_{ij}$ (orthonormal)
$\underline{e}_1 \cdot (\underline{e}_2 \times \underline{e}_3) = 1$ (right-handed)

Suppose $B = (\underline{e}_1, \underline{e}_2, \underline{e}_3)$ and $\hat{B} = (\underline{\hat{e}}_1, \underline{\hat{e}}_2, \underline{\hat{e}}_3)$ are 2 **orthonormal** triads moving relative to each other.

Defn: H is the transition matrix from $\hat{B}$ to B, where

$$H_{ij} = \underline{e}_i \cdot \underline{\hat{e}}_j$$

Note, that the transition matrix from B to $\hat{B}$ is H^T.

$$H^*_{ij} = \underline{\hat{e}}_i \cdot e_j \quad \text{(by definition)}$$

$$= \underline{e}_j \cdot \underline{\hat{e}}_i$$

$$= H_{ji}$$

$$= (H^T)_{ij} \qquad \Rightarrow \quad H^* \equiv H^T$$

The H_{ij} are 9 functions of time which determine the **relative** orientation of the two triads.

—/—

The $\hat{B}$-component of $\underline{e}_i$ are $\underline{e}_i \cdot \underline{\hat{e}}_j$ for $j = 1, 2, 3$ or H_{i1}, H_{i2}, H_{i3}. Similarly H_{1j}, H_{2j}, H_{3j} are the B-component of $\underline{\hat{e}}_j$ (eg. $\underline{\hat{e}}_j \cdot \underline{e}_1 = H_{1j}$ etc).

Ex: $\underline{\hat{e}}_1 = \cos\theta \; \underline{e}_1 - \sin\theta \, \underline{e}_2$
$\underline{\hat{e}}_2 = \sin\theta \; \underline{e}_1 + \cos\theta \, \underline{e}_2$

Then $H_{ij} = \underline{e}_i \cdot \underline{\hat{e}}_j$

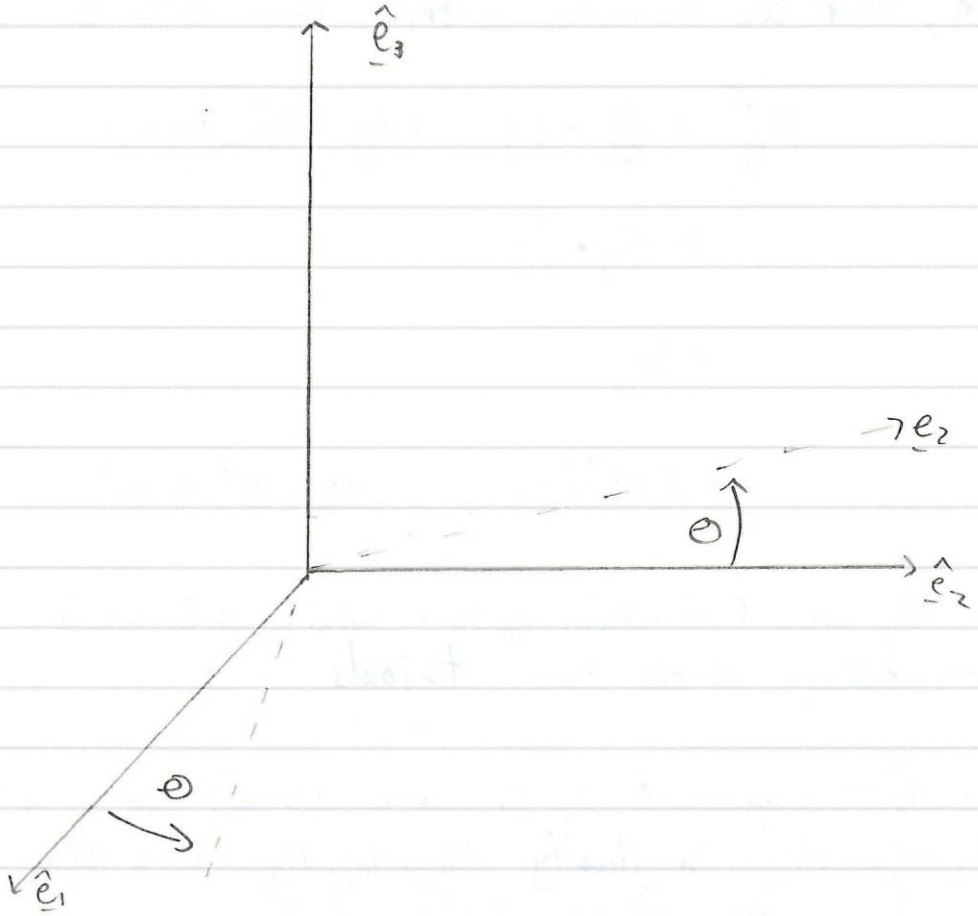
$$H = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So, $\underline{e}_1 = H_{1j} \hat{\underline{e}}_j$ (sum over j).

$= \cos\theta \ \hat{\underline{e}}_1 + \sin\theta \ \hat{\underline{e}}_2$.

$e_2 = -\sin\theta \ \hat{\underline{e}}_1 + \cos\theta \ \hat{\underline{e}}_2$

$\underline{e}_3 = \hat{\underline{e}}_3$



Let $\underline{x}$ be an arbitrary vector.

$x_i = \underline{x} \cdot \underline{e}_i$

$= H_{ij} \underline{x} \cdot \hat{\underline{e}}_j$ (recall $\underline{e}_i = H_{ij} \hat{\underline{e}}_j$)

$= H_{ij} \hat{x}_j$

and $\hat{x}_i = \underline{x} \cdot \hat{\underline{e}}_i$ $\qquad\left(\hat{\underline{e}}_i = H_{ji}\,\underline{e}_j\right)$

$= H_{ji}\,\underline{x} \cdot \underline{e}_j$

$= H_{ji}\,x_j$

$= (H^T)_{ij}\,x_j$

In matrix notation we have

$$\underline{x} = H\hat{\underline{x}} \text{ and } \hat{\underline{x}} = H^T\underline{x}.$$

Since this holds $\forall \underline{x} \Rightarrow \underline{x} = HH^T\underline{x}$

$\Rightarrow HH^T = I$

$\Rightarrow H$ is orthogonal.

Alternatively...

$$\delta_{jk} = \underline{e}_j \cdot \underline{e}_k$$
$$= (H_{jl}\,\hat{\underline{e}}_l)\cdot(H_{km}\,\hat{\underline{e}}_m)$$

—/—

Hint on Homework 1

Q2) Assume

H is 3×3

$$\det(-H) = -\det(H)$$

—/—

18/1/12

Last lecture:

$$B = (\underline{e}_1, \underline{e}_2, \underline{e}_3) \quad , \hat{B} = (\hat{\underline{e}}_1, \hat{\underline{e}}_2, \hat{\underline{e}}_3)$$

$$H_{ij} = \underline{e}_i \cdot \hat{\underline{e}}_j$$

"transition matrix from $\hat{B}$ to B"

$$\underline{x} = H\hat{\underline{x}} \quad , \quad \hat{\underline{x}} = H^T\underline{x}$$

$$\Rightarrow HH^T = \underline{I}$$

$$\underline{e}_j = H_{ji}\hat{\underline{e}}_i$$

$$\hat{\underline{e}}_i = H_{ki}\,\underline{e}_k$$

Alt:
$$\begin{aligned} \delta_{jk} = \underline{e}_j \cdot \underline{e}_k &= (H_{jl}\hat{\underline{e}}_l) \cdot (H_{km}\hat{\underline{e}}_m) \\ &= H_{jl}H_{km}\hat{\underline{e}}_l \cdot \hat{\underline{e}}_m. \\ &= H_{jl}H_{km}\delta_{lm} \\ &= H_{jm}H_{km} \\ &= H_{jm}(H^T)_{mk} \\ &= (HH^T)_{jk} \end{aligned}$$

$$\Rightarrow HH^T = \underline{I}$$

Now consider the rate of rotation of B relative to $\hat{B}$.

$$HH^T = \underline{I}.$$

Diff wrt time

$$\dot{H}H^T + H(\dot{H}^T) = 0$$

$$\dot{H}H^T = -H\dot{H}^T$$

$$= -(\dot{H}H^T)^T$$

Note: B^TA^T
$= (AB)^T$

Define $\Omega = \dot{H}H^T$ is skew-symmetric

i.e: $\Omega^T = \Omega$

$\rightarrow \Omega_{ij} = \dot{H}_{ik}H_{jk}$

$$\Omega = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ \omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix} \quad \text{(wlog)}$$

Note $\Omega_{jk} = \varepsilon_{ijk}\omega_i$

Defn: The angular velocity of <u>B relative to $\hat{B}$</u> is the vector $\underline{\omega} = \omega_i \underline{e}_i$

<u>Ex</u>: $H = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Aim: to find $\underline{\omega}$ via $\Omega = \dot{H}H^T$

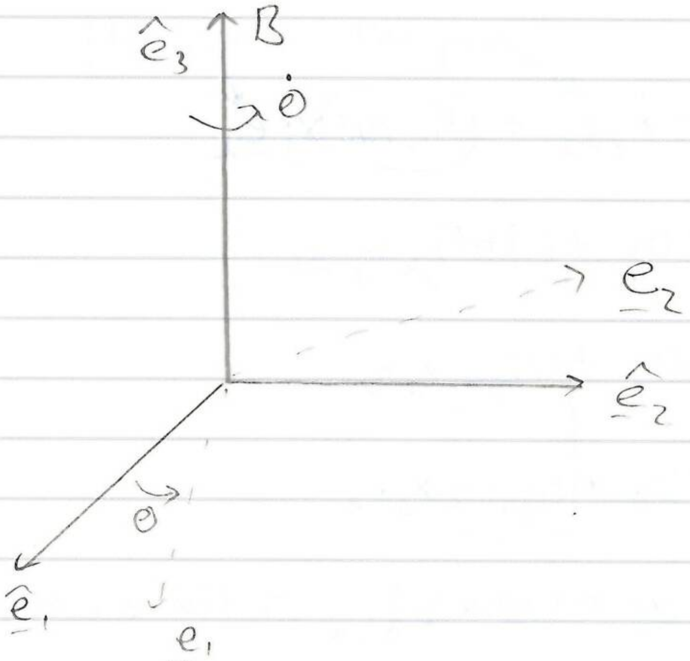
$$\dot{H} = \dot{\theta}\begin{pmatrix} -\sin\theta & \cos\theta & 0 \\ -\cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Omega = \dot{H}H^T$$

$$= \dot{\theta}\begin{pmatrix} -\sin\theta & \cos\theta & 0 \\ -\cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \dot{\theta}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus $\omega_3 = \dot{\theta}$, $\omega_1 = \omega_2 = 0$
i.e the angular velocity of B relative to $\hat{B}$ is $\underline{\omega} = \dot{\theta}\underline{e}_3$



Defn: The time-derivative of the vector $\underline{x} = x_i \underline{e}_i$ w.r.t $B = (\underline{e}_1, \underline{e}_2, \underline{e}_3)$ is the vector $D\underline{x} = \dot{x}_i \underline{e}_i$

—/—

The Coriolis Thm

The time-derivative $D\underline{x}$ and $\hat{D}\underline{x}$ of $\underline{x}$ w.r.t B and $\hat{B}$ are related by

$$\hat{D}\underline{x} = D\underline{x} + \underline{\omega} \times \underline{x}$$

Proof:

$$\hat{D}\underline{x} = \hat{x}_i \hat{\underline{e}}_i \quad (\text{defn})$$

(i) $\hat{x}_i = H_{ji} x_j$

$$= (H_{ji} \dot{x}_j + \dot{H}_{ji} x_j) \hat{\underline{e}}_i$$

$$= H_{ji} \dot{x}_j \hat{\underline{e}}_i + \dot{H}_{ji} x_j \underline{e}_i$$

(ii) $\underline{e}_j = H_{ji} \hat{\underline{e}}_i$

$\hat{\underline{e}}_i = H_{ki} \underline{e}_k$

Using (ii) Using (iii)

$$= \dot{x}_j \underline{e}_j + \boxed{H_{ji} H_{ki}} x_j \underline{e}_k$$

$$= D\underline{x} + (\dot{H} H^T)_{jk} x_j \underline{e}_k$$

$$= D\underline{x} + \boxed{\Omega_{jk}} x_j \underline{e}_k$$

$$= D\underline{x} + \boxed{\varepsilon_{ijk} \omega_i} x_j \underline{e}_k$$

$$= D\underline{x} + (\underline{\omega} \times \underline{x})_k \underline{e}_k = D\underline{x} + w \times \underline{x}$$

Corollary 1: If the angular velocity of B relative to $\hat{B}$ is $\underline{\omega}$ then the angular velocity of $\hat{B}$ relative to B is $-\omega$.

Proof:
$$\hat{D}\underline{x} = D\underline{x} + \underline{\omega} \times \underline{x}$$

$$\Rightarrow D\underline{x} = \hat{D}x + (-\underline{\omega}) \times \underline{x}.$$

Hence, the angular velocity of $\hat{B}$ relative to B is $-\underline{\omega}$.

Corollary 2:
If B has ang. vel. $\underline{\omega}$ relative to $\hat{B}$
and $\hat{B}$ has ang. vel. $\underline{\hat{\omega}}$ relative to B'
then B has ang. vel. $\underline{\omega} + \underline{\hat{\omega}}$ relative to B'.

Proof:
$$\hat{D}\underline{x} = D\underline{x} + \underline{\omega} \times \underline{x}$$

$$D'\underline{x} = \hat{D}\underline{x} + \underline{\hat{\omega}} \times \underline{x}$$

$$= D\underline{x} + (\underline{\omega} + \underline{\hat{\omega}}) \times \underline{x}$$

Hence result.

23/1/12

Last lecture:
H trans. matrix
from $\hat{B}$ to B
$\Omega = \dot{H}H^T$
$\Omega_{jk} = \varepsilon_{ijk}\boxed{w_i}$

$\underline{w} \equiv$ ang. vel. of $\underline{B \text{ rel. } \hat{B}}$

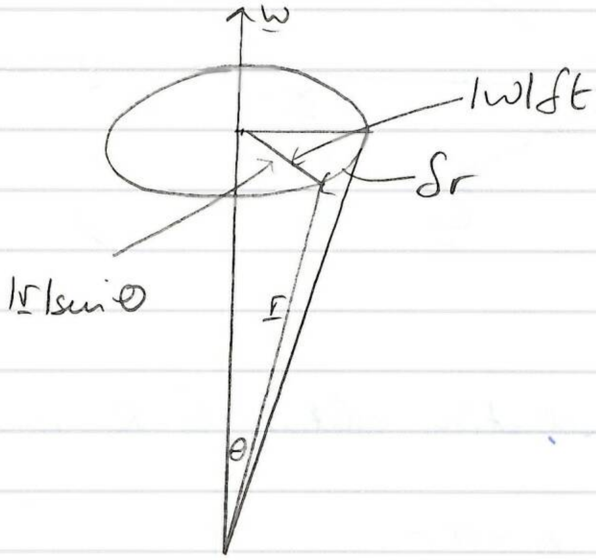
$$\hat{D}\underline{x} = D\underline{x} + \underline{w} \times \underline{x}$$

Coriolis Thm (physical interpretation)

Let $\underline{r}$ be a position vector. Suppose $D\underline{r}$ is the rate of change of the vector relative to axes B, and $\hat{D}\underline{r}$ is the rate of change relative to axes $\hat{B}$ (fixed). B has angular velocity $\underline{w}$ relative to $\hat{B}$.

If $D\underline{r} = 0$, then $\underline{r}$ is constant in B and from $\hat{B}$'s point of view.

In small time δt:



$$|\delta \underline{r}| = |\underline{\omega}| \delta t \, |\underline{r}| \sin\theta .$$

In fact:

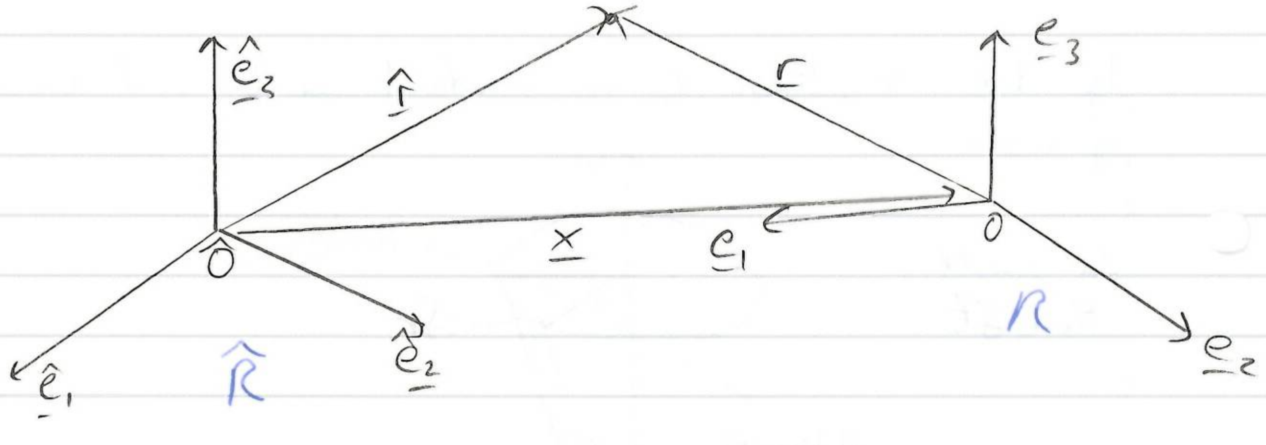
$$\delta \underline{r} = \underline{\omega} \times \underline{r} \, \delta t$$

Dividing by δt and letting $\delta t \to 0$

$$\Rightarrow \lim_{\delta t \to 0} \frac{\delta \underline{r}}{\delta t} = \hat{D}\underline{r} = \underline{\omega} \times \underline{r}$$

which is a term arising in the Coriolis Thm.

Consider the motion of a particle in 2 frames of reference with $\underline{r}$ being the location of a particle from O and $\hat{\underline{r}} = \underline{r} + \underline{x}$ being its location form $\hat{O}$. i.e the position vetor of O from $\hat{O}$ is $\underline{x}$.



The particle's accelation relative to R is

$$\underline{a} = D^2 \underline{r}$$

and relative to $\hat{R}$ it is $\hat{\underline{a}} = \hat{D}^2 \hat{\underline{r}}$

$$\underline{\hat{a}} = \hat{D}^2(\underline{r}+\underline{x})$$
$$= \hat{D}^2\underline{r} + \hat{D}^2\underline{x}$$
$$= \hat{D}(D\underline{r} + \underline{w}\times\underline{r}) + \hat{D}^2\underline{x} \quad \text{(Coriolis theorem)}$$

$$= D(D\underline{r}+\underline{w}\times\underline{r}) + \underline{w}\times(D\underline{r}+\underline{w}\times\underline{r}) + \hat{D}^2\underline{x}.$$

$$= D^2\underline{r} + (D\underline{w})\times\underline{r} + 2\underline{w}\times D\underline{r} + \underline{w}\times(\underline{w}\times\underline{r}) + D\underline{x}$$

$$\underline{\hat{a}} = \underline{a} + (D\underline{w})\times\underline{r} + 2\underline{w}\times(D\underline{r}) + \underline{w}\times(\underline{w}\times\underline{r}) + \underline{A}.$$

where $\underline{A} = \hat{D}^2\underline{x}$ is the acceleration of O from $\hat{R}$.

Consider the special case $\underline{w}=0$ and $\underline{A}=0$ then $\underline{\hat{a}} = \underline{a}$.
The particle has the same acceleration in both frames.

Rotational Effects

Now let $\hat{R}$ be an inertial frame. And R some other frame

Then $m\underline{\hat{a}} = \underline{F}$
where $\underline{F}$ = force acting on the particle.

$$m\underline{\hat{a}} = m(D^2\underline{r} + (D\underline{w})\times\underline{r} + 2\underline{w}\times D\underline{r} + \underline{w}\times(\underline{w}\times\underline{r}) + A) = \underline{F}$$

Or, from R's point of view.

$$m\underline{a} = F - \underbrace{m(D\underline{w})\times\underline{r}}_{(1)} - \underbrace{2m\underline{w}\times D\underline{r}}_{(2)} - \underbrace{m\underline{w}\times(\underline{w}\times\underline{r})}_{(3)} - \underbrace{m\underline{A}}_{(4)}$$

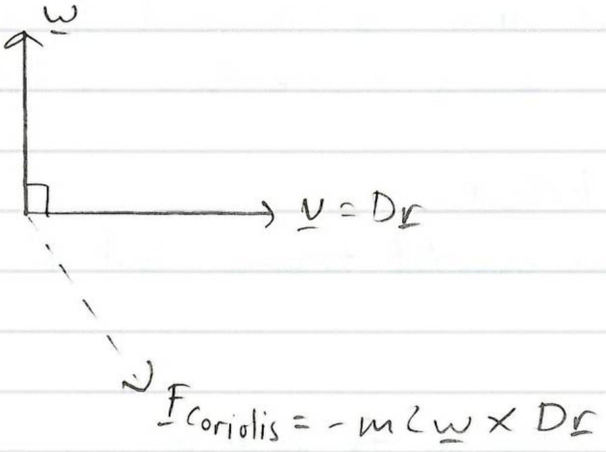
which is the same as if R was an inertial frame, with additional "fictitious" $\underline{F}_1, \underline{F}_2, \underline{F}_3$ and $\underline{F}_4$.

These forces are not real: they are needed to compensate for the rotation and acceleration of R relative to $\hat{R}$.

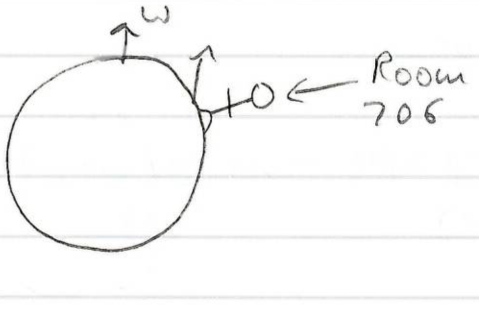
(worth remembering has been in a exam)

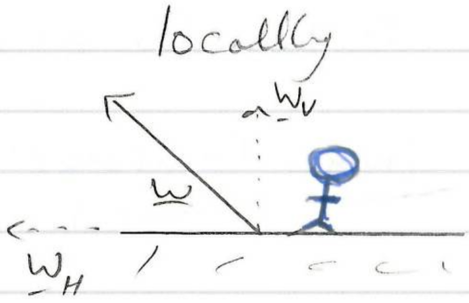
$\underline{F}_1 = -m(D\underline{\omega}) \times \underline{r}$ arises because of the angular <u>acceleration</u> (i.e $D\underline{\omega}$) of R.

$\underline{F}_2 = -2m\underline{\omega} \times D\underline{r}$ is the <u>Coriolis</u> frame and is velocity dependent (i.e $D\underline{r} \neq 0$). It is orthogonal to $\underline{\omega}$ and $D\underline{r}$.

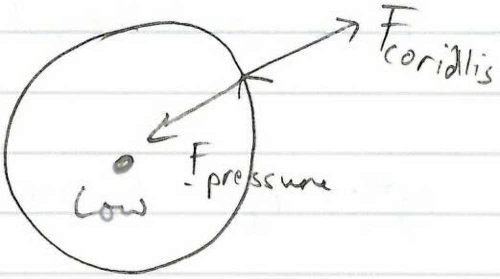


On the rotating earth:





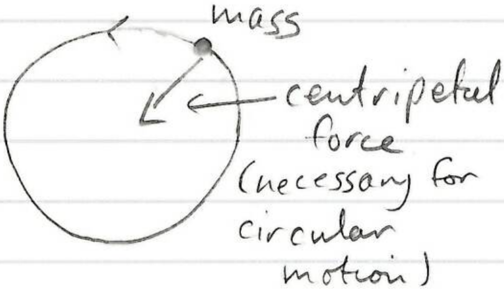
$\underline{\omega}_v$ causes the anti-clockwise circulation (i.e winds) about a low pressure cell in the northern hemisphere.



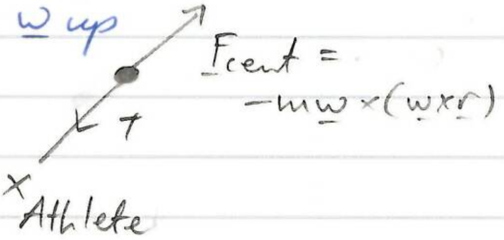
$\underline{F}_{coriolis} = -2m\underline{\omega}_H \times D\underline{r}$ – Either . zero or too week

$\underline{F}_3 = -m\underline{\omega}\times(\underline{\omega}\times\underline{r})$ is the centrifugal force (not velocity dependent).

Ex : Hammer throw.



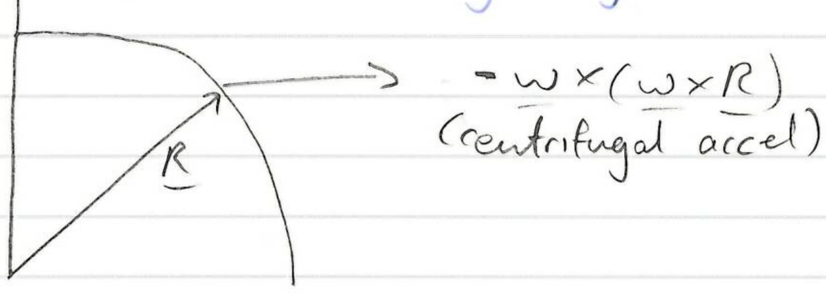
Inertial frame



Rotation frame.

Ex :

$$g^* = g - \underline{\omega}\times(\underline{\omega}\times\underline{R})$$



$|\underline{\omega}| = 7.3\times10^{-5}s^{-1}$, $|\underline{R}| = 6320$ km

Max. magnitude $|\underline{\omega}|^2|\underline{R}| = 34\,mms^{-2} << g$

At the poles there is no centrifigal force ($\underline{\omega} \times \underline{R} = 0$)
$\underline{g}^* = \underline{g}$.

At the equator $\underline{|g^*|} = |\underline{g}| - |\underline{\omega}|^2 |\underline{R}|$

i.e we expect $|\Delta g| = |\underline{g}^*_{pole}| - |g_{eq}|$

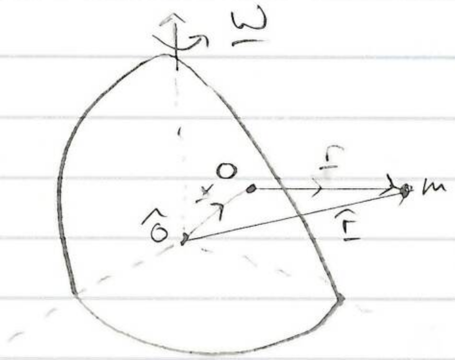
$= 34 \text{mms}^{-2}$

In fact $|\Delta \underline{g}| = 52 \text{mms}^{-2}$.

It is larger owing to the fact the earth is oblate.

25/1/12

Reference frame attached to earth's surface



O is on the earth's surface (e.g room 706)

$\hat{O}$ ≡ Centre of the Earth

According to a non-inertial observer at O with $\dot{\underline{\omega}} = 0$ ("$\cdot$" ≡ $\frac{d}{dt}$ ≡ D as measured by observer at O).

$$m\ddot{\underline{r}} = \underline{F} + m\underline{g} - m\underline{A} - 2m\underline{\omega}\times\dot{\underline{r}} - m\underline{\omega}\times(\underline{\omega}\times\underline{r})$$

But $\underline{A} = \hat{D}^2\underline{x} = \underline{\omega}\times(\underline{\omega}\times\underline{x})$ using Coriolis thm twice and noting $\dot{\underline{x}} = 0$.

$$\Rightarrow m\ddot{\underline{r}} = m\underline{g} + \underline{F} - m\underline{\omega}\times(\underline{\omega}\times\hat{\underline{r}}) - 2m\underline{\omega}\times\dot{\underline{r}} \quad (\hat{\underline{r}} = \underline{r} + \underline{x})$$

$$= m\underline{g}^* + F - 2m\underline{\omega}\times\dot{\underline{r}}.$$

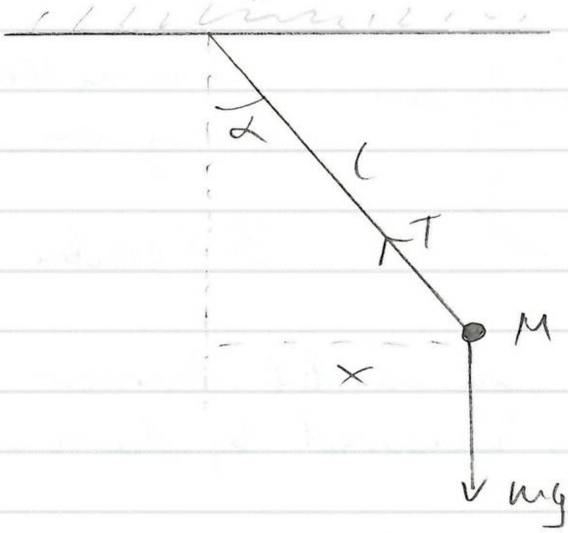
where $\underline{g}^* = \underline{g} - \underline{\omega}\times(\underline{\omega}\times\hat{\underline{r}})$
and if the motion stays near the earth's surface $\hat{\underline{r}} \simeq \underline{x}$ and so:

$$\underline{g}^* = \underline{g} - \underline{\omega}\times(\underline{\omega}\times\underline{x})$$
$$\simeq \underline{g}$$

Foucault's Pendulum

—/—

Aside: "normal" pendulum.



horiz : $m\ddot{x} = -T\sin\alpha$

vert : $m\ddot{z} = T\cos\alpha - mg$

Small amplitude oscillation :

$$\sin\alpha = \frac{x}{l}$$

$$\cos\alpha \simeq 1$$

Also, $\ddot{z} \simeq 0$ (also $\dot{z} \simeq 0$)

$$\Rightarrow T \simeq mg$$

$$\Rightarrow \cancel{m}\ddot{x} = -\frac{\cancel{m}gx}{l} \quad \Rightarrow \ddot{x} = -\frac{g}{l}x$$

$$\Rightarrow \quad x = A\cos\left(\sqrt{\frac{g}{l}}\,t + \delta\right)$$

—/—

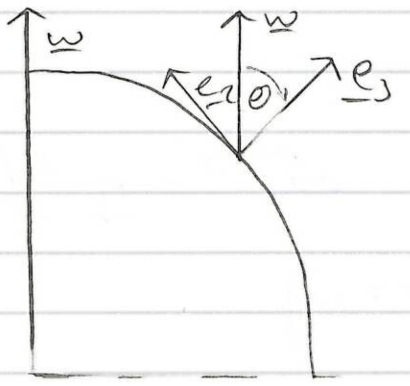
Named after Leon Foucault's 1851 demonstration of the earth's rotation.

- Simple pendulum which is free to oscillate in any vertical plane.
- required to oscillate for long periods.
- Note gravity is much more important in determining the pendulum's motion, but Coriolis force perpendicular to the plane of motion, thus despite its relatively small magnitude it has an observable cumulative effect.

We have

$$m\ddot{\underline{r}} = m\underline{g} + \underline{T} - 2m\underline{\omega}\times\dot{\underline{r}} \qquad (*)$$

Choose axes:



($\underline{e}_1$ into the paper)

$\underline{e}_1$ east
$\underline{e}_2$ north
$\underline{e}_3$ up.

$\theta \equiv$ co-latitude

In this coordinate system

$$\underline{\omega} = (0, \omega\sin\theta, \omega\cos\theta) \qquad \omega \equiv |\underline{\omega}|$$

Let $\underline{\dot{r}} = (\dot{x}, \dot{y}, \dot{z})$, the Coriolis force

is $\underline{F_c} = -2m\underline{\omega}\times\underline{\dot{r}} = 2m\omega(\dot{y}\cos\theta - \dot{z}\sin\theta, -\dot{x}\cos\theta, \dot{x}\sin\theta)$

Note that the vertical component of $F_c (= 2m\omega\dot{x}\sin\theta)$ is small compared to $|g|$. (Since $2\omega\dot{x} \simeq 1\,\mathrm{mms^{-2}}$ for a typical pendulum $\dot{x}$).

Taking $\dot{z} \simeq 0$

(*) becomes:

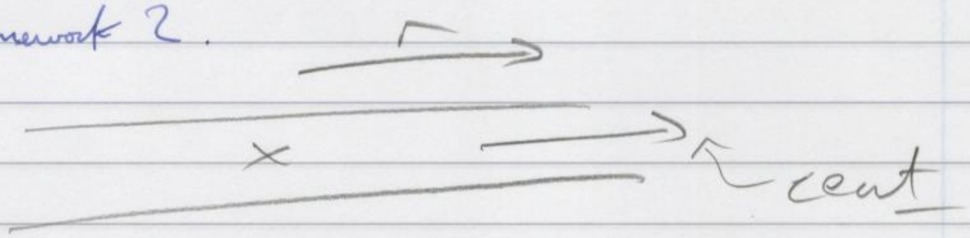
$$\ddot{x} = -\frac{g}{l}x + 2\omega\dot{y}\cos\theta \qquad (1)$$

$$\ddot{y} = \underbrace{-\frac{g}{l}y}_{\text{Usual pendulum}} \underbrace{- 2\omega\dot{x}\cos\theta}_{\text{Coriolis.}} \qquad (2)$$

where (*) is $m\underline{\ddot{r}} = m\underline{g} + \underline{T} - 2m\underline{\omega}\times\underline{\dot{r}}$

$$\phi = x + iy.$$

Hint for homework 2.



$$m\ddot{\underline{r}} = \underline{F} - m\cancel{\dot{\underline{\omega}}\times\underline{r}} - 2m\underline{\omega}\times\underline{r}' - m\underline{\omega}\times(\underline{\omega}\times\underline{r})$$

$$\ddot{r} = \alpha r$$

Second Question

choose ω to simplif

Foucault's Pendulum : supplement

The equation

$$\ddot{\phi} + 2i\Omega\dot{\phi} + \omega_0^2\phi = 0,$$

has a solution of the form

$$\phi = a\, e^{i(\omega_0 - \Omega)t} + b e^{-i(\omega_0 + \Omega)t} \; (*),$$

where the approximation $\omega_0^2 >> \Omega^2$ has been used. This gives, upon taking real and imaginary parts, (taking a and b to be real here)

$$x = a\cos(\omega_0 - \Omega)t + b\cos(\omega_0 + \Omega)t$$

$$y = a\sin(\omega_0 - \Omega)t - b\sin(\omega_0 + \Omega)t.$$

Case 1 The choice $a = b = \frac{A}{2}$ reproduces the case given previously in lectures, namely

$$x = A\cos\Omega t\cos\omega_0 t$$

$$y = -A\sin\Omega t\cos\omega_0 t.$$

The choice $\omega_0 = 5$, $\Omega = 0{\cdot}5, A = 1$ gives the trajectories shown overleaf (Note, these numbers for ω_0 and Ω are not realistic for the Earth!) In this case the initial conditions are $x = A$, $y = 0$, $\dot{x} = 0$, $\dot{y} = -A\Omega$ at $t = 0$.

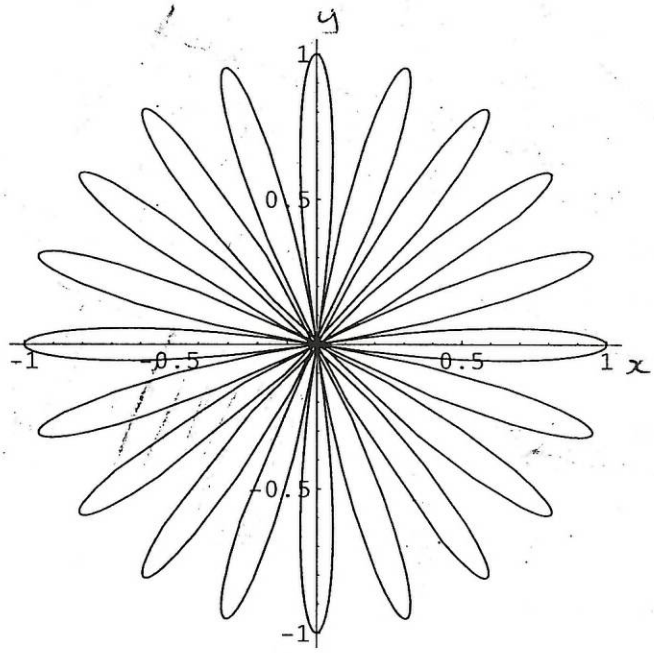
Case 2 Here the choice

$$a = -\frac{1}{2} - \frac{\Omega}{2\omega_0}, \quad b = -\frac{1}{2} + \frac{\Omega}{2\omega_0}$$

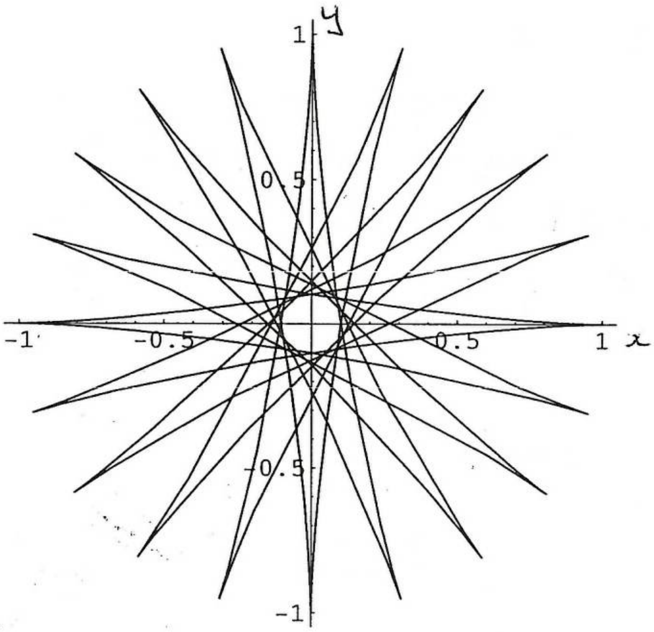
gives rise to the arguably more realistic initial conditions $x = -1$, $\dot{x} = 0$, $y = \dot{y} = 0$ at $t = 0$, (i.e. pendulum released from rest). The trajectories are shown overleaf.

---

* Note a and b are, in general, complex. Effectively there are four arbitrary constants which is consistent with the solution of two (coupled) 2nd-order ODE's for x and y.

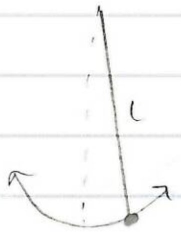


case 1



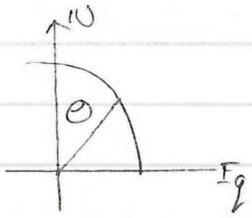
case 2

30/1/12



$$\ddot{x} = -\frac{g}{l}x + 2\omega\dot{y}\cos\theta \qquad (1)$$

$$\ddot{y} = \frac{-g}{l} - 2\omega\dot{x}\cos\theta \qquad (2)$$



$x \equiv$ West → east
$y \equiv$ south → west

$$\varphi = x + iy$$

(1) + (2) ⇒

$$\ddot{\varphi} + 2i\Omega\dot{\varphi} + \omega_0^2\phi = 0.$$

where $\Omega = \omega\cos\theta$, $\omega_0^2 = g/l$

Let $\phi = e^{\lambda t}$ i.e $\lambda^2 + 2i\Omega\lambda + \omega_0^2 = 0$.

$\Rightarrow \lambda = -i\Omega + i\omega_1$

where $\omega_1^2 = \omega_0^2 + \Omega^2$

$\simeq \omega_0^2$ (the pendulum oscillates back and forth many times in one day)

$\Rightarrow \lambda \simeq -i\Omega + i\omega_0$.

$\Rightarrow \phi = e^{-i\Omega t}(A\cos\omega_0 t + B\sin\omega_0 t)$, $A, B \in \mathbb{C}$
$\Rightarrow$ 4 arb. const.

For simplicity, choose $B=0$, $A \in \mathbb{R}$.

$$\Rightarrow \quad x = A\cos\Omega t \cos\omega_0 t$$
$$y = -A\sin\Omega t\cos\omega_0 t. \qquad (\Omega \ll \omega_0)$$

Initially ($t \simeq 0$), the oscillation is in the x-direction since $\cos\Omega t \simeq 1$ and $\sin\Omega t \simeq 0$.

As $t\uparrow$, $\cos\Omega t\downarrow$ and $\sin\Omega t\uparrow$ (meanwhile the pendulum continuous to oscillate owing to the rapid $\cos\omega_0 t$). Thus the amplitude of oscillation decreases in the east-west direction, but increases in the north-south direction. That is, the plane as oscillation rotates at $t\uparrow$.

The solution represents oscillations of amplitude A in a plane rotating with angular velocity $\Omega = \omega\cos\theta$.

Let T be the period of rotation of the plane of oscillation.

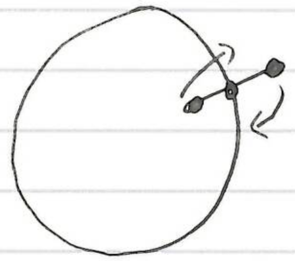
$$T = \frac{2\pi}{\omega\cos\theta} \qquad \Bigg\| \qquad \omega = \frac{2\pi}{24 \text{ hours}}$$

At $\theta = 0 \Rightarrow T = 1$ day.

$\theta = \pi/4 \Rightarrow T = \dfrac{2\pi}{\omega\cos(\frac{\pi}{4})} \simeq 34$ hours.

$\theta = \frac{\pi}{2} \Rightarrow T \longrightarrow \infty$ (no horizontal component of Coriolis at the equator)

Hint for Q2



$$L = R \times M\underline{v} + \sum_i \underline{r}_i' \times m_i \underline{v}_i'$$

Inertial Observer

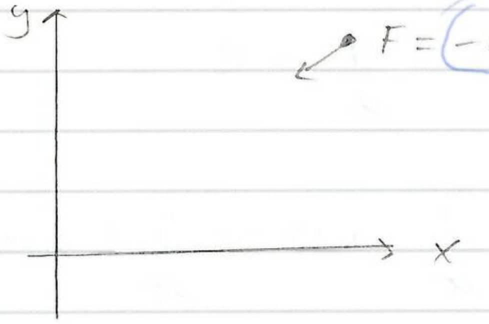
Consider a Foucault pendulum at the North Pole ($\theta = 0$).
For an inertial observer, there is no Coriolis force.
$\Rightarrow$ the plane of oscillation is invariant.

But to the inertial observer, someone standing at the North pole rotates with the Earth

This explains the relative motion between pendulum at earth-bound (non-inertial) observer.

—/—

Homework 2: Hint for question 2:



$\underline{F} = -kr\hat{\underline{r}}$

$$m\ddot{\underline{r}} = \underline{F} - \cancelto{0}{m\dot{\underline{\omega}}\times\underline{r}} - 2m\underline{\omega}\times\dot{\underline{r}} - m\underline{\omega}\times(\underline{\omega}\times\underline{r}) - \cancelto{0}{m\underline{A}}$$

$\omega = \text{const}$

—/—

§.2 Systems of particles

Quick revision of a single particle.

1. Newton's 2nd law

$$\underline{F} = \frac{d\underline{p}}{dt}$$

where $\underline{p} = m\underline{v}$ is the momentum.

If $\underline{F}=0$, then $\underline{p}$ is const.

2. The angular momentum $\underline{L}$ about O is

$$\underline{L} = \underline{r} \times \underline{p}$$

where $\underline{r}$ is the position vector of the point particle from O.

The torque about O is $\underline{N} = \underline{r} \times \underline{F}$

Observe,

$\underline{r} \times$ (Newton's 2nd law) gives

$$\underline{r} \times \underline{F} = \underline{r} \times \frac{d\underline{p}}{dt} = \underline{N}$$

but, $r \times \frac{d\underline{p}}{dt} = \underline{r} \times \frac{d}{dt}(m\underline{v}) = \frac{d}{dt}\underline{r} \times m\underline{v} = \frac{d\underline{L}}{dt}$

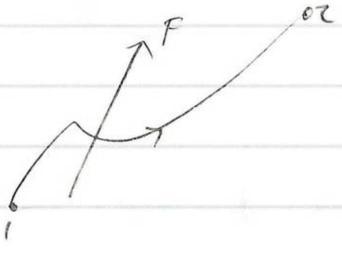
i.e $\underline{N} = \frac{d\underline{L}}{dt}$

Torque = rate of change of angular momentum.

Note, $N = 0 \Rightarrow \underline{L} = \text{const}$.

3. The work done by an external force $\underline{F}$ upon a particle going from pos$^{\underline{n}}$ 1 to pos$^{\underline{n}}$ is

$$W_{12} = \int_1^2 \underline{F} \cdot d\underline{s}$$



If m is const,

$$\int \underline{F}\cdot d\underline{s} = m\int \frac{d\underline{v}}{dt}\cdot\underline{v}\,dt$$

$$= \frac{m}{2}\int\left(\frac{d}{dt}\,\underline{v}\cdot\underline{v}\right)dt$$

$$\Rightarrow W_{12} = \frac{1}{2}m\left(v_2^1 - v_1^2\right) \qquad \Big\| \qquad v_2^{\,2} = \underline{v}\cdot\underline{v}\Big|_{\textcircled{2}}$$

$$= T_2 - T_1$$

where T = kinetic energy.

If $\underline{F}$ is st

$$\oint \underline{F}\cdot d\underline{s} = 0$$

(i.e $\underline{F} \equiv$ conservative)

this implies (by Stoke's Thm)

$$\iint (\underline{\nabla}\times\underline{F})\cdot d\underline{A} = 0$$

$$\Rightarrow \underline{\nabla}\times\underline{F} = 0$$

$$\Rightarrow \underline{F} = -\underline{\nabla}V$$

where V (scalar field) is the potential

Thus $W_{12} = -\int \underline{\nabla} V \cdot d\underline{s}$ (exact integral)

$$= -(V_2 - V_1)$$

Hence $T_2 - T_1 = V_1 - V_2$
or $T_1 + V_1 = T_2 + V_2$

i.e conservation of energy.

Many Particles

We distingish between external force acting on the particles due to sources outside the system; and **internal forces** on, say, some particles i due to all the other particles.

Eq. of motion of i^{th} particle:

$$\sum_j \underline{F}_{ji} + \underline{F}_i^{(e)} = \dot{\underline{p}}_i \quad \Big\| \quad \left(\sum_{j=1}^{N}\right) \equiv \sum_j$$

where $\underline{F}_i^{(e)} \equiv$ external force

$\underline{F}_{ji} \equiv$ internal force on ith particle due to the j^{th}

$$\left(\underline{F}_{ii} = 0\right)$$

We sum this eq. over all particles

$$\Rightarrow \sum_i \dot{\underline{p}}_i = \frac{d^2}{dt^2}\sum_i m_i \underline{r}_i = \sum_i \underline{F}_i^{(e)} + \sum_{i,j} \underline{F}_{ji}$$

We assume that the $\underline{F}_{ji}$ obey Newton's 3rd law $\Rightarrow \underline{F}_{ij} = -\underline{F}_{ji}$

Note:

$$\sum_{i,j} \underline{F}_{ji} = \sum_{j,i} \underline{F}_{ij} = \underset{\substack{\nearrow \\ \text{Newton } \underline{\text{III}}}}{-\sum_{j,i} \underline{F}_{ji}} = -\sum_{i,j} \underline{F}_{ji} = 0 .$$

1/2/12

Last lecture:

$$\underline{\dot{P}}_i = \underline{F}_i^{(e)} + \sum_j \underline{F}_{ji}$$

$$\downarrow$$

Sum

$$\downarrow$$

$$\frac{d^2}{dt^2}\sum_i m_i r_i = \sum_i \underline{F}_i^{(e)} + \cancel{\sum_{i,j} \underline{F}_{ji}}^{\,0}$$

— / —

Define: $\underline{R} = \frac{\sum_i m_i \underline{r}_i}{\sum_i m_i} = \frac{1}{m}\sum_i m_i \underline{r}_i$

is the centre of mass vector

$$\Rightarrow M\frac{d^2\underline{R}}{dt^2} = \sum_i \underline{F}_i^{(e)} = \underline{F}^{(e)}$$

i.e the C of M moves as if the total external force were acting on the entire mass of the system concentrated at the C of M

Note $\frac{d\underline{P}}{dt} = \underline{F}^{(e)}$ $\left(\underline{P} = M\frac{d\underline{R}}{dt}\right)$

So, if $\underline{F}^{(e)} = 0 \Rightarrow \underline{P} \equiv \text{const}$

The total angular momentum $\underline{L} = \sum_i \underline{r}_i \times \underline{p}_i$
and note:

$$\frac{d\underline{L}}{dt} = \frac{d}{dt} \sum_i \underline{r}_i \times \underline{p}_i$$

$$= \sum_i \underline{r}_i \times \dot{\underline{p}}_i$$

$$= \sum_i \underline{r}_i \times \left(\underline{F}_i^{(e)} + \sum_j \underline{F}_{ji} \right)$$

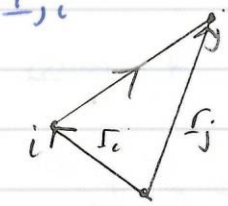
$$= \sum_i \underline{r}_i \times \underline{F}_i^{(e)} + \sum_{i,j} \underline{r}_i \times \underline{F}_{ji}$$

Consider the last term which involves pairs of terms of the form

$$\underline{r}_i \times \underline{F}_{ji} + \underline{r}_j \times \underline{F}_{ij}$$

$$= (\underline{r}_i - \underline{r}_j) \times \underline{F}_{ji}$$

$\underline{F}_{ij} = -\underline{F}_{ji}$
(By Newton's 3rd law)



But $\underline{r}_i - \underline{r}_j$ is parallel to $\underline{F}_{ji}$ since the force acting **between** particles i and j is along the line joining i and j. (strong form of Newton's 3rd law)

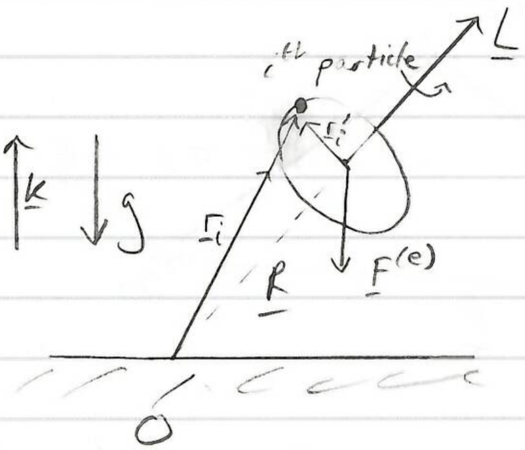
Thus $(\underline{r}_i - \underline{r}_j) \times \underline{F}_{ji} = 0$
and hence

$$\frac{d\underline{L}}{dt} = \underline{N}^{(e)}$$

where $\underline{N}^{(e)} = \sum_i \underline{r}_i \times \underline{F}_i^{(e)} = \sum_i \underline{N}_i^{(e)}$

Gyroscope or Top

Many be considered as a collection of many particles symmetrically placed about the rotation axis.



$$\underline{N}^{(e)} = \sum_i \underline{r}_i \times \underline{F}_i^{(e)}$$

$$= \sum_i (\underline{r}_i' + \underline{R}) \times (-m_i g \underline{k})$$

$$= \underline{R} \times F^{(e)} - g\Big(\sum_i m_i \underline{r}_i'\Big) \times \underline{k}$$

$$\uparrow$$

$$\underline{F}^{(e)} = -\sum_i m_i g \underline{k}$$

But $\sum_i m_i \underline{r}_i' = \sum_i m_i(-\underline{R} + \underline{r}_i)$

$$= -M\underline{R} + \sum_i m_i \underline{r}_i$$

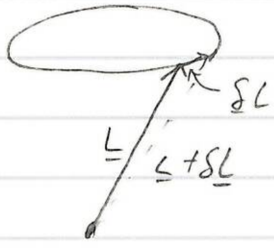
$$= -M\underline{R} + M\underline{R} = 0$$

$$\Rightarrow \quad \underline{N}^{(e)} = \underline{R} \times \underline{F}^{(e)}$$

But $\dfrac{d\underline{L}}{dt} = \underline{N}^{(e)}$

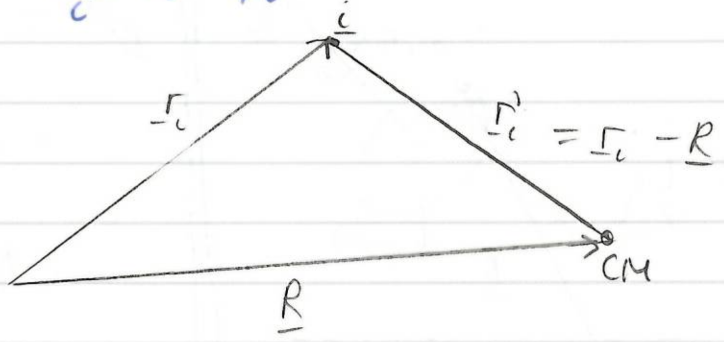
or $d\underline{L} \simeq dt\,\underline{N}^{(e)}$

But $\underline{N}^{(e)}$ is into the board and so perpendicular to $\underline{L}$



Thus the vector $\underline{L}$ precesses, sweeping out a cone under the action $\underline{F}^{(e)}$

Recall $\underline{L} = \sum_i \underline{r}_i \times \underline{p}_i$



Also $\underline{v}_i = \underline{v}_i' + \underline{V}$ where $\underline{V} = \dfrac{d\underline{R}}{dt}$, $\underline{v}_i' = \dfrac{d}{dt}\underline{r}_i'$

$$\underline{L} = \sum_i \underline{r}_i \times \underline{p}_i$$

$$= \sum_i (\underline{r}_i' + \underline{R}) \times m_i (\underline{v}_i' + \underline{v})$$

$$= \sum_i \underline{R} \times m_i \underline{v} + \sum_i \underline{r}_i' \times m_i \underline{v}_i' + \cancelto{0}{\left(\sum_i m_i \underline{r}_i'\right)} \times V$$

$$+ \underline{R} \times \frac{d}{dt} \cancelto{0}{\left(\sum_i m_i \underline{r}_i'\right)}$$

Hence: $$\underline{L} = \underbrace{\underline{R} \times M \underline{v}}_{\text{ang. mom. of the CM}} + \underbrace{\sum \underline{r}_i' \times m_i \underline{v}_i'}_{\text{ang. mom. motion about C of M}}$$

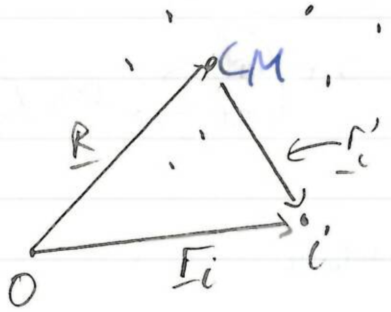
6/2/12

Last lecture

$$L = \sum_i \underline{r}_i \times b_i$$

$$\vdots$$

$$= \underline{R} \times M\underline{v} + \sum_i \underline{r}'_i \times \underline{p}'_i$$



* * *

Energy

Work done by all forces moving system of N particles from configuration 1 to configuration 2:

$$W_{12} = \sum_i \int_1^2 \underline{F}_i \cdot d\underline{s}_i$$

$$= \sum_i \int_1^2 M_i \dot{\underline{v}}_i \cdot \underline{v}_i \, dt$$

$$= \sum_i \int_1^2 d\left(M_i \underline{v}_i \cdot \underline{v}_i\right)$$

$$= T_2 - T_1$$

where $T = \sum_i \frac{1}{2} m_i \overset{\underline{V}_i \cdot \underline{V}_i}{\boxed{v_i^2}} \equiv |\underline{v}_i|$

is the total energy of the system.

Note:

$$T = \frac{1}{2}\sum_i m_i (\underline{v} + \underline{v}_i') \cdot (\underline{v} + \underline{v}_i')$$

$$= \frac{1}{2}\sum_i m_i v_i^2 + \boxed{\sum_i m_i \underline{v} \cdot \underline{v}_i'} + \frac{1}{2}\sum_i m_i v_i'^2$$

$$\hookrightarrow v\left(\frac{d}{dt}\cancelto{0}{\sum_i m_i \underline{r}_i'}\right)$$

$$= \frac{1}{2} M v^2 + \frac{1}{2}\sum_i m_i v_i'^2$$

$$= T_{CM} + T_{rel}$$

The total kinetic energy is the kinetic energy of CM motion plus kinetic energy of motion relative to the CM (T_{rel})

<u>Potential</u>

Recall $$W_{12} = \sum_i \int_1^2 \underline{F}_i \cdot d\underline{s}_i$$

$$= \sum_i \int_1^2 \underline{F}_i^{(e)} \cdot d\underline{s}_i + \sum_{i,j} \int_1^2 \underline{F}_{ji} \cdot d\underline{s}_i$$

For conservative external forces:

$$\underline{F}_i^{(e)} = -\underline{\nabla}_i V_i^{(e)} \quad \Bigg\| \quad \nabla_i = \frac{\partial}{\partial x_i}\underline{i} + \frac{\partial}{\partial y_i}\underline{j} + \frac{\partial}{\partial z_i}\underline{k}$$

Thus $$\sum_i \int_1^2 \underline{F}_i^{(e)} \cdot d\underline{s}_i = -\sum_i \int_1^2 \underline{\nabla}_i V_i^{(e)} \cdot d\underline{s}_i$$

$$= -\sum_i V_i^{(e)} \Big|_1^2$$

—/—

Eg: $V_i = + mgz_i$

$$F = -\nabla_i$$
$$= mg\underline{k}.$$

—/—

Internal forces?

If the internal forces are conservative they are derivable from a potential.

$\exists\ V_{int}$ s.t.

$$V_{int} = V_{int}\left[\underline{r}_1(t),\ \underline{r}_2(t),\ \underline{r}_3(t), \ldots, \underline{r}_N(t)\right]$$

and $\underline{\nabla}_i V_{int} = -\sum_j \underline{F}_{ji}$.

Hence $V = \sum_i V_i^{(e)} + V_{int}$.

$$= V^{(e)} + V_{int}$$

E conservation

$$T + V = \text{const}.$$

i.e $\frac{1}{2}Mv^2 + \frac{1}{2}\sum_i m_i v_i'^2 + \sum_i V_i^{(e)} + V_{int} = \text{const}.$

or, $T_{cm} + T_{rel} + V^{(e)} + V_{int} = \text{const}.$

— / —

Hint for question (1) Ex 3.

$$\frac{d}{dt}\left[T_{rel} + V_{int} = 0 \right]$$

$$\frac{d}{dt} V_{int} = \frac{d}{dt} V_{int}\left[r_1(t), r_2(t), \ldots, r_N(t) \right]$$

$$= \sum_j \nabla_j V_{int} \frac{dr_j}{dt}$$

— / —

Lagrangian Mechanics

Consider N particles with m_α and position vectors:

$$r_\alpha = (r_{\alpha 1}, r_{\alpha 2}, r_{\alpha 3}) \quad , \quad \alpha = 1, \dots, N.$$

Each particle is subject to a force $\underline{F}_\alpha$ 3N degree of freedom.

Configuration space (C: 3N coordinates)

$$(r_{11}, r_{12}, r_{13}, r_{21}, r_{22}, r_{23}, \dots, r_{N1}, r_{N2}, r_{N3})$$

Each point of C corresponds to a particlar configuration of the system.

To emphasise C is a single space

$$q_1 = r_{11} \, , \, q_2 = r_{12} \, , \, q_3 = r_{13} \, , \dots , q_n = r_{N3} \quad (n = 3 \times N)$$

Further, let:

$$\underbrace{\mu_1 = m_1 \, , \, \mu_2 = m_1 \, , \, \mu_3 = m_1 \, ,}_{\text{particle 1}} \quad \underbrace{\mu_4 = m_4 \, , \dots ,}_{\text{particle 2}} \mu_n = m_N$$

Hence, the eq's of motion are

$$\mu_1 \ddot{q}_1 = F_1 \quad , \mu_2 \ddot{q}_2 = F_2 \, , \dots , \mu_n \ddot{q}_n = F_n$$

where F_1, F_2, F_3 are the components of $\underline{F}_1$
F_4, F_5, F_6 are the components of $\underline{F}_2$ etc.

Phase space (P)
Use as coords:

$$(q_1, q_2, \ldots, q_n, v_1, v_2, \ldots, v_n)$$

Eq'n of motion

$$\mu_1 \dot{v}_1 = F_1, \ldots, \mu_n \dot{v}_n = F_n$$

$$\dot{q}_1 = v_1, \ldots, \dot{q}_n = v_n$$

Ex: Harmonic oscillator.

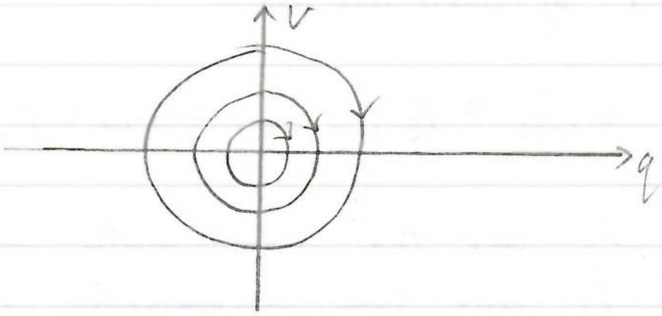
$$\ddot{x} = -x$$

C: one-dimensional $q \equiv x$
P: two-dimensional (q, v)

Eqs of motion : $\dot{q} = v$
$\dot{v} = -q$.

Note: $q^2 + v^2 = \text{const}$.

$\frac{d}{dt}(q^2+v^2)$ (To check)
$= 2\dot{q}q + 2v\dot{v}$
$= 2v(q) + 2v(-q)$
$= 0$ ✓



Coordinate Transformation

Suppose we introduce new coords $\tilde{q}_a = \tilde{t}$ and CT (config-time space) which are related to q_n and t by

$$\tilde{q}_a = \tilde{q}_a(q_1, q_2, q_3, \ldots, q_n, t) \ , \ \tilde{t} = t .$$

By the chain rule:

$$\tilde{v}_a = \dot{\tilde{q}}_a = \frac{\partial \tilde{q}_a}{\partial q_1} \dot{q}_1 + \frac{\partial \tilde{q}_a}{\partial q_2} \dot{q}_2 + \ldots + \frac{\partial \tilde{q}_a}{\partial q_n} \dot{q}_n + \frac{\partial \tilde{q}_a}{\partial t}$$

$$= \frac{\partial \tilde{q}_a}{\partial q_b} \dot{q}_b + \frac{\partial \tilde{q}_b}{\partial t} \quad \text{(sum over } b\text{)}$$

In PT (phase-time) the coord. transform is of the form

$$\tilde{q}_a = \tilde{q}_a(\underset{\sim}{q}, t)$$

$$\tilde{v}_a = \tilde{v}_b(\underset{\sim}{q}, \underline{v}, t)$$

$$= \frac{\partial \tilde{q}_a}{\partial q_b} v_b + \frac{\partial \tilde{q}_b}{\partial t} \quad \text{(as above)}$$

$$\tilde{t} = t .$$

Ultimately, our task is to re-write the e.q. of motion in the new coords.

Theorem (see handout) N.F.E (Not for exam)

The theorem shows that the combination of derivative:

$$\frac{d}{dt}\left(\frac{\partial F}{\partial v_a}\right) - \frac{\partial F}{\partial q_a}$$

transforms in a straightforward way under a change of coords.

Let $T = \frac{1}{2}\left[\mu_1 v_1^2 + \mu_2 v_2^2 + \dots + \mu_n v_n^2\right]$

Note: $\frac{\partial T}{\partial v_1} = \mu_1 v_1, \quad \frac{\partial T}{\partial v_2} = \mu_2 v_2, \quad \dots, \quad \frac{\partial T}{\partial v_n} = \mu_n v_n.$

Also: $\frac{\partial T}{\partial q_a} = 0 \quad , \quad a = 1, \dots, n$

Eqs of motion:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial v_a}\right) - \frac{\partial T}{\partial q_a} = F_a \; ; \; \frac{d}{dt} q_a = v_a .$$

Now transform to new coords: Using the theorem:

$$\frac{\partial \tilde{q}_b}{\partial q_a}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \tilde{v}_b}\right) - \frac{\partial T}{\partial \tilde{q}_b}\right] = F_a .$$

Multiply both sides by $\frac{\partial q_a}{\partial \tilde{q}_c}$, sum over a, and make use of:

$$\frac{\partial \tilde{q}_b}{\partial q_a} \cdot \frac{\partial q_a}{\partial \tilde{q}_c} = \frac{\partial \tilde{q}_b}{\partial \tilde{q}_c} = \delta_{bc}.$$

$$\delta_{bc}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \tilde{v}_b}\right) - \frac{\partial T}{\partial \tilde{q}_b}\right] = \frac{\partial q_a}{\partial \tilde{q}_c} F_a.$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \tilde{v}_c}\right) - \frac{\partial T}{\partial \tilde{q}_c} = \tilde{F}_c$$

where $\tilde{F}_c = \frac{\partial q_a}{\partial q_c} F_a$.

Hence in the coords the eqs of motion are:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \tilde{v}_a}\right) - \frac{\partial T}{\partial \tilde{q}_a} = \tilde{F}_a \quad , \tilde{v}_a = \dot{\tilde{q}}_a.$$

or $$\frac{d}{dt}\left(\frac{\partial T}{\partial \tilde{q}_a}\right) - \frac{\partial T}{\partial q_a} = \tilde{F}_a.$$

Def^n: The $\tilde{F}_a$ are the generalised forces; the $\tilde{q}_a$ the generalised coords and $\dot{\tilde{q}}_a \equiv \tilde{v}_a$ are the generlised velocities.

Ex : Motion of a single particle.

1. Cartesian $T=\frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)$

generalise forces F_x, F_y, F_z.
" coords x, y, z
" velocities $\dot{x}, \dot{y}, \dot{z}$

Observe $\frac{\partial T}{\partial \dot{x}} = m\dot{x}$, $\frac{\partial T}{\partial \dot{y}} = m\dot{y}$, $\frac{\partial T}{\partial \dot{z}} = m\dot{z}$.

Also $\frac{\partial T}{\partial x} = \frac{\partial T}{\partial y} = \frac{\partial T}{\partial z} = 0$.

Hence : $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} = F_x \Rightarrow \frac{d}{dt}(m\dot{x}) = F_x$.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) - \frac{\partial T}{\partial y} = F_y \Rightarrow \frac{d}{dt}(m\dot{y}) = F_y$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{z}}\right) - \frac{\partial T}{\partial z} = F \Rightarrow \frac{d}{dt}(m\dot{z}) = F_z.$$

2. Plane polar coords

Write T in polar coords
Start with
$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) \text{ and use}$$

$$x = r\cos\theta, \quad y = r\sin\theta.$$

$$\Rightarrow \quad \dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta .$$

$$\dot{y} = \dot{r}\cos\theta + r\dot{\theta}\cos\theta .$$

$$\Rightarrow \quad T = \frac{m}{2} T \left(\dot{r}^2 + r^2\dot{\theta}^2 \right)$$

8/2/12

2) Plane polar coords

$$\frac{d}{dt}\left(\frac{\partial \dot{T}}{\partial \tilde{q}_a}\right) - \frac{\partial T}{\partial \tilde{q}_a} = \tilde{F}_a$$

$$\tilde{F}_a = \frac{\partial q_b}{\partial \tilde{q}_a} F_b$$

Last lecture

$$T = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2\right)$$

$$= \frac{1}{2} m \left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

generalised forces : F_r , rF_θ

radial torque.

generalised coords : r, θ .

generalised coords : $\dot{r}, \dot{\theta}$

— / —

Equ's of motion $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) - \frac{\partial T}{\partial r} = F_r$

$$\frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 = F_r$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = rF_\theta$$

$$\frac{d}{dt}(mr^2\dot{\theta}) - 0 = rF_\theta .$$

i.e $\frac{d}{dt}\left(\text{angular mom.}\right) = \text{torque}$

Suppose the forces are conservative
i.e $\exists \quad V = V(\underset{\sim}{q}, t)$

then $F_b = -\frac{\partial V}{\partial q_b}$

Now $\tilde{F}_a = -\frac{\partial q_b}{\partial \tilde{q}_a}\frac{\partial V}{\partial q_b} = -\frac{\partial V}{\partial \tilde{q}_a}$

and the transformed eq. of motion.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\tilde{q}}_a}\right) - \frac{\partial T}{\partial \tilde{q}_a} = -\frac{\partial V}{\partial q_a}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\tilde{q}}}\right) - \frac{\partial L}{\partial \tilde{q}_a} = 0 .$$

where $L = T - V$

L is known as the <u>Lagrangian</u>

Omitting the tildes, we have Lagrange's eqs:

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_a}\right) - \frac{\partial L}{\partial q_a} = 0} \qquad a = 1, \ldots, n$$

Ex: a particle of const. mass m moving in a plane subject to an attractive force $F = -\mu m / r^2$ directed toward the ogin.

Use r, θ coords.

$$T = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

$$V = \frac{-\mu m}{r}$$

$$F = -\frac{\partial V}{\partial r}$$

$$L = T - V = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{\mu m}{r}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_a}\right) - \frac{\partial L}{\partial q_a} = 0$$

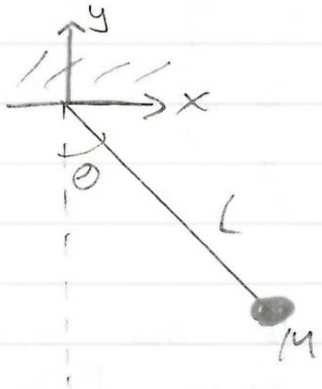
$q \equiv r \Rightarrow$

$$\frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 + \frac{\mu m}{r^2} = 0.$$

$q \equiv \theta$ $\quad \frac{d}{dt}(mr^2\dot{\theta}) = 0$

$\Rightarrow$ ang. momentum is conserved.

Ex: Pendulum.



$x = l \sin\theta$
$y = -l\cos\theta.$
$l = \text{const}.$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$
$$= \frac{1}{2} m l^2 \dot{\theta}^2$$

— / —

Another way: $T = \frac{1}{2} m (\cancel{\dot{r}^2} + r^2 \dot{\theta}^2)$ due to $\dot{r} \equiv \dot{l} = 0$
$l = \text{const}$

— / —

V = due to external force only

= "mgh" - type potential.

$= -mgl\cos\theta$

$$L = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl\cos\theta.$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0.$$

$$\frac{d}{dt}(ml^2\dot{\theta}) + mgl\sin\theta = 0 \Rightarrow \ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

20/2/12

Last lecture:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

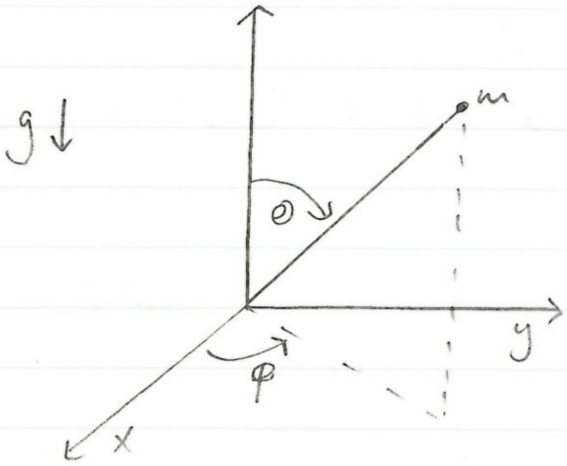
$$L = T - V.$$

—/—

Lagrangian Mechanics and forces of constraint (NFE)
- see handout.

It can be shown that the generalised forces associated with contra
forces vanish.

Ex: Particle moving on a sphere under gravity.



$0 \leq \phi < 2\pi$
$0 \leq \theta \leq \pi.$

—/—

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

—/—

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2).$$

$$x = a\cos\phi\sin\theta$$
$$y = a\sin\phi\cos\theta.$$
$$z = a\cos\theta.$$

$$\dot{x} = -a\dot{\phi}\sin\phi\sin\theta + a\cos\phi\,\dot{\theta}\cos\theta.$$
$$\dot{y} = a\dot{\phi}\cos\phi\sin\theta + a\sin\phi\,\dot{\theta}\cos\theta.$$
$$\dot{z} = -a\dot{\theta}\sin\theta.$$

$$\Rightarrow T = \frac{1}{2}ma^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta).$$

$$V = mga\cos\theta$$

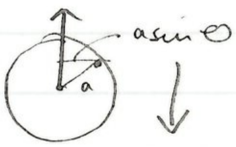
$$L = T - V$$
$$= \frac{1}{2}ma^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) - mga\cos\theta$$

—/—

$q \equiv \theta$ $\quad \frac{\partial L}{\partial \dot{\theta}} = ma^2\dot{\theta}, \quad \frac{\partial L}{\partial \theta} = ma^2\dot{\phi}^2\sin\theta\cos\theta + mga\sin\theta$

$$\frac{d}{dt}(ma^2\dot{\theta}) - ma^2\dot{\phi}^2\sin\theta\cos\theta - mga\sin\theta = 0$$

$q \equiv \phi$ $\quad \frac{\partial L}{\partial \dot{\phi}} = ma^2\sin^2\theta\,\dot{\phi} \quad , \quad \frac{\partial L}{\partial \phi} = 0$

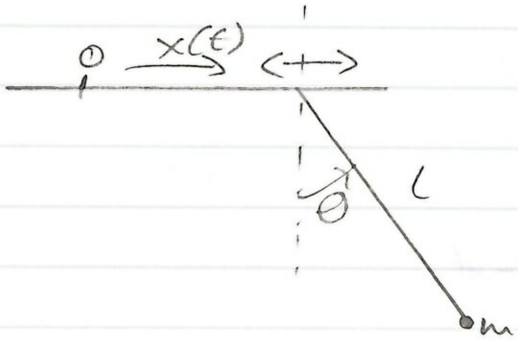


$$\Rightarrow \frac{d}{dt}(ma^2\sin^2\theta\,\dot{\phi}) = 0$$
or $ma^2\sin^2\theta\,\dot{\phi} = \text{const}$

"Recipe" : 1. Choose coords $q_1, \dots, q_m$ that label the configurations of the system.

— / —

Homework hint   Question 1 (Excersice 4)



1 degree of freedom.
i.e θ.

— / —

2. Express T in terms of

$$\dot{q}_1, \dot{q}_2, \dots, \dot{q}_m, q_1, q_2, \dots, q_m$$

on the assumption that the **constraints are satisfied**

3. If the non-constraint forces are conservative find $V = V(q_1, \dots, q_m, t)$.

4. Write down Lagrange's eqn:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 .$$

Constants of the motion and **ignorable** coordinates.

Suppose a Lagrangian has no explicit dependence on q_k ie $\frac{\partial L}{\partial q_k} = 0$.

Then Lagrange implies:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = 0.$$

or $\frac{\partial L}{\partial \dot{q}_k} \overset{\text{def}^n}{\equiv} p_k = \text{const}$

where p_k is called the generalised momentum i.e wherever q_k does not occur explicitly in L, the **corresponding** generalised momentum is constant.

Such a q_k (i.e $\partial L / \partial q_k$) is called ig^{n}orable (sometime "cyclic")

Ex $\quad L = \frac{1}{2} m(\dot{r}^2 + r^2 \dot{\theta}^2) - V(r)$

that is, the particle moves in a plane subject to a radically symmetric potential.

θ is ignorable $\quad \frac{\partial L}{\partial \theta} = 0.$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0.$$

or $p_\theta \equiv \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta} = \text{const}.$ $\quad$ (i.e angular momentum conservation $h = r^2\dot{\theta}$)

If $\frac{\partial L}{\partial t} = 0$ then:

$$\dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L = \text{const}$$ and is equivalent to energy conservatio
(T+V = E = const) provided.

$$T = \frac{1}{2} T_{ab}\dot{q}_a \dot{q}_b .$$

If $\partial L / \partial t = 0$, but $T \neq \frac{1}{2} T_{ab}\dot{q}_a \dot{q}_b$ then $\dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L = \text{const}$, but it is not $T + V = \text{const}$.

Symmetry

1. A system which is invarient under translation along a given direction i.e $\partial L / \partial q_k = 0$ ($q_k \equiv x$) conserves linear momentum in that direction.

2. A system invariant to rotation about an axis (i.e $\frac{\partial L}{\partial q_a} = 0$, $q_k \equiv \theta$) conserves angular momentum about that axis.

3. A system invarient in time ($\frac{\partial L}{\partial t} = 0$) conserves "energy ($\dot{q}_k \, \partial L / \partial \dot{q}_k - L = \text{const}$).

— / —

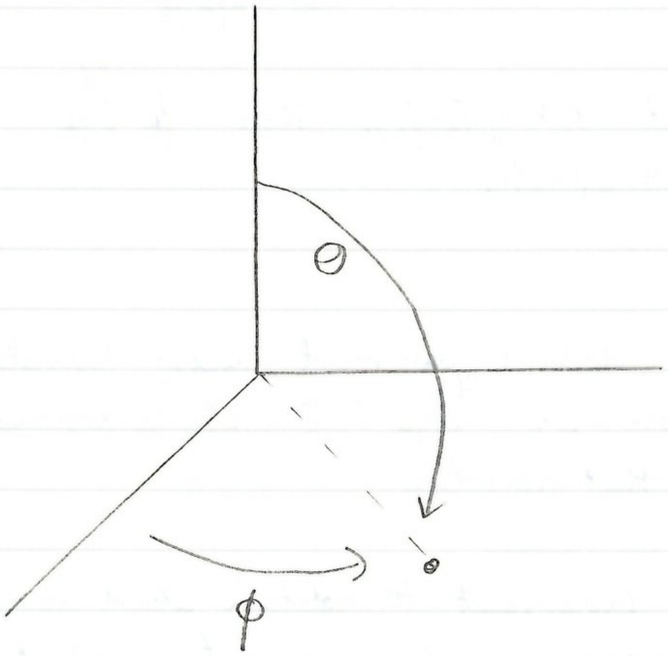
Feynmann
"Character of Physical Law"

— / —

Ex : Pendulum (i.e particle on sphere).

Recall:

$$L = \underbrace{\tfrac{1}{2}ma^2\dot{\theta}^2 + \tfrac{1}{2}ma^2\sin^2\theta\,\dot{\phi}^2}_{T} - \underbrace{mga\cos\theta}_{V}.$$



ϕ is ignorable $\left(\frac{\partial L}{\partial \phi} = 0\right)$

$$P_\phi = \frac{\partial L}{\partial \dot{\phi}} = ma^2\sin^2\theta\,\dot{\phi} = \text{const}.$$

t is ignorable :

$$\frac{\partial L}{\partial t} = 0.$$

$$\Rightarrow \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L = \text{const}.$$

Now consider

$$\frac{d}{dt}\left(\dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L\right) \qquad \left(L = L(\underset{\sim}{q}, \dot{\underset{\sim}{q}}, t)\right)$$

$$= \cancel{\ddot{q} \frac{\partial L}{\partial \dot{q}_k}} + \underbrace{\dot{q}_k \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k}\dot{q}_k}_{=0 \text{ by Lagrange}} - \cancel{\frac{\partial L}{\partial \dot{q}_k}\ddot{q}_k} - \frac{\partial L}{\partial t}$$

$$= -\frac{\partial L}{\partial t}$$

Thus, if L does not depend explicitly on time $\left(\frac{\partial L}{\partial t} = 0\right)$ then

$$\dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L = \text{const.}$$

But, what is it?

Suppose:

$$T = \frac{1}{2} T_{ab}(\underset{\sim}{q})\, \dot{q}_a \dot{q}_b \qquad (\text{sum over } a \text{ and } b).$$

i.e T is a quadratic form of the generalised velocities.

Observe $\dot{q}_k \dfrac{\partial T}{\partial \dot{q}_k} = \dot{q}_k \dfrac{\partial}{\partial \dot{q}_k}\left(\frac{1}{2} T_{ab}(\underset{\sim}{q})\, \dot{q}_a \dot{q}_b\right)$

$$= \dot{q}_k \frac{1}{2} T_{ab} \left(\frac{\partial \dot{q}_a}{\partial \dot{q}_k} \dot{q}_b + \dot{q}_a \frac{\partial \dot{q}_b}{\partial \dot{q}_k} \right)$$

$$= \dot{q}_k \frac{1}{2} T_{ab} \left(\delta_{ak} \dot{q}_b + \dot{q}_a \delta_{bk} \right)$$

$$= \dot{q}_k \frac{1}{2} T_{kb} \dot{q}_b + \dot{q}_k \frac{1}{2} T_{ak} \dot{q}_a$$

$$= \frac{1}{2} T_{ab} \dot{q}_a \dot{q}_b + \frac{1}{2} T_{ab} \dot{q}_a \dot{q}_b$$

$$= T_{ab} \dot{q}_a \dot{q}_b = 2T.$$

Thus $\dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L = \dot{q}_k \frac{\partial T}{\partial \dot{q}_k} - L \qquad (V = V(q, t))$

$= 2T - L \quad (\text{if } T = \frac{1}{2} T_{ab} \dot{q}_a \dot{q}_b)$

$= T + V.$

$= E.$

$$\frac{\partial L}{\partial t} = 0 \Rightarrow \frac{d}{dt}\left(\dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L \right) = 0$$

$T + V.$

If $T = \frac{1}{2} T_{ab} \dot{q}_a \dot{q}_b$

$$\dot{\theta}\frac{\partial L}{\partial \dot{\theta}} + \dot{\phi}\frac{\partial L}{\partial \dot{\phi}} - L = \text{const}.$$

$$ma^2\dot{\theta}^2 + ma^2\sin^2\theta\,\dot{\phi}^2 - \frac{1}{2}ma^2\dot{\theta}^2 - \frac{1}{2}ma^2\sin^2\theta\,\dot{\phi}^2$$
$$+ mga\cos\theta = \text{const}.$$

$$\Rightarrow \frac{1}{2}ma^2\dot{\theta}^2 + \frac{1}{2}ma^2\sin^2\theta\,\dot{\phi}^2 + mga\cos\theta = \text{const}.$$

$$\Rightarrow T + V = \text{const}.$$

Alternatively, not

$$T = \frac{1}{2}ma^2\dot{\theta}^2 + \frac{1}{2}ma^2\sin^2\theta\,\dot{\phi}^2.$$

is purely quadratic in $\dot{\theta}$ and $\dot{\phi}$

$$\Rightarrow \dot{q}_k\frac{\partial L}{\partial \dot{q}_k} - L = T + V = \text{const}.$$

(Also for q2 Ex 4 homework):

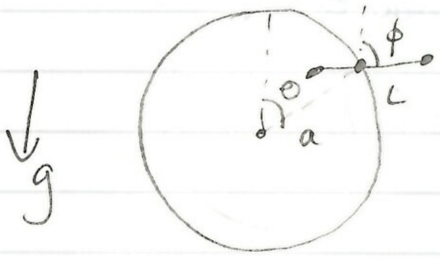
Note, that $p_\phi = ma^2\sin^2\theta\,\dot{\phi} = \text{const}.$
and subst into energy conservation.

$$\frac{1}{2}ma^2\dot{\theta}^2 + \frac{p_\phi^2}{2ma^2\sin^2\theta} + mga\cos\theta = \text{const}.$$

(a single ODE for θ).

22/2/12

Ex:



$$L = ma^2\dot{\theta}^2 + ml^2\dot{\phi}^2 - 2mga\cos\theta$$

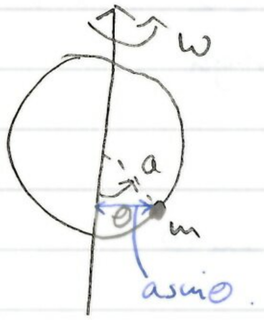
ϕ is ignorable $\left(\frac{\partial L}{\partial \phi} = 0\right)$

$$\Rightarrow \frac{\partial L}{\partial \dot{\phi}} = 2ml^2\dot{\phi} = \text{const}.$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \Rightarrow \frac{d}{dt}\left(2ma^2\dot{\theta}\right) \quad - 2mga\sin\theta = 0$$

$$\ddot{\theta} = \frac{g}{a}\sin\theta.$$

Ex:



A bead of mass m slides without friction on a massless hoop of radius a. The hoop rotates about the vertical diameter with given ang. vel ω.

One degree of freedom since ω is given and knowing θ determines location of bead.

$$T = T_{\substack{\text{motion} \\ \text{around} \\ \text{vertical}}} + T_{\substack{\text{motion} \\ \text{around} \\ \text{hoop.}}}$$

$$T = \frac{1}{2} m\omega^2 a^2 \sin^2\theta + \frac{1}{2} m a^2 \dot{\theta}^2$$

$$V = -mga\cos\theta$$

$$L = T - V$$

$$L = \frac{1}{2} m\omega^2 a^2 \sin^2\theta + \frac{1}{2} m a^2 \dot{\theta}^2 + mga\cos\theta$$

Observe $\frac{\partial L}{\partial t} = 0$

$$\boxed{\dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L = \text{const}}$$

$$\Rightarrow \dot{\theta} \frac{\partial L}{\partial \dot{\theta}} - L = \text{const}.$$

$$m a^2 \dot{\theta}^2 - \frac{1}{2} m\omega^2 a^2 \sin^2\theta - \frac{1}{2} m a^2 \dot{\theta}^2 - mga\cos\theta = \text{const}.$$

$$\frac{1}{2} m a^2 \dot{\theta}^2 - \frac{1}{2} m a^2 \omega^2 \sin^2\theta - mga\cos\theta = \text{const}.$$

Note this is energy conservation ($E = T + V$) since the middle term has the wrong sign.

Note

$$T = \frac{1}{2} m a^2 \dot{\theta}^2 + \frac{1}{2} m\omega^2 a^2 \sin\theta.$$

is not a pure quadratic form in q_i

— / —

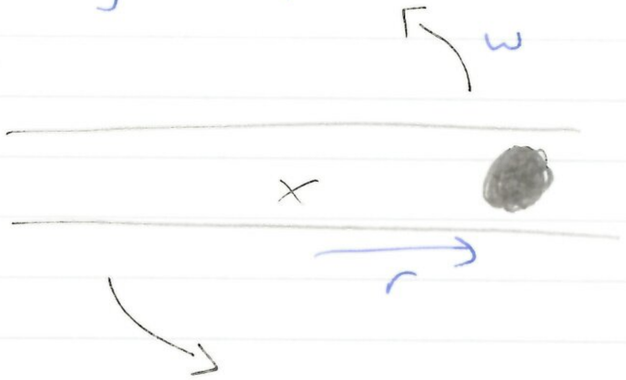
Hint Question 3:

L and L' generates same dynamics

L satisifies Lang.eq $\Leftrightarrow$ L' satisifies Lang.eq

— / —

Ex: Recall "rotating tube problem" (Ex 2).



In inertial frame

$$T = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\omega^2$$

$$V = 0$$

$$L \equiv T = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m\omega^2 r^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

$$\Rightarrow \frac{d}{dt}(\cancel{m}\dot{r}) - \cancel{m}\omega^2 r = 0$$

$$\boxed{\ddot{r} = \omega^2 r}$$

27/2/12

Velocity-dependent potentials (see handout, NFE)

Recall, our derivation of Lagrange's eqs started with:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = F_i \qquad (*)$$

If $F = -\frac{\partial V}{\partial q_i}$, $V = V(\underline{q}, t)$ then

we get $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$.

For the special case $V = V(\underline{q}, \dot{\underline{q}}, t)$ with:

$$F_i = -\frac{\partial V}{\partial q_i} + \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{q}_i}\right)$$

then substituting into $(*)$ still gives Langrange's eqs in the usual form.

Note: In Electromagnetism, the Lorentz force is velocity dependent and can be written

$$F_i = -\frac{\partial V}{\partial q_i} + \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{q}_i}\right)$$

Always ask to derive the equation.

Hamilton's Equations (Back to examinable).

Lagrange's eqs use generalised coords and velocities. Alternatively we could use generalised coords and momenta.

The total differential of $L = L(q, \dot{q}, t)$.

$$dL = \frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i + \frac{\partial L}{\partial t} dt.$$

$$= \dot{p}_i dq_i + p_i dq_i + \frac{\partial L}{\partial t} dt.$$

Since $p_i = \frac{\partial L}{\partial \dot{q}_i}$ (by defn)

and $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$ or $\frac{d}{dt} p_i = \frac{\partial L}{\partial q_i}$

Also $p_i d\dot{q}_i = d(p\dot{q}_i) - \dot{q}_i dp_i$

(by the product rule).

$$\Rightarrow d(p_i \dot{q}_i - L) = \dot{q}_i dp_i - \dot{p}_i dq_i - \frac{\partial L}{\partial t} dt$$

Def^n: The Hamiltonian H is

$$H = p_i \dot{q}_i - L .$$

where H is a function of q, p and t ie
$H = H(q, p, t)$.

Always Ask in exam

Now

$$dH = \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial t} dt$$

But eariler

$$dH = -\dot{p}_i dq_i + \dot{q}_i dp_i - \frac{\partial L}{\partial t} dt .$$

Comparing:

$$\boxed{\begin{aligned} \frac{\partial H}{\partial p_i} &= \dot{q}_i \\ \frac{\partial H}{\partial q_i} &= -\dot{p}_i \end{aligned}}$$

Hamilton's eqn's

A good thing to remember.

Also

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

Note, the total time derivative of

$$H = H(q, p, t)$$

$$\frac{dH}{dt} = \frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i + \frac{\partial H}{\partial t}$$

$$= \cancel{\frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i}} - \cancel{\frac{\partial H}{\partial p_i}\frac{\partial H}{\partial q_i}} + \frac{\partial H}{\partial t}$$

$$\Rightarrow \frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

— / —

"Recipe" for constructing the Hamiltonian

(i) find $L = L(q, \dot{q}, t)$.

(ii) $p_i = \dfrac{\partial L}{\partial \dot{q}_i}$

(iii) Construct $H = p_i \dot{q}_i - L$

(iv) Write $H = H(q, p, t)$.

Ex: Find H for a single particle in 3D

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x,y,z).$$

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}\ ,\ p_y = m\dot{y}\ ,\ p_z = m\dot{z}$$

$$\Rightarrow\ H = p_i\dot{q}_i - L = p_x\dot{x} + p_y\dot{y} + p_z\dot{z} - L$$

$$= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + \frac{1}{2}m\dot{z}^2 + V(x,y,z$$

(Or, observe T is a pure quadratic in $\dot{x}$, $\dot{y}$ and $\dot{z}$ $\Rightarrow$ $H = T - V$)

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) + V(x,y,z)$$

(3)

has the correct dependencies.

Ex: Find the Hamiltonian corresponding to the Lagrangian

$$L = (1 - \dot{q}^2)^{\frac{1}{2}}$$

$$p = \frac{\partial L}{\partial \dot{q}} = -\dot{q}(1-\dot{q})^{-\frac{1}{2}}$$

$$H = p\dot{q} - L = -\dot{q}^2(1-\dot{q}^2)^{-\frac{1}{2}} - (1-\dot{q})^{\frac{1}{2}}$$

$$= \frac{-1}{(1-\dot{q}^2)^{\frac{1}{2}}}$$

but $H = H(q,p)$.

Note: $p^2 = \frac{\dot{q}^2}{1-\dot{q}^2}$

$\Rightarrow 1-\dot{q}^2 = (1-p^2)^{-1}$

$\Rightarrow H = -(1+p^2)^{\frac{1}{2}}$

Ex: Find the Langrangian from the Hamiltonian

$$H = \frac{1}{2}p^2 + p\sin q.$$

first note $\dot{q} = \frac{\partial H}{\partial p} = p + \sin q$.

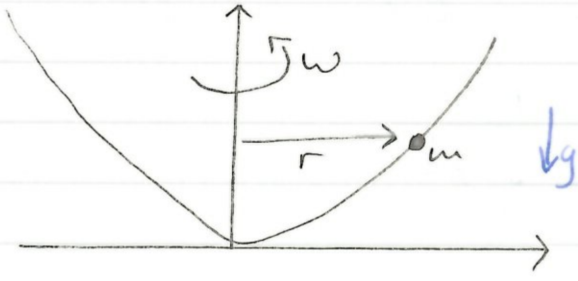
$$L = \dot{q}p - H = p^2 + \cancel{p\sin q} - \frac{1}{2}p^2 - \cancel{p\sin q}.$$

$$= p^2$$

$$L = \frac{1}{2}(\dot{q} - \sin q)^2$$

— / —

Ex [Exam '96] A bead of mass m slides under gravity on a parabolic wire $z = \frac{1}{2}\alpha^2 x^2$. The wire rotates with const. angular velocity ω about the vertical.



$V = mgz$ (Use x as generalized coord.)

$= \frac{mg}{2}\alpha^2 x^2$

$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2)$ $\left\{\begin{array}{l}\text{T is} \\ \text{cylindrical polar}\end{array}\right.$

$r \equiv x, \ \dot{\theta} = \omega, \ \dot{z} = \alpha^2 x\dot{x}$

$= \frac{1}{2}m(\dot{x}^2 + x^2\omega^2 + \alpha^4 x^2\dot{x}^2)$

$L = \frac{1}{2}m(\dot{x}^2 + x^2\omega^2 + \alpha^4 x^2\dot{x}^2) - \frac{mg}{2}\alpha^2 x^2$

$p = \frac{\partial L}{\partial \dot{x}} = m(\dot{x} + \alpha^4 x^2\dot{x})$

$= m\dot{x}(1 + \alpha^4 x^2)$.

$H = p\dot{x} - L$.

$= m\dot{x}^2(1 + \alpha^4 x^2) - L$.

$= \frac{1}{2}m\dot{x}^2(1 + \alpha^4 x^2) - \frac{1}{2}mx^2\omega^2 + \frac{1}{2}mg\alpha^2 x^2$.

$\Rightarrow H = \frac{1}{2m}\frac{p^2}{(1+\alpha^4 x^2)} - \frac{1}{2}mx^2\omega^2 + \frac{1}{2}mg\alpha^2 x^2$.

Observe : $H \neq T + V$.

$\frac{\partial H}{\partial t} = 0$ but $\frac{dH}{dt} = \frac{\partial H}{\partial t} = 0 \Rightarrow H \equiv \text{const}$.

Poisson brackets.

Let X and Y be dynamical variables which are function of q, p and t.

Def^n: The Poisson bracket of X and Y is

$$[X, Y] = \frac{\partial X}{\partial q_i}\frac{\partial Y}{\partial p_i} - \frac{\partial X}{\partial p_i}\frac{\partial Y}{\partial q_i}$$

Properties: (i) $[X, Y] = [Y, X]$.

(ii) $[X, Y_1 + Y_2] = [X, Y_1] + [X, Y_2]$

(iii) $[X, Y_1 Y_2] = [X, Y_1] Y_2 + [X, Y_2] Y_1$

(iv) $[[X,Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$.

Jacobi's identity.

Let $Y \equiv H$ (i.e the Hamiltonian)

$$[X, H] = \frac{\partial X}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial X}{\partial p_i}\frac{\partial H}{\partial q_i}$$

$$= \frac{\partial X}{\partial \dot{q}_i} + \frac{\partial X}{\partial p_i}\dot{p}_i$$

$$= \frac{dX}{dt} - \frac{\partial X}{\partial t}.$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}$$

$$\Rightarrow \frac{dx}{dt} = [X, H] + \frac{\partial X}{\partial t}.$$

So if X does not depend explicitly on t (i.e. $\frac{\partial X}{\partial t} = 0$) then:

$$\frac{dX}{dt} = [X, H]$$

Revisit Jacobi with $Z \equiv H$:

$$\left[[X,Y],H\right] + \left[[Y,H],X\right] + \left[[H,X],Y\right] = 0.$$

Suppose X and Y are constant of the motion

$$\text{i.e } \frac{dX}{dt} = 0, \quad \frac{dY}{dt} = 0.$$

$$\Rightarrow [X, H] = [Y, H] = 0.$$

$$\Rightarrow \left[[X,Y], H\right] = 0.$$

$$\Rightarrow \frac{d}{dt}[X,Y] = 0. \quad \text{i.e } [X,Y] \equiv \text{const of motion.}$$

5/3/11

$$[X, Y] = \frac{\partial X}{\partial q_j}\frac{\partial Y}{\partial p_j} - \frac{\partial X}{\partial p_j}\frac{\partial Y}{\partial q_j}$$

$$\frac{\partial X}{dt} = \frac{\partial X}{\partial t} + [X, H]$$

— / —

Poisson brackets

Some more facts

$$\frac{\partial H}{\partial p_j} = \frac{dq_j}{dt} = [q_j, H]$$

$$-\frac{\partial H}{\partial q_j} = \frac{dp_j}{\partial t} = [p_j, H]$$

$$\frac{\partial X}{\partial p_j} = [q_j, X]$$

$$[q_j, X] = \frac{\partial q_j}{\partial q_i}\frac{\partial X}{\partial p_i} - \cancel{\frac{\partial q_j}{\partial p_i}\frac{\partial X}{\partial q_i}} \rightarrow 0$$

$$= \delta_{ij}\frac{\partial X}{\partial p_i}$$

$$= \frac{\partial X}{\partial p_j}$$

$$-\frac{\partial X}{\partial q_j} = [p_j, X]$$

Also, $[q_i, q_k] = 0$

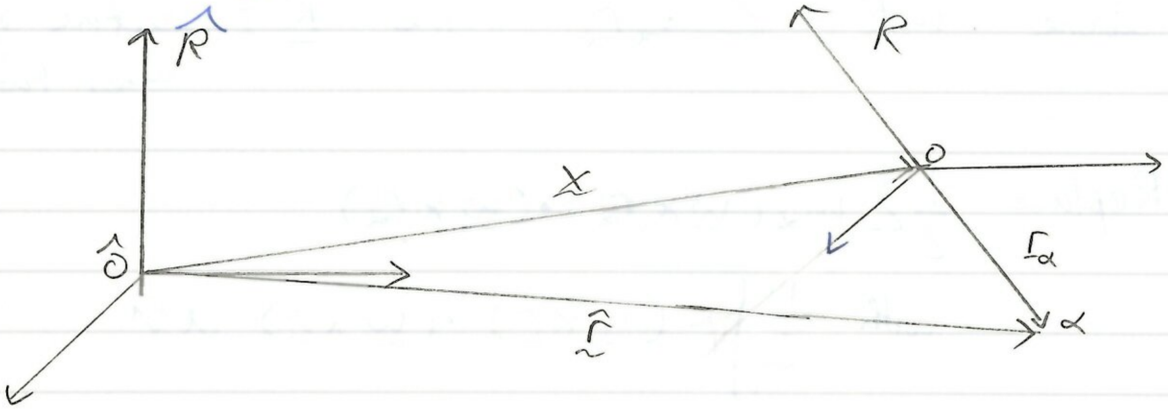
$[p_i, p_k] = 0$

$[q_i, p_k] = \delta_{ik}$

§4 Rigid Bodies

A rigid body require 6 generalized coords: 3 to specify a point on the body (eg CM) and a further 3 to describe its orientation about that point.

Let $\hat{R} = (\hat{O}, \hat{B})$ be an inertial frame and $R = (O, B$ be a rest frame of the body (i.e it is fixed relative to the body)



Let $\underline{r}_\alpha$ be the vector of particle α of the rigid body from O and $\underline{\hat{r}} = \underline{x} + \underline{r}_\alpha$ (see diagram)

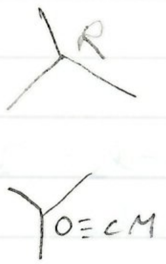
$$\text{Then } \underline{\hat{v}}_\alpha = \hat{D}\underline{\hat{r}} = \hat{D}(\underline{x} + \underline{r}_\alpha)$$
$$= \hat{D}x + \hat{D}\underline{r}_\alpha$$
$$= \underline{\dot{x}} + \cancel{D\underline{r}_\alpha}^{\,0} + \underline{w} \times \underline{r}_\alpha \quad \text{(Coriolis Thm)}$$

(since we are in the rest frame)

where $\underline{w}$ is the angular velocity of the body relative to $\hat{R}$

$$T = \sum_\alpha \tfrac{1}{2} m_\alpha \underline{\hat{v}}_\alpha \cdot \underline{\hat{v}}_\alpha$$

$$= \sum_\alpha (\underline{\dot{x}} + \underline{w} \times \underline{r}_\alpha) \cdot (\underset{\sim}{\dot{x}} + \underline{w} \times \underline{r}_\alpha)$$



$$= \frac{1}{2} m \,\dot{\underline{x}}\cdot\dot{\underline{x}} + \dot{\underline{x}}\cdot\left(\underline{\omega}\times\sum_\alpha m_\alpha \underline{r}_\alpha\right)$$

$$+ \frac{1}{2}\sum_\alpha m_\alpha (\underline{\omega}\times\underline{r}_\alpha)\cdot(\underline{\omega}\times\underline{r}_\alpha)$$

The second term can be simplified to

$$m\dot{\underline{x}}\cdot(\underline{\omega}\times\underline{R})$$

where $m\underline{R} = \sum m_\alpha \underline{r}_\alpha$ i.e $\underline{R} \equiv$ centre of mass from 0.

Replace $\frac{1}{2}\sum m_\alpha(\underline{\omega}\times\underline{r}_\alpha)\cdot(\underline{\omega}\times r_\alpha)$

with $\frac{1}{2}\int_V \rho(\underline{\omega}\times\underline{r})\cdot(\underline{\omega}\times\underline{r})\, dV$

$$= \frac{1}{2}\int_V \rho\left[(\underline{\omega}\cdot\underline{\omega})(\underline{r}\cdot\underline{r}) - (\underline{\omega}\cdot\underline{r})^2\right] dV$$

Note $\underset{\sim}{\omega} \neq \underline{\omega}(\underline{r})$

$$= \frac{1}{2}\omega_i\omega_j \int \rho(r_k r_k \delta_{ij} - r_i r_j)\, dV.$$

Def^n: The inertia matrix of a rigid body in the rest frame R is 3x3 matrix (symmetric) $J(R)$ with entries

$$J_{ij} = \int \rho(r_k r_k \delta_{ij} - r_i r_j)\, dV \quad (= J_{ji})$$

$$(\underline{\omega}\times\underline{r})\cdot(\underline{\omega}\times r)$$
$$= |\underline{\omega}\times\underline{r}|^2$$
$$= |\underline{\omega}|^2|\underline{r}|^2\sin^2\theta$$
$$= |\underline{\omega}|^2|\underline{r}|^2(1-\cos^2\theta)$$
$$= (\underline{\omega}\cdot\underline{\omega})(\underline{r}\cdot\underline{r}) - (\underline{\omega}\cdot\underline{r})^2$$

— / —

$$r_k r_k = \underline{r}\cdot r$$
$$\omega_i\omega_j r_i r_j = (\underline{\omega}\cdot\underline{r})^2$$

Indentifying $(r_1, r_2, r_3) = (x, y, z)$

$$J_{11} = A = \int \rho (y^2 + z^2) \, dV \qquad F = \int \rho y z \, dV = -J_{23}$$

$$J_{22} = B = \int \rho (x^2 + z^2) \, dV \qquad G = \int \rho x z \, dV = -J_{13}$$

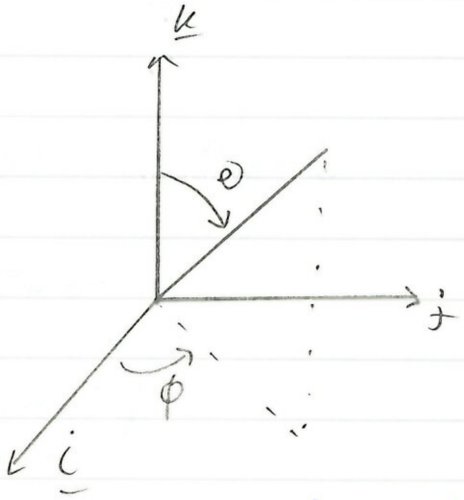
$$J_{33} = C = \int \rho (x^2 + y^2) \, dV \qquad H = \int \rho x y \, dV = -J_{12}$$

$$J(R) = \begin{pmatrix} A & -H & -G \\ -H & B & -F \\ -G & -F & C \end{pmatrix}$$

Examples of inertia martices

① A 1-D rod of length l and mass m.

$$\rho = \frac{m}{l}$$



$x = r \sin\theta \cos\phi$
$y = r \sin\theta \sin\phi$
$z = r \cos\theta$.

$$A = \int_0^l \rho (y^2 + z^2) \, dr$$

$$= \int_0^l \rho (r^2 - x^2) \, dr$$

$$= (1 - \sin^2\theta \cos^2\phi) \rho \int_0^l r^2 \, dr.$$

$$= \frac{ml^2}{3} (1 - \sin^2\theta \cos^2\phi)$$

$$B = \int_0^l (r^2 - y^2)\rho \, dr = \frac{ml^2}{3}(1 - \sin^2\theta \sin^2\phi)$$

$$C = \int_0^l (r^2 - z^2)\rho \, dr = \frac{ml^2}{3}\sin^2\theta .$$

$$F = \int_0^l \rho yz \, dr = \frac{ml^2}{3}\sin\theta\cos\theta\sin\phi$$

$$G = \frac{ml^2}{3}\sin\theta\cos\theta\cos\phi$$

$$H = \frac{ml^2}{3}\sin^2\theta\sin\phi\cos\phi$$

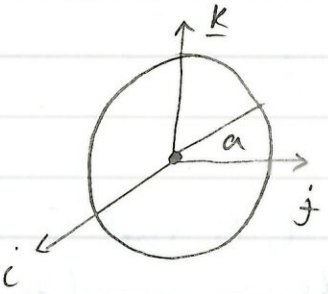
$$J_{ij} = \int \rho(r_k r_k \delta_{ij} - r_i r_j)\, dV.$$

$$J = \frac{ml^2}{3}\begin{pmatrix} 1 - \sin^2\theta\cos^2\phi & -\sin^2\theta\sin\phi\cos\phi & -\sin\theta\cos\theta\cos\phi \\ -\sin^2\theta\sin\phi\cos\phi & 1 - \sin^2\theta\sin^2\phi & -\sin\theta\cos\theta\sin\phi \\ -\sin\theta\cos\theta\cos\phi & -\sin\theta\cos\theta\sin\phi & \sin^2\theta \end{pmatrix}$$

Observe if i, j and k are chosen st $\phi = 0$, $\theta = \frac{\pi}{2}$ i.e $\underline{c_i}$ lies along the rod.

$$J = \frac{ml^2}{3}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

② A 2-D circular disc (2-D lamina i.e a thin, plane body).



$\rho = \dfrac{m}{\pi a^2}$ (= const)

(out $\perp$ to disc)

$$A = \iint (r^2 - x^2)\,\rho\, dA$$

$$= \frac{m}{\pi a^2}\int_0^{2\pi}\int_0^{a} R^2 R \;\; dR\, d\theta$$

$$= \frac{m}{\cancel{\pi a^2}}\, 2\cancel{\pi}\, \frac{a^4}{4} = \frac{ma^2}{2}$$

On disc

$x = 0$

$dA = R\,dR\,d\theta$.

$R^2 = y^2 + z^2$

$y = R\cos\theta$

$z = R\sin\theta$

$$B = \iint (r^2 - y^2)\, dA = \frac{m}{\pi a^2}\int_0^{2\pi}\int_0^{a} R^2(1-\cos^2\theta)\, R\, dR\, d\theta.$$

$$= \frac{ma^2}{4}$$

$$C = \frac{ma^2}{4}$$

$G = H = 0$. (since $x = 0$)

$$F = \iint \rho\, yz\, dA = 0$$

$$J = \frac{ma^2}{4}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Def^n: The diagonal matries A, B and C are called moments of inertia about the x, y and z axes respectively.
F, G and H are called products of inertia.

Def^n: Let $J(R)$ be the inertia matrix in a rest frame $R = (O, B)$. A prinicple axis at O is a line through O in the direction of an eigenvector of $J(R)$. The corresponding eigenvalue is the principal moment of inertia

Remarks:

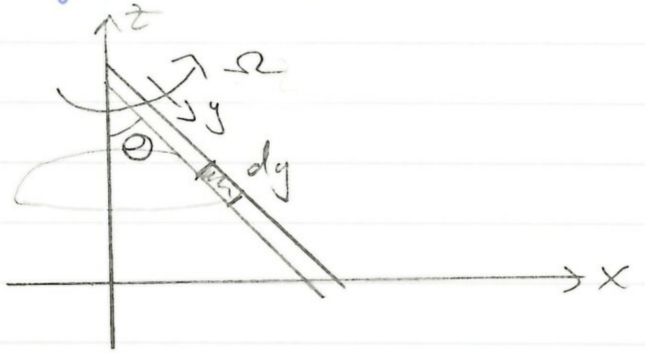
1) If $J(R')$ is diagonal then the coord axes of R' are principal axes (since $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ are eigenvectors) and the diagonal entries are principal moment of inertia.

Theorem: Let $R(O, B)$ and $R'(O, B')$ be two rest frames with the same origin. Let H be the transition matrix from B' to B.

Let J_{ij} and J_{ij}' be entries in the inertia matrices $J(R)$ and $J(R')$

Then $J_{ij} = H_{ip} J_{pq}' H_{jq}$
(or, $J = HJ'H^T$)

Hint for Q2 Ex 5



$\downarrow g$

$$\int_0^{2a} \rho\, dy$$

7/3/12

$$T = \frac{1}{2} m \underline{\dot{x}} \cdot \underline{\dot{x}} + m \underline{\dot{x}} \cdot (\underline{\omega} \times \underline{R}) + \frac{1}{2} \omega_i \omega_j J_{ij}$$

$$J_{ij} = \int_V \rho (r_k r_k \delta_{ij} - r_i r_j) \, dV$$

— / —

Thm: $J_{ij} = H_{ip} J'_{pq} H_{jq}$ ($H \equiv$ trans matrix from R' to R)

We have $r_i = H_{ip} r'_p$, $r_j = H_{jq} r'_q$

$$J_{ij} = \int_V \rho (r_k r_k \delta_{ij} - r_i r_j) \, dV$$

$$= \int_V \rho H_{ip} H_{jq} (r'_k r'_k \delta_{pq} - r'_p r'_q) \, dV$$

since $r_k r_k = r'_k r'_k$ because magnitude^2 (i.e $r_k r_k$) is preserved under rotation of axes $R \longleftrightarrow R'$

OR $r_k r_k = H_{kp} r'_p H_{kq} r'_q$

$$= H_{kp} H_{kq} r'_p r'_q$$

$$= (H^T)_{pk} H_{kq} r'_p r'_q$$

$$= (H^T H)_{pq} r'_p r'_q$$

$$= \delta_{pq} r'_p r'_q = r'_p r'_p \qquad (I = H^T H)$$

Also $H_{ip} H_{jp} \delta_{pq} = H_{ip} H_{jp}$

$$= \delta_{ij}$$

and $H_{ip} H_{jq} r'_p r'_q = r_i r_j$

Thus $$J_{ij} = \int \rho H_{ip} H_{jq} (r'_k r'_k \delta_{pq} - r'_p r'_q) \, d\mathcal{V}$$

$$= H_{ip} \int \rho (r'_k r'_k \delta_{pq} - r'_p r'_q) \, d\mathcal{V} \, H_{jq}$$

$$= H_{ip} J'_{pq} H_{jq}$$

In matrix

$$J = H J' H^T$$

or $$J' = H^T J H$$

But for any symmetric matrix J $\exists$ an othogonal matrix H s.t $H^T J H$ is diagonal.

Suppose that the rest frame has its axes aligned principal axes (i.e $J \equiv$ diagonal). Then $F = G = H = 0$ and

$$\frac{1}{2} J_{ij} \omega_i \omega_j = \frac{1}{2} (A\omega_1^2 + B\omega_2^2 + C\omega_3^2)$$

1. If 0 of the rest frame is the CM.
i.e $\underline{R} = 0$ and

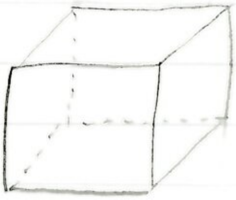
$$T = \underbrace{\frac{1}{2} \underline{\dot{x}} \cdot \underline{\dot{x}}}_{\text{CM KE}} + \underbrace{\frac{1}{2} (A\omega_1^2 + B\omega_2^2 + C\omega_3^2)}_{\text{KE about CM.}}$$

2. O is at rest relative to $\hat{R}$. This is the case when the body is rotating about a fixed point

Here $\underline{\dot{x}} = 0$ and

$$T = \frac{1}{2}(A\omega_1^2 + B\omega_2^2 + C\omega_3^2)$$

Ex :



Solid cube
unifrom density
side-length 2a.

$$J = \frac{2}{3}ma^2 I$$ (with chosen as in pic)

—/—

(Ex: try showing this)

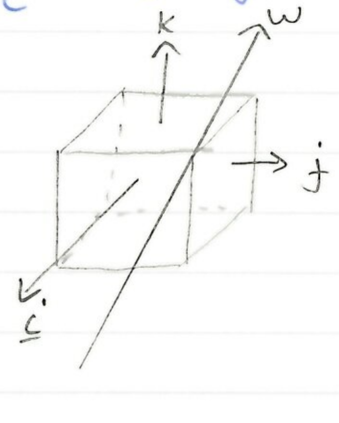
—/—

The body rotates about an axis which goes through oppsite vertices. Its ang vel is ω. Find T.

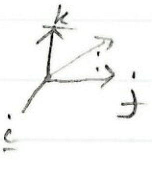
Need to use

$$T = \frac{1}{2}\omega_i \omega_j J_{ij}$$

Note



$$\underline{\omega} = \frac{\omega}{\sqrt{3}}(\underline{i} + \underline{j} + \underline{k})$$



$$\Rightarrow T = \frac{1}{2}\left(A\omega_1^2 + B\omega_2^2 + C\omega_3^2\right)$$

$$= \frac{1}{2}\cancel{3} \times \frac{2}{3} ma^2 \frac{\omega^2}{\cancel{3}}$$

$$= \frac{1}{3} ma^2\omega^2$$

Angular momentum

12/3/12

Angular Momentum

Let a point P be fixed in both the rest frame and the inertial frame (i.e $\dot{\underline{x}} = 0$). We say "the body moves about a fixed point"

The angular momentum about P

$$\underline{L}_P = \int_V \rho \underline{r} \times \underline{v} \, dV \qquad \left(\overset{c.f}{\underline{L} = \sum_i r_i \times m\underline{v}_i} \right)$$

where $\underline{r}$ is the position vector of a volume element of the body from P. $\underline{v} = \hat{D}\underline{r}$

But $\hat{D}\underline{r} = \cancel{\dot{\underline{r}}}^{\,0} + \underline{\omega} \times \underline{r}$

$$= \underline{\omega} \times \underline{r} \qquad \text{(Coriolis Thm)}$$

$$\Rightarrow \underline{L}_P = \int_V \rho(\underline{r} \times (\underline{\omega} \times \underline{r})) \, dV$$

$$= \int \rho((\underline{r} \cdot \underline{r})\underline{\omega} - (\underline{r} \cdot \underline{\omega})\underline{r}) \, dV.$$

— / —

$$a \times (b \times c) = (\underline{a} \cdot \underline{c})\underline{b} - (\underline{a} \cdot \underline{b})\underline{c}$$

— / —

$$= \underline{L}_{P_i} = \int_V \rho(r_k r_k \omega_i - r_j \omega_j r_i) \, dV$$

$$= \omega_j \int_V \rho (r_k r_k \delta_{ij} - r_i r_j)\, dV$$

$$= L_{p_i} = J_{ij} \omega_j$$

OR, $$\underline{L}_p = J \underline{\omega}$$

Thus $\underline{L}_p$ is determined once we know the angular velocity $\underline{\omega}$ and the inertia matrix at P.

Take axes of R to be principal axes. Then $\underline{L}_p = J\underline{\omega}$ becomes

$$\begin{aligned} L_{p_1} &= A\omega_1 \\ L_{p_2} &= B\omega_2 \\ L_{p_3} &= C\omega_3 \end{aligned} \qquad J = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix}$$

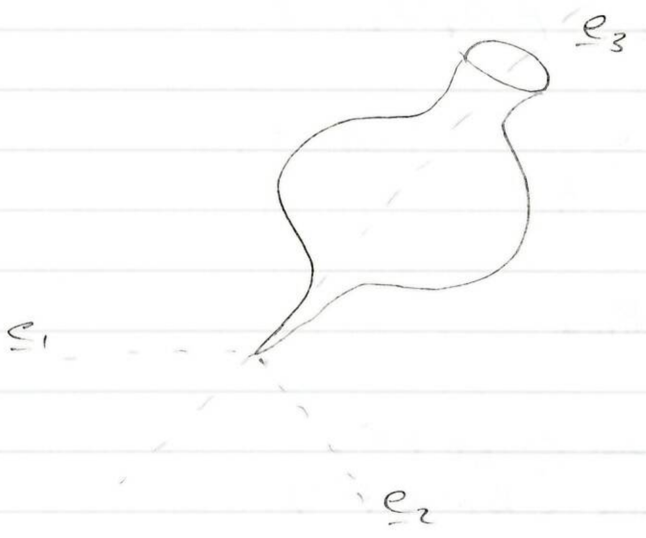
Now $\hat{D}\underline{L}_p = \dot{\underline{L}}_p + \underline{\omega} \times \underline{L}_p = \underline{N}$ (torque)

$$\Rightarrow \left.\begin{aligned} A\dot{\omega}_1 + (C - B)\,\omega_2\omega_3 &= N_1 \\ B\dot{\omega}_2 + (A - C)\,\omega_1\omega_3 &= N_2 \\ C\dot{\omega}_3 + (B - A)\,\omega_1\omega_2 &= N_3 \end{aligned}\right\} \text{Euler's Eqs for rigid-body motion.}$$

Euler's-equs determine the time-dependent behaviour of $\underline{\omega}$ and, indirectly, the orientation of the body.

The symmertical (froce - free) top.

Consider a freely rotating, symmetrical body (i.e $\underline{P} = 0$)
Let the symmertry axis be $\underline{e}_3 \Rightarrow A = B$.



$$A = \int_V \rho(y^2 + z^2)\, dV$$

$$B = \int_V \rho(x^2 + z^2)\, dV \qquad C = \int_V \rho(x^2 + y^2)\, dV \neq 0.$$

$$= A.$$

$$A\dot{\omega}_1 + (C - B)\omega_2\omega_3 = 0$$
$$B\dot{\omega}_2 + (A - C)\omega_1\omega_3 = 0$$
$$C\omega_3 + (B - A)\omega_1\omega_2 = 0$$

$$A = B \Rightarrow C\dot{\omega}_3 = 0 \quad \text{or} \quad \omega_3 = \text{const}.$$

Other 2 Euler s eq's become

$$\dot{\omega}_1 + \beta \omega_3 \omega_2 = 0$$

$$\dot{\omega}_2 - \beta \omega_2 \omega_1 = 0.$$

$$\beta = \frac{C - B}{A} = \text{const}.$$

General Soultion

$$\omega_1 = D\cos(\beta\omega_3 t + \Theta)$$
$$\omega_2 = D\sin(\beta\omega_3 t + \Theta)$$

D, Θ are arb. const.

The angular velocity vector $\underline{\omega}$ precesses in a circle of radius D about the $\underline{e}_3$-axis with angular velocity $\beta\omega_3$.

The instantaneous axis of rotation ($\underline{\omega}$) traces out a cone in the body as it precesses

i.e

To find the motion of the body in space (i.e find $\underline{e}_3$ relative to some fixed direction) we first find $\underline{\omega}$ relative to the <u>fixed</u> direction of $\underline{L}$ (since $\underline{N} = 0$)

The angle α_s between $\underline{\omega}$ and $\underline{L}$ is

$$\cos \alpha_s = \frac{\underline{\omega} \cdot \underline{L}}{|\underline{\omega}||\underline{L}|}$$

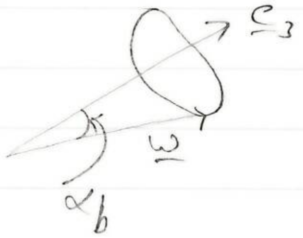
$$= \frac{\omega_i J_{ij} \omega_j}{|\underline{\omega}||\underline{L}|}$$

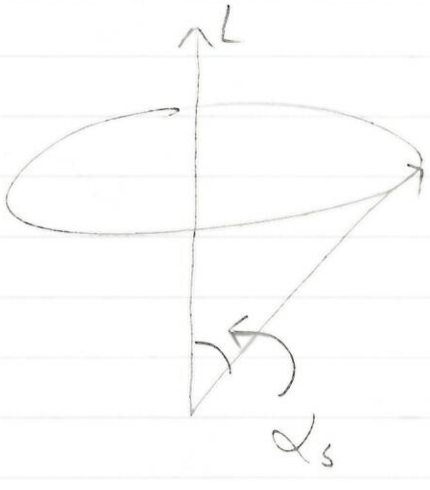
But $|\underline{\omega}| = \sqrt{D^2 + \omega_3^2} = \text{const}.$

$$|\underline{L}| = \text{constant} \quad (\underline{N} = 0)$$

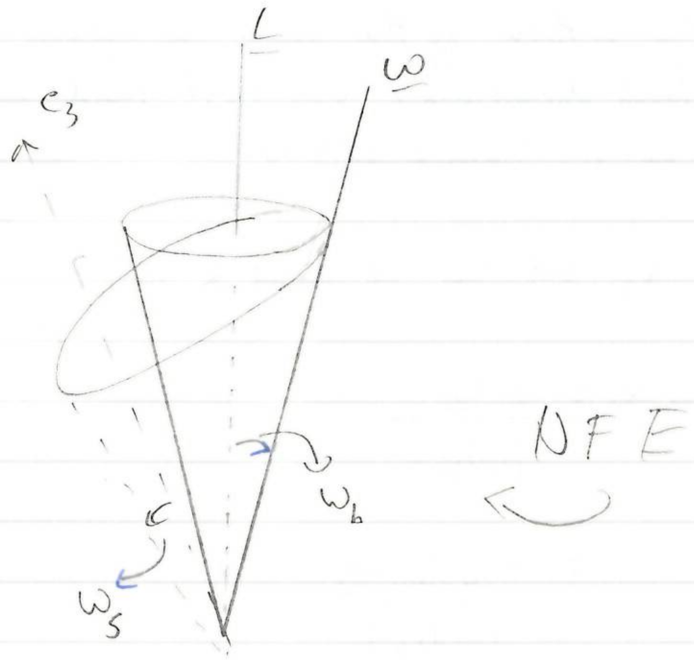
and $\omega_i \omega_j J_{ij} = 2T = \text{const}$

i.e $\alpha_s = \text{const}.$





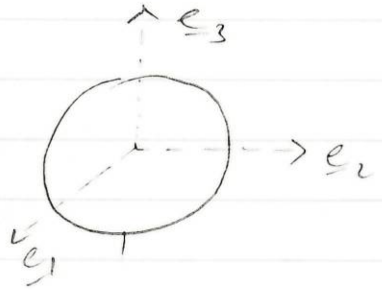
i.e $\underline{\omega}$ traces out a cone about $\underline{L}$ with half-angle α_s.



—/—

Chandler Wobble: precession of the earth

The earth is symmetrical about the polar (N−S) axis, but is "fatter" at the equator i.e



$$A = \int_V \rho(y^2+z^2)\,dV = B$$

$$C = \int_V \rho(x^2+y^2)\,dV \quad > A$$

For the earth,

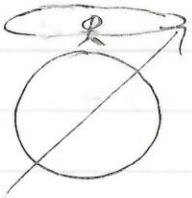
$$\beta = \frac{C-A}{A} = 0.0033 \simeq \frac{1}{300}$$

Hence, the precession frequency of $\underline{\omega}$ about $\underline{e}_3$ is

$$\Omega = \beta\omega_3 = {}^{\omega_3}/_{300}$$

But $\omega_3 = 1 \text{ day}^{-1}$

Period of precession of $\underline{\omega}$ about $\underline{e}_3$ is 300 days



i.e. 300 days i.e the axis of rotation should trace out a circle about N pole every 10 months.

The precession is indeed observed. It has small amplitude with $\underline{\omega}$ never wondering more than 5m from N pole.

The period is actually about 14 months

Stability of rotation about principle axes.
Consider a body in which all 3 principal moments are distinct $A \neq B \neq C$.

Let the body spin about the $\underline{e}_3$ principle axis s.t $\omega_3 = \omega = \text{const}$, $\omega_1 = \omega_2 = 0$.

We perturb the motion slightly so that

$$\omega_1 = \xi_1$$
$$\omega_2 = \xi_2$$
$$\omega_3 = \omega + \xi_3$$

where ξ_1, ξ_2, ξ_3 are all functions of ϵ and are small compared ω.

From Euler's eqs:

$$A\dot{\xi}_1 + (C-B)\xi_2(\xi_3+\omega) = 0$$
$$B\dot{\xi}_2 + (A-C)\xi_1(\xi_3+\omega) = 0$$

Ignoring quadratic terms in ξ_i (they are v. small)

$$A\dot{\xi}_1 + (C-B)\xi_2\omega = 0.$$
$$B\dot{\xi}_2 + (A-C)\xi_1\omega = 0.$$

$$\Rightarrow A\ddot{\xi}_1 + (C-B)\omega\dot{\xi}_2 = 0.$$

and subst 2nd eq for $\dot{\xi}_2$ above

$$\boxed{AB\ddot{\xi}_1 - (C-B)(A-C)\omega^2\xi_1 = 0.}$$

Subst $\xi_1 = e^{pt}$ into the above give:

$$p^2 = \frac{(C-B)(A-C)\omega^2}{AB}$$

Note $\omega^2 > 0$, $AB > 0$ hence

i) If $C > B$ and $A < C$
or $C < B$ and $A > C$
then $p^2 < 0$
i.e p is pure imag i.e $p = \pm i\alpha$.
$\Rightarrow$ oscillatory motion stable

ii) If $A > C > B$ or $A < C < B$

$\Rightarrow p^2 > 0 \Rightarrow$ p are real

i.e $p = \pm \alpha$

$\Rightarrow$ one solution grows exponentially

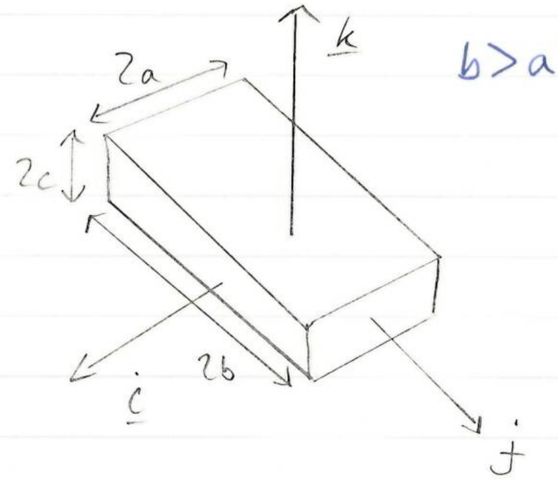
unstable

If the moment of inertia about the spinning axis (i.e C) is greatest or least $\Rightarrow$ stable.

It is unstable otherwise.

14/3/12

A "book"-shaped object:



$b > a > c$

Recall
(Stability)
If the moment
of inertia about
the axis of spin
is greastest or
least ⇒ stable

$$J = \frac{m}{3}\begin{pmatrix} b^2+c^2 & 0 & 0 \\ 0 & \underset{\text{least}}{\boxed{a^2+c^2}} & 0 \\ 0 & 0 & \underset{\text{largest.}}{\boxed{a^2+b^2}} \end{pmatrix}$$

Predict: Stable rotation about $\underline{j}$ and $\underline{k}$ axes
unstable about $\underline{i}$ axis.

~~Lagrangian description of rigid body motion~~

Proposition:

NFE

H from $\hat{B}$ to B

$H = (\quad)(\quad)(\quad)$

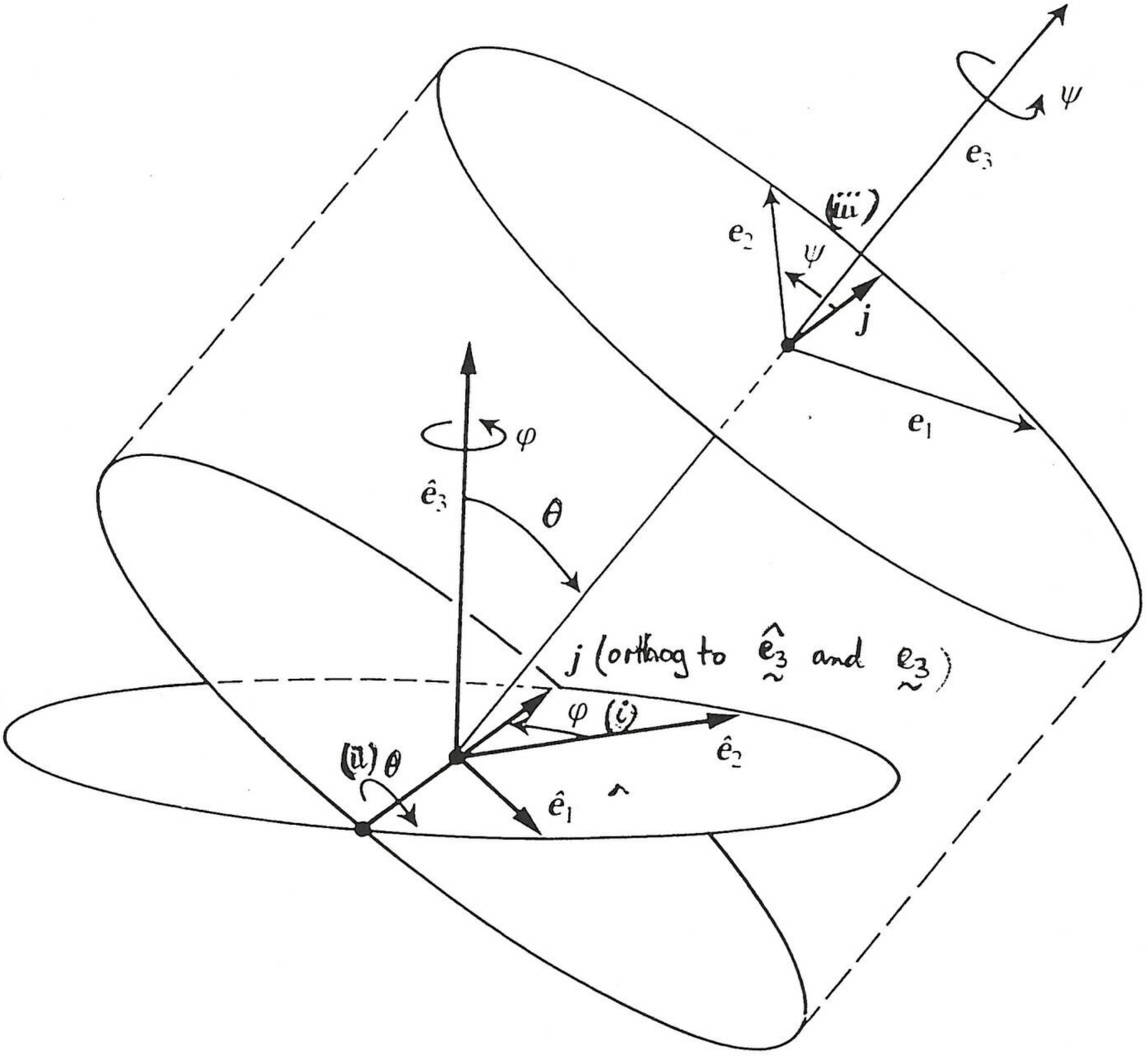


Fig. 3.3.1 Euler angles.

19/3/12

(i) M: A rotation about $\hat{\underline{e}}_3$ through ϕ which brings $\hat{\underline{e}}_2$ into coincidence with $\underline{j}$.

(ii) L: A rotation about $\underline{j}$ through θ which brings $\hat{\underline{e}}_3$ into coincidence with $\underline{e}_3$

(iii) K: A rotation about $\underline{e}_3$ through ψ which brings $\underline{j}$ into coincidence with $\underline{e}_2$.

Def^n: ψ, θ and ϕ are known as the Euler angles.

$H = KLM$.

—/—

We've shown that the transition matrix from $\hat{B}$ to B, H, can be written $H = KLM$.

$$M : \hat{B} \longrightarrow B''$$
$$L : B'' \longrightarrow B'$$
$$K : B' \longrightarrow B$$

$\dot{H}H^T$

Recall if H is the transition matrix from $\hat{B}$ to B, the angular velocity of B relative to $\hat{B}$ is the vector $\underline{\omega} = \omega_i \underline{e}_i$ and is constructed from $\Omega = \dot{H}H^T$ and $\Omega_{jk} = \varepsilon_{ijk}\,\omega_i$

Ex: Since M is the transition matrix from $\hat{B}$ to B'', the angular velocity of B'' relative to $\hat{B}$ is found from $\dot{M}M^T$

$$M = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\dot{M}M^T = \dot{\phi}\begin{pmatrix} -\sin\phi & \cos\phi & 0 \\ -\cos\phi & -\sin\phi & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \dot{\phi}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \omega_3 = \dot{\phi}$$

$$\text{or } \underline{\omega} = \dot{\phi}\underline{e}_3''$$
$$= \dot{\phi}\hat{\underline{e}}_3$$

Similarly for matrices K and L with the result.

$$\underline{\omega} = \dot{\psi}\underline{e}_3 + \dot{\theta}\underline{j} + \dot{\phi}\underline{e}_3$$

(Recall : can add angular velocities "linearly")

Now, $\underline{\omega} = \psi\underline{e}_3 + \dot{\theta}(\sin\psi\,\underline{e}_1 + \cos\psi\, e_3)$ (see handout)
$+\phi(-\sin\theta\cos\psi\,\underline{e}_1 + \sin\theta\sin\psi\,\underline{e}_3 + \cos\theta\,\underline{e}_3)$

$\hat{\underline{e}}_3 = H_{j3}\underline{e}_j$ etc.

The components of $\underline{\omega}$ are

$$\omega_1 = \dot{\theta}\sin\psi - \dot{\phi}\sin\theta\cos\psi$$
$$\omega_2 = \dot{\theta}\sin\psi - \dot{\phi}\sin\theta\sin\psi$$
$$\omega_3 = \dot{\psi} - \dot{\phi}\cos\theta$$

—/—

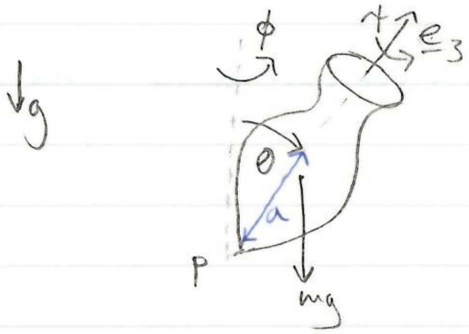
For rotation about fixed point P:

$$T = \frac{1}{2}\left(A\omega_1^2 + B\omega_2^2 + C\omega_3^2\right)$$

$$T = \frac{1}{2}A(\dot{\theta}\sin\psi - \dot{\phi}\sin\theta\cos\psi)^2 - \frac{1}{2}B(\dot{\theta}\cos\psi + \dot{\phi}\sin\theta\sin\psi)$$
$$+\frac{1}{2}C(\dot{\psi} + \dot{\phi}\cos\theta)^2 .$$

Consider a top in a gravitational field

Consider a top with axial symmetry ($A = B$) rotating about about fixed point P.



$\theta \equiv$ angle of nutation.
$\phi \equiv$ angle of precession.
$V = mga\cos\theta$.

$$L = \frac{1}{2}A(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2) + \frac{1}{2}C(\dot{\psi} + \dot{\phi}\cos\theta)^2 - mga\cos\theta .$$

Conservation Laws

ψ ignorable

$$\frac{\partial L}{\partial \dot{\psi}} = \text{const} .$$

i.e $\boxed{\dot{\psi} + \dot{\phi}\cos\theta = \text{const} = n}$ ← ang. mom. about body symm. axis

ϕ ignorable:

$$\frac{\partial L}{\partial \phi} = \text{const}$$

$\Rightarrow \boxed{A\dot{\phi}\sin^2\theta + Cn\cos\theta = \text{const} = h}$. ← ang. mom about vertical axis.

t is ignorable AND $T \equiv$ pure quardratic $\dot{\theta}, \dot{\psi}$ and $\dot{\phi}$.

$$\Rightarrow T + V = E = \text{const}.$$

$$\boxed{A\dot{\theta}^2 + A\dot{\phi}^2\sin^2\theta + Cn^2 + 2mga\cos\theta = 2E.}$$

$$\Rightarrow A\dot{\theta}^2 + A\dot{\phi}^2\sin^2\theta + 2mga\cos\theta = 2E - Cn^2.$$

Let $u = \cos\theta$, the above conservation laws can be written (try yourself)

$$\dot{\phi} = \frac{h - Cnu}{A(1-u^2)}$$

and $A\dot{u}^2 = F(u)$

$\Big\|$ $\dot{u} = -\sin\theta\,\dot{\theta}$.
$\dot{u}^2 = (1-u^2)\dot{\theta}^2$

$$F(u) = (2E - Cn^2 - 2mgau)(1-u^2) - \frac{(h - Cnu)^2}{A}.$$

Suppose initially $\theta = \cos^{-1}(u_1)$
and $\dot{\theta} = 0$ (i.e. no initial nutation)
h and n are fixed with
$n > 0$ and $0 < \frac{h}{Cn} < 1$.
This gives **more** interesting behaviour.

Consider the cubic $F(u)$

(i) cubic s.t. $F \rightarrow +\infty$ as $u \rightarrow +\infty$

(ii) $$F(-1) = -\frac{(h+Cn)^2}{A}$$

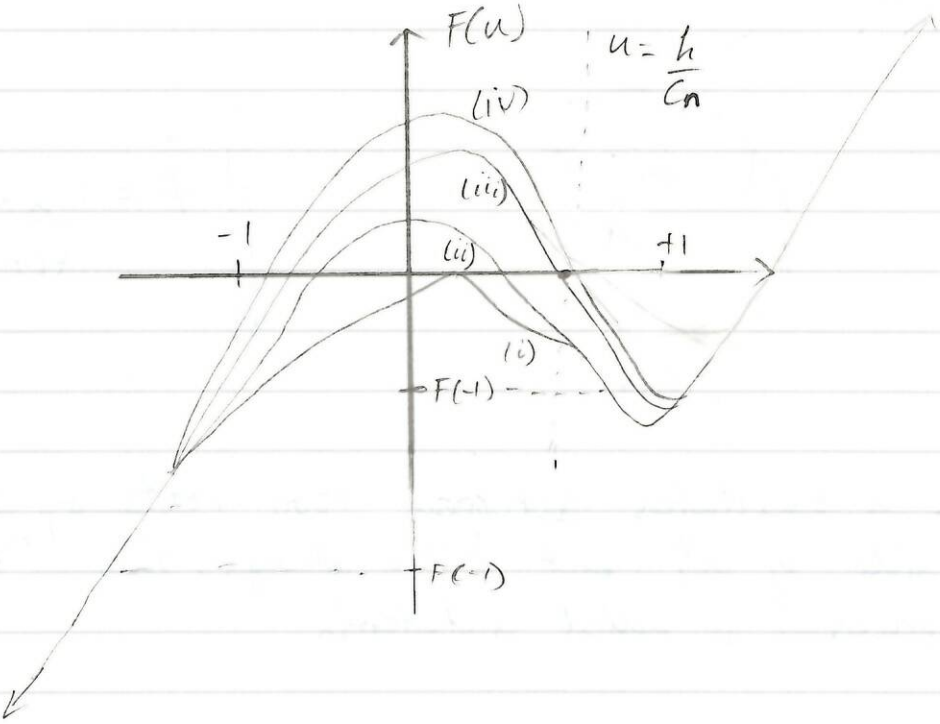
$$F(+1) = -\frac{(h-Cn)^2}{A}$$

(iii) **For** realistic motion:

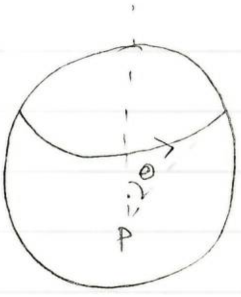
$$\begin{cases} F(u) \geqslant 0 \\ -1 \leqslant u \leqslant +1 \end{cases}$$

iv) $\dot{\phi} = 0$ when $u = \frac{h}{Cn}$.

$$\dot{\phi} = \frac{h - Cnu}{A(1-u^2)}$$



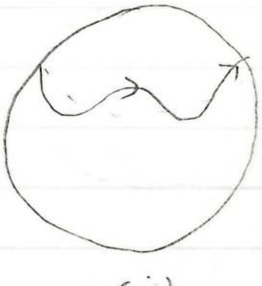
As the top moves its symmetry axis traces out a curve on the unit sphere centred at P.



(i)

i) $\exists$ a critical value $u = u_1^*$ s.t $F(u_1^*) = 0$ and $F'(u_1^*) = 0$ i.e u_1^* is a turning pt. of $F(u)$, and $u = u_1^*$ is the only allowable value of u.

The body axis precesses steadily $\dot{\phi} = \text{const}$. A circle is traced out on the unit sphere.

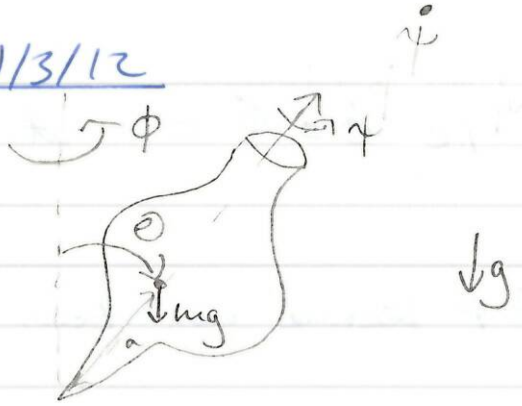


(ii)

$$\dot{\phi} = \frac{Cnu}{A(1-u^2)}$$

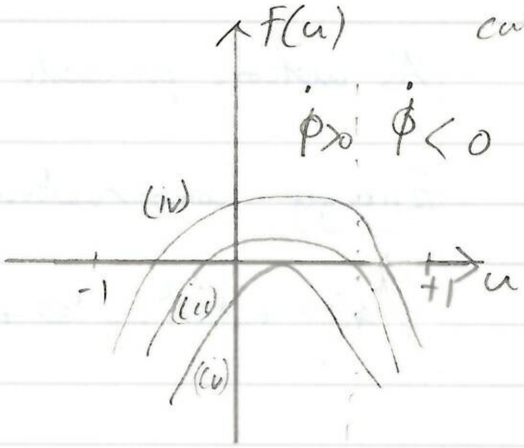
ii) Here u oscillates between two roots of $F(u)$ either side of u_1^*, but $\dot{\phi} > 0$ always. This "bobbing" as it precesses is called nutation.

21/3/12



$u = \cos\theta$

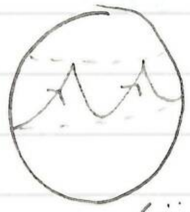
$A\dot{u}^2 = F(u)$ ← cubic



$$\dot\phi = \frac{h - Cnu}{A(1-u^2)}$$

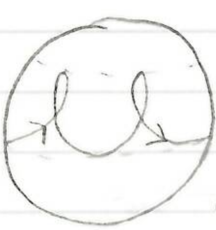
(i)

(ii)



(iii)

iii) Here $\dot\phi = 0$ when θ reaches its minimum $(u = h/Cn)$
But $\dot\phi$ never become negative. This is what happens when a top is relased from "rest". (see later example).



(iv)

iv) Now $\dot\phi < 0$ for part of the motion, and the trajectory loops back on itself.

Ex: Top released from "rest" i.e $\dot{\theta} = \dot{\phi} = 0$ at $t = 0$ (but $\dot{\psi} \neq 0$)

As motion proceeds $\dot{\theta}$ and $\dot{\phi}$ become non-zero.

Energy conservation demands

$$A\dot{\theta}^2 + A\dot{\phi}^2\sin^2\theta + 2mga\cos\theta = 2E - Cn^2$$
$$= \text{const}.$$

$\Rightarrow \cos\theta$ decreases since $A\dot{\theta}^2$ and $A\dot{\phi}^2\sin^2\theta \uparrow$
$\Rightarrow \theta \uparrow$

i.e the top falls just after $t = 0$.

Ex: A top is released with initial conds.
$\theta = \pi/3$, $\dot{\theta} = 0$, $\dot{\psi} = \left(\frac{3A}{C} - 1\right)\left(\frac{mga}{3A}\right)^{1/2}$ and
$\dot{\phi} = 2\left(\frac{mga}{3A}\right)^{\frac{1}{2}}$. Show that

$$2u = 1 + \tanh^2\left(\frac{\gamma t}{2}\right)$$

where $\gamma = \left(\frac{mga}{A}\right)^{1/2}$. What happens as $t \to \infty$?

Hint:

$$A\dot{\theta}^2 + A\dot{\phi}^2\sin^2\theta + 2mga\cos\theta = 2E - Cn^2$$
$$\dot{\psi} + \dot{\phi}\cos\theta = n$$
$$A\dot{\phi}\sin^2\theta + Cn\cos\theta = h$$
$$A\dot{u}^2 = F(u) = (2E - Cn^2 - 2mgau)(1-u^2) - \frac{(h - Cnu)^2}{A}$$

First find a o.d.e for $\dot{u}$ from $A\dot{u}^2 = F(u)$

At $t=0$, $n = \dot{\psi} + \dot{\phi}\cos\theta$

$$= \frac{3A}{C}\left(\frac{mga}{3A}\right)^{1/2}$$

$$h = A\dot{\phi}\sin^2\theta + Cn\cos\theta = 3A\left(\quad\right)^{\frac{1}{2}} = Cn$$

$$3E - Cn^2 = 2mga$$

$$\boxed{A\dot{u}^2 = mga(1-u)^2(2u-1)}$$

$$\Rightarrow \frac{du}{dt} = \gamma(1-u)(2u-1)^{1/2}$$

$$\int \frac{du}{(1-u)(2u-1)^{\frac{1}{2}}} = \int \gamma \, dt .$$

$$z^2 = 2u - 1$$

$$\int \frac{2}{1-z^2} \, dz = \gamma t + c$$

$$2\tanh^{-1} z = \gamma t + c$$

$$2\tanh^{-1}\left(\sqrt{2u-1}\right) = \gamma t + c$$

At $t = 0$, $\theta = \frac{\pi}{3}$ i.e $u = \frac{1}{2}$ $\Rightarrow c = 0$.

$$\Rightarrow \boxed{2u = \tanh^2\left(\frac{\gamma t}{2}\right) + 1}$$

He will return on 25 April.