# 7302 Analytical Dynamics Notes

Based on the 2013 spring lectures by Prof N R McDonald

MATH7302 – Analytical Dynamics

7 January 2013
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Maths 706.

Outline of course provided as a handout at the beginning of class.

Section 0.
SUMMATION CONVENTION.

The $i^{th}$ component of a vector $\underline{a}$ is denoted $a_i$, $i=1,2,3$ in three dimensions.

The $ij^{th}$ element of a matrix H is denoted $H_{ij}$ (i.e. the $i^{th}$ row and $j^{th}$ column).

Whenever an index $i, j, k$ etc is repeated in some term, a summation over 1, 2, and 3 is understood.

[Ex] Let $\underline{a} = a_1\underline{e}_1 + a_2\underline{e}_2 + a_3\underline{e}_3$, and let $\underline{b} = b_1\underline{e}_1 + b_2\underline{e}_2 + b_3\underline{e}_3$ where $\underline{e}_1, \underline{e}_2, \underline{e}_3$ is a suitable set of orthonormal basis vectors.

Then $\underline{a}\cdot\underline{b} = a_1b_1 + a_2b_2 + a_3b_3$. Express $\underline{a}\cdot\underline{b}$ with the summation convention.

soln. $\underline{a}\cdot\underline{b} = a_ib_i = a_kb_k$ (etc) //

[Ex] Explain the meaning of $c_i = H_{ij}a_j$.

soln. $c_i = H_{ij}a_j = H_{i1}a_1 + H_{i2}a_2 + H_{i3}a_3 \Rightarrow$ dot product of $i^{th}$ row of H with $\underline{a}$ $\Rightarrow$ $\underline{c} = H\underline{a}$ //

Note: the <u>free indices</u> (in this case i) on both sides must match!

[Ex] Recall that $\operatorname{div}\underline{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$. Express this with the convention.

soln. $\operatorname{div}\underline{u} = \frac{\partial u_i}{\partial x_i}$ (or $\frac{\partial u_\ell}{\partial x_\ell}$ etc...)

Note: other common notations for this include $\partial_\ell u_\ell$ or $u_{\ell,\ell}$.

Note: Warning! An expression such as $c_ja_jb_j$ (i.e. three repeated indices) has no meaning in this convention. An index may occur a maximum of two times in a given term.

i.e. something like $a_jb_j + c_j$ is acceptable: $a_jb_j + c_j = a_kb_k + c_j = a_1b_1 + a_2b_2 + a_3b_3 + c_j$

Moreover, an index which is not repeated in a term is known as a <u>free index</u>

[Ex] What are the free indices of the sum $a_iH_{ij}J_{jk}c_\ell$?

soln. Free indices are k and $\ell$ $\Rightarrow$ 2 free indices, indicates matrix.

[Ex] Express the Laplacian, $\nabla^2\phi = \frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_2^2} + \frac{\partial^2\phi}{\partial x_3^2}$ using the convention.

soln. $\nabla^2\phi = \frac{\partial^2\phi}{\partial x_i\partial x_i}$ // (not $\frac{\partial^2\phi}{\partial x_i^2}$, which makes no sense as there are no repeated indices).

Some special symbols:

(i) Kronecker delta: $\delta_{ij} = \begin{cases} 0 & i\neq j \\ 1 & i=j \end{cases}$ e.g. $\delta_{23}=0$, $\delta_{33}=1$.

By summation convention, $\delta_{ii} = 3$.

[Ex] What is $\delta_{ij}\delta_{ik}$?

soln. $\delta_{ij}\delta_{ik} = \delta_{1j}\delta_{1k} + \delta_{2j}\delta_{2k} + \delta_{3j}\delta_{3k} = \delta_{jk}$ // (if $j=1$, RHS=1 only if $k=1$. if $j=2$, RHS=1 only if $k=2$, ...)

In fact, for any $T_j$, we have $\delta_{ij}T_j = T_i$ (free indices match). We call this the <u>substitution property</u>.

[Ex] Simplify $\delta_{ij}\Gamma_{ik\ell m}$.

soln. $\delta_{ij}\Gamma_{ik\ell m} = \Gamma_{jk\ell m}$ //

(ii) Permutation symbol: $\varepsilon_{ijk}$, where $i, j, k$ take the values 1, 2, 3.

By definition, $\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = +1$ (even permutations) $\varepsilon_{132} = \varepsilon_{321} = \varepsilon_{213} = -1$ (odd permutations) It is 0 otherwise; e.g. $\varepsilon_{122} = \varepsilon_{333} = 0$.

Then we claim that $\boxed{(\underline{a}\times\underline{b})_k = \varepsilon_{ijk}a_ib_j}$, giving us a formulation for the vector cross product. We check this:

$\underline{a}\times\underline{b} = \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \underline{e}_1(a_2b_3 - a_3b_2) - \underline{e}_2(a_1b_3 - a_3b_1) + \underline{e}_3(a_1b_2 - a_2b_1)$. Using our shorthand, consider where $k=1$.

RHS $= \varepsilon_{ij1}a_ib_j = \varepsilon_{231}a_2b_3 + \varepsilon_{321}a_3b_2 + 0$ (other permutation terms) $= a_2b_3 - a_3b_2$ (verified).

We use these to prove some identities in vector calculus.

[Ex] Prove that $\operatorname{div}(\operatorname{curl}\underline{u}) = 0$.

soln. Note that $(\operatorname{curl}\underline{u})_k = (\underline{\nabla}\times\underline{u})_k = \varepsilon_{ijk}\frac{\partial}{\partial x_i}u_j$. Then $\operatorname{div}(\operatorname{curl}\underline{u}) = \frac{\partial}{\partial x_k}(\operatorname{curl}\underline{u})_k = \frac{\partial}{\partial x_k}\varepsilon_{ijk}\frac{\partial}{\partial x_i}u_j$ (free indices do not exist — match).

$\therefore \operatorname{div}(\operatorname{curl}\underline{u}) = \underset{\substack{\uparrow \\ \text{just a number,} \\ \text{rearrangement valid}}}{\varepsilon_{ijk}}\frac{\partial^2 u_j}{\partial x_k\partial x_i} = \underset{\text{re-labelling } \substack{i\to k \\ k\to i}}{\varepsilon_{kji}\frac{\partial^2 u_j}{\partial x_i\partial x_k}} = \varepsilon_{kji}\frac{\partial^2 u_j}{\partial x_k\partial x_i} = -\varepsilon_{ijk}\frac{\partial^2 u_j}{\partial x_k\partial x_i}$. Hence, $\varepsilon_{ijk}\frac{\partial^2 u_j}{\partial x_k\partial x_i} = -\varepsilon_{ijk}\frac{\partial^2 u_j}{\partial x_k\partial x_i} = 0 \Rightarrow \operatorname{div}(\operatorname{curl}\underline{u}) = 0$ // q.e.d.

$\boxed{\text{Ex}}$ Prove that $[\nabla\times(\nabla\phi)]_k = 0$.

Soln.

Note: the following is useful. $K_{ij}L_{jk} = M_{ik} \Rightarrow M_{ik} = (i^{th}$ row of $K)\cdot(k^{th}$ column of $L)$. Hence $KL = M$.

Observe that $K_{ij}L_{jk} = L_{jk}K_{ij} = M_{ik}$ as well. Think: what is $A_{kj}B_{ij}$ in terms of matrix multiplication? $A_{kj}B_{ij} = A_{kj}(B^T)_{ji} = (AB^T)_{ki}$.

Or, alternatively, $(A^T)_{jk}B_{ij} = B_{ij}(A^T)_{jk} = (BA^T)_{ik} = [(BA^T)^T]_{ki} = [(A^T)^T B^T]_{ki} = (AB^T)_{ki}$

Section 1

## FRAMES OF REFERENCE.

In order to describe a system, it is essential to introduce a coordinate system which labels (describes) the possible configuration of the system.

The number of coordinates required is called the number of degrees of freedom.

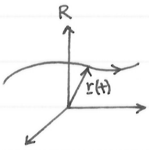
$\boxed{\text{Ex}}$ A single particle moving freely in 3D has 3 degrees of freedom. N particles in 3D has 3N degrees of freedom.

Two particles connected by a rigid rod in 3D has 5 degrees of freedom (three for the first particle, two more for the other – confined to surface of sphere).

How many degrees of freedom does a rigid body in 3D have?

Soln. It has 6 degrees of freedom.// Generalise the rigid body to 3 coplanar (not collinear) points.

### Newton's Laws.




The motion of a particle in space is represented by a curve $\underline{r} = \underline{r}(t)$, where $\underline{r}$ is the position vector from the origin of some Cartesian frame, R.

$\boxed{\text{Definition}}$ The velocity relative to R is $\underline{v} = (\dot{r}_1, \dot{r}_2, \dot{r}_3) = \dot{\underline{r}}$, and the acceleration relative to R is $\underline{a} = (\ddot{r}_1, \ddot{r}_2, \ddot{r}_3) = \ddot{\underline{r}}$

$\exists$ special frames of reference R s.t. Newton's 2nd law holds i.e. $\underline{F} = m\underline{a}$ or $F_i = m\ddot{r}_i$. We call these Newtonian frames.

A frame of reference is represented by an origin O, and $B = \{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$ which are a triad of unit vectors along coordinate axes s.t. $\underline{e}_i\cdot\underline{e}_j = \delta_{ij}$.

We also require that $\underline{e}_1\cdot(\underline{e}_2\times\underline{e}_3) = 1$ (right-handed system).

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Suppose we have two orthonormal triads $B = (\underline{e}_1, \underline{e}_2, \underline{e}_3)$ and $\hat{B} = (\hat{\underline{e}}_1, \hat{\underline{e}}_2, \hat{\underline{e}}_3)$. These are completely general – could be rotating, accelerating w.r.t. each other. Then $\underline{e}_i\cdot\underline{e}_j = \delta_{ij} = \hat{\underline{e}}_i\cdot\hat{\underline{e}}_j$. We define the matrix H with $H_{ij} = \underline{e}_i\cdot\hat{\underline{e}}_j$

$\boxed{\text{Definition}}$ H is the transition matrix from $\hat{B}$ to B.

$\boxed{\text{Proposition}}$ The transition matrix from B to $\hat{B}$ is given by $H^T$.

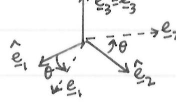
Proof – Let $H^*$ be the transition matrix from B to $\hat{B}$. Then $H^*_{ij} = \hat{\underline{e}}_i\cdot\underline{e}_j = \underline{e}_j\cdot\hat{\underline{e}}_i = H_{ji} = H^T_{ij} \Rightarrow H^* = H^T$// q.e.d.

The $\hat{B}$-components of $\underline{e}_i$ are $\underline{e}_i\cdot\hat{\underline{e}}_j$ for $j=1,2,3$ ($\because$ $\hat{\underline{e}}_j$ are unit vectors), or $H_{i1}, H_{i2}, H_{i3}$. Similarly, we claim $H_{1j}, H_{2j}, H_{3j}$ are B-components of $\hat{\underline{e}}_j$.

e.g. $\hat{\underline{e}}_j\cdot\underline{e}_1 = H_{1j}$ etc. i.e. for $i=1,2,3$, $\underline{e}_i = H_{i1}\hat{\underline{e}}_1 + H_{i2}\hat{\underline{e}}_2 + H_{i3}\hat{\underline{e}}_3$, and similarly $\hat{\underline{e}}_i = H_{1i}\underline{e}_1 + H_{2i}\underline{e}_2 + H_{3i}\underline{e}_3$. (or by summation convention, $\underline{e}_i = H_{ij}\hat{\underline{e}}_j$)

$\boxed{\text{Ex}}$ Take the system $\hat{\underline{e}}_1 = \cos\theta\,\underline{e}_1 - \sin\theta\,\underline{e}_2$, $\hat{\underline{e}}_2 = \sin\theta\,\underline{e}_1 + \cos\theta\,\underline{e}_2$, $\hat{\underline{e}}_3 = \underline{e}_3$. Find the transition matrix H from $\hat{B}$ to B, and interpret it geometrically.

Soln. $H_{ij} = \underline{e}_i\cdot\hat{\underline{e}}_j$. Then, $H = \begin{pmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}$ e.g. because $H_{11} = \underline{e}_1\cdot\hat{\underline{e}}_1 = \cos\theta\ldots$ Also, $\underline{e}_1 = H_{1j}\hat{\underline{e}}_j = \cos\theta\,\hat{\underline{e}}_1 + \sin\theta\,\hat{\underline{e}}_2$, $\underline{e}_2 = -\sin\theta\,\hat{\underline{e}}_1 + \cos\theta\,\hat{\underline{e}}_2$, $\underline{e}_3 = \hat{\underline{e}}_3$




We draw this system as a model, claiming WLOG that the origins are coincident. Then we have $\underline{e}_3 = \hat{\underline{e}}_3$.

The coordinate axes are related by a rotation in the $\hat{\underline{e}}_1$-$\hat{\underline{e}}_2$ plane through an angle of $\theta$ from $\hat{\underline{e}}_i$ to $\underline{e}_i$, anti-clockwise.//

We know that $\delta_{jk} = \underline{e}_j\cdot\underline{e}_k = (H_{jl}\hat{\underline{e}}_l)\cdot(H_{km}\hat{\underline{e}}_m)$ (use dummy indices $l, m$; but make sure they are distinct!) $= H_{jl}H_{km}\hat{\underline{e}}_l\cdot\hat{\underline{e}}_m = H_{jl}H_{km}\delta_{lm}$

Use the substitution property of the Kronecker delta, then $\delta_{jk} = H_{jl}H_{km}\delta_{lm} = H_{jm}H_{km} = H_{jm}H^T_{mk} = (HH^T)_{jk}$ $\therefore$ $I = HH^T$. Likewise, we substitute the other dummy variable: $\delta_{jk} = H_{jl}H_{km}\delta_{lm} = H_{jl}H_{kl}$ $\Rightarrow I = HH^T = H^TH \Rightarrow H$ is orthogonal//.

Furthermore, we differentiate $HH^T = I$ w.r.t. time $\Rightarrow \dot{H}H^T + H(\dot{H}^T) = 0 \Rightarrow \dot{H}H^T + H\dot{H}^T = 0 \Rightarrow \dot{H}H^T = -H\dot{H}^T = -(\dot{H}H^T)^T$. Define $\Omega = \dot{H}H^T$, and so $\Omega = -\Omega^T$.

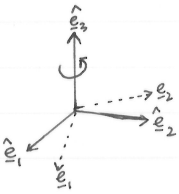
i.e. $\Omega$ is skew-symmetric (also called antisymmetric). Or, in terms of elements, $\Omega_{ij} = \dot{H}_{ik}H^T_{kj} = \dot{H}_{ik}H_{jk}$. Since $\Omega$ is skew-symmetric, WLOG, we let $\Omega = \begin{pmatrix}0 & w_3 & -w_2\\ -w_3 & 0 & w_1\\ w_2 & -w_1 & 0\end{pmatrix}$ – diagonal entries must be 0. Hence, $\Omega_{jk} = \varepsilon_{ijk}w_i$

**Definition** The <u>angular velocity of B relative to $\hat{B}$</u> is the vector $\underline{w} = w_i \underline{e}_i$

**Ex** Recall that in our previous example, $H = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Find angular velocity of B relative to $\hat{B}$.

<u>Soln</u>. $\dot{H} = \dot{\theta}\begin{pmatrix} -\sin\theta & \cos\theta & 0 \\ -\cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and $H^T = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then $\Omega = \dot{H}H^T = \dot{\theta}\begin{pmatrix} -\sin\theta & \cos\theta & 0 \\ -\cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \dot{\theta}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Thus, $w_3 = \dot{\theta}$, $w_1 = w_2 = 0$ i.e. angular velocity of B relative to $\hat{B}$ is $\underline{w} = \dot{\theta}\,\underline{e}_3$ //

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Recall: if $B = (\underline{e}_i)$, $\hat{B} = (\hat{\underline{e}}_i)$ are right-handed orthonormal triads, $H_{ij} = \underline{e}_i \cdot \hat{\underline{e}}_j$, we call H the transition matrix of $\hat{B}$ to B. Why?

Let $\underline{x} \in \mathbb{R}^3$. Then $x_i = \underline{x}\cdot e_i = \underline{x}\cdot H_{ij}\hat{\underline{e}}_j = H_{ij}\,\underline{x}\cdot\hat{\underline{e}}_j = H_{ij}\hat{x}_j$ ($\hat{x}_j$ is $\hat{\underline{e}}_j$ component of $\underline{x}$) $\Rightarrow \underline{x} = H\hat{\underline{x}} \Rightarrow \hat{\underline{x}} = H^T\underline{x}$. (since $H^TH = I$).

**Definition** The time-derivative of vector $\underline{x} = x_i\underline{e}_i$ with respect to $B = (\underline{e}_1, \underline{e}_2, \underline{e}_3)$ is the vector $D\underline{x} = \dot{x}_i\underline{e}_i$.

**Theorem** (Coriolis theorem).

The time derivatives $D\underline{x}$ and $\hat{D}\underline{x}$ of $\underline{x}$, w.r.t. B and $\hat{B}$ respectively, are related by $\hat{D}\underline{x} = D\underline{x} + (\underline{w}\times\underline{x})$.

$\underline{e}_j = H_{ji}\hat{\underline{e}}_i$ $\quad$ $\hat{\underline{e}}_i = H_{ki}\underline{e}_k$

Proof – $\hat{D}\underline{x} = \dot{\hat{x}}_i\hat{\underline{e}}_i = (H_{ji}\dot{x}_j + \dot{H}_{ji}x_j)\hat{\underline{e}}_i$ ($\because \hat{x}_i = H_{ji}x_j$, apply product rule) $= H_{ji}\dot{x}_j\hat{\underline{e}}_i + \dot{H}_{ji}x_j\hat{\underline{e}}_i = \dot{x}_j\underline{e}_j + \dot{H}_{ji}H_{ki}x_j\underline{e}_k$

$= D\underline{x} + \dot{H}_{ji}H^T_{ik}x_j\underline{e}_k = D\underline{x} + (\dot{H}H^T)_{jk}x_j\underline{e}_k = D\underline{x} + \Omega_{jk}x_j\underline{e}_k = D\underline{x} + \varepsilon_{ijk}w_ix_j\underline{e}_k = D\underline{x} + (\underline{w}\times\underline{x})_k e_k = D\underline{x} + (\underline{w}\times\underline{x})$ // q.e.d.

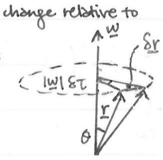
**Corollary** If the angular velocity of B relative to $\hat{B}$ is $\underline{w}$, then the angular velocity of $\hat{B}$ relative to B is $-\underline{w}$.

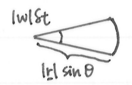
Proof – $\hat{D}\underline{x} = D\underline{x} + \underline{w}\times\underline{x} \Rightarrow D\underline{x} = \hat{D}\underline{x} + (-w)\times\underline{x}$. Hence, angular velocity of $\hat{B}$ relative to B is $-\underline{w}$.

**Corollary** If B has angular velocity $\underline{w}$ relative to $\hat{B}$, and $\hat{B}$ has angular velocity $\hat{\underline{w}}$ relative to $B'$; then B has angular velocity $\underline{w}+\hat{\underline{w}}$ relative to $B'$!

Proof – By linearity of angular velocity. $\hat{D}\underline{x} = D\underline{x} + \underline{w}\times\underline{x}$, $D'\underline{x} = \hat{D}\underline{x} + \hat{\underline{w}}\times\hat{\underline{x}} = D\underline{x} + (\underline{w}+\hat{\underline{w}})\times\underline{x} \Rightarrow$ B has angular velocity $\underline{w}+\hat{\underline{w}}$ relative to $B'$ // q.e.d

Physical interpretation of $\underline{w}\times\underline{x}$ term: let $\underline{r}$ be a position vector. Suppose time rate of change of $\underline{r}$ relative to B is $D\underline{r}$, and $\hat{D}\underline{r}$ the time rate of change relative to fixed axes $\hat{B}$. B has angular velocity $\underline{w}$ relative to $\hat{B}$. If $D\underline{r} = 0$, $\underline{r}$ is constant in B. So from $\hat{B}$'s point of view, with coincident origins: In time $\delta t$, the particle sweeps out an angle of $|\underline{w}|\,\delta t$. Distance of $\underline{r}$ from vertical axis is a constant $|\underline{r}|\sin\theta$, so if $\delta\underline{r}$ is the arc swept out by $\underline{r}$, then $|\delta\underline{r}| = |\underline{r}|\sin\theta\,|\underline{w}|\,\delta t \Rightarrow |\delta\underline{r}| = |\underline{w}\times\underline{r}|\,\delta t$. By right-hand rule, $\delta\underline{r}$ is in direction of $\underline{w}\times\underline{r}$, so this gives $\delta\underline{r} = \underline{w}\times\underline{r}\,\delta t$. Dividing by $\delta t$, and as $\delta t \to 0$, $\frac{\delta\underline{r}}{\delta t} = \underline{w}\times\underline{r} \Rightarrow \hat{D}\underline{r} = \lim_{\delta t\to 0}\frac{\delta\underline{r}}{\delta t} = \underline{w}\times\underline{r}$.

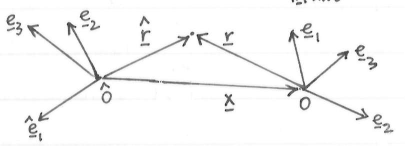



Consider motion of a particle in 2 frames of reference. Let $\underline{r}$ and $\hat{\underline{r}}$ be the positions of the particle w.r.t R and $\hat{R}$.

Also, let $\underline{X}$ be a vector connecting the origin of $\hat{B}$ to the origin of B, s.t. $\hat{\underline{r}} = \underline{r} + \underline{X}$.

Acceleration relative to R is $\underline{a} = D^2\underline{r}$, and relative to $\hat{R}$ is $\hat{\underline{a}} = \hat{D}^2\hat{\underline{r}}$. i.e. $\hat{\underline{a}} = \hat{D}^2(\underline{r}+\underline{X})$




$\therefore\ \hat{\underline{a}} = \hat{D}^2\underline{r} + \hat{D}^2\underline{X} = \hat{D}(\hat{D}\underline{r}) + \hat{D}^2\underline{X} = \hat{D}(D\underline{r} + \underline{w}\times\underline{r}) + \hat{D}^2\underline{X} = D(D\underline{r} + \underline{w}\times\underline{r}) + \underline{w}\times(D\underline{r} + \underline{w}\times\underline{r}) + \hat{D}^2\underline{X} = D^2\underline{r} + D(\underline{w}\times\underline{r}) + \underline{w}\times(D\underline{r} + \underline{w}\times\underline{r}) + \hat{D}^2\underline{X}$

$= \underline{a} + (D\underline{w})\times\underline{r} + \underline{w}\times(D\underline{r}) + \underline{w}\times(D\underline{r}) + \underline{w}\times(\underline{w}\times\underline{r}) + \hat{D}^2\underline{X} = \underline{a} + (D\underline{w})\times\underline{r} + 2\underline{w}\times D\underline{r} + \underline{w}\times(\underline{w}\times\underline{r}) + \hat{D}^2\underline{X} = \underline{a} + (D\underline{w})\times\underline{r} + 2\underline{w}\times(D\underline{r}) + \underline{w}\times(\underline{w}\times\underline{r}) + \underline{A}$,

where $\underline{A}$ is the acceleration of O relative to frame $\hat{R}$. This is a completely general case: $\boxed{\hat{\underline{a}} = \underline{a} + (D\underline{w})\times\underline{r} + 2\underline{w}\times(D\underline{r}) + \underline{w}\times(\underline{w}\times\underline{r}) + \underline{A}}$

Special case: $\underline{w} = 0$ and $\underline{A} = 0 \Rightarrow \hat{\underline{a}} = \underline{a}$. We say that R and $\hat{R}$ are <u>equivalent</u>. A special class of equivalent frames are equivalent <u>inertial frames</u>, if F=ma works in both.

Rotational effects:

Now let $\hat{R}$ be an inertial frame (Newton's laws hold), and R be some other non-inertial frame. Then $m\hat{\underline{a}} = \underline{F} \Rightarrow$ by our substitution,

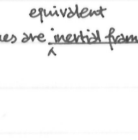
$m(D^2\underline{r} + (D\underline{w})\times\underline{r} + 2\underline{w}\times D\underline{r} + \underline{w}\times(\underline{w}\times\underline{r}) + \underline{A}) = \underline{F}$; or from R's point of view, $\boxed{m\underline{a} = m(D^2\underline{r}) = \underline{F} \overset{①}{-} m(D\underline{w})\times\underline{r} \overset{②}{-} 2m\underline{w}\times D\underline{r} \overset{③}{-} m\underline{w}\times(\underline{w}\times\underline{r}) \overset{④}{-} m\underline{A}}$

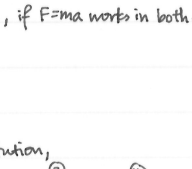
which is the case as if R was inertial, and 4 "fictitious" external forces act on it due to state of motion. We label these additional forces $F_1, F_2, F_3, F_4$.

These forces $F_i$ are not "real": they are owing to the state of motion (i.e. $\underline{w}, \underline{A}$) of R relative to $\hat{R}$; they lack a physical counterpart.

① $\underline{F}_1 = -m(D\underline{w})\times\underline{r}$ arises from the angular acceleration of R i.e. $D\underline{w}$. Not typically important, e.g. earth rotates at near-constant velocity.

② $\underline{F}_2 = -2m\,\underline{w}\times D\underline{r}$ is the <u>Coriolis force</u>. It is velocity-dependent (particle must be moving for force to have effect, depends on $D\underline{r}$), and orthogonal to both $\underline{w}$ and $D\underline{r}$. To a large extent, it determines the motion of the atmosphere and oceans.



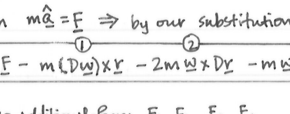




On the earth, we assume north-south pole is axis of rotation. Earth spins west-to-east.

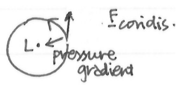
Imagine locally, standing in London (northern hemisphere). $\underline{w}$ points diagonally out of ground, towards N-pole.

We split it into two components. Consider effect on $\underline{w}_r$: $\underline{F}_C = -2m\,\underline{w}_r\times\underline{v}$

What about $\underline{w}_H$? No components walking N-S. Walking any other direction on surface, force is normal to ground; but this has negligible magnitude compared to gravity.




In the atmosphere, $\underline{w}_v$ causes the anti-clockwise circulation (winds) about low pressure in the northern hemisphere.




How about in Sydney?

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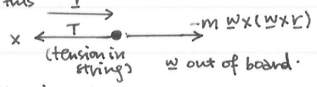
③ $\underline{F}_3 = -m\,\underline{w}\times(\underline{w}\times\underline{r})$ is the <u>centrifugal force</u>. It is not velocity dependent; function of position.

e.g. "Hammer throw" event in athletics: heavy ball on a string. In a rotating frame (i.e. moving with the hammer thrower), there is a tension in the string (need to exert a force to hold on with it). However in this frame, the ball has no net movement. There must be a balancing force, which is the centrifugal force. However, from an inertial point of view (observer), $\underline{w}=\underline{0}$, the ball is accelerating inwards, supplied by a centripetal force, the tension ⇒ yielding circular motion. (no balance of force!). There is no longer a centrifugal force.




Note: We only have centrifugal forces in rotating frames!

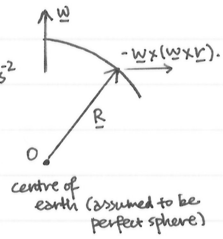
e.g. centrifugal force on the rotating earth. Due to the rotation of the earth, we measure <u>apparent gravity</u>, $g^*$.

Then we have $g^* = g - \underline{w}\times(\underline{w}\times\underline{R})$. $|\underline{w}| = 7.3\times10^{-5}\,s^{-1} = \frac{2\pi}{24\,h}$. $|\underline{R}| = 6320\,km \Rightarrow |\underline{w}|^2|\underline{R}| \simeq 34\times10^{-3}\,ms^{-2}$

i.e. $|\underline{w}\times(\underline{w}\times\underline{r})| \simeq 34\,ms^{-2} << g \simeq 9.81\,ms^{-2}$. At the pole there is no centrifugal force, i.e. $g^* = g$.

At the equator there is a maximum of this force: $|g^*| = |g| - |\underline{w}|^2|\underline{R}|$. We expect $\Delta g = g^*_{pole} - g^*_{equator} \simeq 34\times10^{-3}\,ms^{-2}$.

In fact, physicists have done the experiment and obtained $\Delta g \simeq 52\times10^{-3}\,m$ as earth is oblate, as earth is not rigid and particles near the equator are flung out.

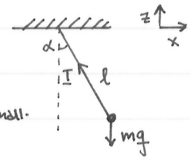



④ $\underline{F}_4 = -m\underline{A}$, $\underline{A} = D^2\underline{x}$, acceleration of one frame relative to the other. This has the same effect as a uniform gravitational field.

For instance, imagine an inertial fixed observer watching a man and piece of chalk falling under gravity. Relative to observer, chalk is accelerating downwards, relative to the falling man's frame of reference, chalk is stationary.

To demonstrate the effect of the Coriolis force, we analyse the example of Foucault's pendulum.

<u>Foucault's pendulum.</u>




As an aside: we first consider a simple pendulum oscillating in xz-plane, set up as in diagram on right.

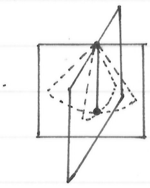
Resolve forces into components: $m\ddot{x} = -T\sin\alpha$, $m\ddot{z} = T\cos\alpha - mg$. Assume small amplitude oscillations — $\cos\alpha\simeq1$, $\sin\alpha = \frac{x}{\ell}$. $\ddot{z}$ small.

$\Rightarrow T \simeq mg \Rightarrow \ddot{x} = -\frac{g}{\ell}x \Rightarrow x = A\cos\sqrt{\frac{g}{\ell}}t + B\sin\sqrt{\frac{g}{\ell}}t$.

Returning now to Foucault's (Leon Foucault's 1851 demonstration) pendulum — a simple pendulum which is free to oscillate in any plane vertically.

It is required to oscillate for long periods (overcoming friction and air resistance).

Over time, the plane of oscillation varies due to the rotation of the earth. We will analyse its motion in a frame of reference rotating with the earth, with deviations accumulated over time.



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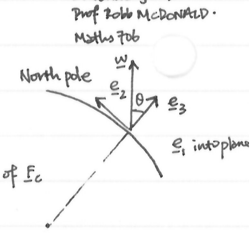
In a frame of reference rotating with the earth: $m\ddot{\underline{r}} = m\underline{g} + \underline{T} - 2m\,\underline{w}\times\dot{\underline{r}}$ ⊛ where $\underline{T}$ is the tension in the string. (rather than $g^*$, as $g^*\simeq g$ and we can approximate.)

We choose coordinate axes s.t. $\underline{e}_1$ is to the east, $\underline{e}_2$ is to the north, $\underline{e}_3$ is upwards.




Take $\theta$ to be the colatitude, $0\le\theta\le\pi$. Thus in this coordinate system $\underline{w} = (0, w\sin\theta, w\cos\theta)$. Let $\dot{\underline{r}} = (\dot{x}, \dot{y}, \dot{z})$.

Then the Coriolis force is $-2m\,\underline{w}\times\dot{\underline{r}} = \underline{F}_c$, $\underline{F}_c = 2mw(\dot{y}\cos\theta - \dot{z}\sin\theta, -\dot{x}\cos\theta, \dot{x}\sin\theta)$. Note that the vertical ($\underline{e}_3$) component of $\underline{F}_c$ is extremely small compared to gravity, since $2w\dot{x}\simeq 1\times10^{-3}\,ms^{-2}$, so we can ignore it, and $\underline{F}_c = 2mw(\dot{y}\cos\theta, -\dot{x}\cos\theta, 0)$.

Also taking $\ddot{z}\approx0$, then ⊛ becomes the system $\ddot{x} = -\frac{g}{\ell}x + 2w\dot{y}\cos\theta$ — ①, $\ddot{y} = -\frac{g}{\ell}y - 2w\dot{x}\cos\theta$ — ② where $T_x = -\frac{mg}{\ell}x$, $T_y = -\frac{mg}{\ell}y$.

This gives a system of linear coupled ODEs. (Note: the ordinary pendulum is governed by $\ddot{x} = -\frac{g}{\ell}x$, $\ddot{y} = -\frac{g}{\ell}y$).

Put $\phi = x + iy$. Then ①, ② ⇒ $\boxed{\ddot{\phi} + 2i\Omega\dot{\phi} + w_0^2\phi = 0}$ where $\Omega = w\cos\theta$, $w_0 = \frac{g}{\ell}$; which is a linear constant coefficient ODE. Try $\phi = e^{\lambda t}$.

Solve quadratic auxillary equation $\lambda^2 + 2i\Omega\lambda + w_0^2 = 0 \Rightarrow \lambda = -i\Omega \pm iw_1$ where $w_1^2 = w_0^2 + \Omega^2 \Rightarrow w_1 = (w_0^2+\Omega^2)^{\frac{1}{2}} \simeq w_0 + \frac{\Omega^2}{w_0} + \cdots$ (for $w_0 >> \Omega$).

Hence, $\lambda \simeq -i\Omega \pm iw_0$ (up to $\Omega$ order). ⇒ $\phi = e^{-i\Omega t}(A\cos w_0t + B\sin w_0t)$ where $A, B\in\mathbb{C}$.

For the general cases of this, refer to the handout on Foucault's pendulum. For simplicity, here we choose $B=0$, $A\in\mathbb{R}$. ⇒ $\boxed{x = A\cos\Omega t\cos w_0t \quad y = -A\sin\Omega t\cos w_0t \quad (\Omega << w_0).}$

$$y = \underbrace{-A\sin\Omega t}_{\text{slowly changing oscillations. (modulation-like amplitude)}}\ \underbrace{\cos w_0t}_{\text{rapid oscillation}}$$

Initially $x \approx A\cos w_0t$, $y\approx0$, at $t=0$. Here, oscillations occur in the x-direction, since $\cos\Omega t\approx1$, $\sin\Omega t\approx0$.

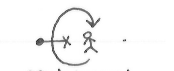
As t increases, $\cos\Omega t$ decreases and $\sin\Omega t$ increases. Meanwhile, the pendulum continues to oscillate, owing to the rapid $\cos w_0t$ term. Thus, the amplitude of oscillation decreases in the x-direction but increases in the y-direction.

Net result: Plane of oscillation appears to rotate as t increases. (provided there is an initial velocity provided).

The solution represents oscillations of amplitude $A = \sqrt{A^2\cos^2\Omega t + A^2\sin^2\Omega t}$ in a plane rotating with angular speed $\Omega = w\cos\theta$

At $\theta=0$ (north pole), $T = \frac{2\pi}{\Omega} = \frac{2\pi}{w} = 24$ hours. At $\theta = \frac{\pi}{4}$, $T = \frac{2\pi}{\Omega} = \frac{2\pi}{w\cos\frac{\pi}{4}} \simeq 34$ hours. At $\theta = \frac{\pi}{2}$ (equator), $T = \infty$.

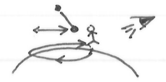
However, how would an inertial observer explain this phenomenon? For an observer not attached to the earth, looking at a pendulum at the north pole.



pendulum only rocks back and forth

For such an oscillation, there is no Coriolis force $\Rightarrow$ plane of oscillation is invariant.

To an inertial observer, someone standing at the north pole rotates once every 24 hours accounting for the relative motion between pendulum and non-inertial observer.

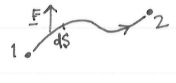


Section 2

SYSTEMS OF PARTICLES.

Quick revision: We look at some laws/ definitions for a single particle:

1. Newton's $2^{nd}$ law: $\mathbf{F} = \frac{d\mathbf{p}}{dt}$ where $\mathbf{p} = m\mathbf{v}$ is the <u>linear momentum</u>. If $\mathbf{F} = \mathbf{0}$, $\mathbf{p}$ is constant.
2. The <u>angular momentum</u> $\mathbf{L}$ about O is $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ where $\mathbf{r}$ is the position vector of the particle from O.
   The <u>torque (moment)</u> about O is $\mathbf{N} = \mathbf{r} \times \mathbf{F}$. $\Rightarrow$ $\mathbf{r} \times$ (Newton's $2^{nd}$ law) gives $\mathbf{r}\times\mathbf{F} = \mathbf{r} \times \frac{d\mathbf{p}}{dt} = \mathbf{N}$. But $\mathbf{r} \times \frac{d\mathbf{p}}{dt} = \mathbf{r} \times \frac{d}{dt}(m\mathbf{v}) = \frac{d}{dt}(\mathbf{r} \times m\mathbf{v})$,
   since $\mathbf{v} \parallel m\mathbf{v} \Rightarrow \frac{d\mathbf{r}}{dt} \times m\mathbf{v} = \mathbf{0}$. $\Rightarrow$ $\mathbf{N} = \frac{d\mathbf{L}}{dt}$; or torque is equal to rate of change of angular momentum. If $\mathbf{N} = 0$, $\mathbf{L}$ is conserved (constant).
3. <u>Work</u> done by a force $\mathbf{F}$ upon a particle moving from position 1 to position 2 is $W_{12} = \int_1^2 \mathbf{F} \cdot d\mathbf{s}$



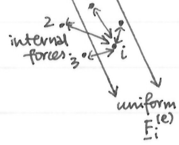

   If $m$ is constant, then $\int \mathbf{F} \cdot d\mathbf{s} = m \int \frac{d\mathbf{v}}{dt} \cdot \mathbf{v}\, dt$ ($\because \frac{d\mathbf{s}}{dt} = \mathbf{v}$) $= m \int \frac{d}{dt}(\frac{1}{2}|\mathbf{v}|^2)\, dt$ i.e. $W_{12} = \frac{1}{2} m (v_2^2 - v_1^2)$ where
   $v_2^2 = (\mathbf{v} \cdot \mathbf{v})_2$. Then $W_{12} = T_2 - T_1$ where $T = \frac{1}{2}mv^2$ is the <u>kinetic energy</u>.
   Suppose $\mathbf{F}$ is such that $\oint \mathbf{F} \cdot d\mathbf{s} = 0$; then $\mathbf{F}$ is known as a <u>conservative force</u>. (Friction is an example of a non-conservative force).
   By Stoke's theorem, $\iint (\nabla \times \mathbf{F}) \cdot d\mathbf{A} = 0. \Rightarrow \nabla \times \mathbf{F} = 0$ since our closed curve is arbitrary. $\Rightarrow$ $\mathbf{F} = -\nabla V$ where $V$ is a scalar function called the <u>potential</u>.
   Thus, work done is $W_{12} = -\int_1^2 \nabla V \cdot d\mathbf{s} \overset{\text{exact integral}}{=} -[V_2 - V_1] = V_1 - V_2$. Hence, $W_{12} = T_2 - T_1 = V_1 - V_2 \Rightarrow T_1 + V_1 = T_2 + V_2$
   i.e. If forces are conservative, then total energy in form $T+V$ is constant.

<u>Many particles</u>

We distinguish between <u>external forces</u> acting on the particles due to sources outside the system; and <u>internal forces</u> on some particle $i$ due to all other particles.

Equation of motion for particle $i$: $\boxed{\dot{\mathbf{p}}_i = \sum_j \mathbf{F}_{ji} + \mathbf{F}_i^{(e)}}$, where $\mathbf{F}_i^{(e)}$ is the force on $i^{th}$ particle due to external forces,

and $\mathbf{F}_{ji}$ is the internal force on $i^{th}$ particle due to $j^{th}$ particle. Note that $\mathbf{F}_{ii} = \mathbf{0}$.

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For N particles, we sum our vector equation over them to get the following: $\sum_i \dot{\mathbf{p}}_i = \frac{d^2}{dt^2} \sum_i m_i \mathbf{r}_i = \sum_i \mathbf{F}_i^{(e)} + \sum_{i,j} \mathbf{F}_{ji}$

Assume Newton's $3^{rd}$ Law: $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$, and so $\sum_{\substack{i,j \\ (i \neq j)}} \mathbf{F}_{ji} \overset{\text{reindex}}{=} \sum_{j,i} \mathbf{F}_{ij} \overset{\text{N3L}}{=} -\sum_{j,i} \mathbf{F}_{ji} \overset{\text{swap order}}{=} -\sum_{i,j} \mathbf{F}_{ji} = \mathbf{0}$. We lose some information (internal forces), but can see what happens systemwide.

We define $\frac{\sum_i m_i \mathbf{r}_i}{\sum m_i} = \mathbf{R}$, or $\frac{\sum_i m_i \mathbf{r}_i}{M}$ where $M = \sum_i m_i$ is the total system mass. Then $\mathbf{R}$ is the <u>centre of mass</u> of the system.

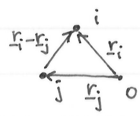
Hence, $M \frac{d^2\mathbf{R}}{dt^2} = \frac{d^2}{dt^2} \sum_i m_i \mathbf{r}_i$, and $M \frac{d^2 \mathbf{R}}{dt^2} = \sum_i \mathbf{F}_i^{(e)} = \mathbf{F}^{(e)}$, i.e. the centre of mass (CM) moves as if the total external force acts on it.

Note: $\frac{d\mathbf{p}}{dt}$ is the total linear momentum of the system. If $\mathbf{F}^{(e)} = 0$, then $\mathbf{p}$ = constant.

The <u>total angular momentum</u> $\mathbf{L} = \sum_i \mathbf{r}_i \times \mathbf{p}_i$ of the system has time-derivative $\frac{d}{dt}\mathbf{L} = \frac{d}{dt} \sum_i \mathbf{r}_i \times \mathbf{p}_i = \sum_i (\mathbf{v}_i \times \mathbf{p}_i) + (\mathbf{r}_i \times \dot{\mathbf{p}}_i) = \sum_i \mathbf{r}_i \times \dot{\mathbf{p}}_i = \sum_i \mathbf{r}_i \times \mathbf{F}_i$

Then $\frac{d}{dt}\mathbf{L} = \sum_i \mathbf{r}_i \times (\mathbf{F}_i^{(e)} + \sum_j \mathbf{F}_{ji}) = \sum_i \mathbf{r}_i \times \mathbf{F}_i^{(e)} + \sum_{i,j} \mathbf{r}_i \times \mathbf{F}_{ji}$. Consider the last term: $\sum_{i,j} \mathbf{r}_i \times \mathbf{F}_{ji}$. This involves pairs of terms of form $\mathbf{r}_i \times \mathbf{F}_{ji} + \mathbf{r}_j \times \mathbf{F}_{ij}$.

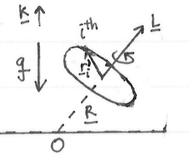
Then $\mathbf{r}_i \times \mathbf{F}_{ji} + \mathbf{r}_j \times \mathbf{F}_{ij} = (\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ji}$ by Newton's $3^{rd}$ law. We assume the strong form of Newton's $3^{rd}$ law (does not always hold): force acting between particles is assumed to act along the line joining them. Then $\mathbf{r}_i - \mathbf{r}_j \parallel \mathbf{F}_{ji}$ and $(\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ji} = 0 = \mathbf{r}_i \times \mathbf{F}_{ji} + \mathbf{r}_j + \mathbf{F}_{ij}$. Each pair sums to 0, hence $\sum_{i,j} \mathbf{r}_i \times \mathbf{F}_{ji} = 0$, and $\frac{d}{dt}\mathbf{L} = \sum_i \mathbf{r}_i \times \mathbf{F}_i^{(e)} = \mathbf{N}^{(e)}$, where $\mathbf{N}^{(e)}$ is the <u>total external torque</u>. $\mathbf{N}^{(e)} = \sum_i \mathbf{N}_i^{(e)} = \sum_i \mathbf{r}_i \times \mathbf{F}_i^{(e)}$.




The total angular momentum $\mathbf{L}$ = const if the total external torque is zero.

We apply some of these ideas to the example of a gyroscope.

<u>Gyroscope</u>

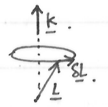



This may be considered as a system with many particles about some (rotation) axis. Let $\mathbf{r}_i = \mathbf{R} + \mathbf{r}_i'$ where $\mathbf{R}$ is the centre of mass and $\mathbf{r}_i$ the position vector of the $i^{th}$ particle. We use $\frac{d\mathbf{L}}{dt} = \mathbf{N}^{(e)} = \sum_i \mathbf{r}_i \times \mathbf{F}_i^{(e)} = \sum_i (\mathbf{r}_i' + \mathbf{R}) \times (-m_i g \mathbf{k}) = \sum_i (\mathbf{r}_i' \times (-m_i g \mathbf{k})) + \mathbf{R} \times (-m_i g \mathbf{k})$

Then $\frac{d\mathbf{L}}{dt} = \mathbf{R} \times \mathbf{F}^{(e)} - g(\sum_i \mathbf{r}_i' m_i) \times \mathbf{k}$. However, $\sum_i m_i \mathbf{r}_i' = \sum_i m_i (\mathbf{r}_i - \mathbf{R}) = \sum_i m_i \mathbf{r}_i - (\sum_i m_i)\mathbf{R} = M\mathbf{R} - M\mathbf{R} = 0$, so finally,

$\frac{d\mathbf{L}}{dt} = \mathbf{N}^{(e)} = \mathbf{R} \times \mathbf{F}^{(e)}$. Here, $\mathbf{N}^{(e)}$ acts into the page. $\Rightarrow$ $\delta\mathbf{L} = \mathbf{N}^{(e)} \delta t$.

Hence, $\delta \mathbf{L}$ is also into the board $\Rightarrow$ $\mathbf{L}$ sweeps out a circle; i.e. $\mathbf{L}$ precesses around the vertical.

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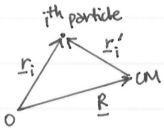
there is a subtlety associated with this problem: $\underline{L}$ does not point exactly in the direction of the symmetric axis: this is because the precession of $\underline{L}$ generates its own angular momentum in turn, affecting the system. We return to this problem later, using Lagrangian dynamics.

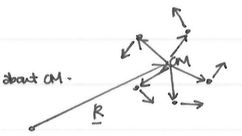



Recall that $\underline{L} = \sum_i \underline{r}_i \times \underline{p}_i$. then $\underline{r}_i = \underline{R} + \underline{r}_i'$, also $\underline{v}_i = \underline{v} + \underline{v}_i'$, where $\underline{v}$ is the velocity of the centre of mass, $\underline{v}_i$ is the velocity of $i$th particle about the centre of mass. Substitute to get:

$$\underline{L} = \sum_i \underline{r}_i \times \underline{p}_i = \sum_i (\underline{R} + \underline{r}_i') \times m_i (\underline{v} + \underline{v}_i') = \sum_i \underline{R} \times m_i \underline{v} + \overbrace{(\sum_i m_i \underline{r}_i') \times \underline{v}}^{0} + \underline{R} \times \overbrace{\sum_i m_i \underline{v}_i'}^{\frac{d}{dt}\sum_i m_i \underline{r}_i' = \frac{d}{dt}(0) = 0} + \sum_i \underline{r}_i' \times m_i \underline{v}_i' = \sum_i \underline{R} \times m_i \underline{v} + \sum_i \underline{r}_i' \times m_i \underline{v}_i'$$

$$= \underline{R} \times M\underline{v} + \sum_i \underline{r}_i' \times m_i \underline{v}_i'.$$

$R \times M\underline{v}$ gives the angular momentum of the centre of mass.




i.e. total angular momentum about O = angular momentum of the system concentrated at CM + angular momentum contribution about CM.

Energy

Consider the work done by all forces moving the system from configuration 1 to configuration 2. Then $W_{12} = \sum_i \int_1^2 \underline{F}_i \cdot d\underline{s}_i = \sum_i \int_1^2 m_i \dot{\underline{v}}_i \cdot \underline{v}_i \, dt$

Then $W_{12} = \sum_i \int_1^2 d(\frac{1}{2} m_i |\underline{v}_i|^2) = \sum_i (\frac{1}{2} m_i |\underline{v}_i|^2)\Big|_1^2 = T_2 - T_1$; where $T = \sum_i (\frac{1}{2} m_i |\underline{v}_i|^2)$ is the total kinetic energy (sum of individual KEs).

As before, let $\underline{v}_i = \underline{v} + \underline{v}_i'$, then $T = \sum_i \frac{1}{2} m_i (\underline{v} + \underline{v}_i') \cdot (\underline{v} + \underline{v}_i') = \frac{1}{2} M \underline{v} \cdot \underline{v} + \underline{v} \cdot \overbrace{\sum_i m_i \underline{v}_i'}^{0} + \frac{1}{2} \sum_i m_i \underline{v}_i' \cdot \underline{v}_i' = \frac{1}{2} M |\underline{v}|^2 + \frac{1}{2} \sum_i m_i |\underline{v}_i'|^2$.

$\Rightarrow$ total kinetic energy = KE of motion of CM ($T_{CM}$) + motion relative to CM ($T_{rel}$).

We then consider potential. For a many-particle system, we know $W_{12} = \sum_i \int_1^2 \underline{F}_i \cdot d\underline{s}_i = \sum_i \int_1^2 \underline{F}_i^{(e)} \cdot d\underline{s}_i + \sum_{i,j} \int_1^2 \underline{F}_{ji} \cdot d\underline{s}_i$

For conservative external forces, $\underline{F}_i^{(e)} = -\underline{\nabla}_i V_i^{(e)}$. Here, "i" in $\underline{\nabla}_i$ indicates that the derivatives are w.r.t. the components of $\underline{r}_i$. [e.g. $V_i^{(e)} = m_i g z_i$, $\underline{F}_i^{(e)} = -m_i g \underline{k}$

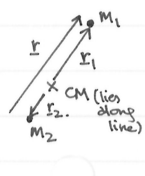
then $\sum_i \int_1^2 \underline{F}_i^{(e)} \cdot d\underline{s}_i = -\sum_i \int_1^2 \underline{\nabla}_i V_i^{(e)} \cdot d\underline{s}_i = -\sum_i V_i^{(e)}\Big|_1^2$. then, we consider the internal forces: if they are conservative (assumed to be), then they can be derived from a scalar function, the internal potential, $V_{int}$. Then $V_{int} = V_{int}[\underline{r}_1(t), \underline{r}_2(t), \dots, \underline{r}_n(t)]$. Then $\underline{\nabla}_i V_{int} = -\sum_j \underline{F}_{ji}$.

Hence, $V = \sum_i V_i^{(e)} + V_{int} = V^{(e)} + V_{int}$. This gives us a relation for the conservation of energy: $T + V = \text{const.} \Rightarrow \frac{1}{2} M v^2 + \frac{1}{2} \sum_i m_i |v_i'|^2 + V^{(e)} + V_{int} = \text{const}$, or alternatively we represent it as $T_{CM} + T_{rel} + V^{(e)} + V_{int} = \text{const}$.

$\boxed{\text{Ex}}$ (2-body dynamics)

Consider two particles, of masses $m_1$ and $m_2$. Show that $T_{rel} = \frac{1}{2} \mu \dot{\underline{r}} \cdot \dot{\underline{r}}$, where $\underline{r}$ is the vector between them and $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is the reduced mass.

Soln. Let CM be the origin O. Then $\underline{R} = \frac{m_1 \underline{r}_1 + m_2 \underline{r}_2}{m_1 + m_2} = 0. \Rightarrow m_1 \underline{r}_1 = -m_2 \underline{r}_2$.




$T_{rel} = \frac{1}{2} m_1 |\dot{\underline{r}}_1|^2 + \frac{1}{2} m_2 |\dot{\underline{r}}_2|^2$. But $\underline{r} = \underline{r}_1 - \underline{r}_2 \Rightarrow \underline{r}_1 = \underline{r} + \underline{r}_2$, and $\underline{r}_1 = \underline{r} - \frac{m_1}{m_2} \underline{r}_1 \Rightarrow \underline{r}_1 = \frac{m_2}{m_1 + m_2} \underline{r}$. Likewise,

$\underline{r}_2 = \underline{r}_1 - \underline{r} = -\underline{r} - \frac{m_2}{m_1} \underline{r}_2 \Rightarrow \underline{r}_2 = -\frac{m_1}{m_1 + m_2} \underline{r}$. Hence, $T_{rel} = \frac{1}{2} m_1 (\frac{m_2^2}{(m_1 + m_2)^2}) |\dot{\underline{r}}|^2 + \frac{1}{2} m_2 \frac{m_1^2}{(m_1 + m_2)^2} |\dot{\underline{r}}|^2 = \frac{1}{2} \frac{m_1 m_2}{(m_1 + m_2)^2} (m_1 + m_2) |\dot{\underline{r}}|^2$

i.e. $T_{rel} = \frac{m_1 m_2}{2(m_1 + m_2)} |\dot{\underline{r}}|^2 = \frac{1}{2} \mu \dot{\underline{r}} \cdot \dot{\underline{r}}$, q.e.d.

Remark: We can write $\underline{r}$ as unit vectors: $\underline{r} = s \cos\theta \, \underline{e}_1 + s \sin\theta \, \underline{e}_2. \Rightarrow \dot{\underline{r}} \cdot \dot{\underline{r}} = \dot{s}^2 + s^2 \dot{\theta}^2$ in two dimensions.

section 3.

## LAGRANGIAN MECHANICS.

Consider N particles with masses $m_\alpha$ and position vectors $\underline{r}_\alpha = (r_{\alpha 1}, r_{\alpha 2}, r_{\alpha 3})$, $\alpha = 1, \dots, N$. Each particle is subject to a force $\underline{F}_\alpha$.

There are 3N degrees of freedom in this system $\Rightarrow$ the <u>configuration space</u> (C) has 3N coordinates: $r_{11}, r_{12}, r_{13}, r_{21}, r_{22}, r_{23}, \dots\dots, r_{N1}, r_{N2}, r_{N3}$.

Each point of C corresponds to a particular configuration/arrangement of the system. To emphasise that C is a single space, we introduce new coordinates:

$q_1 = r_{11}$, $q_2 = r_{12}$, $q_3 = r_{13}$, $q_4 = r_{21}$, $q_5 = r_{22}$, ……, $q_n = r_{3N}$ where $n = 3N$.

Further, let $\mu_1 = m_1$, $\mu_2 = m_1$, $\mu_3 = m_1$, $\mu_4 = m_2$, …, $\mu_i = m_{\lceil i/3 \rceil}$, $\mu_n = m_n$. Then we define $\mu_1 \ddot{q}_1 = F_1$, $\mu_2 \ddot{q}_2 = F_2$, …, $\mu_n \ddot{q}_n = F_n$.

Note here that these are scalar equations! (i.e. $F_1, F_2$, etc are components $\Rightarrow F_1, F_2, F_3$ are components of $\underline{F}_1$, $F_4, F_5, F_6$ are components of $\underline{F}_2$ etc.)

Then we also have the <u>phase space</u> (P). As coordinates, we use $q_1, \dots, q_n, v_1, \dots, v_n \Rightarrow$ contains at each point twice the information of C (position and velocity).

Our equations of motion become $\left\{ \begin{matrix} \mu_1 \dot{v}_1 = F_1 & & \mu_n \dot{v}_n = F_n \\ \dot{q}_1 = v_1 & , \dots, & \dot{q}_n = v_n \end{matrix} \right.$. This is equivalent to the earlier statement, where we've reduced one second-order ODE to two first-order ODEs that are coupled. i.e. C has n $2^{nd}$ order ODEs, P has 2n $1^{st}$ order ODEs.

$\boxed{\text{Ex}}$ Consider a harmonic oscillator, $\ddot{x} + x = 0$. Find C and P.




Soln. C: one-dimensional, $q = x$. equation of motion: $\ddot{q} = -q$. P: two-dimensional $(q, v)$. equations of motion are $\{\dot{v} = -q, \dot{q} = v\}$.

Note: $q^2 + v^2 = \text{const}$. Show this by differentiating: $\frac{d}{dt}(q^2 + v^2) = 2q\dot{q} + 2v\dot{v} = 2(qv - vq) = 0 \Rightarrow q^2 + v^2 = \text{const}$.

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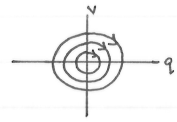
With this information, we can draw curves in phase space: $q^2+v^2=\text{const}$.




For $v>0$, $\dot{q}>0$ $\Rightarrow$ curves circulate clockwise.

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<u>coordinate transforms</u>

Suppose we introduce new coordinates $\tilde{q}_a$ and $\tilde{t}$ on <u>configuration-time space (CT)</u>, which are related to $q_a$ and $t$ by

$\tilde{q}_a=\tilde{q}_a(q_1,q_2,\cdots,q_n,t)$, a function of $q_i$ and $t$ (old coordinates); and $\tilde{t}=t$.

By the chain rule, $\tilde{v}_a=\dot{\tilde{q}}_a=\frac{\partial\tilde{q}_a}{\partial q_1}\dot{q}_1+\cdots+\frac{\partial\tilde{q}_a}{\partial q_n}\dot{q}_n+\frac{\partial\tilde{q}_a}{\partial t}=\frac{\partial\tilde{q}_a}{\partial q_b}\dot{q}_b+\frac{\partial\tilde{q}_a}{\partial t}$ $a=1,\cdots,n$.

Thus in <u>phase-time space (PT)</u>, the coordinate transform is of the form $\tilde{q}_a=\tilde{q}_a(q,t)$ and $\tilde{v}_a=\tilde{v}_a(q,\underline{v},t)=\frac{\partial\tilde{q}_a}{\partial q_b}v_b+\frac{\partial\tilde{q}_a}{\partial t}$, $\tilde{t}=t$.

**Theorem** Let $F:PT\to\mathbb{R}$, then $\frac{d}{dt}\left(\frac{\partial F}{\partial v_a}\right)-\frac{\partial F}{\partial q_a}=\frac{\partial\tilde{q}_b}{\partial q_a}\left[\frac{d}{dt}\left(\frac{\partial F}{\partial\tilde{v}_b}\right)-\frac{\partial F}{\partial\tilde{q}_b}\right]$.

Proof – see handout 2 for one degree of motion.

The theorem shows that the combination of derivatives $\frac{d}{dt}\left(\frac{\partial F}{\partial v_a}\right)-\frac{\partial F}{\partial q_a}$ transforms in a relatively simple way under change of variables.

Let $T=\frac{1}{2}mv^2=\frac{1}{2}(\mu_1v_1^2+\mu_2v_2^2+\cdots+\mu_nv_n^2)$. then note $\frac{\partial T}{\partial v_1}=\mu_1v_1$, $\frac{\partial T}{\partial v_2}=\mu_2v_2,\cdots$, $\frac{\partial T}{\partial v_n}=\mu_nv_n$. Also, $\frac{\partial T}{\partial q_a}=0$ $\forall a=1,\cdots,n$ $\Rightarrow$ $\frac{d}{dt}(\text{linear momentum})+0=\text{force (KE)}$

Equation of motion can we written $\frac{d}{dt}\left(\frac{\partial T}{\partial v_a}\right)-\frac{\partial T}{\partial q_a}=F_a$. We then perform coordinate transform:

In new coordinates, using our theorem with $F=T$, kinetic energy, then $\frac{\partial\tilde{q}_b}{\partial q_a}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial\tilde{v}_b}\right)-\frac{\partial T}{\partial\tilde{q}_b}\right]=F_a$. Multiply both sides by $\frac{\partial q_a}{\partial\tilde{q}_c}$, sum over $a$.

We also use $\frac{\partial\tilde{q}_b}{\partial q_a}\frac{\partial q_a}{\partial\tilde{q}_c}=\frac{\partial\tilde{q}_b}{\partial\tilde{q}_c}=\delta_{bc}$ $\Rightarrow$ $\delta_{bc}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial\tilde{v}_b}\right)-\frac{\partial T}{\partial\tilde{q}_b}\right]=\frac{d}{dt}\left(\frac{\partial T}{\partial\tilde{v}_c}\right)-\frac{\partial T}{\partial\tilde{q}_c}=\tilde{F}_c$ where $\tilde{F}_c=\frac{\partial q_a}{\partial\tilde{q}_c}F_a$.

Hence, the equations of motion in the new coordinates are (replacing index $c$ with $a$): $\frac{d}{dt}\left(\frac{\partial T}{\partial\tilde{v}_a}\right)-\frac{\partial T}{\partial\tilde{q}_a}=\tilde{F}_a$ or $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\tilde{q}}_a}\right)-\frac{\partial T}{\partial\tilde{q}_a}=\tilde{F}_a$.

**Definition** The $\tilde{F}_a$ are the generalised forces, the $\tilde{q}_a$ are the generalised coordinates, and the $\tilde{v}_a=\dot{\tilde{q}}_a$ are the generalised velocities.

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For the motion of a single particle, in Cartesian coordinates: $T=\frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)$.

Our generalised forces are $F_x,F_y,F_z$; generalised coordinates are $x,y,z$; generalised coordinates are $\dot{x},\dot{y},\dot{z}$.

Note that $\tilde{q}_1=x$, $\dot{\tilde{q}}_1=\dot{x}$. Then $\frac{\partial T}{\partial\dot{x}}=m\dot{x}$, $\frac{\partial T}{\partial x}=0$. $\Rightarrow$ $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{x}}\right)-\frac{\partial T}{\partial x}=F_x$ $\Rightarrow$ $\frac{d}{dt}(m\dot{x})=F_x$ indeed. Similarly, $\frac{d}{dt}(m\dot{y})=F_y$, $\frac{d}{dt}(m\dot{z})=F_z$.

Then in plane polar coordinates for a 2D particle, generalised coordinates are $r,\theta$; velocities are $\dot{r},\dot{\theta}$. We know $x=r\cos\theta$, $y=r\sin\theta$. Then

$\dot{x}=\dot{r}\cos\theta-r\dot{\theta}\sin\theta$, $\dot{y}=\dot{r}\sin\theta+r\dot{\theta}\cos\theta$. Recall $T=\frac{1}{2}m(\dot{x}^2+\dot{y}^2)=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)$

Two equations of motion: (i) $\tilde{q}_1\equiv r$. $\dot{\tilde{q}}_1\equiv\dot{r}$. Then $\frac{d}{dt}(m\dot{r})-\underbrace{mr\dot{\theta}^2}_{\text{centripetal force}}=F_r$ ← generalised force in radial direction. Also,

(ii) $\tilde{q}_2=\theta$, $\dot{\tilde{q}}_2=\dot{\theta}$. Then $\frac{d}{dt}(mr^2\dot{\theta})-0=rF_\theta$ ← generalised torque in azimuthal direction.

Suppose the forces are conservative; i.e. $\exists V=V(q,t)$ s.t. $F_b=-\frac{\partial V}{\partial q_b}$, then $\tilde{F}_a=F_b\frac{\partial q_b}{\partial\tilde{q}_a}=-\frac{\partial V}{\partial q_b}\frac{\partial q_b}{\partial\tilde{q}_a}=-\frac{\partial V}{\partial\tilde{q}_a}$

Hence, the equations of motion become $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\tilde{q}}_a}\right)-\frac{\partial T}{\partial\tilde{q}_a}=-\frac{\partial V}{\partial\tilde{q}_a}$ $\Rightarrow$ $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\tilde{q}}_a}\right)-\frac{\partial L}{\partial\tilde{q}_a}=0$ where $L=T-V$, since $V=V(q,t)$. $L$ is called the <u>Lagrangian</u>.

Now that they have served their purpose, we can drop the tildes, and we get Lagrange's equations: $\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_a}\right)-\frac{\partial L}{\partial q_a}=0}$, $a=1,\cdots,n$;

where $q_a$ are the generalised coordinates, $\dot{q}_a$ are the generalised velocities.

**Ex** Consider a particle moving in 2D under an attractive force $\underline{F}=-\mu m/r^2$ directed to the origin of polar $(r,\theta)$. Find $L$, and equations of motion.

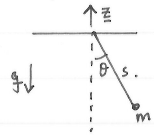
Soln. (i) $T=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)$, (ii) $V=-\frac{\mu m}{r}$. Hence $L=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)+\frac{\mu m}{r}$. Equations of motion are ① $q=r$, $\dot{q}=\dot{r}$, $\frac{d}{dt}(m\dot{r})-mr\dot{\theta}^2+\frac{\mu m}{r^2}=0$,

② $q=\theta$, $\dot{q}=\dot{\theta}$. $\frac{d}{dt}(mr^2\dot{\theta})=0$.

Remark: How do we know that angular momentum is conserved? $V=V(r)$ and $L=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)-V(r)$ $\Rightarrow$ $\frac{d}{dt}(mr^2\dot{\theta})=0$ i.e. angular momentum is constant.

Hence, symmetry $\Leftrightarrow$ conservation law (more to follow later).

**Ex** Lagrange's equations for a plane pendulum. Find $L$.



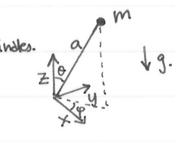

Soln. $x=s\sin\theta$, $z=-s\cos\theta$. or $T=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)=\frac{1}{2}ms^2\dot{\theta}^2$, $r=s$, $\dot{s}=0$. $V\equiv\text{GPE}=-mgs\cos\theta$.

$L=\frac{1}{2}ms^2\dot{\theta}^2+mgs\cos\theta$. $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right)-\frac{\partial L}{\partial\theta}=0$ $\Rightarrow$ $\frac{d}{dt}(ms^2\dot{\theta})+mgs\sin\theta=0$ $\Rightarrow$ $\boxed{\ddot{\theta}+\frac{g}{s}\sin\theta=0}$, which is the standard pendulum equation.

Remark: Why were we able to ignore tension? See Handout 3. Constraint forces need not be considered in formulating the Lagrangian.

For the pendulum, tension is ignored and $V$ is due to gravity. This is because the constraint forces do no work.

**Ex** Particle moving on a sphere under uniform gravity. Let $\theta\in[0,\pi]$ be colatitude, $\varphi\in[0,2\pi]$ be azimuthal angle be our coordinates.




Note: This is a 3D pendulum. Find $L$, and obtain equations of motion.

Soln. $x=a\sin\theta\cos\phi$, $y=a\sin\theta\sin\phi$, $z=a\cos\theta$. $T=\frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)=\frac{1}{2}ma^2(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta)$.

$L=T-V=\frac{1}{2}ma^2(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta)-mga\cos\theta$. ① Take $q=\theta$, $\dot{q}=\dot{\theta}$ $\frac{d}{dt}(ma^2\dot{\theta}^2)-[ma^2\dot{\phi}^2\sin\theta\cos\theta+mga\sin\theta]=0$

② $q=\phi$, $\dot{q}=\dot{\phi}$. $\frac{d}{dt}(ma^2\sin^2\theta\,\dot{\phi})=0$. $\Rightarrow ma^2\dot{\phi}\sin^2\theta=\text{const}$ (angular momentum conservation about the vertical).

A recipe for solving these type of problems.

(a) choose coordinates $q_1, \dots, q_n$ that label the configurations of the system which satisfy constraint(s). $n$ = degree of freedom.

(b) Express $T$ in terms of $\dot{q}_1, \dots, \dot{q}_n$ and $t$.

(c) If non-constraint forces are conservative, find $V = V(q_1, \dots, q_n, t)$.

(d) Obtain Lagrangian and differentiate to get equations. $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_a}\right) - \frac{\partial L}{\partial q_a} = 0, \quad a = 1, \dots, n.$

Constants of the motion and ignorable coordinates.

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Suppose a Lagrangian $L$ has no explicit dependence on $q_k$ ($\dot{q}_k$ might be there) i.e. $\frac{\partial L}{\partial q_k} = 0$. Then Lagrange gives $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = 0$;

or $\frac{\partial L}{\partial \dot{q}_k} = p_k$ = constant, where $p_k \equiv$ generalised momentum (sometimes "conjugate momentum"). i.e. whenever $q_k$ does not appear explicitly in $L$, the corresponding generalised momentum $p_k$ is conserved and we say $q_k$ is ignorable (sometimes called "cyclic".). Normally this gives rise to a conservation law.

Ex A particle in a plane moving with radially symmetric potential $V = V(r)$. Show that angular momentum is conserved.

Soln. $L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$. $\theta$ is ignorable i.e. $\frac{\partial L}{\partial \theta} = 0 \Rightarrow p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$ = const. Angular momentum is conserved, q.e.d.

Consider $\frac{d}{dt}\left(\dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L\right) = \ddot{q}_k \frac{\partial L}{\partial \dot{q}_k} + \dot{q}_k \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \left\{\dot{q}_k \frac{\partial L}{\partial q_k} + \ddot{q}_k \frac{\partial L}{\partial \dot{q}_k} + \frac{\partial L}{\partial t}\right\} \quad \because L = L(q_k, \dot{q}_k, t)$

then we get that LHS. $= \dot{q}_k\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k}\right] - \frac{\partial L}{\partial t} = \underbrace{\dot{q}_k[0]}_{\text{(Lagrange)}} - \frac{\partial L}{\partial q_k} = -\frac{\partial L}{\partial t}$. If $L$ does not depend explicitly on $t$, then $\frac{\partial L}{\partial t} = 0$.

$\Rightarrow \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L$ = const. But what is this quantity $\dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L$ that is conserved? Suppose that $T = \frac{1}{2}T_{ab}(q)\dot{q}_a\dot{q}_b$ (sum over $a, b$).

i.e. $T$ is a quadratic form in generalised velocities. e.g. $T = \frac{1}{2}m(\underline{\dot{r}^2} + r^2\underline{\dot{\theta}^2})$ generalised coordinates in quadratic form.

Then with that assumption, $\dot{q}_k \frac{\partial T}{\partial \dot{q}_k} = \dot{q}_k \frac{\partial}{\partial \dot{q}_k}\left(\frac{1}{2}T_{ab}\dot{q}_a\dot{q}_b\right) = \dot{q}_k \frac{1}{2}T_{ab}\frac{\partial}{\partial \dot{q}_k}(\dot{q}_a\dot{q}_b)$ since $T_{ab}(q)$ is a function of space only and not velocity.

$\dot{q}_k \frac{\partial T}{\partial \dot{q}_k} = \dot{q}_k \frac{1}{2}T_{ab}\left(\frac{\partial \dot{q}_a}{\partial \dot{q}_k}\dot{q}_b + \dot{q}_a\frac{\partial \dot{q}_b}{\partial \dot{q}_k}\right) = \dot{q}_k\frac{1}{2}T_{ab}(\delta_{ak}\dot{q}_b + \dot{q}_a\delta_{bk}) = \frac{1}{2}T_{ab}(\dot{q}_a\dot{q}_b + \dot{q}_a\dot{q}_b) = T_{ab}\dot{q}_a\dot{q}_b = 2\left(\frac{1}{2}T_{ab}\dot{q}_a\dot{q}_b\right) = 2T.$

Hence, $\dot{q}_k\frac{\partial L}{\partial \dot{q}_k} - L = 2T - (T - V) = T + V = E$, const because $\frac{\partial L}{\partial \dot{q}_k} = \frac{\partial T}{\partial \dot{q}_k}$ since $L = T - V$ and $\frac{\partial V}{\partial \dot{q}_k} = 0$.

This gives us the equation for the conservation of energy, if $\frac{\partial L}{\partial t} = 0$, and $T$ is a quadratic form of generalised velocities.

If $T$ is not of the form $T = \frac{1}{2}T_{ab}\dot{q}_a\dot{q}_b$, the quantity $\dot{q}_k\frac{\partial L}{\partial \dot{q}_k} - L$ remains conserved, but is not energy. In general, we define $\dot{q}_k\frac{\partial L}{\partial \dot{q}_k} - L$ as the Hamiltonian.

Symmetry.

1. A system which is invariant under translation along a given direction (i.e. $\frac{\partial L}{\partial q_k} = 0$, $q_k \equiv x$) conserves linear momentum in that direction.
2. A system invariant to rotation about an axis ($\frac{\partial L}{\partial q_k} = 0$ ; $q_k \equiv \theta$) conserves angular momentum about that axis.
3. A system invariant in time $t$ (i.e. $\frac{\partial L}{\partial t} = 0$) conserves "energy" (specifically the Hamiltonian).

These ideas don't simply apply to Newtonian mechanics, but are powerful notions extending to quantum mechanics, relativity etc.

Refer to Richard Feynman: "the character of physical law".

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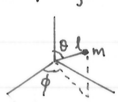
Ex (Spherical Pendulum).

Recall $L = \frac{1}{2}ml^2\dot{\theta}^2 + \frac{1}{2}ml^2\sin^2\theta\,\dot{\phi}^2 - mgl\cos\theta$. $\phi$ does not appear explicitly here, so $\frac{\partial L}{\partial \dot{\phi}}$ = const.

$\Rightarrow ml^2\sin^2\theta\,\dot{\phi}^2$ = const (1). This is conservation of angular momentum. Note $\phi$ was the ignorable coordinate about the vertical.

$t$ is also ignorable, so we have $\left[\dot{q}_k\frac{\partial L}{\partial \dot{q}_k} - L = \text{const}\right] \Rightarrow \dot{\theta}\frac{\partial L}{\partial \dot{\theta}} + \dot{\phi}\frac{\partial L}{\partial \dot{\phi}} - L = \text{const} \Rightarrow ml^2\dot{\theta}^2 + ml^2\sin^2\theta\,\dot{\phi}^2 - \frac{1}{2}ml^2\dot{\theta}^2 - \frac{1}{2}ml^2\sin^2\theta\,\dot{\phi}^2 + mgl\cos\theta = \text{const}.$

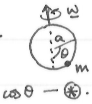
$\Rightarrow \left(\frac{1}{2}ml^2\dot{\theta}^2 + \frac{1}{2}ml^2\sin^2\theta\,\dot{\phi}^2\right) + mgl\cos\theta = T + V = E$ (energy) (2).




We can also state this, since KE is a quadratic form in $(\dot{\theta}, \dot{\phi})$.

Note: $p_\phi = ml^2\sin^2\theta\,\dot{\phi}$. substitute into (2) $\therefore \frac{1}{2}ml^2\dot{\theta}^2 + \frac{p_\phi}{2ml^2\sin^2\theta} + mgl\cos\theta = E$ is a single ODE for $\theta$.

Ex Consider a circular hoop of radius $a$ spinning in its plane, with a bead of mass $m$ sliding frictionally on the hoop. The hoop rotates about a vertical diameter with given angular velocity $\omega$. Here, we have just one degree of freedom, so we know the position of the hoop at any given time.




We need only to locate the particle on the hoop. $T = \frac{1}{2}ma^2\dot{\theta}^2 + \frac{1}{2}m(a\sin\theta)^2\omega^2$. $V = mga\cos\theta$. $L = T - V = \frac{1}{2}ma^2\dot{\theta}^2 + \frac{1}{2}ma^2\sin^2\theta\,\omega^2 + mga\cos\theta$ — (*).

$t$ is ignorable, $\dot{\theta}\frac{\partial L}{\partial \dot{\theta}} - L = \text{const} \Rightarrow ma^2\dot{\theta}^2 - \frac{1}{2}ma^2\dot{\theta}^2 - \frac{1}{2}ma^2\sin^2\theta\,\omega^2 - mga\cos\theta = \frac{1}{2}ma^2\dot{\theta}^2 - \frac{1}{2}ma^2\sin^2\theta\,\omega^2 - mga\cos\theta = \text{const}.$

Note that this not $T+V$. Recall the rotating tube problem on sheet 2. $T = \frac{1}{2}m($

## Hamilton's Equations.

Lagrange's equations use generalised coordinates and velocities. Alternatively, we could use generalised coordinates and generalised momenta.

The total differential of $L = L(q, \dot{q}, t)$ is $dL = \frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i + \frac{\partial L}{\partial t} dt = \dot{p}_i\, dq_i + p_i\, d\dot{q}_i + \frac{\partial L}{\partial t} dt$ and $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{p}_i}\right) - \frac{\partial L}{\partial p_i} = 0$

$p_i = \frac{\partial L}{\partial \dot{q}_i}$ is the generalised momentum. Also $p_i\, d\dot{q}_i = d(p_i \cdot \dot{q}_i) - \dot{q}_i\, dp_i$ (product rule). Hence, the expression for $dL$ yields

$d(p_i \dot{q}_i - L) = \dot{q}_i\, dp_i - \dot{p}_i\, dq_i - \frac{\partial L}{\partial t} dt$. $\dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L = \text{const}$ when $L$ is $t$-independent.

$\boxed{\text{Definition}}$ The Hamiltonian, $H = p_i \dot{q}_i - L = H(q, p, t)$.

Note: this does not depend on $\dot{q}$, generalised velocities. At nowhere in $H$ should we have an explicit time derivative.

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Recall: we have shown that $d(p_i \dot{q}_i - L) = \dot{q}_i\, dp_i - \dot{p}_i\, dq_i - \frac{\partial L}{\partial t} dt$. $H = p_i \dot{q}_i - L = H(q, p, t)$ (function of space, generalised momenta, possibly time).

Hence, $dH = \dot{q}_i\, dp_i - \dot{p}_i\, dq_i - \frac{\partial L}{\partial t} dt$. If $H = H(q, p, t)$ then $dH = \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial t} dt$. Compare the two expressions:

We get the following – $\boxed{\frac{\partial H}{\partial p_i} = \dot{q}_i, \quad \frac{\partial H}{\partial q_i} = -\dot{p}_i}$ $i = 1, \dots, n$ by setting some quantities/components to zero and varying them.

These are called Hamilton's equations. Of course, we also have $-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t}$.

Recall that in Lagrangian mechanics, we had $n$ 2nd order DEs; in Hamiltonian mechanics we have $2n$ 1st order DEs.

The total time derivative of $H(q, p, t)$ is $\frac{dH}{dt} = \frac{\partial H}{\partial q_i} \dot{q}_i + \frac{\partial H}{\partial p_i} \dot{p}_i + \frac{\partial H}{\partial t} = -\dot{p}_i \dot{q}_i + \dot{q}_i \dot{p}_i + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$.

In fact, we already know this — we investigated this earlier, showing that a quantity (now introduced as $H$) is constant if $t$ is ignorable in $L$ (and hence $H$).

A recipe for constructing $H$:

(a) Find $L(q, \dot{q}, t)$.

(b) Find generalised momenta: $p_i = \frac{\partial L}{\partial \dot{q}_i}$

(c) Construct $H = p_i \dot{q}_i - L$ (this will have terms with $\dot{q}_i$ present).

(d) Write in correct units: $H = H(q, p, t)$ only $\Rightarrow$ i.e. replacing $\dot{q}_i$ in terms of $p_k$ and $q_k$.

Remark: If $T$ is purely quadratic in $\dot{q}_i$s, then $H = T + V$. (not true in general!)

$\boxed{\text{Ex}}$ Consider single particle in 3D cartesians, of mass $m$. Find $H$, and obtain equations of motion.

Soln. $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z)$. Take $p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$, $p_y = m\dot{y}$, $p_z = m\dot{z}$. $T$ is purely quadratic, so here $H = T + V$.

$H = T + V = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + V(x, y, z) = \frac{1}{2}m\left(\frac{p_x^2}{m^2} + \frac{p_y^2}{m^2} + \frac{p_z^2}{m^2}\right) + V(x, y, z) = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + V(x, y, z)$ //

From Hamilton's equations, $\dot{p}_i = -\frac{\partial H}{\partial q_i}$, $\dot{q}_i = \frac{\partial H}{\partial p_i}$. For $i = $ "1", $\dot{p}_x = -\frac{\partial V}{\partial x}$, $\dot{x} = \frac{p_x}{m}$. // Analogously, $\dot{p}_y = -\frac{\partial V}{\partial y}$, $\dot{y} = \frac{p_y}{m}$ // and $\dot{p}_z = -\frac{\partial V}{\partial z}$, $\dot{z} = \frac{p_z}{m}$.

Note: $\dot{p}_x = -\frac{\partial V}{\partial x}$, $\dot{x} = \frac{p_x}{m}$ is equivalent to $m\ddot{x} = -\frac{\partial V}{\partial x}$.

$\boxed{\text{Ex}}$ Find the Hamiltonian of a 1-D harmonic spring oscillator: $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$.

Soln. $p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$. $T$ is purely quadratic, so $H = T + V = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \frac{1}{2m}p^2 + \frac{1}{2}kx^2$. Then $\dot{p} = -kx$, $\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m}$. (i.e. $m\ddot{x} = -kx$).

$\boxed{\text{Ex}}$ Find the Hamiltonian given Lagrangian $L = (1 - \dot{q}^2)^{1/2}$.

Soln. $p = \frac{\partial L}{\partial \dot{q}} = \frac{1}{2}(1 - \dot{q}^2)^{-\frac{1}{2}}(-2\dot{q}) = -\frac{\dot{q}}{\sqrt{1 - \dot{q}^2}}$. Then $H = p\dot{q} - L = \frac{-\dot{q}^2}{\sqrt{1 - \dot{q}^2}} - \frac{1 - \dot{q}^2}{\sqrt{1 - \dot{q}^2}} = -\frac{1}{\sqrt{1 - \dot{q}^2}}$. We see that $p^2 = \frac{\dot{q}^2}{1 - \dot{q}^2} \Rightarrow \dot{q}^2 = p^2 - p^2\dot{q}^2 \Rightarrow \dot{q}^2 = \frac{p^2}{1 + p^2}$

then $1 - \dot{q}^2 = \frac{1}{1 + p^2}$, so $H = -(1 + p^2)^{\frac{1}{2}}$ //.

$\boxed{\text{Ex}}$ Find $L$ given $H = \frac{1}{2}p^2 + p \sin q$.

Soln. $H = \dot{q}p - L \Rightarrow L = \dot{q}p - H$, but $L = L(q, \dot{q}, t)$. Also, $\dot{q} = \frac{\partial H}{\partial p} = p + \sin q$ (we need this to construct $L = L(q, \dot{q})$).

Then $L = \dot{q}p - H = p^2 + p \sin q - \frac{1}{2}p^2 - p \sin q = \frac{1}{2}p^2 = \frac{1}{2}(\dot{q} - \sin q)^2$ //

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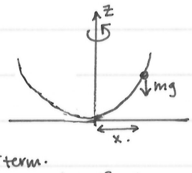
$\boxed{\text{Ex}}$ (Exam 1996) A bead of mass $m$ (frictionless) slides under gravity on a parabolic wire, $z = \frac{1}{2}\alpha^2 x^2$. The wire rotates about the $z$-axis with constant angular velocity $w$. Show that $H = \frac{p^2}{2m(1 + \alpha^4 x^2)} + \frac{mx^2}{2}(g\alpha^2 - w^2)$.




Soln. First construct Lagrangian: $V = mgz = mg\frac{1}{2}\alpha^2 x^2$. $T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2)$ in general, so $r = x$, $\dot{r} = \dot{x}$, $\dot{\theta} = w$.

Also, $z = \frac{1}{2}\alpha^2 x^2 \Rightarrow \dot{z} = \alpha^2 x\dot{x}$. Putting it all together, $T = \frac{1}{2}m(\dot{x}^2 + x^2w^2 + \alpha^4 x^2 \dot{x}^2)$. Then, our Lagrangian is

$L = \frac{1}{2}m[\dot{x}^2 + x^2w^2 + \alpha^4 x^2 \dot{x}^2] - \frac{1}{2}mg\alpha^2 x^2$. We see that $T + V \neq H$, because it is not a pure quadratic form: look at the $x^2w^2$ term.

$w$ was given (fixed).

$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x} + \alpha^4 x^2 \dot{x}) = m\dot{x}(1 + \alpha^4 x^2)$. $H = p\dot{x} - L = m\dot{x}^2(1 + \alpha^4 x^2) - \left[\frac{1}{2}m\dot{x}^2(1 + \alpha^4 x^2) + \frac{1}{2}mx^2w^2 - \frac{1}{2}mg\alpha^2 x^2\right]$

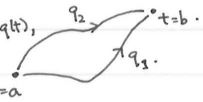
$H = \frac{1}{2}m\dot{x}^2(1 + \alpha^4 x^2) - \frac{1}{2}mx^2w^2 + \frac{1}{2}mg\alpha^2 x^2 = \frac{1}{2} \cdot \frac{p^2}{m(1 + \alpha^4 x^2)} + \frac{mx^2}{2}(g\alpha^2 - w^2)$ // q.e.d.

Remark: $\frac{\partial H}{\partial t} = 0 \Rightarrow \frac{dH}{dt} = 0 \Rightarrow H = \text{const}$.

Hamilton's Principle (NFE).

Consider a system with one degree of freedom (we can later generalise). Then $L = L(q, \dot{q}, t)$. We seek the extrema of the integral $J = \int_a^b L(q, \dot{q}, t)\,dt$ over all smooth functions $q = q(t)$ s.t. $q(a) = c$, $q(b) = d$ $(b > a)$. $a, b, c, d$ are given constants.

considering a dynamical system




i.e. we look for "critical functions" $q(t)$; that is, functions such that the variation $\delta J = 0$ when $q(t)$ is replaced by neighbouring path $q(t) + \delta q(t)$, s.t. $\delta q(a) = \delta q(b) = 0$. Analyse this:

Let $\delta q(t) = \varepsilon\, u(t)$ s.t. $u(a) = u(b) = 0$, $\varepsilon \ll 1$. Then $\delta J = \int_a^b L(q + \varepsilon u, \dot{q} + \varepsilon \dot{u}, t)\,dt - \int_a^b L(q, \dot{q}, t)\,dt$.

By Taylor's theorem, $\delta J = \int_a^b \left(\frac{\partial L}{\partial q} u + \frac{\partial L}{\partial \dot{q}} \dot{u}\right) dt + O(\varepsilon^2)$ [zeroth order terms cancel]. $= \int_a^b \left(\frac{\partial L}{\partial q} u - u \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right) dt + O(\varepsilon^2)$

This is integration by parts. Boundary contributions disappear because $u(a) = u(b) = 0$. Then $\delta J = \varepsilon \int_a^b u \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right) dt + O(\varepsilon^2)$.

For critical $q(t)$, $\delta J = 0$ to $O(\varepsilon)$. Since $u$ is arbitrary, $\int_a^b \frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) dt = 0 \Rightarrow \frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$, which is Lagrange's equation.

This ties in our dynamical system understanding with the calculus of variations:
(Hamilton's Principle).
In the evolution of a dynamical system from time $t_1 \to t_2$, the action $J = \int_{t_1}^{t_2} L(q, \dot{q}, t)\,dt$ is unchanged under small variation.

Poisson Brackets.

Let $X$ and $Y$ be dynamical variables which depend on $q, p$ and $t$. (e.g. Hamiltonian).

$\boxed{\text{Definition}}$ The Poisson bracket of $X$ and $Y$ is $[X, Y] = \frac{\partial X}{\partial q_j}\frac{\partial Y}{\partial p_j} - \frac{\partial X}{\partial p_j}\frac{\partial Y}{\partial q_j}$.

This gives us some properties –

(i) Poisson brackets anti-commute: $[X, Y] = -[Y, X]$.

(ii) $[X, Y_1 + Y_2] = [X, Y_1] + [X, Y_2]$ (linearity).

(iii) $[X, Y_1 Y_2] = [X, Y_1]Y_2 + [X, Y_2]Y_1$ (product rule)

(iv) $[[X,Y], Z] + [[Y,Z], X] + [[Z,X], Y] = 0$. (Jacobi's identity).

Set $Y = H$, and we have $[X, H] = \frac{\partial X}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial X}{\partial p_i}\frac{\partial H}{\partial q_i} = \frac{\partial X}{\partial q_i}\dot{q}_i + \frac{\partial X}{\partial p_i}\dot{p}_i = \frac{dX}{dt} - \frac{\partial X}{\partial t}$ (by chain rule).

$\Rightarrow \frac{dX}{dt} = \frac{\partial X}{\partial t} + [X, H]$. If $X$ does not depend explicitly on $t$, then $\frac{dX}{dt} = [X, H]$.

Other facts based on Poisson brackets:

Let $X = q_j$ and $X = p_j$ in turn, then we get $\frac{\partial H}{\partial p_j} = \frac{dq_j}{dt} = [q_j, H]$; and respectively $\frac{\partial H}{\partial q_j} = \frac{dp_j}{dt} = [p_j, H]$.

In fact, for any $X$, $\frac{\partial X}{\partial p_j} = [q_j, X]$ and $-\frac{\partial X}{\partial q_j} = [p_j, X]$.

For instance, $[q_j, X] = \frac{\partial q_j}{\partial q_k}\frac{\partial X}{\partial p_k} - \frac{\partial q_j}{\partial p_k}\frac{\partial X}{\partial q_k} = \delta_{jk}\frac{\partial X}{\partial p_k} - 0 \cdot \frac{\partial X}{\partial q_k} = \delta_{jk}\frac{\partial X}{\partial p_k} = \frac{\partial X}{\partial p_j}$ (beware: do not use $j$ in the denominator on the first step).

Also, $[q_j, q_k] = [p_j, p_\ell] = 0$. Then $[q_j, p_k] = \delta_{jk}$, because $[q_j, p_k] = \frac{\partial q_j}{\partial q_\alpha}\frac{\partial p_k}{\partial p_\alpha} - \frac{\partial q_j}{\partial p_\alpha}\frac{\partial p_k}{\partial q_\alpha} = \frac{\partial q_j}{\partial q_\alpha}\frac{\partial p_k}{\partial p_\alpha} = \delta_{j\alpha}\delta_{k\alpha} = \delta_{jk}$

Recall: Jacobi's identity with $Z = H$. $[[X,Y], H] + [[Y,H], X] + [[H,X], Y]$. Suppose $X, Y$ do not depend explicitly on $t$ and are constants of the motion.

Thus $[X, H] = [Y, H] = 0$. $\Rightarrow [[X,Y], H] + 0 + 0 = 0 \Rightarrow [[X,Y], H] = 0$.

However, $[X, Y]$ does not depend explicitly on $t$ (since neither of them do explicitly in themselves). i.e. $\frac{d}{dt}[X, Y] = 0$

$\Rightarrow [X, Y]$ is another constant of the motion.

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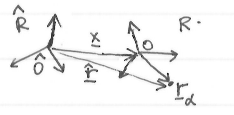
Section 4.

RIGID BODY MOTION.

A rigid body requires 6 generalised coordinates to describe its configuration:




3 to specify a point on the body (typically CM) + 3 more to specify orientation of body.

Let $\hat{R} = (\hat{O}, \hat{B})$ be an inertial frame and $R = (O, B)$ be a rest frame of the body (i.e. fixed relative to the body).

Let $\underline{r}_\alpha$ be the position vector of particle $\alpha$ from $O$, and $\hat{\underline{r}} = \underline{r}_\alpha + \underline{x}$ be its position vector from $\hat{O}$.

Then $\hat{\underline{v}}_\alpha = \hat{D}(\hat{\underline{r}}) = \hat{D}(\underline{x} + \underline{r}_\alpha) = \hat{D}\underline{x} + \hat{D}\underline{r}_\alpha = \dot{\underline{x}} + \hat{D}\underline{r}_\alpha = \dot{\underline{x}} + \widetilde{D\underline{r}_\alpha} + (\underline{\omega} \times \underline{r}_\alpha)$. by Coriolis theorem. Since observer at $O$ and $\alpha$ are fixed relative to each other, $D\underline{r}_\alpha = 0 \Rightarrow \hat{\underline{v}}_\alpha = \dot{\underline{x}} + (\underline{\omega} \times \underline{r}_\alpha)$ where $\underline{\omega}$ is the angular velocity of rigid body relative to $\hat{R}$.

Hence, total KE relative to inertial frame $\hat{R}$ is $T = \sum_\alpha \frac{1}{2} m_\alpha \hat{\underline{v}}_\alpha \cdot \hat{\underline{v}}_\alpha = \sum_\alpha \frac{1}{2} m_\alpha (\dot{\underline{x}} + \underline{w} \times \underline{r}_\alpha) \cdot (\dot{\underline{x}} + \underline{w} \times \underline{r}_\alpha) = \frac{1}{2} m \dot{\underline{x}} \cdot \dot{\underline{x}} + \dot{\underline{x}} \cdot (\underline{w} \times \sum_\alpha m_\alpha \underline{r}_\alpha) + \frac{1}{2} \sum_\alpha m_\alpha (\underline{w} \times \underline{r}_\alpha) \cdot (\underline{w} \times \underline{r}_\alpha)$.

Here, $m = \sum_\alpha m_\alpha$. The second term can be written as $\dot{\underline{x}} \cdot (\underline{w} \times \sum_\alpha m_\alpha \underline{r}_\alpha) = \dot{\underline{x}} \cdot (\underline{w} \times m\underline{R})$, where $\underline{R} = \frac{\sum_\alpha m_\alpha \underline{r}_\alpha}{m}$ is the position vector of CM.

Suppose we have a continuous distribution of mass, and we repace $\frac{1}{2} \sum_\alpha m_\alpha (\underline{w} \times \underline{r}_\alpha) \cdot (\underline{w} \times \underline{r}_\alpha)$ with $\frac{1}{2} \int_V \rho (\underline{w} \times \underline{r}) \cdot (\underline{w} \times \underline{r}) \, dV$.

This is because $m = \rho V \Rightarrow dm = \rho \, dV$ where $\rho$ is the density.

We claim that $\frac{1}{2} \int_V \rho (\underline{w} \times \underline{r}) \cdot (\underline{w} \times \underline{r}) \, dV = \frac{1}{2} \int_V \rho [(\underline{w} \cdot \underline{w})(\underline{r} \cdot \underline{r}) - (\underline{w} \cdot \underline{r})^2] \, dV \; (= I)$. Recall that $(\underline{w} \times \underline{r}) \cdot (\underline{w} \times \underline{r}) = |\underline{w} \times \underline{r}|^2 = |\underline{w}|^2 |\underline{r}|^2 \sin^2\theta$.

Then $(\underline{w} \times \underline{r}) \cdot (\underline{w} \times \underline{r}) = |\underline{w}|^2 |\underline{r}|^2 (1 - \cos^2\theta) = (\underline{w} \cdot \underline{w})(\underline{r} \cdot \underline{r}) - (\underline{w} \cdot \underline{r})^2$. Now $\underline{w} = \underline{w}(t)$ i.e. does not depend on space.

Moreover, $I = \frac{1}{2} w_i w_j \int_V \rho (r_k r_k \delta_{ij} - r_i r_j) \, dV$; since $\overset{(i)}{r_k r_k = \underline{r} \cdot \underline{r}}$, $\overset{(ii)}{w_i w_j \delta_{ij} = w_i w_i = \underline{w} \cdot \underline{w}}$, $\overset{(iii)}{w_i w_j r_i r_j = (w_i r_i)(w_j r_j) = (\underline{w} \cdot \underline{r})^2}$.

$\boxed{\text{Definition}}$ The <u>inertia matrix</u> (or inertia tensor) of a rigid body in the rest frame R is the 3x3 matrix (which is symmetric):

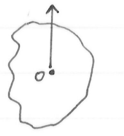
$J(R)$ with entries $J_{ij} = \int_V \rho (r_k r_k \delta_{ij} - r_i r_j) \, dV$. $(= J_{ji})$

$J(R)$ is purely a geometric property of the object, not any of its velocities.

Or, identifying $r_1 \equiv x$, $r_2 \equiv y$, $r_3 \equiv z$, we get $J_{11} = A = \int \rho (y^2 + z^2) \, dV$, $J_{22} = B = \int \rho (x^2 + z^2) \, dV$, $J_{33} = C = \int \rho (x^2 + y^2) \, dV$.

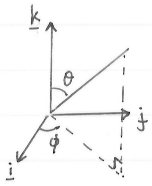
$F = \int \rho yz \, dV = -J_{23}$, $G = \int \rho xz \, dV = -J_{13}$, $H = \int \rho xy \, dV = -J_{12}$. This is all information we need, since J symmetric.

Then $J(R) = \begin{pmatrix} A & -H & -G \\ -H & B & -F \\ -G & -F & C \end{pmatrix}$.



$\boxed{\text{Definition}}$ The <u>diagonal entries</u> $A, B, C$ are the <u>moments of inertia</u> about the $x, y, z$ axes respectively, while the terms $F, G, H$ are the <u>products of inertia</u>.

We examine a few examples:



$\boxed{\text{Ex}}$ Consider the 1-D rod of length $l$ and mass $m$, with uniform density $\rho = \frac{m}{l}$. Mass of a "small element" of rod is $\rho \, dr$, and $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r \cos\theta$. Find inertia matrix.

<u>Soln</u>. $A = \int_0^l \rho (y^2 + z^2) \, dr = \frac{m}{l} \int_0^1 r^2 - x^2 \, dr = \frac{m}{l} \int_0^l r^2 (1 - \sin^2\theta \cos^2\phi) \, dr = \frac{m}{l} (1 - \sin^2\theta \cos^2\phi) \left[\frac{r^3}{3}\right]_0^l = \frac{ml^2}{3} (1 - \sin^2\theta \cos^2\phi)$.

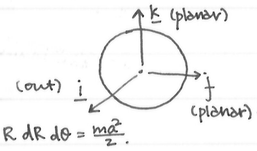
Likewise, $B = \int_0^l \rho (r^2 - y^2) \, dr = \frac{ml^2}{3} (1 - \sin^2\theta \sin^2\phi)$; $C = \int_0^l \rho (r^2 - z^2) \, dr = \frac{ml^2}{3} \sin^2\theta$. We also seek products of inertia:

$F = \int_0^l \rho y \, dr = \frac{ml^2}{3} \sin\theta \cos\theta \sin\phi$, $G = \frac{ml^2}{3} \sin\theta \cos\theta \cos\phi$, $H = \frac{ml^2}{3} \sin^2\theta \sin\phi \cos\phi$. Then, our inertia matrix is:

$$J = \frac{ml^2}{3} \begin{pmatrix} 1 - \sin^2\theta \cos^2\phi & -\sin^2\theta \sin\phi \cos\phi & -\sin\theta \cos\theta \cos\phi \\ -\sin^2\theta \sin\phi \cos\phi & 1 - \sin^2\theta \sin^2\phi & -\sin\theta \cos\phi \sin\phi \\ -\sin\theta \cos\theta \cos\phi & -\sin\theta \cos\theta \sin\phi & \sin^2\theta \end{pmatrix}.$$

Note: moments should always have units mass x length².

Note however that we could have simplified this, since this is a rest frame and we can choose orientations. For instance, if we set rod in $\underline{i}$ direction $\phi = 0$, $\theta = \frac{\pi}{2}$. Then $J = \frac{ml^2}{3} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.



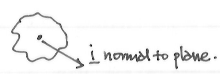

$\boxed{\text{Ex}}$ Consider the 2-D lamina (e.g. a disc) ~~with~~ constant uniform density. Pick axes $\underline{i}$ normal, $\underline{j}, \underline{k}$ coplanar in disc.

Let $\rho = \frac{m}{\pi a^2}$. Find inertia matrix.

<u>Soln</u>. $y = R\cos\theta$, $z = R\sin\theta$ where $R^2 = y^2 + z^2$. Also $x = 0$ is fixed. Then $A = \iint \rho (y^2 + z^2) \, dA = \frac{m}{\pi a^2} \int_0^{2\pi} \int_0^a R^2 \cdot R \, dR \, d\theta = \frac{ma^2}{2}$. ($dA = R \, dR \, d\theta$)

$B = \iint \rho (r^2 - y^2) \, dA = \int_0^{2\pi} \int_0^a z^2 \, dA = \int_0^{2\pi} \int_0^a R^3 \sin^2\theta \, dR \, d\theta = \int_0^{2\pi} \sin^2\theta \, d\theta \int_0^a R^3 \, dR = \frac{ma^2}{4}$.

$C = \iint \rho (r^2 - z^2) \, dA = B$ (symmetric case) $= \frac{ma^2}{4}$. For products of inertia, $G = H = 0 \because x = 0$. This leaves us with $F = \iint \rho (yz) \, dA = 0$, because $\int_0^{2\pi} \sin\theta \cos\theta \, d\theta = 0$. Then $J = \frac{ma^2}{4} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ //.




Remark: We can show that for any plane lamina, $A = B + C$ if $\underline{i}$ is normal to plane.

We can write $\boxed{T = \frac{1}{2} m \dot{\underline{x}} \cdot \dot{\underline{x}} + m \dot{\underline{x}} \cdot (\underline{w} \times \underline{R}) + \frac{1}{2} J_{ij} w_i w_j}$. The rest frame is not unique, and a suitable choice of R can simplify the last term.

$\boxed{\text{Theorem}}$ Let $R = (0, B)$ and $R' = (0, B')$ be two rest frames with the same origin. Let H be the transition matrix from $B'$ to $B$.

Let $J_{ij}$ and $J'_{ij}$ be the entries of inertia matrices $J(R)$ and $J(R')$. Then $J_{ij} = H_{ip} J'_{pq} H_{jq}$. i.e. $J = HJ'H^T$ (indeed H is orthogonal).

Proof – $r_i = H_{ip} r'_p$ by consequence of transition matrix. $r_j = H_{jq} r'_q$. Then $J_{ij} = \int_V \rho (r_k r_k \delta_{ij} - r_i r_j) \, dV$. Hence

$J_{ij} = \int_V \rho H_{ip} H_{jq} (r'_k r'_k \delta_{pq} - r'_p r'_q) \, dV = H_{ip} J'_{pq} H_{jq}$. It just remains to show that:

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$\int_V \rho (r_k r_k \delta_{ij} - r_i r_j) \, dV = \int_V \rho H_{ip} H_{jq} (r'_k r'_k \delta_{pq} - r'_p r'_q) \, dV$. We claim that $r_k r_k = r'_k r'_k$ since $|\underline{r}|^2 = |\underline{r}'|^2$ relative to same origin, i.e. length is preserved under rotation of axes. [Alternative proof: $r_k r_k = H_{kp} r'_p H_{kq} r'_q = H_{kp} H_{kq} r'_p r'_q = (H^T H)_{pq} r'_p r'_q$. Then since H is adjoint, $(H^T H)_{pq} = \delta_{pq}$, so $r_k r_k = \delta_{pq} r'_p r'_q = r'_p r'_p = r'_k r'_k$ ]. Then we have, justifying algebraically, $H_{ip} H_{jq} \delta_{pq} = H_{ip} H_{jp} = (HH^T)_{ij} = \delta_{ij}$.

Thus, RHS $= \int_V \rho H_{ip} H_{jq} (r'_k r'_k \delta_{pq} - r'_p r'_q) \, dV = \int_V \rho [r_k r_k (H_{ip} H_{jq}) \delta_{pq} - H_{ip} r'_p H_{jq} r'_q] \, dV = \int_V \rho (r_k r_k \delta_{ij} - r_i r_j) \, dV$

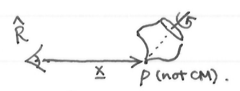
$\Rightarrow$ identity holds //, q.e.d.

In matrix notation, $J(R) = H\,J'(R')\,H^T$, or $J' = H^T J H$. Since $J$ is a real symmetric matrix, then $\exists$ an orthogonal matrix $H$ s.t. $H^T J H$ is diagonal.
Thus, it follows that it is always possible to choose axes such that $J'$ is diagonal. (or $J$ is diagonal)

$\boxed{\text{Definition}}$ Let $J(R)$ be the inertia matrix in a rest frame $R(O,B)$. A <u>principal axis</u> at $O$ is a line through $O$ in the direction of an eigenvector of $J(R)$.
The corresponding <u>eigenvalue</u> is called a <u>principal moment of inertia</u>.

Remark: If $J(R')$ is diagonal, then the coordinate axes are in fact principal axes. (since $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\0\\1\end{pmatrix}$ are eigenvectors). and the diagonal entries are principal moments of inertia.

Suppose that the rest frame has its axes aligned with principal axes (we can always manipulate to get this). Then $F=G=H=0$, and $\frac{1}{2}w_i w_j J_{ij}$ simplifies to become $\frac{1}{2}w_i w_j J_{ij} = \frac{1}{2}(w_1^2 J_{11} + w_2^2 J_{22} + w_3^2 J_{33}) = \frac{1}{2}(Aw_1^2 + Bw_2^2 + Cw_3^2)$. We consider two cases:




1. If $O$ of rest frame is the centre of mass, $T$ simplifies to $T = \frac{1}{2}m\dot{\underline{x}}\cdot\dot{\underline{x}} + \frac{1}{2}(Aw_1^2 + Bw_2^2 + Cw_3^2)$.
2. The origin $O$ is at rest relative to $\hat{R}$ (such an observer noticing the base of a spinning top). This is the case when the body moves about a <u>fixed point</u>.
   Then $\dot{\underline{x}} = 0$, so $\frac{1}{2}m\dot{\underline{x}}\cdot\dot{\underline{x}} = 0$ and $m\dot{\underline{x}}\cdot(\underline{w}\times\underline{R}) = 0 \Rightarrow T = \frac{1}{2}(Aw_1^2 + Bw_2^2 + Cw_3^2)$.

<u>Angular momentum</u>.

Let a point $p$ be fixed in both the rest frame and inertial frame. Then the angular momentum about $P$ is $\underline{L}_P$, where $\underline{L}_P = \int_V \underline{r}\times\underline{v}\,dm = \int_V \rho\,\underline{r}\times\underline{v}\,dV$.
But $\underline{v} = \hat{D}\underline{r}$ (rate of change of $\underline{r}$ as measured by an inertial observer). By Coriolis theorem, $\hat{D}\underline{r} = \dot{\underline{r}} + \underline{w}\times\underline{r} = 0 + \underline{w}\times\underline{r}$ ($\dot{\underline{r}}$ is measured in rest frame) $= \underline{w}\times\underline{r}$.
Then $\underline{L}_P = \int_V \rho\,\underline{r}\times(\underline{w}\times\underline{r})\,dV = \int_V \rho\,[(\underline{r}\cdot\underline{r})\underline{w} - (\underline{r}\cdot\underline{w})\underline{r}]\,dV$. Then $(L_P)_i = \int_V \rho\,[(\underline{r}\cdot\underline{r})w_i - (\underline{r}\cdot\underline{w})r_i]\,dV = \int_V \rho\,(r_k r_k w_i - r_j w_j r_i)\,dV$.
Hence $(L_P)_i = \int_V \rho\,(r_k r_k \delta_{ij} - r_j r_i)\,dV \cdot w_j = J_{ij}w_j$ (by definition). Hence, $\underline{L}_P = J\underline{w}$.
Remark: the total angular momentum about $P$, $\underline{L}_P$, is determined once we know angular velocity $\underline{w}$ and inertia matrix at $P$.

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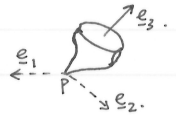
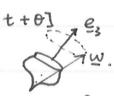
Take axes of $R$ to be principal axes at $P$, i.e. $J = \begin{pmatrix}A&0&0\\0&B&0\\0&0&C\end{pmatrix}$. Then $L_{P1} = Aw_1$, $L_{P2} = Bw_2$, $L_{P3} = Cw_3$. Now $\hat{D}\underline{L}_P = \underline{N} \Rightarrow \dot{\underline{L}}_P + \underline{w}\times\underline{L}_P = \underline{N}$ (Coriolis);
where $\dot{\underline{L}}_P$ is the time derivative in $R$. $\Rightarrow A\dot{w}_1 + (C-B)w_2 w_3 = N_1$ for $\underline{e}_1$ component. Likewise, $B\dot{w}_2 + (A-\cancel{B}^{\,C})w_1 w_3 = N_2$, $C\dot{w}_3 + (B-A)w_1 w_2 = N_3$.
Since $\underline{w}\times\underline{L}_P = \begin{vmatrix}\underline{e}_1 & \underline{e}_2 & \underline{e}_3\\ w_1 & w_2 & w_3\\ Aw_1 & Bw_2 & Cw_3\end{vmatrix} = \underline{e}_1(C-B)w_2 w_3 + \cdots$ These relations are called <u>Euler's equations</u>.
Euler's equations determine the time-dependence of angular velocity, and (indirectly) the orientation of the rigid body.

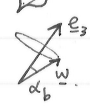
<u>The symmetric (force-free) top</u>.



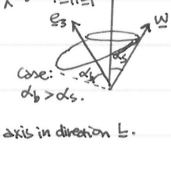

Let the top be spinning about fixed point $P$. By symmetry about $\underline{e}_3$-axis, $A=B$ (degeneracy of $\underline{e}_1, \underline{e}_2$ axes).
[Algebraically, $A = \int_V \rho(y^2+z^2)\,dV = B = \int_V \rho(x^2+z^2)\,dV$]. Since top is force free, $\underline{N} = 0$.
Euler's equations give $C\dot{w}_3 = 0 \Rightarrow w_3$ is constant. Also, $\dot{w}_1 + (\beta w_3)w_2 = 0$, $\dot{w}_2 - \beta w_3 w_1 = 0$ where constant $\beta = \frac{C-A}{A} \Rightarrow$
$\ddot{w}_1 + (\beta w_3)\dot{w}_2 = 0$ (since $w_3$ is constant) $\Rightarrow \ddot{w}_1 + (\beta w_3)^2 w_1 = 0 \Rightarrow w_1 = D\cos[\beta w_3 t + \theta]$ where $D, \theta$ are constants. $\dot{w}_1 = -\beta w_3 D\sin[\beta w_3 t + \theta]$
$\Rightarrow w_2 = D\sin[\beta w_3 t + \theta]$. Hence, $\underline{w} = \begin{pmatrix}D\cos[\beta w_3 t + \theta]\\ D\sin[\beta w_3 t + \theta]\\ w_3\end{pmatrix}$ i.e. about the $\underline{e}_3$ axis, the top precesses in a circle.




The angular velocity vector $\underline{w}$ precesses in a circle of radius $D$ about the $\underline{e}_3$ axis. It sweeps out a cone, called the <u>body cone</u>.
The cone has half-angle $\alpha_b$, where $\tan\alpha_b = \frac{D}{w_3}$.




To find the motion of the body in space, (i.e. find $\underline{e}_3$ relative to some direction) we need to first locate $\underline{w}$ with respect to a fixed direction in space.
If there is zero torque, then we know that $\underline{L}$ is constant, and in particular its direction is constant. The angle between $\underline{w}$ and $\underline{L}$ is $\alpha_s$, where $\cos\alpha_s = \frac{\underline{w}\cdot\underline{L}}{|\underline{w}||\underline{L}|}$.
$\cos\alpha_s = \frac{w_i L_i}{|\underline{w}||\underline{L}|} = \frac{w_i J_{ij} w_j}{|\underline{w}||\underline{L}|}$.




Case: $\alpha_b > \alpha_s$.

Remarks: (i) $|\underline{L}|$ is constant (ii) $|\underline{w}|^2 = D^2 + w_3^2 = \text{const.}$ (iii) $w_i w_j J_{ij}$ is related to (up to constant) total KE about $P \Rightarrow$ also constant.
Thus, $\alpha_s$ is constant; $\cos\alpha_s = \frac{2T}{|\underline{w}||\underline{L}|} = \text{const.}$ The axis of rotation ($\underline{w}$) traces out a cone in space (<u>space cone</u>) with half-angle $\alpha_s$, and axis in direction $\underline{L}$.
The line of contact between the cones at any instant gives the direction of $\underline{w}$.

<u>Stability of rotation about a principal axes</u>.

Consider a rigid body in which all three principal moments of inertia are different. Let the body spin about the $\underline{e}_3$ principal axis s.t. $w_3 = w = \text{const}$ and $w_1 = w_2 = 0$; and $\underline{N} = 0$. Clearly, this is an exact solution of Euler's equations. However, is it stable?
We perturb the motion slightly, such that $w_1 = \varepsilon_1$, $w_2 = \varepsilon_2$, $w_3 = w + \varepsilon_3$; where $\varepsilon_i$ ($i=3$) are small functions of time i.e. $|\varepsilon_i| \ll w$.

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From Euler's equations, $A\dot{\varepsilon}_1 + (C-B)\varepsilon_2(w+\varepsilon_3)=0$, $B\dot{\varepsilon}_2 + (A-C)\varepsilon_1(w+\varepsilon_3)=0$. To a good approximation, ignoring quadratic terms in $\varepsilon_i$, we get the following two evolution equations: $\begin{matrix} A\dot{\varepsilon}_1 + (C-B)w\varepsilon_2 = 0 \\ B\dot{\varepsilon}_2 + (A-C)w\varepsilon_1 = 0 \end{matrix}$ $\Rightarrow AB\ddot{\varepsilon}_1 - (C-B)(A-C)w^2\varepsilon_1 = 0$. Substitute $\varepsilon_1 = e^{pt}$. $\Rightarrow p^2 = \frac{(C-B)(A-C)}{AB}w^2$

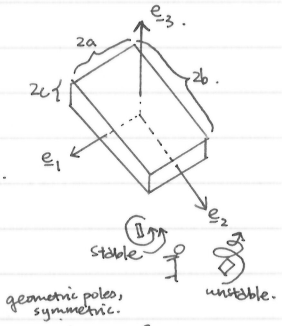
Now, $w^2>0$ and $AB>0$ (positive definite integral). We then get two cases:

Case 1: If $C>B, A<C$ or $C<B, A>C$, then $p^2<0$. $\Rightarrow$ $p$ are purely imaginary. $\Rightarrow$ solutions are oscillatory (sines/cosines) $\Rightarrow$ motion is bounded $\Rightarrow$ stable.

(intermediate)

case 2: If $A>C>B$ or $A<C<B$, then $p^2>0$. $\Rightarrow$ $p$ are real $\Rightarrow$ one of the solutions grows exponentially $\Rightarrow$ system is unstable.

Conclusion — if moment of inertia about rotation axes ($C$) is greatest or least, the motion is stable. If it is otherwise, it is unstable.



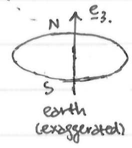

consider the following "book-shaped" object that is homogenous. Pick origin to be at centre of book.

If $b>a>c$, let $J = \frac{M}{3}\begin{pmatrix} b^2+c^2 & 0 & 0 \\ 0 & a^2+c^2 & 0 \\ 0 & 0 & a^2+b^2 \end{pmatrix}$ since $a^2+c^2$ is the most and $a^2+b^2$ is the least, we predict:

stable motion about $e_2$ and $e_3$ axes, with unstable motion otherwise. Do a quasi-experiment: toss a book! (and ignore the falling motion).

Indeed, this does hold (within classical mechanics)

<u>Chandler wobble.</u>



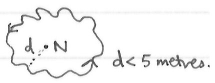

The chandler wobble describes the precession of the earth. The Earth is symmetrical about the polar axis, but fatter at the equator.

Here then, $A=B$, $C=\int_V \rho(y^2+x^2)\,dV > A$. Then for the earth, $\beta = \frac{C-A}{A} \approx \frac{1}{300}$. Hence, from earlier example, we have that

$\Omega \equiv$ precession frequency of $\underline{w}$ about $e_3$ is $\Omega = \beta w_3 = \frac{w_3}{300}$, where $w_3 = 1\ \text{day}^{-1}$ (angular frequency of rotation).

Thus, period of procession $\simeq$ 300 days $\simeq$ 10 months. i.e. in about 10 months, $\underline{w}$ will go around the north pole once.

This precession does actually occur (hence, D is non-zero — but really small); with amplitude < 5m!




However, the period is closer to 14 months than to 10! why?

The main reason is that the Earth is not a completely rigid body — it can buckle, strain and deform accordingly.

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<u>Lagrangian description of rigid body motion.</u>

$\boxed{\text{Proposition}}$ Let $H$ be the transition matrix from $\hat{B}$ to $B$. Then $H=KLM=\begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Proof – (see handout) (NFE).

Sketch of proof. $H: \hat{B} \to B$, $\hat{B} \xrightarrow{M} B'' \xrightarrow{L} B' \xrightarrow{K} B$ (rotations about well-defined directions).

Remark: $\hat{e}_3$ and $e_3$ are not parallel, so $\hat{e}_3 \times e_3 \neq 0$. Let $\underline{j}$ be a unit vector in direction $\hat{e}_3 \times e_3$. $\Rightarrow$ $\underline{j}$ is perpendicular to $\hat{e}_3$ and $e_3$.

e.g. $\underline{j} = \sin\psi\, e_1 + \cos\psi\, e_2$ s.t. $|\underline{j}| = \sqrt{\sin^2\psi + \cos^2\psi} = 1$.

We observe that $H=KLM$, where

(i) $M$ is a rotation about $\hat{e}_3$ through $\phi$, which brings $\hat{e}_2$ into coincidence with $\hat{\underline{j}}$. (see handout).

(ii) $L$ is a rotation about $\underline{j}$ through $\theta$, which brings $\hat{e}_3$ into coincidence with $e_3$.

(iii) $K$ is a rotation about $\hat{e}_3$ through $\psi$, which brings $\hat{\underline{j}}$ into coincidence with $e_2$.

We can always decompose a transition matrix $H$ into its three rotational matrices.

We have shown the transition matrix of $\hat{B}$ to $B$ ($H$) can be written $H=KLM$.

Now recall if $H$ is the transition matrix from $\hat{B}$ to $B$, then the <u>angular velocity</u> of $B$ relative to $\hat{B}$, $\underline{w} = w_i e_i$, is constructed by $\Omega = \dot{H}H^T$, and $\Omega_{jk} = \varepsilon_{ijk} w_i$

i.e. $\Omega = \begin{pmatrix} 0 & w_3 & -w_2 \\ -w_3 & 0 & w_1 \\ w_2 & -w_1 & 0 \end{pmatrix}$. Since $M$ is the transition matrix from $\hat{B}$ to $B''$, the angular velocity of $B''$ relative to $B$ is determined by $\dot{M}M^T$:

$M = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \dot{M}M^T = \begin{pmatrix} -\sin\phi & \cos\phi & 0 \\ -\cos\phi & -\sin\phi & 0 \\ 0 & 0 & 0 \end{pmatrix}\dot{\phi}\begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} = \dot{\phi}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow w_1 = w_2 = 0$, $w_3 = \dot{\phi}$ and hence, $\underline{w} = \dot{\phi}\, e_3'' = \dot{\phi}\,\hat{e}_3$ (look through proof, this is not immediately obvious).

We can do the same thing with matrices $K$ and $L$. This gives us the end-result: $\underline{w} = \dot{\psi}\, e_3 + \dot{\theta}\,\underline{j} + \dot{\phi}\,\hat{e}_3 = \dot{\psi}\, e_3 + \dot{\theta}(\sin\psi\, e_1 + \cos\psi\, e_2) + \dot{\phi}(-\sin\theta\cos\psi\, e_1 + \sin\theta\sin\psi\, e_2 + \cos\theta\, e_3)$.

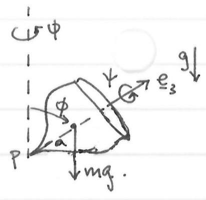
Last term comes from $\hat{e}_3 = H_{j3} e_j$. The components of $\underline{w}$ are thus: $w_1 = \dot{\theta}\sin\psi - \dot{\phi}\sin\theta\cos\psi$; $w_2 = \dot{\theta}\cos\psi + \dot{\phi}\sin\theta\sin\psi$, $w_3 = \dot{\psi} + \dot{\phi}\cos\theta$.

For rigid body motion about a fixed point, recall $T = \frac{1}{2}(Aw_1^2 + Bw_2^2 + Cw_3^2)$ by choosing principal axes. Thus, this gives us —

$T = \frac{1}{2}\left[A(\dot{\theta}\sin\psi - \dot{\phi}\sin\theta\cos\psi)^2 + B(\dot{\theta}\cos\psi + \dot{\phi}\sin\theta\sin\psi)^2 + C(\dot{\psi} + \dot{\phi}\cos\theta)^2\right]$.

This all leads us to the ultimate problem of this course:

<u>Symmetric top in a gravitational field.</u>




Consider a symmetric top $(A=B)$ rotating about a fixed point P under action of gravity. Here, let $a$ be the distance from P to CM.

Then, $T=\frac{1}{2}A[(\dot\theta\sin\psi-\dot\phi\sin\theta\cos\psi)^2+(\dot\theta\cos\psi+\dot\phi\sin\theta\sin\psi)^2]+\frac{1}{2}C(\dot\psi+\dot\phi\cos\theta)^2=\frac{1}{2}A(\dot\phi^2\sin^2\theta+\dot\theta^2)+\frac{1}{2}C(\dot\psi+\dot\phi\cos\theta)^2$

Hence, $L=\frac{1}{2}A(\dot\phi^2\sin^2\theta+\dot\theta^2)+\frac{1}{2}C(\dot\psi+\dot\phi\cos\theta)^2-mga\cos\theta$.

Here, $\phi$, $\psi$ and $t$ are ignorable coordinates, so we should get corresponding laws of conservation. Before we do that, we define

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$\theta\equiv$ <u>angle of nutation</u> (angle of inclination from vertical), $\phi\equiv$ <u>angle of precession</u>.

- $\psi$ is ignorable $\Rightarrow \frac{\partial L}{\partial\psi}=0 \Rightarrow \frac{\partial L}{\partial\dot\psi}=\text{const} \Rightarrow \boxed{\dot\psi+\dot\phi\cos\theta=\text{const}=n}$ ① This is the law of conservation of angular momentum about body's symmetry axis (makes sense as torque has no component in this direction). $\dot\psi$ is due to self-spin, $\dot\phi\cos\theta$ is the component of the precession.
- $\phi$ is ignorable $\Rightarrow \frac{\partial L}{\partial\phi}=0 \Rightarrow \frac{\partial L}{\partial\dot\phi}=\text{const} \Rightarrow \boxed{A\dot\phi\sin^2\theta+C(\dot\psi+\dot\phi\cos\theta)\cos\theta=A\dot\phi\sin^2\theta+Cn\cos\theta=\text{const}=h}$ ②. This is the law of conservation of angular momentum about the vertical axis.
- $t$ is ignorable $\Rightarrow \frac{\partial L}{\partial t}=0$. We note that $T=\frac{1}{2}A[\dot\phi^2\sin^2\theta+\dot\theta^2]+\frac{1}{2}C(\dot\psi+\dot\phi\cos\theta)^2$ is a pure quadratic form in generalised velocities $\dot\phi$, $\dot\psi$, $\dot\theta \Rightarrow T+V$ is constant.

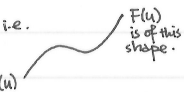
$\therefore$ $\boxed{A\dot\theta^2+A\dot\phi^2\sin^2\theta+2mga\cos\theta=\underbrace{2E-Cn^2}_{\text{const.}}}$ ③. This is the law of conservation of energy.

Without solving the equations, we try to understand (generally) what these equations can yield:

Let $u=\cos\theta$. Then ②: $\dot\phi=\frac{h-Cnu}{A(1-u^2)}$. Also, we can show that ③: $A\dot u^2=F(u)=(2E-Cn^2-2mga\,u)-\frac{(h-Cnu)^2}{A}$ [$\because u=\cos\theta$, $\dot u=-\sin\theta\cdot\dot\theta$, $\dot u^2=(1-u^2)\dot\theta^2$]

Suppose initially that $\theta=\cos^{-1}(u_1)$ and $\dot\theta=0$ (i.e. no initial nutation). $h$ and $n$ are constants with $n>0$ and $0<\frac{h}{Cn}<1$. (this gives more interesting behaviour).

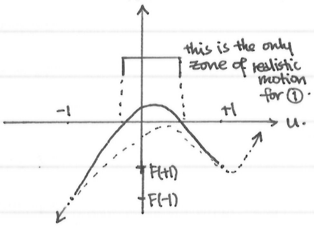
We note that $F(u)$ is cubic in $u$, with $F(u)\to+\infty$ as $u\to+\infty$. i.e. (i) $F(u)\sim u^3$ (+ve) as $u\to\infty$. While this is true, $u$ is bounded. i.e.



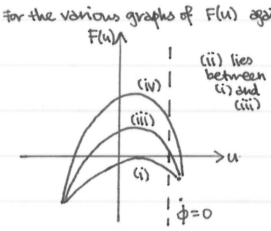

However, as $u=\cos\theta$, $u$ is restricted to values between 1 and $-1$. (ii) $F(+1)=-\frac{(h-Cn)^2}{A}\leq 0$. Similarly, we can find $F(-1)$:

(iii) $F(-1)=-\frac{(h+Cn)^2}{A}\leq 0$. We know that for realistic motion, (iv) $A\dot u^2\geq 0\Rightarrow F(u)\geq 0$. Also $-1\leq u\leq -1$ for it to make sense.

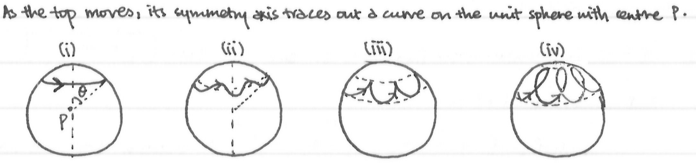
(v) $\dot\phi=0$ when $u=\frac{h}{Cn}<1$




For the various graphs of $F(u)$ against $u$, refer to handout. We have four cases (as seen on handout).




As the top moves, its symmetry axis traces out a curve on the unit sphere with centre P.




(i) $\exists$ a critical value $u=u_1^*$ s.t. $F(u_1^*)=F'(u_1^*)=0$. (touching the u-axis, and is a local maximum turning point)

Hence $u=u_1^*$ is the only allowable value of $u\Rightarrow$ the body precesses steadily ($\because \dot\phi>0$ is constant), tracing out a circle on the unit sphere.

(ii) there is an allowable band for $u$ (and consequently $\theta$), but $\dot\phi>0$ always. Here, $u$ oscillates between the two roots of $F(u)$, at either side of $u_1^*$. Crucially, $\dot\phi>0\ \forall$ motion $\Rightarrow$ the angle of nutation oscillates between two fixed values of $\theta$.

(iii) It can move in a wider band of $\theta$ than case (ii), but $u_{max}$ (i.e. $\theta_{min}$) is critical with $\dot\phi=0$; here $\dot\phi$ remains non-negative. This is what happens when a spinning top is released from rest.

(iv) Here $\dot\phi<0$ for part of the motion, and the trajectory on the sphere "loops".

Ex: A top released from rest (i.e. $\dot\theta=\dot\psi=0$, $\psi\neq 0$). As motion proceeds, $\dot\theta$ and $\dot\phi$ become non-zero in general.

Energy conservation demands that $A\dot\theta^2+A\dot\phi^2\sin^2\theta+2mga\cos\theta=\text{const.}$ Initially, $\dot\theta=\dot\psi=0\Rightarrow 2mga\cos\theta=\text{const.}$

As $t$ increases from 0, $A\dot\theta^2+A\dot\phi^2\sin^2\theta$ becomes positive $\therefore$ $2mga\cos\theta$ decreases $\Rightarrow\cos\theta$ decreases $\Rightarrow\theta$ increases. $\because\theta\in[0,\pi]$. i.e. the top "falls".

And each time it reattains its minimum value of $\theta$ again, it would fall.

END OF SYLLABUS.

7302-14.