7304 Electromagnetism Notes

Based on the 2015 spring lectures by Dr R Bowles

7304 (Electromagnetism)

Year:	2014–2015
Code:	MATH7304
Level:	Intermediate
Value:	Half unit (= 7.5 ECTS credits)
Term:	2
Structure:	3 hours lectures per week. Weekly assessed coursework.
Assessment:	The final weighted mark for the module is given by: 90% examination, 10% coursework. In order to pass the module you must have at least 40% for both the examination mark and the final weighted mark.
Normal Pre-requisites:	MATH2401
Lecturer:	Dr R Bowles

Course Description and Objectives

The course aims to provide students who have an interest in mathematical physics with an introduction to classical electromagnetism and relativistic mechanics. The course should also be of interest to students wishing to see further application of the ideas covered in mathematical methods courses. The course will start with Maxwell's equations and a brief discussion of their historical development and will proceed to study their solution illustrating classical electrostatics and magnetostatic phenomena, together with electromagnetic phenomena including wave propagation. The final part of the course looks at Einstein's special theory of relativity and the generalisation of Newtonian mechanics that follows, together with the insight it gives into our understanding of the relationship between electricity and magnetism.

By the end of this course students should have

- An understanding of steady and time-varying electric and magnetic fields and their description through Maxwell's equations, both in integral and differential form and scalar and vector potentials.
- The ability to calculate steady solutions to these equations for simple geometries and as far-field expansions for more general situations. The ability to calculate electrostatic and magnetic energy, capacitance and inductance for simple geometries.
- An understanding of electromagnetic wave propagation in a vacuum and of energy and momentum flow within time-varying fields and a description of the fields in terms of retarded potentials.
- An understanding of special theory of relativity, space-time, relativistic mechanics and the behaviour of magnetic and electric fields under Lorentz transformation.

should have a copy - all on line

Recommended Texts

- The Feynman Lectures on Physics - Volume II, R.P. Feynman, R.B. Leighton, M.L. Sands, and M.A. Gottlieb, ISBN: 9780805390476, Pearson/Addison-Wesley.
- Special Relativity, N.M.J Woodhouse, ISBN: 1852334266, Springer Undergraduate Mathematics Series.

acts as intro to special/general relativity applications to physics - not needed

↓

– Electricity and Magnetism, W.N. Cottingham and D.A. Greenwood, ISBN: 9780521368032, Cambridge University Press

Griffiths - Electromagnetism excellent book, but hard

Detailed Syllabus

– Electric charge and field. Superposition. Electric current. Magnetic fields. Lorentz force on a moving charge.

– A statement of Maxwell's equations in a vacuum. Lack of magnetic monopoles. Charge conservation. The displacement current. Integral forms of Maxwell's equations.

– Electrostatics. Gauss' theorem. Electric Potential. Green's functions for the Laplace equation. The steady electric field for discrete and continuous distribution of charge. Multipole expansions. Conductors. Surface charge. Boundary conditions at a surface. Energy. Capacitance.

– Electric Currents. Magnetostatics. The Coulomb Gauge. Magnetic Potential. Biot-Savart Law. Boundary conditions at a surface. Magnetic force on conductors. Ampere's Law. Electromagnetic Induction. Magnetic Energy. Self-inductance. Relaxation of a charge distribution within a conductor.

– Electromagnetic waves. Energy and momentum transport in an electromagnetic field. The Poynting vector. The Lorentz Gauge. Wave equations for the electric and magnetic potential. Retarded time.

– Special relativity. Frame invariance. Tensors and metrics. Invariance of $dx^2 - c^2dt^2$. Lorentz transformations, transformation of velocities. Proper time. Relativistic mechanics. Equations of electromagnetism in space-time.

Electromagnetism

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Electromagnetism papers from 6 years ago

Recent papers not relevant

Closest analogue is 3rd year electromagnetism physics course

Office hour: 8-9am Tues + Wed

Room 603

o/w by appt

Revision of vectors

Differentiation of scalar (temp) & vector (velocity) fields

The gradient of a scalar field $\varphi(x,y,z) = \varphi(\underline{r})$

is a vector $\underline{u} = \nabla\varphi$ & in Cartesian coordinates

$$\underline{u} = \begin{pmatrix} \varphi_x \\ \varphi_y \\ \varphi_z \end{pmatrix}$$

The divergence of a vector field $\underline{V}$ is the scalar

$$S = \underline{\nabla} \cdot \underline{V}$$

& if $\underline{V} = \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}$, $\underline{\nabla} \cdot \underline{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$

The curl of a vector $\underline{V}$ is the vector $\underline{u} = \underline{\nabla} \wedge \underline{V}$

$$= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_1 & V_2 & V_3 \end{vmatrix}$$ in Cartesian coordinates

The Laplacian of a scalar φ is $\nabla^2\varphi = \varphi_{xx} + \varphi_{yy} + \varphi_{zz}$

$$= \nabla \cdot (\nabla\varphi)$$

The Laplacian of a vector $\nabla^2 \underline{V}$ is the vector made up of the Laplacian of each component $\begin{pmatrix} \nabla^2 V_1 \\ \nabla^2 V_2 \\ \nabla^2 V_3 \end{pmatrix}$

The directional derivative of a scalar φ in the direction $\underline{n}$ (not necess. unit vector)

is $\hat{\underline{n}} \cdot \underline{\nabla}\varphi$

no underline to give length

The expression $\underline{u} \cdot \underline{\nabla}\varphi$ is $|\underline{u}| = u$ times $\hat{\underline{u}} \cdot \underline{\nabla}\varphi$

$$= u_1 \frac{\partial \varphi}{\partial x} + u_2 \frac{\partial \varphi}{\partial y} + u_3 \frac{\partial \varphi}{\partial z}$$

The expression $(\underbrace{\underline{u} \cdot \underline{\nabla}}_{\text{vector}})\underline{v}$ is $\underline{u} \cdot \underline{\nabla}$ acting on the three components of $\underline{V}$ in turn

of we have among others, the following

vector field

$$\underline{\nabla} \cdot (\underline{\nabla} \wedge \underline{F}) = 0 \qquad \underline{\nabla} \wedge (\underline{\nabla}\varphi) = \underline{0}$$

$$\underline{\nabla}(\varphi \chi) = \chi(\underline{\nabla}\varphi) + \varphi(\underline{\nabla}\chi)$$

$$\underline{\nabla} \wedge (\underline{\nabla} \wedge \underline{F}) = \underline{\nabla}(\underline{\nabla} \cdot \underline{F}) - \nabla^2 \underline{F}$$

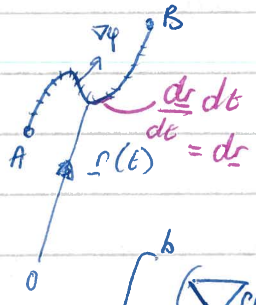
vector field scalar $\underline{\nabla} \cdot (\varphi \underline{u}) = (\underline{\nabla}\varphi) \cdot \underline{u} + \varphi \, \underline{\nabla} \cdot \underline{u}$

$$\underline{\nabla} \cdot (\varphi \underline{u}) = \frac{\partial}{\partial x}(\varphi u_1) + \frac{\partial}{\partial y}(\varphi u_2) + \frac{\partial}{\partial z}(\varphi u_3)$$

$$= \varphi\left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}\right) + \varphi_x u_1 + \varphi_y u_2 + \varphi_z u_3$$

$$= \varphi(\underline{\nabla} \cdot \underline{u}) + \underline{u} \cdot \underline{\nabla}\varphi$$

We also have integral theorems



Might describe path by saying how pv of a point on path varies

Find tangent vector to path which is $\frac{d\underline{r}}{dt}$

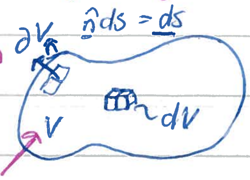
$$\int_a^b \underbrace{(\underline{\nabla}\varphi) \cdot d\underline{r}}_{d\varphi} = [\varphi]_a^b = \varphi(b) - \varphi(a)$$

proof is just chain rule

This is the line integral of a gradient

vector w/ direction normal
magnitude of the bit it represents

Divergence Theorem

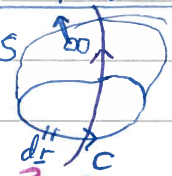


$$\int_V (\underline{\nabla} \cdot \underline{u})\, dV = \int_{\partial V} \underline{u} \cdot d\underline{S}$$

sometimes ~~might~~ might write $\iiint_V$

vector field $\underline{u}$ → $\underline{u}$

Stokes Theorem



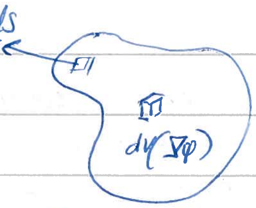
$r(t)$, $\quad d\underline{r} = \dfrac{d\underline{r}}{dt}\, dt$

$$\oint_C \underline{u} \cdot d\underline{r} = \iint_S (\underline{\nabla} \wedge \underline{u}) \cdot d\underline{S}$$

$$(\underline{\nabla} \wedge \underline{u}) \cdot d\underline{S}$$

For any surface that closed loop acts as bdry find curl u
and direction of curl u w/ normal to surface
Vector field happens to be curl of something
Result is independent of surface but has same edge c
Will either be that or minus that
Need to think of it in terms of right handed corkscrew

These can be extended (more examples in notes + HW)



$$\int_V \nabla\varphi \, dV = \int_S \varphi \, d\underline{S}$$

this is a vector $\hat{\underline{n}}\, dS$

answer is vectors since adding up lots of vectors

constant scalar

Consider the divergence of $\underline{a}\varphi = \underline{u}$
Then the divergence theorem gives

$$\int_V \underline{\nabla} \cdot \underline{u}\, dV = \int_V \nabla \cdot (\underline{a}\,\varphi)\, dV$$

$$= \int_S \underline{a}\,\varphi \cdot dS$$

but $\underline{\nabla}\cdot(\underline{a}\varphi) = \underline{a}\cdot\underline{\nabla}\varphi + \varphi\,\underline{\nabla}\cdot\underline{a}$

If $\underline{a}$ is a constant vector then this is → w.r.t space i.e. const. function of position

$\underline{a}\cdot\underline{\nabla}\varphi$

& we have

$$\underline{a}\cdot\int \underline{\nabla}\varphi\, dV = \underline{a}\cdot\int_S \varphi\, \underline{dS}$$

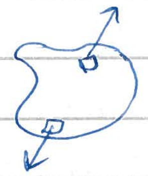
This is true independent of a, i.e. a is arbitrary

so

$$\int_V \nabla\varphi\, dV = \int_S \varphi\, dS$$

Sum of vectors over whole surface is 0

i.e. if $\varphi = 1$ then $\oint_S \varphi\, dS = 0$



they cancel each other out

Helmholtz' Theorem

$\underline{u}(r)$ as $r\to$

If we know $\underline{\nabla}\cdot\underline{u} = Q(\underline{r})$ & we know

scalar

$\underline{\nabla}\wedge\underline{u} = W(r)$

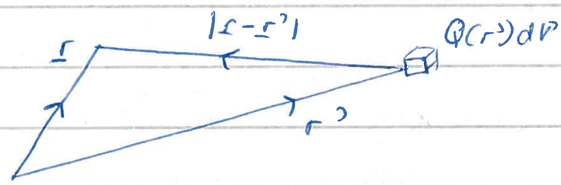
then if $u(\underline{r})\to 0$ as $r\to\infty$ then we can reconstruct $\underline{u}$ uniquely

1) Define $\varphi(\underline{r}) = \dfrac{1}{4\pi}\displaystyle\int_{V'} \frac{Q(\underline{r}')}{|\underline{r}-\underline{r}'|}\, dV'$

dashed variables are variables of integration

Note that $r = r'$

Choose $\underline{r}$ & integrate w/ dashed variables



' variables are variables of integration

$\underline{r}$ is a parameter in the integral

2) Define $\underline{A}(\underline{r}) = \frac{1}{4\pi} \int_{V'} \frac{\underline{W}(\underline{r}')}{|\underline{r} - \underline{r}'|} \, dV'$

singularity at $r = r'$? No problem, will see why later - there is a zero at top as well

Then $\underline{u} = -\underbrace{\nabla\varphi}_{\text{scalar potential}} + \nabla \wedge \underbrace{\underline{A}}_{\text{vector potential}}$

In terms of fluids: div represents source or sink
curl represents swirl

If $\nabla \cdot \underline{u} = 0 = Q$, $\varphi = 0$ & $\underline{u} = \nabla \wedge \underline{A}$ — no sources or sinks

Then $\underline{u}$ is called divergence less or solenoidal
If $\nabla \wedge \underline{u} = \underline{0} = \underline{W}$, $\underline{A} = \underline{0}$ & $\underline{u} = -\nabla\varphi$
u is said to be irrotational or conservative

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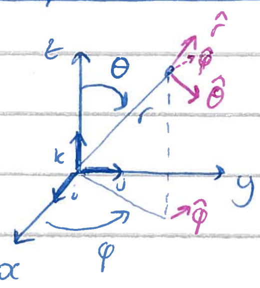
Spherical polar coordinates



← actual directions vary from point to point

$\underline{a} = \hat{\varphi} = x\underline{i} + y\underline{j} + z\underline{k}$ (y: $\hat{\varphi}\cdot j$, z: $\hat{\varphi}\cdot k$)

Looking for projection of vector

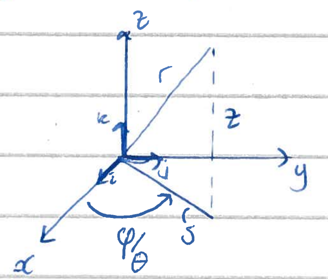
So $i\cdot\hat{\varphi} = x\cdot 1 + 0 + 0$ ← projection of $\hat{\varphi}$ on to i

$i\cdot\hat{\varphi} = -\sin\varphi$

Want to go from cartesian to spherical - need to find r, θ, φ

$r^2 = x^2+y^2+z^2$ $\quad \cos\theta = z/r$ $\quad \tan\varphi = y/x$

Cylindrical polars



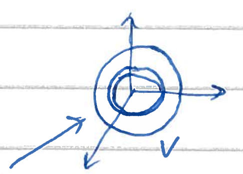
$r^2 = z^2+s^2$

$\tan\varphi = y/x$

if this happens i.e. as you rotate around value does not change

$$\int_V f(\underline{r})\,dV = \int f(r)\,dV$$

$$= \int_0^a f(r)\cdot 4\pi r^2 dr$$



spherical shell, made up of multiple ones this can be our volume rather than ~~□~~

$dv = dS\cdot dr$
$4\pi r^2\cdot dr$

Index Notation

We can express a vector

$\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$, as a_i

& if $\underline{a} = \underline{b}$ then $a_i = b_i$ – 3 equations

In 1402: $\underline{a} = a_1\underline{e}_1 + a_2\underline{e}_2 + a_3\underline{e}_3$

We are just going to use a_i component

Remember that when diff. polar you get contributions from a_i and $\underline{e}_i$ since $\underline{e}_i$ not fixed

We will be implicitly supposing that $\underline{e}_i$ are constant

Summation Convention

If an index is repeated we sum over it.

$\underline{a} \cdot \underline{b} = a_i b_i = a_j b_j$

free index – should have it on either side of equal sign

$\{(\underline{a} \cdot \underline{b})\underline{c}\}_i = (a_j b_j)c_i$

$= (a_k b_k)c_i \neq a_i b_i c_i$

Means i^{th} component

Also $\{\underline{\underline{M}}\,\underline{a}\}_i = M_{ij} a_j$ talking about i^{th} component

$\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$

$= x_1\underline{i} + x_2\underline{j} + x_3\underline{k}$

$\{\underline{r}\}_i = x_i$

$\underline{\nabla}$ can be expressed as $\frac{\partial}{\partial x_i}$

$\{\underline{\nabla}\varphi\}_i = \frac{\partial \varphi}{\partial x_i}$

$\underline{a} \cdot \nabla\varphi = a_i \frac{\partial}{\partial x_i} \varphi$ so sum over i: $a_1 \frac{\partial \varphi}{\partial x} + a_2 \frac{\partial \varphi}{\partial y} + a_3 \frac{\partial \varphi}{\partial z}$

$\nabla \cdot \underline{u} = \frac{\partial u_i}{\partial x_i}$ may start with writing ∇ as $\frac{\partial}{\partial x_i}$
can write ∂_i or $,i$

$= \partial_i x_i$

$= x_{i,i}$ ← where we are diff. w.r.t.

↑ what we are differentiating

If $\varphi = x_i$ & I want $\nabla\varphi$ I will need to know $\frac{\partial}{\partial x_j} x_i$
i.e. diff x wrt x, y or z so either 0 or 1

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & i \neq j \end{cases}$$

Consider the function $\varphi(\underline{r}) = r$
(Note $\hat{r} = \underline{r}/r$)
Then $r^2 = x^2 + y^2 + z^2 = x_i x_i = x_j x_j$ ← changed since we want i as free index
Now differentiate w.r.t. x_i

$$2r \underbrace{\frac{\partial r}{\partial x_i}}_{\{\nabla\varphi\}_i} = 2x_j \frac{\partial x_j}{\partial x_i} = 2x_j \delta_{ij} = 2x_i$$

↑ summing over j
only $\neq 0$ when $j = i$

$\{\nabla\varphi\}_i = x_i/r \longrightarrow \nabla r = \underline{r}/r = \hat{\underline{r}}$

Could have done $r = \sqrt{x^2+y^2+z^2}$ and worked out $\partial r/\partial x$, $\partial r/\partial y$, $\partial r/\partial z$

$\nabla \cdot \underline{r} = \frac{\partial}{\partial x_i} x_i = \delta_{ii} = 1 + 1 + 1 = 3$

↑ sum!

$$\{\nabla(r^n)\}_i = \frac{\partial}{\partial x_i}(x_j x_j)^{n/2} = \frac{n}{2}(x_j x_j)^{n/2 - 1}\left(x_j \frac{\partial x_j}{\partial x_i} + \frac{\partial x_j}{\partial x_i} x_j\right)$$

since we had δ_{ij} from $\frac{\partial x_j}{\partial x_i}$

$$= \frac{n}{2}(x_j x_j)^{\frac{n}{2}-1} \cdot 2x_i \quad = n(r)^{n-2} x_i$$

So $\underline{\nabla}(r^n) = n r^{n-2} \underline{r} = n r^{n-1} \hat{\underline{r}}$

$\nabla^2 r^n = n(n+1) r^{n-2}$ Take gradient then divergence

$(= 0 \text{ if } n = -1)$

$\nabla^2\left(\frac{1}{r}\right) = 0$ ie $\frac{1}{\sqrt{x^2+y^2+z^2}}$ is Harmonic

away from the origin - they don't exist when these derivatives are zero

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Levi - Civita Symbol $\varepsilon_{ijk} = \begin{cases} 1 & \text{if } ijk \text{ even permutation of } (1,2,3) \\ -1 & \text{" odd "} \\ 0 & \text{if at least two of } i,j,k \text{ are same} \end{cases}$

even: $(1,2,3)$ $3,1,2$ or $2,3,1$

odd: $1,3,2$

then

$\{\underline{a} \wedge \underline{b}\}_i = \varepsilon_{ijk} a_j b_k$ 9 sums, lots of them will be 0

20/01/15

$$\varepsilon_{ijk} = \begin{cases} 1 \\ -1 \\ 0 \end{cases}$$

$\underline{\nabla} \wedge \underline{\nabla}\varphi = \underline{0}$

$\{\underline{\nabla} \wedge \underline{\nabla}\varphi\}_i = \varepsilon_{ijk} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} \varphi = \varepsilon_{ijk} \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_j} \varphi$

j & k are summed & can be replaced by k & j $\longrightarrow$ $= \varepsilon_{ikj} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} \varphi = -\varepsilon_{ijk} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} \varphi$

So $\underline{\nabla} \wedge \underline{\nabla}\varphi = \underline{0}$

$\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$

$\{\underline{\nabla} \wedge (\underline{\nabla} \wedge \underline{F})\}_i = \varepsilon_{ijk} \frac{\partial}{\partial x_j} \underbrace{\varepsilon_{klm} \frac{\partial}{\partial x_l} F_m}_{\{\underline{\nabla} \wedge \underline{F}\}_k}$

$= \varepsilon_{kij}\varepsilon_{klm} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_l} F_m$

$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_l} F_m$

$= \underbrace{\frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} F_j}_{\frac{\partial}{\partial x_i}(\underline{\nabla} \cdot \underline{F})} - \underbrace{\frac{\partial}{\partial x_j} \frac{\partial}{\partial x_j} F_i}_{\nabla^2 F_i}$

So $\underline{\nabla} \wedge \underline{\nabla} \wedge \underline{F} = \underline{\nabla}(\underline{\nabla} \cdot \underline{F}) - \nabla^2 \underline{F}$

inverse is transpose

translation

$$\underline{r} = \underline{\underline{H(t)}}\,\underline{r}' + \underline{T}(t)$$
$$\dot{\underline{r}} = \dot{\underline{\underline{H}}}\,\underline{r}' + 2\dot{\underline{\underline{H}}}\,\dot{\underline{r}}' + \underline{\underline{H}}\,\ddot{\underline{r}}' + \ddot{\underline{T}}$$
$$\ddot{\underline{r}} = \underline{\underline{H}}\,\ddot{\underline{r}}'$$

GALILEAN RELATIVITY

e

1) Particles have charge, positive or negative, & the strength of a particle's interaction with electric or magnetic fields is proportional to the charge.
Measured in Coulombs C

Electrostatic far more powerful than gravity ~ 10^{30}
Net charge force often zero

2) Electric $\underline{E}$ & magnetic $\underline{B}$ are time dependent vector fields

3) The force felt by a charge e in a field is
$$\underline{f} = e(\underline{E} + \underline{v} \wedge \underline{B})$$
where $\underline{v}$ the velocity of the charge e
E has units $kgms^{-2}/C$, or Newtons / Coulomb
charge ~~B~~ Volts / metre

B has units $kgms^{-2}/cms^{-1} = kg\,s^{-1}c^{-1}$
Tesla

Magnetic fields don't do any work
Take dot product of v w/ mB & get nothing

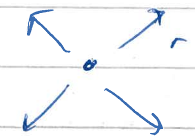
4) Moving or static charges generate $\underline{E}$ & $\underline{B}$

We will take as read

EM1 The fields generated depend linearly on the charges

EM2 A stationary charge generates an electric field only which drops off in strength with an inverse square law

$$\underline{E} = \frac{e}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{\underline{r}}$$



More than one charge - attract or repel so don't really find stationary

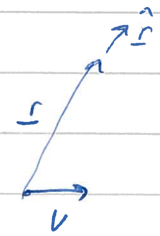
permittivity of free space $8.9 \times 10^{-12}\ C^2 s^2 m^{-3} kg^{-1}$

$$\underline{f} = e\underline{E}$$

Charges different - attractive so radially inward

EM3 A moving charge with "slow" velocity $\underline{v}$ generates a magnetic field

$$\underline{B} = \frac{\mu_0}{4\pi}\frac{1}{r^2}(\underline{v} \wedge \hat{\underline{r}})$$



$\underline{v} \wedge \underline{r}$ is coming round out on paper

μ_0 is permeability of free space
$1.3 \times 10^{-6}\ kg\ m\ C^{-2}$

$$\frac{1}{\sqrt{\mu_0\varepsilon_0}} = c \text{ speed of light}$$

$$\rho = \lim_{V \to 0} \frac{\sum_V e}{V}$$

charge density

volume to zero

$$\underline{J} = \lim_{V \to 0} \frac{\sum_V e\underline{v}}{V}$$

current density

$de = \rho\, dV$

Electric Field

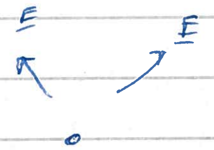
$$\underline{E} = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\underline{r}$$

$$\nabla\cdot\underline{E} = \frac{1}{4\pi\varepsilon_0}\nabla\cdot\left[\underline{r}\,\frac{1}{r^3}\right] = \frac{1}{4\pi\varepsilon_0}\left[\underline{r}\cdot\nabla\left(\frac{1}{r^3}\right) + \frac{1}{r^3}\nabla\cdot\underline{r}\right]$$

$$\left(\nabla\cdot(\underline{v}f) = \underline{v}\cdot\nabla f + f\,\nabla\cdot\underline{v}\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\underline{r}\left(\frac{-3}{r^4}\right)\cdot\underbrace{\nabla r}_{\hat{r}} + \frac{1}{r^3}\underbrace{\nabla\cdot\underline{r}}_{3}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{-3}{r^3} + \frac{3}{r^3}\right) = 0$$



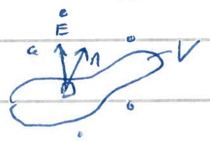
$$\nabla\cdot\underline{E} = 0$$

Regions where no charge $\nabla\cdot E = 0$

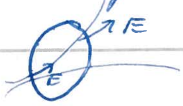
If V is a volume free of charge

$$\int_V \nabla\cdot E\,dV = \int_V 0\,dV = 0$$

from divergence theorem $\displaystyle\int_{\partial V} E\cdot dS = 0$



Total flux is zero



← if there are no charges

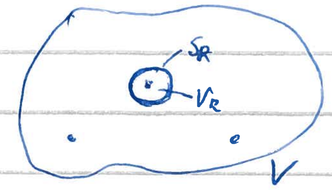
i.e. no sources or sinks

incompressible

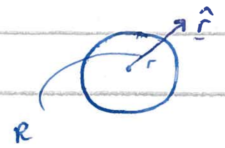
E flows like a fluid

1, the volume contains a single charge e

We consider the integral of $\int_{S_R} \underline{E}\cdot d\underline{S}$



with S_R the surface of a spherical volume V_R, radius R, centre the charge



$$\underline{E} = \frac{e}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{\underline{r}} = \frac{e}{4\pi\varepsilon_0}\frac{1}{R^2}\hat{\underline{r}}$$

so $$\underline{E}\cdot\hat{\underline{n}} = \frac{e}{4\pi\varepsilon_0}\frac{1}{R^2}\hat{\underline{r}}\cdot\hat{\underline{r}} = \frac{e}{4\pi\varepsilon_0}\frac{1}{R^2}$$

$$\int_{S_R} \underline{E}\cdot\hat{\underline{n}}\, dS = \frac{e}{4\pi\varepsilon_0}\frac{1}{R^2}\int_{S_R} dS$$

$$= \frac{e}{4\pi\varepsilon_0}\frac{1}{R^2}4\pi R^2 = \frac{e}{\varepsilon_0}$$

amount of charge / permittivity

$\cancel{\oint \underline{E}\cdot d\underline{S}}$

$$\boxed{\int \underline{\nabla}\cdot\underline{E}\, dV = \frac{Q}{\varepsilon_0}}$$ with Q the total charge inside V

Gauss Theorem

But $Q = \int_V \rho\, dV$ by definition

So $$\int_V \left[\underline{\nabla}\cdot\underline{E} - \rho/_{\varepsilon_0}\right] dV = 0$$

independent of V

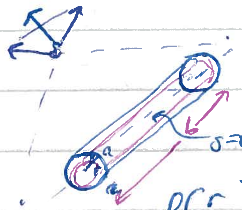
$$\Longrightarrow \boxed{\underline{\nabla}\cdot\underline{E} = \rho/_{\varepsilon_0}}$$ Coulomb's Law

says exactly the same as Gauss but in terms of differentiation electric field points outwards

charge causes electric field that is radial

purple bits cancel

Example - use of Gauss Theorem



infinitely long

insulator so charges don't flow

fixed amount of charge at particular point

$\rho(r) = \frac{q}{\lambda} r$ for $0 \le r \le a$

$= 0$ for $r > a$

Symmetry says $\underline{E} = \hat{\underline{s}}\, E(s)$ ← since cannot tell where you are rel. to another point since infinitely long

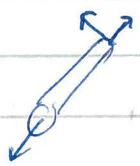
Gauss Theorem

$$\int_{\partial V} \underline{E} \cdot \underline{dS} = Q/\varepsilon_0 = \int_V \underline{\nabla} \cdot \underline{E}\, dV$$

Taking chunk of cylinder of length L

Choose the cylindrical volume radius s, length L aligned with the centre of the cylinder of charge

$$\int_{\partial V} \underline{E} \cdot \underline{dS} = Q/\varepsilon_0$$



End: normal points out & E is radial

Over the endcaps $\int \underline{E} \cdot dS = 0$ as $\underline{E} \perp \underline{dS}$

On cylindrical part $\underline{E} \parallel dS$ (parallel) & $\underline{E}$ is constant

& $\underline{dS} = \hat{\underline{s}}\, dS$ & $\underline{E} \cdot \underline{dS} = E(s)\, \hat{\underline{s}} \cdot \hat{\underline{s}}\, dS$

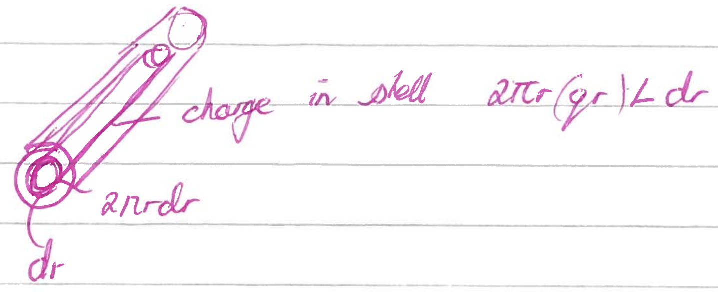
constant since s const

$$\text{So} \int_{\partial V} \underline{E} \cdot \underline{dS} = E(s) \int dS$$

length · circumference

$$= E(s)\, 2\pi s L$$

$$Q/\varepsilon_0 = \frac{1}{\varepsilon_0} \int_V \rho\, dV$$ since $Q = \int_V \rho\, dV$



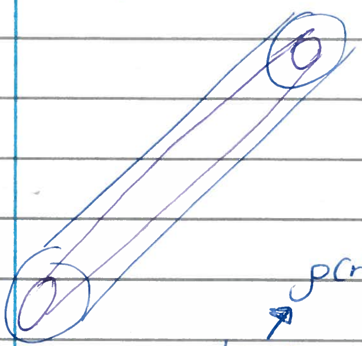
$$Q/\varepsilon_0 = \int_0^s 2\pi r(qr)L\,dr = \frac{1}{\varepsilon_0} 2\pi q L \frac{s^3}{3}$$

Gauss Theorem

$$\int_{\partial V} \underline{E} \cdot d\underline{S} = Q/\varepsilon_0 = \int_V \underline{\nabla} \cdot \underline{E} \, dV$$

where Q is total charge within V

Example



- infinitely long
- insulator so charges don't flow
- fixed amount of charge at particular point

$$\rho(r) = \begin{cases} qr & 0 \leq r \leq a \\ 0 & r > a \end{cases}$$

depends on radius

charge density

Symmetry says $\underline{E} = \hat{\underline{s}} \, E(s)$

Take chunk of cylinder length L radius s aligned with centre of cylinder of charge

At the end: normal points out & E is radial

i.e. $\int \underline{E} \cdot d\underline{S} = 0$ as $\underline{E} \perp d\underline{S}$

On cylinder: $E \parallel dS$ and $\underline{E}$ is constant, $d\underline{S} = \hat{\underline{s}} \, dS$

$$\underline{E} \cdot d\underline{S} = E(s) \, \hat{\underline{s}} \cdot \hat{\underline{s}} \, dS$$

So $\int_{\partial V} \underline{E} \cdot dS = E(s) \int dS = E(s) \, 2\pi s \, L$

circumference ← length

$$Q/\varepsilon_0 = \frac{1}{\varepsilon_0} \int_V \rho \, dV \quad \text{since} \quad Q = \int_V \rho \, dV$$

Charge in shell: $2\pi r(qr) L \, dr$

$$Q/\varepsilon_0 = \frac{1}{\varepsilon_0} \int_0^s 2\pi r (qr) L \, dr = \frac{1}{\varepsilon_0} 2\pi q \frac{L s^3}{3}$$

since depends on radius

Equating both sides

$$E(s) = \frac{q s^2}{3 \varepsilon_0}$$

For the field strength outside use a cylinder radius $s > a$

$$\rho(r) = \begin{cases} qr & r < a \\ 0 & r > a \end{cases}$$

$$\int \underline{E} \cdot dS = 2\pi s L E(s)$$
$$= \int_0^r L\, 2\pi r \rho \, dr / \varepsilon_0$$
$$= \int_0^a L\, 2\pi r\, q r \, dr / \varepsilon_0$$
$$= L\, 2\pi q a^3 / 3\varepsilon_0$$

$$\Longrightarrow \underline{E} = \hat{s} E(s) = \frac{q a^3}{3 \varepsilon_0} \frac{1}{s} \hat{\underline{s}}$$

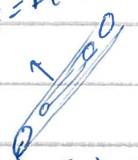
The total charge / unit length is $2\pi q a^3 / 3$

So $\underline{E} = \frac{Q}{2\pi \varepsilon_0} \frac{1}{s} \hat{s}$ independent of $\rho(r)$
- symmetry which arises from fact that field just depends on radius

21/01/15

Gauss ~~Theorem~~ Law $\int_{\partial V} \underline{E}\cdot d\underline{S} = Q/\varepsilon_0$, Q is charge in V

$\underline{E} = E(s)\hat{s}$



$\rho(r) = qr$

cylinder aligns w/ centre $\Longrightarrow$ electric field does not depend on position

$$\int \underline{E}\cdot d\underline{s} = \underbrace{2\pi s}_{\text{circumference}} L E(s) = \int_0^s L 2\pi r (qr)\, dr/\varepsilon_0$$

$dV = L 2\pi r\, dr$

$$= L\cdot 2\pi q s^3/3\varepsilon_0$$

$$E(s) = \frac{q s^2}{3\varepsilon_0}$$

Charge distribution in radial direction

Field strength outside, use a cylinder radius $s > a$

$\rho(r) = qr \quad r < a$

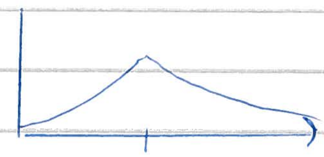
$\quad\;\; 0 \quad r > a$

Similarly $\int \underline{E}\cdot d\underline{s} = 2\pi s L E(s)$

$$= \int_0^a L 2\pi r (qr)\, dr/\varepsilon_0$$

$$= L 2\pi q a^3/3\varepsilon_0$$

$$\Longrightarrow \underline{E} = \frac{q a^3}{3\varepsilon_0}\frac{1}{s}\hat{s}$$



Note the total charge / unit length inside distribution is $2\pi q a^3/3$

So $\underline{E} = \frac{Q}{2\pi\varepsilon_0}\frac{1}{s}\hat{s}$ independent of $\rho(r)$

symmetry which arises from fact that field just depends on radius

Would be expected to do arguments with spherically symmetric

$$\underline{E} = \frac{e}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2} = \frac{e}{4\pi\varepsilon_0}\frac{\underline{r}}{r^3}$$

Note $\nabla(1/r) = -1/r^2\, \nabla r = \frac{-\hat{r}}{r^2}$

So $\underline{E} = \frac{-e}{4\pi\varepsilon_0}\nabla(1/r)$ & so $\underline{E} = -\nabla\varphi$

$$\varphi(\underline{r}) = \frac{e}{4\pi\varepsilon_0}\frac{1}{r}$$

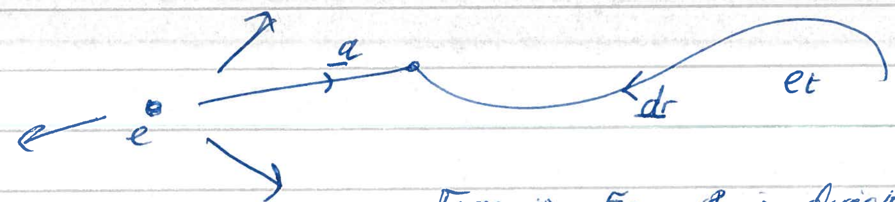
φ is Electric Potential measured in Volts

We say $\varphi(r) \to 0$ as $r \to \infty$ ← this is how we choose our constant of integration

Can say φ on some other surface is zero which fixes const of integration

Since $e\underline{E}$ is the repulsive force acting on a charge e

$\varphi(\underline{r})$ is The work done in bringing a test charge e_t from infinity to a point $\underline{a}$ along a path $\underline{r}(t)$



Force is $\underline{E}e_t$ & is directed outwards

& work done in moving the distance $\underline{dr}$ is $dw = -e_t\,\underline{E}\cdot\underline{dr}$

Total work is

$$W = -\int_\infty^a e_t\,\underline{E}\cdot\underline{dr}$$

$$= e_t\int_\infty^a \nabla\varphi\cdot\underline{dr}$$

$$= e_t[\varphi(a) - \varphi(\infty)]$$
$$= e_t\,\varphi(a)$$

$\varphi(\underline{r})$ is the work done per unit charge in bringing charge to a position $\underline{r}$ in the field $\underline{E}(\underline{r})$, $\underline{E} = -\underline{\nabla}\varphi$

$$\varphi(\underline{r}) = \frac{e}{4\pi\varepsilon_0}\frac{1}{r}$$

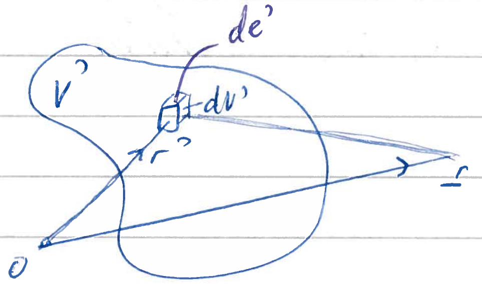
Since $\underline{E} = -\underline{\nabla}\varphi$, $\underline{\nabla}\wedge\underline{E} = \underline{0}$

However for a situation with time varying magnetic fields

$\underline{\nabla}\wedge\underline{E} = -\partial B/\partial t$

Given a charge distribution determined by a charge density $\rho(\underline{r})$, the charge at a position $\underline{r}'$ is $\rho(\underline{r}')\,dV' = de'$

small bit of charge

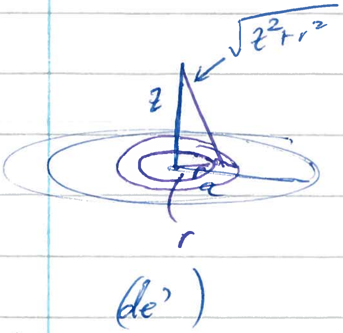


The potential at a point $\underline{r}$ due to this bit of charge de' is

$$d\varphi = \frac{1}{4\pi\varepsilon_0}\frac{de'}{|\underline{r}-\underline{r}'|}$$

So $$\varphi(\underline{r}) = \frac{1}{4\pi\varepsilon_0}\int_{V'}\frac{\rho(\underline{r}')}{|\underline{r}-\underline{r}'|}\,dV'$$

$\underline{E} = -\underline{\nabla}\varphi$



Flat disk, radius a, with a uniform distribution of charge σ per unit area

What is the electric potential at a height z above the centre of the disk & what is the electric field?

Lots of rings making up disk so add all φ

$$\frac{\sigma 2\pi r\, dr}{\sqrt{z^2+r^2}} \quad \frac{1}{4\pi\varepsilon_0}$$

$\uparrow$

$(|r-r'|)$

so integrating

$$\varphi(z) = \frac{2\pi\sigma}{4\pi\varepsilon_0}\int_0^a \frac{r\,dr}{(z^2+r^2)^{1/2}}$$

$$= \frac{\sigma}{2\varepsilon_0}\Big[(z^2+r^2)^{1/2}\Big]_0^a$$

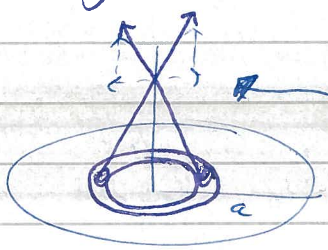
$$= \frac{\sigma}{2\varepsilon_0}\Big((z^2+a^2)^{1/2} - z\Big)$$

$$Q = \sigma\cdot\pi a^2 \Rightarrow \sigma = \frac{Q}{\pi a^2}$$

$\underline{E} = -\nabla\varphi \leftarrow$ only depends on one of x, y, z

$$= -\underline{k}\frac{\partial}{\partial z}\varphi = \frac{\underline{k}\,\sigma}{2\varepsilon_0}\left[1 - \frac{z}{\sqrt{a^2+z^2}}\right]$$

Other way of doing this is by directly trying to find E, by seeing what E generated by ring of charge is.



E·k is only bit that matters

Horizontals cancel

Verticals add

The component of the vector $\underline{dE}$ from an element of charge de in the $\underline{k}$ direction is $\underline{dE}\cdot\underline{k} = \underline{dE}\cos\theta$, θ as in diagram

The contributions from a ring of charge have components that cancel in the radial direction

All that survives is the sum of contributions to $\underline{E}$ in the $\underline{k}$ direction

$$\underline{dE} = \frac{\underline{k}}{4\pi\varepsilon_0}\,\frac{2\pi r\sigma}{z^2+r^2}\,dr\cos\theta$$

but $\cos\theta = \dfrac{z}{\sqrt{z^2 + r^2}}$

$$\underline{E} = \frac{k\sigma}{2\varepsilon_0} z \int_0^a \frac{r}{(z^2+r^2)^{3/2}} dr$$

$$= \frac{k\sigma z}{2\varepsilon_0}\left[-\frac{1}{(z^2+r^2)^{1/2}} \right]_0^a$$

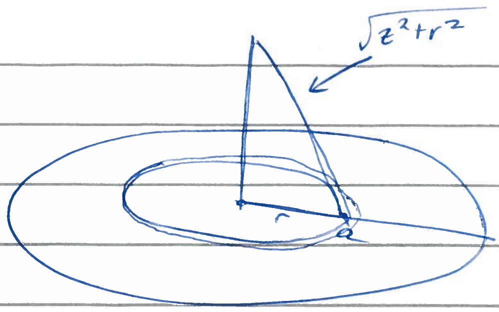
$$= \frac{k\sigma}{2\varepsilon_0}\left[1 - \frac{z}{\sqrt{a^2+z^2}} \right]$$

If $\underline{E} = -\underline{\nabla}\varphi$ & $\underline{\nabla}\cdot\underline{E} = \rho/\varepsilon_0$ then

$$\underline{\nabla}\cdot(\underline{\nabla}\varphi) = \nabla^2\varphi = -\underline{\nabla}\cdot\underline{E} = -\rho/\varepsilon_0$$

$$\boxed{\nabla^2\varphi = -\rho/\varepsilon_0}$$ Poisson Equation for Potential

Will later find out solution



Flat disk, radius a with a uniform distribution of charge σ per unit

What is the electric potential at a height z above the centre of the disk & what is the electric field?

We have $\dfrac{\sigma 2\pi r\,dr}{\sqrt{z^2+r^2}} \quad \dfrac{1}{4\pi\varepsilon_0}$

$|r - \vec{r}'|$

lots of rings make up this disk so add all up

$$\varphi(z) = \frac{2\pi\sigma}{4\pi\varepsilon_0}\int_0^a \frac{r\,dr}{(z^2+r^2)^{1/2}}$$

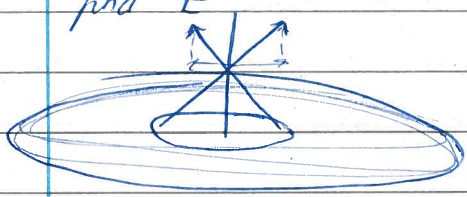
$$= \frac{\sigma}{2\varepsilon_0}\left[(z^2+r^2)^{1/2}\right]_0^a$$

$$= \frac{\sigma}{2\varepsilon_0}\left((z^2+a^2)^{1/2} - z\right)$$

Note: $Q = \sigma \cdot \pi a^2 \implies \sigma = \dfrac{Q}{\pi a^2}$

$$\underline{E} = -\nabla\varphi = -\underline{k}\frac{\partial\varphi}{\partial z} = \frac{\underline{k}\,\sigma}{2\varepsilon_0}\left[1 - \frac{z}{\sqrt{a^2+z^2}}\right]$$

The other way of doing this is by directly trying to find $\underline{E}$



The component of the vector $d\underline{E}$ from an element of charge dq in the $\underline{k}$ direction is $d\underline{E} \cdot \underline{k} = dE \cos\theta$ as in diagram

The contributions from a ring of charge have components that cancel in the radial direction

All that survives is the sum of contributions to $\underline{E}$ in the $\underline{k}$ direction

$$\underline{dE} = \frac{k}{4\pi\varepsilon_0} \frac{2\pi r \sigma \, dr \cos\theta}{z^2+r^2}$$

But $\cos\theta = \dfrac{z}{\sqrt{z^2+r^2}}$

$$\underline{E} = \frac{k\sigma}{2\varepsilon_0} z \int_0^a \frac{r}{(z^2+r^2)^{3/2}} \, dr$$

$$= \frac{k\sigma z}{2\varepsilon_0} \left[-\frac{1}{(z^2+r^2)^{1/2}} \right]_0^a$$

$$= \frac{k\sigma}{2\varepsilon_0} \left[1 - \frac{z}{\sqrt{a^2+z^2}} \right]$$

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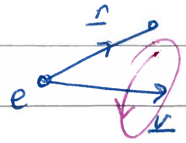
Magnetic field

$\underline{B} = \frac{e\mu_0}{4\pi} \frac{\underline{v} \wedge \underline{r}}{r^3}$

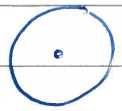


$\nabla \cdot B = -\frac{e\mu_0}{4\pi} \underline{\nabla} \cdot (\underline{v} \wedge \underline{\nabla} {}^1/_r)$

$\underline{\nabla} \cdot (u \wedge v) = \underline{v} \cdot \nabla \wedge \underline{u} - \underline{u} \cdot (\underline{\nabla} \wedge \underline{v})$

constant so curl is zero

$= \frac{e\mu_0}{4\pi} \underline{v} \cdot \underbrace{\underline{\nabla} \wedge \underline{\nabla}}_{=0} ({}^1/_r) = 0$



$\int_{\partial V} \underline{B} \cdot d\underline{S} = \int_{\partial V} \frac{e\mu_0}{4\pi} \underbrace{\frac{\underline{v} \wedge \underline{r}}{R^3} \cdot \frac{\underline{r}}{R}}_{=0} dS = 0$

$\hat{\underline{r}}\, dS = \frac{\underline{r}}{r} dS$

could have lots of charges around v. close to each charge everything has radial symmetry

small volume V, radius R

$\int_{\partial V} \underline{B} \cdot d\underline{S} = 0$ for any volume V

$\underline{\nabla} \cdot \underline{B} = 0$ everywhere

Since $\underline{\nabla} \cdot \underline{B} = 0$ we know there exists a vector A such that

$\underline{B} = \underline{\nabla} \wedge \underline{A}$

$\underline{A}$ is called the vector potential for $\underline{B}$

$\underline{A}$ is not unique $\underline{A}' = \underline{A} + \nabla\varphi$ then

$\underline{\nabla} \wedge \underline{A}' = \underline{\nabla} \wedge \underline{A} + \underline{\nabla} \wedge \underline{\nabla}\varphi = \underline{B} + \underline{0} = \underline{B}$

We can use this arbitrariness in $\underline{A}$ to impose another condition on $\underline{A}$. Here we choose $\underline{\nabla} \cdot \underline{A} = 0$ the so called Coulomb Gauge

With this choice, we will see that

$\nabla^2 \underline{A} = -\mu_0 \underline{J}$

$\nabla^2 \varphi = -\rho/\varepsilon_0$

& this has solution

$$A(r) = \frac{\mu_0}{4\pi} \int_{V'} \frac{J(r')}{|\underline{r} - \underline{r}'|} \, dV'$$

will see where this comes from

$$\varphi(\underline{r}) = \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{\sigma(r')}{|r - r'|} \, dV'$$

Measure a field $f = e(\underline{E} + \underline{v} \wedge \underline{B})$

test charge e moving with speed v
Stationary observer O sees a test charge e moving w/ speed v & measures force $\underline{E} + \underline{v} \wedge \underline{B}$

Observer O' moving with the test charge speed $\underline{v}$ measures only an electric field $\underline{E}'$

$$\underline{E} + \underline{v} \wedge \underline{B} = \underline{E}' \implies \underline{E}' - \underline{E} = \underline{v} \wedge \underline{B}$$

But interchange role of observers, we see E

he sees $E' - v \wedge B'$

$$E' - E = v \wedge B'$$

Subtract one from other $\underline{v} \wedge (\underline{B} - \underline{B}') = \underline{0}$

$\implies \underline{B} - \underline{B}' \parallel \underline{v}$

Read pech 3.4 part 5. page 16 to get

$$\underline{B} - B' \propto \underline{v} \wedge \underline{r} \quad \perp \underline{v}$$

This is alright if we change EM 3

$$\underline{B} = \frac{\mu_0}{4\pi} \frac{\underline{v} \wedge \underline{r}}{r^3} + O(v^2/c^2) \qquad c^2 = \frac{1}{\mu_0 \varepsilon_0}$$

Then what we said about no sources, being divergence free is correct

Faraday's Law of induction

If $\underline{B}$ is time varying it generates an $\underline{E}$

$\longrightarrow v$

Frame R sees a test charge moving with speed v & measures a force $\underline{E} + \underline{v} \wedge \underline{B}$ only got $\nabla \wedge E$ for steady fields

Frame R' moving with the test charges measures E'

$\nabla \wedge \underline{E}' = 0$

$\underline{E}' = \underline{E} + \underline{v} \wedge \underline{B} \qquad \nabla \wedge \underline{E}' = 0$ ← from before (if just measuring electric field + not magnetic)

which tells us

$$\nabla \wedge E' = 0 = \nabla \wedge \underline{E} + \nabla \wedge (\underline{v} \wedge \underline{B})$$ ← expands to 4 terms but $\underline{v}$ is constant so some are zero
$$= \underline{\nabla \wedge E} + \underline{v}(\nabla \cdot \underline{B}) - \underline{v} \cdot \underline{\nabla B}$$

$$\nabla \wedge E' = \underline{v} \cdot \underline{\nabla B}$$

curl of E is dir^n deriv of B in direction charges are moving

$\overset{\underline{v}}{\longrightarrow}$

mag field generated by charges moving w/ speed v

$$\underline{B}(r, t) = \underline{B}(\underline{r} + \underline{v}\tau, t + \tau)$$

$$\underline{E}(\underline{r}, t) = \underline{E}(\underline{r} + \underline{v}\tau, t + \tau)$$

diff wrt τ

diff w.r.t second variable

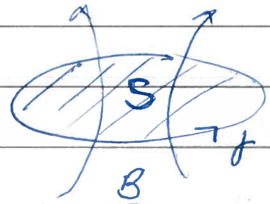
$$\underline{0} = \underline{v} \cdot \underline{\nabla}\,\underline{B} + \frac{\partial \underline{B}}{\partial t}$$ & evaluating at $t=0$

$$0 = \underline{v} \cdot \underline{\nabla}\,\underline{E} + \frac{\partial \underline{E}}{\partial t}$$

Combining our two equations

$$\underline{\nabla} \wedge \underline{E} = -\frac{\partial \underline{B}}{\partial t}$$ Faraday's law

shows we can generate electricity by moving magnet



Consider $$\int_\gamma \underline{E} \cdot d\underline{r} \overset{\text{STOKES}}{=} \int_S (\underline{\nabla} \wedge \underline{E}) \cdot d\underline{S}$$

$$\overset{\text{FARADAY}}{=} \int_S -\frac{\partial \underline{B}}{\partial t} \cdot d\underline{S} = -\frac{d}{dt} \int_S \underline{B} \cdot d\underline{S}$$

since loop + surface fixed

$$= -\frac{d}{dt} \mathcal{F}$$

$\mathcal{F}$ is flux of $\underline{B}$ through S

$$\int \underline{E} \cdot d\underline{r} = \int -\nabla \varphi \cdot d\underline{r}$$

= Potential at start − potential at end

$\mathcal{E}$ electromotive force

$$\mathcal{E} = -\frac{d\mathcal{F}}{dt}$$ rate of change of flux $\propto$ voltage generated around loop

VITALLY important as we will see

no runaway current

The displacement current in a frame R

We have seen, EM3', that a charge moving with speed $\underline{v}$ generates a magnetic field

$$\underline{B} = \frac{\mu_0 e}{4\pi} \frac{\underline{v} \wedge \underline{r}}{r^3} + O(v^2/c^2)$$

If R' is moving with the charge an observer in R' sees only an electric field $E' = \frac{e}{4\pi\varepsilon_0} \frac{r}{r^3}$

So we have

$$\underline{B} = \mu_0 \varepsilon_0 \, \underline{v} \wedge \underline{E}' + O(v^2/c^2)$$

relationship between magnetic field you see & electric field I see

$$= \frac{1}{c^2} \underline{v} \wedge \underline{E}' + O(v^2/c^2)$$

$$c\underline{B} = \left(\frac{\underline{v}}{c}\right) \wedge \underline{E}' \left(1 + O\left(v/c\right)\right)$$

but $E' = \underline{E} + \underline{v} \wedge \underline{B} = \underline{E} + \frac{\underline{v}}{c} \wedge (c\underline{B})$

$$= \underline{E}\left(1 + O\left(v/c\right)\right)$$

$$c\underline{B} = \underline{v}/c \wedge \underline{E}\left(1 + O\left(v/c\right)\right)$$

$\underline{\nabla} \wedge$ this $= \underline{\nabla} \wedge (c\underline{B}) = \frac{1}{c} \underline{\nabla} \wedge (\underline{v} \wedge \underline{E})$ $\quad$ v const

$$= \frac{1}{c}\Big(\underline{v}\underbrace{(\underline{\nabla} \cdot \underline{E})}_{\rho/\varepsilon_0} - \underbrace{\underline{v} \cdot \underline{\nabla}\,\underline{E}}_{-\frac{\partial E}{\partial t}}\Big)$$

$$\underline{\nabla} \wedge (c\underline{B}) - \frac{1}{c}\frac{\partial \underline{E}}{\partial t} = \frac{1}{c} \underline{v}\rho/\varepsilon_0$$

$$= \frac{1}{c\varepsilon_0} \underline{J} = \frac{\mu_0}{c\underbrace{\mu_0 \varepsilon_0}_{\frac{1}{c^2}}} \underline{J}$$

$$= \mu_0 c \, \underline{J}$$

$$\underline{\nabla} \wedge \underline{B} - \underbrace{\frac{1}{c^2} \frac{\partial E}{\partial t}} = \mu_0 \underline{J}$$ last Maxwell equation

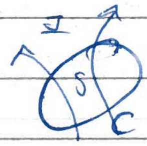
displacement current

tiny since $\frac{1}{c^2}$ - have to move quickly to have impact

$\underline{\nabla} \wedge \underline{B} = \mu_0 \underline{J}$ Ampere's Law or Oersted's Law

this is what we will use

This has an integral form



Consider $\oint_C \underline{B} \cdot \underline{dr} = \int_S \underline{\nabla} \wedge \underline{B} \cdot d\underline{S}$

$= \int \mu_0 \underline{J} \cdot \underline{dS} = \mu_0 J_S$

J_S is flux of $\underline{J}$ through S

$$\oint_C \underline{B} \cdot \underline{dr} = \mu_0 J_S$$

Infinitely long wire carrying current I

$\underline{B} = B(s)\hat{\theta}$ only, where s distance from wire

coming out of board

Choose C to be an 'Amperian loop' centre the wire radius s

$\int \underline{B} \cdot \underline{dr} = \int_0^{2\pi} B(s)\hat{\theta} \cdot \hat{\theta}\, dr$ r is const

$= B(s)\, 2\pi s$

$= \mu_0 J_S = \mu_0 I$

$$B(s) = \frac{\mu_0 I}{2\pi s}$$

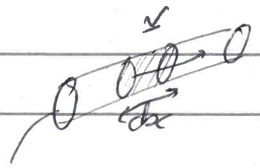
$$B(s) = \frac{\mu_0 I \hat{\theta}}{2\pi s}$$

28/01/15

Current $\hat{I}$? Amount of charge passing a particular point in unit time

It has units Amp i.e. a Coulomb / Sec Cs^{-1}

$\underline{J} = \rho \underline{v}$ In a wire with cross-sectional area dA, charges moving with speed v will travel a distance $dx = v\,dt$ in a time interval dt



dA area of cross section

The amount of charge in the volume length dx of area dA is

$$\rho\, dx\, dA = \rho v\, dt\, dA$$

So current is $\rho v\, dA$

We can define a vector current $\underline{I} = \rho \underline{v}\, dA = \underline{J}\, dA$

$\underline{J}$ is the current per unit area

Force on a current carrying wire

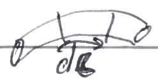
A small bit of moving charge de experiences a force

$\underline{dF} = \underline{v}\, de \wedge \underline{B}$

where $\underline{B}$ is the externally applied magnetic field

This is $\underline{dF} = \rho \underline{v}(dA\,dL) \wedge \underline{B}$ force due to magnetic field only

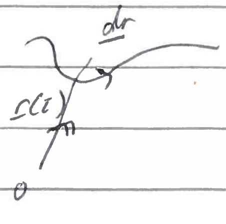
$\rho v = \underline{J}$, $\rho \underline{v}\, dA = \underline{I}$ So $\underline{dF} = (\underline{I} \wedge \underline{B})\, dl$



Total force is $\int \underline{dF} = \int_{\text{length of wire}} (\underline{I} \wedge \underline{B})\, dl$

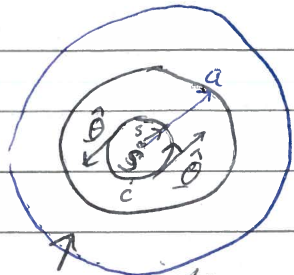
If the wire is given by the curve $\underline{r}(t)$ then

$$\underline{I}\, dl = I\, \underline{dr}$$



$$\int \underline{dF} = I \int_{\text{length of wire}} \underline{dr} \wedge \underline{B}$$

Ampere's law



wire of circular cross-section with uniform current density J, directed out of the paper

$$\int_C \underline{B} \cdot d\underline{r} = \mu_0 J_S = \mu_0 \int_S \underline{J} \cdot d\underline{S}$$

Symmetry suggests that $\underline{B} = B(s)\hat{\theta}$

$\hat{\theta}\, B(s) \cdot \hat{\theta}\, dl = B(s) \int dl = B(s) 2\pi s$

The LHS is $B(s) \cdot 2\pi s$

The RHS is $\mu_0 \underbrace{\pi s^2}_{\text{area}} \underbrace{J}_{\text{uniform current density}}$

$$B(s) = \frac{\mu_0}{2} J s$$

If outside, still have same symmetry

Outside the wire $\underline{B} = B(s)\hat{\theta}$

$$B(s) = \frac{\pi a^2 J \mu_0}{2\pi s}$$

I - current density x area

$$= \frac{\mu_0}{2\pi s} \cdot I$$

Ampere's law $\Rightarrow$ equation for $\underline{A}$ where $\underline{B} = \nabla \wedge \underline{A}$

$\nabla \cdot \underline{A} = 0$ ← choosing this i.e. A divergence free

$$\nabla \wedge \underline{B} \overset{-\frac{1}{c^2}\frac{\partial E}{\partial t}}{=} \mu_0 \underline{J}$$

$$\nabla \wedge (\nabla \wedge \underline{A}) = \nabla(\underbrace{\nabla \cdot \underline{A}}_{=0}) - \nabla^2 \underline{A} = -\nabla^2 \underline{A}$$

so $\nabla^2 \underline{A} = -\mu_0 \underline{J}$ so A satisfies Poisson's equation

so $\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi} \int_{V'} \frac{\underline{J}(\underline{r}')\, dV'}{|\underline{r} - \underline{r}'|}$

$$\underline{B} = \nabla \wedge \underline{A}$$

We derived MEs by neglecting terms $O(v/c)$ but doesn't matter

Maxwell's Equations

1 $\underline{\nabla} \cdot \underline{E} = \rho / \varepsilon_0$ ①

2 $\underline{\nabla} \cdot \underline{B} = 0$ ②

3,4,5 $\underline{\nabla} \wedge \underline{E} = -\frac{\partial B}{\partial t}$ ③

6,7,8 $\underline{\nabla} \wedge \underline{B} - \underbrace{\mu_0 \varepsilon_0}_{\frac{1}{c^2},\ c\text{ - speed of light}} \frac{\partial \underline{E}}{\partial t} = \mu_0 \underline{J}$ ④

LEARN THESE FOR EXAM

will need to derive at least parts of each.

8 equations

Given ρ & $\underline{J}$ we have 6 unknowns won't be able to do this w/o more information

①,③ ⇒ $\underline{\nabla} \cdot (\underline{\nabla} \wedge \underline{E}) = 0 = -\underline{\nabla} \frac{\partial}{\partial t} \underline{B} = -\frac{\partial}{\partial t} \underline{\nabla} \cdot \underline{B} = 0$

④ ⇒ $\underline{\nabla} \cdot (\underline{\nabla} \wedge \underline{B}) - \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \underline{\nabla} \cdot \underline{E} = \mu_0 \underline{\nabla} \cdot \underline{J}$

$\rightarrow \nabla \cdot E = \rho$

$-\mu_0 \varepsilon_0 \frac{1}{\varepsilon_0} \frac{\partial \rho}{\partial t} = \mu_0 \underline{\nabla} \cdot \underline{J}$

$$\underline{\nabla} \cdot \underline{J} + \frac{\partial \rho}{\partial t} = 0$$

CONTINUITY EQUATION

MEs only have chance of soln if this equation holds

This is the equation of charge conservation, or continuity

Integrate this equation over a volume V

$$\int_V \underline{\nabla} \cdot \underline{J}\, dV + \int_V \frac{\partial \rho}{\partial t}\, dV = 0$$

$$\int_{\partial V} \underline{J} \cdot d\underline{S} = -\frac{\partial}{\partial t} \int_V \rho\, dV$$

← can do this since volume doesn't change with time

i.e. $\underbrace{\int_{\partial V} \rho \underline{v} \cdot d\underline{S}}_{\text{flux of charge across boundary of } V} = \underbrace{-\frac{d}{dt} \underbrace{\int_V \rho\, dV}_{\text{charge in } V}}_{\text{rate of change}}$

which states that the rate of change of charge in a volume is given by the ~~volume~~ rate at which charge flows across the surface of the volume.

03/02/15

Recall: $\nabla \cdot \left(\frac{e\hat{r}}{r^3}\right) = 0 \Rightarrow \nabla \cdot E = 0$

But what about $r = 0$?

charge density

$de = \rho dV$ = density × vol = de (amount of charge)

$\rho(r) = 0$ if $r \neq 0$
$= \infty$ $r = 0$ ← this is only way of putting finite amount of charge in a small area

Several ways of making sense of this — these are generalised functions

not satisfactory, what does it mean?

$\delta(r) = \begin{cases} \infty & r = 0 \\ 0 & r \neq 0 \end{cases}$ always have inside integral

DIRAC DELTA FUNCTION

Integral over all space is total amount of charge

$\int_{V'} \delta(r') \, dV' = 1$ this normalises ∞

Point charge at origin has $\rho = e\delta(r)$

$\int_V \rho \, dV = \int e\delta(r) \, dV = e \int \delta(r) \, dV = e \cdot 1$

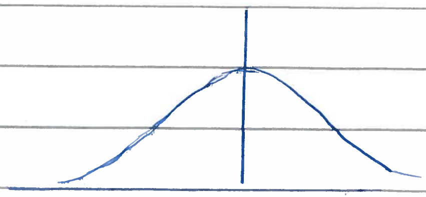
Can make sense of this in applied way

Theory - work in 1d

Think of δ as limit of sequence of functions, functions differentiable, limit not

Pick any function $g(x)$ where $\int_{-\infty}^{\infty} g(x) \, dx = 1$

e.g. $\frac{1}{\pi(1+x^2)}$



Define $g_L(x) = \frac{1}{L} g\left(\frac{x}{L}\right)$

We see $\int_{-\infty}^{\infty} g_L(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{L} g\left(\frac{x}{L}\right) dx$

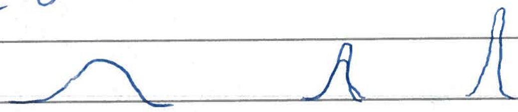
Let $u = x/L$ get $\int_{-\infty}^{\infty} g_L(x)\,dx = \int_{-\infty}^{\infty} g(u)\,du = 1$

which we said

Fix $a \neq 0$ $g_L(a) = \underbrace{\underset{\to\infty}{\frac{1}{L}}\, \underset{\to 0}{g\left(\frac{a}{L}\right)}}_{\to 0} \to 0$ as $L \to 0$

as g is integrable

but at zero $g_L(0) = \frac{1}{L} g(0) \longrightarrow \infty$

as $L \to 0$ g_L 

narrower, taller, integral still 1

$\delta(x) = \lim_{L\to 0} g_L(x)$

$\delta(x-y) = \begin{matrix} 0 & \\ \infty & x=y \end{matrix}$

$\delta(r-r')$ is point source at $r=r'$

What about

$f(y)\,\delta(x-y) = f(x)$

zero everywhere apart from when $x=y$

The product is $f(x)$

This is zero except where $y=x$ since

element of sequence that tends to it

$$\int_{-\infty}^{\infty} f(y)\,\delta(x-y)\,dy = \lim_{L\to 0} \int_{-\infty}^{\infty} f(y)\,\delta_L(x-y)\,dy$$

$L \to 0$ this $\to 0$ except near where $y=x$

Approximate $f(y)$ by $f(x)$

Replace $f(y)$ by its approximation $f(x)$

$$= f(x) \lim_{L\to 0} \int_{-\infty}^{\infty} \delta_L(x-y)\,dy$$

$$= f(x)\cdot 1 = f(x)$$

This integral is saying $f(x) = \int_{-\infty}^{\infty} f(y)\, \delta(x-y)\, dy$

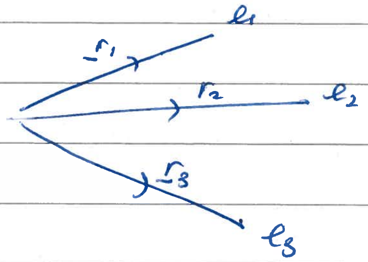
strength $\quad$ sum of δ functions

Lots of peaks w/ height $f(y)$ s.t. sum of them $= f(x)$

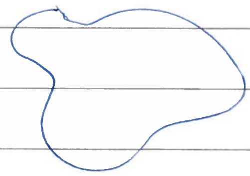
$\delta(\underline{r}) = \delta(x)\delta(y)\delta(z)$

Warning: if using polar coordinates it is not $\delta(z)\delta(r)\delta(\theta)$
since $\int r\, dr\, d\theta\, dz$ ← i.e. extra r

Have got: load of point charges strengths e_i



$$\rho(\underline{r}) = \sum_i e_i\, \delta(\underline{r} - \underline{r}_i)$$



$$\rho(\underline{r}) = \int_V \rho(\underline{r}')\, \delta(\underline{r} - \underline{r}')\, dV'$$

charge density made up of sum of little point charges everywhere

May write down charge dist. $\delta(\underline{r})$
to mean unit charge at origin

Laplace $\downarrow$ Poisson (i.e. w/ forcing) $\downarrow$

Green's function $\quad \nabla^2\varphi = 0 \quad \nabla^2\varphi = \nu(r)$

$G(r;r')$

The Green's function for the Laplace operator ∇^2 is the solution to the equation $\nabla^2 G(\underline{r};\underline{r}') = \delta(\underline{r}-\underline{r}')$

differentiation wrt $\underline{r}$ (not $\underline{r}'$)

Have differential equation & am forcing it by a point charge at $\underline{r}'$

Must say: equation
geometry (1D, 2D, sphere, box, inside or out)
BC (does $G \to 0$ as $r \to \infty$)

We are looking at 3D space where $G \to 0$ as $r \to \infty$

$\nabla^2 G(\underline{r};\underline{0}) = \delta(\underline{r})$

Consider the function $1/r$

We have seen $\nabla^2(1/r) = 0$

Recall: $\underline{\nabla}\cdot\underline{E} = \rho/\varepsilon \qquad \underline{E} = -\nabla\varphi$

$\nabla^2\varphi = -\rho/\varepsilon_0 \quad = 0$ where no charge

So it is a candidate for Green's function

In fact $G(\underline{r},\underline{0}) = -\frac{1}{4\pi}\frac{1}{r}$ since we can show

$$\int_V \nabla^2 G \, dV = \int_V \delta(\underline{r}) = 1$$

Consider $\int_V \nabla^2 G \, dV = \int_V \underline{\nabla}\cdot(\nabla G)\, dV = \int_{\partial V} \underline{\nabla} G \cdot \underline{dS}$

spherical volume radius R centre O

$$= \frac{-1}{4\pi}\int \underline{\nabla}(1/r)\cdot \underline{dS}$$

$\nabla(1/r) = -\frac{\hat{r}}{r^2}$

$dS\,\hat{r}$

so we have $\int_{\partial V} \frac{1}{4\pi} \frac{\hat{r}\cdot dS}{r^2}$

$= \frac{1}{4\pi r^2} \int dS = \frac{1}{4\pi r^2} 4\pi r^2 = 1$

The solution to $\nabla^2 G(\underline{r};\underline{r}') = \delta(\underline{r} - \underline{r}')$

is $G(\underline{r}, \underline{r}') = \frac{-1}{4\pi|\underline{r}-\underline{r}'|}$ solution in free space

~~The~~ We want sol^n to $\nabla^2 G = \nu(\underline{r})$

$= \int_{V'} \nu(r')\,\delta(r-r')\,dV'$

G is a scalar not a vector

$\Longrightarrow$ using linearity $G = \frac{-1}{4\pi} \int \frac{\nu(r')}{|\underline{r}-\underline{r}'|} dV'$

Close to r' volume goes like $(|r-r'|)^3$

Electrostatic Energy

$\underline{E} = \frac{1}{4\pi\varepsilon_0} \frac{\underline{r}}{r^3}$ $\quad e$

$\exists\ \varphi$ $\ni$ $\underline{E} = -\nabla\varphi$ $\quad \varphi = \frac{1}{4\pi\varepsilon_0} \frac{1}{r}$

work done in taking unit charge distance r up to point charge

$\nabla\cdot\underline{E} = \rho/\varepsilon_0$ if have charge distribution $\quad \nabla^2\varphi = -\rho/\varepsilon_0$

$\varphi(\underline{r}) = \frac{1}{4\pi\varepsilon_0} \int_{V'} \frac{\rho(r')}{|r-r'|} dV'$

$e_1 \quad e_2 \quad e_3$

If $\rho(\underline{r}) = \sum e_i\,\delta(\underline{r} - \underline{r}_i)$

Substitute in

e_i correspond to point charges

giving $\varphi = \sum_i \frac{e_i}{4\pi\varepsilon_0} \frac{1}{|r - r_i|}$ $(\underline{r}' \rightarrow \underline{r}_i)$

work required to bring test charge q to position vector, not interested in this - we want to know how much work we have to do to construct our distribution of charge e_1 e_2 e_3

Energy required to construct a system of point charges

1) first charge e_1 placed without doing any work giving a potential $\varphi_1 = \frac{e_1}{4\pi\varepsilon_0 |\underline{r} - \underline{r}_1|}$

2) second charge placed at r_2 requires work

$e_2 \varphi_1(\underline{r}_2)$, $\quad w_{12} = e_2 \varphi_1(\underline{r}_2) = \frac{e_1 e_2}{4\pi\varepsilon_0 |\underline{r}_2 - \underline{r}_1|}$

charge × unit

Now $\varphi = \varphi_{12}(\underline{r}) = \frac{e_1}{4\pi\varepsilon_0 |\underline{r} - \underline{r}_1|} + \frac{e_2}{4\pi\varepsilon_0 |\underline{r} - \underline{r}_2|}$

The work done in placing e_3 at $r = r_3$ is

$e_3 \varphi_{12}(r_3)$

Total work done so far is

$$w_{123} = \frac{1}{4\pi\varepsilon_0} \left(\frac{e_2 e_1}{|\underline{r}_2 - \underline{r}_1|} + \frac{e_3 e_1}{|\underline{r}_3 - \underline{r}_1|} + \frac{e_3 e_2}{|\underline{r}_3 - \underline{r}_2|} \right)$$

The total work done in assembling a system of point charges is $w = \frac{1}{4\pi\varepsilon_0} \sum_{i>j} \sum_j \frac{e_j e_i}{|\underline{r}_j - \underline{r}_i|}$

$$w = \frac{1}{2} \frac{1}{4\pi\varepsilon_0} \sum_{i \neq j} \sum_j \frac{e_i e_j}{|\underline{r}_j - \underline{r}_i|}$$

$$w = \frac{1}{2} \sum_{i \neq j} e_i \sum_j \frac{e_j}{4\pi\varepsilon_0 |\underline{r}_i - \underline{r}_j|}$$

$$= \frac{1}{2} \sum_i e_i \sum_{i \neq j} \frac{e_j}{4\pi\varepsilon_0 |\underline{r}_i - \underline{r}_j|}$$

swapped over sum

← potential due to all charges $\neq i$

$$= \frac{1}{2} \sum_i e_i \varphi_i^-(\underline{r}_i),$$

φ^- is potential due to all charges other than i

φ_i^- is different potential for each i

Will turn this in to an integral

Imagine all e_i's being made up of $\rho\, dV = e_i \delta(\underline{r} - \underline{r}_i)$

This looks like, for a continuous distribution of charge

$$w = \frac{1}{2} \int_V \rho(\underline{r}) \varphi(\underline{r})\, dV$$

Putting charge in place, potential is potential due to all other charges

charge density × potential = work done per unit volume

Above is formula that will give us what we need for continuous distribution of charge  instead of 

Integral has product over last piece you are putting in

φ^- is potential due to <u>all</u> charges (not last piece)

difference between energies doesn't depend on self energy

Far away, no charge

We can write $\underline{\nabla} \cdot \underline{E} = \rho / \varepsilon_0$ $\Rightarrow$ use $\varepsilon_0 \underline{\nabla} \cdot \underline{E} = \rho$

$$w = \frac{1}{2} \varepsilon_0 \int_V (\underline{\nabla} \cdot \underline{E}) \varphi\, dV$$

$\nabla(\varphi \underline{E}) = (\underline{\nabla} \cdot E)\varphi + \underline{E} \cdot \underline{\nabla}\varphi$

$$= \frac{\varepsilon_0}{2} \int_V \underline{\nabla} \cdot (\underline{E}\varphi) - \underline{E} \cdot \underline{\nabla}\varphi \, dV$$

but $-\nabla\varphi = \underline{E}$

$$= \frac{\varepsilon_0}{2} \int_V \underline{E}^2 dV + \underbrace{\frac{\varepsilon_0}{2} \int_V \underline{\nabla} \cdot (\underline{E}\varphi) \, dV}_{\text{claim this is } 0}$$

If V is all space then this last integral is zero

Consider the integral over a spherical volume radius R

V_R $(R \to \infty)$

$$\int_{V_R} \underline{\nabla} \cdot (\underline{E}\varphi) \, dV = \int_{\partial V'} \varphi \underline{E} \cdot \underline{dS}$$

Far away, all we see is point charge

E decays like $1/r^2$

Potential goes like $1/r$

Product " $1/r^3$

Surface " r^2

Far from the region where $\rho \neq 0$ $\quad \underline{E} \sim \frac{\hat{r}}{R^2}$

$\varphi \sim \frac{1}{R}$ $\quad \hat{r} \cdot \underline{dS} \sim \underbrace{(4\pi)}_{\text{not important so (")}} R^2$

This integral goes like $\sim \frac{1}{R^3} R^2 \sim \frac{1}{R}$ as $R \to \infty$

$$\omega = \frac{\varepsilon_0}{2} \int_V \underline{E}^2 \, dV$$

$\frac{\varepsilon_0}{2} E^2$ — call ~~charge~~ energy density

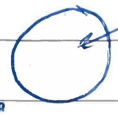
energy per unit volume.

integral gives total charge

Will explicitly work out integral next time

04/02/15

$U = \frac{\varepsilon_0}{2} \int_V E^2 \, dV$



Ball is spherical bunch of charge

1) Imagine a ball radius R charge q & find the work done in bringing charge dq from infinity & spreading it over the ball to increase the radius by an amount dR

$\underline{E} = E(r)\,\hat{\underline{r}}$

Gauss' Theorem gives

$4\pi r^2 E(r) = q/\varepsilon_0$, $\qquad E(r) = \frac{q}{4\pi\varepsilon_0 r^2}$

so $\varphi(r) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r}\right)$ $\qquad \underline{E} = -\nabla\varphi$

The work done in bringing φ charge dq is ~~$\varphi\, dq$~~ $dw = \varphi(r)\,dq$

$$dw = \frac{1}{4\pi\varepsilon_0} q(R) \frac{dq}{R}$$

If the charge density is ρ

$$q = \frac{4\pi}{3} R^3 \rho$$

So $dq = 4\pi R^2 \rho \, dR$ $\qquad$ amount of charge needed to increase radius which is currently dR

$$\& \quad dw = \frac{1}{4\pi\varepsilon_0} \underbrace{\frac{4\pi}{3} R^3 \rho}_{q} \; \underbrace{\frac{4\pi R^2 \rho}{R}}_{dq} dR$$

$$= \frac{4\rho^2 \pi R^4}{3\varepsilon_0} dR$$

The total work done is the integral of this from $R=0$ to a

$$\omega = \int_0^a \frac{4\rho^2\pi}{3\varepsilon_0} R^4 \, dR$$

$$= \frac{4}{15} \frac{\rho^2 \pi}{\varepsilon_0} a^5$$

total charge in ball radius a write as e

but $\rho = \frac{3q}{4\pi} \frac{1}{a^3}$

$$\omega = \frac{3}{5} \frac{e^2}{4\pi\varepsilon_0} \frac{1}{a}$$

$a \to 0$ this tends to point charge

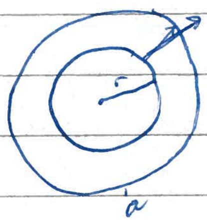
That is one way of doing it -
Alternatively we can integrate the square of $\underline{E}$

$$\frac{\varepsilon_0}{2} \int_V E^2 \, dV$$

For $r > a$ already found E for $r > R$

$$E = \frac{e}{4\pi\varepsilon_0} \frac{1}{r^2} \hat{r}$$

For $r < a$



$\underline{E} = E(r)\hat{r}$

charge

$$\underbrace{\frac{4\pi}{3} r^3}_{\text{volume}} \underbrace{\frac{e}{\frac{4\pi}{3} a^3}}_{\text{charge density}} = \frac{er^3}{a^3}$$

Gauss gives

$$\underline{E}(r) \cdot 4\pi r^2 = er^3/a^3/\varepsilon_0$$

$$\underline{E} = \hat{r} \frac{e}{4\pi\varepsilon_0 a^3} r$$

We need to find $\frac{\varepsilon_0}{2}\int E^2 \, dV$

shell radius r, thickness dr

Write $dV = \underline{4\pi r^2} dr$

~~surface area~~

$$w = \frac{\varepsilon_0}{2}\left\{\int_0^a \frac{e^2}{(4\pi\varepsilon_0)^2}\frac{r^2}{a^6}\cdot 4\pi r^2 \, dr\right.$$

$$+ \int_a^\infty \frac{e^2}{(4\pi\varepsilon_0)^2}\frac{1}{r^4} 4\pi r^2 \, dr$$

from $E = \frac{e}{4\pi\varepsilon_0}\frac{1}{r^2}$ for $r > R$

$$= \frac{3}{5}\frac{e^2}{4\pi\varepsilon_0}\frac{1}{a}$$ same expression!

In a conductor - electrons move but not v far since they bump in to atoms

Heat causes Brownian motion of electrons

Ohm's Law

$\underline{J} = \sigma \underline{E}$

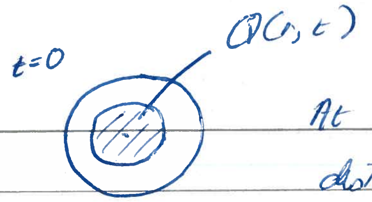
conductivity

$\underline{E} = \rho \underline{J}$

resistivity

If conductors have a net charge, that net charge lives on surface of conductor

But how quickly does this occur? V. quickly



At $t=0$ a total charge q uniformly distributed inside a conducting sphere radius a.

Let $Q(r,t)$ be the charge inside a ball, radius r at time t
B_r

Charge flowing: use cont. eqn
r.o.c. of charge = ~~flux~~ $\int$ current dS = flux of charge out

The continuity equation

$$\frac{\partial}{\partial t}\int_{B_r}\rho\, dV = -\int_{\partial B_r} \underline{J}\cdot \underline{dS}$$

$$\frac{\partial Q}{\partial t} \qquad\qquad J(r,t)\,\hat{\underline{r}}$$

$$= -4\pi r^2 J(r,t)$$

$$\frac{\partial Q}{\partial t} = -4\pi r^2 J(r,t) = -4\pi r^2 \overbrace{\sigma E(r,t)}^{J}$$

where $\underline{E} = E(r,t)\,\hat{\underline{r}}$

everything is radially symmetric i.e. just depends on radius + time

We have seen

$$E(r,t) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}Q(r,t)$$

$$\frac{\partial Q}{\partial t} = \frac{-\sigma}{\varepsilon_0} Q$$

(4π)'s disappear

At $t=0$ $Q = q\, r^3/a^3$

$$Q(r,t) = \frac{q r^3}{a^3} e^{\frac{-\sigma}{\varepsilon_0}t}$$

and $\underline{E} = \hat{r}\frac{1}{4\pi\varepsilon_0}\frac{Q(r,t)}{r^2} = q\frac{r}{4\pi\varepsilon_0}\frac{1}{a^3}e^{-t\sigma/\varepsilon_0}$

Claim this charge ends up at surface

The total charge at the surface $Q_s(t)$

cont. equation: $\frac{dQ_s}{dt} = \int_{\text{surface}} \underline{J}\cdot d\underline{S} = 4\pi a^2\sigma E(a,t)$

σE

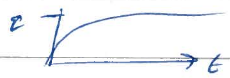
J at surface

$= q\frac{\sigma}{\varepsilon_0}e^{-t\sigma/\varepsilon_0}$

At $t=0$ $Q_s = 0$

So $Q_s = q(1-e^{-t\sigma/\varepsilon_0})$

rapidly goes up to q



Charge / unit area - a surface charge density is

$\frac{q}{4\pi a^2}(1-e^{-t\sigma/\varepsilon_0}) \longrightarrow \frac{q}{4\pi a^2}$

$= \sigma$ — not conductivity

charge per unit area

Now can make claim: generally in a conductor $E=0$

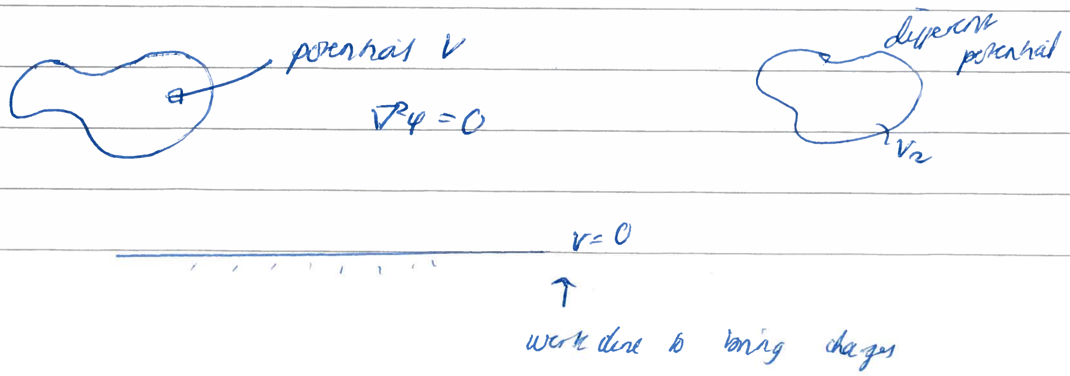
So $\underline{E}=\underline{0}$ in a conductor

$\Longrightarrow$ since $\underline{\nabla}\cdot\underline{E} = \rho/\varepsilon_0$, $\rho = 0$ in a conductor

charge density

Also since $\underline{\nabla}\varphi = -\underline{E} = \underline{0}$

So the potential φ is constant in a conductor



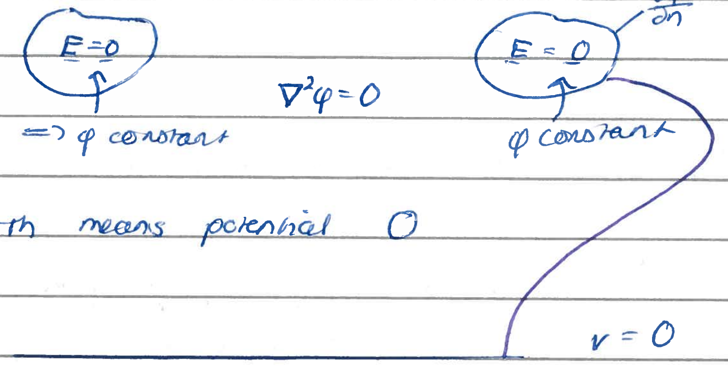
If solving for potential, have geometry & eq^n is
$\nabla^2\varphi = 0$ & we know what φ is on surfaces

Conductors

10/02/15

Have conductor, bringing charges close to it to interact with it
All the time you are doing it, $\underline{E}=0$ inside conductor

$\frac{\partial \varphi}{\partial n}$ normal to surface



$\underline{E} = -\nabla\varphi$

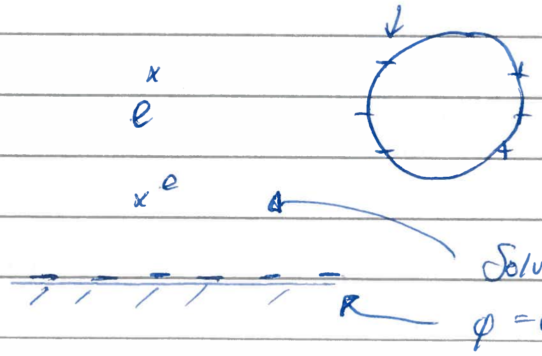
have attached body to earth w/ wire
if connected they have exactly same potential since it is conducting body

Earth means potential 0

Outside $\nabla^2\varphi = 0$
Once found φ, can find $\frac{\partial\varphi}{\partial n}$

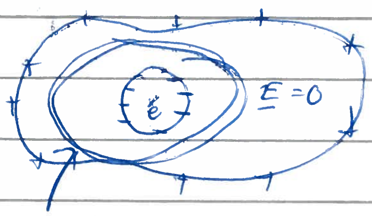
It turns out $\frac{\partial\varphi}{\partial n}$, found after solving $\nabla^2\varphi=0$ with φ given on boundaries, is related to the electric field $\underline{E}$ on the surface of the conductors & the charge density is related to $\underline{E}$

-ve ones moved over so +ve deficit
net charge zero
so get +ve on RHS



these -ve charges are induced charges

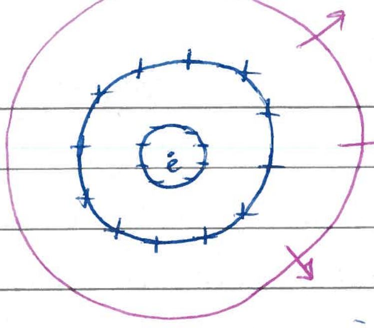
Solving $\nabla^2\varphi = 0$
$\varphi = 0$



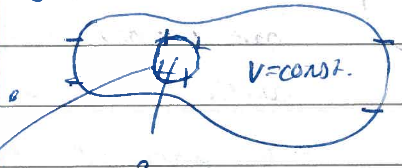
S
gaussian surface

$$\int_S \underline{E}\cdot d\underline{s} = \int_S \underline{0}\cdot d\underline{s} = 0$$

so -ve charges on surface to balance e
charge density comes in to surface to counteract
+ all around outside to compensate



axisymmetric case
charge uniformly distributed so remains axisymmetric
Draw Gaussian surface S – +
is as if conductor isn't there at all



+ charges to compensate for fact that $E = 0$
V is voltage, φ potential same thing

$\varphi = V$
const. fixed

$\nabla^2 \varphi = 0$
since no charge in here

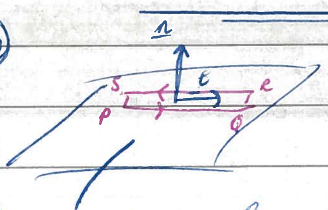
Solution $\varphi = V$

$\underline{E} = -\underline{\nabla}\varphi = \underline{0}$ so $\underline{E} = \underline{0}$ inside circle

So influence of charges outside doesn't get inside

Think elevator – lift is conducting shell, inside $E = 0$, outside charges electric forces

② E_2



thin region of charge

E_1, E_2 separated by surface charge density

① σ charge density / unit area E_1

Discontinuities in $\underline{E}$ in crossing a charged surface

$= (E_2 - E_1)\cdot t$

$[\underline{t}\cdot\underline{E}]_1^2 = 0$ i.e. no jump in tangential component of electric field

$[\underline{n}\cdot\underline{E}]_1^2 = \sigma/\varepsilon_0$

$\boxed{[\underline{E}]_1^2 = \dfrac{\sigma \underline{n}}{\varepsilon_0}}$ *

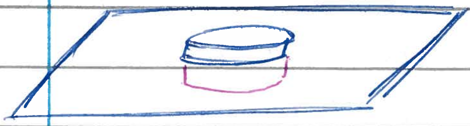
Consider $\int_{PQRS} \underline{E}\cdot d\underline{r} = 0$

$\uparrow$
E is conservative

l length
d thickness

something doesn't matter since small

$$\int \underline{E} \cdot d\underline{r} = \underline{E}_1 \cdot t l + ?d + \underline{E}_2 \cdot -t L + ?d$$
$$= (\underline{E}_1 - \underline{E}_2) \cdot t L \quad \text{if} \quad d/L \longrightarrow 0$$



pink bit below surface
much shallower than wide

$$\int \underline{E} \cdot d\underline{S} = Q/\varepsilon_0$$

forgotten flux around edges since small

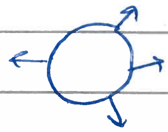
surface area x charge density per unit area

$$\underline{E}_2 \cdot \hat{n} S - \underline{E}_1 \cdot \hat{n} S = S\sigma/\varepsilon_0$$

$$[\underline{n} \cdot \underline{E}]_1^2 = \sigma/\varepsilon_0 \qquad [\underline{E}]_1^2 = \frac{\sigma \underline{n}}{\varepsilon_0}$$

If n points out of a conductor then $\underline{E}_1 = \underline{0}$ because E inside is nothing

$$\underline{E}_2 = \frac{\sigma}{\varepsilon_0}\hat{n}$$



strength of E proportional to charge density
charge density great, electric field strength strong

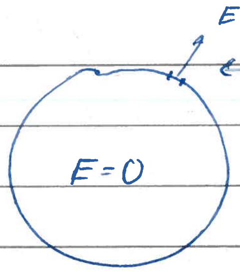
Also $\underline{E} = -\frac{\partial \varphi}{\partial n}\hat{n} \leftarrow \nabla\varphi \cdot \hat{n}$

So $\sigma = -\varepsilon_0 \frac{\partial \varphi}{\partial n}$

Induced charge sits in electric field so there is a force on it

force per unit area acting on surface - pressure

when think about force on a charge - it is due to another charge.



Two bits - charge here generating E above + below $\hat{e}$

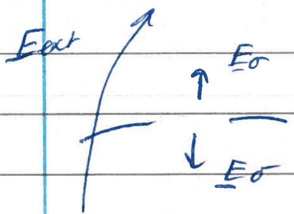
field due to rest needs to cancel inward field

Force per unit area is σE no wrong

$$\underline{E} = \frac{\sigma}{\varepsilon_0}\hat{n}$$

E is sum of E_{local} $\underline{E}_\sigma$

E_{global} $\underline{E}_{ext}$



outside $\underline{E}_{ext} + \underline{E}_\sigma = \frac{\sigma}{\varepsilon_0}\hat{n}$

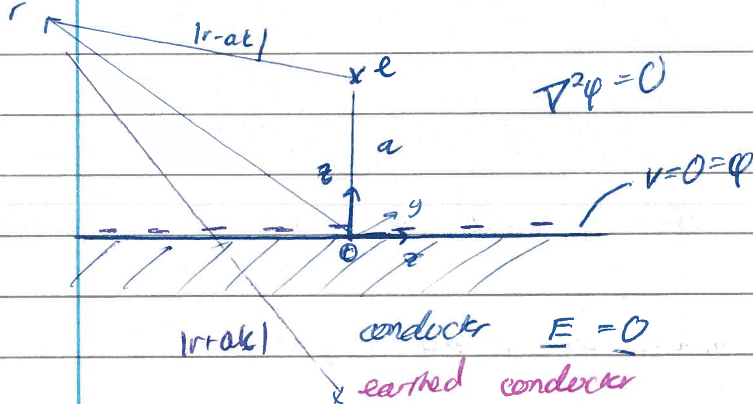
inside $\underline{E}_{ext} - \underline{E}_\sigma = 0$

$$\underline{E}_{ext} = \frac{1}{2}\frac{\sigma\hat{n}}{\varepsilon_0}$$

Force / unit arc due to σ charge / unit area

$$\sigma\underline{E}_{ext} = \frac{1}{2}\frac{\sigma^2}{\varepsilon_0}\hat{n}$$

Can put electric charge on soap bubble - makes it grow + suck air in



$\nabla^2\varphi = 0$

+ve charge induces -ves on conductor

So by method of images

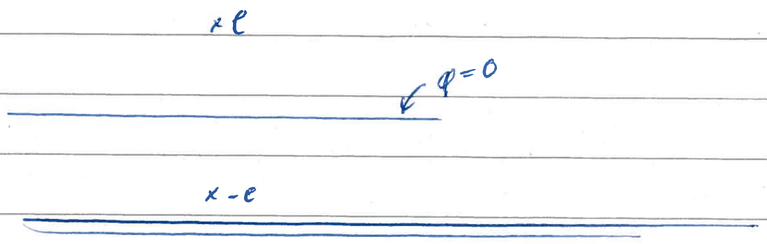
$$\varphi = \frac{e}{4\pi\varepsilon_0}\frac{1}{|\underline{r} - a\underline{k}|} - \frac{e}{4\pi\varepsilon_0}\frac{1}{|r + a\underline{k}|}$$

$= \varphi(\underline{r})$

IMAGE

Method of images

Idea - stick an image in such a place s.t. BC is satisfied

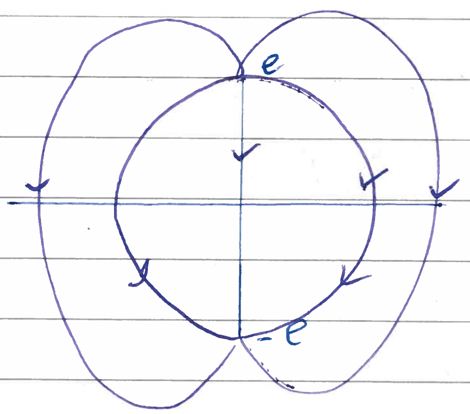


$$\frac{1}{\sqrt{x^2+y^2+(z-a)^2}}$$

and on $\underline{r} = x\underline{i} + y\underline{j}$ $(z=0)$

$$\varphi(\underline{r}) = \frac{e}{4\pi\varepsilon_0}\left(\frac{1}{|x\underline{i}+y\underline{j}-a\underline{k}|} - \frac{1}{|x\underline{i}+y\underline{j}+a\underline{k}|}\right) = 0$$

$$\underline{E} = -\nabla\varphi = \frac{e}{4\pi\varepsilon_0}\left(\frac{x\underline{i}+y\underline{j}+(z-a)\underline{k}}{(x^2+y^2+(z-a)^2)^{3/2}} - \frac{x\underline{i}+y\underline{j}+(z+a)\underline{k}}{(x^2+y^2+(z+a)^2)^{3/2}}\right)$$



On $z=0$ this is

$$\underline{E} = -\frac{2a}{(x^2+y^2+a^2)^{3/2}}\frac{e}{4\pi\varepsilon_0}\underline{k}$$

$\underline{E}$ points in $-$ve $\underline{k}$

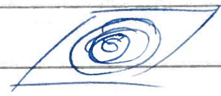
We can find induced charge σ as $\underline{E} = \sigma\hat{\underline{n}}/\varepsilon_0$ $\leftarrow \underline{k}$

Think of charges as sources + sinks

So $\sigma = \frac{-e}{4\pi\varepsilon_0} \frac{2a}{(x^2+y^2+a^2)^{3/2}} \varepsilon_0$

Total charge is $\int \sigma \, dA$

$$= \int_0^\infty \frac{-e}{4\pi} \frac{2a \cdot 2\pi r \, dr}{(r^2+a^2)^{3/2}}$$



$$= -ea \left[\frac{-1}{(r^2+a^2)^{1/2}} \right]_0^\infty$$

$$= -ea \cdot \frac{1}{a} = -e$$

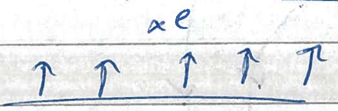
what we expected! total charge induced is -ve

Force / unit area is $\hat{n}\, \varepsilon_0 E^2/2$ ← this is changed in online notes

$$= \frac{e_0^2}{(4\pi)^2 \varepsilon_0} \frac{2a^2}{(x^2+y^2+a^2)^3} \underline{k}$$

$e_0 = e$

Note: force is normal to surface, <u>doesn't</u> point towards charge



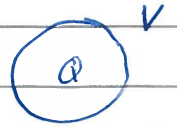
Total force

$$\int_0^\infty \frac{e_0^2}{(4\pi)^2} \frac{1}{\varepsilon_0} \frac{2a^2 \, 2\pi r \, dr}{(r^2+a^2)^3}$$

$$= \frac{e_0}{4\pi\varepsilon_0} e \frac{1}{(2a)^2}$$

↑ distance between image + original charge

Capacitance



conductor potential V

No charge Q

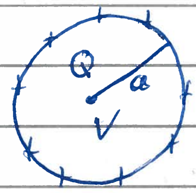
put a charge Q + potential changes

Q/V is constant + depends only on geometry

$\overset{\shortparallel}{C}$ capacitance

Can have situations when conductor has charge + other conductors brought near induce charge.

Capacitance C measured in C/V (coulombs / volt) = Farad



charge here, electric field outside

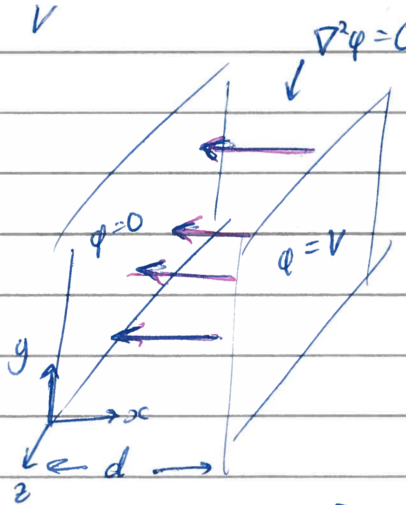
$$\underline{E} = E(r)\,\hat{r}$$

$$4\pi r^2 E(r) = Q/\varepsilon_0 \implies E = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r^2}$$

$$\underline{E} = -\underline{\nabla}\varphi \quad \text{so} \quad \varphi = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r}$$

$$V = \varphi(a) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{a}$$ relates potential to charge

$$\frac{Q}{V} = C = 4\pi\varepsilon_0 a$$ amount of charge sphere will hold per given potential



Plates held at $\varphi = 0$, $\varphi = V$

$$\nabla^2\varphi = 0 \implies \varphi_{xx} = 0$$

we are in gap - only x dep.

$\varphi = \dfrac{xV}{d}$ easy to solve

$$\underline{E} = -\nabla\varphi = -\underline{i}\,\frac{V}{d}$$ - see arrows $\leftarrow$

The charge per unit area σ is related to $\underline{E}$

$$\underline{E} = \sigma\hat{n}/\varepsilon_0$$

here $\hat{n} = -\underline{i}$ since $\hat{n}$ goes from conductor out

$$\implies \frac{\sigma}{\varepsilon_0} = \frac{V}{d} \implies \sigma = \frac{V\varepsilon_0}{d}$$

On LHS $\hat{n} = \underline{i}$

$\underline{E}$ is the same

So $\sigma = -\frac{V \varepsilon_0}{d}$

Area of capacitance

The total charge is $\sigma A = \frac{V \varepsilon_0}{d} A$

This is area of one of the planes - they are infinite, just using A for convenience

Capacitance is $\frac{\text{charge}}{\text{voltage}} = \frac{V\varepsilon_0 A}{d} \cdot \frac{1}{V} = \frac{\varepsilon_0 A}{d}$

Capacitance / unit area $= \frac{\varepsilon_0}{d}$

Force on plates tries to pull them together

Force on plates

Force / unit area is $\frac{1}{2} \varepsilon_0 E^2$

But $E = \frac{V}{d}$ $\frac{\text{potential}}{\text{distance}}$ $= \frac{\sigma}{\varepsilon_0} = \frac{q}{A \varepsilon_0}$ total charge / total area

Force / unit area $\frac{1}{2} \varepsilon_0 \left(\frac{q}{A\varepsilon_0}\right)^2$

Total force $\frac{1}{2} \varepsilon_0 \left(\frac{q}{A\varepsilon_0}\right)^2 A = \frac{q^2}{2 \varepsilon_0 A}$

11/02/15

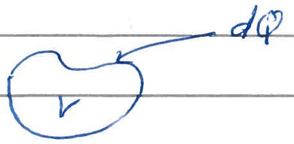
Capacitance - have a conductor in some geometric

$C = Q/V$ coulombs / volt

↑ constant of proportionality

↖ always proportional, since equations linear

The work done in bringing a charge dQ to a body with potential V is VdQ



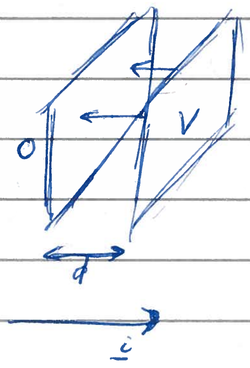
The total work done is

$$\int V\,dQ = \int_0^Q \tilde{Q}/C \; d\tilde{Q} \quad \text{as} \quad V = Q/C$$

$$= \frac{1}{2}\,Q^2/C$$

$$= \frac{1}{2}QV \quad \text{as} \quad \left(\frac{1}{C} = \frac{V}{Q}\right)$$

$$= \frac{1}{2}\frac{(CV)^2}{C} = \frac{1}{2}CV^2$$



The energy density is $\frac{\varepsilon_0 E^2}{2} = \frac{\varepsilon_0 V^2}{2d^2}$

∴ the energy/unit area $\frac{\varepsilon_0 V^2}{2d^2} \cdot d$

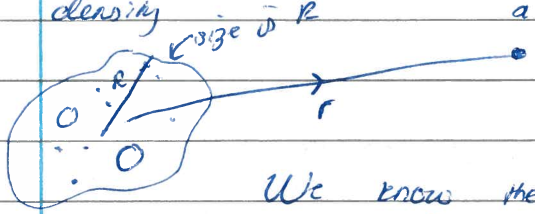
↑ $\frac{\text{volume}}{\text{area}} = \frac{\text{area} \times d}{\text{area}} = d$

$$= \frac{\varepsilon_0 V^2}{2d}$$

which is consistent with capacitance ε_0/d

Multipole expansions

Here charge, might be point charge or regions of charge density



a long way away

$r/R << 1$

We know the potential is

$$\varphi(\underline{r}) = \frac{1}{4\pi\varepsilon_0}\int_{V'} \frac{\rho(\underline{r}')\,dV'}{|\underline{r}-\underline{r}'|}$$

want to examine when size $(r) >>$ size (r')

We examine $\dfrac{1}{|\underline{r}-\underline{r}'|}$ when $|\underline{r}'|/|\underline{r}| << 1$

We consider $|\underline{r}-\underline{r}'|^2 = (\underline{r}-\underline{r}')\cdot(\underline{r}-\underline{r}')$

$$= \underline{r}\cdot\underline{r} - 2\underline{r}'\cdot\underline{r} + \underline{r}'\cdot\underline{r}'$$

$$= \underset{|r|}{r^2}\left(\underbrace{\hat{\underline{r}}\cdot\hat{\underline{r}}}_{1} - 2\hat{\underline{r}}\cdot\left(\frac{\underline{r}'}{r}\right) + \left(\frac{\underline{r}'}{r}\right)\cdot\left(\frac{\underline{r}'}{r}\right)\right)$$

$\underline{r}' = \hat{\underline{r}}'(r')$

$$= r^2\left(1 \underbrace{- 2\hat{\underline{r}}\cdot\hat{\underline{r}}'\varepsilon}_{a} + \varepsilon^2\underbrace{\hat{\underline{r}}'\cdot\hat{\underline{r}}'}_{b}\right) \qquad \varepsilon = r'/r$$

$$(1 + a\varepsilon + b\varepsilon^2)^{-1/2} = 1 - \frac{1}{2}a\varepsilon + \left(-\frac{1}{2}b\varepsilon^2 + \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)a^2\varepsilon^2\right)$$

$$= 1 - \frac{1}{2}a\varepsilon + \frac{\varepsilon^2}{2}\left(\frac{3}{4}a^2 - b\right)$$

$$|r-r'|^{-1} = (\text{above})^{-1/2}$$

$$= \frac{1}{r}\left(1 - \frac{1}{2}(-2\hat{\underline{r}}\cdot\hat{\underline{r}}')\varepsilon + \frac{\varepsilon^2}{2}\left(\frac{3\cdot 4(\hat{\underline{r}}\cdot\hat{\underline{r}}')^2}{4} - \hat{\underline{r}}'\cdot\hat{\underline{r}}'\right)\right)$$

← notice change from $\hat{r}'$ to r'

$$= \frac{1}{r} + \frac{1}{r^2}\hat{\underline{r}}\cdot\underline{r}' + \frac{1}{r^3}\left(\frac{3}{2}(\hat{\underline{r}}\cdot\underline{r}')^2 - \frac{\underline{r}'\cdot\underline{r}'}{2}\right)\ldots$$

Recall: $$\varphi(\underline{r}) = \frac{1}{4\pi\varepsilon_0}\int_{V'}\frac{\rho(\underline{r}')}{|\underline{r}-\underline{r}'|}\,dV$$

constant since integrating wrt r'

$4\pi\varepsilon_0$

$$\text{So } \varphi(\underline{r}) = \frac{1}{r}\int_{V'} \rho(\underline{r}')\, dr' \qquad \leftarrow \text{SLOWEST DECAYING PART} \quad \text{sum 1}$$

$$+ \frac{\hat{\underline{r}}}{r^2} \cdot \int_{V'} \underline{r}'\rho(\underline{r}')\, dV' \qquad \text{sum of 3}$$

sum over i
sum over j

$$+ \frac{1}{2}\frac{1}{r^3}\hat{r}_i\hat{r}_j \int (3r'_i r'_j - r'^2\delta_{ij})\rho(\underline{r}')\, dV' \qquad \text{sum of 9}$$

$$\hat{r}_i\hat{r}_j\delta_{ij} = \hat{r}_i\hat{r}_i = |\hat{r}|^2 = 1$$

So we see that for large r

$$\varphi \sim \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{r} + \frac{1}{r^2}\hat{\underline{r}}\,\underline{p} + \frac{1}{2}\frac{1}{r^3}\hat{r}_i\hat{r}_j Q_{ij} \dots\right)$$

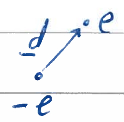
$Q = \int_{V'} \rho(\underline{r}')\, dV' =$ total charge

$\underline{p} = \int_{V'} \underline{r}'\rho(\underline{r}')\, dV'$ – the first moment of the charge distribution (or dipole moment)

$Q_{ij} = \int_{V'} (3r'_i r'_j - \delta_{ij} r'^2)\rho(\underline{r}')\, dV'$ – the second moment of the charge distribution or the quadrupole moment (a tensor of rank 2)

Charge e at $\underline{r}' = \underline{a} + \underline{d}$

$-e$ at $\underline{r}' = \underline{a}$



$\underline{d}$ points from –ve to +ve charge

$$\rho = e\,\delta(\underline{r} - \underline{a} - \underline{d}) - e\,\delta(\underline{r} - \underline{a})$$

So $Q = 0$ think $Q_{ij} = 0$ as well

Taken ~~dashess~~ dashes off

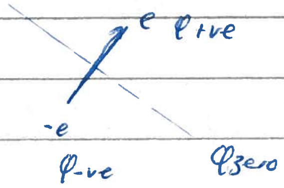
$$\underline{p} = \int \rho\, \underline{r}\, dV = \int [e\,\delta(\underline{r}-\underline{a}-\underline{d}) - e\,\delta(\underline{r}-\underline{a})]\,\underline{r}\, dV$$

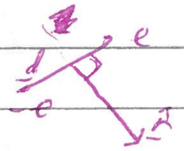
$$= e(\underline{a}+\underline{d}) - e\underline{a} = e\underline{d}$$

δ 0 everywhere except $r=a$ or $r=a+d$

$$\varphi = \frac{e}{4\pi\varepsilon_0}\,\frac{1}{r^2}\,\hat{r}\cdot\underline{d}$$

$\varphi = 0$ when $\hat{r} \perp d$





An ideal dipole has $d \to 0$ but $e \to \infty$ such that $ed = p$, the strength of the dipole

Quadrupole moment corresponds to 2 dipole moments brought close to eachother
Octopole moment ... two quadrupole moments

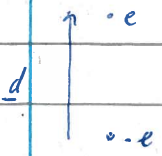
Why are we interested? Dipole's v. important
All MEs are in a vacuum but they are different in material ← (made up of spherical charged atoms - apply E which pulls at charge in atoms + pulls away from centre
Electron cloud + nucleus slightly separated - this is a dipole)

i.e. Put in an electric field, generates dipole field, which generates another electric field.

free space ↓

ε_0, ε ← generates different ε

24/02/15



$$\underline{d}\,e = \underline{p}$$

$$\varphi = \frac{\hat{r}\cdot p}{4\pi\varepsilon_0 r^2} = \frac{r\cdot p}{4\pi\varepsilon_0 r^3}$$

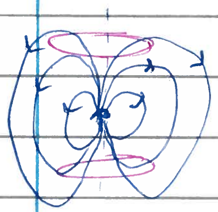
$$\underline{E} = -\underline{\nabla}\varphi = \frac{-1}{4\pi\varepsilon_0}\nabla\left(\frac{r\cdot p}{r^3}\right)$$

Note that $\frac{\partial}{\partial x_i}x_j = \delta_{ij}$

$$\left\{\underline{\nabla}\left(\frac{r\cdot p}{r^3}\right)\right\}_i = \frac{\partial}{\partial x_i}\left(\frac{p_j x_j}{(x_k x_k)^{3/2}}\right) = p_j\left\{\underbrace{\frac{\partial x_j}{\partial x_i}}_{\delta_{ij}}\frac{1}{r^3} + x_j\left(\frac{-3}{2}\right)\frac{2x_k}{r^5}\underbrace{\frac{\partial x_k}{\partial x_i}}_{\delta_{ki}}\right\}$$

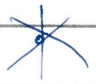
$$= \frac{p_i}{r^3} - 3(r\cdot p)\frac{x_i}{r^5}$$

$$\underline{E} = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\left(3\hat{r}(\hat{r}\cdot p) - p\right)$$

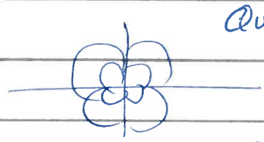


have dipole that points in particular direction
axis runs round middle

on circles field has same strength + point in direction outwards

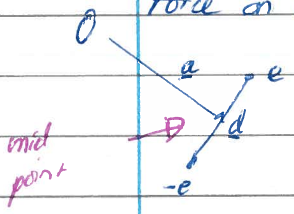


point charge



Quadrupole - can be made of two dipoles interfering like
$\uparrow\downarrow$

Force on a dipole in a field $\underline{E}$



Taylor expansion

Net force

$$e\underline{E}(\underline{a}+\underline{d}/2) - e\underline{E}(\underline{a}-\underline{d}/2)$$

$$= e\left(\underline{E}(a) + \frac{\underline{d}}{2}\cdot\nabla\underline{E}|_a \dots\right) - e\left(\underline{E}(a) - \frac{\underline{d}}{2}\cdot\nabla\underline{E}|_{\underline{a}} \dots\right)$$

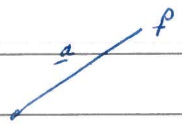
Think of this as $\underline{d}\cdot\nabla$ acting on each component of $\underline{E}$

Force is zero if E is constant

$$= e\underline{d}\cdot\nabla\underline{E} + \text{terms in } e|\underline{d}|^2 + \dots$$

$$\longrightarrow \underline{p}\cdot\nabla\underline{E} \text{ as } |\underline{d}|\to 0 \ \& \ e\to\infty \quad e\underline{d} = \underline{p}$$

Moment on a dipole at a about a point O



$$\underline{G} = \sum_{i=1}^{2} \underline{r}_i \times \underline{F}_i = \underline{p}\wedge\underline{E} + \underline{a}\wedge(\underline{p}\cdot\nabla\underline{E})$$

Compasses have little dipoles induced in them

Energy of a dipole is $u = \sum_{i=1}^{2} e_i\varphi_i = e\underline{d}\cdot\nabla\varphi|_a$

$$= -\underline{p}\cdot\underline{E}$$

Min when parallel not perp.

Magnetism

Magnetism

Recall $\nabla \wedge \underline{B} - \underbrace{\mu_0 \varepsilon_0}_{\frac{1}{c^2}} \frac{\partial \underline{E}}{\partial t} = \mu_0 \underline{J}$

Steady or slowly varying means we can forget $\mu_0 \varepsilon_0 \frac{\partial \underline{E}}{\partial t}$

$\underline{\nabla} \times \underline{B} = \mu_0 \underline{J}$ Ampere's Law

$\underline{\nabla} \cdot \underline{B} = 0 \Rightarrow \underline{B} = \underline{\nabla} \wedge \underline{A}$

Many choices of A so we insist $\underline{\nabla} \cdot \underline{A} = 0$ called taking Coulomb gauge

$\nabla^2 \underline{A} = -\mu_0 \underline{J}$

$$\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi} \int_{V'} \frac{J(\underline{r}')}{|\underline{r} - \underline{r}'|} \, dV'$$

r' integration variable

But $\underline{B} = \underline{\nabla} \wedge \underline{A}$

this differentiation is wrt r rather than r'

$$\underline{B} = \frac{\mu_0}{4\pi} \int_{V'} \underline{\nabla} \wedge \left(\frac{J(r')}{|\underline{r} - \underline{r}'|} \right) dV'$$

$J(r')$ const. here since diff wrt r not r'

$\underline{\nabla} \wedge (u \underline{F}) = (\underline{\nabla} u) \wedge \underline{F} + u(\cancel{\underline{\nabla} \wedge \underline{F}})$ (const; $= 0$)

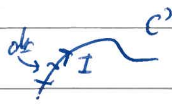
$$= \frac{\mu_0}{4\pi} \int_{V'} \underline{\nabla} \left(\frac{1}{|\underline{r} - \underline{r}'|} \right) \wedge \underline{J}(r') \, dV'$$

$\nabla\left(\frac{1}{r}\right) = -\frac{\underline{r}}{r^3}$

$$= \frac{\mu_0}{4\pi} \int_{V'} -\frac{(\underline{r} - \underline{r}')}{|\underline{r} - \underline{r}'|^3} \wedge \underline{J}(\underline{r}') \, dV'$$

$$= \frac{\mu_0}{4\pi} \int_{V'} J(r') \times \frac{(\underline{r} - \underline{r}')}{|\underline{r} - \underline{r}'|^3} \, dV'$$

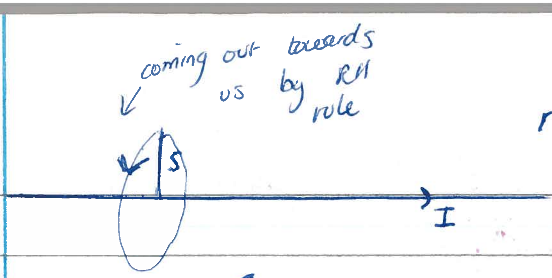
If current is in a wire then we have seen $\underline{J} dV = \underset{\text{current}}{\underline{I}} d\underline{r}$



We did this a while ago

$$\underline{B} = \frac{I \mu_0}{4\pi} \int_{C'} \frac{d\underline{r}' \wedge (\underline{r} - \underline{r}')}{|\underline{r} - \underline{r}'|^3}$$

This is the Biot-Savart Law



$r' = t\underline{i}$

$\underline{B} = B(s)\,\hat{\underline{\theta}}$ Ampere's Law $\int \underline{B}\cdot d\underline{r} = \mu_0 I$

$= B(s)\,2\pi s$

$B = \dfrac{\mu_0 I}{2\pi s}$, $\underline{B} = \dfrac{\mu_0 I}{2\pi s}\,\hat{\underline{\theta}}$

Done this in multiple ways in notes

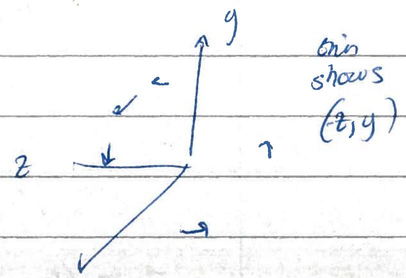
$d\underline{r}' = dt\begin{pmatrix}1\\0\\0\end{pmatrix}$ $\underline{r} = \begin{pmatrix}x\\y\\z\end{pmatrix}$

$\underline{r} - \underline{r}' = \begin{pmatrix}x-t\\y\\z\end{pmatrix}$ $d\underline{r}' \times (\underline{r} - \underline{r}') = \begin{pmatrix}1\\0\\0\end{pmatrix} \times \begin{pmatrix}x-t\\y\\z\end{pmatrix}$

$= \begin{pmatrix}0\\-z\\y\end{pmatrix} dt$

$|\underline{r} - \underline{r}'|^3 = \left((x-t)^2 + y^2 + z^2\right)^{3/2}$

$\underline{B}(\underline{r}) = \dfrac{I\mu_0}{4\pi}\displaystyle\int_{-\infty}^{\infty} \underbrace{\begin{pmatrix}0\\-z\\y\end{pmatrix}}_{\text{this is } s\hat{\theta}} \frac{dt}{\left((x-t)^2+y^2+z^2\right)^{3/2}}$



this is $s\hat{\underline{\theta}}$

$s = \sqrt{z^2+y^2}$

$= \dfrac{\mu_0 I}{4\pi}\, s\hat{\underline{\theta}} \displaystyle\int_{-\infty}^{\infty} \frac{dt}{(t^2+s^2)^{3/2}}$

x may as well be 0 y^2+z^2

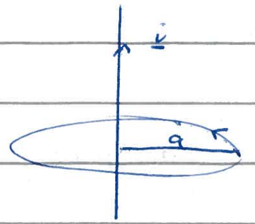
Put $t = s\tan\theta$ $dt = s\sec^2\theta\, d\theta$

$= \dfrac{\mu_0}{4\pi} I s\,\hat{\underline{\theta}} \displaystyle\int_{-\pi/2}^{\pi/2} \frac{s\sec^2\theta}{s^3\sec^3\theta}\, d\theta$

$= \dfrac{\mu_0}{4\pi} I \dfrac{1}{s}\,\hat{\theta}\cdot 2$ which is what we got before

$= \dfrac{\mu_0 I}{2\pi s}\,\hat{\theta}$

Circular wire loop



$\underline{r}' = a\cos\theta\,\underline{j} + a\sin\theta\,\underline{k} \qquad \theta \in [0, 2\pi]$

Restrict attention to points on x-axis

$x. \quad \underline{r} = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} \qquad \underline{r} - \underline{r}' = \begin{pmatrix} x \\ -a\cos\theta \\ -a\sin\theta \end{pmatrix}$

$|\underline{r} - \underline{r}'|^3 = (x^2 + a^2)^{3/2}$

$\underline{dr}' = \begin{pmatrix} 0 \\ -a\sin\theta \\ a\cos\theta \end{pmatrix} d\theta$

$$\underline{dr}' \wedge (\underline{r} - \underline{r}') = \begin{pmatrix} 0 \\ -\sin\theta \\ \cos\theta \end{pmatrix} \wedge \begin{pmatrix} x \\ -a\cos\theta \\ -a\sin\theta \end{pmatrix} a\,d\theta$$

$$= a\,d\theta \begin{pmatrix} a \\ x\cos\theta \\ x\sin\theta \end{pmatrix}$$

$$\underline{B} = \frac{\mu_0 I a}{4\pi} \int_0^{2\pi} \begin{pmatrix} a \\ x\cos\theta \\ x\sin\theta \end{pmatrix} \frac{d\theta}{(x^2+a^2)^{3/2}}$$

$$= \frac{\mu_0 I a}{4\pi} \frac{2\pi a}{(x^2+a^2)^{3/2}} \underline{i} = \frac{\mu_0 I a^2}{2(x^2+a^2)^{3/2}} \underline{i}$$

At $x = 0$ this is max. magnitude at $\frac{\mu_0 I}{2a} \underline{i}$

If $a \to 0$ $\quad \underline{B} = \frac{\mu_0 I}{2} \frac{a^2}{|x|^3} \underline{i}$ $\quad$ & we write $\quad S = \pi a^2$ (area inside loop)

and let $I \to \infty$ so $SI = m$ fixed value

$$\Longrightarrow \quad \underbrace{\underline{B} = \frac{\mu_0 m}{2\pi} \frac{1}{|x|^3} \underline{i}}$$

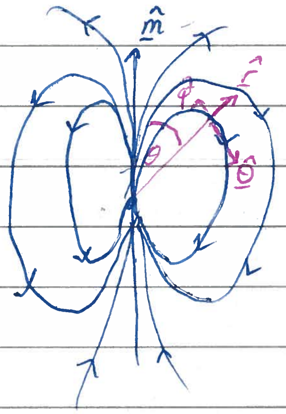
↳ leads to idea of magnetic dipole

The field generated by a vanishingly small loop with normal $\hat{\underline{m}}$ gives a magnetic dipole of strength $\underline{m}$

$$\underline{A} = \frac{\mu_0 m}{4\pi} \frac{\hat{\underline{m}} \wedge \hat{\underline{r}}}{r^2} = \frac{\mu_0}{4\pi} \frac{\underline{m} \wedge \underline{r}}{r^3}$$ is associated potential

$$\underline{B} = \frac{\mu_0 m}{4\pi r^3} \left(3(\hat{\underline{m}} \cdot \hat{\underline{r}})\hat{\underline{r}} - \hat{\underline{m}} \right) \Leftarrow \underline{\nabla} \wedge \underline{A}$$

$$= \frac{\mu_0}{4\pi} \left(\frac{3(\underline{m} \cdot \underline{r})\underline{r}}{r^5} - \frac{\underline{m}}{r^3} \right)$$



$$\underline{B} \cdot \hat{\underline{r}} = \frac{\mu_0}{4\pi} \frac{m}{r^3} \left(3(\hat{\underline{m}} \cdot \hat{\underline{r}})(\hat{\underline{r}} \cdot \hat{\underline{r}}) - \hat{\underline{m}} \cdot \hat{\underline{r}} \right)$$

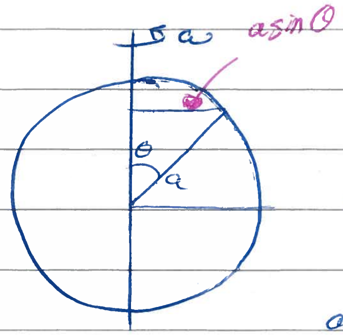
$$= \frac{\mu_0}{4\pi} \frac{m}{r^3} 2(\hat{\underline{m}} \cdot \hat{\underline{r}})$$

$$= \frac{\mu_0}{4\pi} m \frac{2\cos\theta}{r^3}$$

$$\underline{B} \cdot \hat{\underline{\theta}} = \frac{\mu_0 m}{4\pi} \frac{1}{r^3} \left(3(\hat{\underline{m}} \cdot \hat{\underline{r}})(\hat{\underline{r}} \cdot \hat{\underline{\theta}}) - \hat{\underline{m}} \cdot \hat{\underline{\theta}} \right)$$

$$= -\frac{\mu_0 m}{4\pi r^3} \sin\theta$$

Example exam question

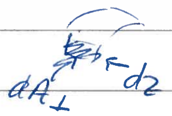


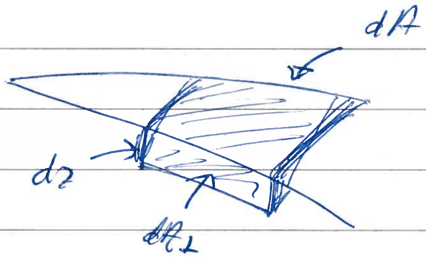
Sphere, radius a with a uniform surface charge density σ rotating with angular velocity ω
What is ~~magnitude~~ magnetic field at centre of sphere

Normally $\underline{J} = \rho \underline{v}$
Here $\underline{v} = a\omega \sin\theta \, \hat{\underline{\phi}}$
Take a small part of charge $dq = \rho dV$
$= \sigma dA$ (in this case)

$\underline{I} = \underline{J} dA_\perp$ ← area of cross section of wire

Take surface area





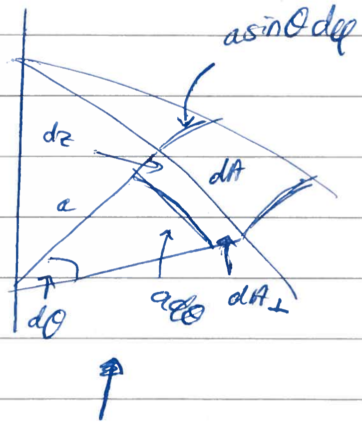
$\rho dV = \sigma dA \implies \rho = \sigma \frac{dA}{dV}$

So $\underline{I} = \rho \underline{v} dA_\perp = \sigma \frac{dA}{dV} dA_\perp \underline{v}$

$= \frac{\sigma (a\sin\theta d\varphi)(a d\theta)(dz\, a d\theta)}{(a\sin\theta d\varphi)(a d\theta) dz} \underline{v}$

$= a\sigma d\theta a\omega \sin\theta \hat{\underline{\varphi}}$

So $dI = \sigma a^2 \sin\theta \hat{\underline{\phi}} \, d\theta$



Check this diagram

charge all over surface

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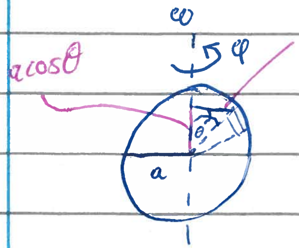
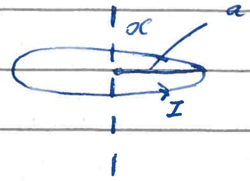


surface charge density σ
rotating angular velocity ω
$I = \sigma\omega a^2 \sin\theta \, d\theta \, \hat{\varphi}$

$\omega a\sin\theta$ is velocity of surface as it goes round



this is from 10.3 : the field of a circular loop of wire

$\underline{B} = \dfrac{I\mu_0 a^2}{2(x^2+a^2)^{3/2}} \, \underline{i}$

$\sigma a d\theta$ is amount of charge
Working out this current will be put on Moodle
Goes in azimuthal direction

This is only along axis

Want value of $\underline{B}$ at centre
$\underline{B}$ at centre can be found by integrating contributions $d\underline{B}$, each generated by a current loop, indexed by θ with current $I = \sigma\omega a^2 \sin\theta \, d\theta$ and taking $x = a\cos\theta$
$\underline{B}$ with radius $a\sin\theta$
Radius of loop changes

changed to $\underline{k}$ since less confusing

$\underline{k}$

So $d\underline{B}\big|_{centre} = \dfrac{\mu_0}{2} \dfrac{\sigma\omega a^2 \sin\theta (a\sin\theta)^2 d\theta}{(\underbrace{a^2\cos^2\theta + a^2\sin^2\theta}_{x^2})^{3/2}} \, \underline{k}$ from above

$=$

$\underline{B}\big|_{centre} = \dfrac{\mu_0}{2} \dfrac{\sigma\omega a^4}{a^3} \displaystyle\int_0^{\pi} \sin^3\theta \, d\theta \, \underline{k}$

θ runs from 0 to π

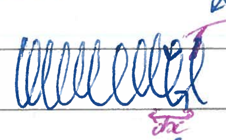
$= \dfrac{2}{3} \mu_0 \sigma\omega a \, \underline{k}$

Integrating up gives value at centre since x is distance from centre of loop

Turns out $\underline{B}$ is const inside all of this rotating sphere

Solenoid

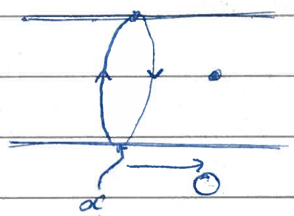
narrow $\angle$ $d\underline{r}$



I current
$d\underline{r}$ curve that current flowing in
dx distance along contour

Number of loops / unit length is N
Then the current density $\underline{J}\,dV$ is replaced by $I\,d\underline{r}\,N\,dx$

Can reach this by similar argument as yesterday



$$dB\Big|_{\substack{\text{along}\\\text{axis}}} = \frac{\underline{i}\,\mu_0 a^2 I N\,dx}{2(a^2+x^2)^{3/2}}$$

at origin, say

at centre

$$\underline{B} = \frac{\underline{i}\,\mu_0 a^2 I N}{2}\int_{-\infty}^{\infty}\frac{dx}{(a^2+x^2)^{3/2}} \qquad x = a\tan\theta$$

$$= \mu_0 N I\,\underline{i}$$

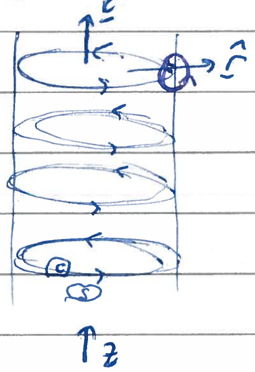
no of turns & current

independent of radius
B is constant

Turns out that this is magnetic field at centre but also anywhere inside

In fact $\underline{B} = \mu_0 I N \underline{i}$ at any point inside the solenoid, not just at centre.

different $\hat{\underline{r}}$!



Solenoid turned round

Magnetic field will have $\underline{k}$ comp. or $\hat{\underline{r}}$
$\underline{B} = \underline{I} \wedge \hat{\underline{r}}$
Not in φ or θ direction
B doesn't depend on θ or z so only r

$$\underline{B} = B_z(r)\,\underline{k} + B_r(r)\,\hat{\underline{r}}$$

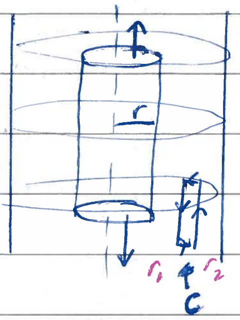
assumed turns are not angled - all vertical

$\underline{\nabla} \cdot \underline{B} = 0$

$\underline{\nabla} \wedge \underline{B} = \mu_0 \underline{J} = 0$ inside ($\&$ outside) solenoid

$$\int_S [B_z(r)\underline{k} + B_r(r)\underline{\hat{r}}] \cdot \underline{dS} = 0$$ by divergence theorem

$$\int_C [B_z(r)\underline{k} + B_r(r)\underline{\hat{r}}] \cdot \underline{dr} = 0$$



Cylinder centred on centre, radius r length L

Value of B dS at top + bottom cancel since normal equal + opposite

$\underline{dS}$ points in dir^n of $\underline{\hat{r}}$ but $\underline{\hat{r}} \cdot \underline{k} = 0$

So we get

$B_r(r)\, 2\pi r L = 0 \implies B_r(r) = 0$

So field must point in $\underline{k}$ dir^n but still could depend on position

$$\int_C B_z(r)\underline{k} \cdot \underline{dr} = 0$$ C is closed contour

length of contour not cylinder — goes down at 1 value of r, back up at another

$B_z(r_1)(-L) + B_z(r_2)(L) = 0$ cancels in $\rightleftarrows$ directions

$B_z(r_1) = B_z(r_2)$ but r_1, r_2 totally arbitrary

$\implies B_z$ is constant, B

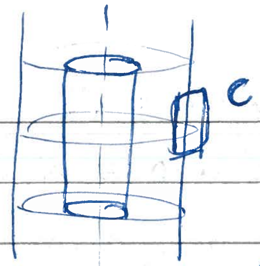
$\underline{B} = B\underline{k}$

Outside $B = B_e \underline{k}$

We will say magnetic field outside is zero

Relate magnetic to current by choosing Amperian loop to straddle current

$$\int_C B\,\underline{t}\cdot d\underline{r} = \mu_0 J_s$$
$$= \mu_0 NIL$$



so contains some current

One side of loop inside cylinder + one side outside

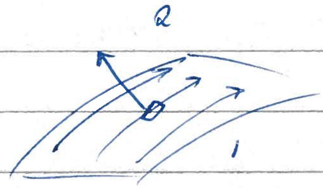
$$B_e L - B L = \mu_0 NIL$$

↑ external × length ↑ internal × length

$B_e = 0 \qquad B = -\mu_0 NI$ have got directions wrong ~~sign~~ - this mistake is not made in notes

Modelled solenoid as current hoops

Using Ampere loop found an expression for discontinuity in magnetic field - similar to pillbox example



Vector goes from inside to outside

Can show $\hat{n}\cdot[\underline{B}]_1^2 = 0$

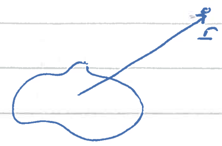
Use any contour, current density doesn't need to be const.

$\hat{n}\wedge[B]_1^2 = \mu_0 \underline{J}_s$ ← surface current density

03/03/15

$\underline{A} = \frac{\mu_0}{4\pi} \int \frac{\underline{J}(\underline{r}')\, dV}{|\underline{r} - \underline{r}'|}$ $\qquad |\underline{r}| \to \infty$

↑ not $\underline{r}'$



$$\frac{1}{|\underline{r} - \underline{r}'|} = \frac{1}{r} + \frac{\hat{\underline{r}} \cdot \underline{r}'}{r^2} + \ldots$$

So if $|\underline{r}| \to \infty$

$$\underline{A}(\underline{r}) \approx \frac{\mu_0}{4\pi} \left[\frac{1}{r} \int_{V'} \underline{J}(\underline{r}')\, dV' + \frac{1}{r^2} \int \underline{J}(\underline{r}')\, \hat{\underline{r}} \cdot \underline{r}'\, dV' + \ldots \right]$$

before, in electric context
this gave us monopole
Will show $= 0$

Recall cont. eq^n: $\frac{d\rho}{dt} + \underline{\nabla} \cdot \underline{J} = 0$ $\qquad$ only works if currents steady $\frac{d\rho}{dt} = 0 \Rightarrow \nabla \cdot J = 0$

Consider $\underline{J} \cdot \underline{\nabla} f$ & recall: $\underline{\nabla} \cdot (\underline{J} f) = \underbrace{f \underline{\nabla} \cdot \underline{J}}_{=0} + \underline{J} \cdot \underline{\nabla} f$

Integrate to get $\int_{V'} \underline{\nabla} \cdot (\underline{J} f)\, dV' = \int_{V'} \underline{J} \cdot \underline{\nabla} f\, dV'$

$$\int_{\partial V'} \underline{J} f \cdot d\underline{S} = \int_{V'} \underline{J} \cdot \underline{\nabla} f\, dV'$$

↑ 0 away from current $\qquad$ big enough sphere so J is zero, then extend to all space

$\int_{V'} \underline{J} \cdot \underline{\nabla} f\, dV' = 0$ $\qquad$ if V' is all space

Surprising, since we have said nothing about r

$f = \underline{a} \cdot \underline{r}$, $\quad \underline{\nabla} f = \underline{a}$ $\qquad$ since if $f = a_1 x + a_2 y + a_3 z$, $\nabla f = \begin{matrix} a_1 \\ a_2 \\ a_3 \end{matrix}$

$$\int_{V'} \underline{J} \cdot \underline{a}\, dV' = 0 \Rightarrow \underline{a} \cdot \int_{V'} \underline{J}\, dV' = 0$$

True for any $\underline{a}$ $\Rightarrow \boxed{\int_{V'} \underline{J}\, dV' = 0}$

$f(\underline{r}) = (\underline{a}\cdot\underline{r})(\underline{b}\cdot\underline{r})$

$\nabla f = \underline{a}(\underline{b}\cdot\underline{r}) + (\underline{a}\cdot\underline{r})\underline{b}$

So $\int_{V'} \underline{J}(\underline{r}')\cdot[\underline{a}(\underline{b}\cdot\underline{r}') + (\underline{a}\cdot\underline{r}')\underline{b}]\, dV' = 0$ int. wrt r'

As we are integrating wrt $\underline{r}'$ & $\underline{r}$ is constant

so can choose b const as far as integration concerned

$$\int_{V'} \underline{J}\cdot[\underline{a}(\hat{\underline{r}}\cdot\underline{r}') - (\underline{a}\cdot\underline{r}')\hat{\underline{r}}]\, dV' = 0$$

So $\underline{a}\int_{V'} \underline{J}(\hat{\underline{r}}\cdot\underline{r}') + \underline{r}'(\underline{J}\cdot\hat{\underline{r}})\, dV = 0$

$$\underbrace{\int_{V'} \underline{J}(\hat{\underline{r}}\cdot\underline{r}')\, dV'}_{\text{this is our second term}} = -\int_{V'} \underline{r}'(\underline{J}\cdot\hat{\underline{r}})\, dV'$$

where we have done $A = \frac{1}{2}\cdot 2 \ldots$

$$\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^2}\frac{1}{2}\left(\int_{V'} \underline{J}(\hat{r}\cdot r') - r'(\underline{J}\cdot\hat{r})\, dV\right)$$

using identity

$$= \frac{\mu_0}{4\pi}\frac{1}{r^2}\frac{1}{2}\int_{V'} (\underline{r}'\wedge\underline{J})\wedge\hat{\underline{r}}\, dV$$

$$= \frac{\mu_0}{4\pi}\frac{1}{r^2}\, \underline{m}\wedge\hat{\underline{r}} = \frac{\mu_0}{4\pi}\, \underline{m}\wedge\underline{r}$$ where $\underbrace{\underline{m} = \frac{1}{2}\int_{V'} \underline{r}\wedge\underline{J}\, dV}_{\text{Magnetic dipole moment}}$

similar to electric one

If the current is in a wire (for multiple wires use linearity + add them all up) $\underline{r}(t)$ then $\underline{J}\, dV = I\, d\underline{r}$ &

$$\underline{m} = \frac{I}{2}\int_C \underline{r}\wedge d\underline{r}$$

Then $\frac{1}{2}(\underline{r}\wedge d\underline{r}) = dA$

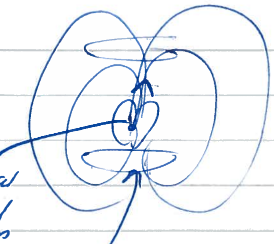
$= I\underline{S}$ $\quad$ $\underline{S}$ the vector area of wire loop

$$\underline{A} = \frac{\mu_0}{4\pi}\frac{1}{r^3}\, \underline{m}\wedge\underline{r}$$

$\underline{B} = \nabla\wedge\underline{A}$ $\quad$ $B_i = \varepsilon_{ijk}\frac{\partial}{\partial x_j}\left(\frac{\varepsilon_{kpq} m_p x_q}{(x_l x_l)^{3/2}}\right)\left(\frac{\mu_0}{4\pi}\right)$

$$\underline{B} = \frac{\mu_0}{4\pi}\left(\frac{3(\underline{m}\cdot\underline{r})\underline{r}}{r^5} - \frac{\underline{m}}{r^3}\right)$$

$$\underline{B} = \frac{\mu_0}{4\pi}\frac{m}{r^3}\left(3(\hat{m}\cdot\hat{r}) - \hat{m}\right)$$

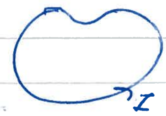


horizontal loop, normal straight up

symmetry rotating about axis

If we have magnetic field + loop carrying current, then what is force on loop due to B

$\underline{B}$



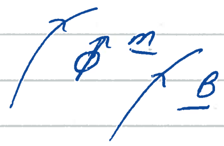
$$\overbrace{e\underline{v}\wedge\underline{B}}^{\text{force}}$$

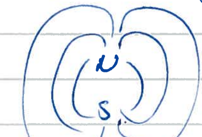
$$d\underline{F} = I d\underline{r}\wedge\underline{B}$$

$$\underline{F} = \int d\underline{F} = I\int d\underline{r}\wedge\underline{B}$$

If $\underline{B}$ is constant this is $\underline{F} = I\underbrace{\left(\int d\underline{r}\right)}_{=0 \text{ if loop is closed}}\wedge\underline{B}$

If you have tiny loop w/ magnetic dipole m





+compass

$$\underline{F} = \nabla(\underline{m}\cdot\underline{B})$$

won't go through since algebra horrible

Moment on a wire loop (about the origin)

r is position vector of point on loop → $\underline{r}$

$$\underline{G} = \oint \underline{r}\wedge d\underline{F}$$

$$= \oint \underline{r}\wedge(I d\underline{r}\wedge\underline{B})$$

$$= I\oint\left[d\underline{r}(\underline{r}\cdot\underline{B}) - \underline{B}\,\underline{r}\cdot d\underline{r}\right]$$

If wire loop tiny, $\underline{B}$ approx constant

If loop is small $\underline{B}$ is approximately constant

$$\underline{G} = I\oint(\underline{r}\cdot\underline{B})\,d\underline{r} - I\underline{B}\oint\underline{r}\cdot d\underline{r}$$

dr^2

$d\underline{r}\cdot\underline{r}$

$$= \frac{1}{2}\left(\oint\underline{r}\wedge d\underline{r}\right)\wedge\underline{B}$$

Take as proved

$$\underline{G} = \underline{m}\wedge\underline{B}$$

stops turning when $\underline{m}$ parallel to $\underline{B}$ so this is why compasses line up w/ lines of magnetic force

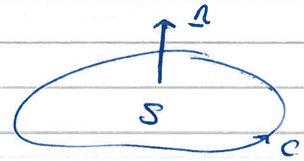
2

Exam: what is force on wire loop
won't ask: what is moment on tiny loop
moment / force on magnetic dipole
BUT it is fair to ask moment / force on electric dipole
like in HW

Missing out inductance (measure of flux linkage)
It is in notes - not going to be examined

Faraday's Law: $\underline{\nabla} \wedge \underline{E} + \frac{\partial B}{\partial t} = 0$

$\int_S \underline{\nabla} \wedge \underline{E}\, \underline{dS} + \int_S \frac{\partial B}{\partial t} \cdot \underline{dS} = 0$



$\underbrace{\int_C \underline{E} \cdot \underline{dr}}_{\mathcal{E}} = -\frac{\partial}{\partial t} \underbrace{\int_S \underline{B} \cdot \underline{dS}}_{\Phi}$

← here we can do this by saying our loop is fixed

$\mathcal{E}$ Electromotive force

Flux of magnetic field through surface Φ

$\mathcal{E} = -\frac{d}{dt}\Phi$

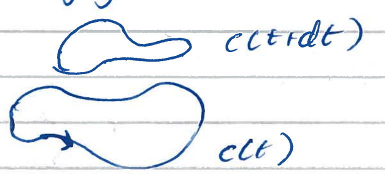
$\mathcal{E} = -\oint \nabla\varphi \cdot dr = -\int_{start}^{end} \nabla\varphi \cdot \underline{dr}$

$= -(\varphi_{end} - \varphi_{start})$

$= \varphi_{start} - \varphi_{end}$ difference between voltage at start + end of loop

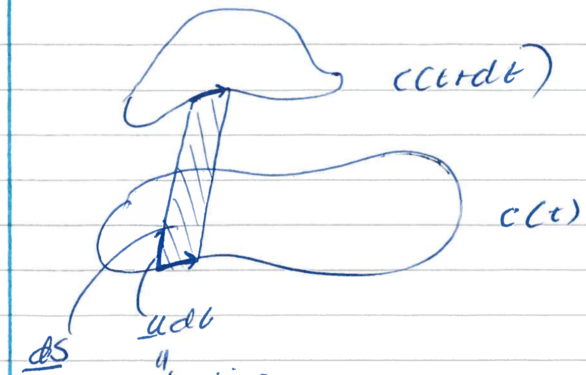
so could use this as a generator

Loop varying: how does this generate magnetic field



moving position and/or changing shape

Moving wire loop carries a fixed current I through a time independent field $\underline{B}$



$\underline{dS} = \underline{dr} \wedge \underline{u}\, dt$

Lorentz force acting on charge in wire

$\underline{F} = e\underline{u} \wedge \underline{B}$

This is seen as a force generated by an electric field by the charges $\underline{E}$

Consider the component of the electric field along the wire

$\underline{E} \cdot \underline{dr} = (\underline{u} \wedge \underline{B}) \cdot \underline{dr}$

$\underline{F} = e\underline{u} \wedge \underline{B}$

$= e\underline{E}$ ← charges see this as electric field

this is looking at it in two different frames

Integrate around C

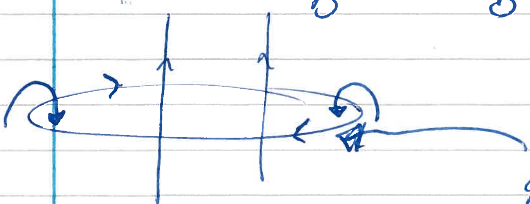
$\mathcal{E} = \int \underline{E} \cdot \underline{dr} = \int_c (\underline{u} \wedge \underline{B}) \cdot \underline{dr}$

$= -\int_C \underline{B} \cdot (\underline{u} \wedge \underline{dr})$

$\underline{dr} \wedge \underline{u} = -\dfrac{\underline{dS}}{dt}$ (since $\underline{dS} = \underline{dr} \wedge \underline{u}\, dt$)

$\mathcal{E} = -\dfrac{d}{dt} \oint \underline{B} \cdot \underline{dS}$ ← we have fudged this (we want minus sign that is there but don't actually have it)

will find out what went wrong maybe next lecture



B increasing $\Rightarrow \dfrac{d\Phi}{dt}$ +ve

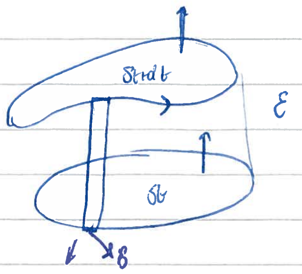
$\mathcal{E} < 0$

which means current goes opposite way!

magnetic field generated by current generated by changing magnetic field acts to reduce current

This is Lenz' Law

04/03/15



$d\varepsilon = dr \wedge u\,dt$

$\varepsilon = \oint \underline{E} \cdot \underline{dr}$

$\underline{E} \cdot \underline{dr} = (\underline{u} \wedge \underline{B}) \cdot dr$

$\nabla \cdot B = 0$

('change in flux is minus flux through sigma')

$\Rightarrow \int \underline{B} \cdot n \, dS = 0$

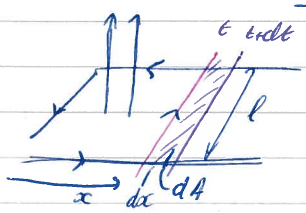
$\Rightarrow \int_{c(t+dt)} B \cdot dS + \int_{c(t)} \underline{B} \cdot dS + \int_{\varepsilon} \underline{B} \cdot \underline{d\varepsilon} = 0$

$\Phi(t+dt) - \Phi(t) = -\int_{\varepsilon} \underline{B} \cdot d\varepsilon$

must be minus because of direction of unit normal

$d\Phi = -\int_{\varepsilon} \underline{B} \cdot d\varepsilon$

now we have the lost minus sign from last lecture



Imagine two parallel rails joined by a conductor at one end in the presence of a constant magnetic field B, normal to the plane containing the rails.
The rails are joined at left end & we complete a circuit by placing a bar across the rails.
The bar is given a velocity v in the rightwards direction.

In a time interval dt, the change of magnetic flux through the loop is

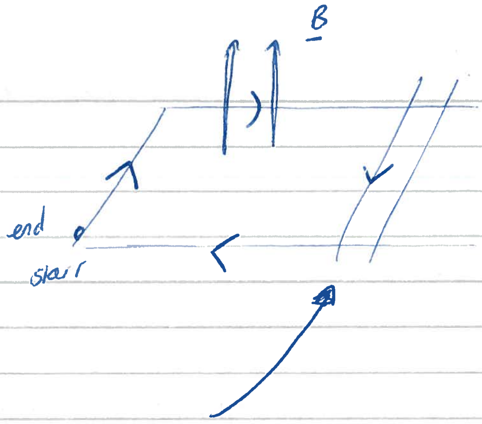
$d\Phi = B \cdot dA = B dx\, l$

$\frac{d\Phi}{dt} = Bl\frac{dx}{dt} = Blv$

$\varepsilon = -\frac{d\Phi}{dt} = -Blv$

$\varepsilon = \oint \underline{E} \cdot \underline{dr} = \varphi_{start} - \varphi_{end} = -Blv$ so $\varphi_{end} > \varphi_{start}$ (since -ve)

$\underline{E} = -\nabla\varphi$ ~~&~~ other way around because of minus



so a current goes through the loop in a clockwise direction

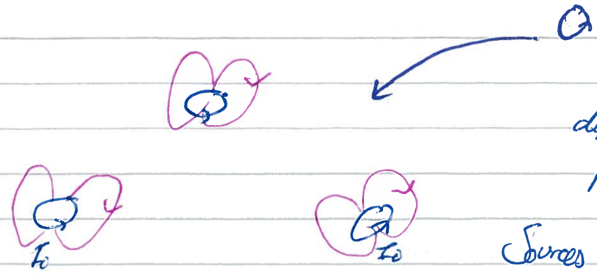
$V = IR$

$I = \frac{BLv}{R}$ ← resistance in this example

Wire is moving through magnetic field so what is force
As the moving wire has a current within it, it feels a force
$\underline{F} = I \int \underline{dr} \times \underline{B}$

This force has a direction to left (use RH rule)
& so the wire is subject to a force that slows it down

Energy lost as heat (after giving a push)
Can put in to $F=ma$ to get diff. eq^n for v & see it decay exponentially to zero



Current loops, or magnetic dipoles, generate a magnetic field

Sources of magnetism are, in effect, any loops of current

An individual loop sits in the magnetic field generated by the other loops

The flux through the ith loop is $\int_{S_i} \underline{B}\cdot d\underline{S} = \Phi_i$

$$\int \underline{B}\cdot d\underline{S} = \int \underline{\nabla}\wedge\underline{A}\cdot dS = \oint \underline{A}\cdot d\underline{r}$$

Bringing in another loop, carries own magnetic field, adds to magnetic field already there + changes flux

As a new dipole is brought in from infinity it brings a magnetic field & all the Φ_i potentially alter as $\underline{B}$ alters

This change in flux generates a potential potential drop $\mathcal{E}$ about the loops already in place.

In order to keep the current in a particular dipole at the same level, work needs to be done in keeping the current constant against this $\mathcal{E}$

Reminder

$\underset{\text{force}}{e\underline{E}} \cdot \underset{\text{distance}}{d\underline{r}}$ = work done

To generate current must work at particular rate

Rate = power

$\cancel{\underline{I}\cdot d\underline{r}}$

$\underline{I}\cdot\underline{E}$ is power required to keep current flowing through field $\underline{E}$

The rate at which this work needs to be done to keep the current I_i fixed in the loop i is

$$\mathcal{E}_i I_i = \frac{d\Phi_i}{dt} I_i$$

Total work required to bring in a new loop is

$$\sum_i \int_0^t I_i \frac{d\Phi_i}{dt}\, dt = \sum_i I_i \Phi_i$$

as current is constant

product gives us how much work required to bring up another one

We can identify the magnetic energy in an arrangement of dipoles as

$$\frac{1}{2}\sum_i I_i \Phi_i = \frac{1}{2}\sum_i I_i \oint_{C_i} \underline{A}\cdot d\underline{r}$$

$$= \frac{1}{2}\sum_i \int_{C_i} \underline{A}\cdot(\underbrace{I_i d\underline{r}}_{\underline{J}\,dV})$$

← all current in wires
← general currents moving about
} big leap

$$= \frac{1}{2}\int_V \underline{A}\cdot\underline{J}\,dV$$

Ampere's law (i.e. ignoring displacement current) gives $\mu_0 \underline{J} = \nabla\wedge\underline{B}$

So the energy $u = \frac{1}{2\mu}\int_V \underline{A}\cdot\nabla\wedge\underline{B}\,dV$

Recall:

$$\nabla\cdot(\underline{A}\wedge\underline{B}) = \underline{B}\cdot\overbrace{\nabla\wedge\underline{A}}^{\underline{B}} - \underline{A}\cdot\nabla\wedge\underline{B}$$

so $u = \frac{1}{2\mu_0}\left\{\int_V \underline{B}\cdot\underline{B}\,dV - \int_V \nabla\cdot(\underline{A}\wedge\underline{B})\,dV\right\}$

↑ will show this is zero as done before

$$\int_{\partial V}(\underline{A}\wedge\underline{B})\cdot d\underline{S}$$

↑ drops like $\sim\frac{1}{r^2}$ ↑ drops like $\sim\frac{1}{r^3}$ ↖ $\sim r^2$

so product goes like $\frac{1}{r^3}$ as $r\to\infty$

$$u = \frac{1}{2\mu_0}\int_V \underline{B}\cdot\underline{B}\,dV$$

expected to know not derive

~~der~~ magnetic case here is not examinable

derivation of electric field is examinable

10/03/15

No currents, charges

$\underline{E}$ and $\underline{B}$ interacting w/o presence of charges (although may still be generated by one)

Want to look at effect of displacement current term

$\nabla\cdot\underline{E} \qquad \nabla\cdot\underline{B} = 0 \qquad \nabla\wedge\underline{E} + \underline{B}_t = \underline{0}$

(*) $\nabla\wedge\underline{B} = \underbrace{\mu_0\varepsilon_0}_{1/c^2}\underline{E}_t$

Taking curl: $\nabla\wedge(\nabla\wedge\underline{E}) + (\nabla\wedge\underline{B})_t = \underline{0}$

$\nabla(\underbrace{\nabla\cdot\underline{E}}_{0}) - \nabla^2\underline{E} + \left(\frac{1}{c^2}\underline{E}_t\right)_t = \underline{0}$

$\implies \quad c^2\nabla^2\underline{E} = \underline{E}_{tt}$

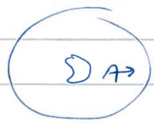
$\cancel{c^2\frac{\partial^2 E}{\partial x^2}} \quad c^2\frac{\partial^2 E}{\partial x^2} = E_{tt}$

Could have taken $\nabla\wedge$ (*) to get $c^2\nabla^2\underline{B} = \underline{B}_{tt}$

1D

no change of form moving w/ speed c
$\leftarrow$ here $c>0$ but could move $\leftarrow$ with $c<0$

In 3D moves out w/o change of form but amplitude decays like 1/distance



Electrical energy density (energy in $\underline{E}$ per unit volume)

$\frac{1}{2}\varepsilon_0|\underline{E}|^2$ and consider its time variation

$\varepsilon_0\mu_0\varepsilon_0 = \frac{1}{\mu_0}$

$\frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon_0\underline{E}\cdot\underline{E}\right) = \varepsilon_0\underline{E}\cdot\frac{\partial\underline{E}}{\partial t} = \varepsilon_0 c^2\,\underline{E}\cdot(\nabla\wedge\underline{B})$

Consider $\nabla\cdot(\underline{E}\wedge\underline{B}) = \underline{B}\cdot(\nabla\wedge\underline{E}) - \underline{E}\cdot(\nabla\wedge\underline{B})$

$= -\underline{B}\cdot\underline{B}_t - \underline{E}\cdot(\nabla\wedge\underline{B})$

In magnetic field energy density is $\frac{1}{2\mu_0}\underline{B}\cdot\underline{B}$

Divide by $\frac{1}{\mu_0}$ $\quad = -\frac{1}{2\mu_0}\frac{\partial}{\partial t}|\underline{B}|^2 - \frac{\underline{E}\cdot(\nabla\wedge\underline{B})}{\mu_0}$

$$\frac{\partial}{\partial t}\underbrace{\left(\frac{1}{2}\varepsilon_0|\underline{E}|^2 + \frac{1}{2\mu_0}|\underline{B}|^2\right)}_{\text{total energy density } = u} + \nabla\cdot\underbrace{\frac{(\underline{E}\wedge\underline{B})}{\mu_0}}_{\underline{S}} = 0$$

$$\frac{\partial u}{\partial t} + \nabla \cdot \underline{S} = 0$$ (this looks like charge conservation $\frac{\partial \rho}{\partial t} + \nabla \cdot J$)

$$\underline{S} = \underline{E} \wedge \underline{B} / \mu_0$$ Poynting vector

Given a volume V

$$\underbrace{\frac{\partial}{\partial t} \int_V u \, dV}_{\text{rate of change of energy in volume}} + \int \nabla \cdot \underline{S} \, dV = 0$$

$\int_{\partial V} \underline{S} \cdot d\underline{\Sigma}$

Σ 'sigma' is edge of volume

$$\frac{\partial}{\partial t} \int_V u \, dV = -\int_{\partial V} \underline{S} \cdot d\underline{\Sigma}$$

The Poynting vector gives the direction and magnitude of energy flux / unit area

rate of change of energy in volume is flux of S outside of volume

If did same exercise in place of current get

$$\frac{\partial u}{\partial t} + \nabla \cdot S + \underline{E} \cdot \underline{J} = 0$$

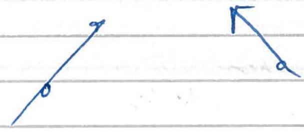
r.o.c of energy

flux out of volume

corresponds to current flowing against electric field - requires work
work done in moving in volume

This leads to damping of waves

Momentum per unit volume $\underline{S}/c^2$

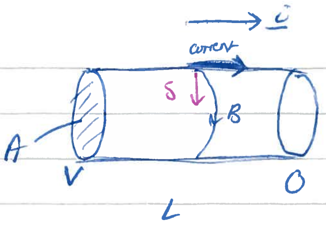


charge one due to other are ~~not~~ equal + opposite

when you add in momentum, it is conserved (same as saying equal + opposite)

Power station → house
Energy carried in $\underline{E}$ that surrounds wire
Poynting vector does not point along wire but instead points into.

potential only varying in one direction



$\underline{E} = \underline{i}\, V/L$

$\underline{J} = \sigma \underline{E} = \frac{V\sigma}{L}\, \underline{i}$

$J\, dV = I\, dr$

$J(AL) = IL \implies I = JA$

(Recall $V = IR$ where R resistance)

$I = \frac{A\sigma V}{L}$, $\frac{A\sigma}{L} = \frac{1}{R}$

$\underline{B} = \frac{\mu_0 I}{2\pi a} \cdot \underline{\hat{\theta}}$

$\underline{S} = \frac{1}{\mu_0} \underline{E} \wedge \underline{B}$ & $\underline{S}$ points in to the wire & has magnitude ~~EB~~

$S = \frac{1}{\mu_0} \frac{V}{L} \frac{\mu_0 I}{2\pi a}$

Total flux of energy is $2\pi a L \frac{1}{\mu_0} \frac{V}{L} \frac{\mu_0 I}{2\pi a} = VI$

which is the rate at which work is done to move the current I through the potential V
Can read more in Feynman

Plane waves

We look for a solution to the wave equation $\varphi_{tt} = c^2 \nabla^2 \varphi$ of the form $\varphi = f(\underline{e} \cdot \underline{r} - ct)$

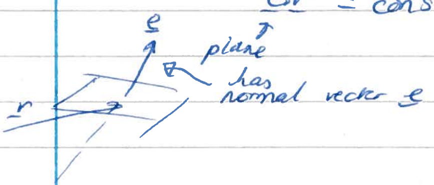
$(e_1, e_2, e_3) \cdot (x, y, z)$

We see φ is constant on the plane $\underline{e} \cdot \underline{r} - ct = \text{const}$.

$e_1 x + e_2 y + e_3 z$

$e_1 x - ct$ (if axis done right)

$\underline{e} \cdot \underline{r} = \text{const} + ct$



The wave travels in the direction of $\underline{e}$ with speed c

Want to verify that our sol^n satisfies wave equation

$\nabla^2 = \nabla \cdot \nabla$

$\nabla f(\underline{e} \cdot \underline{r} - ct) = \underline{e} f'(\underline{e} \cdot \underline{r} - ct)$

$\nabla \cdot (\nabla f) = \underbrace{\underline{e} \cdot \underline{e}}_{\underline{e} \cdot \underline{e} = 1 \text{ since } \underline{e} \text{ is unit vector}} f''(\underline{e} \cdot \underline{r} - ct)$

$c^2 \nabla^2 f = c^2 f'' = f_{tt}$

Now consider $\overset{\varphi}{\shortparallel}$ $f(\underline{e}\cdot\underline{r} - ct) = \begin{matrix}\sin\\ \cos\end{matrix}\left(\omega t - \frac{\omega}{c}\,\underline{r}\cdot\underline{e}\right)$

so we have mult. by $-1 + \frac{\omega}{c}$ $\quad = \begin{matrix}\sin\\ \cos\end{matrix}\left(\omega t - k\,\underline{r}\cdot\underline{e}\right)$

$\frac{2\pi}{\omega}$ is period, ω is frequency of wave

$\frac{2\pi}{k}$ is wavelength of the wave, k is wave number
$k\underline{e}$ " vector

Can write $e^{i(\omega t - k\underline{r}\cdot\underline{e})}$ but understood always to take real part

$\varphi = \mathrm{Re}\{\varphi_0\, e^{i(\omega t - k(\underline{r}\cdot\underline{e}))}\}$

Consider $\underline{E} = \underline{\alpha}\cos\Omega + \underline{\beta}\sin\Omega$ $\quad \Omega = \omega t - k\underline{e}\cdot\underline{r}$
$\qquad = \omega t - \underline{k}\cdot\underline{r}$
How much α, β you have relates to polarisation $\qquad$ where $k\underline{e} = \underline{k}$

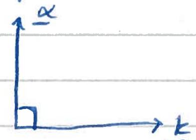
This is a solution to the wave equation
$c^2\nabla^2\underline{E} = \underline{E}_{tt}$ for any $\underline{\alpha}$ & $\underline{\beta}$

But $\underline{\alpha}$ & $\underline{\beta}$ must satisfy some constraints

$\underline{\nabla}\cdot\underline{E} = 0$

So $\frac{\partial}{\partial x}\Omega = \frac{\partial}{\partial x}(\omega t - \underline{k}\cdot\underline{r}) = -k_1$

$\underline{\nabla}\cdot\underline{E} = -\underline{\alpha}\,\underline{k}\,(-\sin\Omega) \mp \underline{\beta}\,\underline{k}\cos\Omega$
$\quad = \underline{\alpha}\cdot\underline{k}\sin\Omega - \underline{\beta}\cdot\underline{k}\cos\Omega = 0$ for all Ω
So $\underline{k} \perp$ to $\underline{\alpha}$ & $\underline{\beta}$



~~Time dependent E generates a B through~~
~~$\nabla\wedge\underline{B} = \frac{1}{c^2}\frac{\partial \underline{E}}{\partial t}$ no current but~~

$\underline{B}_t = -\underline{\nabla}\wedge\underline{E}$ $\qquad$ should be $\underline{k}$ not $\underline{e}$

integrate

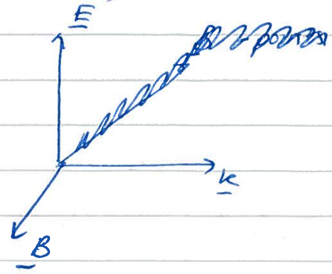
$= \underline{k}\wedge\underline{\alpha}\left(\frac{\omega}{c}\right)\sin\Omega - \underline{k}\wedge\underline{\beta}\cos\Omega\left(\frac{\omega}{c}\right)$

$B = \frac{1}{c}\left(\underline{k}\wedge\underline{\alpha}\cos\Omega + \underline{k}\wedge\underline{\beta}\sin\Omega\right)$ $\qquad$ notes have $\underline{e}$ rather than

$$B_t = -\nabla \wedge \underline{E} = -\underline{k} \wedge \underline{\alpha} \sin \Omega + \underline{k} \wedge \underline{\beta} \cos \Omega$$

$$B = \frac{1}{c}\left(\underline{e} \wedge \underline{\alpha} \cos \Omega + \underline{e} \wedge \underline{\beta} \sin \Omega\right)$$

$$= \frac{1}{c}\, \underline{e} \wedge \underline{E}$$



Feature of plane waves:
$\underline{B}$ always at right angles to $\underline{E}$

$$u = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2}\frac{1}{\mu_0} B^2$$ ← Recall

energy density

$$|B| = \frac{1}{c}|E|$$

$$B^2 = \frac{1}{c^2}|E|^2 \quad \text{so} \quad \frac{1}{2\mu_0}B^2 = \frac{1}{2\mu_0 c^2}E^2 = \frac{1}{2\mu_0}\mu_0\varepsilon_0 E^2 = \frac{\varepsilon_0}{2}E^2$$

so energy in magnetic field is the same as that in electric field

$$\underline{\nabla} \cdot \underline{E} = \rho / \varepsilon_0 \qquad \underline{\nabla} \cdot \underline{B} = 0 \qquad \nabla \wedge \underline{E} + \underline{B_t} = \underline{0}$$

$$\nabla \wedge B - c^{-2}\underline{E_t} = \mu_0 \underline{J}$$

You need to know these!

$$\nabla \cdot \underline{B} = 0 \implies \underline{B} = \nabla \wedge \underline{A} \implies B_t = \underline{\nabla} \wedge \underline{A_t}$$

diff. wrt to t + diff. wrt to space commute

$$\text{so} \quad \underline{\nabla} \wedge \underline{E} + \underline{\nabla} \wedge \underline{A_t} = 0 \implies \underline{\nabla} \wedge (\underline{E} + \underline{A_t}) = 0$$

so $\exists\, \varphi$ so that
$$\underline{E} + \underline{A_t} = -\underline{\nabla}\varphi$$
$$\underline{E} = -\underline{\nabla}\varphi - \underline{A_t}$$

$$\underline{E} = -\underline{\nabla}\varphi - \underline{A_t} \quad \& \quad \underline{B} = \underline{\nabla} \wedge \underline{A}$$

$$\varphi = \varphi(r,t) \qquad \underline{A} = A(r,t)$$

$$\bar{A} = A + \underline{\nabla}\chi \qquad \bar{\varphi} = \varphi - \chi_t$$

← some arbitrariness in choice of $\underline{A}$ and φ

so can see gauge transformation doesn't change fields

$$\bar{B} = \nabla \wedge \bar{A} = \underline{\nabla} \wedge (A + \nabla\chi) = \underline{\nabla} \wedge \underline{A} + \underline{\nabla} \wedge \cancel{\nabla\chi}^{\,0}$$
$$= \underline{B}$$

curl of grad $= 0$

$$\underline{\bar{E}} = -\nabla\bar{\varphi} - \bar{A_t} = -\nabla(\varphi - \cancel{\chi_t}) - A_t - \cancel{(\nabla\chi)_t}$$
$$= -\nabla\varphi - A_t$$
$$= \underline{E}$$

← $\nabla(\chi_t)$ and $(\nabla\chi)_t$ cancel

Coulomb Gauge

We saw this way before & chose $\overbrace{\underline{\nabla} \cdot \underline{A} = 0}$

$\nabla^2 \varphi = -\rho/\varepsilon_0 \qquad \underline{\nabla}^2 \underline{A} = -\mu_0 \underline{J}$ ← satisfy Laplace not wave

$$\underline{A} = \frac{\mu_0}{4\pi} \int_V \frac{\underline{J}(\underline{r}')}{|\underline{r} - \underline{r}'|} \, dV'$$

suggests instantaneous action at a distance

$$\underbrace{\underline{\nabla} \cdot \underline{A} + \frac{1}{c^2} \frac{\partial \varphi}{\partial t}}_{\text{Lorenz Gauge}} = 0$$

↑ not Lorentz

We have $\underline{E} = -\nabla\varphi - \underline{A}_t$

$$\underline{\nabla} \cdot \underline{E} = \rho/\varepsilon_0 = -\underline{\nabla} \cdot \underline{\nabla} \varphi - \underline{\nabla} \cdot \underline{A}_t = -\nabla^2 \varphi - (\underline{\nabla} \cdot \underline{A})_t$$

$$= -\nabla^2 \varphi - \left(\frac{-1}{c^2} \varphi_t \right)_t$$

so φ satisfies wave equation with a forcing ρ/ε_0

$$\nabla^2 \varphi - \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} = -\rho/\varepsilon_0$$

$$\underline{B} = \underline{\nabla} \wedge \underline{A} \quad \text{so} \quad \underline{\nabla} \wedge (\underline{\nabla} \wedge \underline{A}) = \underline{\nabla} \wedge \underline{B} = \mu_0 \underline{J} + \frac{1}{c^2} \underline{E}_t$$

$$\nabla(\underline{\nabla} \cdot \underline{A}) - \underline{\nabla}^2 \underline{A} = \mu_0 \underline{J} + \frac{1}{c^2} (-\underline{\nabla}\varphi - \underline{A}_t)_t$$

$$\nabla \underbrace{(\underline{\nabla} \cdot \underline{A} + \frac{1}{c^2} \varphi_t)}_{=0} = \mu_0 \underline{J} + \nabla^2 \underline{A} - \frac{1}{c^2} \underline{A}_{tt}$$

$$\nabla^2 \underline{A} - \frac{1}{c^2} \underline{A}_{tt} = -\mu_0 \underline{J}$$

$\underline{J}$ + ρ vary, information travels outwards w/ speed c, so can construct $\underline{E}$, and $\underline{B}$. No instantaneous action

If you choose Coulomb Gauge have instantaneous β then something later

If include displacement current, all satisfy wave equation

11/03/15

$$\left.\begin{aligned} B &= \nabla \wedge A \\ \underline{E} &= -A_t - \nabla\varphi \end{aligned}\right\} \quad \nabla \cdot A + \frac{1}{c^2}\varphi_t = 0$$

$$\nabla^2 A - \frac{A_{tt}}{c^2} = -\mu_0 J$$

$$\nabla^2 \varphi - \frac{\varphi_{tt}}{c^2} = -\rho/\varepsilon_0$$

retarded time

only difference in sol^n to Laplace's equation

$$\varphi(r,t) = \frac{1}{4\pi\varepsilon_0}\int_{V'} \frac{\rho(r', t-(r-r')/c)}{|r-r'|}\, dV'$$

← our solution is of this form

$$A(r,t) = \frac{\mu_0}{4\pi}\int J(\qquad)\, dV$$

$|r - r'|$

where you are — where forcing is

$(r-r')/c$ gives time taken for wave to get to you

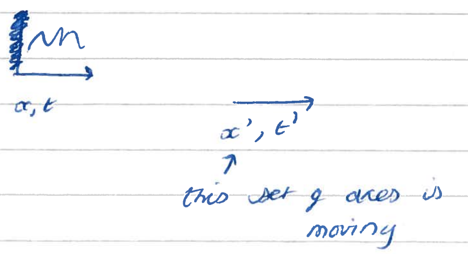
0) We want the laws of physics to be the same in each inertial reference frame

1) Maxwell's equations have the same form in all reference frame
We wrote MEs in vectors + vectors are indep. of reference frames
⟹ speed of light the same in all reference frames

2) Newton / Galilean relativity, however predicts that different inertial frames measure different relative velocities

1) + 2) contradict each other, so keep 0) + 1) & throw away 2) - which means throwing away absolute time

$\varphi_{tt} = c^2\varphi_{xx}$ 1-D wave equation



$$x' = x - vt$$
$$y' = y$$
$$z' = z$$
$$t' = t$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'}\frac{\partial t'}{\partial t} + \frac{\partial}{\partial x'}\frac{\partial x'}{\partial t}$$

$$= \frac{\partial}{\partial t'} - \frac{v\,\partial}{\partial x'}$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial t'}\frac{\partial t'}{\partial x} + \frac{\partial}{\partial x'}\frac{\partial x'}{\partial x} = \frac{\partial}{\partial x'}$$

$$\left(\frac{\partial}{\partial t'} - \frac{v\partial}{\partial x'}\right)^2 \varphi = c^2 \frac{\partial^2}{\partial x'^2} \varphi$$

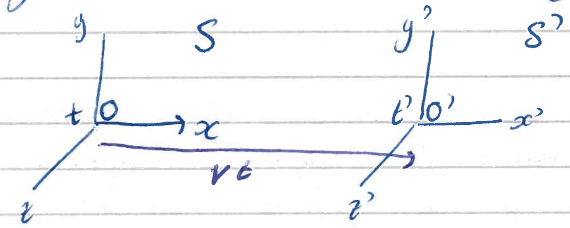
$$\varphi_{t't'} - 2v\varphi_{t'x'} + v^2 \varphi_{x'x'} = c^2 \varphi_{x'x'}$$

Try $\varphi = f(x - qt)$

$$q^2 + 2vq + v^2 = c^2 \qquad (q+v)^2 = c^2$$
$$q = -v \pm c$$

This is Newtonian mechanics - what we expect
Now we say speed of light in any frame is indep. of speed of that inertial frame

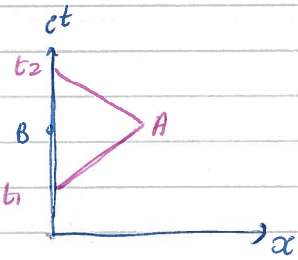
Inertial frames are coordinate systems in uniform relative motion to eachother. Such frames are in standard configuration if their origins coincide at $t=0$ (measured in both frames)



The frame S' is moving along the x axis of the frame S with speed v
O' seen in S is at $x = vt$
O' seen in S' is at $x' = 0$

This is standard relative motion

we call this our world line



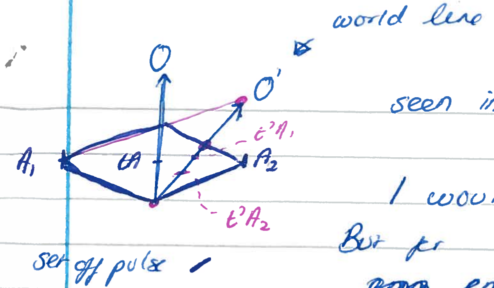
ct is time axis

light is sent from where I am it then hits a mirror & returns back to me

The event of the emitted light hitting the mirror must be simultaneous with the event B at $x=0$ & $t = \frac{t_2 - t_1}{2}$

We also know that the distance of the event A from $x=0$ is $x = c\left(\frac{t_2 - t_1}{2}\right)$

Here, observers can tell the time as well as send + receive light



/ travels at speed of light

↗ travels slower so steeper

seen in S

I would say $tA_1 = tA_2$

But for O', 'yellow travelling by' would see light ~~emm~~ emitted at same time but return at t'

We can't see light hitting mirror we just know it travels at speed c so we say it is half the time it takes to return

Also A_1 will return & hit yellow elsewhere & halfway point is $t'A_1$

$tA_1 = tA_2$ BUT $t'A_1 > t'A_2$

& using Newt. Mech. light ray in S' would not have same slope as S

Time is no longer separate from space & we need to have a four dimensional object describing the position & time in a particular frame (ct, x, y, z)

(ct', x', y', z')

some authors would write $ict = \sqrt{-1}\, ct$

Space in frames not Euclidean, i.e. bent.

Conserved quantity as you look between different frames $c^2t^2 - \underbrace{(x^2 + y^2 + z^2)}_{\text{this would be distance in Euclidean space}}$

At $t=0$, $S + S'$ sat on top of eachother, but then they move apart

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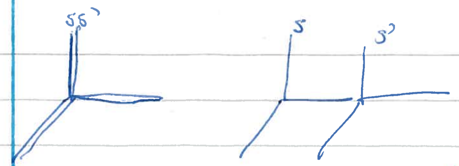


$\underline{x}$ $\qquad$ $\underline{x}'$

(ct, x, y, z) $\qquad$ (ct', x', y', z')

$t' = t \quad y' = y \quad z' = z \quad x' = x - vt$ $\qquad$ this is Galilean relativity

Later we may write x_α for $\underline{x}$, x'_α for $\underline{x}'$

Standard configuration - moving at const. speed rel. to another

$\underline{x}' = \underline{\underline{L}}\,\underline{x}$

We will work in x and ct, y and z do not change

$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}\begin{pmatrix} ct \\ x \end{pmatrix}$ $\quad$ 1) First physical event - origin of S'. Consider the origin of S' $\underline{x}' = \begin{pmatrix} ct' \\ 0 \end{pmatrix}$, $\underline{x} = \begin{pmatrix} ct \\ vt \end{pmatrix}$ ← this is where person in x sees origin of S'

Bottom row $\quad x' = 0 = b_1(ct) + b_2 x = b_1(ct) + b_2(vt) \implies \boxed{b_1 = -\frac{v}{c} b_2 = -\beta b_2}$ since $\beta c = v$

2) Look at ratio $\quad \frac{x'}{ct'} = \frac{b_1 ct + b_2 x}{a_1 ct + a_2 x} = \frac{b_2(x - \beta ct)}{a_1 ct + a_2 x} = \frac{b_2(x - vt)}{a_1 ct + a_2 x}$

3) Consider the origin of S - this has $x = 0$, $x' = -vt'$

$\frac{x'}{ct'} = \frac{-vt'}{ct'} = -\beta = \frac{b_2(-\beta ct)}{a_1 ct} \overset{x=0}{=} \frac{b_2(-vt)}{a_1 ct} = \frac{-\beta b_2}{a_1} \implies \boxed{a_1 = b_2}$

Light pulse released as frames pass at $t = t' = 0$ along x and x' axes

in S $x = ct$ $\quad S'$ $x' = ct'$ $\quad$ (doesn't matter if frame moving - speed of light is same)

$1 = \frac{x'}{ct'} \overset{\text{from 2)}}{=} \frac{b_2(ct - vt)}{a_1 ct + a_2 ct} \overset{\div ct}{=} \frac{b_2(1-\beta)}{a_1 + a_2} \overset{3)}{=} \frac{b_2(1-\beta)}{b_2 + a_2} \implies b_2 + a_2 = b_2 - \beta b_2 \implies \boxed{a_2 = -\beta b_2}$

Transformation $S \to S' = S' \to S$ but with one $v > 0$ β for other $v < 0$

$\underline{\underline{L}} = b_2 \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \qquad \underline{\underline{L}}^{-1} = b_2 \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix}$

$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = L^{-1}L = b_2^2 \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix}\begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} = b_2^2 \begin{pmatrix} 1-\beta^2 & 0 \\ 0 & 1-\beta^2 \end{pmatrix}$

$\implies b_2 = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1 - v^2/c^2}} = \gamma(v) = \gamma \qquad$ Lorentz γ factor > 1

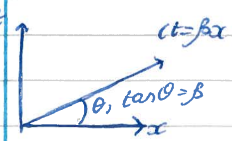
$$\boxed{\underline{\underline{L}}^{-1} = \gamma\begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} \qquad \underline{\underline{L}} = \gamma\begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \qquad \underline{x}' = \underline{\underline{L}}\,\underline{x} \qquad \underline{x} = \underline{\underline{L}}^{-1}\underline{x}'}$$

This leaves the wave equation invariant - unlike Galilean - will see this in HW

$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & \mp\gamma\beta & 0 & 0 \\ \mp\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$ $\qquad$ In pink shows reverse transformation

$\underline{x}$ $\qquad$ $\underline{x}'$

corresponds to $t' = 0$ at diff. values of x' →



What is the image in the S coordinates system of points on the x' axis of the S' coordinates i.e. those with $ct' = 0$ $\quad$ so $\underline{x} = \underline{\underline{L}}^{-1}\underline{x}'$

$\begin{pmatrix} ct \\ x \end{pmatrix} = \gamma\begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix}\begin{pmatrix} 0 \\ x' \end{pmatrix}$ $\quad$ x' is a parameter

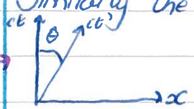
$\begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} \gamma\beta x' \\ \gamma x' \end{pmatrix}$ $\quad$ i.e. $\frac{ct}{x} = \frac{\gamma\beta x'}{\gamma x'} = \beta$

$ct = \gamma\beta x'$, $\quad x = \gamma x'$

so $ct = \beta x$, and $\tan\theta = \beta < 1$ because $\beta = v/c$

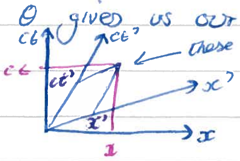
corresponds to $x' = 0$ at diff. values of t' →

Similarly the image of t' axis can be found $\begin{pmatrix} ct \\ x \end{pmatrix} = \gamma\begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix}\begin{pmatrix} ct' \\ 0 \end{pmatrix}$ giving $ct = x/\beta$



where θ gives us our new β

Putting both together:



← these lines are parallel to our x' and ct' lines

as $\frac{v}{c} \to 1$ everything gets a bit squashed

Be aware: 1m along x does not correspond to 1m along x'

If have two coordinate systems there is one quantity that is conserved - distance of point from origin $x^2 + y^2$. Here the conserved quantity is $(ct)^2 - x^2$

$$\boxed{(ct)^2 - x^2 = (ct')^2 - x'^2}$$

this is a space time interval - sign + magnitude are invariant

$\underline{x}' = \underline{\underline{L}}\,\underline{x}$

$\swarrow\ \underline{x}'^{T} = \underline{x}^{T}\underline{\underline{L}}^{T}$

Before: $x'^2 + y'^2 = x^2 + y^2$ i.e. $(x\ y)\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$

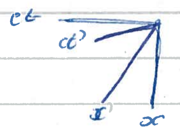
Here: $(ct\ x)\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}ct\\x\end{pmatrix}$ $\quad (ct'\ x')\underbrace{\begin{pmatrix}1&0\\0&-1\end{pmatrix}}_{G}\begin{pmatrix}ct'\\x'\end{pmatrix} = \underline{x}'^{T}\underline{\underline{G}}\,\underline{x}' = \underline{x}^{T}\underbrace{\underline{\underline{L}}^{T}\underline{\underline{G}}\,\underline{\underline{L}}}\,\underline{x}$

this would prove equality → can we show $= G$?

$$\underline{\underline{L}}^{T}\underline{\underline{G}}\,\underline{\underline{L}} = \gamma\begin{pmatrix}1&-\beta\\-\beta&1\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\gamma\begin{pmatrix}1&-\beta\\-\beta&1\end{pmatrix} = \gamma^2\begin{pmatrix}1&\beta\\-\beta&-1\end{pmatrix}\begin{pmatrix}1&-\beta\\-\beta&1\end{pmatrix}$$

$$= \gamma^2\begin{pmatrix}1-\beta^2&0\\0&\beta^2-1\end{pmatrix} = \begin{pmatrix}1&0\\0&-1\end{pmatrix} = \underline{\underline{G}}$$

$\frac{1}{1-\beta^2}$ ↗ this is right here but wrong in notes



$(ct)^2 - x^2 = (ct')^2 - (x')^2$ from previous graph

s is distance in ct frame → $s^2 = (ct)^2 - x^2$ this is space time interval

$= (ct')^2 - x'^2$ however it can be +ve or −ve & it is conserved

If we have two events can look at difference in times & difference in x

$(\Delta s)^2 = (\Delta ct)^2 - (\Delta x)^2$ recall that some authors will write ict

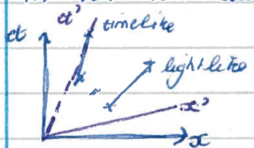
$(ds)^2 = (dct)^2 - (dx)^2$ Pythagoras would suggest $(dct)^2 + (dx)^2$ but this is not Euclidean space. $(dct)^2 = c^2(dt)^2$ this is curved space

$\underline{x} = \begin{pmatrix}x\\y\\z\end{pmatrix}$ It is confusing to have $\underline{x}$ as spatial coordinate & the coordinates we are using here so we might start writing X w/ spatial + temporal parts

$c^2(\Delta t)^2 = (\Delta x)^2$ → $(\Delta s)^2 = 0$

If ~~$s^2 = 0$~~ the interval between the two events is said to be light-like; we have two events, one of which can be caused by the other but only through a signal travelling at light speed

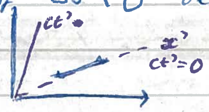
If $\Delta s^2 > 0$ i.e. $c^2\Delta t^2 > \Delta x^2$, the slope joining the events is greater than 45°; the interval is said to be time like



one can be caused by another through a signal travelling with speed $v < c$

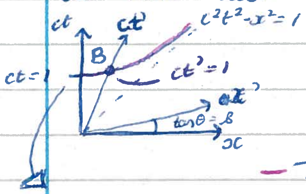
It is possible to find a frame in which the two events occur at the same position but different times shown in purple

If $\Delta s^2 < 0$ i.e. $\Delta x^2 > c^2\Delta t^2$ then the interval is space-like



It is impossible to find a frame in which the two events occur at the same position But there does exist a frame where the events occur simultaneously but at different positions i.e. it is possible to look at one point, 100 years look at another & if in a frame travelling relative to Bowles you will see this happening at the same time

We have lost ideas of simultaneity



Consider an event with $x = 0$, $ct = 1$ in S

The interval of this event from the origin is $s^2 = c^2t^2 - x^2 = 1$

— This is a locus of possible positions in which we can see this event

This is a locus of all possible image points of the event $X = (1, 0)$ in different Lorentz transformations $X' = \underline{\underline{L}}\,X$ (or should this be x)

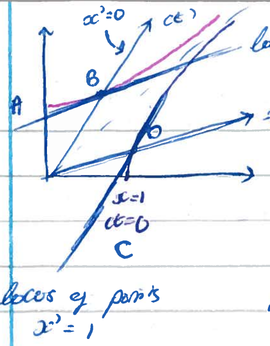
$(ct)'^2 - x'^2 = 1$ & if $x' = 0$ then $(ct')^2 = 1$ $\quad ct' = 1$

Given $(c^2t^2) - x^2 = 1$, $\quad 2ct\,d(ct) - 2x\,dx = 0$

so $\dfrac{d(ct)}{dx} = \dfrac{x}{ct}$

Event B has $x' = 0$ & so has $x = vt$

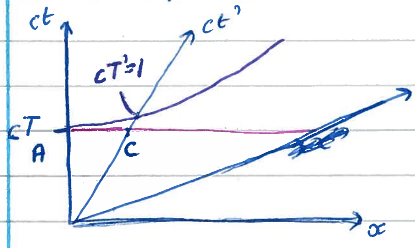
& so slope of hyperbola at B is $\dfrac{vt}{ct} = \beta$ & so is // to x' axis

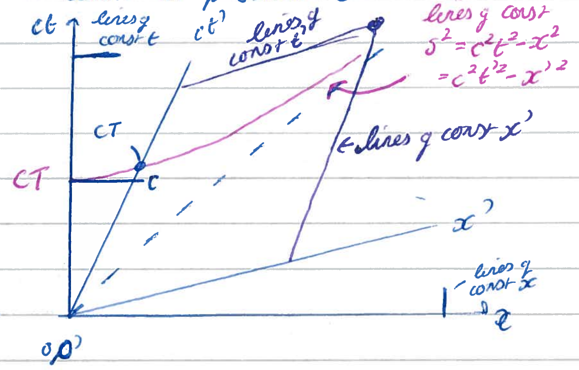


If start with $x=1$ $ct=0$ at C
Tangent // to ct' in same way

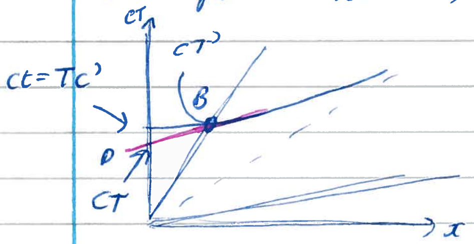
Up until now, online notes are fine, now move on to Bowles handwritten notes which cover p150 onwards.

Time dilation At $t=t'=0$ frames are aligned
Two frames S and S' are in standard configuration. After a time T in S the observer O in S looks at his clock (event A) & reads a time T. Simultaneously as far as he is concerned, i.e. in frame S he looks at the clock in S' which is positioned at $x = vt = vT$: call this event C





C - event of him looking at clock in S'
event C has coordinates $\underline{x} = \begin{pmatrix} cT \\ vT \end{pmatrix}$, $\underline{x}'_c = \begin{pmatrix} cT' \\ 0 \end{pmatrix}$ ← sitting at origin in S' so $T' < T$
since have labelled cT which occurs after event
$c^2t^2 - x^2 = c^2t'^2 - x'^2$ on event C $c^2T^2 - v^2T^2 = c^2T'^2 - 0^2$ so $T'^2 = (1 - v^2/c^2)T^2$
$T' = T/\gamma < T$ so $T' < T$
or we can say $\underline{x}' = \underline{L}\,\underline{x}$, $\begin{pmatrix} cT' \\ 0 \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} cT \\ vT \end{pmatrix}$
$cT' = \gamma cT - \gamma\beta vT$, $\beta = v/c$, $T' = \gamma(1 - \frac{\beta^2}{c^2})T = T\gamma/\gamma^2 = T/\gamma$

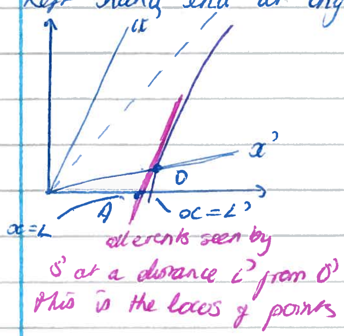


B - fellow in S' looking at his clock

D is event of observer in S' looking at clock in S at the same time, T' as he looks at clock in S'
$s^2 = c^2T'^2$ which we set equal to c^2t^2
and we get $ct = T'c$

Event D has coordinates $\underline{x}_D' = \begin{pmatrix} cT' \\ -vT' \end{pmatrix}$ $\underline{x}_D = \begin{pmatrix} cT \\ 0 \end{pmatrix}$
$\begin{pmatrix} cT' \\ -vT' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} cT \\ 0 \end{pmatrix}$
Top row: $cT' = \gamma cT - 0$ so $T = T'/\gamma < T'$

Lorentz or length contraction A rod of length L' measured in a frame S' in which it is at rest is viewed from a frame S in which S' is moving with speed v
Left hand end at origin - right hand end is what we are looking at



Event D is the observer in S' measuring the right end of the rod $\underline{x}_D' = \begin{pmatrix} 0 \\ L' \end{pmatrix}$
Event A is the observer in S looking at $t=0$, at the right end of a rod of length L' as measured in S'
Let A have coordinates $\underline{x}_A = \begin{pmatrix} 0 \\ L \end{pmatrix}$ (as seen by me) $\underline{x}_A' = \begin{pmatrix} ct'_A \\ L' \end{pmatrix}$
event of me looking at rod not same as event of you looking at rod; $\underline{x}_A'$ is as seen by S'

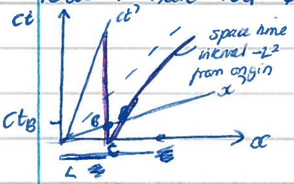
$= c^2t'^2 - x'^2$

$L < L'$; $s^2 = c^2t^2 - x^2 = -L'^2$ (done at $t' = 0$) hyperbola drawn in purple

$\Longrightarrow x = L'$ (drawn on diagram)

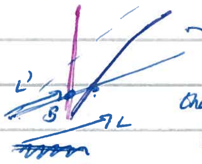
$\begin{pmatrix} ct'_A \\ L' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} 0 \\ L \end{pmatrix}$ bottom row: $L' = \gamma L$, $L = L'/\gamma < L'$ $\boxed{L < L'}$

Now I have rod + bowles moving past in S'; C: S measures right end of rod



B: event of observer in S' looking at $t'=0$, of the right end of a rod measured to have length L in S

$x'_B = \begin{pmatrix} 0 \\ L' \end{pmatrix}$



Thus $L' < L$

$x_B = \begin{pmatrix} ct_B \\ L \end{pmatrix}$

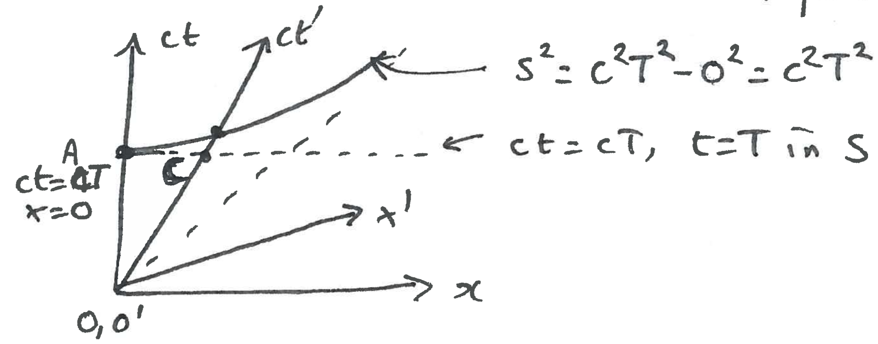
$\begin{pmatrix} 0 \\ L' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} ct_B \\ L \end{pmatrix}$ Top row: $0 = \gamma(ct_B - \beta L)$; Bottom: $L' = \gamma(-\beta ct_B + L)$

$\Longrightarrow ct_B = \beta L \Longrightarrow L' = \gamma(1-\beta^2)L$

$L' = \gamma/\gamma^2 L = L/\gamma \Longrightarrow L' = L/\gamma < L$ $\boxed{L' < L}$

TIME DILATION.

Two frames S & S' are in standard configuration. After a time T_1 in S, the observer O_1 in S, looks at his clock$_1$ (Event A). It reads time T. Simultaneously, as far as he is concerned, he looks at the clock in S', positioned at $x = vT$. This is event C. We have



$$\underline{x}_c = \begin{pmatrix} cT \\ vT \end{pmatrix}, \quad \underline{x}'_c = \begin{pmatrix} cT' \\ 0 \end{pmatrix}$$

where T' is the time elapsed in S' since the two frames S&S' were coincident.

However $T' < T$ since the point on the ct' axis where it intersects with the hyperbola $s^2 = c^2T^2$ corresponds to $t' = T$ as $s^2 = c^2t'^2 - x'^2 = c^2t'^2 = c^2T^2$ ie $t' = T$. As C is closer to O' along the ct' axis $\underline{T' < T}$.

To find T' we can

a) Use $c^2t^2 - x^2 = c^2t'^2 - x'^2$ on event C giving

$$c^2T^2 - v^2T^2 = c^2T'^2 - 0^2$$

$$\Rightarrow T' = (1 - v^2/c^2)^{1/2}T = 1/\gamma\, T \qquad \boxed{T' = 1/\gamma\, T < T}$$

or

b) Use the Lorentz transformation on event C, $\underline{x}'_c = \underline{\underline{L}}\,\underline{x}_c$

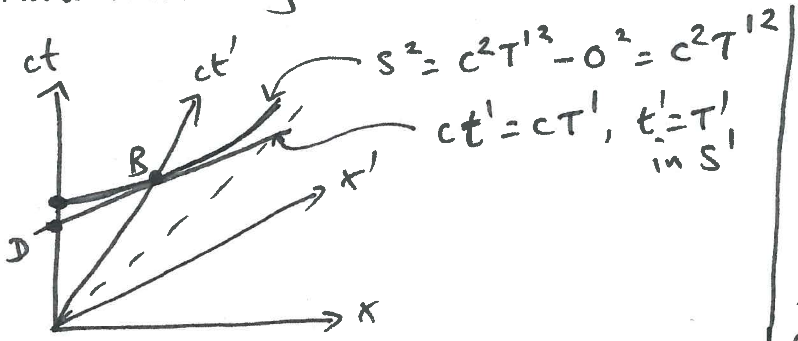
$$\& \begin{pmatrix} cT' \\ 0 \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} cT \\ vT \end{pmatrix} = \gamma \begin{pmatrix} cT - \beta vT \\ -\beta cT + vT \end{pmatrix}$$

$$= \gamma cT \begin{pmatrix} 1 - \beta^2/c^2 \\ 0 \end{pmatrix} = \begin{pmatrix} cT/\gamma \\ 0 \end{pmatrix}$$

Time in S' is seen to run slower, or be dilated, by an observer in S

$$\Rightarrow \boxed{T' = T/\gamma}$$

The same effect is seen by an observer in S' looking at his clock, event B, & the clock in S at the same time ie simultaneously as measured in S', event D



We have $\underline{x}'_D = \begin{pmatrix} cT' \\ -vT' \end{pmatrix}$, $\underline{x}_D = \begin{pmatrix} cT \\ 0 \end{pmatrix}$

where T is the time elapsed in S since the two frames S&S' were coincident

However $T < T'$ since the point on the ct axis where it intersects with the hyperbola $s^2 = c^2T'^2$ corresponds to $t = T'$ as $s^2 = c^2t^2 - x^2 = c^2t^2 - 0^2 = c^2T'^2$ ie $t = T'$. As D is closer to O along the ct axis than is this point $T < T'$

To find T we can

a) Use $c^2t^2 - x^2 = c^2t'^2 - x'^2$ on D giving

$$c^2T^2 - 0^2 = c^2T'^2 - (-vT')^2$$

$$\Rightarrow cT = cT'(1 - v^2/c^2)^{1/2} = cT'/\gamma \qquad \boxed{T = 1/\gamma\, T' < T'}$$

or b) Use the Lorentz transformation on event D, $\underline{x}'_D = \underline{\underline{L}}\, \underline{x}_D$

& $\begin{pmatrix} cT' \\ -vT' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} cT \\ 0 \end{pmatrix} = \gamma \begin{pmatrix} cT \\ -\beta cT \end{pmatrix}$ &

So $T' = \gamma T$ & $T = 1/\gamma\, T' < T'$

& then $-vT' = -\gamma\beta cT = -\beta cT' = -vT'$ ✓

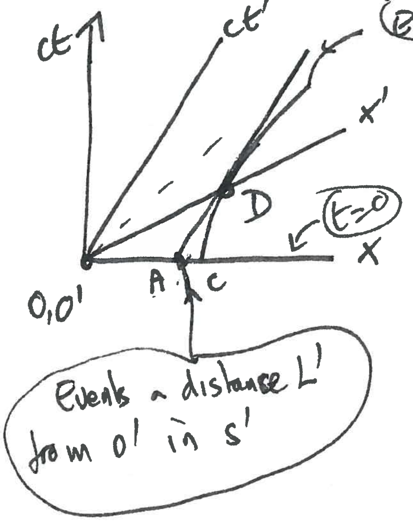
Time in S is seen to run slower, or be dilated, by an observer in S'

LORENTZ or LENGTH CONTRACTION

If a rod has a given length L' measured in a frame where it is at rest, then it has a length $L = L'/\gamma < L'$ measured in a frame in which it is seen to be moving.

Let the measurements take place when the origins of the two frames S & S' ie O & O' are coincident & let the left hand end of the rod coincide with the origin O & O' at the point of measurement in the frames S & S'

Events a space time interval L' from the origin O'



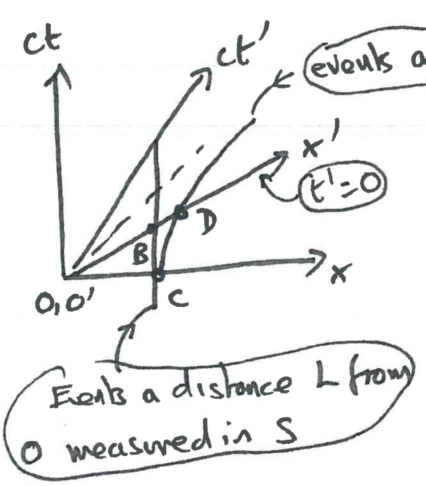
D is the event of the observer in S' measuring the right end of the rod. He makes this observation at the same time, $t'=0$ as he observes the left end at the origin. D has coordinates $\underline{x}'_D = \begin{pmatrix} 0 \\ L' \end{pmatrix}$

A is the event of an observer in S measuring the right end of a rod, a distance L' from O' measured in S', at the same time as he measures the left end of the rod at O, ie $t=0$.

Since OA is less than OC which is a distance L' from the origin O, the observer in S measures a length of the rod $L < L'$

We have $\underline{x}'_A = \begin{pmatrix} ct'_A \\ L' \end{pmatrix}$ & $\underline{x}_A = \begin{pmatrix} 0 \\ L \end{pmatrix}$ with t'_A the time that the observer in S' sees the observer in S measuring the right end of the rod

$\underline{x}' = \underline{\underline{L}}\,\underline{x} \Rightarrow \begin{pmatrix} ct'_A \\ L' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} 0 \\ L \end{pmatrix} \Rightarrow ct'_A = -\gamma\beta L$ and $L' = \gamma L \Rightarrow \boxed{L = L'/\gamma < L'}$



Ⓚ

A similar observation is made by an observer in S' measuring a rod of length L in S. He sees a length $L' < L$.

C is the event of the observer in S measuring the right end of the rod. He makes this observation at the same time, $t=0$, as he observes the left end of the rod at the origin. $\underline{x}_C = \begin{pmatrix} 0 \\ L \end{pmatrix}$

B is the event of an observer in S' measuring the right end of the rod, a distance L from O measured in S, at the same time as he measures the left end of the rod at O', ie at $t'=0$

Since O'B is less than O'D which is a distance L from the origin O', the observer in S' measures a length of the rod $L' < L$.

We have $\underline{x}_B = \begin{pmatrix} ct_B \\ L \end{pmatrix}$ & $\underline{x}'_B = \begin{pmatrix} 0 \\ L' \end{pmatrix}$ with t_B the time that the observer in S sees the observer in S measuring the right end of the rod $\underline{x}' = \underline{\underline{L}}\,\underline{x} \Rightarrow \begin{pmatrix} 0 \\ L' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} ct_B \\ L \end{pmatrix}$

$$\Rightarrow L' = \gamma(L - \beta c t_B) \quad \& \quad 0 = \gamma(ct_B - \beta L) \Rightarrow ct_B = \beta L$$

$$\text{so} \quad L' = \gamma L(1-\beta^2) = \gamma L/\gamma^2 = L/\gamma < L \qquad \boxed{L' = \frac{L}{\gamma} < L}$$

OR $\underline{x} = \underline{\underline{L}}^{-1}\underline{x}' \Rightarrow \begin{pmatrix} ct_B \\ L \end{pmatrix} = \gamma \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} \begin{pmatrix} 0 \\ L' \end{pmatrix} \Rightarrow L = \gamma L'$

$$\boxed{L' = L/\gamma < L}$$

We can define a proper time τ for a particle, or along a world line, 24/03/15
as being the time measured in a frame in which the particle is at rest
We have $c^2(dt)^2 - dx^2 = c^2(dt')^2 - (dx')^2$ & if S' is in a frame where the particle is at rest, $(dx')^2 = 0$ - doesn't change position. Then $dt' = d\tau$.
Then $c^2 dt^2 - dx^2 = c^2 d\tau^2$, $\left(1 - \frac{v^2}{c^2}\right) dt^2 = d\tau^2$
where $v = dx/dt$ the velocity of frame S' in which particle is at rest, is moving relative to S

$$d\tau = (1 - v^2/c^2)^{1/2} dt = \frac{dt}{\gamma} < dt \; ; \quad \tau = \int \frac{dt}{\gamma} \; ; \quad \frac{d\tau}{dt} = \frac{1}{\gamma}$$

Physical quantities must transform according to a Lorentz transformation. We have seen that the 4-vector $(ct, x, y, z) = X_\mu = (\underset{\text{temporal}}{ct}, \underset{\text{space}}{\underline{x}})$; $X' = LX$ with $X' = (ct', \underline{x}')$
goes through & each element
The space time distance $s^2 = c^2t^2 - x^2$ is scalar which is invariant under Lorentz transformation. Proper time τ is also a scalar (rest mass m_0, charge q)
~~The~~

4-dot products: the dot product $A_\mu B_\mu = A_0 B_0 - (A_1 B_1 + A_2 B_2 + A_3 B_3)$
this is just defin $\rightarrow$ or $A_t B_t - (A_x B_x + A_y B_y + A_z B_z)$
$A_\mu B_\mu = A^T G B$ $\quad A'_\mu B'_\mu = A_\mu B_\mu$ $\quad G = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

The 4-gradient vector; one might think that $\begin{pmatrix} \frac{1}{c}\partial/\partial t \\ \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} = \nabla^*_\mu$ $\leftarrow$ * means wrong
is a good candidate for the 4-gradient operator. However is $\nabla^{*'} = L\nabla^*$ where $X' = LX$
Have to use chain rule to get $\partial/\partial x'$ etc. e.g.
$\left(\frac{\partial}{\partial x} = \frac{\partial}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial}{\partial y'}\frac{\partial y'}{\partial x} + \frac{\partial}{\partial z'}\frac{\partial z'}{\partial x} + \frac{\partial}{\partial t'}\frac{\partial t'}{\partial x} \right.$ can write as $\nabla^*\varphi = L\nabla^{*'}\varphi$

It turns out that $\nabla^* = L\nabla^{*'}$; but we know $LGL = G$ $\Rightarrow GL = L^{-1}G$
$G\nabla^* = GL\nabla^{*'} = L^{-1}G\nabla^{*'}$ so $L(G\nabla^*) = (G\nabla^*)'$
So $G\nabla^*$ is a 4 vector & so

$$\nabla_\mu = \begin{pmatrix} \frac{1}{c}\partial/\partial t \\ -\partial/\partial x \\ -\partial/\partial y \\ -\partial/\partial z \end{pmatrix} = \left(+\frac{1}{c}\frac{\partial}{\partial t}, -\underline{\nabla} \right) \text{ is a 4-vector}$$

- The 4-divergence; if $B_\mu = (cB_t, B_x, B_y, B_z) = (cB_t, \underline{B})$
not derivative

$$\nabla_\mu B_\mu = \frac{1}{c}\frac{\partial}{\partial t}(cB_t) - (-\underline{\nabla})\cdot\underline{B}$$
dy'g a product

We get $\frac{\partial B_t}{\partial t} + \underline{\nabla}\cdot\underline{B} = 0$

- The velocity 4-vector: we have $X_\mu = (ct, \underline{x})$
We define the velocity-4 vector to be $\frac{d}{d\tau}X_\mu = U_\mu$ $\leftarrow$ proper time
It is a 4 vector as $X' = LX$ &
as L is independent of τ $U' = LU$ (differentiating wrt τ)
We have $U_\mu = \frac{d}{d\tau}(ct, \underline{x}) = \gamma\frac{d}{dt}(ct, \underline{x}) = \gamma(c, \underline{v})$ $\leftarrow$ v is 3-velocity

& $\gamma(c, \underline{v})' = L\gamma(c, \underline{v})$

$$\begin{pmatrix} \gamma c \\ v_1' \\ v_2' \\ v_3' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma c \\ v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

will only talk about x, t, not y, z

Another example of velocity transformation If a particle is moving with speed $-v$ relative to
(5) O, then $x = -vt + a$, with t as a parameter in a frame moving with velocity u relative to O (S')

$\gamma(u) = 1/\sqrt{1-u^2/c^2}$

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \gamma(u) \begin{pmatrix} 1 & -u/c \\ -u/c & 1 \end{pmatrix} \begin{pmatrix} ct \\ -vt+a \end{pmatrix}$$

This is just using Lorentz transformation

Mult. out: $t' = \gamma[(1 + uv/c^2)t - au/c^2$
$x' = \gamma(-(u+v)t + a)$ ⟵ position + time in S'

An observer in S' sees a velocity $dx'/dt' = \frac{dx'/dt}{dt'/dt} = \frac{u+v}{1+uv/c^2}$

Nonlinear mess - ensues

$U_\alpha = \gamma(c, \underline{v})$

Note for example $U_\alpha U_\alpha = \gamma^2 c^2 - \gamma^2 v^2 = \gamma^2 c^2 \left(1 - \frac{v^2}{c^2}\right) = \gamma^2 c^2 / \gamma^2 = c^2$

m_0 ⟵ scalar

Mass Let the rest mass of a particle be the mass measured in a frame in which it is stationary. We define the 4-vector $P_\alpha = m_0 U_\alpha$ to be the 4-momentum of a particle

$$P_\alpha = \begin{pmatrix} m_0\gamma c \\ m_0\gamma v_x \\ m_0\gamma v_y \\ m_0\gamma v_z \end{pmatrix} = (m_0\gamma c, m_0\gamma \underline{v})$$

It turns out that $m_0\gamma = \frac{m_0}{\sqrt{1-v^2/c^2}}$ is the mass of a particle, rest mass m_0, moving with velocity v

If p is the relativistic 3-momentum $= m\underline{v} = m_0\gamma\underline{v}$ then $P_\alpha = (m_0\gamma c, p)$

What is $m_0\gamma c = \frac{m_0 c}{\sqrt{1-v^2/c^2}} = m_0 c\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \dots\right)$

We recognise $\frac{1}{2}m_0 v^2$ as the kinetic energy of the particle & $\gamma m_0 c = \frac{1}{c}\underbrace{\left(m_0 c^2 + \frac{1}{2}m_0 v^2 \dots\right)}_{E}$

We identify $m_0\gamma c = E/c$ with E the energy of the particle. The energy of a particle at rest is $m_0 c^2$ - $P_\alpha = \left(\frac{E}{c}, p\right)$

dot product

$P_\alpha P_\alpha = \frac{E^2}{c^2} - p^2$ scalar: should be same whichever frame!

$P'_\alpha P'_\alpha = \frac{m_0^2 c^4}{c^2}$ if S' is a frame in which particle is at rest (mom'm = 0)

& as $P_\alpha P_\alpha$ is invariant we get

$$\frac{E^2}{c^2} - p^2 = m_0^2 c^2, \quad E^2 - c^2 p^2 = m_0^2 c^4$$

(If $p = 0$, $E = mc^2$; photons have no mass, $E^2 = c^2 p^2$)

Doing this with exam question in mind

$$\underline{E} = \frac{e}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2} \quad (1) \qquad \underline{B} = \mu_0 \underline{v} \wedge \frac{e}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2} = \mu_0\varepsilon_0 \underline{v} \wedge \underline{E} = \frac{\underline{v} \wedge \underline{E}}{c^2} \quad (2)$$

Did this before, found it didn't agree w/ Galilean relativity - (2) is right still but now (1) is wrong (still right for particle at rest)

$F = e(\underline{E} + \underline{v} \wedge \underline{B})$ ⟵ this also contradicted Galilean relativity

4-current We define a 4-current to be $J_\mu = (c\rho, \underline{J})$

If know charge density + current in one frame, can find in another since it ~~obeys~~ obeys Lorentz transition. To show this,

We note that charge conservation is $\frac{\partial}{\partial t}\rho + \nabla \cdot \underline{J} = 0$

i.e. $\frac{\partial}{\partial ct}(c\rho) + \nabla \cdot \underline{J} = \nabla_\mu J_\mu = 0$ ⟵ 4 vector?

$\frac{\partial}{\partial t}$ or $\frac{\partial}{\partial ct}$?

So $J_\mu' = L J_\mu$; so if S' is a frame in which a charge distribution is at rest $J_\mu' = (c\rho, \underline{0})$ ← up here

here drop dimensions, using 2D transformations since lazy

The 4 current in a frame S, in which S' is moving with speed V is given by

$$\begin{pmatrix} c\rho \\ \underline{J} \end{pmatrix} = \gamma \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} \begin{pmatrix} c\rho' \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma c\rho' \\ \gamma \beta c \rho' \end{pmatrix}$$

(J no longer vector since gone from 3 spatial dim's → 1)

So $\rho = \gamma\rho'$, $\quad J = v\rho$

charge conserved but charge density is not! due to contraction

4-potential We define an electromagnetic 4-potential to be $A_\mu = (\varphi/c, \underline{A})$ where φ is electric potential, A is magnetic vector potential

$$\nabla_\mu A_\mu = \frac{1}{c}\frac{\partial}{\partial t}\frac{\varphi}{c} + \nabla\cdot\underline{A} = \frac{1}{c^2}\frac{\partial\varphi}{\partial t} + \underline{\nabla}\cdot\underline{A} = 0$$

if we use the Lorentz Gauge.

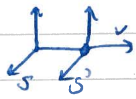
(As move from one frame to another, time can become space + v.v - so $\underline{E}$ can become B and v.v.)

$$\nabla_\mu \nabla_\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - (-\nabla)(-\nabla) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$$ the wave operator

$$= \Box^2$$ square squared, say

$\Box^2 A_\mu = \mu_0 J_\mu$ ← check this!

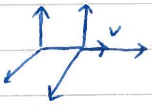
The electric field of a moving point charge



A point charge ~~the~~ e is moving along the x axis of S with velocity V

In a frame S' moving with the charge we have a 4-potential

$A_\mu' = (\varphi'/c, 0)$ ← no magnetic field in S' so $\underline{A}' = 0$ & $\varphi' = \dfrac{e}{4\pi\varepsilon_0}\dfrac{1}{r'}$



moving along x axis with speed v 25/03/15

note: $r' = (x'^2 + y'^2 + z'^2)^{1/2}$

$$\left.\begin{array}{l} \varphi' = \dfrac{e}{4\pi\varepsilon_0}\dfrac{1}{r'} \\ A' = 0 \end{array}\right\} \quad A_\mu' = (\varphi'/c, A') = (\varphi'/c, 0)$$

$A = L^{-1}A'$ inverse Lorentz trans

$$A = L^{-1}A' = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \varphi'/c \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \varphi/c \\ A \\ 0 \\ 0 \end{pmatrix}$$

$\beta = \frac{v}{c}$, $\varphi = \gamma\varphi'$

First row: $\varphi = \gamma\varphi'$

Subsequent rows: $\underline{A} = \begin{pmatrix} \gamma\beta\varphi'/c \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \varphi v/c^2 \\ 0 \\ 0 \end{pmatrix}$

$= \underline{v}\,\varphi/c^2$ since moving w/ speed v in x direction

$\underline{E} = -\nabla\varphi - \underline{A}_t$ ← for unsteady field, $\quad B = \underline{\nabla}\wedge\underline{A}$

We need to find X_μ in terms of X_μ'. We will say

$\beta = \frac{v}{c}$

$$X' = \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = LX = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

so $y' = y$, $z' = z$, $x' = -\beta\gamma ct + \gamma x = \gamma(x - vt)$

$$E = -\nabla\underbrace{\left\{\frac{1}{4\pi\varepsilon_0}\frac{\gamma}{\sqrt{\gamma^2(x-vt)^2 + y^2 + z^2}}\right\}}_{\varphi} - \frac{\partial}{\partial t}\begin{pmatrix} \frac{1}{4\pi\varepsilon_0}\frac{\gamma}{\sqrt{(x-vt)^2 + y^2 + z^2}}\frac{v}{c^2} \\ 0 \\ 0 \end{pmatrix}$$

There is a neater way to do this - recall $\underline{E} = -\nabla\varphi - \frac{\partial A}{\partial t}$ $\quad A = v\,\varphi/c^2$

$$\underline{E} = \begin{pmatrix} -\varphi_x + \frac{v^2}{c^2}\varphi_x \\ -\varphi_y \\ -\varphi_z \end{pmatrix}$$ ← since $\dfrac{\partial\varphi}{\partial t} = -v\dfrac{\partial\varphi}{\partial x}$

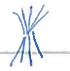
E is not 4 vector, it is 3 vector

$$= \begin{pmatrix} -\varphi_x/\gamma^2 \\ -\varphi_y \\ -\varphi_z \end{pmatrix} = \underbrace{\frac{e\gamma}{4\pi\varepsilon_0}}_{\text{change in charge density}} \underbrace{\frac{((x-vt), y, z)^T}{(\gamma^2(x-vt)^2 + y^2 + z^2)^{3/2}}}_{\text{Lorentz contraction}} \neq \frac{e(x-vt, y, z)^T}{4\pi\varepsilon_0((x-vt)^2 + y^2 + z^2)^{3/2}}$$

① ②

← we would have said this was our $\underline{E}$ at start of course!

① This is what they look like ②

 ← much more focused in vertical 

Our two results agree if $\gamma \approx 1$

$\underline{B} = \nabla \wedge \underline{A} = \nabla \wedge (\underline{v}\varphi/c^2) = \frac{\nabla\varphi}{c^2} \wedge \underline{v}$ ← from formula at start of course

$E = -\nabla\varphi - A_t$ → $= \frac{(-\underline{E} - \underline{A}_t)}{c^2} \wedge \underline{v} = \underline{v} \wedge \frac{(\underline{E} + \underline{A}_t)}{c^2}$ but $\underline{A} \parallel \underline{v}$ so $\underline{A}_t \parallel \underline{v}$ & $\underline{v} \wedge \underline{A}_t = 0$

so $\underline{B} = \underline{v} \wedge \underline{E}/c^2$

so $c\underline{B} = \frac{\underline{v}}{c} \wedge \underline{E}$ This is the same as what we had before

Exam: lots of bookwork

broadly same structure as model exam

sufficient similarity between non-assessed HW questions + exam i.e. do them!