MATH0056 Mathematical Methods 4 Notes

Based on the 2019 spring lectures by Dr I Smears

Methods 4 8/01/19

Chapter 1 Separation of variables

We will be interested in solving many partial differential equations (PDE)

Several examples:

① Laplace's equation $\boxed{\Delta u = 0}$

In 2D Cartesian co-ordinates

$\Delta u = u_{xx} + u_{yy}$ where the solution $u = u(x,y)$

Notation partial derivatives can be written in several ways $u_x = \frac{\partial u}{\partial x}$

Also $u_{xx} = \frac{\partial^2 u}{\partial x^2}$ $u_{xy} = \frac{\partial^2 u}{\partial y \partial x}$

In 3D Cartesian co-ordinates

$\Delta u = u_{xx} + u_{yy} + u_{zz}$ where $u = u(x,y,z)$

② Heat equation $\boxed{u_t = \Delta u}$

In 2D, $u = u(x,y,t)$, $u_t = \frac{\partial u}{\partial t}$

In 3D, $u = u(x,y,z,t)$

The heat equation models the flow of heat in a solid body.

③ The wave equation $\boxed{u_{tt} = \Delta u}$

This models the vibrations/waves in elastic bodies. It also models acoustic/electromagnetic waves

④ Helmholtz equation $\boxed{\Delta u + k^2 u = 0}$

where $k \in \mathbb{R}$ is a parameter

and $u = u(x,y)$ in 2D or $u = u(x,y,z)$ in 3D

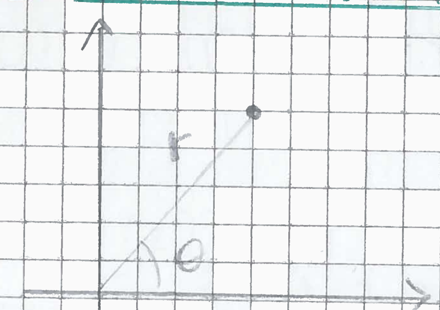
The Helmholtz equation is related to the wave equation, since if u solves the Helmholtz equation, then $e^{ikt} u$ will solve the wave eqn.

- Co-ordinate system -

We will often consider these P.D.E in a domain: a subset of $\mathbb{R}^n$ ($n \in \{1,2,3\}$) where the P.D.E is satisfied, and where we seek its solution. Typical examples will be discs, cylinders, squares, cubes etc...

Therefore it will be often more convenient to express P.D.E into diff. co-ordinate systems

① Polar co-ordinates

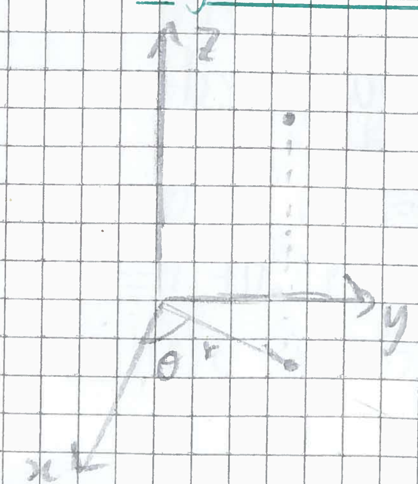


$x = r\cos\theta$

$y = r\sin\theta$

$$\boxed{\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}}$$

② Cylindrical co-ordinates



$x = r\cos\theta$

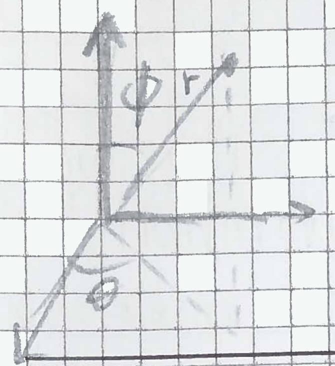
$y = r\sin\theta$

$z = z$

$$\boxed{\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} + u_{zz}}$$

③ Spherical coordinates

In this course, we will use the following conversion



$$x = r\cos\theta\sin\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\phi$$

$$\boxed{\Delta u = u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2\sin^2\phi}u_{\theta\theta} + \frac{1}{r^2\tan\phi}u_\phi + \frac{1}{r^2}u_{\phi\phi}}$$

Example 1 Separation of variables

The heat equation in 1D

Consider the equation $u_t = u_{xx}$ in the following domain $\{(x,t) \; x\in[0,L], t>0\}$ where $L>0$ is a constant.

The idea of separation of variables is to find the general solution of the problem in terms of the infinite series of simpler solutions called separated solutions.

Separated functions are functions that can be written as the product of functions of the different variables

e.g. $u(x,y) = F(x)G(y)$

for instance; $u(x,y) = xy$ is a separated function, but $v(x,y) = x+y$ is not.

Let us seek all separated solutions of

$$u_t = u_{xx}$$

Try: $u(x,t) = F(x)G(t)$ with F and G to be determined. Insert this into the equation, we get:

$$F(x)G'(t) = F''(x)G(t)$$

Assuming we can divide by $F(x)$ and $G(t)$ we can re-arrange this to find

$$\frac{F''(x)}{F(x)} = \frac{G'(t)}{G(t)}$$

The L.H.S. is independent of t and the R.H.S. is independent of x $\Rightarrow$ both sides are equal to a constant: there must be a constant λ s.t.

$$\frac{F''(x)}{F(x)} = \frac{G'(t)}{G(t)} = \lambda$$

Depending on the sign of λ, we get different solutions

ⓐ Case where $\lambda > 0$. Then we obtain 2 O.D.E.

$$F''(x) = \lambda F(x) \qquad G'(t) = \lambda G(t)$$
$$\Rightarrow F(\ ) = A\cosh(\sqrt{\lambda}x) + B\sinh(\sqrt{\lambda}x)$$
$$G(t) = Ce^{\lambda t}$$

where A, B, C are generic constants.

ⓑ Case where $\lambda = 0$

$$\Rightarrow F''(x)\ 0 \qquad G'(t) = 0$$
$$\Rightarrow F(x) = Ax + B \quad , \quad G(t) = C$$

where A, B, C are generic constants.

(c) Cas where $\lambda < 0$

$F''(x) - \lambda F(x) = 0$, $G'(t) = \lambda G(t)$

$F(x) = A\cos(\sqrt{-\lambda}\, x) + B\sin(\sqrt{-\lambda}\, x)$

$G(t) = Ce^{\lambda t}$

Generally, P.D.E are supplemented by additional conditions, such as, conditions on the solution that hold on the boundary of the domain. For example, let us consider the additional boundary conditions:

$u(0,t) = 0 \quad \forall t > 0$

$u(L,t) = 0 \quad \forall t > 0$

We seek separated solutions that satisfy these boundary conditions.

We are primarily interested in non trivial (i.e. not 0) solutions. We check in each case if there are non trivial solutions.

(a) Case $\lambda > 0$; Recall $u(x,t) = F(x)G(t)$

we get $F(0)G(t) = 0 \quad \forall t > 0$

$F(L)G(t) = 0 \quad \forall t > 0$

Assuming $G(t)$ is non-trivial $\Rightarrow G(t) \neq 0 \ \forall t > 0$

$\Rightarrow F(0) = 0$, $F(L) = 0$

$F(0) = 0 \Rightarrow A\cosh(0) = 0 \iff A = 0$

$F(L) = 0 \Rightarrow B\underbrace{\sinh(\sqrt{\lambda}L)}_{>0} = 0 \iff B = 0$

This shows that for $\lambda > 0$, the only solution (separated) is that solves the P.D.E + the boundary conditions is the trivial solution.

(b) $\lambda = 0$ $F(0) = 0$ $F(L) = 0$

$F(0) = 0 \Rightarrow B = 0$ $F(L) = AL = 0 \Rightarrow A = 0$

Again there is only the trivial solution in this case

(c) $\lambda < 0$ $F(x) = A\cos(\sqrt{-\lambda}\, x) + B\sin(\sqrt{-\lambda}\, x)$

As before, the boundary conditions imply

$F(0) = 0$, $F(L) = 0$

$F(0) = 0 \iff A\cos(0) = 0 \iff A = 0$

$F(L) = 0 \Rightarrow B\sin(\sqrt{-\lambda}\, L) = 0$

This is satisfied iff $B = 0$ (trivial case) or $\sqrt{-\lambda}\, L = k\pi$ (k is a positive integer) (as LHS > 0)

$\iff \boxed{\lambda = \frac{k^2\pi^2}{L^2}}$

Therefore, the only non trivial solution (separated) are (up to a constant multiple) of the form:

$\boxed{\sin\left(\frac{k\pi}{L}x\right) e^{-\frac{k^2\pi^2}{L^2}t}}$ (i.e. $F(x)G(t)$)

case where $B = 1$

The heat equation $u_t = u_{xx}$ has the property of being linear, this means that linear combinations of solutions are also solutions

Under some technical assumptions on the convergence of series, any function

$$u(x,t) = \sum_{k=1}^{\infty} a_k \sin\left(\frac{k\pi x}{L}\right) e^{-\frac{k^2\pi^2}{L^2}t}$$

will also be a solution of the P.D.E. and the B.C.

Consider now the heat equation
$u_t = u_{xx}$, $u(0,t) = 0 = u(L,t)$
with the additional condition
$u(x,0) = f(x)$ where $f(x)$ is a given function
Try to find coefficients $\{a_k\}_{k=1}^{\infty}$ to satisfy $u(x,0) = f(x)$ - (initial condition)
evaluating the series at $t=0$

$$u(x,0) = \sum_{k=1}^{\infty} a_k \sin\left(\frac{k\pi}{L}x\right) = f(x)$$

using result on Fourier series, it is known that

$$a_k = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{k\pi}{L}x\right) dx$$

Example 2 Laplace's equation in a disc

In example 1, we were given the explicit B.C.
Sometimes geometry gives implicit conditions.
Find all non-trivial separated solutions of $\Delta u = 0$ in the unit disc in $\mathbb{R}^2$.
$u(r,\theta) = F(r)G(\theta)$
The implicit condition will be that $G(\theta)$ must be 2π periodic.

10/01/19

Recall: Example II Laplace's Eqn in the unit disc

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

We seek all non-trivial separated solns of the form $u(r,\theta) = F(r)G(\theta)$
$\Rightarrow G(\theta)$ must be 2π-periodic

Inserting $u(r,\theta) = F(r)G(\theta)$ into Laplace's eqn

$$F''(r)G(\theta) + \frac{1}{r}F'(r)G(\theta) + \frac{1}{r^2}F(r)G''(\theta) = 0$$

$$F''(r)G(\theta) + \frac{1}{r}F'(r)G(\theta) = -\frac{1}{r^2}F(r)G''(\theta)$$

$$\left(F'' + \frac{1}{r}F'\right)G(\theta) = -\frac{1}{r^2}F\,G''(\theta)$$

$$\frac{F'' + \frac{1}{r}F'}{\frac{1}{r^2}F} = -\frac{G''}{G} = \lambda \text{ constant}$$

The LHS depends only on r, RHS only on θ
Therefore both LHS and RHS are equal to a constant $\lambda \in \mathbb{R}$.

Case a: If $\lambda > 0$

$G'' + \lambda G = 0 \qquad G(\theta) = A\cos(\sqrt{\lambda}\,\theta) + B\sin(\sqrt{\lambda}\,\theta)$
G is 2π periodic iff $\sqrt{\lambda} = n$ a positive integer
Equivalently, $\lambda = n^2$

Turning to equation for $F(r)$:

$$F'' + \frac{1}{r}F' = \frac{n^2}{r^2}F \quad \text{(Euler Equation)}$$

Try a solution $F(r) = r^c$, c a constant

$$c(c-1)r^{c-2} + \frac{1}{r}cr^{c-1} = \frac{n^2}{r^2}r^c = n^2 r^{c-2}$$

$c^2 = n^2 \iff c = n$ or $c = -n$

The solution $F(r)$ is generally of the form

$$F(r) = Ar^n + Br^{-n}$$

Case b: $\lambda = 0$

Equation for G. $G'' = 0 \implies G(\theta) = A\theta + B$

G is 2π-periodic iff $A = 0 \iff G(\theta) = B$

$$F'' + \frac{1}{r}F' = 0 \iff \frac{1}{r}\frac{d}{dr}(rF') = 0$$

$\implies rF' = C$ constant $F(r) = C\log r + D$

where C, D are generic constants

Case c: $\lambda < 0$

$$G'' + \lambda G = 0 \implies G(\theta) = A\cosh(\sqrt{-\lambda}\theta) + B\sinh(\sqrt{-\lambda}\theta)$$

2π periodicity of $G(\theta)$ is only satisfied if $A = B = 0$

Therefore there is only the trivial soln in this case.

Exercise: Prove that $G(\theta) = 0$ under 2π periodicity. [Hint

$G(0) = G(2\pi)$ use]

$G'(0) = G'(2\pi)$

Summary: The non-trivial separated solutions are $u(r,\theta) = C\log r + D$

$$u(r,\theta) = (A\cos n\theta + B\sin n\theta)(C'r^n + D'r^{-n})$$

for $n \geq 1$ a positive integer.

The functions $\log r$, r^{-n} have singularities at $r = 0$. We are usually looking for non-singular solutions so we shall set the associated constants to zero before forming the general series solution. Therefore the general series solution is

$$u(r,\theta) = D + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)$$

Bessel's equation

Consider Helmholtz's equation in polar coord.

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + k^2 u = 0$$

where $k \in \mathbb{R}$, $k > 0$

We seek nontrivial separated solutions

$u(r,\theta) = F(r)G(\theta)$

$$F''G + \frac{1}{r}F'G + \frac{1}{r^2}FG'' + k^2FG = 0$$

$$\left(F'' + \frac{1}{r}F' + k^2F\right)G = -\frac{1}{r^2}FG''$$

$$\frac{F'' + \frac{1}{r}F' + k^2F}{\frac{1}{r^2}F} = -\frac{G''}{G} = \lambda^2$$

where $\lambda \geq 0$ constants

The equation for F becomes

$$F'' + \frac{1}{r}F' + k^2F = \frac{\lambda^2}{r^2}F$$

$$F'' + \frac{1}{r}F' + \left(k^2 - \frac{\lambda^2}{r^2}\right)F = 0$$

To remove the dependency on k, we shall make the change of variables $z = kr$

$\implies f(z) = F(r)$

Inserting $F(r) = f(z) = f(kr)$ into the eqn gives

$$k^2 f''(z) + \frac{k^2 f'}{z} + \left(k^2 - \frac{\lambda^2 k^2}{z^2}\right) f = 0$$

since $k \neq 0$, we can simplify to find

$$\boxed{f''(z) + \frac{1}{z} f'(z) + \left(1 - \frac{\lambda^2}{z^2}\right) f(z) = 0}$$ Bessel's equation

The parameter λ is called the index of the eqn.

Legendre's Equation

Consider Laplace's equation is spherical coordinates

$$\Delta u = 0,$$

$$u_{rr} + \frac{2}{r} u_r + \frac{1}{r^2 \sin^2\phi} u_{\theta\theta} + \frac{1}{r \tan\phi} u_\phi + \frac{1}{r^2} u_{\phi\phi} = 0$$

Let us consider separated solutions $u(r,\phi) = F(r) H(\phi)$

$$F'' H + \frac{2}{r} F' H + \frac{1}{r^2 \tan\phi} F H' + \frac{1}{r^2} F H'' = 0$$

$$\left(F'' + \frac{2}{r} F'\right) H + \frac{1}{r^2} F \left(\frac{1}{\tan\phi} H' + H''\right) = 0$$

$$\frac{F'' + \frac{2}{r} F'}{\frac{1}{r^2} F} = - \frac{H'' + \frac{1}{\tan\phi} H'}{H}$$

The LHS depends only on r; RHS on ϕ

Legendre's Equation

15/01/19

$$\frac{F'' + \frac{2}{r} F'}{\frac{1}{r^2} F} = \frac{-\left(H'' + \frac{1}{\tan\phi} H'\right)}{H} = \text{constant}$$

Equation for $F(r)$:

$$F'' + \frac{2}{r} F' = \frac{\text{constant}}{r^2} F$$

Similarly to before we try as a solution $F(r) = r^\nu$, with $\nu \in \mathbb{R}$ a parameter.

Inserting this into the equation:

$$\nu(\nu-1) r^{\nu-2} + \frac{2\nu}{r} r^{\nu-1} = \frac{\text{const}}{r^2} r^\nu$$

$$\iff \left[\nu(\nu+1) - \text{const}\right] r^{\nu-2} = 0$$

$$\implies \text{const} = \nu(\nu+1)$$

The equation for H becomes

$$H'' + \frac{1}{\tan\phi} H' + \nu(\nu+1) H = 0 \qquad (1)$$

To simplify the equation, we introduce the change of variable $u = \cos\phi$

we set $h(u) = H(\phi) = h(\cos\phi)$

$$\frac{dH}{d\phi} = \frac{du}{d\phi}\frac{dh}{du} = -\sin\phi \frac{dh}{du} = -\sqrt{1-u^2}\,\frac{dh}{du} \qquad (2)$$

as $0 \leqslant \phi < \pi$

$$\frac{d^2H}{d\phi^2} = \frac{d}{d\phi}\left(-\sqrt{1-u^2}\,\frac{dh}{du}\right) = \frac{du}{d\phi}\frac{d}{du}\left(-\sqrt{1-u^2}\,\frac{dh}{du}\right)$$

$$\frac{d^2H}{d\phi^2} = -\sqrt{1-u^2}\,\frac{d}{du}\left(-\sqrt{1-u^2}\,\frac{dh}{du}\right)$$

$$= (1-u^2)\frac{d^2h}{du^2} - \frac{2u}{2\sqrt{1-u^2}}\sqrt{1-u^2}\,\frac{dh}{du}$$

$$= (1-u^2)\frac{d^2h}{du^2} - u\frac{dh}{du} \qquad (3)$$

therefore, by putting together ①, ②, ③ we obtain

$$(1-u^2)\frac{d^2h}{du} - u\frac{dh}{du} + \frac{u}{\sqrt{1-u^2}}\left(-\sqrt{1-u^2}\frac{dh}{du}\right) + \nu(\nu+1)h = 0$$

$$\Leftrightarrow \boxed{(1-u^2)\frac{d^2h}{du^2} - 2u\frac{dh}{du} + (\nu)(\nu+1)h = 0}$$

↳ Legendre's Equation

Chapter 2
The Frobenius method of series solutions to ODEs

In applying the method of separation of variables, we often encounter ODEs that can be put in the general form

$$w''(z) + p(z)w'(z) + q(z)w(z) = 0 \quad ⊛$$

Where $p(z)$ and $q(z)$ are some functions of z.

If p and q are just constants, this can be solved using the characteristic polynomial. However, for $p(z)$ and $q(z)$ being functions, we need a more general method of solution.

Remarks: since ⊛ is a second order ODE, we expect to find two linearly independent solutions, say $w_1(z)$ and $w_2(z)$, so the general solution is
$w(z) = Aw_1(z) + Bw_2(z)$ (A, B constants)

The main idea is to try a series solution of the form:

$$\boxed{w(z) = \sum_{k=0}^{\infty} a_k z^{k+c}}$$

with $a_0 \neq 0$, $c \in \mathbb{R}$
(by renumbering can always take $a_0 \neq 0$)

We set $a_0 \neq 0$, w.l.o.g., since otherwise we can simply relabel the coefficients and change the value of c.

Example: $zw'' + \frac{1}{2}w' + \frac{1}{4}w = 0$

This can be put in the form of ⊛ with $p(z) = 1/2z$, $q(z) = 1/4z$

Let's insert $w(z) = \sum_{k=0}^{\infty} a_k z^{k+c}$ into the equation

$$z\sum_{k=0}^{\infty} a_k(k+c)(k+c-1)z^{k+c-2}$$
$$+\frac{1}{2}\sum_{k=0}^{\infty} a_k(k+c)z^{k+c-1} + \frac{1}{4}\sum_{k=0}^{\infty} a_k z^{k+c} = 0$$

The first few terms are:

$$a_0(c)(c-1)z^{c-1} + \frac{1}{2}a_0 c z^{c-1} + a_1(c+1)(c)z^c + \frac{1}{2}a_1(c+1)z^c + \frac{1}{4}a_0 z^c + \ldots = 0$$

$$z^{c-1}\left[a_0\left(c(c-1) + \frac{1}{2}c\right)\right] + z^c\left[a_1\left(c(c+1) + \frac{c(c+1)}{2}\right) + \frac{a_0}{4}\right] + \ldots = 0$$

Setting these coefficients to zero, we get

$$c(c-1) + \frac{1}{2}c = 0 \Leftrightarrow \boxed{c\left(c-\frac{1}{2}\right) = 0}$$

Indicial equation (I.E)

The equation for z^c gives

$$\boxed{a_1 = -\frac{a_0}{4\left(c+\frac{1}{2}\right)(c+1)}}$$

Recurrence relation for a_1

Returning to full equation:

$$\sum_{k=0}^{\infty} a_k \left((k+c)(k+c-1)+\frac{1}{2}(k+c)\right) z^{k+c-1}$$

$$+\sum_{k=0}^{\infty} \frac{1}{4} a_k z^{k+c} = 0$$

$$\Longrightarrow a_0 c\left(c-\frac{1}{2}\right) z^{c-1} + \sum_{k=1}^{\infty} a_k (k+c)\left(k+c-\frac{1}{2}\right) z^{k+c-1}$$

$$+\sum_{k=0}^{\infty} \frac{1}{4} a_k z^{k+c} = 0$$

Setting the new index $k' = k-1$ we get

$$a_0 (c)\left(c-\frac{1}{2}\right) z^{c-1} + \sum_{k'=0}^{\infty} a_{k'+1}\left((k'+1+c)\left(k'+c+\frac{1}{2}\right)\right) z^{k'+c}$$

$$+\sum_{k=0}^{\infty} \frac{1}{4} a_k z^{k+c} = 0$$

Therefore:

$$a_0 (c)\left(c-\frac{1}{2}\right) z^{c-1} + \sum_{k=0}^{\infty} \left[a_{k+1}(k+c+1)\left(k+c+\frac{1}{2}\right)+\frac{1}{4}a_k\right] z^{k+c} = 0$$

- Indicial equation $(c)\left(c-\frac{1}{2}\right) = 0$
- Recurrence relation

$$a_{k+1}(k+c+1)(k+c+1/2) = -\frac{a_k}{4} \quad \forall k \geq 0$$

The indicial equation shows that either $c=0$ or $c=\frac{1}{2}$

Case 1: $c=0$

RR becomes $a_{k+1} = -\dfrac{a_k}{4(k+1)(k+1/2)} \quad \forall k \geq 0$

for $a_0 \neq 0$

$$a_1 = -\frac{a_0}{4\cdot 1\cdot 1/2} = -\frac{a_0}{2}$$

$$a_2 = \frac{a_1}{4\cdot 3\cdot 3/2} = -\frac{a}{4\cdot 3} = \frac{a_0}{4\cdot \ 2\cdot 1}$$

It is easy to check by induction that

$$a_k = \frac{(-1)^k a_0}{(2k)!} \quad \forall k \geq 1$$

solves the Recurrence Relation.

The solution for $c=0$ is then

$$w(z) = \sum_{k=}^{\infty} a_k z^{k+c} = \sum_{k=0}^{\infty} \frac{(-1)^k a_0}{(2k)!} z^k$$

$$= a_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} (\sqrt{z})^{2k}$$

$$= a_0 \cos(\sqrt{z})$$

Where a_0 is a generic constant (and can be taken arbitrarily as the solution is linear).

Case II: $c = 1/2$

Then the Recurrence Relation becomes

$$a_{k+1} = \frac{-a_k}{4(k+3/2)(k+1)} = -\frac{a_k}{(2k+3)(2k+2)}$$

The solution is

$$a_k = \frac{(-1)^k a}{(2k+1)!}$$

So, the solution for $c=\frac{1}{2}$ is

$$w(z) = \sum_{k=0}^{\infty} \frac{(-1)^k a_0}{(2k+1)!} z^{k+\frac{1}{2}} = a_0 \sin(\sqrt{z})$$

The general solution of $zw'' + \frac{1}{2}w' + \frac{1}{4}w = 0$

is $w(z) = A\cos(\sqrt{z}) + B\sin(\sqrt{z})$

where A, B are generic constants. ▨

- Fuch's Theorem

Under which conditions does method work on $p(z)$ and $q(z)$:

$$w''(z) + p(z)w'(z) + q(z)w(z) = 0 \quad (*)$$

It turns out that for reasons of convergence of power series it is helpful to consider z possibly being complex. need some definitions

Defn. A point $z_0 \in \mathbb{C}$ is called an ordinary point if both $p(z)$ and $q(z)$ are complex analytic around z_0.

The point z_0 is called a regular singular point if it is not an ordinary point and if $(z-z_0)p(z)$ and $(z-z_0)^2 q(z)$ are both analytic around z_0.

In our previous example, $p(z) = \frac{1}{2z}$, $q(z) = \frac{1}{4z}$, $p(z)$ and $q(z)$ have poles of order 1 at $z_0 = 0$, so $z_0 = 0$ is a regular singular point.

Fuch's Theorem: A solution of $(*)$ can be expressed as a generalized power series of the form:

$$w(z) = \overset{A(z)}{A}\log(z-z_0) + \sum_{k=0}^{\infty} a_k (z-z_0)^{k+c}$$

provided that z_0 is either an ordinary point or a regular point of $(*)$

Partial Proof of Fuch's Theorem

(We won't consider issues of convergence of the Power series)

Let us consider, w.l.o.g., $z_0 = 0$, and suppose that it is at least a regular singular point. Then $zp(z)$, $z^2q(z)$ are analytical at 0 implies that:

$$zp(z) = \sum_{k=0}^{\infty} p_k z^k, \quad z^2 q(z) = \sum_{k=0}^{\infty} q_k z^k$$

Inserting the series for $w(z)$, $zp(z)$ and $z^2q(z)$ we get:

(step 1 Insert $w(z) = \sum_k a_k z^{k+c}$)

$$\sum_{k=0} a_k (k+c)(k+c-1) z^{k+c-2}$$

$$+ p(z)\sum_{k=0}^{\infty} a_k (k+c) z^{k+c-1} + q(z)\sum_{k=0}^{\infty} a_k z^{k+c} = 0$$

Borrowing power of z:

$$\sum_{k=0}^{\infty} a_k (k+c)(k+c-1) z^{k+c-2} + zp(z)\sum_{k=0}^{\infty} a_k (k+c) z^{k+c-2}$$

$$+ z^2 q(z) \sum_{k=0}^{\infty} a_k z^{k+c-2} = 0$$

Step 2: Insert series of $zp(z)$, $z^2q(z)$

$$\sum_{k=0}^{\infty} a_k (k+c)(k+c-1) z^{k+c} + \left(\sum_{j=0}^{\infty} p_j z^j\right)\left(\sum_{k=0}^{\infty} a_k (k+c) z^{k+c-2}\right)$$

$$+ \left(\sum_{j=0}^{\infty} q_j z^j\right)\left(\sum_{k=0}^{\infty} a_k z^{k+c-2}\right) = 0$$

Using the expansion formula

$$\left(\sum_{j=0}^{\infty} f_j z^j\right)\left(\sum_{k=0}^{\infty} g_k z^k\right) = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} g_j f_{k-j}\right) z^k, \text{ we get}$$

$$\left(\sum_{j=0}^{\infty} p_j z^j\right)\left(\sum_{k=0}^{\infty} a_k (k+c) z^{k+c-2}\right) = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} a_j (j+c) p_{k-j}\right) z^{k+c-2}$$

$$\left(\sum_{j=0}^{\infty} q_j z^j\right)\left(\sum_{k=0}^{\infty} a_k z^{k+c-2}\right) = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} a_j q_{k-j}\right) z^{k+c-2}$$

Therefore

$$\sum_{k=0}^{\infty}\left[a_k (k+c)(k+c-1) + \sum_{j=0}^{\infty} a_j \left((j+c)p_{k-j} + q_{k-j}\right)\right] z^{k+c-2} = 0$$

To obtain a solution, we then require that

$$a_k (k+c)(k+c-1) + \sum_{j=0}^{k} a_j \left((j+c)p_{k-j} + q_{k-j}\right) = 0 \quad \forall k \geq 0$$

for $k=0$

$$a_0 \left(c(c-1) + cp_0 + q_0\right) = 0 \iff a_0\left(c^2 + c(p_0 - 1) + q_0\right) = 0$$

For $a_0 \neq 0 \implies F(c) = c^2 + (p_0 - 1)c + q_0 = 0$ (I.E.)

Correction to last lecture: 17/01/19

Generalised power series $w(z) = A(z)\ln(z-z_0) + \sum_{k=0}^{\infty} a_k (z-z_0)^{k+c}$

$$\sum_{k=0}\left[a_k (k+c)(k+c-1) + \sum_{j=0}^{k} a_j \left((j+c)p_{k-j} + q_{k-j}\right)\right] z^{k+c-2} = 0$$

Setting the coefficients to zero gives

For $k=0 \implies$ I.E. $c^2 + (p_0 - 1)c + q_0 = 0$

For $k \geq 1 \implies a_k (k+c)(k+c-1) + \sum_{j=0}^{k} a_j \left((j+c)p_{k-j} + q_{k-j}\right) = 0$

Extracting terms in a_k:

$$a_k\left[(k+c)(k+c-1) + p_0(k+c) + q_0\right] + \sum_{j=0}^{k-1} a_j \left((j+c)p_{k-j} + q_{k-j}\right)$$

Introduce the quadratic function

$$F(\lambda) = \lambda^2 + (p_0 - 1)\lambda + q_0$$

Then we can rewrite

$$a_k F(k+c) = -\sum_{j=0}^{k-1} a_j \left((j+c)p_{k-j} + q_{k-j}\right)$$

If $F(k+c) \neq 0$

$$a_k = -\frac{1}{F(k+c)} \sum_{j=0}^{k-1} a_j \left((j+c)p_{k-j} + q_{k-j}\right) \quad \text{(RR)}$$

There are 3 main cases to consider

Case I) (I.E.) $F(c) = 0$ has two distinct roots c_1, c_2 (possibly complex)
$c_1 - c_2$ is not an integer.
$\implies$ It is possible to solve the RR for $c = c_1$ and $c = c_2$ since $F(k+c_1)$ and $F(k+c_2)$ are nonzero for $k \geq 1$

Case II) (I.E.) $F(c) = 0$ has a double root
$c = c_1$ i.e. $F(c) = (c - c_1)^2$

Case III) $F(c) = 0$ has two distinct roots c_1, c_2
with $c_1 - c_2 = n$, $n \in \mathbb{Z}$ an integer ($n \neq 0$)
$\implies F(k + c_1) = 0$ for $k = c_2 - c_1$ (assuming $c_2 \geq c_1$)
$\implies$ RR breaks down

How to find a second particular solution in case II

Let us introduce $w(z,c) = \sum_{k=0}^{\infty} a_k(c) z^{k+c}$

where $a_k(c)$ solves the RR $\forall k \geq 1$ and $a_0 \neq 0$ constant.

Inserting $w(z,c)$ into the equation,

$$\left(\frac{d^2}{dz^2}+p(z)\frac{d}{dz}+q(z)\right)w(z,c)=a_0(c-c_1)^2 z^{c-2}$$

Differentiate this equation w.r.t. c, and then set $c=c_1$

$$\left(\frac{d^2}{dz^2}+p(z)\frac{d}{dz}+q(z)\right)\frac{\partial w}{\partial c}\Big|_{c=c_1}=\Big[2a_0(c-c_1)z^{c-2}$$
$$+a_0(c-c_1)^2\ln z\, z^{c-2}\Big]\Big|_{c=c_1}$$

Therefore

$$\left(\frac{d^2}{dz^2}+p(z)\frac{d}{dz}+q(z)\right)\frac{\partial w}{\partial c}\Big|_{c=c_1}=0$$

So $\frac{\partial w}{\partial c}\Big|_{c=c_1}$ is also a solution of the equation.

$$\frac{\partial w(z,c)}{\partial c}\Big|_{c=c_1}=\sum_{k=0}^{\infty}\frac{da_k}{dc}\Big|_{c=c_1}z^{k+c_1}+a_k(c_1)z^{k+c_1}\ln z$$

$$=\sum_{k=0}^{\infty}\frac{da_k}{dc}\Big|_{c=c_1}z^{k+c}+\ln z\, w(z,c_1)$$

<u>Example of case II Bessel's Eqn with index 0</u>

$$w''+\frac{1}{z}w'+w=0$$

let $w(z,c)=\sum_{k=0}^{\infty}a_k(c)z^{k+c}$

and insert this into the eqn to get

$$\sum_{k=0}^{\infty}a_k(k+c)(k+c-1)z^{k+c-2}+\sum_{k=0}^{\infty}a_k(k+c)z^{k+c-2}$$
$$+\sum_{k=0}^{\infty}a_k z^{k+c}=0$$

$$\sum_{k=0}^{\infty}a_k\left[(k+c)(k+c-1)+k+c\right]z^{k+c-2}+\sum_{k=0}^{\infty}a_k z^{k+c}=0$$

$$\Longleftrightarrow \sum_{k=0}^{\infty}a_k(k+c)^2 z^{k+c-2}+\sum_{k=0}^{\infty}a_k z^{k+c}=0$$

Extracting the first two terms on the first series gives $a_0c^2z^{c-2}+a_1(c+1)^2z^{c-1}$

$$+\sum_{k=2}^{\infty}a_k(k+c)^2z^{k+c-2}+\sum_{k=0}^{\infty}a_kz^{k+c}=0$$

The I.E. is $c^2=0 \Rightarrow c=0$ is a double root (case II)

The coefficients of z^{c-1} is

$$a_1(c+1)^2$$

So $a_1(c+1)^2=0 \Longleftrightarrow a_1=0$ (for c close to 0)

To find a RR set

$$\sum_{k=2}^{\infty}a_k(k+c)^2z^{k+c-2}+\sum_{k=0}^{\infty}a_kz^{k+c}=0$$

$$\Longleftrightarrow \sum_{k=0}^{\infty}\left[a_{k+2}(k+c+2)^2+a_k\right]z^{k+c}=0$$

$$\Rightarrow a_{k+2}(k+c+2)^2=-a_k \quad \forall k\geq 0$$

$$\Longleftrightarrow a_k=-\frac{a_{k-2}}{(k+c)^2} \quad \text{for } k\geq 2 \quad \text{RR}$$

We solve the RR for general value of c

The RR implies that $a_k=0$ for all k odd

For $k=2m$ even number

$$a_k=a_{2m}=\frac{-a_{2(m-1)}}{(2m+c)^2}=\frac{+a_{2(m-2)}}{(2m+c)^2(2(m-1)+c)^2}$$

$$\Rightarrow a_{2m}=\frac{(-1)^m a_0}{\prod_{j=0}^{m}(2j+c)^2}$$

$$\left(\frac{d^2}{dz^2} + p(z)\frac{d}{dz} + q(z)\right) w(z,c) = a_0 (c-c_1)^2 z^{c-2}$$

Differentiate this equation w.r.t. c, and then set $c=c_1$

$$\left(\frac{d^2}{dz^2} + p(z)\frac{d}{dz} + q(z)\right)\frac{\partial w}{\partial c}\Big|_{c=c_1} = \Big[2a_0(c-c_1)z^{c-2} + a_0(c-c_1)^2 \ln z \, z^{c-2}\Big]\Big|_{c=c_1}$$

Therefore

$$\left(\frac{d^2}{dz^2} + p(z)\frac{d}{dz} + q(z)\right)\frac{\partial w}{\partial c}\Big|_{c=c_1} = 0$$

So $\frac{\partial w}{\partial c}\Big|_{c=c_1}$ is also a solution of the equation.

$$\frac{\partial w(z,c)}{\partial c}\Big|_{c=c_1} = \sum_{k=0}^{\infty} \frac{da_k}{dc}\Big|_{c=c_1} z^{k+c_1} + a_k(c_1) z^{k+c_1} \ln z$$

$$= \sum_{k=0}^{\infty} \frac{da_k}{dc}\Big|_{c=c_1} z^{k+c} + \ln z \, w(z,c_1)$$

Example of case II Bessel's Eqn with index 0

$$w'' + \frac{1}{z}w' + w = 0$$

let $w(z,c) = \sum_{k=0}^{\infty} a_k(c) z^{k+c}$

and insert this into the eqn to get

$$\sum_{k=0}^{\infty} a_k(k+c)(k+c-1) z^{k+c-2} + \sum_{k=0}^{\infty} a_k(k+c) z^{k+c-2} + \sum_{k=0}^{\infty} a_k z^{k+c} = 0$$

$$\sum_{k=0}^{\infty} a_k\big[(k+c)(k+c-1) + k + c\big] z^{k+c-2} + \sum_{k=0}^{\infty} a_k z^{k+c} = 0$$

$$\iff \sum_{k=0}^{\infty} a_k(k+c)^2 z^{k+c-2} + \sum_{k=0}^{\infty} a_k z^{k+c} = 0$$

Extracting the first two terms on the first series gives $a_0 c^2 z^{c-2} + a_1(c+1)^2 z^{c-1}$

$$+ \sum_{k=2}^{\infty} a_k(k+c)^2 z^{k+c-2} + \sum_{k=0}^{\infty} a_k z^{k+c} = 0$$

The I.E. is $c^2 = 0 \Rightarrow c = 0$ is a double root (case II)

The coefficients of z^{c-1} is

$a_1(c+1)^2$

so $a_1(c+1)^2 = 0 \iff a_1 = 0$ (for c close to 0)

To find a RR set

$$\sum_{k=2}^{\infty} a_k(k+c)^2 z^{k+c-2} + \sum_{k=0}^{\infty} a_k z^{k+c} = 0$$

$$\iff \sum_{k=0}^{\infty} \big[a_{k+2}(k+c+2)^2 + a_k\big] z^{k+c} = 0$$

$$\Rightarrow a_{k+2}(k+c+2)^2 = -a_k \quad \forall k \geq 0$$

$$\iff a_k = -\frac{a_{k-2}}{(k+c)^2} \quad \text{for } k \geq 2 \quad \text{RR}$$

We solve the RR for general value of c

The RR implies that $a_k = 0$ for all k odd

For $k = 2m$ even number

$$a_k = a_{2m} = \frac{-a_{2(m-1)}}{(2m+c)^2} = \frac{+a_{2(m-2)}}{(2m+c)^2(2(m-1)+c)^2}$$

$$\Rightarrow a_{2m} = \frac{(-1)^m a_0}{\prod_{j=0}^{m} (2j+c)^2}$$

To find the first particular solution we set $c=c_1$

$$w_1(z) = w(z,0) = \sum_{m=0}^{\infty} \frac{(-1)^m a_0}{2^{2m}(m!)^2} z^{2m} = a_0 \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2}\left(\frac{z}{2}\right)^{2m}$$

The second particular solution

$$w_2(z) = \frac{\partial w}{\partial c}\Big|_{c=0} = w_1(z)\ln z + \sum_{m=0}^{\infty} \frac{\partial}{\partial c}\Big|_{c=0}\left(\frac{(-1)^m a_0}{\prod_{j=0}^{m}(2j+c)^2}\right) z^{2m}$$

Exercise: show that

$$\frac{\partial}{\partial c}\Big|_{c=0} \frac{1}{\prod_{j=0}^{m}(2j+c)^2} = -\frac{1}{2^{2m}(m!)^2} S_m, \quad S_m = \sum_{j=0}^{m} \frac{1}{j}$$

Therefore

$$w_2(z) = w_1(z)\ln z - a_0 \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} S_m \left(\frac{z}{2}\right)^{2m}$$

For the choice $a_0 = 1$, then we call

$$w_1(z) = J_0(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2}\left(\frac{z}{2}\right)^{2m}$$

the Bessel function of the first kind with index 0

$$w_2(z) = J(z)\ln(z) - \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2}\left(\frac{z}{2}\right)^{2m} S_m = Y_0(z)$$

is the Bessel function of the second kind with index 0

22/01/19

- Bessel's equation of index $\nu > 0$

$$w'' + \frac{1}{z}w' + \left(1 - \frac{\nu^2}{z^2}\right) w = 0$$

$p(z) = \frac{1}{z}$, $q(z) = 1 - \frac{\nu^2}{z^2}$

$z=0$ is a regular singular point

We will use Frobenius' method about $z=0$

Insert solution $\sum_{k=0}^{\infty} a_k z^{k+c}$ into the equation

$$\sum_{k=0}^{\infty} a_k \left((k+c)(k+c-1) + (k+c)\right) z^{k+c-2}$$

$$+ \sum_{k=0}^{\infty} -a_k \nu^2 z^{k+c-2} + \sum_{k=0}^{\infty} a_k z^{k+c} = 0$$

$$\sum_{k=0}^{\infty} a_k \left((k+c)^2 - \nu^2\right) z^{k+c-2} + \sum_{k=0}^{\infty} a_k z^{k+c} = 0$$

Separating the first two terms:

$$a_0(c^2 - \nu^2) z^{c-2} + a_1\left((c+1)^2 - \nu^2\right) z^{c-1}$$

$$+ \sum_{k=0}^{\infty} \left[a_{k+2}\left((k+c+2)^2 - \nu^2\right) \right] z^{k+c} = 0$$

Therefore the indicial equation is $c^2 = \nu^2$

hence $c = \nu$ or $c = -\nu$

This is case I, if 2ν is not an integer, otherwise it is a case III.

In either case, we can at least find a particular solution for the larger root $c = \nu$

Setting $c = \nu$, the RR becomes:

$a_{k+2}((k+\nu+2)^2 - \nu^2) = -a_k \quad \forall k \geq 0$

the coeff for z^{-1} gives $a_1 = 0$

It then follows that $a_k = 0 \ \forall k$ odd.

Reindexing the RR,

$$a_k = \frac{-a_{k-2}}{((k+\nu)^2 - \nu^2)} \quad \forall k \geq 2$$

for $k = 2m$

$$a_{2m} = -\frac{a_{2(m-1)}}{(2m+\nu)^2 - \nu^2} = -\frac{a_{2(m-1)}}{(2m+2\nu)(2m)} = -\frac{a_{2(m-1)}}{4(m+\nu)m}$$

By induction:

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m} m! \prod_{j=1}^{m}(j+\nu)}$$

Notice: $\Gamma(a+1) = a\Gamma(a) \Rightarrow \prod_{j=1}^{m}(j+\nu) = \frac{\Gamma(m+\nu+1)}{\Gamma(\nu+1)}$

$$a_{2m} = \frac{(-1)^m a_0 \Gamma(\nu+1)}{2^{2m} m! \Gamma(m+\nu+1)}$$

first particular solution is then

$$w(z) = a_0 \Gamma(\nu+1) \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m} m! \Gamma(m+\nu+1)} z^{2m+\nu}$$

multiplying and dividing by 2^{ν}

$$w(z) = 2^{\nu} a_0 \Gamma(\nu+1) \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+\nu+1)} \left(\frac{z}{2}\right)^{2m+\nu}$$

If we set $a_0 = \frac{1}{2^{\nu}\Gamma(\nu+1)}$, then we obtain

$$J_{\nu}(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+\nu+1)} \left(\frac{z}{2}\right)^{2m+\nu} \quad (*)$$

Question 3 2018 Exam

The generating function formula is

$$G(x,t) = \exp\left(\frac{x}{2}\left(t - \frac{1}{t}\right)\right) = \sum_{n=-\infty}^{\infty} t^n J_n(x)$$

(a) Show that

$$n J_n(x) = \frac{x}{2}\left(J_{n-1}(x) + J_{n+1}(x)\right)$$

and $J_n'(x) = \frac{1}{2}\left(J_{n-1}(x) - J_{n+1}(x)\right)$

Differentiate w.r.t x. We obtain:

$$\frac{\partial G}{\partial x} = \sum_{n=-\infty}^{\infty} t^n J_n'(x) = \frac{1}{2}\left(t - \frac{1}{t}\right)\exp\left(\frac{x}{2}\left(t - \frac{1}{t}\right)\right)$$

Substituting for the series again, we get:

$$\sum_{n=-\infty}^{\infty} t^n J_n'(x) = \frac{1}{2}\left(t - \frac{1}{t}\right)\sum_{n=-\infty}^{\infty} t^n J_n(x)$$

$$= \frac{1}{2}\sum_{n=-\infty}^{\infty} t^{n+1} J_n(x) - \frac{1}{2}\sum_{n=-\infty}^{\infty} t^{n-1} J_n(x)$$

Changing the indices of the series on the RHS, and re-arranging gives:

$$\sum_{n=-\infty}^{\infty} t^n \left[J_n'(x) - \frac{1}{2}\left(J_{n-1}(x) - J_{n+1}(x)\right)\right] = 0$$

For this to hold for general values of t, it is necessary that:

$$J_n'(x) = \frac{1}{2}\left(J_{n-1}(x) - J_{n+1}(x)\right)$$

For the second identity differentiate w.r.t t:

$$\frac{\partial G}{\partial t} = \sum_{n=-\infty}^{\infty} n t^{n-1} J_n(x) = \frac{\partial}{\partial t}\left(\exp\left(\frac{x}{2}\left(t - \frac{1}{t}\right)\right)\right)$$

$$= \frac{x}{2}\left(1 + \frac{1}{t^2}\right)\exp\left(\frac{x}{2}\left(t - \frac{1}{t}\right)\right)$$

$$\sum_{n=-\infty}^{\infty} n t^{n-1} J_n(x) = \frac{x}{2}\left(1 + \frac{1}{t^2}\right)\sum_{n=-\infty}^{\infty} t^n J_n(x)$$

$$\sum_{n=-\infty}^{\infty} n t^{n-1} J_n(x) = \frac{x}{2}\left[\sum_{n=-\infty}^{\infty} t^n J_n(x) + \sum_{n=-\infty}^{\infty} t^{n-2} J_n(x)\right]$$

Re-indexing the series on RHS gives

$$\sum_{n=-\infty}^{\infty} n t^{n-1} J_n(x) = \frac{x}{2}\left[\sum_{n=-\infty}^{\infty} t^{n-1}(J_{n-1}(x) + J_{n+1}(x)\right]$$

Therefore, we obtain:

$$\sum_{n=-\infty}^{\infty} t^{n-1}\left[n J_n(x) - \frac{x}{2}\left[J_{n-1}(x) + J_{n+1}(x)\right]\right] = 0$$

Therefore, $\forall n$ $\quad n J_n(x) = \frac{x}{2}\left[J_{n-1}(x) + J_{n+1}(x)\right]$

(b) Show that

$$J_{n-1}(x) J_{n+1}(x) = \frac{n^2}{x^2} J_n^2(x) - J_n'(x)^2$$

Solution:

$$\left[\frac{n}{x} J_n(x)\right]^2 = \frac{1}{4}\left(J_{n-1}(x) + J_{n+1}(x)\right)^2$$

$$\frac{n^2}{x^2} J_n^2 - (J_n')^2 = \frac{1}{4}\left[(J_{n-1} + J_{n+1})^2 - (J_{n-1} - J_{n+1})^2\right]$$

$$= J_{n-1} J_{n+1}$$

(c) Using part (b) and Bessel's equation

$$x^2 J_n'' + x J_n' + (x^2 - n^2) J_n = 0$$

Show that $\int_0^x \alpha J_n(\alpha)^2 \, d\alpha = \frac{x^2}{2}\left(J_n^2(x) - J_{n-1}(x) J_{n+1}(x)\right)$

Re-write the RHS as

$$\frac{x^2}{2}\left(J_n^2 - \frac{n^2}{x^2} J_n^2 + (J_n')^2\right)$$

$$= \frac{1}{2}\left[(x^2 - n^2) J_n^2 + x^2 (J_n')^2\right]$$

diff w.r.t. x

$$\frac{d}{dx}\text{RHS} = \frac{1}{2}\left(2x J_n^2 + 2(x^2 - n^2) J_n J_n' + 2x (J_n')^2 + 2x^2 J_n' J_n''\right)$$

$$= x J_n^2 + (x^2 - n^2) J_n J_n' + x J_n' J_n' + x^2 J_n'' J_n'$$

$$= x J_n^2 + J_n' \underbrace{\left[x^2 J_n'' + x J_n' + (x^2 - n^2) J_n\right]}_{\text{Bessel's equation}}$$

so, $\frac{d}{dx}\text{RHS} = x J_n^2$

$$\Rightarrow \int_0^x \alpha J_n^2(\alpha)\, d\alpha = \frac{x^2}{2}\left(J_n(x)^2 - J_{n-1}(x) J_{n+1}(x)\right)$$

(d) Let $j_{nk} > 0$ denote the k^{th} root of $J_n(x)$. Show that

$$\int_0^{j_{nk}} \alpha J_n(\alpha)^2 \, d\alpha = \left(\frac{j_{nk}}{\sqrt{2}}\right)^2 \left(J_{n+1}(j_{nk})\right)^2$$

Solution: since j_{nk} is a ~~[illegible]~~ root, $J_n(j_{nk}) = 0$.

Hence, using (c)

$$\int_0^{j_{nk}} \alpha J_n(\alpha)\, d\alpha = \frac{(j_{nk})^2}{2}\left(0 - J_{n-1}(j_{nk}) J_{n+1}(j_{nk})\right)$$

Recall that

$$n J_n(x) = \frac{x}{2}\left(J_{n-1}(x) + J_{n+1}(x)\right)$$

Therefore $\quad 0 = \frac{j_{nk}}{2}\left(J_{n-1}(j_{nk}) + J_{n+1}(j_{nk})\right)$

$\Rightarrow J_{n-1}(j_{nk}) = -J_{n+1}(j_{nk}) \quad (j_{nk} \neq 0)$

Substituting for J_{n+1} gives

$$\int_0^{\alpha j_n} \alpha J_n(\alpha)^2 \, d\alpha = \frac{(j_{nk})^2}{2}\left[J_{n+1}(j_{nk})\right]^2 . \blacksquare$$

24/01/19

Ordinary points

(This is a special case of a case III, where $c_1 - c_0 =$ integer, works because of ordinary points).

In applying Frobenius method about an ordinary point z_0, there are some simplications that occur. (w.l.o.g, assume $z_0 = 0$)

The series expansion for $zp(z) = \sum_{j=0}^{\infty} p_j z^j$ and $z^2 q(z) = \sum_{j=0}^{\infty} q_j z^j$ satisfy $p_0 = 0$, $q_0 = q_1 = 0$ when $z_0 = 0$ is an ordinary point. This is because $p(z)$ and $q(z)$ are analytic at z_0 and therefore don't have any poles at z_0.

Recall, that in general, the indicial equation is

$F(c) \quad c + (p_0 - 1)c + q_0 = 0$

this simplifies to $\quad c(c-1) = 0 \quad$ (as $p_0 = q_0 = 0$)

the root are $c = 0$ and $c = 1$

The first particular solution for $c = 1$ can be found in the usual way.

To find the second particular solution for $c = 0$, we go back to general RR.

$$\text{RR} \quad a_k F(k+c) + \sum_{j=0}^{k-1} a_j\left[(j+c)p_{k-j} + q_{k-j}\right] = 0 \quad (k \geq 1)$$

for $k = 1$, there is a simplification

$$a_1 F(1) + a_0(0 \cdot p_1 + q_1) = 0$$

Since $F(1)$, q_1 are both zero, we see that for $k = 1$, the RR is satisfied by any a_1

We are therefore free to choose $a_1 = 0$ to then obtain a second particular solution $w_2(z)$.

In fact, choosing $a_1 \neq 0$ amount to adding ~~constant~~ a constant multiple of $w_1(z)$ back into $w_2(z)$.

As a result it turns out that in either case, the solutions can be found by trying the ansatz

$$w(z) = \sum_{k=0}^{\infty} a_k z^k$$

The 1st particular solution $w_1(z)$ is then found by setting $a_0 = 0$, $a_1 \neq 0$

$w_1(z) = a_1 z + a_2 z^2 + \ldots$ and the second particular solutions is found by setting $a_0 \neq 0$, $a_1 = 0 \Rightarrow$

$$w_2(z) = a_0 + a_2 z^2 + \ldots$$

Legendre's equation and Legendre's polynomial

Legendre's equation is $(1-z^2)w'' - 2zw' + \nu(\nu+1)w = 0$
where ν is the index of the equation.

The functions: $p(z) = -\frac{2z}{(1-z^2)}$; $q(z) = \frac{\nu(\nu+1)}{(1-z^2)}$

Therefore $z_0 = 0$ is an ordinary point

To find the solution, we will try the simplified version of Frobenius method for ordinary points.

Try $w(z) = \sum_{k=0}^{\infty} a_k z^k$ where $a_0 = 0,\ a_1 \neq 0$ / $a_0 \neq 0,\ a_1 = 0$

We then obtain

$$(1-z^2)\sum_{k=0}^{\infty} a_k (k)(k-1) z^{k-2} - 2\sum_{k=0}^{\infty} a_k k z^{k} + \nu(\nu+1)\sum_{k=0}^{\infty} a_k z^k = 0$$

$$\Longleftrightarrow \sum_{k=0}^{\infty} a_k (k)(k-1) z^{k-2} + \sum_{k=0}^{\infty} \underbrace{[\nu(\nu+1) - k(k-1) - 2k]}_{\substack{\nu(\nu+1) - k(k+1) \\ = (\nu-k)(\nu+k+1)}} z^k = 0$$

We can then isolate the 1st two terms of 1st series

$$a_0(0)(-1)z^{-2} + a_1(1)(0)z^{-1} + \sum_{k=2}^{\infty} a_k (k)(k-1) z^{k-2} + \sum_{k=0}^{\infty} a_k(\nu-k)(k+\nu+1) z^k = 0$$

Re-indexing the first series gives

$$\sum_{k=0}^{\infty} [a_{k+2}(k+2)(k+1) + a_k(\nu-k)(\nu+k+1)] z^k = 0$$

$\Longrightarrow$ the RR becomes

$$a_{k+2}(k+2)(k+1) = a_k(k-\nu)(\nu+k+1)$$

i.e. $a_{k+2} = \frac{(k-\nu)(\nu+k+1)}{(k+2)(k+1)} a_k \quad \forall k \geq 0$

We then see that if we choose $a_0 = 0$, $a_1 \neq 0$, then we get $a_k = 0$ for k even.
So $w_1(z) = a_1 z + a_3 z^3 + a_5 z^5 + \dots$
for $w_2(z)$, we set $a_1 = 0$, so all odd coeff. vanish, an $w_2(z) = a_0 + a_2 z^2 + a_4 z^4 + \dots$

For ν an integer, we see that $a_{\nu+2} = 0$, due to the term $(k-\nu)$ in the RR.
If ν is even, then the series of $w_2(z)$ becomes

$w_2(z) = a_0 + a_2 z^2 + a_4 z^4 + \dots + a_\nu z^{\nu}$
(terminates at $a_\nu z^\nu$ because all terms $a_{\nu+2}, a_{\nu+4}, \dots = 0$)
and if ν is odd, then

$w_2(z) = a_1 z + a_3 z^3 + \dots + a_\nu z^\nu$

In either case, one of w_1 or w_2 is then a polynomial when ν is an integer.

The polynomial solutions are, up to a multiplicative constant, the Legendre's polynomial.

The definition of $P_n(z)$, the n th Legendre polynomial, is then the polynomial solution of Legendre's equation with index n an integer with the normalised condition $P_n(1)=1$

Next time, we will prove a simple formula for $P_n(z)$

Rodriguez Formula

$$P_n(z) = \frac{1}{2^n n!} \frac{d^n}{dz^n}\left[(z^2-1)^n\right]$$

29/01/19

We have found that there are polynomials solutions $P_n(x)$ of Legendre's equation of index n where $n \geq 0$ is an integer, i.e.

$$(1-x^2)P_n''(x) - 2x P_n(\ ) + n(n+1)P_n(x) = 0$$

$P_n(x)$ is defined further by the condition $P_n(1)=1$ $\forall n \geq 0$

Theorem - Rodriguez Formula

An equivalent de inition of $P_n(x)$ is

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[(x^2-1)^n\right]$$

Partial proof We will check that defining $P_n(x)$ by Rodriguez Formula give a solution of Legendre's equation.
Also that $P_n(1)=1$ $\forall n \geq 0$

Part ① Legendre's equation

Let us denote $h(x) = \frac{1}{2^n n!}(x^2-1)^n$

Therefore $P_n(x) = \frac{d^n}{dx^n}h(x)$

We see that

$$h'(x) = \frac{1}{2^n n!}\cdot 2xn(x^2-1)^{n-1}$$

so, we get the identity:

$$(x^2-1)h' = 2nxh \quad ⊛$$

The main idea is to differentiate ⊛ $n+1$ times

$$\frac{d^{n+1}}{dx^{n+1}} LHS = \frac{d^{n+1}}{dx^{n+1}}\left[(x^2-1)h'\right]$$

$$= \sum_{k=0}^{n+1}\binom{n+1}{k}\frac{d^k}{dx^k}(x^2+1)\frac{d^{n+1-k}}{dx^{n+1-k}}h'$$

Since $\frac{d^k}{dx^k}(x^2-1)=0$ $\forall k \geq 3$

$$\frac{d^{n+1}}{dx^{n+1}} LHS = \binom{n+1}{0}(x^2-1)\frac{d^{n+1}}{dx^{n+1}}h' + \binom{n+1}{1}(2x)\frac{d^n}{dx^n}h' + \binom{n+1}{2}(2)\frac{d^{n-1}}{dx^{n-1}}h'$$

$$= (x^2-1)\frac{d^2}{dx^2}\left(\frac{d^n}{dx^n}h\right) + (n+1)(2x)\frac{d}{dx}\left(\frac{d^n}{dx^n}h\right) + \frac{n(n+1)}{2}\cdot 2\cdot\frac{d^n}{dx^n}h \qquad \left[\frac{d^n}{dx^n}h = P_n(x)\right]$$

Simplify and obtain:

$$\frac{d^{n+1}}{dx^{n+1}} LHS = (x^2-1)P_n'' + 2x(n+1)P_n' + n(n+1)P_n$$

Now, differentiate RHS:

$$\frac{d^{n+1}}{dx^{n+1}}(2nxh(x))$$

$$= 2n \sum_{k=0}^{n+1} \binom{n+1}{k} \frac{d^k}{dx^k}(x) \frac{d^{n+1-k}}{dx^{n+1-k}} h$$

$$= 2n\left[\binom{n+1}{0} x \frac{d^{n+1}}{dx^{n+1}} h + \binom{n+1}{1} \frac{d^n}{dx^n} h\right]$$

$$= 2nxP_n' + 2n(n+1)P_n$$

So, we have shown that:

$$(x^2-1)P_n'' + (n+1)2xP_n' + n(n+1)P_n = 2nxP_n' + 2n(n+1)P_n$$

After simplification:

$$(x^2-1)P_n'' + 2xP_n' = n(n+1)P_n$$

$$(1-x^2)P_n'' - 2xP_n' + n(n+1)P_n = 0 \quad \blacksquare$$

<u>Part ②</u> <u>For $P_n(1)=1$</u>

Rodriguez Formula $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}[(x^2-1)^n]$

for $n=0$; $P_0(x) = \frac{1}{2^0(0!)}(x^2-1)^0 = 1$

$\Rightarrow P_n(1) = 1$ for $n=0$

We remark that $(\forall j \le n-1)$

$$\frac{d^j}{dx^j}(x^2-1)^n = (x^2-1)^{n-j} Q_j(x)$$

Where $Q_j(x)$ is some polynomial.

This implies that

$$\left.\frac{d^j}{dx^j}(x^2-1)^n\right|_{x=1} = 0 \quad \forall j \le n-1$$

<u>Inductive step</u>: Suppose $P_{n-1}(1) = 1$, then:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)(x^2-1)^{n-1}$$

$$P_n(x) = \frac{1}{2^n n!} \sum_{k=0}^{n} \binom{n}{k} \frac{d^k}{dx^k}(x^2-1) \frac{d^{n-k}}{dx^{n-k}}[(x^2-1)^{n-1}]$$

Therefore

$$2^n n! P_n(x) = \binom{n}{0}(x^2-1)\frac{d^n}{dx^n}(x^2-1)^{n-1} + \binom{n}{1}(2x)\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^{n-1} + \binom{n}{2}(2)\frac{d^{n-2}}{dx^{n-2}}(x^2-1)^{n-1}$$

For $x=1$; we see that 1st and 3rd term vanish, because of (x^2-1) at $x=1$ and $\frac{d^{n-2}}{dx^{n-2}}(x^2-1)^{n-1}$ as well

$\boxed{\text{Exercise: } P_n(-1) = (-1)^n}$

$$2^n n! P_n(1) = \underbrace{n \cdot 2 \cdot 2^{n-1}(n-1)!}_{= n!\, 2^n} P_{n-1}(1)$$

as $P_{n-1} = \frac{1}{2^{n-1}(n-1)!} \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^{n-1}$

<u>Theorem</u>: The Legendre polynomial $\{P_n(x)\}_{n\ge 0}$ are mutually orthogonal.

$$\int_{-1}^{1} P_n(x) P_j(x)\, dx = \frac{2}{2n+1}\delta_{nj}$$

The polynomials $\{P_0, \dots, P_n\}$ are then a basis of the space of polynomials of degree at most n.

$q(x) = \sum_{k=0}^{n} b_k P_k(x)$, $b_k = \frac{2k+1}{2}\int_{-1}^{1} q(x) P_k(x)\, dx$

for any polynomial $q(x)$ of degree at most n.

Proof. We start by showing that

$$\int_{-1}^{1} P_n(x) P_j(x)\, dx = 0 \quad \text{for } n \neq j$$

w.l.o.g we may assume that $j<n$

$$\int_{-1}^{1} P_j(x) P_n(x)\, dx = \int_{-1}^{1} P_j(x)\left[\frac{(x^2-1)P_n'' + 2xP_n'}{n(n+1)}\right] dx$$

From Legendre's equation $(x^2-1)P_n'' + 2xP_n' = n(n+1)P_n$

$$\int_{-1}^{1} P_j P_n\, dx = \frac{1}{n(n+1)}\int_{-1}^{1} P_j \frac{d}{dx}\left[(x^2-1)P_n'\right] dx$$

We see that $\frac{d}{dx}\left[(x^2-1)P_n'\right] = (x^2-1)P_n'' + 2xP_n'$

We can then integrate by parts to obtain

$$\int_{-1}^{1} P_n P_j\, dx = \frac{1}{n(n+1)}\left[\left[P_j(x)(x^2-1)P_n'(x)\right]_{-1}^{1} - \int_{-1}^{1}(x^2-1)P_n' P_j'\, dx\right]$$

$$= -\frac{1}{n(n+1)}\int_{-1}^{1}(x^2-1)P_n' P_j'\, dx$$

Integrate by parts again:

$$= -\frac{1}{n(n+1)}\left[\left[(x^2-1)P_n P_j'\right]_{-1}^{1} - \int_{-1}^{1}\frac{d}{dx}\left[(x^2-1)P_j'\right]P_n\, dx\right]$$

$$= \frac{1}{n(n+1)}\int_{-1}^{1} P_n \frac{d}{dx}\left[(x^2-1)P_j'\right] dx$$

The polynomial P_j solves

$$\frac{d}{dx}\left[(x^2-1)P_j'\right] = j(j+1)P_j$$

$$\Rightarrow \int_{-1}^{1} P_j P_n\, dx = \frac{j(j+1)}{n(n+1)}\int_{-1}^{1} P_n P_j\, dx$$

since $j \neq n \Rightarrow \int_{-1}^{1} P_n P_j\, dx = 0$

Now, we show that

$$\int_{-1}^{1} P_n(x)^2\, dx = \frac{2}{2n+1} \quad \forall n \geq 0$$

From Rodriguez Formula

$$\int_{-1}^{1} P_n(x)^2\, dx = \frac{1}{2^{2n}(n!)^2}\int_{-1}^{1}\frac{d^n}{dx^n}(x^2-1)^n \frac{d^n}{dx^n}(x^2-1)^n\, dx$$

$$= \frac{1}{2^{2n}(n!)^2}\left(\left[\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n \frac{d^n}{dx^n}(x^2-1)^n\right]_{-1}^{1} - \int_{-1}^{1}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n \frac{d^{n+1}}{dx^{n+1}}(x^2-1)^n\, dx\right)$$

Therefore

$$\int_{-1}^{1} P_n(x)^2\, dx = \frac{1}{2^{2n}(n!)^2}\left(-\int_{-1}^{1}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n \frac{d^{n+1}}{dx^{n+1}}(x^2-1)^n\, dx\right)$$

We can repeat integration by parts n times in total, to obtain:

$$\int_{-1}^{1} P_n(x)^2\, dx = \frac{(-1)^n}{2^{2n}(n!)^2}\int_{-1}^{1}(x^2-1)^n \underbrace{\frac{d^{2n}}{dx^{2n}}(x^2-1)^n}_{=(2n)!}\, dx$$

EXPAND AND DIFFERENTIATE

$$= \frac{(-1)^n(2n!)}{2^{2n}(n!)^2}\int_{-1}^{1}(x^2-1)^n\, dx$$

We have the Known value

$$\int_{-1}^{1}(x^2-1)^n\, dx = \frac{(-1)^n 2^{n+1}(n!)^2}{(2n+1)!}$$

(Related to Beta function)

So, we get:

$$\int_{-1}^{1} P_n(x)^2\, dx = \frac{(-1)^{2n}(2n)!\, 2^{n+1}(n!)^2}{(2n+1)!\, 2^{2n}(n!)^2} = \frac{2}{2n+1}$$

Similarly to Bessel functions, there is a generating function formula for Legendre's polynomials

$$G(x,t) = \frac{1}{(1-2xt+t^2)^{1/2}} = \sum_{n=0}^{\infty} P_n(x)\, t^n$$

Exercise

Use the formula to show that the recurrance relation

$$(n+1)P_{n+1}(x) = (2n+1)\,x P_n(x) - nP_{n-1}(x)$$

Differentiate with respect to ~~x~~ t.

$$\frac{\partial}{\partial t}\text{RHS} = \sum_{n=0}^{\infty} n t^{n-1} P_n(x) = \sum_{n=1}^{\infty} n t^{n-1} P_n(x)$$

$$\frac{\partial}{\partial t}\text{LHS} = \left(-\frac{1}{2}\right)(2t - 2x)(1-2xt+t^2)^{-3/2}$$
$$= \frac{(x-t)}{(1-2xt+t^2)}\, G(x,t)$$

Rearranging, we get

$$(1-2xt+t^2)\sum_{n=1}^{\infty} n t^{n-1} P_n(x) = (x-t)\sum_{n=0}^{\infty} t^n P_n(x)$$

31/01/19

Goal: $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$

Expanding the products

$$\text{LHS} = \sum_{n=1}^{\infty} n t^{n-1} P_n(x) - \sum_{n=0}^{\infty} (2n)\, t^n x P_n(x) + \sum_{n=0}^{\infty} n t^{n+1} P_n(x)$$

$$\text{RHS} = \sum_{n=0}^{\infty} x P_n(x) t^n - \sum_{n=0}^{\infty} t^{n+1} P_n(x)$$

Re-arranging, we get

$$\sum_{n=1}^{\infty} n t^{n-1} P_n(x) = \sum_{n=0}^{\infty} (2n+1)xP_n(x)t^n - \sum_{n=0}^{\infty}(n+1)t^{n+1}P_n(x)$$

$$\sum_{n=0}^{\infty}(n+1)t^n P_{n+1}(x) = \sum_{n=0}^{\infty}(2n+1)xP_n(x)t^n - \sum_{n=0}^{\infty} n t^n P_{n-1}(x)$$

(We use the convention that $0\cdot P_{-1}(x) = 0$)

$$\iff \sum_{n=0}^{\infty}\left[(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x)\right]t^n = 0$$

For all $n \geq 0$, we get

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

Exam 2015

Consider Hermite's equation

$$y'' - 2xy' + 2\lambda y = 0 \quad (*)$$

a) Seek a power series solution of the form ~~form~~

$$y(x) = \sum_{k=0}^{\infty} a_k x^k$$

and show that

$$a_{k+2} = \frac{-2(\lambda-k)}{(k+2)(k+1)}\, a_k \quad \forall k \geq 0$$

b) Explain why for $\lambda = n \geq 0$ an integer, there are polynomial solutions of $(*)$. Find the polynomial solution $H_n(x)$ that satisfies $a_n = 2^n$

Answer a) Inserting the series into the equation

$$\sum_{k=0}^{\infty} a_k (k)(k-1) x^{k-2} - 2\sum_{k=0}^{\infty} a_k k x^k + \sum_{k=0}^{\infty} 2\lambda a_k x^k = 0$$

Isolate the first few terms:

$$a_0\,0\,(-1)x^{-2}+a_1(1)(\,)\cdot x^{-1}+\sum_{k=2}^{\infty}a_k(k)(k-1)x^{k-2}$$
$$+\sum_{k=0}^{\infty}2a_k(\lambda-k)x^k=0$$

Change the indices of the 1^{st} series

$$\sum_{k=0}^{\infty}a_{k+2}(k+2)(k+1)x^k+\sum_{k=0}^{\infty}2a_k(\lambda-k)x^k=0$$

Therefore, we see that the RR

$$a_{k+2}(k+2)(k+1)+2a_k(\lambda-k)=0 \quad \forall k\geq 0$$

$$a_{k+2}=\frac{-2(\lambda-k)}{(k+2)(k+1)}a_k \quad \forall k\geq 0$$

b) If $\lambda=n\geq 0$ (a non negative integer), then $a_{n+2}=0$, and by the RR $a_{n+2j}=0 \ \forall j\geq 1$

So, if n is even, we choose $a_0\neq 0$, and $a_1=0$,
So the solution becomes:

$$y(x)=a_0+a_2x^2+\ldots+a_nx^n$$

Conversely, if n is odd, we set $a_0=0$, $a_1\neq 0$

$$y(x)=a_1x+a_3x^3+\ldots+a_nx^n.$$

$H_n(x)$ is the polynomial solution defined by the condition $a_n=2^n$.

Remark: Since we possibly have $a_0=0$ or $a_1=0$, and we are given a_n, it is advantageous to solve the RR "backwards" starting with a_n

RR can be written as

$$a_k=\frac{-(k+2)(k+1)}{2(n-k)}a_{k+2}$$

So for $k=n-2$, we get

$$a_{n-2}=\frac{-(n)(n-1)}{2(n-(n-2))}a_n$$
$$=-\frac{n(n-1)}{2\cdot 2}a_n$$

and $$a_{n-4}=\frac{-(n-2)(n-3)}{2\cdot(n-(n-4))}a_{n-2}=\frac{n(n-1)(n-2)(n-3)}{2\cdot 2\cdot 2\cdot 2\cdot 2}a_n$$

So, for $j\geq 1$

$$a_{n-2j}=\frac{-(n-2j+2)(n-2j+1)}{2(n-(n-2j))}a_{n-2j+2}$$
$$=\frac{-(n-2j+2)(n-2j+1)}{2^2 j}a_{n-2j+2}$$

By induction,

$$a_{n-2j}=(-1)^j\frac{n!}{(n-2j)!}\frac{1}{2^{2j}}\frac{1}{j!}a_n$$

$$\Rightarrow a_{n-2j}=\frac{(-1)^j n!}{(n-2j)!\,j!}\frac{1}{2^{2j}}a_n$$

So, the solution $H_n(x)$ is given by

$$H_n(x)=\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}\frac{(-1)^j n!}{(n-2j)!\,j!}2^{n-2j}x^{n-2j}$$
$$=\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}\frac{(-1)^j n!}{(n-2j)!\,j!}(2x)^{n-2j}$$

where $\lfloor a\rfloor$ is the floor function of a, i.e. it is the largest integer less than or equal to a

5/02/19

Chapter 3: Orthogonality and Generalised fourier series.

Fourier series expansion:

$$f(x) = \frac{a}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)\ b_n \sin(nx)$$

Where the coefficients

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\ dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\ dx$$

The expansion is derived from the orthogonality properties of: $\{1, \sin x, \cos x, \sin 2x, \dots\}$.
We have seen that Legendre Polynomial $\{P_n(x)\}_{n=0}^{\infty}$ have similar orthogonality propertie: (for the interval $(-1,1)$).

Is there a similar Legendre series expansion for rather general functions $f(x)$?

In this chapter, we see that the answer is positive, and we will see how many more examples of series expansions can be found collectively and these will be called generalised fourier series.

Inner product spaces

There are links between fourier series expansions and linear Algebra: the functions $\{1, \sin x, \cos x, \dots\}$ can be viewed as forming a basis of some space V of functions that admit a Fourier Series (F.S.) expansion.
The coeff. a_n, b_n can be seen as coordinates in this basis.

Unlike $\mathbb{R}^n$ (standard example of a finite dimensional vector space), this space V is infinite dimensional. However, similar to $\mathbb{R}^n$, it has some structures that are similar, eg there is a inner product $\langle \cdot, \cdot \rangle$, defined here by:

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x) g(x)\ dx$$

and the norm:

$\|f\| = \sqrt{\langle f, f \rangle}$, for functions f and g in V.

With these definitions, the basis $\{1, \sin x, \cos x, \dots\}$, is orthogonal, since:

$$\langle \cos nx, \sin nx \rangle = 0$$

and $\langle \cos nx, \cos jx \rangle = \langle \sin nx, \sin jx \rangle = \pi \delta_{nj}$
and $\langle 1, 1 \rangle = 2\pi$

To simplify the following, let us enumerate the basis $\{1, \sin x, \cos x, \dots\}$ as follows:

$$\psi_0 = \frac{1}{\sqrt{2}} \quad \psi_{2j-1}(x) = \sin jx \quad \psi_{2j}(x) = \cos jx \quad \forall j \geq 1$$

$$\psi = \frac{1}{\sqrt{2}} \cdot 1 \quad \psi_1 = \sin(x), \psi_2(x) = \cos x, \ \psi_3(x) = \sin 2x$$
$$\psi_4(x) = \cos 2x \dots$$

The expansion then becomes $\sum_{j=0}^{\infty} c_j \psi_j(x)$

Then, the norm can be found by.

$$\|f\|^2 = \langle f, f \rangle$$
$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} c_j c_k \langle \tau_j, \tau_k \rangle$$
$$= \sum_{j=0}^{\infty} c_j^2 \langle \tau_j, \tau_j \rangle$$

Recall c_j are of the form

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(\ )\begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix} dx, \text{ where } j = \begin{Bmatrix} 2k-1 \\ 2k \end{Bmatrix}$$

So, the coefficients c_j are of the form:

$$c_j = \frac{\langle f, \tau_j \rangle}{\langle \tau_j, \tau_j \rangle}$$

Therefore, $\|f\|^2 = \sum_{j=0}^{\infty} \dfrac{|\langle f, \tau_j \rangle|^2}{|\langle \tau_j, \tau_j \rangle|}$

From the definition of inner product we also have

$$\|f\|^2 = \int_{-\pi}^{\pi} |f(x)|^2 dx$$

A necessary condition for the functions f in V is that $\|f\| < \infty$ for the F.S. to exist.

A deeper analytic shows that this is essentially also sufficient, i.e. V is the space of functions that are integrable (in Lebesgue sense)

and that satisfy $\int_{-\pi}^{\pi} |f(x)|^2 dx < \infty$

This space is known as $L^2(\ \pi, \pi)$

The space $L^2(-\pi, \pi)$ is an example of an infinite dimensional inner product space. It is furthermore a complete space, i.e. it is a Hilbert complete space,

Generally we define the space $L^2(a,b) =$
$\{ f\ (a,b) \to \mathbb{R},\ f \text{ is integrable, } \int_a^b |f(x)|^2 dx < \infty \}$

We will now see how functions from these spaces can be expressed by generalised F.S that come from eigenvalue problems for differential operators.

Eigenvalue problems for differential operators

In Linear Algebra, complete orthogonal bases are related to eigenvalue problems for symmetric matrices. Here, we look at how complete complete orthogonal bases for functions in $L^2(a,b)$ are generated by eigenvalue problems for differential operators and boundary conditions.

Let $\mathcal{L} = -\dfrac{d^2}{dx^2}$ i.e. $\mathcal{L}f = -\dfrac{d^2 f}{dx^2}$

Consider the eigenvalue problem

$\mathcal{L}y = \lambda y$ with $y(0) = y(L) = 0$

We saw that the solutions are:

$$y(x) = y_k(x)\ \ c\sin\left(\frac{k\pi x}{L}\right),\ k \geq 1$$

an integer, so the eigenvalues are

$$\lambda_k = \left(\frac{k\pi}{L}\right)^2$$

Ex. 2 $\mathcal{Y} = -\frac{d^2}{dx^2}$ and consider $\mathcal{Y}\, y = \lambda y$

$y(0) = 0 \quad y'(1) = 0$

If $\lambda > 0$, then the solutions are

$y(x) = A\cos(\sqrt{\lambda}\, x) + B\sin(\sqrt{\lambda}\, x)$

$y(0) = 0 \Rightarrow A = 0$

$y'(1) = 0 \Leftrightarrow B\sqrt{\lambda}\cos(\sqrt{\lambda}) = 0$

Another solution exists for $\sqrt{\lambda} = (2k+1)\pi/2$ for $k \geq 0$ an integer

$\Rightarrow \lambda = \lambda_k = \left[\frac{(2k+1)\pi}{2}\right]^2 \quad \forall k \geq 0$

with eigenfunctions:

$y_k = c\sin\left[\frac{(2k+1)\pi x}{2}\right]$

In these examples, we obtain orthogonal eigenfunctions, e.g. in example 1, we get

$\int_0^L \sin\left(\frac{k\pi x}{L}\right)\sin\left(\frac{j\pi x}{L}\right) dx = \frac{L}{2}\delta_{kj}$

In linear Algebra, orthogonal eigenvectors result from eigenvalue problems for symmetric matrices. Here, we see that the appropriate notion for differential operators is self-adjointness.

Recall that for a matrix $A \in \mathbb{R}^{n\times n}$, two equivalent definitions/conditions for A being symmetric are:

- $A_{ij} = A_{ji} \quad \forall i,j \leq n$
- $(Ax)\cdot y = (Ay)\cdot x \quad \forall x, y \in \mathbb{R}^n$

The second notion can be generalised to differential operators.

A self-adjoint operator $\mathcal{Y}$ on some space of functions with boundary conditions and an inner product is one that satisfies

$$\langle \mathcal{Y}f, g\rangle = \langle \mathcal{Y}g, f\rangle$$

for functions f and g in the space that satisfy the given boundary conditions.

Ex. Show that $\mathcal{Y} = -\frac{d^2}{dx^2}$ is self-adjoint on the space of functions y that satisfy $y(0) = y(L) = 0$, with inner product $\langle f, g\rangle = \int_0^L f(x)g(x)\,dx$

Solution: $\langle \mathcal{Y}f, g\rangle = \int_0^L \left(-\frac{d^2 f}{dx^2}\right) g(x)\,dx$

apply integration by parts:

$\langle \mathcal{Y}f, g\rangle = -\left[\frac{df}{dx} g\right]_0^L + \int_0^L \frac{df}{dx}\frac{dg}{dx}\,dx$

since $g(0) = g(L) = 0$

$\langle \mathcal{Y}f, g\rangle = -g(L)\frac{df}{dx}(L) + g(0)\frac{df}{dx}(0) + \int_0^L \frac{df}{dx}\frac{dg}{dx}\,dx$

$= \int_0^L \frac{df}{dx}\frac{dg}{dx}\,dx$

Integrate by parts again:

$\langle \mathcal{Y}f, g\rangle = \left[f\frac{dg}{dx}\right]_0^L + \boxed{\int_0^L f\frac{d^2 g}{dx^2}\,dx} = \langle f, \mathcal{Y}g\rangle$

$= f(L)\frac{dg}{dx} - f(0)\frac{dg}{dx} + \langle \mathcal{Y}g, f\rangle$

$\langle \mathcal{Y}f, g\rangle = \langle \mathcal{Y}g, f\rangle$

Sometimes, the inner product is important for self-adjointness.

Consider some function $w(x)$ on (a,b) that is positive $\forall x \in (a,b)$. Consider the weighted inner product $\langle f, g\rangle = \int_a^b w(x) f(x) g(x)\, dx$

Ex: let $\mathcal{Y} = -\frac{1}{x}\frac{d}{dx}\left(x\frac{d}{dx}\right)$

$\mathcal{Y}f = \frac{1}{x}\frac{d}{dx}\left(x\frac{df}{dx}\right)$

$= -\frac{1}{x}\frac{df}{dx} - \frac{d^2f}{dx^2}$

Let's consider the weighted inner product

$\langle f, g\rangle_x = \int_0^b x f(x) g(x)\, dx$ for $0 < b$.

We show that $\mathcal{Y}$ is self-adjoint w.r.t. $\langle , \rangle_x$ on the space functions y that vanishes at $x = b$ and that are finite at $x = 0$

$\langle \mathcal{Y}f, g\rangle = \int_0^b x \cdot -\frac{1}{x}\frac{d}{dx}\left(x\frac{df}{dx}\right) g\, dx$

$= \int_0^b -\frac{d}{dx}\left(x\frac{df}{dx}\right) g\, dx$

$= \left[-x\frac{df}{dx} g\right]_0^b + \int_0^b x\frac{df}{dx}\frac{dg}{dx}\, dx$

$= -b\frac{df}{dx}(b) g(b) + 0 \cdot \frac{df}{dx}(0) g(0)$

$+ \int_0^b x\frac{df}{dx}\frac{dg}{dx}\, dx$

$\langle \mathcal{Y}f, g\rangle = \left[xf\frac{dg}{dx}\right]_0^b - \int_0^b f\frac{d}{dx}\left(x\frac{dg}{dx}\right) dx = \langle \mathcal{Y}g, f\rangle$

Sturm Liouville eigenvalue Problems

7/02/19

We now consider eigenvalue problems

$$\mathcal{L}y = -\lambda y$$

where $\mathcal{L} = \frac{1}{w(x)}\left[\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + r(x)\right]$

with BCs of the form $\alpha_1 y(a) + \beta_1 y'(a) = 0$

$\alpha_2 y(b) + \beta_2 y'(b) = 0$

where $a < b$, and $\alpha_i, \beta_i \in \mathbb{R}$ for $i = 1, 2$ and $|\alpha_i| + |\beta_i| > 0$ (i.e. they must not both vanish simultaneously)

There are some conditions on the coefficients of $\mathcal{L}$

- $p(x)$ is differentiable, and positive on $[a, b]$
- $w(x)$ is positive on (a, b)
- $r(x)$ bounded on $[a, b]$ (usually be continuous on $[a, b]$)

Proposition. $\mathcal{L}$ is self-adjoint on the space of functions that satisfy the BCs under the inner-product $\langle \cdot, \cdot\rangle_w$

Recall $\langle f, g\rangle_w = \int_a^b w(x) f(x) g(x)\, dx$

Proof. Need to show $\langle \mathcal{L}f, g\rangle_w = \langle \mathcal{L}g, f\rangle_w$

$\langle \mathcal{L}f, g\rangle_w = \int_a^b w(x)\frac{1}{w(x)}\left[\frac{d}{dx}\left(p(x)\frac{df}{dx}\right) + r(x) f\right] g(x)\, dx$

$= \int_a^b \frac{d}{dx}\left(p(x)\frac{df}{dx}\right) g(x) + \int_a^b r(x) f(x) g(x)\, dx$

Integrating by parts 1^{st} term:

$\int_a^b \frac{d}{dx}\left(p(x)\frac{df}{dx}\right) g(x)\, dx = \left[p(x)\frac{df}{dx} g(x)\right]_a^b - \int_a^b p(x)\frac{df}{dx}\frac{dg}{dx}\, dx$

Integrating by parts again

$$\int_a^b \frac{d}{dx}\left(p(x)\frac{df}{dx}\right)g(x)\,dx = \left[p(x)\frac{df}{dx}g\right]_a^b - \left[p(x)f\frac{dg}{dx}\right]_a^b + \int_a^b f(x)\frac{d}{dx}\left(p(x)\frac{dg}{dx}\right)dx$$

So, we see from this that

$$\langle \mathcal{L}f, g\rangle_w = \langle \mathcal{L}g, f\rangle_w + \left[p(x)\left[\frac{df}{dx}g - f\frac{dg}{dx}\right]\right]_a^b$$

Then, we will have shown the self adjointness property if we show that

$$\left[p(x)\left[\frac{df}{dx}g - f\frac{dg}{dx}\right]\right]_a^b = 0$$

$\Longleftarrow$ $f'(b)g(b) - f(b)g'(b) = 0$

and $f'(a)g(a) - f(a)g'(a) = 0$

(case 1) If $\beta_i = 0$ (e.g. $\beta_2 = 0$)

BC is $\alpha_2 y(b) = 0 \Rightarrow f(b) = g(b) = 0$ since $\alpha_2 \neq 0$

$\Rightarrow f'(b)g(b) - f(b)g'(b) = 0 - 0 = 0$

The same argument holds for $\beta_1 = 0$

(case 2) If $\beta_i \neq 0$ (eg $\beta_2 \neq 0$)

$\Rightarrow y'(b) = -\frac{\alpha_2}{\beta_2}y(b)$

$\Rightarrow f'(b)g(b) - f(b)g'(b) = -\frac{\alpha_2}{\beta_2}\left(f(b)g(b) - f(b)g(b)\right) = 0$

A similar argument holds for $\beta_1 \neq 0$

In both cases we guarantee that

$$\left[p(b)\left(\frac{df}{dx}g - f\frac{dg}{dx}\right)\right]_a^b = 0$$

Main result on Sturn-Liouville Eigenvalue problems

Consider $\mathcal{L}y = -\lambda y$ with same BCs as before

① The eigenvalues are all real, and they form a countable sequence $\lambda_1 < \lambda_2 < \ldots$

with $\lim_{k\to\infty} \lambda_k = \infty$

② Eigenfuctions y_j and y_k for distinct eigenvalues λ_j and λ_k ($\lambda_j \neq \lambda_k$) are orthogonal.

$$\langle y_j, y_k\rangle_w = 0 \quad (\text{i.e } \mathcal{L}y_j = -\lambda_j y_j)$$

③ The eigenfunctions are unique up to a multiplicative constant.

If λ is an eigenvalue, and $y, \tilde{y}$ are eigenfunctions, ($\mathcal{L}y = -\lambda y$, $\mathcal{L}\tilde{y} = -\lambda\tilde{y}$) $\Rightarrow y = c\tilde{y}$

④ (If w is positive on $[a,b]$) then $\{y_j\}_{j=1}^{\infty}$ form a complete orthogonal basis of $L^2(a,b)$ and

$$f(x) = \sum_{j=1}^{\infty} \frac{\langle f, y_j\rangle_w}{\langle y_j, y_j\rangle_w} y_j(x)$$

How to find the SL from an eigenvalue problem

Often, we encounter eigenvalue problems in the form $P(x)y'' + Q(x)y' + [R(x) + \lambda w]y = 0$

How do we re-write it in SL form?

$$\frac{1}{w(x)}\left[\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + r(x)y\right] = -\lambda y$$

Expanding, we see that we need:

$$\frac{p(x)}{w(x)} = P(x), \quad \frac{p'(x)}{w(x)} = Q(x), \quad \frac{r(x)}{w(x)} = R(x)$$

$$\Rightarrow \frac{p'(x)}{p(x)} = \frac{Q(x)}{P(x)} \Longleftrightarrow \frac{d}{dx}\ln p(x) = \frac{Q(x)}{P(x)}$$

$$\Rightarrow p(x) = \exp\left(\int \frac{Q(x)}{P(x)}\,dx\right)$$

From $\frac{p(x)}{w(x)} = P(x)$

$$\Rightarrow w(x) = \frac{p(x)}{P(x)}$$

Then $r(x) = R(x)w(x)$

Example: Convert $x^2y'' - xy' + \lambda y = 0$ into SL form

$P(x) = x^2$, $Q(x) = -x$, $\Rightarrow p(x) = \exp\left(\int -\frac{x}{x^2}\,dx\right)$

$$= \exp(-\ln x) = \frac{1}{x}$$

$w(x) = \frac{p(x)}{P(x)} = \frac{1}{x}\frac{1}{x^2} = \frac{1}{x^3}$ $(R(x) = 0 - r(x))$

So, the SL form is then

$$\frac{1}{w}\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + \lambda y = 0$$

$$\Longleftrightarrow x^3\frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dx}\right) + \lambda y = 0$$

Proofs of ② and ④ 19/02/19

② Orthogonality of eigenfunctions

Let y_k, y_j be eigenfunctions with respective eigenvalues λ_k, λ_j, $\lambda_k \neq \lambda_j \Rightarrow \mathcal{L}y_k + \lambda_k y_k = 0$, $\mathcal{L}y_j + \lambda_j y_j = 0$

We want to show that $\langle y_k, y_j \rangle_w = 0$

i.e. $\int_a^b w(x) y_k(x) y_j(x)\, dx = 0$

Consider: $\langle \mathcal{L}y_k, y_j \rangle_w = -\lambda_k \langle y_k, y_j \rangle_w$

since y_k is an eigenfunction

Also, since $\mathcal{L}$ is self-adjoint w.r.t $\langle \cdot, \cdot \rangle_w$

$\langle \mathcal{L}y_k, y_j \rangle_w = \langle y_k, \mathcal{L}y_j \rangle_w$

$= -\lambda_j \langle y_k, y_j \rangle_w$ since y_j is an eigenfunction.

Together we get:

$(\lambda_k - \lambda_j)\langle y_k, y_j \rangle_w = 0$

Since $\lambda_k \neq \lambda_j \Rightarrow \langle y_k, y_j \rangle_w = 0$. ■

Partial proof ④

We shall admit here that $\{y_k\}_{k \geq 1}$ forms a complete basis of $L^2(a,b)$, i.e. every $f \in L^2(a,b)$ admits an expansion of the form:

$$f(x) = \sum_{k=1}^{\infty} c_k y_k(x)$$

We show here that $c_k = \dfrac{\langle f, y_k \rangle_w}{\langle y_k, y_k \rangle_w}$

Take, inner product of f with y_j:

$$\langle f, y_j \rangle_w = \langle \sum_{k=1}^{\infty} c_k y_k, y_j \rangle_w$$

$$= \sum_{k=1}^{\infty} c_k \langle y_k, y_j \rangle_w$$

Since $\langle y_k, y_j \rangle_w = 0$ for $k \neq j$, we then obtain:

$\langle f, Y_j \rangle_w = c_j \langle Y_j, Y_j \rangle_w$

$$\iff c_j = \frac{\langle f, Y_j \rangle_w}{\langle Y_j, Y_j \rangle_w}. \blacksquare$$

<u>Example</u>: Consider the interval $(1, e^{\pi})$ and the SL problem:

$$x^2 y'' - x y' + \lambda y = 0, \quad y(1) = y(e^{\pi}) = 0$$

Find the SL form and express a general function $f(x)$ in terms of a generalised GFS (Generalised Fourier Series) involving the eigenfunctions of the problem.

We found previously that the SL form is:

$$x^3 \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dx}\right) + \lambda y = 0$$

We can solve the differential equation with Frobenius method, however here it is in fact simpler to perform a substitution $x = e^t$

Set $y(t) = y(x) = y(\ln x)$

$$y'(x) = \frac{y'(\ln x)}{x}, \quad y'' = -\frac{y'(\ln x)}{x^2} + \frac{1}{x^2} y''(\ln x)$$

The equation then becomes

$$Y''(t) - 2Y'(t) + \lambda Y(t) = 0$$

We leave it as an exercise to check that for $\lambda \leq 1$, the only solution $Y(t)$ of $Y'' - 2Y' + \lambda Y = 0$ that also satisfy the BCs are trivial.

We now only consider the case $\lambda > 1$.

In this case, the characteristic polynomial $u^2 - 2u + \lambda = 0$ which has roots of the form $1 \pm \sqrt{\lambda - 1}$, we see that the general form of the solution is

$$Y(t) = Ae^t \cos(\sqrt{\lambda-1}\, t) + Be^t \sin(\sqrt{\lambda-1}\, t)$$

Transforming back to x we obtain:

$$y(x) = Ax\cos(\sqrt{\lambda-1}\ln x) + Bx\sin(\sqrt{\lambda-1}\ln x)$$

Consider now the BCs $y(1) = 0 = y(e^{\pi})$

$y(1) = A\cos(0) = A = 0$

$y(e^{\pi}) = Be^{\pi}\sin(\sqrt{\lambda - 1}\,\pi) = 0$

This has non-trivial solutions when $\sqrt{\lambda-1} = k \geq 1$ with k an integer (i.e. $\lambda = k^2+1$)

Therefore the eigenvalues are $\lambda_k = k^2+1$ for $k \geq 1$ integer with associated eigenfunctions $y_k = Bx\sin(k\ln x)$

(W.l.o.g we can take $B = 1$)

We have the GFS for every $f \in L^2(1, e^{\pi})$

$$f(x) = \sum_{k=1}^{\infty} \frac{\langle f, y_k \rangle_w}{\langle y_k, y_k \rangle_w} y_k(x)$$

Here $\langle f, y_k \rangle_w = \int_1^{e^{\pi}} \frac{1}{x^3} f(x) \cdot x\sin(k\ln x)\, dx$

$$= \int_1^{e^{\pi}} \frac{f(x)\sin(k\ln x)}{x^2}\, dx. \ \blacksquare$$

Singular SL eigenvalue problems

So far we assumed that $p(x)$ was positive on $[a,b]$. This is what we call a regular SL eigenvalue problem.

If instead $p(x)$ vanishes at one or both end points $x = a$ or/and $x = b$, then we call the SL eigenvalue problem singular.

Example: SL eigenvalue problem for Bessel's equation.

Consider the SL problem: $y'' + \frac{1}{x}y' + \left(\lambda - \frac{n^2}{x^2}\right)y = 0 \quad (x > 0)$

(Bessel's equation of index n, with λ a parameter)

To find the SL form:

$$p(x) = \exp\left(\int \frac{Q(x)}{P(x)}\, dx\right) = x$$

$$w(x) = \frac{p(x)}{P(x)} = x \qquad r(x) = R(x)w(x) = -\frac{n^2}{x}$$

So, the SL form is:

$$\frac{1}{x}\left[\frac{d}{dx}\left(x\frac{dy}{dx}\right)-\frac{n^2}{x}y\right]+\lambda y=0$$

Since $p(x)=x$ vanishes at $x=0$, this SL problem will be singular on any interval of the form $(0,R)$. $(R>0)$
For singular SL problems, it is not always possible to find non-trivial eigenfunctions when considering general BCs at the endpoints where $p(x)$ vanishes.

<u>Example</u>: Consider Bessel's equation with index 0.

$$\frac{1}{x}\frac{d}{dx}\left(x\frac{dy}{dx}\right)+\lambda y=0 \quad (y(0)=y(1)=0)$$

Then, the general solution is of the form

$$y(x)=A\,J_0(\ln x)+B\,Y_0(\ln x)$$

Where J_0 and Y_0 are Bessel's functions of first and second kind respectively. Then, the condition $y(0)=0$ implies that $B=0$ because $Y_0(x)$ is singular at $x=0$ (as Y_0 contains log term) and also $A=0$ because $J_0(0)=1$, so there is only the trivial soln to this problem.

For singular SL problems, it turns out that no boundary conditions is required for the self-adjointness of the associated operator $\mathcal{L}$ at endpoints where $p(x)$ vanishes (provided only that the solution and its first derivative are bounded at the endpoint).
Therefore, for singular SL problems, we replace any BC by the condition that the solution must be bounded at the endpoints where $p(x)$ vanishes.

Fourier Bessel series:

Consider the SL problem:

$$\frac{1}{x}\left[\frac{d}{dx}\left(x\frac{dy}{dx}\right)-\frac{n^2}{x}y\right]+\lambda y=0$$

on the interval $(0,1)$ with the B.C. $y(1)=0$ and must be bounded at 0.

Then the general solution is of the form

$$y(x)=A J_n(\sqrt{\lambda}x)+B Y_n(\sqrt{\lambda}x)$$

Where J_n and Y_n are Bessel functions of the 1st and 2nd kind of index n.

Under the conditions that $y(x)$ must be bounded at $x=0$, we require $B=0$.

Then, the BC $Y(1)=0 \Longleftrightarrow A J_n(\sqrt{\lambda})=0$

This has non-trivial solutions when $\sqrt{\lambda}=j_{nk}$, where $\{j_{nk}\}_{k\geq 1}$ denotes the roots of $J_n(x)$.

Therefore the eigenvalues $\lambda_k=j_{nk}^2$, with eigenfunctions

$y_k(x)=J_n(j_{nk}x)$ (Take $A=1$)

The completeness of the eigenfunctions still holds and a general function f satisfies:

$$f(x)=\sum_{k=1}^{\infty}\frac{\langle f, y_k\rangle_w}{\langle y_k, y_k\rangle_w}y_k(x)$$

Where $\langle f, y_k\rangle_w=\int_0^1 x f(x) J_n(j_{nk}x)dx$

and $\langle y_k, y_k\rangle_w=\int_0^1 x J_n(j_{nk}x)^2 dx$

Exercise: Using the substitution $s=j_{nk}x$ and the identity: $\int_0^x sJ_n(s)^2 ds=\frac{x^2}{2}\left[J_n(x)^2-J_{n-1}(x)J_{n+1}(x)\right]$

Show that

$\langle y_k, y_k\rangle_w=\frac{J_{n+1}(j_{nk})^2}{2}$, then $f(x)=\sum_{k=1}^{\infty}a_k J_n(j_{nk}x)$

where $a_k=\frac{2}{J_{n+1}(j_{nk})^2}\int_0^1 x f(x) J_n(j_{nk}x)\,dx$

Fourier Legendre Series

The SL form of Legendre's equation is:

$$\frac{d}{dx}\left[(1-x^2)\frac{dv}{dx}\right]+\nu(\nu+1)y=0$$

Where $\lambda=\nu(\nu+1)$ is the eigenvalue.
Usually we consider this equation on the interval $(-1,1)$, so $p(x)=1-x^2$ vanishes at both endpoints.
Therefore we consider this equation with the conditions that $y(x)$ should be bounded at $x=\pm1$
It can be shown that the only non-trivial ~~solution~~ eigenfunctions that satisfy these conditions are the Legendre polynomials $\{P_n(x)\}_{n\geq0}$, with associated eigenvalues are for $\nu=n\geq0$ with integer n
(i.e. $\lambda_n=n(n+1)$)
Every $f\in L^2(-1,1)$ admits a Fourier Legendre ~~[illegible]~~ series expansion

$$f(x)=\sum_{n=0}^{\infty}a_nP_n(x)$$

where $a_n=\dfrac{\langle f,P_n\rangle_w}{\langle P_n,P_n\rangle_w}$

Since $w(x)=1$, we can simplify

$$\langle P_n,P_n\rangle_w=\int_{-1}^{1}P_n^2(x)\,dx=\frac{2}{2n+1}$$

So, $\boxed{a_n=\dfrac{2n+1}{2}\displaystyle\int_{-1}^{1}f(x)P_n(x)\,dx}$

21/02/19

Chapter 4
Separation of variables revisited

We now apply the methods of Ch 2+3 to solve the PDE considered in Ch 1.
We will consider a range of examples.

Example Wave equation on the unit disc

Small vertical displacements on a circular membrane of radius 1 are modelled by the wave equation: $u_{tt} = \Delta u$
where $u = u(r,\theta,t)$ is the vertical displacement.
The membrane is held fixed at $r=1$, so
$u(1,\theta,t) = 0 \quad \forall \theta \in [0,2\pi),\ t \geq 0$
We are given the initial conditions
$$u(r,\theta,0) = f(r), \quad u_t(r,\theta,0) = g(r)$$
for some given functions $f(r)$ and $g(r)$

a) Assuming that u remains axisymmetric for all times, show that
$$u = u(r,t) = \sum_{k=1}^{\infty} J_0(j_{0k} r)[A_k \cos(j_{0k} t) + B_k \sin(j_{0k} t)]$$
where the coefficients A_k and B_k can be expressed in terms of integrals involving $f(r)$ and $g(r)$

(b) Find u when $f(r) = 0$, and
$$g(r) = \begin{cases} 1 & 0 \leq r < \delta \\ 0 & \delta \leq r \leq 1 \end{cases}, \text{ where } 0 < \delta < 1 \text{ parameter}$$

Step 1 We seek nontrivial separated solutions of the form $u(r,t) = F(r)G(t)$

Inserting into the equation, we get
$u_{tt} = F(r)G''(t)$

$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = (F'' + \frac{1}{r}F')G$

$\Rightarrow (F'' + \frac{1}{r}F')G = F(r)G''$

$\Leftrightarrow \dfrac{F'' + \frac{1}{r}F'}{F} = \dfrac{G''}{G} = -\lambda$ with λ a constant

We then get
$F'' + \frac{1}{r}F' + \lambda F = 0$
and $G'' + \lambda G = 0$

The BC $u(1,t) = 0 \ \forall t \geq 0 \Rightarrow F(1) = 0$
We also require that F be bounded at $r=0$
so we recognize the singular SL eigenvalue problem **Step 2**

$F'' + \frac{1}{r}F' + \lambda F = 0$, $F(1) = 0$ F bounded at $r=0$
From the previous lecture, we see that the solutions are $F(r) = F_k(r) = J_0(\sqrt{\lambda} r)$
with $J_0(\sqrt{\lambda}) = 0 \Leftrightarrow \sqrt{\lambda} = j_{0k}$, for $k \geq 1$
and j_{0k} is the k-th root of J_0.

$\Rightarrow F_k(r) = J_0(j_{0k} r)$, for $k \geq 1$
Hence the equation for G becomes

$G'' + j_{0k}^2 G = 0$

$\Rightarrow G(t) = G_k(t) = A_k \cos(j_{0k} t) + B_k \sin(j_{0k} t)$
where A_k and B_k are constants.

Step 3 The general solution is then found as the series involving all the particular separated solutions
$$u(r,t) = \sum_{k=1}^{\infty} J_0(j_{0k} r)[A_k \cos(j_{0k} t) + B_k \sin(j_{0k} t)]$$

To find the coefficients A_K and B_K we use the initial conditions

$$f(r) = u(r,0) = \sum_{K=1}^{\infty} J_0(j_{0K} r) A_K$$

We recognise here is the Fourier-Bessel series for $f(r)$

Therefore $A_K = \frac{2}{[J_1(j_{0K})]^2} \int_0^1 r f(r) J_0(j_{0K} r)\, dr$

We have $u_t(r,0) = g(r)$

$$\iff g(r) = u_t(r,0) = \sum_{K=1}^{\infty} J_0(j_{0K} r)[j_{0K} A_K(-\sin(j_{0K} t)) + j_{0K} B_K \cos(j_{0K} t)]_{t=0}$$

$$\Rightarrow g(r) = \sum_{K=1}^{\infty} J_0(j_{0K} r) B_K j_{0K}$$

$$\Rightarrow B_K = \frac{1}{j_{0K}} \frac{2}{J_1(j_{0K})^2} \int_0^1 r g(r) J_0(j_{0K} r)\, dr$$

b) $f(r) = 0$,

$$g(r) = \begin{cases} 1 & 0 \leq r \leq \delta \\ 0 & \delta \leq r \leq 1 \end{cases}$$

Recall the RR

$$\frac{d}{dx}[x J_1(x)] = x J_0(x)$$

Since $f(r) = 0$, we see that $A_K = 0 \ \forall k \geq 1$

$$B_K = \frac{2}{J_1(j_{0K})^2} \int_0^{\delta} r J_0(j_{0K} r)\, dr$$

If we set $x = j_{0K} r$,

$$\int_0^{\delta} r J_0(j_{0K} r)\, dr = \frac{1}{j_{0K}^2} \int_0^{j_{0K}\delta} x J_0(x)\, dx$$

$$= \frac{1}{j_{0K}^2} [x J_1(x)]_0^{\delta j_{0K}}$$

$$\Rightarrow \int_0^{\delta} r J_0(j_{0K} r)\, dr = \frac{\delta J_1(j_{0K}\delta)}{j_{0K}}$$

$$\Rightarrow B_K = \frac{2\delta J_1(j_{0K}\delta)}{j_{0K}^2 J_1(j_{0K})^2} \quad \forall k \geq 1$$

General procedure for separation of variables

26/02/19

- Given a PDE on some domain with some b.c.s, choose a coordinate system that is well suited to the problem.
  Typically, choose a system s.t. each part of the boundary is represented by constant values for one (or more) variables.
- Suppose (q,v) are the variables for the chosen co-ordinate system, seek non-trivial separated solutions $u(q,v) = Q(q)V(v)$ by separating variables and finding ODEs for $Q(q)$ and $V(v)$.
  Determine all boundary conditions for Q and V including implicit conditions such as periodicity/boundedness.
- Try to identify a SL eigenvalue problem for either Q or V.
  Suppose for sake of explanation, that Q solves a SL eigenvalue problem, find the eigenfunctions $\{Q_K\}_{K\geq 1}$ with associated eigenvalues λ_K.
  Solve the associated equations for V (typically with same λ_K) to get: $\boxed{V_K(v) = A_K V_{1K}(v) + B_K V_{2K}(v)}$

with A_k, B_k constants, V_{1k}, V_{2k} linearly independent particular solutions.

- Sometimes there are further B.C. on V that can help to simplify V_k further.
- The general solution is then

$$u(q,v) = \sum_{k=1}^{\infty} Q_k(q) V_k(v)$$
$$= \sum_{k=1}^{\infty} Q_k(q) [A_k V_{1k}(v) + B_k V_{2k}(v)]$$

- finally, use remaining BCs to determine $\{A_k\}, \{B_k\}$ in terms of generalised fourier series with the boundary data.

Example: Find the steady state temperature u in the unit sphere with BCs at $r=1$

$$u(1,\theta,\phi) = f(\phi), \text{ where } f(\phi) = \begin{cases} 1, & \phi \in [0, \frac{\pi}{2}] \\ 0, & \phi \in (\frac{\pi}{2}, \pi] \end{cases}$$

Solution: The steady state temperature solves $\Delta u = 0$ (from heat equation $u_t - \Delta u$) with $u_t = 0$ due to the steady state condition).
Since $f(\phi)$ is independent of θ, we seek a solution $u(r,\theta,\phi) = u(r,\phi)$, i.e. no θ dependance.
Recall from Chapter 1 that separated solutions $u(r,\phi) = F(r)H(\phi)$ solve the following ODEs

$F'' + \frac{2}{r}F' = \frac{\lambda}{r^2}F$, and, after setting $u = \cos\phi$, $h(u) = H(\phi)$

$$(1-u^2)h'' - 2uh' + \lambda h = 0$$

(In Chapter 1, we set $\lambda = \nu(\nu+1)$; ν is the index of Legendre's equation).

We consider the implicit conditions that it must be bounded at $\phi = 0$ and $\phi = \pi$, which is equivalent to requiring h be bounded at $u = \pm 1$.
We also require $F(r)$ to be bounded at $r=0$.

We recognise here that $h(u)$ solves the singular S.L. problem for Legendre's equation on $u \in (-1,1)$ with boundedness conditions at $u = \pm 1$.
The only non-trivial solut on (up to a constant) are then:
$h_n(u) = P_n(u)$ with P_n the n-th Legendre polynomial, with $n \geq 0$ an integer.
The eigenvalues are $\lambda_n = n(n+1)$

Therefore, $H(\phi) = H_n(\phi) = P_n(\cos(\phi))$
Now, we solve the equation for F

$$F'' + \frac{2}{r}F' = \frac{n(n+1)}{r^2}F$$

Try Frobenius method: $F(r) = \sum_{k=0}^{\infty} a_k r^{k+c}$, $a_0 \neq 0$

$$\Rightarrow \sum_{k=0}^{\infty} a_k [(k+c)(k+c-1) + 2(k+c) - n(n+1)] r^{k+c-2} = 0$$

The I.E is: $c(c+1) = n(n+1)$
with solution $\boxed{c = n, \ c = -(n+1)}$

The RR can be solved trivially with $a_k = 0 \ \forall k \geq 1$

This gives solution:

$F(r) = F_n(r) = A_n r^n + B_n r^{-(n+1)}$

Since we require F to be bounded at $r=0$, we must set $B_n = 0$ $\forall n$.

So $F_n(r) = A_n r^n$ with A_n a constant

General solution is then

$$u(r,\phi) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos(\phi))$$

We now find the $\{A_n\}$ using the BCs

$$u(1,\phi) = f(\phi) = \begin{cases} 1, & \phi \in [0, \pi/2] \\ 0, & \phi \in (\pi/2, \pi] \end{cases}$$

$$\Rightarrow f(\phi) = \sum_{n=0}^{\infty} A_n P_n(\cos\phi)$$

Changing back to u variable

$$\sum_{n=0}^{\infty} A_n P_n(u) = \begin{cases} 1, & u \in [0,1] \\ 0, & u \in [-1,0) \end{cases}$$

We have the generalised fourier series with

$$A_n = \frac{2n+1}{2} \int_0^1 P_n(u)\,du$$

We use the identity:

$$\int_x^1 P_n(u)\,du = \frac{1}{2n+1}\left(P_{n-1}(x) - P_{n+1}(x)\right) \quad \forall x \in [-1,1]$$

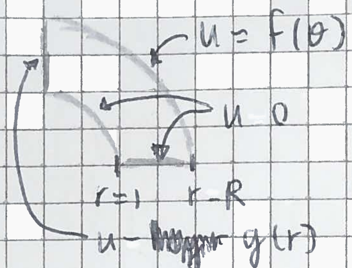
$$\Rightarrow A_n = \frac{2n+1}{2} \cdot \frac{1}{2n+1}\left(P_{n-1}(0) - P_{n+1}(0)\right)$$

Hence $A_n = \frac{1}{2}\left[P_{n-1}(0) - P_{n+1}(0)\right]$

$A_n = 0$ if n is odd.

Example: Question 2 of Extra problems Sheet 4

Consider $\Delta u = 0$ in the region



$1 < r < R$ $\quad 0 < \theta < \pi/2$ with conditions

$u(1,\theta) = 0$ $\quad \forall \theta \in (0, \frac{\pi}{2})$

$u(r,0) = 0$ $\quad$ for $1 < r < R$

$u(R,\theta) = f(\theta)$ $\quad \forall \theta \in (0, \pi/2)$

where $f(\theta)$ and $g(r)$ are some given functions

Try separated solution $u(r,\theta) = F(r)G(\theta)$ (in polar co-ordinates)

$$\Rightarrow F'' + \frac{F'}{r} = \frac{\lambda}{r^2}F \; ; \; G'' + \lambda G = 0$$

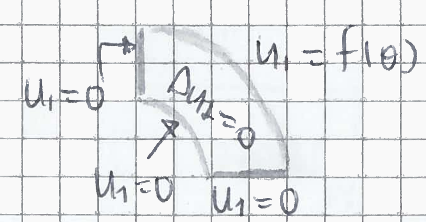
with BC. $F(1) = 0$, $G(0) = 0$

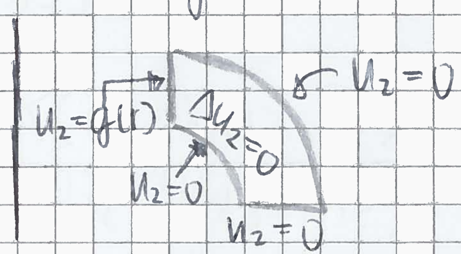
but we don't have any conditions for $r = R$ or $\theta = \pi/2$

It is then not immediately possible to find a SL eigenvalue problem for either F or G.

The way to overcome this is to split the problem in two parts:

We seek the solution $u = u_1 + u_2$ where u_1 and u_2 solve the following subproblems:





Step I: Solve for u_1 $\quad (u_1 = F(r)G(\theta))$

$$F'' + \frac{1}{r}F' = \frac{\lambda}{r^2}F \quad \text{and} \quad G'' + \lambda G = 0$$

$F(1) = 0$, $G(0)$ $G(\pi/2) = 0$

$\Rightarrow$ we recognise the regular SL eigenvalue problem for G.

$G(\theta) = \sin(\sqrt{\lambda}\,\theta)$

$G(\pi/2) = 0 \implies \sqrt{\lambda}\,\pi/2 = k\pi$, $k \geq 1$ an integer

$\iff \sqrt{\lambda} = 2k$

Eigenfunctions are $G(\theta) = G_k(\theta) = \sin(2k\theta)$

Then,

$$F'' + \frac{1}{r}F' = \frac{(2k)^2}{r^2}F$$

The general solution is:

$$F(r) = A_k r^{2k} + B_k r^{-2k}$$

Using $F(1) = 0$, we get:

$$F(r) = F_k(r) = A_k\left(r^{2k} - r^{-2k}\right)$$

so, the general solution is:

$$u_1(r,\theta) = \sum_{k=1}^{\infty} A_k\left(r^{2k} - r^{-2k}\right)\sin(2k\theta)$$

we must find $\{A_k\}$ using $u_1(R,\theta) = f(\theta)$

~~A_k~~

$$\iff f(\theta) = \sum_{k=1}^{\infty} A_k\left(R^{2k} - R^{-2k}\right)\sin(2k\theta)$$

$$A_k\left(R^{2k} - R^{-2k}\right) = \frac{\langle f, \sin(2k\theta)\rangle_w}{\langle \sin(2k\theta), \sin(2k\theta)\rangle_w}$$

$$= \frac{4}{\pi}\int_0^{\pi/2} f(\theta)\sin(2k\theta)\,d\theta$$

$$\implies A_k = \frac{4}{\pi\left(R^{2k} - R^{-2k}\right)}\int_0^{\pi/2} f(\theta)\sin(2k\theta)\,d\theta$$

Step II We now solve the problem for u_2, try $u_2 = F(r)G(\theta)$, we get

$$F'' + \frac{1}{r}F' = \frac{\lambda}{r^2}F, \quad G'' + \lambda G = 0$$

$$F(1) = 0 = F(R), \quad G(0) = 0$$

This time, F solves a regular SL eigenvalue problem on interval $(1, R)$

The general solution for F is:

$$F(r) = \begin{cases} A\cos(\sqrt{-\lambda}\log r) + B\sin(\sqrt{-\lambda}\log r) & \text{if } \lambda < 0 \\ A\log r + B & \text{if } \lambda = 0 \\ A r^{\sqrt{\lambda}} + B r^{-\sqrt{\lambda}} & \text{if } \lambda > 0 \end{cases}$$

Example Show that for $\lambda \geq 0$, there are no non-trivial solutions $F(r)$ solving $F(1) = F(R) = 0$

Therefore, all eigenvalues λ will be negative,

$F(1) = 0 \implies A\cos(\sqrt{-\lambda}\cdot 0) + B\sin(\sqrt{-\lambda}\cdot 0) = 0$

$\implies A = 0$

$F(R) = 0 \implies B\sin(\sqrt{-\lambda}\log R) = 0$

$\iff \sqrt{-\lambda}\log R = k\pi$, $k \geq 1$ an integer

$$\iff \boxed{\sqrt{-\lambda} = \frac{k\pi}{\log R}}$$

So, the eigenfunctions are

$F_k(r) = \sin\left(\frac{k\pi}{\log R}\log r\right)$ and eigenvalues

$\lambda_k = -\left(\frac{k\pi}{\log r}\right)^2$. Solving the equation for G + B.C. $G(0) = 0$ gives

$$G_k(\theta) = B_k \sinh\left(\frac{k\pi}{\log R}\theta\right)$$

The general solution for u_2 is then

$$u_2 = \sum_{1=}^{\infty} B_k \sin\left(\frac{k\pi}{\log R}\log r\right)\sinh\left(\frac{k\pi}{\log R}\theta\right)$$

To find B_k, we fit the B.C. $u_2(r, \pi/2) = g(r)$

$$g(r) = \sum_{k=1}^{\infty} B_k \sinh\left(\frac{k\pi^2}{2\log R}\right)\sin\left(\frac{k\pi}{\log R}\log r\right)$$

Using the G.F.s for eigenfunctions $\{F_k\}$ we have:

$$B_k \sinh\left(\frac{k\pi^2}{2\log R}\right) = \frac{\langle g(r), F_k(r)\rangle_w}{\langle F_k(r), F_k(r)\rangle_w}$$

We can show that the SL form of the equation for F is:

$$r\frac{d}{dr}(rF') - \lambda F = 0 \implies w(r) = \frac{1}{r}$$

$$\langle g, F_k\rangle_w = \int_1^R \frac{1}{r} g(r)\sin\left(\frac{k\pi}{\log R}\log r\right) dr$$

$$\langle F_k, F_k\rangle_w = \int_1^R \frac{1}{r}\sin^2\left(\frac{k\pi}{\log R}\log r\right) dr$$

$$= \frac{\log R}{2} \quad \text{(after substituting } x = \log r\text{)}$$

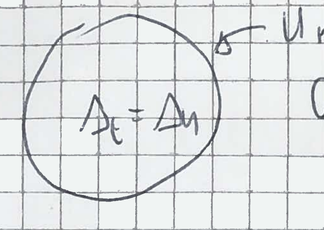
$$\implies B_k = \frac{1}{\sinh\left(\frac{k\pi^2}{2\log R}\right)} \cdot \frac{2}{\log R}\int_1^R \frac{1}{r} g(r)\sin\left(\frac{k\pi}{\log R}\log r\right) dr$$

The full solution is then $u = u_1 + u_2$ with u_1 and u_2 as above.

Q4 Extra Problems Sheet 4

28/02/19

Consider the heat equation $u_t = \Delta u$ in the unit disc in $\mathbb{R}^2$ with the boundary condition $u_r(1,\theta,t) = 0 \quad \forall t > 0 \quad \forall \theta \in [0, 2\pi]$



and with the initial condition $u(r,\theta,0) = f(r)$

By seeking an axisymmetric solution show that

$$u(r,t) = A_0 + \sum_{k=1}^{\infty} A_k J_0(\alpha_k r)e^{-\alpha_k^2 t}$$

where $\{\alpha_k\}$ and $\{A_k\}$ are constants that you should determine. In particular, find the $\{A_k\}$ in terms of $f(r)$

Solution: We seek separated solutions $u(r,t) = F(r)G(t)$

Inserting into the heat equation

$$FG' = (F'' + \tfrac{1}{r}F')G$$

$$\frac{F'' + \frac{1}{r}F'}{F} = \frac{G'}{G} = -\lambda, \quad \text{with } \lambda \text{ a constant}$$

Therefore $F'' + \frac{1}{r}F' + \lambda F = 0$

$$G' = -\lambda G$$

We require $F'(1) = 0$ as a result of $u_r(1,t) = 0$

We recognise F as the solution of the singular SL eigenvalue problem with Bessel's equation of index 0.

We start by showing that any eigenvalue λ with a non-trivial eigenfunction F must be non-negative i.e. $\lambda \geq 0$

First, rewrite the equation in SL form

$\frac{1}{r}\frac{d}{dr}\left(r\frac{dF}{dr}\right)+\lambda F=0$

Multiply by $w(r)F(r)=rF(r)$ and integrate over $(0,1)$

$$\int_0^1 \frac{d}{dr}\left(r\frac{dF}{dr}\right)F+\lambda r F^2 dr=0$$

Integrate by parts

$$\left[\cancel{r\frac{dF}{dr}F}\right]_0^1 - \int_0^1 r\left(\frac{dF}{dr}\right)^2 dr+\lambda\int_0^1 rF(r)^2 dr=0$$

$$\iff \lambda \underbrace{\int_0^1 rF(r)^2 dr}_{\geq 0}=\int_0^1 \underbrace{r\left(\frac{dF}{dr}\right)^2}_{\geq 0} dr$$

For non-trivial $F(r)$ we have that $\int_0^1 rF(r)^2 dr>0$

$$\Rightarrow \lambda=\frac{\int_0^1 r(F'(r))^2 dr}{\int_0^1 rF(r)^2 dr}$$

If $\lambda=0$, then $F''+\frac{1}{r}F'=0$
which has general solution
$F(r)=C+D\log r$
with C,D constants. Since we require F to
be bounded at $r=0$, we must take $D=0 \Rightarrow F(r)=C$
This solves $F'(1)=0$ for any constant C.
We therefore can define $F_0(r)=1$ as the 1^{st}
eigenfunction with eigenvalue $\lambda_0=0$.

Now, consider $\lambda>0$, the general solution of

$$F''+\frac{1}{r}F'+\lambda F=0$$

that is non-singular at $r=0$, is

$F(r)=C J_0(\sqrt{\lambda}r)$, where C is a constant

We determine λ with $F'(1)=0 \iff \sqrt{\lambda}J_0'(\sqrt{\lambda})=0$
$\iff J_0'(\sqrt{\lambda})=0$ (since we are considering $\lambda>0$)

Recall the RR

$$J_n'(x)=\frac{1}{2}(J_{n-1}(x)-J_{n+1}(x))$$

$$\Rightarrow J_0'(x)=\frac{1}{2}(J_{-1}(x)-J_1(x))$$

Recall that $J_{-n}(x)=(-1)^n J_n(x)\ \forall n\geq 0$ integers

Therefore $J_{-1}(x)=-J_1(x)$ and $J_0'(x)=-J_1(x)$

$\Rightarrow J_0'(\sqrt{\lambda})=0 \iff J_1(\sqrt{\lambda})=0$
which holds iff $\sqrt{\lambda}=j_{1k}$, $k\geq 1$, an integer,
where $\{j_{1k}\}$ denote the positive roots of $J_1(x)$.
Therefore, the positive eigenvalues are
$\lambda_k=(j_{1k})^2$, we can set the associated eigenfunctions

$$F_k(r)=J_0(j_{1k}r)\quad \text{for each } k\geq 1$$

The equation for G becomes $G'=-\lambda_k G$

$$G_k(t)=A_k e^{-\lambda_k t}$$

$$G_k(t)\begin{cases}A_0 e^{0}=A_0 & k=0\\ A_k e^{-j_{1k}^2 t} & k\geq 1\end{cases}$$

So, the general solution is then

$$u(r,t)=\sum_{k=0}^{\infty}F_k(r)G_k(t)=F_0(r)G_0(t)+\sum_{k=1}^{\infty}F_k(r)G_k(t)$$

$$=A_0+\sum_{k=1}^{\infty}A_k J_0(j_{1k}r)e^{-j_{1k}^2 t}$$

To find the $\{A_k\}$ we use the initial condition

$$f(r)=u(r,0)=A_0+\sum_{k=1}^{\infty}A_k J_0(j_{1k}r)$$

using the orthogonality of the eigenfunctions,
we get that

$$A_k = \frac{\langle f, F_k \rangle_w}{\langle F_k, F_k \rangle_w} \quad \forall k \geq 0$$

For $k=0$, $\langle f, F_0 \rangle_w = \int_0^1 r f(r) \cdot \underset{F_0(r)\,\forall r}{\overset{}{1}}\, dr = \int_0^1 r f(r)\, dr$

$$\langle F_0, F_0 \rangle_w = \int_0^1 r\, dr = \frac{1}{2}$$

$$\Rightarrow \quad A_0 = 2 \int_0^1 r f(r)\, dr$$

For $k \geq 1$,

$$\langle f, F_k \rangle_w = \int_0^1 r f(r) J_0(j_{1k} r)\, dr$$

$$\langle F_k, F_k \rangle_w = \int_0^1 r J_0(j_{1k} r)^2\, dr$$

Recall

$$\int_0^x s J_n(s)\, ds = \frac{x^2}{2}\left[J_n(x)^2 - J_{n-1}(x) J_{n+1}(x) \right]$$

Take $s = j_{1k} r$

$$\int_0^1 r J_0(j_{1k} r)^2\, dr = \frac{1}{j_{1k}^2} \int_0^{j_{1k}} s J_0(s)^2\, ds$$

$$= \frac{1}{2}\left[J_0(j_{1k})^2 - J_{-1}(j_{1k}) \overset{0}{J_1(j_{1k})} \right]$$

$$= \frac{J_0(j_{1k})^2}{2}$$

Therefore

$$A_k = \frac{2}{J_0(j_{1k})^2} \int_0^1 r f(r) J_0(j_{1k} r)\, dr \quad \forall k \geq 1$$

Chapter 5

5/03/19

Fourier and Laplace transforms

So far, we have considered P.D.E. on bdd domain. To solve problems on unbounded domains, it is no longer possible to use the series solution as for bounded domains. Instead, we will use transform methods.

① Motivation and definition of the fourier transform.

To derive the fourier transform (FT), consider the fourier series on the interval $[-L, L]$, where we eventually take $L \to \infty$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

writing
$$\cos\left(\frac{n\pi x}{L}\right) = \frac{1}{2}\left(e^{\frac{in\pi x}{L}} + e^{-\frac{in\pi x}{L}}\right)$$
$$\sin\left(\frac{n\pi x}{L}\right) = \frac{1}{2i}\left(e^{\frac{in\pi x}{L}} - e^{-\frac{in\pi x}{L}}\right)$$

$$f(x) = \frac{a_0}{2} + \frac{1}{2}\sum_{n=1}^{\infty} \left[\left(a_n + \frac{b_n}{i}\right) e^{\frac{in\pi x}{L}} + \left(a_n - \frac{b_n}{i}\right) e^{-\frac{in\pi x}{L}} \right]$$

This can be equivalently written as

$$f(x) = \frac{1}{2} \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}}$$

where
$$c_n = \begin{cases} a_{-n} - \frac{b_{-n}}{i} & n < 0 \\ a_0 & n = 0 \\ a_n + \frac{b_n}{i} & n > 0 \end{cases}$$

$$A_k = \frac{\langle f, F_k \rangle_w}{\langle F_k, F_k \rangle_w} \quad \forall k \geq 0$$

For $k=0$, $\langle f, F_0 \rangle_w = \int_0^1 r f(r) \cdot \underset{F_0(r)\ \forall r}{1}\, dr = \int_0^1 r f(r)\, dr$

$$\langle F_0, F_0 \rangle_w = \int_0^1 r\, dr = \frac{1}{2}$$

$$\Rightarrow \quad A_0 = 2\int_0^1 r f(r)\, dr$$

For $k \geq 1$,

$$\langle f, F_k \rangle_w = \int_0^1 r f(r) J_0(j_{1k} r)\, dr$$

$$\langle F_k, F_k \rangle_w = \int_0^1 r J_0(j_{1k} r)^2\, dr$$

Recall

$$\int_0^x s J_n(s)^2\, ds = \frac{x^2}{2}\left[J_n(x)^2 - J_{n-1}(x) J_{n+1}(x)\right]$$

Take $s = j_{1k} r$

$$\int_0^1 r J_0(j_{1k} r)^2\, dr = \frac{1}{j_{1k}^2}\int_0^{j_{1k}} s J_0(s)^2\, ds$$

$$= \frac{1}{2}\left[J_0(j_{1k})^2 - J_{-1}(j_{1k}) \underset{0}{J_1(j_{1k})}\right]$$

$$= \frac{J_0(j_{1k})^2}{2}$$

Therefore

$$A_k = \frac{2}{J_0(j_{1k})^2}\int_0^1 r f(r) J_0(j_{1k} r)\, dr \quad \forall k \geq 1$$

Chapter 5
Fourier and Laplace transforms

5/03/19

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writing
$$\cos\left(\frac{n\pi x}{L}\right) = \frac{1}{2}\left(e^{\frac{in\pi x}{L}} + e^{-\frac{in\pi x}{L}}\right)$$
$$\sin\left(\frac{n\pi x}{L}\right) = \frac{1}{2i}\left(e^{\frac{in\pi x}{L}} - e^{-\frac{in\pi x}{L}}\right)$$

$$f(x) = \frac{a_0}{2} + \frac{1}{2}\sum_{n=1}^{\infty}\left[\left(a_n + \frac{b_n}{i}\right)e^{\frac{in\pi x}{L}} + \left(a_n - \frac{b_n}{i}\right)e^{-\frac{in\pi x}{L}}\right]$$

This can be equivalently written as

$$f(x) = \frac{1}{2}\sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}}$$

where
$$c_n = \begin{cases} a_{-n} - \frac{b_{-n}}{i} & n < 0 \\ a_0 & n = 0 \\ a_n + \frac{b_n}{i} & n > 0 \end{cases}$$

Recall now:

$$\begin{Bmatrix} a_n \\ b_n \end{Bmatrix} = \frac{1}{L}\int_{-L}^{L} f(x) \begin{Bmatrix} \cos\left(\frac{n\pi x}{L}\right) \\ \sin\left(\frac{n\pi x}{L}\right) \end{Bmatrix} dx$$

This allows us to simplify c_n

(if $n>0$) $c_n = a_n + \frac{b_n}{i} = \frac{1}{L}\int_{-L}^{L} f(x)\left[\cos\left(\frac{n\pi x}{L}\right) + \frac{1}{i}\sin\left(\frac{n\pi x}{L}\right)\right] dx$

$\cos\left(\frac{n\pi x}{L}\right) + \frac{1}{i}\sin\left(\frac{n\pi x}{L}\right) = \cos\left(\frac{n\pi x}{L}\right) - i\sin\left(\frac{n\pi x}{L}\right)$ $e^{\frac{in\pi x}{L}}$

Therefore for $n>0$

$$c_n = \frac{1}{L}\int_{-L}^{L} f(x) e^{-\frac{in\pi x}{L}} dx$$

Exercise: check that for $n\le 0$, the same formula holds true i.e:

$$c_n = \frac{1}{L}\int_{-L}^{L} f(x) e^{-\frac{in\pi x}{L}} dx \quad \forall n \in \mathbb{Z}$$

If we define the points $k_n = \frac{n\pi}{L}$, $\forall n \in \mathbb{Z}$ these form a partition of the real line $\mathbb{R}$, with spacing $\delta_n = k_{n+1} - k_n = \pi/L$, which tends to zero as $L \to \infty$. We can re-write previous eqn as:

$$f(x) = \frac{1}{2L}\sum_{n=-\infty}^{\infty}\left(\int_{-L}^{L} f(s) e^{-ik_n s} ds\right) e^{ik_n x}$$

multiply and divide by δ_n:

$$f(x) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\left(\int_{-L}^{L} f(s) e^{-ik_n s} ds\right) e^{ik_n s}\,\delta_n$$

We identify here a Riemann sum which in the limit $L\to\infty$, i.e. $\delta_n \to 0$, we expect to converge to the following expression

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(s) e^{-iks} ds\right) e^{ikx} dk$$

This is called the Fourier integral formula.

The inner integral in the fourier integral is a function of k. We define the fourier transform.

$$\boxed{\vec{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) e^{-ikx} dx}$$

Therefore, the fourier integral formula can be written as:

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \vec{f}(k) e^{ikx} dk}$$

This is sometimes called the fourier inversion formula.

Here, $f(x)$ and $\vec{f}(k)$ are generally complex-valued functions defined on all of $\mathbb{R}$

Remark: The derivation of the FT, we have seen is heuristic. Without proof, we mention that a sufficient condition for $\vec{f}(k)$ to exist $\forall k$ is

$$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$$

Key idea of the proof:

$$|\vec{f}(k)| \le \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} |f(x)||e^{-ikx}|\, dx$$
$$\le \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$$

② Calculating FT:

Ex 1] Let $f(x) = \begin{cases} 1-|x|, & x\in[-1,1] \\ 0, & \text{otherwise} \end{cases}$

We calculate $\vec{f}(k)$ directly:

$$\vec{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} f(x)e^{-ikx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-1}^{1} e^{-ikx}\,dx + \int_{-1}^{0} x e^{-ikx}\,dx - \int_{0}^{1} x e^{-ikx}\,dx\right]$$

We leave it as an exercise to check

$$\int_{-1}^{1} e^{-ikx} = \frac{1}{-ik}\left(e^{-ik} - e^{ik}\right)$$

$$\int_{-1}^{0} x e^{-ikx} = -\frac{e^{ik}}{ik} - \frac{1}{(-ik)^2}\left[e^{-ik} - 1\right]$$

Therefore, after simplifying,

$$\vec{f}(k) = \frac{1}{\sqrt{2\pi}}\frac{1}{(-ik)^2}\left(1 - e^{-ik} + 1 - e^{ik}\right)$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{(-ik)^2}\right)\left(2 - e^{-ik} - e^{ik}\right)$$

$$= \frac{1}{\sqrt{2\pi}}\frac{-1}{-k^2}\left(e^{ik/2} - e^{-ik/2}\right)^2$$

$$= \frac{1}{\sqrt{2\pi}}\frac{\sin^2(k/2)}{(k/2)^2}$$

[Ex 2] let $a>0$ parameter. Let $f(x) = e^{-x^2/2a^2}$

We show that $\vec{f}(k) = a e^{-a^2k^2/2}$

$$\vec{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2a^2} e^{-ikx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2a^2 - ikx}\,dx.$$

We can complete the square:

$$\frac{x^2}{2a^2} + ikx = \left(\frac{x}{\sqrt{2}a} + \frac{iak}{\sqrt{2}}\right)^2 + \frac{a^2k^2}{2}$$

$$\vec{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left(\frac{x}{\sqrt{2}a} + \frac{iak}{\sqrt{2}}\right)^2} e^{-\frac{a^2k^2}{2}}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{a^2k^2}{2}}\int_{-\infty}^{\infty} e^{-\left(\frac{x}{\sqrt{2}a} + \frac{iak}{\sqrt{2}}\right)^2}\,dx$$

Introduce $z = \frac{x}{\sqrt{2}a} + \frac{iak}{\sqrt{2}}$

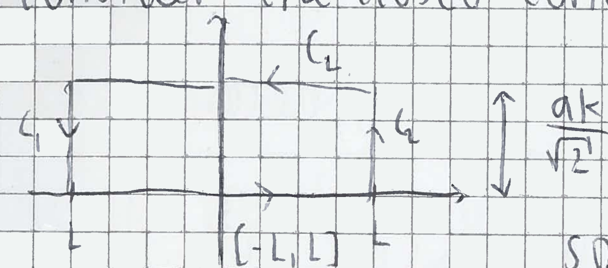
$$\Rightarrow \int_{-\infty}^{\infty} e^{-\left(\frac{x}{a\sqrt{2}} + \frac{iak}{\sqrt{2}}\right)^2}\,dx = \sqrt{2}\,a\int_{C_a} e^{-z^2}\,dz$$

where $C_a = \{z \in \mathbb{C},\ \operatorname{Im}(z) = \frac{iak}{\sqrt{2}}\}$

We will use contour integration to show that

$$\int_{C_a} e^{-z^2}\,dz = \sqrt{\pi}$$

Consider the closed contour $\tilde{C}$ given as follows:



The integrand e^{-z^2} is holomorphic on the whole $\mathbb{C}$ so, Cauchy's Theorem tells us that $\oint_{\tilde{C}} e^{-z^2}\,dz = 0$

$$\Longleftrightarrow \int_{-L}^{L} e^{-z^2}\,dz + \int_{C_2} e^{-z^2}\,dz + \int_{C_L} e^{-z^2}\,dz + \int_{C_1} e^{-z^2}\,dz = 0$$

for $z \in C_1 \cup C_2$

$$e^{-z^2} = e^{-(\pm L + iy)^2} \qquad y \in \left[0, \frac{ak}{\sqrt{2}}\right]$$

$$= e^{-L^2 \mp 2iLy + y^2} = O(e^{-L^2})$$

$$\lim_{L\to\infty}\int_{C_1} e^{-z^2}\,dz = \lim_{L\to\infty}\int_{C_2} e^{-z^2}\,dz = 0$$

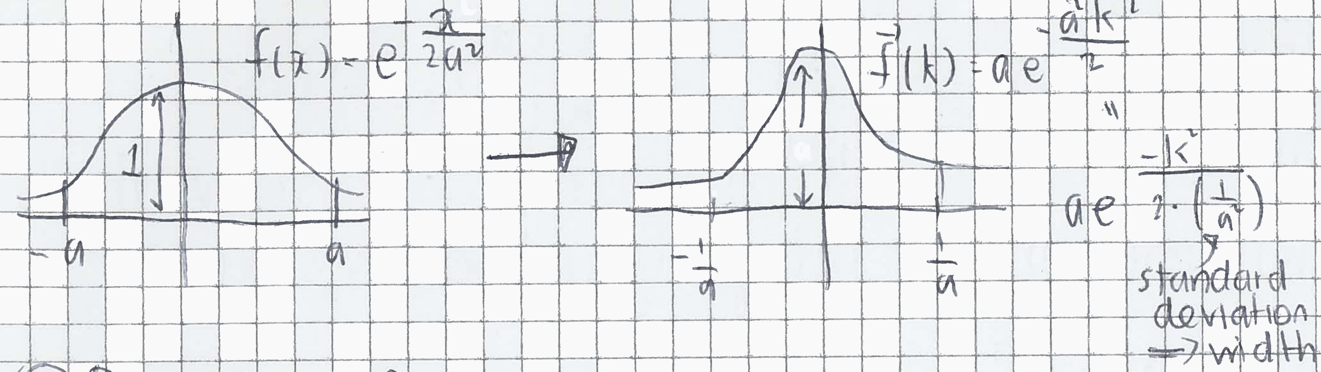
$\Rightarrow$ In the limit $L \to \infty$

$$\lim_{L\to\infty}\int_{C_L} e^{-z^2}\,dz = -\int_{-\infty}^{\infty} e^{-x^2}\,dx = -\sqrt{\pi}$$

Clearly, $\int_{C_a} e^{-z^2}\,dz = -\lim_{L\to\infty}\int_{C_L} e^{-z^2}\,dz = \sqrt{\pi}$

$$\Rightarrow \hat{f}(k) = \frac{1}{\sqrt{2\pi}} e^{-\frac{a^2k^2}{2}} \sqrt{2}\, a \sqrt{\pi} = a e^{-\frac{a^2k^2}{2}}$$

Remarks: This example shows that the FT of a Gaussian of a width "a" is a Gaussian of width $1/a$



③ Properties of FT:

• The FT of derivatives

A function f and its FT $\hat{f}$ are uniquely determined via the fourier integral formula.

A consequence of this is that if we can find some function $g(k)$ s.t

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(k) e^{ikx}\, dk$$

then we deduce $g(k) = \hat{f}(k)$

This allows us to determine the FT of the derivative $f'(x)$ as follows

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx}\, dk$$

Assuming we can interchange differentiation/integration

$$f'(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \hat{f}(k)\, ik\, e^{ikx}\, dk$$

$\Rightarrow g(k) = ik\hat{f}(k)$ must be the FT of $f'(x)$

so, $\widehat{f'}(k) = ik\hat{f}(k)$

More generally $\widehat{f^{(n)}}(k) = (ik)^n \hat{f}(k)$

This formula will be very useful for solving P.D.E.

• The derivative of the transform:

$$\hat{f}'(k) = \frac{d\hat{f}}{dk} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)(-ix) e^{-ikx}\, dx$$

$$\Rightarrow \widehat{(-ix)f(x)} = \hat{f}'(k)$$

more generally,

$$\widehat{(-ix)^n f(x)} = \hat{f}^{(n)}(k)$$

This is often useful for computing transforms of functions of the form $x^n f(x)$.

ex] Compute $\widehat{x e^{-x^2/2}}$

Recall: Transform of the derivative 7/03/19

$$\widehat{f'}(k) = ik\hat{f}(k)$$

Derivative of the transform ②

$$\hat{f}'(k) = \widehat{-ixf(x)}(k)$$

③ Convolution Theorem:

The convolution of two functions $f(x)$ and $g(x)$ that are defined on $\mathbb{R}$ is denoted by $(f*g)(x)$ and is defined by

$$(f*g)(x) = \int_{-\infty}^{\infty} f(x-y) g(y)\, dy$$

Key properties

a) Symmetry $(f*g)(x) = (g*f)(x)$

Proof:

$(f*g)(x) = \int_{-\infty}^{\infty} f(x-y)\, g(y)\, dy$

let $z = x - y \iff y = x - z$

$(f*g)(x) = \int_{-\infty}^{\infty} f(z)\, g(x-z)\, dz$

Therefore $(f*g)(x) = (g*f)(x)$

b) If $f \in L^1(\mathbb{R})$ and $g \in L^1(\mathbb{R})$
then $(f*g) \in L^1(\mathbb{R})$

Recall $f \in L(\mathbb{R})$ means $\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$

Proof

$\int_{-\infty}^{\infty} |(f*g)(x)|\, dx = \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} f(x-y)\, g(y)\, dy \right| dx$

Recall the triangle inequality for integrals

$\left| \int_{-\infty}^{\infty} f(x-y)\, g(y)\, dy \right| < \int_{-\infty}^{\infty} |f(x-y)||g(y)|\, dy$

$\Rightarrow \int_{-\infty}^{\infty} |(f*g)(x)|\, dx \leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x-y)||g(y)|\, dy\, dx$

Change order of integration:

$\Rightarrow \int_{-\infty}^{\infty} |(f*g)(x)|\, dx \leq \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |f(x-y)|\, dx \right) |g(y)|\, dy$

Setting $z = x - y$

$\int_{-\infty}^{\infty} |f(x-y)|\, dx = \int_{-\infty}^{\infty} |f(z)|\, dz$

$\Rightarrow \int_{-\infty}^{\infty} |(f*g)(x)|\, dx \leq \left(\int_{-\infty}^{\infty} |f(z)|\, dz \right) \left(\int_{-\infty}^{\infty} |g(y)|\, dy \right)$

Hence, if f and g are in $L^1(\mathbb{R})$ i.e.

$\int_{-\infty}^{\infty} |f(z)|\, dz$ and $\int_{-\infty}^{\infty} |g(y)|\, dy$ are finite,
then $f*g \in L^1(\mathbb{R})$

Recall that functions in $L^1(\mathbb{R})$ have a well-defined FT, hence $f*g$ has a FT whenever $f, g \in L^1(\mathbb{R})$

Convolution Theorem

$$\widehat{f*g}(k) = \sqrt{2\pi}\, \hat{f}(k)\, \hat{g}(k)$$

Proof:

$\widehat{f*g}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (f*g)(x)\, e^{-ikx}\, dx$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x-y)\, g(y)\, e^{-ikx}\, dy\, dx$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x-y)\, e^{-ikx}\, dx \right) g(y)\, dy$

Multiply/divide by e^{iky}

$\widehat{f*g}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x-y)\, e^{-ik(x-y)}\, dx \right) g(y)\, e^{-iky}\, dy$

Set $z = x - y$

$\int_{-\infty}^{\infty} f(x-y)\, e^{-ik(x-y)}\, dx = \int_{-\infty}^{\infty} f(z)\, e^{-ikz}\, dz = \sqrt{2\pi}\, \hat{f}(k)$

$\Rightarrow \widehat{f*g}(k) = \hat{f}(k) \underbrace{\int_{-\infty}^{\infty} g(y)\, e^{-iky}\, dy}_{= \sqrt{2\pi}\, \hat{g}(k)}$

$= \sqrt{2\pi}\, \hat{f}(k)\, \hat{g}(k)$

④ Parseval's Theorem:

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\hat{f}(k)|^2 dk$$

Proof: Start by combining the Fourier Inversion formula with the convolution Theorem.

$$(f*g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f*g}(k) e^{ikx} dk$$

$$= \int_{-\infty}^{\infty} \hat{f}(k)\hat{g}(k) e^{ikx} dk$$

We now choose $g(x) = \overline{f(-x)}$

The LHS can be evaluated as follows

$$(f*g)(x) = \int_{-\infty}^{\infty} f(x-y) g(y) dy$$

$$= \int_{-\infty}^{\infty} f(x-y) \overline{f(-y)} dy$$

For $x=0$

$$(f*g)(0) = \int_{-\infty}^{\infty} f(-y)\overline{f(-y)} dy = \int_{-\infty}^{\infty} |f(-y)|^2 dy$$

$$= \int_{-\infty}^{\infty} |f(x)|^2 dx$$

The RHS at $x=0$ becomes

$$\int_{-\infty}^{\infty} \hat{f}(k)\hat{g}(k) \cdot 1 \, dk$$

Now $\hat{g}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(-x)}\, e^{-ikx} dx$

$$= \frac{1}{\sqrt{2\pi}} \overline{\int_{-\infty}^{\infty} f(-x) \overline{e^{-ikx}}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \overline{\int_{-\infty}^{\infty} f(-x) e^{ikx} dx}$$

$$\int_{-\infty}^{\infty} f(-x) e^{ikx} dx \overset{z=-x}{=} \int_{-\infty}^{\infty} f(z) e^{-ikz} dz = \sqrt{2\pi}\,\hat{f}(k)$$

$\Rightarrow \hat{g}(k) = \overline{\hat{f}(k)}$

Therefore the RHS at $x=0$ becomes

$$\int_{-\infty}^{\infty} \hat{f}(k)\overline{\hat{f}(k)} dk = \int_{-\infty}^{\infty} |\hat{f}(k)|^2 dk$$

matching LHS and RHS

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\hat{f}(k)|^2 dk \quad \blacksquare$$

Example Evaluate $\int_{-\infty}^{\infty} \frac{\sin^2 k}{k^2} dk$ using FT

Answer: On HW5, check that

$$f(x) = \begin{cases} 1 & \text{if } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

has FT

$$\hat{f}(k) = \sqrt{\frac{2}{\pi}} \frac{\sin k}{k}$$

Then Parseval's Thm gives

$$\int_{-\infty}^{\infty} \frac{\sin^2 k}{k^2} dk = \int_{-\infty}^{\infty} \left|\sqrt{\frac{\pi}{2}} \hat{f}(k)\right|^2 dk$$

$$= \frac{\pi}{2} \int_{-\infty}^{\infty} |\hat{f}(k)|^2 dk$$

$$= \frac{\pi}{2} \int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{\pi}{2} \cdot 2 = \pi$$

Applications of FT to PDEs 12/03/19

Ex 1] Heat equation on the Real axis

Consider the heat equation

$u_t = u_{xx} \quad \forall x \in \mathbb{R}, t > 0$

with the initial conditions $u(x,0) = f(x)$

where $f(x)$ is a given function.

Let us define $\hat{u}(k,t)$ as the FT of $u(x,t)$ w.r.t. x for fixed t:

$$\hat{u}(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t) e^{-ikx} \, dx$$

The idea is to find the equation that is obtained by applying the FT to all the terms in the PDE. Therefore, we seek:

$$\hat{u}_t = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_t(x,t) e^{-ikx} \, dx = \frac{\partial}{\partial t} \hat{u}(k,t)$$

Applying the result on transforms of derivatives

$$\widehat{u_{xx}} = (ik)^2 \hat{u}(k,t)$$

Therefore, the heat equation becomes

$$\frac{\partial}{\partial t} \hat{u}(k,t) = -k^2 \hat{u}(k,t)$$

This is an ODE w.r.t. t, with k acting as a parameter. The solution is then of the form:

$$\hat{u}(k,t) = A(k) e^{-k^2 t}$$

Where $A(\,)$ is the "constant" of the ODE, which can depend on the parameter k

To find $A(k)$, we use the initial conditions:

$$u(x,0) = f(x) \iff \hat{u}(k,0) = \hat{f}(k)$$
$$\iff A(k) = \hat{f}(k)$$

$\Rightarrow$ The solution is $\hat{u}(k,t) = \hat{f}(k) e^{-k^2 t}$

We now seek a function whose transform is $e^{-k^2 t}$, so that we can write $u(x,t)$ as the convolution of that function with $f(x)$.

Recall: $g(x) = e^{-x^2/2a^2} \Rightarrow \hat{g}(k) = a e^{-a^2 k^2/2}$

Using this, let us set $a^2 = 2t$, and let us define

$$G(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}$$

Therefore

$$\hat{G}(k,t) = \frac{1}{\sqrt{4\pi t}} \sqrt{2t}\, e^{-k^2 t} = \frac{1}{\sqrt{2\pi}} e^{-k^2 t}$$

Therefore, the solution $\hat{u}$ can be written as

$$\hat{u}(k,t) = \sqrt{2\pi}\, \hat{f}(k) \frac{1}{\sqrt{2\pi}} e^{-k^2 t} = \sqrt{2\pi}\, \hat{f}(k) \hat{G}(k,t)$$

Recall the Convolution Theorem

$$\widehat{f * g}(k,t) = \sqrt{2\pi}\, \hat{f}(k) \hat{g}(k)$$

Therefore $\hat{u}(k,t) = \widehat{f * G}(k,t)$

by the fourier integral formula

$$u(x,t) = (f * G)(x,t)$$
$$= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} f(x-y) e^{-y^2/4t} \, dy$$

so $u(x,t) = \dfrac{1}{\sqrt{4\pi t}} \displaystyle\int_{-\infty}^{\infty} f(x-y) e^{-y^2/4t} \, dy$

Ex 2] Wave equation on the real axis

Consider $u_{tt} = u_{xx} \quad \forall x \in \mathbb{R}, t > 0$

with initial conditions

$u(x,0) = f(x) \quad u_t(x,0) = g(x)$

Where $f(x)$ and $g(x)$ are given functions
For simplicity, we consider here $g(x)=0$
We define:

$$\hat{u}(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t) e^{-ikx} dx$$

The transform of the derivatives in the PDE are then

$$\hat{u}_{tt} = \frac{\partial^2}{\partial t^2} \hat{u}(k,t)$$
$$\hat{u}_{xx} = -k^2 \hat{u}(k,t)$$

Therefore, the equation becomes

$$\frac{\partial^2}{\partial t^2} \hat{u}(k,t) = -k^2 \hat{u}(k,t)$$

Again, this is an ODE w.r.t. t, 'k' a parameter
The general form of the solution is:

$$\hat{u}(k,t) = A(k)\cos(kt) + B(k)\sin(kt)$$

we determine $A(k)$ and $B(k)$ using the B.Cs

$$u(x,0) = f(x) \iff \hat{u}(k,0) = \hat{f}(k)$$
$$A(k) = \hat{f}(k)$$

So $\hat{u}(k,t) = \hat{f}(k)\cos(kt) + B(k)\sin(kt)$

$$\hat{u}_t(k,t) = -k\hat{f}(k)\sin(kt) + kB(k)\cos(kt)$$

$$u_t(x,0) = 0 \iff \hat{u}_t(k,0) = 0$$
$$\iff kB(k) = 0$$

This is solved by $B(k) = 0 \quad \forall k \in \mathbb{R}$
The solution is then:

$$\hat{u}(k,t) = \hat{f}(k)\cos(kt)$$

It won't be possible to write down $u(x,t)$ as the convolution of two functions

We use the following trick

$$\hat{u}(k,t) = \frac{\partial}{\partial t}\left[\hat{f}(k)\frac{\sin(kt)}{k}\right]$$

If we define the function

$$w(x,t) = \begin{cases} 1 & \text{for } |x| \le t \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow \hat{w}(k,t) = \sqrt{\frac{2}{\pi}} \frac{\sin(kt)}{k}$$

$$\Rightarrow \hat{u}(k,t) = \sqrt{\frac{\pi}{2}} \frac{\partial}{\partial t}\left[\hat{f}(k)\hat{w}(k,t)\right]$$

Multiply and divide by $\sqrt{2\pi}$

$$\hat{u}(k,t) = \frac{1}{2}\frac{\partial}{\partial t}\left[\sqrt{2\pi}\hat{f}(k)\hat{w}(k,t)\right]$$

By the Convolution Theorem:

$$\hat{u}(k,t) = \frac{1}{2}\frac{\partial}{\partial t}\left[\widehat{(f*w)}(k,t)\right]$$
$$= \frac{1}{2}\left[\widehat{\frac{\partial}{\partial t}(f*w)}\right]$$

$\Rightarrow$ By the Fourier integral formula

$$u(x,t) = \frac{1}{2}\frac{\partial}{\partial t}(f*w)$$
$$= \frac{1}{2}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} f(x-y)\, w(y,t)\, dy$$

$$\Rightarrow u(x,t) = \frac{1}{2}\frac{\partial}{\partial t}\int_{-t}^{t} f(x-y)\, dy$$

Using Leibniz's formula we get

$$u(x,t) = \frac{1}{2}\left[f(x-t) + f(x+t)\right]$$

This is (part of) d'Alembert's formula

Ex 3] Laplace's equation in the half plane

Consider $\Delta u = 0$ in the upper half plane

$\{(x,y)\in\mathbb{R}^2, y\geq 0\}$ with B.Cs $u(x,0)=f(x)$
and $u(x,y)\to 0$ as $y\to\infty$

We set $\hat{u}(k,y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,y)e^{-ikx}\,dx$

Therefore

$\hat{u}_{yy}(k,y) = \frac{\partial^2}{\partial y^2}\hat{u}(k,y)$

and $\hat{u}_{x} = -k^2\hat{u}(k,y)$

since $\Delta u = 0$ $\quad u_{xx}+u_{yy}=0$

$-k^2\hat{u}(k,y) + \frac{\partial^2}{\partial y^2}\hat{u}(k,y) = 0$

This is an ODE w.r.t y, with parameter k

We can write the solution as

$\hat{u}(k,y) = A(k)e^{-|k|y} + B(k)e^{|k|y}$

(This is w.l.o.g. equivalent to writing the solution in terms of $\cosh(ky)$ and $\sinh(ky)$ but is easier to handle)

Recall $\lim_{y\to\infty} u(x,y) = 0$

So, we shall require that $\hat{u}(k,y)\to 0$ as $y\to\infty$

Since $\lim_{y\to\infty} e^{|k|y} = \begin{cases} 1 & \text{if } k=0 \\ \infty & \text{if } k\neq 0\end{cases}$

So, we require that $B(k)=0$ $\forall k$ to ensure that $\hat{u}(k,y)\to 0$ as $y\to\infty$

To find $A(k)$, we use $u(x,0)=f(x)$.

$\iff \hat{u}(k,0) = A(k) = \hat{f}(k)$

Therefore, the solution is $\hat{u}(k,y) = \hat{f}(k)e^{-|k|y}$

let $G(x,y) = \sqrt{\frac{2}{\pi}}\frac{y}{x^2+y^2}$

Then $\hat{G}(k,y) = e^{-|k|y}$

So, we can write

$\hat{u}(k,y) = \hat{f}(k)\hat{G}(k,y) = \frac{1}{\sqrt{2\pi}}\sqrt{2\pi}\hat{f}(k)\hat{G}(k)$

$= \frac{1}{\sqrt{2\pi}}\widehat{(f * G)}(k,y)$

Using the fourier integral formula, this shows

$u(x,y) = \frac{1}{\sqrt{2\pi}}(f*G)(x,y)$

$= \frac{1}{\sqrt{2\pi}}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} f(x-t)\frac{y}{t^2+y^2}\,dt$

$$\boxed{u(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(x-t)}{t^2+y^2}\,dt}$$

Convolution of $f(x)$ and $g(x)$

$(f*g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t)\,dt$

$g(x) = G(x,y)$

$(f*G)(x,y) = \int_{-\infty}^{\infty} f(x-t)G(t,y)\,dt$

Using the FT to prove energy identities for the solution of PDE

We aim to prove some basic properties of the solutions of PDE we have studied.

Consider the wave equation:

$u_{tt} = u_{xx}$

with initial conditions $u(x,0)=f(x)$
$u_t(x,0)=0$

Then the energy

$E = \frac{1}{2}\left[\int_{-\infty}^{\infty}|u_t|^2dx + \int_{-\infty}^{\infty}|u_x|^2dx\right]$ is conserved w.r.t. t.

Recall solution $\hat{u}(k,t) = \hat{f}(k)\cos(kt)$
Then, using Parseval's Theorem,

$$\int_{-\infty}^{\infty} |u_t|^2 dx - \int_{-\infty}^{\infty} |u_t| dt = \int_{-\infty}^{\infty} |\hat{f}(k)k(\sin(kt))|^2 dk$$

$$\Rightarrow \int_{-\infty}^{\infty} |u_t|^2 dx \quad \int_{-\infty}^{\infty} |\hat{f}(k)|^2 k^2 \sin^2(kt)\, dt$$

Again by Parseval's Theorem

$$\int_{-\infty}^{\infty} |u_x|^2 dx = \int_{-\infty}^{\infty} |\hat{u}_x|^2 dk = \int_{-\infty}^{\infty} |(ik)\hat{u}(k,t)|^2 dk$$
$$= \int_{-\infty}^{\infty} k^2 |\hat{f}(k)|^2 \cos^2(kt)\, dk$$

$$\Rightarrow E = \frac{1}{2}\int_{-\infty}^{\infty} k^2 |\hat{f}(k)|^2 dk$$

which is independent of t.

Ex] For the heat equation example, show that

$$\frac{d}{dt}\int_{-\infty}^{\infty} |u(x,t)|^2 dx = -2\int_{-\infty}^{\infty} |u_x(x,t)|^2 dx$$

Conclude that $\int_{-\infty}^{\infty} |u(x,t)|^2 dx$ is non-increasing.

The Laplace Transform

14/03/19

The Laplace Transform (LT) is defined for functions $f(t)$ that are defined for the half axis $t \geq 0$ as

$$\mathcal{L}[f](s) = \bar{f}(s) = \int_0^\infty f(t)e^{-st} dt$$

which is defined for all $s \in \mathbb{C}$ for which the integral converges.

Remark We will use the notation $\mathcal{L}[f]$ as much as possible. Sometimes this notation can be cumbersome, in which case the notation $\bar{f}(s)$ will be employed.
DO NOT CONFUSE THIS WITH COMPLEX CONJUGATES.

The Laplace transform exists for a wide class of functions (wider than just $L^1(\mathbb{R})$ which we saw in the context of the FT). For example, if f grows (at most) exponentially as $t \to \infty$
ie $\exists\ C, \beta \in \mathbb{R}_{\geq 0}$ such that $|f(t)| \leq Ce^{\beta t}$
Then $|\mathcal{L}[f](s)| \leq C\int_0^\infty e^{-(s-\beta)t} dt < \infty$ if $Re(s) > \beta$

Examples

$$\mathcal{L}[1](s) = \int_0^\infty e^{-st} dt = \frac{1}{s} \qquad (1)$$

which is well defined for all $s \in \mathbb{C}\setminus\{0\}$

$$\mathcal{L}[t^n](s) = \int_0^\infty t^n e^{-st} dt \qquad (2)$$

If we set $u = st$, then

$$\Rightarrow \mathcal{L}[f](s) = \frac{1}{s}\int_0^\infty u^n e^{-u} du = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}}$$

(if n is an integer)

3) $\mathcal{L}[\cos \omega t]$ and $\mathcal{L}[\sin \omega t]$, $\omega \in \mathbb{R}$ a parameter

Since $e^{i\omega t} = \cos \omega t + i\sin \omega t$

$$\mathcal{L}[e^{i\omega t}](s) = \int_0^\infty e^{i\omega t} e^{-st}\,dt = \int_0^\infty e^{-(s-i\omega)t}\,dt$$
$$= \frac{1}{s-i\omega} = \frac{s+i\omega}{s^2+\omega^2}$$

By taking real and imaginary parts

$$\mathcal{L}[\cos\omega t](s) = \operatorname{Re}[\mathcal{L}[e^{i\omega t}](s)] = \frac{s}{s^2+\omega^2}$$

$$\mathcal{L}[\sin\omega t](s) = \operatorname{Im}[\mathcal{L}[e^{i\omega t}](s)] = \frac{\omega}{s^2+\omega^2}$$

4) $\mathcal{L}[e^{\alpha t}](s) = \int_0^\infty e^{\alpha t} e^{-st}\,dt \qquad (\alpha \in \mathbb{R})$

$$= \int_0^\infty e^{-(s-\alpha)t}\,dt = \frac{1}{s-\alpha} \quad \text{for all } s \in \mathbb{C} \text{ s.t. } \operatorname{Re}(s) > \alpha$$

Properties of the LT

1) Shift results

a) For α a constant

$$\mathcal{L}[e^{-\alpha t} f(t)](s) = \mathcal{L}[f](s+\alpha)$$

Proof.

$$\mathcal{L}[e^{-\alpha t} f(t)](s) = \int_0^\infty e^{-\alpha t} f(t) e^{-st}\,dt$$
$$= \int_0^\infty f(t) e^{-(s+\alpha)t}\,dt$$
$$= \mathcal{L}[f](s+\alpha)$$

b) Typically, for a function $f(t)$ defined for $t \geq 0$ we shall consider f as being extended by zero for all $t < 0$. Using the convention

$$\mathcal{L}[f(t-\alpha)](s) = e^{-\alpha s}\mathcal{L}[f](s) \qquad (\alpha \geq 0)$$

Proof

$$\mathcal{L}[f(t-\alpha)](s) = \int_0^\infty f(t-\alpha) e^{-st}\,dt$$

For $\alpha \geq 0$, $f(t-\alpha)$ is zero for $t \in [0,\alpha)$

So $\mathcal{L}[f(t-\alpha)](s) = \int_\alpha^\infty f(t-\alpha) e^{-st}\,dt$

Setting $z = t-\alpha$, after substitution

$$\mathcal{L}[f(t-\alpha)](s) = \int_0^\infty f(z) e^{-s(z+\alpha)}\,dz$$
$$= e^{-\alpha s}\int_0^\infty f(z) e^{-sz}\,dz$$
$$= e^{-\alpha s}\mathcal{L}[f](s)$$

2) Derivatives of the LT

$$\mathcal{L}[t^n f(t)](s) = \left(-\frac{d}{ds}\right)^n \mathcal{L}[f](s) \quad n \geq 0 \text{ integer}$$

Proof

$$\mathcal{L}[t^n f(t)](s) = \int_0^\infty t^n f(t) e^{-st}\,dt$$

Note that

$$t^n e^{-st} = \left(-\frac{d}{ds}\right)^n e^{-st}$$

Therefore

$$\mathcal{L}[t^n f(t)](s) = \int_0^\infty \left(-\frac{d}{ds}\right)^n [f(t) e^{-st}]\,dt$$

Assuming we can interchange differentiation and integration,

$$\mathcal{L}[t^n f(t)](s) = \left(-\frac{d}{ds}\right)^n \int_0^\infty f(t) e^{-st}\,dt$$
$$= \left(-\frac{d}{ds}\right)^n \mathcal{L}[f](s)$$

Example: $\mathcal{L}[t^n](s) = \frac{n!}{s^{n+1}} = \left(-\frac{d}{ds}\right)^n \frac{1}{s} = \left(-\frac{d}{ds}\right)^n \mathcal{L}[1](s)$

3) Laplace Transform of derivatives

$\mathcal{L}[f'(t)](s) = s\mathcal{L}[f](s) - f(0_+)$

where $f(0_+)$ is the $\lim_{t \to 0^+} f(t)$ the right-limit of f at zero

This is valid for differentiable f, such that $f(t)e^{-st} \to 0$ as $t \to \infty$.

Proof:

$$\mathcal{L}[f'](s) = \int_0^\infty f'(t)e^{-st}\,dt$$

$$= [f(t)e^{-st}]_0^\infty - \int_0^\infty f(t)\frac{\partial}{\partial t}e^{-st}\,dt$$

$$\mathcal{L}[f'](s) = -f(0_+) + s\int_0^\infty f(t)\,e^{-st}\,dt$$

$$= s\mathcal{L}[f](s) - f(0_+)$$

Example: Show that $\mathcal{L}[f''](s) = s^2\mathcal{L}[f](s) - sf(0_+) - f'(0_+)$

$$\mathcal{L}[f''](s) = s\mathcal{L}[f'](s) - f'(0_+)$$

$$= s^2\mathcal{L}[f](s) - sf(0_+) - f'(0_+)$$

$$(= s[s\mathcal{L}[f](s) - f(0_+)] - f'(0_+))$$

Using an induction argument we see

$$\mathcal{L}[f^{(n)}](s) = s^n\mathcal{L}[f](s) - s^{n-1}f(0_+) - s^{n-2}f'(0_+) - \dots - f^{(n-1)}(0_+)$$

19/3/19.

4) Convolution Theorem for the LT.

Let $f(t)$ and $g(t)$ be two functions defined for $t \geq 0$ and consider their zero-extension for $t < 0$

$$(f*g)(t) = \int_{-\infty}^{\infty} f(t-y)g(y)\, dy.$$

If $y>t$, f vanish.

$-\infty \rightarrow$ Can start from zero

Since $g(y) = 0$ for $y<0$, and $f(t-y) = 0$ for $t<y$, we can simplify

$$(f*g)(t) = \int_0^t f(t-y)g(y)\, dy.$$

Convolution Theorem for LT.

$$\mathcal{L}[f*g](s) = \mathcal{L}[f](s)\,\mathcal{L}[y](s)$$

or equivalently $\overline{f*g}(s) = \bar{f}(s)\bar{g}(s)$

Proof:

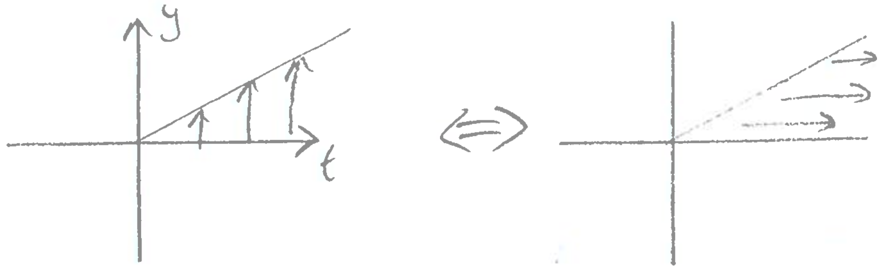
$$\mathcal{L}[f*g](s) = \int_0^\infty (f*g)(t)e^{-st}\, dt$$ (def of LT)

$$= \int_0^\infty \int_0^t f(t-y)g(y)e^{-st}\, dy\, dt$$

Multiply/Divide by e^{-sy}

$$\mathcal{L}[f*g](s) = \int_0^\infty \int_0^t f(t-y)e^{-s(t-y)} \cdot g(y)e^{-sy}\, dt$$

We now swap order of integration:



$$\mathcal{L}[f*g](s) = \int_0^\infty \left(\int_y^\infty f(t-y)e^{-s(t-y)}\, dt \right) g(t)e^{-sy}\, dy$$

If we set $z = t-y$, then we can simply the inner integral.

$$\int_y^\infty f(t-y)e^{-s(t-y)}\, dt = \int_0^\infty f(z)e^{-sz}\, dz = \mathcal{L}[f](s)$$

$$\Rightarrow \mathcal{L}[f*g](s) = \mathcal{L}[f](s) \int_0^\infty g(y)e^{-sy}\, dy$$

$$= \mathcal{L}[f](s)\, \mathcal{L}[g](s)$$

Inversion of the LT

We consider several methods for inverting LT.

1) Using known examples, shift theorems, partial fractions and formulas for derivatives.

Example 1 Find $f(t)$ such that

$$\bar{f}(s) = \frac{e^{-s\pi}}{s^2(1+s^2)}$$

Answer: Real shift result:

$$\mathcal{L}[f(t-\alpha)](s) = e^{-s\alpha}\mathcal{L}[t](s)$$

We therefore see that $f(t) = g(t-\pi)$ where

$$\mathcal{L}[g](s) = \frac{1}{s^2(s^2+1)}$$

Using partial fraction:

$$\mathcal{L}[g](s) = \frac{1}{s^2} - \frac{1}{s^2+1}$$

Recall from previous examples

$$\mathcal{L}[t^n](s) = \frac{n!}{s^{n+1}}$$

$$\Rightarrow \mathcal{L}[t](s) = \frac{1}{s^2}$$

and $\mathcal{L}[\sin \omega t] = \dfrac{\omega}{s^2+\omega^2}$

$\Rightarrow$ For $\omega = 1$, we have $\mathcal{L}[\sin t] = \dfrac{1}{1+s^2}$

$$\Rightarrow g(t) = \begin{cases} t - \sin t & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

satisfing $\mathcal{L}[g](s) = \dfrac{1}{s^2} - \dfrac{1}{s^2+1}$ ...

Therefore the solution is

$t \geq \pi$ $\Updownarrow$

$$f(t) = g(t-\pi) = \begin{cases} (t-\pi) - \sin(t-\pi) & t - \pi \geq 0 \\ 0 & \text{if } t < \pi \end{cases}$$

<u>Example 2</u> : Find $f(t)$ such that

$$\mathcal{L}[f](s) = \frac{1}{(s+1)(s+2)}$$

<u>Answer</u>: Using Partial fractions

$$\mathcal{L}[t](s) = \frac{1}{s+1} - \frac{1}{s+2}$$

Recall the other shift results.

$$\mathcal{L}[e^{-\alpha t} f(t)](s) = \mathcal{L}[f](s+\alpha)$$

$$\Rightarrow \frac{1}{s+1} = \mathcal{L}[e^{-t} g(t)](s) \text{ where } \mathcal{L}[g](s) = \frac{1}{s}$$

Recalling that $\mathcal{L}[1](s) = 1/s$, we see that $g(t) = 1$

$$\Rightarrow \mathcal{L}[e^{-t}](s) = \frac{1}{s+1} \text{ and } \mathcal{L}[e^{-2t}](s) = \frac{1}{s+2}.$$

$$\Rightarrow f(t) = \begin{cases} e^{-t} - e^{-2t} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0. \end{cases}$$

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<u>Example 3.</u>

$$\mathcal{L}[f](s) = \frac{1}{(s+1)^5}$$

We see that $f(t) = e^{t} g(t)$, where

$$\mathcal{L}[g](s) = \frac{1}{s^5}$$

Recalling $\mathcal{L}[t^4](s) = \frac{4!}{s^5} = \frac{24}{s^5}.$

we see that $g(t) = \frac{1}{24} t^4$ so that $\mathcal{L}[g](s) = \frac{1}{s^5}$

$$\Rightarrow f(t) = \begin{cases} \frac{t^4 e^t}{24} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0. \end{cases}$$

2) General Method for inverting LT: The Bromwich integral formula.

Recall the Fourier Integral Formula:

$$F(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} F(y) e^{-iky} \, dy \right) e^{ikt} \, dk$$

Choosing $F(t) = e^{-ct} f(t)$, where c is a constant that will be choosen to generate the convergence of some integral.

$$\Rightarrow e^{-ct} f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(y) e^{-(c+ik)} \, dy \right) e^{ikt} \, dk$$

Multiply by e^{ct}, we get

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(y) e^{-(c+ik)y} \, dy \right) e^{(c+ik)t} \, dk.$$

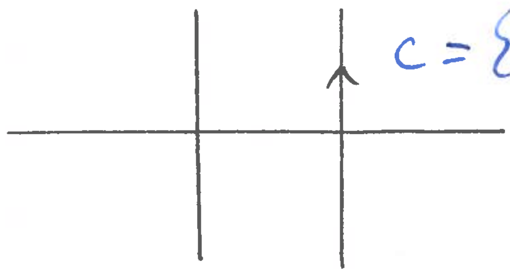
Supposing that $f(t) = 0$ for $t<0$, we have

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_0^{\infty} f(y)e^{-(c+ik)y}\,dy\right)e^{(c+ik)y}\,dy.$$

Observe that $\mathcal{L}[f](c+ik) = \int_0^{\infty} f(y)e^{-(c+ik)t}\,dk$

$$\Rightarrow f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{L}[t](c+ik)e^{(c+ik)t}\,dt.$$

This repersent as integral in the complex plane on a line parallel to the imaginary axis, shifted by c.



$c = \{s = c+ik,\ k\in(-\infty,\infty)\}$

integral transform

Aside: Put $s = c+ik$, $ds = idk$, we have

$$f(t) = \frac{1}{2\pi i}\int_C \mathcal{L}[t](s)e^{st}\,ds$$

Back to the example:

Question: How to choose c?

Example $f(t) = e^{\beta t}$, $\beta \in \mathbb{R}$

$$\Rightarrow \mathcal{L}[f](s) = \begin{cases} \dfrac{1}{s-\beta} & \text{for } Re(s) > \beta \\ +\infty & \text{for } Re(s) < \beta \text{ or } s=\beta \\ \text{undefined} & \text{for } Re(s) = \beta,\ s \neq \beta. \end{cases}$$

For this function, we must pick $c > \beta$, i.e the contour must lie to the right of the singularity of $1/s-\beta$ in the complex plane.

More generally, suppose that $\mathcal{L}[t](s)$ can be extended to a holomorphic function on $\mathbb{C}$ except at finitely many pole of some finite degree. Eg, for $f(t) = e^{\beta t}$, we choose to extend $\mathcal{L}[t](s)$ as

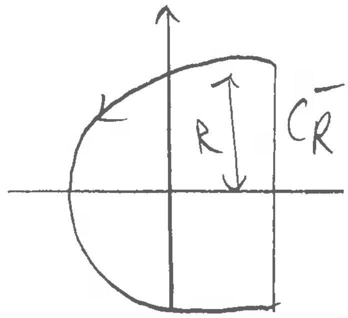
$$\mathcal{L}[f](s) = \frac{1}{s-\beta} \quad \text{for all } s \in \mathbb{C} \setminus \{\beta\}.$$

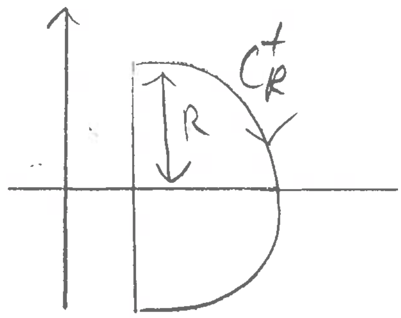
We now choose c such that $C = \{s = c + ik, k \in (-\infty, \infty)\}$ lies to the right of singularity in $\mathcal{L}[f](s)$

$$f(t) = \frac{1}{2\pi} \int_C \mathcal{L}[f](s) e^{st} \, ds$$

Bromwich integral formula

Depending on the functions given, and the value of t, we can view $\int_c \mathcal{L}[f](s)e\hat{}(st)\,ds$ as the limit of either a left contour C_R^- or a right contour C_R^+ (as $R \to \infty$)

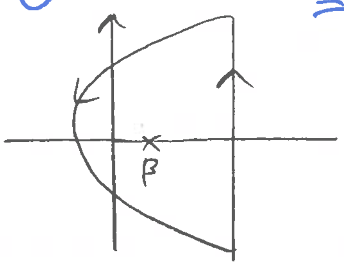




<u>Example</u> For $\mathcal{L}[f](s) = 1/s-\beta$, then for any $c > \beta$, we have:

$$f(t) = \int_c \frac{1}{s-\beta} e^{st}\,ds$$

If $t > 0$



$$\Rightarrow \int_c \frac{1}{s-\beta} e^{st} ds = \lim_{R\to\infty} \int_{C_R^-} \frac{1}{s-\beta} e^{st} ds$$

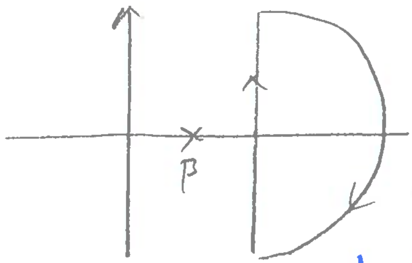
$$= 2\pi \operatorname{Res}\left[\frac{e^{st}}{s-\beta}, s=\beta\right]$$

$$= 2\pi i \left[e^{\beta t}\right]$$

$$\Rightarrow f(t) = e^{\beta t} \text{ for } t > 0.$$

For $t<0$, we have ...

$$\int_C \frac{1}{s-\beta} e^{st}\, ds = \lim_{R\to+\infty} \int_{C_R^+} \frac{1}{s-\beta} e^{st}\, ds$$



$= 0$

$\Rightarrow f(t) = 0 \quad \forall t < 0.$

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<u>Example</u>: $\mathcal{L}[f](s) = \dfrac{e^{-s\pi}}{1+s^2}$

The poles of $\mathcal{L}[f](s)$ are at $s = \pm i$

So, we have

$$f(t) = \frac{1}{2\pi i} \int_C \frac{e^{-s\pi}}{1+s^2} e^{st}\, ds = \frac{1}{2\pi i} \int_C \frac{e^{s(t-\pi)}}{1+s^2}\, ds$$

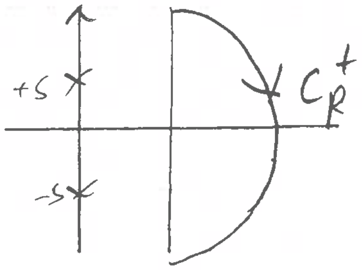
where $C = \{s = c + ik,\ k \in (-\infty, \infty)\}$,

where $c > 0$.

For $t - \pi < 0$, then $e^{s(t-\pi)} \longrightarrow 0$ as $Re(s) \to \infty$

so

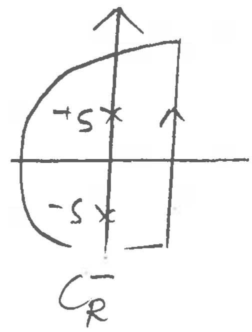
$$f(t) = \frac{1}{2\pi i} \lim_{R\to\infty} \int_{C_R^+} \frac{e^{s(t-\pi)}}{1+s^2}\, ds = 0.$$



$\Rightarrow f(t) = 0 \quad \forall t < \pi.$

For $t > \pi$, $e^{s(t-\pi)} \to 0$ when $Re(s) \to -\infty$

So $f(t) = \dfrac{1}{2\pi i} \lim\limits_{R\to\infty} \displaystyle\int_{C_R^-} \frac{e^{s(t-\pi)}}{1+s^2}\, ds$



Therefore, by the Residue Theorem

$$f(t) = Res\left[\frac{e^{s(t-\pi)}}{1+s^2}, s = +i\right]$$

$$+ Res\left[\frac{e^{s(t-\pi)}}{1+s^2}, s = -i\right]$$

21/3/19

LT For sloving ODEs

The LT is useful for sloving ODE's with non-zero right-hand sides, i.e inhomogenous ODEs, provided that all boundary conditions are imposed at a single point (e.g $t=0$)

Example Q5 of Pb sheet 6 use the LT to show that the soln of

$$x'' + 2x' + 2x = f(x)$$

and $x(0) = x'(0) = 0$

is $x(t) = \int_0^t f(t-y) e^{-y} \sin(y)\, dy$

Solution: Recall the formular of derivatives (using bar notation)

$$\bar{x}''(s) = s^2 \bar{x}(s) - x'(0_+) - s x(0_+)$$

$$\bar{x}'(s) = s\bar{x}(s) - x(0_+)$$

Assuming $x(t)$ is contiuous on $[0, +\infty)$, we can use the I.C. to simplify

$$\bar{x}''(s) = s^2 \bar{x}(s), \quad \bar{x}'(s) = s\bar{x}(s)$$

Therefore, the transformation of the eqn is

$$s^2 \bar{x}(s) + 2s\bar{x}(s) + 2\bar{x}(s) = \bar{f}(s)$$

$$\Leftrightarrow \bar{x}(s)(s^2+2s+2) = \bar{f}(s)$$

$$\Leftrightarrow \bar{x}(s) = \frac{\bar{f}(s)}{s^2+2s+2}$$

Let $g(t)$ be the function that satisfies

$$\bar{g}(s) = \frac{1}{s^2+2s+2}$$

$$\Rightarrow \bar{x}(s) = \bar{f}(s)\bar{g}(s) = \overline{(f*g)}(s)$$

By the convolution Thm

Thus, $x(t) = (f*g)(t)$

It remains only to find $g(t)$

a) Shift theorem method

$$\bar{g}(s) = \frac{1}{s^2+2s+2} = \frac{1}{(s+1)^2+1} \leftarrow \frac{1}{s^2+1} \text{ shifted to } s+1$$

Recall shift result

$$\mathcal{L}[e^{-\alpha t} f(t)](s) = \mathcal{L}[t](s+\alpha)$$

Since $\mathcal{L}[\sin \omega t](s) = 1/s^2+1$, it follows that

$$\mathcal{L}[e^{-t}\sin t] = \frac{1}{(s+1)^2+1} = \bar{g}(s)$$

$$\Rightarrow g(t) = e^{-t}\sin t \quad \text{for } t \geq 0$$

b) Bromwich Integral Formula method

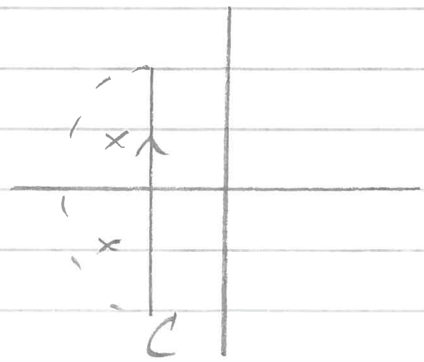
Find the poles of $\bar{g}(s)$, i.e. roots of s^2+2s+2 which are $\alpha_\pm = -1 \pm i$

$$\Rightarrow \bar{g}(s) = \frac{1}{(s-\alpha_+)(s-\alpha_-)}$$

the Bromwich Integral Formula is

$$g(t) = \frac{1}{2\pi i}\int_C \frac{e^{st}}{(s-\alpha_+)(s-\alpha_-)}\, ds$$

where C is a line parallel to the imaginary axis, to the right of $\alpha_\pm$



We can evaluate the integral for $t \geq 0$

$$\int_C \frac{e^{st}}{(s-\alpha_+)(s-\alpha_-)}\, ds = \lim_{R\to\infty}\int_{C_R^-} \frac{e^{st}}{(s-\alpha_+)(s-\alpha_-)}\, ds$$

Therefore, by the Residue Theorem, for $t \geq 0$

$$\int_C \frac{e^{st}}{(s-\alpha_+)(s-\alpha_-)}\,ds = 2\pi i\left[\mathrm{Res}\left[\frac{e^{st}}{(s-\alpha_+)(s-\alpha_-)}, s=\alpha_+\right] + \mathrm{Res}\left[\frac{e^{st}}{(s-\alpha_+)(s-\alpha_-)}, s=\alpha_-\right]\right]$$

$$\Rightarrow g(t) = \left[\frac{e^{\alpha_+ t}}{\alpha_+-\alpha_-} + \frac{e^{\alpha_- t}}{\alpha_- - \alpha_+}\right]$$

$$= \frac{1}{2i}\left[e^{-t+it} - e^{-t+it}\right] = e^{-t}\operatorname{sen} t$$

Therefore:

$$x(t) = (f*g)(t) = \int_0^t f(t-y)g(y)\,dy$$

$$= \int_0^t f(t-y)e^{-y}\operatorname{sen} y\,dy$$

<u>Excercise</u>: Derive $g(t)$ using partial fractions

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Q6, Pb Sheet 6

Slove

$$y'' + y = f(t) \quad , \quad \begin{array}{l} y(0) = 0 \\ y'(0) = 1 \end{array}$$

Solution:

Transform of the eqn is:

$$s^2\bar{y}(s) - s\underbrace{y(0)}_{=0} - \underbrace{y'(0)}_{=1} + \bar{y}(s) = \bar{f}(s)$$

$$\Leftrightarrow (s^2+1)\bar{y}(s) = 1 + \bar{f}(s)$$

$$\Leftrightarrow \bar{y}(s) = \frac{1}{s^2+1} + \frac{\bar{f}}{s^2+1}$$

Therefore, using $\mathcal{L}[\sin t](s) = 1/s^2+1$, and using convolution Thm

$$y(t) = \sin t + (f * \sin t)(t)$$

$$= \sin t + \int_0^t f(t-y)\sin y \, dy .$$