7402 Mathematical Methods 4 Notes

Based on the 2013 spring lectures by Dr G Esler

MATH7402– Mathematical Methods 4.

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Chapter 1
FROBENIUS' METHOD AND SPECIAL FUNCTIONS.

Consider the 2^{nd}-order ODE $\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0$ – (*). This equation is <u>linear</u> and <u>homogeneous</u>.

Special cases:

① <u>Constant coefficients</u>, $p(x) = a$, $q(x) = b$. $a, b \in \mathbb{R}$ (constants)

Try $y = e^{nx}$. This gives <u>auxillary equation</u>, $m^2 + am + b = 0$. The solutions are given by

$$y(x) = \begin{cases} Ae^{m_1x} + Be^{m_2x}, & m_1, m_2 \text{ are distinct real roots} \\ (Ax+B)e^{m_1x}, & m_1 \text{ is a double root} \\ e^{m_rx}(A\cos m_ix + B\sin m_ix), & \text{complex conjugate roots: } m_{1,2} = m_r \pm im_i \text{ } (m_r \text{ real part, } m_i \text{ imaginary part}) \end{cases}$$

② <u>Euler equations</u>, where $p(x) = \frac{a}{x}$, $q(x) = \frac{b}{x^2}$. This gives $x^2y'' + axy' + by = 0$

Use the substitution $x = e^t$. Let $Y(t) = y(e^t) = y(x)$. Then $\frac{dY}{dt} = \frac{d}{dt}y(e^t) = \frac{dy}{dx}\cdot\frac{dx}{dt} = x\frac{dy}{dx}$

Likewise, $\frac{d^2Y}{dt^2} = \frac{d}{dt}\left(e^t\frac{dy}{dx}\right) = x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx}$. This leads to $\frac{d^2Y}{dt^2} + (a-1)\frac{dY}{dt} + bY = 0$. Then our auxillary equation is $m^2 + (a-1)m + b = 0$.

This leads to the first case, with appropiate substitutions

$$y(x) = \begin{cases} Ax^{m_1} + Bx^{m_2} & m_1, m_2 \text{ real and distinct} \\ (A\log x + B)x^{m_1} & m_1 = m_2 \\ x^{m_r}[A\cos(m_i\log x) + B\sin(m_i\log x)] & \text{complex conjugates } m_{1,2} = m_r \pm im_i \end{cases}$$

There is a quick method for this: insert $y = x^m$ directly into the equation (be careful though, complex solutions can be tricky).

$\boxed{\text{Ex}}$ Solve $x^2y'' - 2xy' + 2y = 0$.

Soln. Try $y = x^m$. Then $y' = mx^{m-1}$, $y'' = m(m-1)x^{m-2}$. $\Rightarrow$ $x^2y'' - 2xy' + 2y = 0$ reduces to $m(m-1)x^m - 2mx^m + 2x^m = 0 \Rightarrow [m^2 - m - 2m + 2]x^m = 0$.

$m^2 - 3m + 2 = 0 \Rightarrow m = 1$ or $2 \Rightarrow y(x) = Ax + Bx^2$ //

What happens if $p(x)$ and $q(x)$ have a more general form? For example, $p(x) = \frac{P(x)}{Q(x)}$ for P, Q polynomials etc. We could try series solutions as a naïve method.

Insert <u>ansatz</u> $y(x) = \sum_{k=0}^{\infty} a_kx^k$ and work from there. We illustrate this with a general worked example.

$\boxed{\text{Ex}}$ 1 Solve $y'' - y = 0$ with the power series solution method.

Soln. Take $y(x) = \sum_{k=0}^{\infty} a_kx^k$, $y' = \sum_{k=1}^{\infty} ka_kx^{k-1}$, $y'' = \sum_{k=2}^{\infty} k(k-1)a_kx^{k-2}$. Insert in equation: $\sum_{k=2}^{\infty} k(k-1)a_kx^{k-2} - \sum_{k=0}^{\infty} a_kx^k = 0$. We reindex first term in series, write $j = k-2$. Then $\sum_{k=2}^{\infty} k(k-1)a_kx^{k-2} = \sum_{j=0}^{\infty} (j+2)(j+1)a_{j+2}x^j$. Hence, relabelling $j \to k$, our sums become $\sum_{k=0}^{\infty}\left((k+2)(k+1)a_{k+2} - a_k\right)x^k = 0$.

We know that all these coefficients must, by definition of equality, be identically equal to 0. Then we rearrange $(k+2)(k+1)a_{k+2} - a_k = 0$ to get the <u>recurrence relation</u>: $a_{k+2} = \frac{a_k}{(k+2)(k+1)}$. As we are solving a 2^{nd} order ODE, we expect two linearly independent solutions.

Generate one solution by setting $a_0 = 1$, $a_1 = 0$ $\Rightarrow$ series in even powers of x, even function. The other is found by setting $a_0 = 0$, $a_1 = 1 \Rightarrow$ series in odd powers, odd fn.

The solution space is the linear span of these two solutions. If $a_0 = 1, a_1 = 0$, then $a_2 = \frac{1}{1\cdot 2}$, $a_4 = \frac{1}{1\cdot2\cdot3\cdot4}$, $\dots$, $a_{2k} = \frac{1}{(2k)!}$. $y_1(x) = \sum_{k=0}^{\infty} a_{2k}x^{2k} = \sum_{k=0}^{\infty}\frac{x^{2k}}{(2k)!} = \cosh x$

If $a_0 = 0$, $a_1 = 1$, then $a_3 = \frac{1}{3\cdot2}$, $a_5 = \frac{1}{5\cdot4\cdot3\cdot2}$, $a_{2k+1} = \frac{1}{(2k+1)!}$. Then $y_2(x) = \sum_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)!} = \sinh x$. Hence, $y(x) = a\cosh x + b\sinh x$ //

It turns out to be (sometimes) dangerous to proceed as above:

- in general, we do not know where the power series should begin.
- we know nothing about the existence or nature of solutions
- could also extend to the complex plane to use familiar results about power series and analytic functions.

We address these using the Frobenius' method.

<u>Frobenius' method for solution of ODEs.</u>

We work in $\mathbb{C}$. Our equation is $w'' + p(z)w' + q(z)w = 0$ for $w, z \in \mathbb{C}$, the primes now indicate derivatives $\frac{d}{dz}$.

Use the following ansatz: $w(z) = \sum_{k=0}^{\infty} a_kz^{k+c}$, $a_0 \neq 0$. We force $a_0 \neq 0$, so that power series begins for $k = 0$ i.e. at z^c.

c is an unknown to be determined (may be different for solutions $w_1(z)$, $w_2(z)$.). We illustrate this with an example.

$\boxed{\text{Ex}}$ 2 $zw'' + \frac{1}{2}w' + \frac{1}{4}w = 0$. Solve for $w(z)$.

Soln. $w = \sum_{k=0}^{\infty} a_kz^{k+c}$, $a_0 \neq 0$. $w' = \sum_{k=0}^{\infty} a_k(k+c)z^{k+c-1}$, $w'' = \sum_{k=0}^{\infty} a_k(k+c)(k+c-1)z^{k+c-2}$. Inserting into equation, we get:

$\sum_{k=0}^{\infty} a_k(k+c)(k+c-1)z^{k+c-1} + \frac{1}{2}\sum_{k=0}^{\infty} a_k(k+c)z^{k+c-1} + \frac{1}{4}\sum_{k=0}^{\infty} a_kz^{k+c} = 0$. By convention, we re-index final term downwards. Also demand $a_{-1} = a_{-2} = \dots = 0$.

This gives $\sum_{k=0}^{\infty}\left[a_k\left[(k+c)(k+c-1) + \frac{1}{2}(k+c)\right] + \frac{1}{4}\underbrace{a_{k-1}}_{\text{acceptable, as } a_{-1}=0}\right]z^{k+c-1} = 0$. Power series identically equal to zero $\Rightarrow$ coefficients are zero.

Set $k = 0$, then $a_0\left(c(c-1) + \frac{1}{2}c\right) = 0 \Rightarrow \boxed{c(c-\frac{1}{2}) = 0}$, which is the <u>indicial equation</u>. Then $c = 0$ or $\frac{1}{2}$. For $k \geq 1$, $\boxed{(k+c)(k+c-\frac{1}{2})a_k + \frac{1}{4}a_{k-1} = 0}$, the recurrence.

To find the two solutions, insert $c = 0$ and $c = \frac{1}{2}$ into recurrence relation. For $c = 0$, $k(k-\frac{1}{2})a_k = -\frac{1}{4}a_{k-1} \Rightarrow a_k = -\frac{1}{2k(2k-1)}a_{k-1} = (-1)^k\frac{1}{(2k)!}a_0$. Then, WLOG, set $a_0 = 1$ – arbitrary constant in solution. First solution is $w_1(z) = \sum_{k=0}^{\infty} a_k z^{k+0} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^k = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} (z^{\frac{1}{2}})^{2k} = \cos(z^{\frac{1}{2}})$.

For $c = \frac{1}{2}$, $a_k = \frac{-a_{k-1}}{4(k+\frac{1}{2})k} = -\frac{a_{k-1}}{(2k+1)(2k)} = (-1)^k \frac{a_0}{(2k+1)!}$ (set $a_0 = 1$). Thus, other solution is $w_2(z) = \sum_{k=0}^{\infty} a_k z^{k+\frac{1}{2}} = \sum_{k=0}^{\infty} a_k (z^{\frac{1}{2}})^{2k+1} = \sin(z^{\frac{1}{2}})$. Thus, our final solution is $w(z) = a \cos z^{1/2} + b \sin z^{1/2}$ //

Theory of the Frobenius' method.

Under what conditions (on $p(z), q(z)$) will the Frobenius method work? Consider the equation $w''(z) + p(z)w'(z) + q(z)w(z) = 0$ — $\dagger$

Definition A point $z = z_0$ is said to be an <u>ordinary point</u> of $\dagger$ if $p(z)$ and $q(z)$ are both analytic at $z = z_0$.

e.g. In Ex 1, $w'' - w = 0$, $p = 0$ and $q = -1$ are analytic everywhere.

Definition A point $z = z_0$ is said to be a <u>regular singular point</u> of $\dagger$ if $(z-z_0)p(z)$, $(z-z_0)^2 q(z)$ are both analytic there.

Note: In other words, $p(z)$ can have at worst a simple pole at $z = z_0$. $p(z) = \frac{C_{-1}}{z-z_0} + C_0 + C_1(z-z_0) + \cdots \Rightarrow (z-z_0)p(z)$ is analytic.

And by analogy, $q(z)$ can have at worst a pole of order 2: $q(z) = \frac{C_{-2}}{(z-z_0)^2} + \frac{C_{-1}}{z-z_0} + C_0 + C_1(z-z_0) + \cdots$

e.g. In Ex 2, $zw'' + \frac{1}{2}w' + \frac{1}{4}w = 0$, $p = \frac{1}{2z}$, $q = \frac{1}{4z}$. About $z = 0$, $p(z), q(z)$ are not analytic; but $zp(z)$, $z^2 q(z)$ are $\Rightarrow$ $z = 0$ is a regular singular point.

Theorem (Fuchs's theorem)

The general solution of $\dagger$ is obtainable by the method of Frobenius, in the form of a <u>generalised</u> power series about $z = z_0$, provided that $z = z_0$ is an ordinary or regular singular point of $\dagger$.

Proof – Rigorous proof omitted, but partial proof will be derived subsequently.

This gives us a general rule of thumb: If $z = z_0$ is an ordinary point, use naïve power series. If it is an ordinary singular point, use Frobenius. If neither, give up!

Corollary Further, when $z = z_0$ is an ordinary point, the solution is analytic, with radius of convergence (at least) as large as the minimum of the radii of convergence of $p(z), q(z)$.

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Proof – (Fuchs's) Note that we need only consider $z_0 = 0$ (we could make the change of variables $\tilde{z} = z - z_0$ in $\dagger$).

If $z_0 = 0$ is a regular singular point, then we know that the functions $zp(z)$ and $z^2 q(z)$ have Taylor series:

$zp(z) = \sum_{k=0}^{\infty} p_k z^k$, $z^2 q(z) = \sum_{k=0}^{\infty} q_k z^k$ with coefficients p_k, q_k. Now insert the Frobenius ansatz in $\dagger$.

$w(z) = \sum_{k=0}^{\infty} a_k z^{k+c}$, $a_0 \neq 0$. $w'(z) = \sum_{k=0}^{\infty} (k+c)a_k z^{k+c-1}$, $w''(z) = \sum_{k=0}^{\infty} a_k (k+c)(k+c-1) a_k z^{k+c-2}$. With that, $\dagger$ becomes:

$$w'' + pw' + qw = \sum_{k=0}^{\infty} a_k (k+c)(k+c-1) z^{k+c-2} + \underbrace{\left(\sum_{k=0}^{\infty} p_k z^k\right)}_{zp} \underbrace{\left(\sum_{k=0}^{\infty} a_k (k+c) z^k\right)}_{z^{1}w'} z^{c-2} + \underbrace{\left(\sum_{k=0}^{\infty} q_k z^k\right)}_{z^2 q} \underbrace{\left(\sum_{k=0}^{\infty} a_k z^k\right) z^{c-2}}_{z^{-2}w} = 0.$$

Recall that we have an identity for multiplying power series, which in the general statement form is given by $\left(\sum_{k=0}^{\infty} f_k z^k\right)\left(\sum_{k=0}^{\infty} g_k z^k\right) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} f_{k-j} g_j\right) z^k$.

Using this, $\dagger$ becomes $\sum_{k=0}^{\infty} \left[a_k (k+c)(k+c-1) + \sum_{j=0}^{k} p_{k-j} a_j (j+c) + \sum_{j=0}^{k} q_{k-j} a_j \right] z^{k+c-2} = 0$.

We can set coefficients to 0, taking out the $j=k$ from the nested sum: $a_k[(k+c)(k+c-1) + p_0(k+c) + q_0] + \sum_{j=0}^{k-1} a_j (p_{k-j}(j+c) + q_{k-j}) = 0$.

For $k=0$ coefficient, $a_0[(c)(c-1) + p_0 c + q_0] = 0$. Indicial equation is $F(c) = c^2 + (p_0 - 1)c + q_0 = 0$. This determines the coefficients c, i.e. where the power series begins. We note that $(k+c)(k+c-1) + p_0(k+c) + q_0 = F(k+c)$, so for $k \geq 1$,

$a_k = -\frac{1}{F(k+c)} \sum_{j=0}^{k-1} a_j (p_{k-j}(j+c) + q_{k-j})$. This is a recurrence relation for a_k, in terms of known $\{a_0, a_1, \ldots, a_{k-1}\}$.

In principle then, we can use the recurrence relation to construct two LI solutions ... but not always! Two things could theoretically go wrong:

- Case I: Two distinct roots c_1, c_2 of indicial equation, $c_1 - c_2$ is not an integer (this case is good)
- Case II: Double roots, $c_1 = c_2$
- Case III: Two distinct roots c_1, c_2 differing by an integer i.e. $c_1 - c_2 = n \in \mathbb{Z}$ (WLOG $c_1 > c_2$)

} can prevent us from getting 2 LI solutions with outlined method.

Each of these cases will have to be treated separately. In order to deal with the trickier cases II and III, we introduce the function of two variables $W(z,c) = \sum_{k=0}^{\infty} a_k(c) z^{k+c}$. Now c is instead allowed to vary freely as the argument of W; and $a_k(c)$ all satisfy the recurrence relation above: $a_k = -\frac{1}{F(k+c)} \sum_{j=0}^{k-1} a_j (p_{k-j}(j+c) + q_{k-j})$. Insert $W(z,c)$ into equation: exactly as above, we get

$\left(\frac{d^2}{dz^2} + p(z)\frac{d}{dz} + q(z)\right) W(z,c) = \sum_{k=0}^{\infty} \left[a_k F(k+c) + \sum_{j=0}^{k-1} a_j (p_{k-j}(j+c) + q_{k-j})\right] z^{k+c-2}$. So for $k \geq 1$, these coefficients all equal to 0 as $\{a_k\}$ satisfies recurrence relation. Thus, only remaining term is $k=0$ term: $a_0 F(c) z^{c-2} = a_0 (c-c_1)(c-c_2) z^{c-2}$ for roots c_1, c_2.

For case I, this is simple, and things are exactly as before $\Rightarrow$ the 2 solutions are $w_1(z) = W(z, c_1) = \sum_{k=0}^{\infty} a_k(c_1) z^{k+c_1}$, $w_2(z) = W(z, c_2)$, as in Ex 2.

For case II, we have $c_1 = c_2$. Then $\left(\frac{d^2}{dz^2} + p(z)\frac{d}{dz} + q(z)\right) W(z,c) = a_0 (c-c_1)^2 z^{c-2}$ for $c_1 = c_2$. We differentiate this w.r.t c, to get a second solution in addition to $w_1 = W(z, c_1)$. The differentiated statement is $\left(\frac{d^2}{dz^2} + p(z)\frac{d}{dz} + q(z)\right)\left(\frac{\partial W}{\partial c}(z,c)\right) = 2a_0(c-c_1) z^{c-2} + a_0 (c-c_1)^2 (\log z) z^{c-2}$

because $\frac{d}{dc} z^c = \frac{d}{dc} e^{c \log z} = \log z \cdot z^c$. Set $c = c_1$, then $\left(\frac{d^2}{dz^2} + p(z)\frac{d}{dz} + q(z)\right) \frac{\partial W}{\partial c}(z, c_1) = 0$,

so our second solution is $\frac{\partial W}{\partial c}(z, c_1) = w_2(z) = \sum_{k=0}^{\infty} \frac{da_k}{dc}(c_1) z^{k+c_1} + \left(\sum_{k=0}^{\infty} a_k(c_1) z^{k+c_1}\right) \log z = \sum_{k=0}^{\infty} \frac{da_k}{dc}(c_1) z^{k+c_1} + w_1(z) \log z$.

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Ex (Bessel's equation of index 0)

solve the ODE $zw'' + w' + zw = 0$.

sdn. insert $w = \sum_{k=0}^{\infty} a_k z^{k+c}$ (a_k is a function of c). $\underbrace{\sum_{k=0}^{\infty} a_k (k+c)(k+c-1) z^{k+c-1}}_{zw''} + \underbrace{\sum_{k=0}^{\infty} a_k (k+c) z^{k+c-1}}_{w'} + \underbrace{\sum_{k=0}^{\infty} a_k z^{k+c+1}}_{zw} = 0$. Reindexing downwards, $\sum_{k=0}^{\infty} [a_k (k+c)^2 + a_{k-2}] z^{k+c-1} = 0$. set coefficients to zero: $k=0$, $a_0 c^2 = 0$; $a_0 \neq 0 \Rightarrow c^2 = 0$ is indicial equation. $c_1 = 0$ is a double root.

k odd: $a_k (k+c)^2 + a_{k-2} = 0$. Since $a_{-1} = 0$ by construction. otherwise: $a_k = -\frac{a_{k-2}}{(k+c)^2}$ by recurrence relation. For k even, this is equivalent to $a_{2k} = -\frac{a_{2k-2}}{(2k+c)^2}$. Introduce $b_k = a_{2k}$, then $b_k = -\frac{b_{k-1}}{2^2(k+\frac{c}{2})^2}$ (for book-keeping reasons, $\{b_k\}$ picks out even $\{a_k\}$). Solution is now $w = \sum_{k=0}^{\infty} b_k z^{2k+c}$.

Then from recurrence relation, $b_k = \frac{(-1)^k}{2^{2k}} \left(\frac{1}{(k+\frac{c}{2})^2 (k-1+\frac{c}{2})^2 \cdots (1+\frac{c}{2})^2} \right) b_0$. First solution: set $c = 0 \Rightarrow b_k(0) = \frac{(-1)^k}{2^{2k}} \cdot \frac{1}{(k!)^2} b_0$, we can set $b_0 = 1$ WLOG.

$w_1(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{z}{2}\right)^{2k}$. This solution is denoted $J_0(z)$: Bessel function of first kind, with index 0.

From our theory, we know that the second solution is given by $w_2(z) = \left[\sum_{k=0}^{\infty} \frac{db_k}{dc} z^{2k+c} + \sum_{k=0}^{\infty} b_k z^{2k+c} \log z \right]_{c=0} = \left[\sum_{k=0}^{\infty} \frac{db_k}{dc} z^{2k+c} \right]_{c=0} + w_1(z) \log z$.

We need to find $\frac{db_k}{dc}$: in particular, what is $\frac{d}{dc}\left[\frac{1}{(k+\frac{c}{2})^2 (k-1+\frac{c}{2})^2 \cdots (1+\frac{c}{2})^2} \right]$? Use logarithmic differentiation: $\frac{d}{dx} \log f(x) = \frac{f'(x)}{f(x)}$.

So $\frac{df}{dx} = f(x) \frac{d}{dx} \log f(x)$. let $f(c) = \frac{1}{(k+\frac{c}{2})^2 (k-1+\frac{c}{2})^2 \cdots (1+\frac{c}{2})^2}$, then $\log f(x) = \sum_{j=1}^{k} -2 \log (j + \frac{c}{2})$. $\Rightarrow \frac{d}{dc} f(c) = f(c) \frac{d}{dc} \left(\sum_{j=1}^{k} -2 \log (j+\frac{c}{2}) \right)$.

$\frac{d}{dc} f(c) = f(c) \cdot \sum_{j=1}^{k} \frac{-1}{j+\frac{c}{2}}$, and $\frac{db_k}{dc} = \frac{1}{(k+\frac{c}{2})^2 \cdots (1+\frac{c}{2})^2} \left(-\sum_{j=1}^{k} \frac{1}{j+\frac{c}{2}} \right) \left(\frac{(-1)^k}{2^{2k}} \right)$. Now set $c=0$, then $\frac{db_k}{dc}(0) = \frac{(-1)^{k+1} S_k}{2^{2k} (k!)^2}$, $S_k = \sum_{j=1}^{k} \frac{1}{j}$

Hence the second solution is $w_2(z) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} S_k}{(k!)^2} \left(\frac{z}{2}\right)^{2k} + w_1(z) \log(z)$: define $Y_0(z)$ as the Bessel function of the second kind (index 0).

Then the second solution is a linear combination of $J_0(z)$ and $Y_0(z)$ //

The Gamma Function.

This function is used for book-keeping in Frobenius problems: $\boxed{\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt}$ $\Gamma(1) = \int_0^{\infty} e^{-t} dt = [-e^{-t}]_0^{\infty} = 1$.

We can evaluate these using integration by Parts — $\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt = \underbrace{[-t^{x-1} e^{-t}]_0^{\infty}}_{0} + \int_0^{\infty} (x-1) t^{x-2} e^{-t} dt = \int_0^{\infty} (x-1) t^{x-2} e^{-t} dt = (x-1)\Gamma(x-1)$.

We can express products in terms of Γ. consider $(k+c)(k-1+c)\cdots(1+c)$. $\Gamma(k+c+1) = (k+c)\Gamma(k+c) = (k+c)(k-1+c)\Gamma(k-1+c) = \cdots = (k+c)(k-1+c)\cdots(1+c)\Gamma(1+c)$.

then $(k+c)(k-1+c)\cdots(1+c) = \frac{\Gamma(k+1+c)}{\Gamma(1+c)}$. On integers (specific case), if $c=0$, $k! = \frac{\Gamma(k+1)}{\Gamma(1)} = \frac{\Gamma(k+1)}{1} = \Gamma(k+1)$. so $\boxed{k! = \Gamma(k+1)}$

Of course, the gamma function can also take on non-integer values; interpolates factorial to all $x \in \mathbb{R}$ (except where singular).

Special Functions: Bessels.

Special functions are defined by their complex power series, often obtained by solving ODEs (such as (†)).

An example of this is Bessel's equation with index $\nu \in \mathbb{C}$ (constant), $\boxed{z^2 w'' + zw' + (z^2 - \nu^2) w = 0}$. which often arises in mathematical physics.

this comes up particularly in problems with axisymmetry (e.g. cylindrical geometry).

The solutions to this equation are given by, as ν varies across a range of values, $w(z) = \begin{cases} A J_\nu(z) + B J_{-\nu}(z), & \nu \notin \mathbb{Z} \\ A J_0(z) + B Y_0(z), & \nu = 0 \\ A J_m(z) + B Y_m(z) & \nu = m \in \mathbb{Z} \setminus \{0\}. \end{cases}$

These correspond to our earlier three cases to solutions in Fuchs's Theorem.

We refer to $J_\nu(z)$ as the Bessel function of the first kind with index $\nu \in \mathbb{C}$; $Y_\nu(z)$ as the Bessel function of the second kind with index $\nu \in \mathbb{C}$.

For specific properties of the Bessel functions, refer to the printed handout.

We also examine the behaviour of Bessel functions as $z \to 0$: $J_\nu(z) \sim z^\nu$, $\overbrace{Y_0(z) \sim \log z, \; Y_m(z) \sim z^{-m}}^{\text{singular.}}$

To get relationships between the Bessel functions, particularly the $J_m(z)$, $m \in \mathbb{Z}$ which are particularly important in applications, we can derive them from the Bessel generating function: $G(x,t) = \exp\left(\frac{x}{2}(t - \frac{1}{t})\right) = \sum_{m=-\infty}^{\infty} J_m(x) t^m$. This formula helps us find certain relationships:

Ex By differentiating both sides with respect to t, find a relation between $J_{m-1}(x)$, $J_m(x)$ and $J_{m+1}(x)$. Also, by differentiating w.r.t. x, find a relation between J_{m-1}, J_{m+1}, J'_m

sdn. $\exp\left(\frac{x}{2}(t-\frac{1}{t})\right) = \sum_{m=-\infty}^{\infty} J_m(x) t^m \Rightarrow \frac{x}{2}(1+\frac{1}{t^2}) \exp\left(\frac{x}{2}(t-\frac{1}{t})\right) = \sum_{m=-\infty}^{\infty} J_m(x) m t^{m-1} \Rightarrow \frac{x}{2}(1+\frac{1}{t^2}) \sum_{m=-\infty}^{\infty} J_m(x) t^m = \sum_{m=-\infty}^{\infty} J_m(x) m t^{m-1}$

$\Rightarrow \frac{x}{2}\left(\sum_{m=-\infty}^{\infty} J_m(x) t^m + \sum_{m=-\infty}^{\infty} J_m(x) t^{m-2} \right) = \sum_{m=-\infty}^{\infty} J_m(x) m t^{m-1} \Rightarrow \sum_{m=-\infty}^{\infty} \left[\frac{x}{2} J_{m-1}(x) + \frac{x}{2} J_{m+1}(x) - m J_m(x) \right] t^{m-1} = 0$.

Set coefficients equal to 0, then $\frac{x}{2}[J_{m-1}(x) + J_{m+1}(x)] - m J_m(x) = 0 \Rightarrow \frac{x}{2}(J_{m-1}(x) + J_{m+1}(x)) = m J_m(x)$ // (the recurrence relation)

Differentiate w.r.t. , we get $J'_m(x) = \frac{1}{2}(J_{m-1}(x) - J_{m+1}(x))$ //, the differentiation relation.

Note: We know $J_{-1}(x) = -J_1(x)$, hence $J'_0(x) = -J_1(x)$.

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Legendre's equation and Legendre functions.

The Legendre's differential equation is the equation $\boxed{(1-z^2)w'' - 2zw' + \nu(\nu+1)w = 0}$ (⊛), where $\nu \in \mathbb{R}$ is the index. This appears regularly in mathematical physics, particularly in problems with spherical symmetry.

We notice that $z=0$ is an ordinary point of (⊛), while $z = \pm 1$ are regular singular points, hence we expand about the ordinary point using the naïve ansatz.

Then we have $w(z)=\sum_{k=0}^{\infty} a_k z^k$ (no constraint on a_0), and we insert this into ⊛ to get $(1-z^2)w''-2zw'+\nu(\nu+1)w=0 \Rightarrow$

$$\sum_{k=2}^{\infty} a_k k(k-1) z^{k-2} - \sum_{k=0}^{\infty} a_k k(k-1) z^k - 2\sum_{k=1}^{\infty} a_k k z^k + \sum_{k=0}^{\infty} \nu(\nu+1)a_k z^k = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)z^k - \sum_{k=0}^{\infty} a_k k(k-1) - \sum_{k=0}^{\infty} 2a_k k z^k + \sum_{k=0}^{\infty} \nu(\nu+1)a_k z^k = 0.$$

$\therefore \sum_{k=0}^{\infty}\left[a_{k+2}(k+2)(k+1) - a_k(k(k+1)-\nu(\nu+1))\right]z^k = 0$. Hence, set coefficients all to 0, which yields recurrence relation $a_{k+2} = a_k\left(\frac{(k-\nu)(k+\nu+1)}{(k+1)(k+2)}\right)$.

Two solutions will include one with even powers of z: non-zero $a_0, a_2, \cdots$; the other with odd powers of z: non-zero $a_1, a_3, \cdots$

Even $\{a_k\}$ are related by $a_{2k+2} = a_{2k}\frac{(2k-\nu)(2k+\nu+1)}{(2k+1)(2k+2)}$. Write $b_k = a_{2k}$, then $b_{k+1} = b_k\frac{(2k-\nu)(2k+\nu+1)}{(2k+1)(2k+2)}$, $k \to 2k$ in recurrence relation.

Odd $\{a_k\}$ are related by $a_{2k+3} = a_{2k+1}\frac{(2k+1-\nu)(2k+2+\nu)}{(2k+2)(2k+3)}$. Write $\tilde{b}_k = a_{2k+1}$, then $\tilde{b}_{k+1} = \tilde{b}_k\frac{(2k+1-\nu)(2k+2+\nu)}{(2k+2)(2k+3)}$, $k \to 2k+1$ in recurrence relation.

So the two solutions are $w_1(z) = \sum_{k=0}^{\infty} b_k z^{2k}$, $w_2(z) = \sum_{k=0}^{\infty} \tilde{b}_k z^{2k+1}$.

Usually, the general solution of ⊛ is written $w(z) = A P_\nu(z) + B Q_\nu(z)$; where $P_\nu(z)$, $Q_\nu(z)$ are known as the <u>Legendre functions</u>.

Each is a (different) linear combination of $w_1(z)$ and $w_2(z)$.

What is the radius of convergence of these functions? We try d'Alembert's ratio test with $w_1(z)$: $\lim_{k\to\infty}\left|\frac{b_{k+1}z^{k+2}}{b_k z^{2k}}\right| = \lim_{k\to\infty}\left|\overbrace{\frac{(2k-\nu)(2k+\nu+1)}{(2k+1)(2k+2)}}^{\to 1} z^2\right| = \lim_{k\to\infty}|z^2|$.

If $w_1(z)$ converges, $|z^2|<1 \Rightarrow |z|<1 \Rightarrow R=1$ is the radius of convergence; same is true for w_2 where $R=1$ is also the radius of convergence $\Rightarrow$ and also for $P_\nu(z), Q_\nu(z)$.

We do not yet know if the series solution actually converges at $z=\pm1$. In general, they do not. However, in applications, it turns out that we care only about the Legendre functions that are regular at $z=\pm1$. Are there special situations when this occurs? Yes, where $\nu = n \in \mathbb{Z}$.

Examine $b_{k+1} = b_k\left(\frac{(2k-\nu)(2k+1+\nu)}{(2k+1)(2k+2)}\right)$. If at some point the numerator is 0, the series terminates. We call such solutions <u>Legendre polynomials of order n</u>.

If $\nu = 2n$, $n \in \mathbb{Z}$, then we have $b_{n+1} = \frac{(2n-2n)(2n+1+\nu)}{(2n+1)(2n+2)} = 0$, and all subsequent coefficients are 0. This gives a Legendre polynomial of order $2n$.

i.e. $w_1(z) = \sum_{k=0}^{n} b_k z^{2k} \propto P_{2n}(z)$ ← proportional to, multiply by constant

The odd polynomials $P_{2n+1}(z)$ come from $w_2(z)$ when $\nu = 2n+1$. $P_\nu(z) = P_n(z)$ whenever $\nu = n$.

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In practice, it is easier to use <u>Rodrigues's formula</u>: $\boxed{P_n(x) = \frac{1}{2^n n!}\frac{d^{(n)}}{dx^{(n)}}\left[(x^2-1)^n\right]}$ to find the Legendre polynomial (although it can be calculated from recurrence relation).

Note: the formula differentiates a polynomial of order $2n$, $(x^2-1)^n$, n times; yielding a polynomial of order n.

$P_0(x)=1$, $P_1(x) = \frac{1}{2}\frac{d}{dx}(x^2-1) = x$, $P_2(x) = \frac{1}{8}\frac{d^2}{dx^2}(x^2-1)^2 = \frac{1}{8}\frac{d^2}{dx^2}(x^4-2x^2+1) = \frac{3}{2}x^2 - \frac{1}{2}$.

These coefficients $\frac{1}{2^n n!}$ have been chosen in order to satisfy the property that $P_n(1)=1$. This is the standard <u>normalisation</u> for the Legendre polynomials.

To show that Rodrigues's formula gives the Legendre polynomials, it suffices to show that $y = h^{(n)}(x)$ is a solution of $(1-x^2)y'' - 2xy' + n(n+1)y = 0$, where $h(x) = (x^2-1)^n$, $h^{(n)} \equiv \frac{d^n}{dx}$

Notice that $h'(x) = 2xn(x^2-1)^{n-1} \Rightarrow (x^2-1)h'(x) = 2xn\,h(x) \Rightarrow (1-x^2)h'(x) + 2nx\,h(x) = 0$. — ⊛. We differentiate ⊛ $n+1$ times, using Leibniz's Rule:

$(fg)^{(n+1)} = \sum_{k=0}^{n+1}\binom{n+1}{k} f^{(k)} g^{(n+1-k)}$. First choose $f(x) = 1-x^2$, $g(x) = h'(x)$. Then $(fg)^{(n+1)} = \binom{n+1}{0}(1-x^2)h^{(n+2)}(x) + \binom{n+1}{1}(-2x)h^{(n+1)}(x) + \binom{n+1}{2}(-2)h^{(n)}(x) + 0$ ($\because f^{(3)} = f^{(4)} = \cdots = 0$)

Then choose $f = 2nx$, $g = h$; $(fg)^{(n+1)} = \binom{n+1}{0}(2nx)h^{(n+1)}(x) + \binom{n+1}{1}(2n)h^{(n)}(x)$. Thus, ⊛ becomes: $(1-x^2)h^{(n+2)}(x) + (n+1)(-2x)h^{(n+1)}(x) + \frac{n(n+1)}{2}(-2)h^{(n)}(x) + 2nxh^{(n+1)}(x) + (n+1)(2n)h^{(n)}$

i.e. $(1-x^2)h^{(n+2)} + (-2x)h^{(n+1)} + n(n+1)h^{(n)} = 0$. Thus $y = h^{(n)}$ is a solution of Legendre's equation, so $h^{(n)}(x)$ is proportional to the Legendre polynomials.

Alternatively, we use the <u>Legendre generating function</u> formula: $\boxed{G(x,t) = \frac{1}{(1-2xt+t^2)^{1/2}} = \sum_{m=0}^{\infty} t^m P_m(x)}$.

As was the case with the Bessel function, this gives us some relations between the polynomials.

Take partial derivatives w.r.t t: $\frac{x-t}{(1-2xt+t^2)^{3/2}} = \sum_{m=0}^{\infty} m t^{m-1} P_m(x) = G(x,t)\left(\frac{x-t}{1-2xt+t^2}\right)$. Then $(x-t)\sum_{m=0}^{\infty} t^m P_m(x) = (1-2xt+t^2)\sum_{m=0}^{\infty} m t^{m-1} P_m(x)$.

$$x\sum_{m=0}^{\infty} t^m P_m(x) - \sum_{m=0}^{\infty} t^{m+1} P_m(x) = \sum_{m=0}^{\infty} m t^{m-1} P_m(x) - 2x\sum_{m=0}^{\infty} m t^m P_m(x) + \sum_{m=0}^{\infty} m t^{m+1} P_m(x) \Rightarrow \sum_{m=0}^{\infty} m t^{m-1} P_m(x) - (2m+1)x\sum_{m=0}^{\infty} t^m P_m(x) + \sum_{m=0}^{\infty} (m+1) t^{m+1} P_m(x) = 0.$$

i.e. $\sum_{m=0}^{\infty}\left[(m+1)P_{m+1}(x) - (2m+1)xP_m(x) + mP_{m-1}(x)\right]t^m = 0$. Set coefficients to 0, which gives $\boxed{(m+1)P_{m+1}(x) - (2m+1)xP_m(x) + mP_{m-1}(x) = 0}$. which is <u>Bonnet's recursion formula</u>.

Where $m=0$, $P_1(x) - xP_0(x) = 0 \Rightarrow P_1(x) = x$. Where $m=1$, $2P_2(x) - 3xP_1(x) + P_0(x) = 0 \Rightarrow 2P_2(x) = 3x^2 - 1 \Rightarrow P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$ indeed.

We could also have done the partial differentiation w.r.t. x, which would have yielded $\boxed{P'_{m+1}(x) - P'_{m-1}(x) = (2m+1)P_m(x)}$, $m \geq 1$, which is the differentiation identity.

Chapter 2

ORTHOGONALITY AND GENERALISED FOURIER SERIES.

Let $f(x)$ be defined on the (periodic) interval $-\pi < x \leq \pi$. Then recall the formula for Fourier series: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx)$, where $\begin{Bmatrix} a_n \\ b_n \end{Bmatrix} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\begin{Bmatrix} \cos(nx) \\ \sin(nx) \end{Bmatrix} dx$

New idea: We think of the set V of functions $f(x)$ defined on $-\pi < x \leq \pi$ as a <u>vector space</u>. Then,

the functions $\{\psi_1(x) = 1,\ \psi_{2j}(x) = \cos(jx),\ \psi_{2j+1}(x) = \sin(jx)$ for $j \geq 1\}$ can be regarded as a basis for the vector space V, since any $f(x)$ in V can be expressed as a linear combination of the $\{\psi_j(x)\}$. (expressed up to equality "almost everywhere", this is a casual statement).

V is equipped with a "natural" inner product: $\langle f,g\rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx$. Under this inner product, notice that $\langle\psi_j,\psi_k\rangle = 0$ if $j \neq k$ as it gives Fourier integrals.

Also, our definition gives that $\langle \psi_j, \psi_j \rangle = \begin{cases} 2\pi & j=1 \\ \pi & j>1 \end{cases}$. i.e. the set $\{\psi_j\}$ is orthogonal.

Notice then that the Fourier series formula can be written $f(x) = \sum_{j=1}^{\infty} \frac{\langle f, \psi_j \rangle}{\langle \psi_j, \psi_j \rangle} \psi_j(x)$, which is the form for Gram-Schmidt orthogonalisation.

We can generalise it further, extending it to other orthogonal bases.

Consider eigenvalue problem $A\underline{x} = \lambda \underline{x}$. If $\mathcal{L} \equiv \frac{d^2}{dx^2}$ is the differential operator. Take $\mathcal{L}y = -\lambda y$, $y(0)=0$, $y(1)=0$ as boundary conditions. $\exists$ non-trivial solutions for $\lambda = k^2\pi^2$.

Then we see that complete orthogonal bases are generated by eigenvalue problems, and occur for differential equations.

The role of the matrix is taken by a linear differential operator: plus boundary conditions.

Ex Solve $\mathcal{L}y = -\lambda y$, $y(0)=y(1)=0$ for $\mathcal{L} \equiv \frac{d^2}{dx^2}$. (This gives equation $y'' + \lambda y = 0$, $y(0)=0, y(1)=0$), to get non-trivial values of λ.

Soln. Our solutions are $y = \begin{cases} A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x) & \lambda>0 \\ Ax+B & \lambda=0 \\ A\cosh(\sqrt{-\lambda}x) + B\sinh(\sqrt{-\lambda}x) & \lambda<0 \end{cases}$. We only have non-zero solutions for first case, by plugging in boundary conditions.

$A=B=0$ for $\lambda \le 0 \Rightarrow$ no eigenvalues. For $\lambda>0$: $y(0)=0 \Rightarrow A=0$, $y(1)=0 \Rightarrow B\sin(\sqrt{\lambda})=0 \Rightarrow \sqrt{\lambda}=k\pi \Rightarrow \lambda = k^2\pi^2$ are non-trivial solutions.

Then $\lambda_k = k^2\pi^2$ are the eigenvalues ($k \ge 1$). These eigenvalues correspond to eigenfunctions $y_k(x) = \sin(k\pi x)$.

Of course, we leave out the arbitrary constant B as eigenfunctions are defined only up to a constant, i.e. we can set $B=1$ WLOG.

Are these found eigenfunctions orthogonal? We use $\langle f, g \rangle = \int_0^1 f(x)\overline{g(x)}\,dx$, then $\langle y_j, y_k \rangle = \int_0^1 \sin(j\pi x)\sin(k\pi x)\,dx = 0$ for $j \ne k$.

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Differential equation eigenvalue problems arise naturally in the method of separation of variables. Boundary conditions play a key role.

Ex Find the eigenvalues and eigenfunctions of $\mathcal{L}y = -\lambda y$ for $\mathcal{L} = \frac{d^2}{dx^2}$, $y(0)=0$ and $y'(1)=0$.

Soln. As always, our only non-trivial solution is $y = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x) \Rightarrow y'(x) = B\sqrt{\lambda}\cos(\sqrt{\lambda}x)$ $\because A=0$. $y'(1)=0 \Rightarrow \cos(\sqrt{\lambda})=0$, for non-trivial solns.

$\therefore \sqrt{\lambda} = \frac{2k+1}{2}\pi$ for $k \in \mathbb{Z}^+ \cup \{0\}$. Then $\lambda_k = \left(\frac{2k+1}{2}\right)^2 \pi^2$ are eigenvalues. Corresponding eigenfunctions are $\sin\left(\frac{2k+1}{2}\pi x\right)$.

Inner product on $L^2(a,b)$

$L^2(a,b)$ is the space of functions f s.t. $\int_a^b |f|^2\,dx$ exists. Then we define $\langle f, g \rangle_w = \int_a^b w(x) f(x)\overline{g(x)}\,dx$ with a real-valued weight-function $w(x)>0$ on (a,b).

Ex Show that $\langle y_j, y_k \rangle_w = 0$ for the previous example ($j \ne k$) for $w(x)=1$.

Soln. $\langle y_j, y_k \rangle_w = \int_0^1 \sin\left(\frac{2j+1}{2}\pi x\right)\overline{\sin\left(\frac{2k+1}{2}\pi x\right)}$

i.e. this example also generates an orthogonal set of eigenfunctions $\{y_k\}$.

What makes these eigenfunctions orthogonal? In linear algebra, real symmetric (or complex Hermitian) matrices generate orthogonal bases.

For a linear operator $\mathcal{L}$ (plus boundary conditions), the corresponding property is that it is self-adjoint.

Definition The adjoint, $\mathcal{L}'$, of a linear operator $\mathcal{L}$ with respect to an inner product $\langle -,- \rangle$ is another linear operator satisfying

$\langle f, \mathcal{L}'g \rangle_w = \langle \mathcal{L}f, g \rangle_w \quad \forall f, g \in L^2(a,b)$.

$\mathcal{L}$ is self-adjoint if $\mathcal{L}' = \mathcal{L}$.

Consider our first example, with $f, g \in L^2(0,1)$ both satisfying $f(0)=f(1)=0$, then $\langle \mathcal{L}f, g \rangle = \int_0^1 f''\bar{g}\,dx$ for $w(x)=1$.

We integrate this by parts, then $\langle \mathcal{L}f, g \rangle = \int_0^1 f''g\,dx = -\int_0^1 f'\bar{g}'\,dx + [f'\bar{g}]_0^1 = -\int_0^1 f'\bar{g}'\,dx$ ($\because g$ satisfies b.c.) $= \int_0^1 f\bar{g}''\,dx + [f\bar{g}']_0^1 = \int_0^1 f\bar{g}''\,dx = \langle f, \mathcal{L}g \rangle$.

Then since $\langle f, \mathcal{L}'g \rangle = \langle \mathcal{L}f, g \rangle$, $\langle f, \mathcal{L}'g \rangle = \langle f, \mathcal{L}g \rangle \Rightarrow \mathcal{L}' = \mathcal{L} \Rightarrow \mathcal{L}$ is self-adjoint.

What is the most general second order linear self-adjoint differential operator there is? The operator is known as the Stürm-Liouville operator, with boundary conditions.

Consider $\mathcal{L}y = -\lambda y$; with boundary conditions $\begin{cases} \alpha_1 y(a) + \beta_1 y'(a) = 0 \\ \alpha_2 y(b) + \beta_2 y'(b) = 0 \end{cases}$ (†) having real coefficients $\alpha_1, \alpha_2, \beta_1, \beta_2$; α_1, β_1 not both zero, for

$\boxed{\mathcal{L} \equiv \frac{1}{w(x)}\left[\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + r(x)\right]}$ with $p(x)$ real, differentiable on $[a,b]$, $p(x)>0$; $r(x)$ real, continuous on (a,b); weight function $w(x)>0$ real continuous on (a,b).

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Proposition $\mathcal{L}$ is self-adjoint with boundary conditions stated above (†)

Proof - NTP: $\langle \mathcal{L}f, g \rangle_w = \langle f, \mathcal{L}g \rangle_w$. LHS $= \langle \mathcal{L}f, g \rangle_w = \int_a^b w(x)\left(\frac{1}{w(x)}\frac{d}{dx}\left(p(x)\frac{df}{dx}\right) + r(x)f(x)\right)\overline{g(x)}\,dx = \int_a^b \left(\frac{d}{dx}\left[p(x)\frac{df}{dx}\right] + r(x)f(x)\right)\overline{g(x)}\,dx$.

Then $\langle \mathcal{L}f, g \rangle_w = \int_a^b ((pf')' + rf)\bar{g}\,dx$. Integrate by parts: $\langle \mathcal{L}f, g \rangle_w = [pf'\bar{g}]_a^b + \int_a^b -pf'\bar{g}' + rf\bar{g}\,dx = [pf'\bar{g} - pf\bar{g}']_a^b + \int_a^b f((p\bar{g}')' + r\bar{g})\,dx$

$= [pf'\bar{g} - pf\bar{g}']_a^b + \int_a^b w(x)f(x)\left[\frac{1}{w(x)}\frac{d}{dx}\left(p(x)\frac{d\bar{g}(x)}{dx}\right) + r(x)\bar{g}(x)\right]dx = [pf'\bar{g} - pf\bar{g}']_a^b + \langle f, \mathcal{L}g \rangle_w$.

Hence $\langle \mathcal{L}f, g \rangle_w = \langle f, \mathcal{L}g \rangle_w$ if the boundary term is 0. Check boundary term at $x=b$. $[\dots]^b = p(b)\left(f'(b)\bar{g}(b) - f(b)\bar{g}'(b)\right)$ for $\alpha_1, \alpha_2, \beta_1, \beta_2$ all non-zero.

i.e. $[\dots]^b = p(b)\Big[\underbrace{\left(-\tfrac{\alpha_2}{\beta_2}f(b)\right)}_{f'(b)}\bar{g}(b) - f(b)\underbrace{\left(-\tfrac{\alpha_2}{\beta_2}\bar{g}(b)\right)}_{\bar{g}'(b)}\Big] = 0$. Likewise, $[\dots]_a = 0$, so we get $[\dots]_a^b = 0$, $\langle \mathcal{L}f, g \rangle_w = \langle f, \mathcal{L}g \rangle_w \Rightarrow \mathcal{L}$ is self-adjoint, q.e.d.

Much theory follows from the self-adjointness of $\mathcal{L}$ concerning the eigenvalue problem $\mathcal{L}y = -\lambda y$ subject to (†).

This gives us the general Stürm-Liouville eigenvalue problem.

Stürm-Liouville eigenvalue problems.

We can solve the eigenvalue problem: $\boxed{\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + (r(x)+\lambda w(x))y = 0 \text{ subject to } (\dagger)\begin{cases}\alpha_1 y(a)+\beta_1 y'(a)=0\\ \alpha_2 y(b)+\beta_2 y'(b)=0\end{cases}}$ $(*)$

Key results; which partial proofs of will be given:

[1] The eigenvalues of $(*)$ are real, and form an infinite unbounded set e.g. if $\lambda_1<\lambda_2<\lambda_3<\cdots$, then $\lambda_k \to \infty$.

[2] Two eigenfunctions y_j and y_k associated with eigenvalues λ_j and λ_k are orthogonal (under b.c. $(\dagger)$ and inner product $\langle\cdot,\cdot\rangle_w$)

i.e. $\langle y_j, y_k\rangle_w = 0$ if $j\neq k$.

[3] The eigenfunction y_j associated with eigenvalue λ_j is unique up to a multiplicative constant.

[4] Functions $f(x)\in L^2(a,b)$ can be expanded in general Fourier series involving the $\{y_k(x)\}$, which form an orthogonal basis for $L^2(a,b)$.

Our generalised Fourier series is given by $\boxed{f(x)=\sum_{k=1}^{\infty}\frac{\langle f,y_k\rangle_w}{\langle y_k,y_k\rangle_w}y_k(x)}$, with equality holding almost everywhere.

(Partial) Proofs — [1] Consider $\langle \mathcal{L}y_k, y_k\rangle_w = \langle -\lambda_k y_k, y_k\rangle_w = -\lambda_k\langle y_k,y_k\rangle_w$. ← y_k is eigenfunction. Also, $\langle \mathcal{L}y_k, y_k\rangle_w \overset{\text{self-adjoint}}{=} \langle y_k, \mathcal{L}y_k\rangle_w = \langle y_k, -\lambda_k y_k\rangle_w = -\bar{\lambda}_k\langle y_k,y_k\rangle_w$

Hence, $\lambda_k=\bar{\lambda}_k \Rightarrow \lambda_k$ is real // q.e.d. [Note: we can cancel $\langle y_k,y_k\rangle_w$ because eigenfunctions are non-zero solutions].

[2] Consider $y_j(x)$ and $y_k(x)$ corresponding to λ_j and λ_k. Then $\langle \mathcal{L}y_j, y_k\rangle_w = \langle -\lambda_j y_j, y_k\rangle_w = -\lambda_j\langle y_j,y_k\rangle_w$.

$\langle \mathcal{L}y_j, y_k\rangle_w = \langle y_j, \mathcal{L}y_k\rangle_w = \langle y_j, -\lambda_k y_k\rangle_w = -\lambda_k\langle y_j,y_k\rangle_w \Rightarrow \lambda_j\langle y_j,y_k\rangle_w = \lambda_k\langle y_j,y_k\rangle_w \Rightarrow (\lambda_j-\lambda_k)\langle y_j,y_k\rangle_w=0$. But $\lambda_j\neq\lambda_k$, so

$\langle y_j,y_k\rangle_w=0 \Rightarrow$ eigenfunctions are orthogonal // q.e.d.

[3] See printed lecture notes.

[4] We assume that the $\{y_k\}$ form a complete basis for $L^2(a,b) \Rightarrow f(x)=\sum_{k=1}^{\infty}a_k y_k(x)$ i.e. we can write any $f(x)\in L^2(a,b)$ as a linear combination of basis vectors

Take inner product with y_j: $\langle f,y_j\rangle_w = \langle \sum_{k=1}^{\infty}a_k y_k, y_j\rangle_w = \sum_{k=1}^{\infty}a_k\langle y_k,y_j\rangle_w = 0 + a_j\langle y_j,y_j\rangle_w \Rightarrow$ for arbitrary index j,

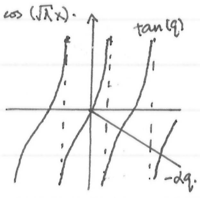
$\langle f,y_j\rangle_w = a_j\langle y_j,y_j\rangle_w \Rightarrow a_k = \frac{\langle f,y_k\rangle_w}{\langle y_k,y_k\rangle_w}$. Hence, $f(x)=\sum_{k=1}^{\infty}\frac{\langle f,y_k\rangle_w}{\langle y_k,y_k\rangle_w}y_k(x)$. // q.e.d.

[Ex] Solve $y''+\lambda y=0$, with $\begin{matrix}y(0)=0\\ y(1)+\alpha y'(1)=0\end{matrix}$ $(\alpha>0)$. Then, prove that $\sum_{k=1}^{\infty}\frac{4(1-\cos q_k)}{2q_k-\sin(2q_k)}\sin(q_k x)=1$ for all x in $0<x<1$.

Soln. This is a Stürm-Liouville eigenvalue problem: $\mathcal{L}=\frac{1}{w}\left(\frac{d}{dx}\left(p\frac{d}{dx}\right)+r\right)$ with $w(x)=1$, $p(x)=1$, $r(x)=0$. Then $\mathcal{L}y=-\lambda y$; with S-L b.c. at $x=0, x=1$.

$y(x)=A\cos(\sqrt{\lambda}x)+B\sin(\sqrt{\lambda}x)$ for $\lambda>0$ (non-trivial solutions). $y(0)=0\Rightarrow A=0$. $y(x)=B\sin(\sqrt{\lambda}x)$, $y'(x)=B\sqrt{\lambda}\cos(\sqrt{\lambda}x)$.

Then $y(1)+\alpha y'(1) = B(\sin(\sqrt{\lambda})+\alpha\sqrt{\lambda}\cos(\sqrt{\lambda}))=0$. Consider $B\neq0$, then consider $\tan(q)+\alpha q=0$, with $q=\sqrt{\lambda}$.



Roots q_k occur at intersections of curves $\tan q$ and $-\alpha q$. There are infinitely many.

Our system has eigenvalues $\lambda_k=q_k^2$, $k=1,2,\ldots$ where $\{q_k\}$ are the roots of $\tan q+\alpha q=0$.

Then, the corresponding eigenfunctions are $y_k=\sin(q_k x)$, which form a complete basis for $L^2(a,b)$.

Above is our generalised Fourier series for $f(x)=1$. $f(x)=1=\sum_{k=1}^{\infty}\frac{\langle 1,y_k\rangle_w}{\langle y_k,y_k\rangle_w}y_k(x)$. $\quad \frac{\int_0^1\sin(q_k x)\,dx}{\int_0^1|\sin(q_k x)|^2dx} = \frac{-\frac{1}{q_k}[\cos(q_k x)]_0^1}{\frac{1}{2}\int_0^1 1-\cos(2q_k x)\,dx} = \frac{-\frac{1}{q_k}(\cos q_k-1)}{\frac{1}{2}[x-\frac{1}{2q_k}\sin(2q_k x)]_0^1}$

$f(x)=\sum_{k=1}^{\infty}\frac{\frac{1}{q_k}(1-\cos q_k)}{\frac{1}{2}[1-\frac{1}{2q_k}\sin(2q_k)]}\sin(q_k x) = \sum_{k=1}^{\infty}\frac{4(1-\cos q_k)}{2q_k-\sin(2q_k)}\sin(q_k x)$ // q.e.d.

General 2^{nd} order problems (from separation of variables).

A typical separation of variables problem leads to $P(x)y''(x)+Q(x)y'(x)+(R(x)+\underbrace{\lambda}_{\text{separation constant}})y(x)=0$. We compare this to Stürm-Liouville equation $\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right)+(r(x)+\lambda w(x))y=0$.

The Stürm-Liouville equation can be re-expressed as $\frac{p(x)}{w(x)}y''+\frac{p'(x)}{w(x)}y'+\left(\frac{r(x)}{w(x)}+\lambda\right)y=0$. We need to choose p, w, r s.t. $\boxed{\frac{p(x)}{w(x)}=P(x),\ \frac{p'(x)}{w(x)}=Q(x),\ \frac{r(x)}{w(x)}=R(x)}$

We get $\frac{p'(x)}{p(x)}=\frac{Q(x)}{P(x)} \Rightarrow \frac{d}{dx}(\log p(x))=\frac{Q(x)}{P(x)}$, and $p(x)=\exp\left(\int^x\frac{Q(t)}{P(t)}dt\right)$ is the integrating factor. For w, r, we get $w=\frac{p}{P}$, $r=\frac{Rp}{P}$.

Conclusion: We can always convert our typical separation of variables problem into a Stürm-Liouville problem provided that $P(x)$ has no roots in $a<x<b$.

(s.t. $w>0$). i.e. the separation of variables equation has no singular points on $a<x<b$.

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There are almost unlimited possibilities for expanding functions $f(x)\in L^2(a,b)$ in different generalised Fourier series.

[Ex] Use the eigenvalue problem $x^2y''-xy'+\lambda y=0$ $\quad y(1)=y(e^{\pi})=0$. to expand $f(x)$ defined on $1<x<e^{\pi}$ in a generalised Fourier series.

Recall: Write $Y(t)=y(e^t)$, $x=e^t$. $\frac{dY}{dt}=e^t y'(e^t)=xy'$, $\frac{d^2Y}{dt^2}=e^{2t}y''(e^t)+e^t y'(e^t)=x^2y''+xy'$. Equation becomes $\frac{d^2Y}{dt^2}-2\frac{dY}{dt}+\lambda Y=0$.

Soln. $\frac{d^2Y}{dt^2}-2\frac{dY}{dt}+\lambda Y=0$, $\quad Y(t)=\begin{cases}e^t(A\cos\sqrt{\lambda-1}t)+B\sin(\sqrt{\lambda-1}t) & \lambda>1\\ e^t(At+B) & \lambda=1\\ Ae^{(1+\sqrt{1-\lambda}t)}+Be^{(1-\sqrt{1-\lambda}t)} & \lambda<1.\end{cases}$ By our boundary conditions, we only get non-trivial solutions

for $\lambda>1$. Then, $y(1)=0\Rightarrow A=0$. $y(e^{\pi})=0 \Rightarrow B\sin(\sqrt{\lambda-1}\pi)=0 \Rightarrow \sqrt{\lambda-1}=k\in\mathbb{Z}$, $\lambda_k=k^2+1$ are our eigenvalues.

The corresponding eigenfunctions are $y_k(x)=x\sin(k\log x)$.

For a generalised Fourier series, we need the $\{y_k(x)\}$ to be orthogonal with respect to an inner product — need to convert system into S-L form

Find integrating factor: $IF=e^{\int\frac{Q}{P}}=e^{\int-\frac{1}{x}dx}=e^{-\log x}=\frac{1}{x}=p(x) \Rightarrow$ S-L form must be $x^3\frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dx}\right)+\lambda y=0 \Rightarrow w(x)=\frac{1}{x^3}, p(x)=\frac{1}{x}, r(x)=0$

Hence, $f(x)=\sum_{k=1}^{\infty}\frac{\langle f,y_k\rangle_w}{\langle y_k,y_k\rangle_w}y_k(x)$, where $\langle f,y_k\rangle_w=\int_1^{e^{\pi}}\frac{1}{x^3}f(x)\,x\sin(k\log x)\,dx$, $\quad\langle y_k,y_k\rangle_w=\int_1^{e^{\pi}}\frac{1}{x^3}x^2\sin^2(k\log x)\,dx$.

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$\langle y_k, y_k \rangle_w = \int_1^{e^\pi} \sin^2(k \log x) \frac{dx}{x} = \int_0^\pi \sin^2(kt)\, dt = \frac{\pi}{2}$. then $f(x) = \sum_{k=1}^{\infty} \frac{2}{\pi} \left(\int_1^{e^\pi} \frac{f(q)}{q^2} \sin(k \log q)\, dq \right) \times \sin(k \log x)$ //.

<u>Singular Sturm-Liouville systems.</u>

We return to the proof of the self-adjointness of $\mathcal{L}$. We had:

$\langle \mathcal{L}f, g \rangle_w = \langle f, \mathcal{L}g \rangle_w + [p(f\bar{g}' + f'\bar{g})]_a^b$. In a regular Sturm-Liouville system, we had boundary conditions (†) that made the boundary terms vanish i.e. $[p(f\bar{g}' + f'\bar{g})]_a^b = 0$. We also had $a \le p(x) \le b$.

In a singular Sturm-Liouville system, we have $p(a)=0$ and/or $p(b)=0$. The boundary condition (†) is then not needed at $x=a$ (or $x=b$).

It is replaced by the condition: $y(a)$ finite and/or $y(b)$ finite.

Ex $(1-x^2)y'' - 2xy' - \lambda y = 0$ is Legendre's equation. Analyse solutions at $x = \pm 1$.

soln. $(1-x^2)y'' + (1-x^2)'y - \lambda y = 0$ (✱) $\Rightarrow \frac{d}{dx}((1-x^2)y') + \lambda y = 0$, which is in Sturm-Liouville form with $w(x)=1$, $p(x) = 1-x^2$, $r(x) = 0$.

Recall that our solutions are $y(x) = AP_\nu(x) + BQ_\nu(x)$, where P_ν, Q_ν are Legendre functions.

In general, at $x=\pm1$ solution is singular (i.e. $x = \pm 1$ are R.S.P. of (✱)) $\Rightarrow (1-x^2)y'' - 2xy' + \lambda y = 0$, $y(\pm1)$ finite is a singular S-L system with eigenvalues $\lambda_k = k(k+1)$ $(k \in \mathbb{Z})$, $y_k(x) = P_k(x)$. $\because$ the $P_k(x)$ are the only finite solutions of (✱).

Orthogonality: We know that $\int_{-1}^1 P_j(x) P_k(x) = 0$ for $j \neq k$. follows immediately. then we have our generalised Fourier series:

For $-1 \le x \le 1$, $f(x) = \sum_{k=0}^\infty \frac{\langle f, y_k \rangle}{\langle y_k, y_k \rangle} y_k(x) = \sum_{k=0}^\infty a_k P_k(x)$ where $a_k = \frac{\int_{-1}^1 f(x) P_k(x)\, dx}{\int_{-1}^1 (P_k(x))^2\, dx}$. We seek value of $\int_{-1}^1 f(x) P_k(x)\, dx$.

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Use the generating function $G(x,t) = \frac{1}{(1-2xt+t^2)^{1/2}} = \sum_{k=0}^\infty t^k P_k(x)$. Square both sides, then integrate to get:

$\int_{-1}^1 \frac{dx}{1-2xt+t^2} = \int_{-1}^1 \left(\sum_{k=0}^\infty t^k P_k(x) \right)\left(\sum_{k=0}^\infty t^k P_k(x) \right) dx = \sum_{k=0}^\infty t^{2k} \int_{-1}^1 (P_k(x))^2\, dx$ (by orthogonality). LHS $= \left[-\frac{1}{2t} \log(1-2xt+t^2) \right]_{x=-1}^{x=1} = \frac{1}{2t}\left[\log(1+t)^2 - \log(1-t)^2 \right]$

LHS $= \frac{1}{t}[\log(1+t) - \log(1-t)] = \frac{1}{t}\left[(t + \frac{t^2}{2} + \frac{t^3}{3} + \cdots) - (-t + \frac{t^2}{2} - \frac{t^3}{3} + \cdots) \right] = \frac{1}{t}\left[2t + \frac{2t^3}{3} + \frac{2t^5}{5} + \cdots \right] = \sum_{t=0}^\infty \frac{2t^{2k}}{2k+1}$. Hence,

comparing both sides, $\int_{-1}^1 (P_k(x))^2 = \frac{2}{2k+1} \Rightarrow \boxed{f(x) = \frac{2k+1}{2}\left(\int_{-1}^1 f(x) P_k(x)\, dx \right) P_k(x)}$, which is the <u>Fourier-Legendre series</u>.

Ex Consider the eigenvalue problem $x^2 y'' + xy' + (-m^2 + \lambda x^2) y = 0$, $m \in \mathbb{Z}$, with $y(0)$ finite and $y(1) = 0$.

soln. Convert this into Sturm-Liouville form. Integrating factor is $e^{\int \frac{x}{x^2} dx} = e^{\log x} = x$. We get $x(xy')' - (m^2 - \lambda x^2) y = 0$

Divide by x^2: $\frac{1}{x}\frac{d}{dx}(x \frac{dy}{dx}) - \frac{m^2}{x^2} y = -\lambda y$, or $\mathcal{L}y = -\lambda y$, with $p(x) = x$, $w(x) = x$, $r(x) = -\frac{m^2}{x}$ $\quad p(0) = 0 \Rightarrow$ system is singular at $x=0$

$\therefore$ We have a Sturm-Liouville eigenvalue problem. This is similar to Bessel's equation if $\lambda = 1$.

To solve this, we need to perform a change of variables: Write $q = \sqrt{\lambda} x$ and $Y(q) = y(x(q))$, $x(q) = \frac{q}{\sqrt{\lambda}}$. Use the chain rule:

$\frac{dY}{dq} = y' \frac{dx}{dq} = \frac{1}{\sqrt{\lambda}} y'$, $\quad \frac{d^2Y}{dq^2} = \frac{1}{\lambda} y''$. then we have: $\lambda x^2 \frac{d^2Y}{dq^2} + \sqrt{\lambda} x \frac{dY}{dq} + (-m^2 + \lambda x^2) Y = 0 \Rightarrow q^2 \frac{d^2Y}{dq^2} + q \frac{dY}{dq} + (-m^2 + q^2) Y = 0$

This is Bessel's equation of index m: $Y(q) = AJ_m(q) + BY_m(q)$ is the general solution for $m \in \mathbb{Z}$. Substitute to get

$y(x) = AJ_m(\sqrt{\lambda}x) + BY_m(\sqrt{\lambda}x)$. At $x=0$, $y(0)$ is finite $\Rightarrow B = 0$ since $Y_m(0)$ is singular at $x=0 \Rightarrow y(x) = AJ_m(\sqrt{\lambda}x)$.

At $x=1$, $y(1) = 0 \Rightarrow AJ_m(\sqrt{\lambda}) = 0$. Bessel functions have infinitely many zeros, so $J_m(\sqrt{\lambda}) = 0$ i.e. $A \neq 0$ at $\sqrt{\lambda} = j_{mk}$, $k \in \mathbb{N}$

where $\{j_{mk}\}$ are the zeros of J_m. $\Rightarrow$ eigenvalues are $\lambda_k = j_{mk}^2$, eigenfunctions are $y_k(x) = J_m(j_{mk}x)$.

Orthogonality: $\langle y_j, y_k \rangle_w = \int_0^1 x J_m(j_{mj}) J_m(j_{mk})\, dx = 0 \quad \forall j \neq k$. then we obtain our generalised Fourier series:

$f(x)$ on $0 \le x \le 1$ is given by $f(x) = \sum_{k=1}^\infty \frac{\langle f, y_k \rangle_w}{\langle y_k, y_k \rangle_w} y_k(x) = \sum_{k=1}^\infty a_k J_m(j_{mk}x)$ where $a_k = \frac{\int_0^1 x f(x) J_m(j_{mk}x)\, dx}{\int_0^1 x (J_m(j_{mk}x))^2 dx}$

$\int_0^1 x(J_m(j_{mk}x))^2 dx = \frac{(J_{m+1}(j_{mk}))^2}{2}$ (verify on Sheet 3, Q2(d)). Hence, $\boxed{a_k = \frac{2}{(J_{m+1}(j_{mk}))^2} \int_0^1 x f(x) J_m(j_{mk}x)\, dx}$, the <u>Fourier-Bessel series</u>.

<u>Periodic Sturm-Liouville systems.</u>

Our ordinary trigonometric Fourier series are neither regular nor singular Sturm-Liouville systems. In fact, they derive from the <u>periodic Sturm-Liouville system</u> defined by $y'' + \lambda y = 0$, $\boxed{\begin{cases} y(\pi) = y(-\pi) \\ y'(\pi) = y'(-\pi) \end{cases}}$ (periodic boundary conditions).

This leads to $\lambda_k = k^2$, $k \ge 0$. eigenfunctions are $y_0(x) = 1$; and two eigenfunctions each for every $k \ge 1$: $y_{k_1}(x) = \cos(kx)$, $y_{k_2} = \sin(kx)$

We call the property of having two eigenfunctions for an eigenvalue: <u>degeneracy</u>.

Degeneracy does not occur for regular or singular problems: see $\boxed{3}$ in theory on the printed lecture notes.

Chapter 3

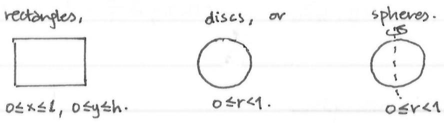
SEPARATION OF VARIABLES, REVISITED...

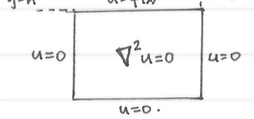
In this chapter, we aim to solve linear partial differential equations: such as $\nabla^2 u = 0$ (Laplace's), $u_t = \nabla^2 u$ (heat), $u_{tt} = \nabla^2 u$ (wave).

We will be operating on a few geometries: such as — rectangles, discs, or spheres.

To begin with, we return to a simpler problem:

Laplace's equation in a rectangle.





Ex. Consider Laplace's equation on the rectangle. Find the steady temperature distribution $u(x,y)$ in a rectangle when the sides are held at fixed temperature. (Dirichlet boundary conditions).

Soln. Write $u(x,y) = X(x)Y(y)$ $\quad \nabla^2 u = X''Y + XY'' = 0 \Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$ (-ve sign leads to positive eigenvalues).

Then $\begin{cases} X'' + \lambda X = 0 \\ Y'' - \lambda Y = 0. \end{cases}$ We solve the first equation as it is a Sturm-Liouville problem (has Sturm-Liouville boundary conditions).

$X'' + \lambda X = 0,\ \overset{u(0,y)=0,\ u(L,y)=0}{X(0) = 0,\ X(L) = 0.}$ is our eigenvalue problem. $\quad X(x) = \begin{cases} A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x) & \lambda > 0. \\ \text{trivial solutions} & \lambda \le 0. \end{cases}$

Using $X(0) = 0 \Rightarrow A = 0$. then $X(L) = 0$ gives $0 = B\sin(\sqrt{\lambda}L) \Rightarrow \sqrt{\lambda_k}L = k\pi \Rightarrow \lambda_k = \frac{k^2\pi^2}{L^2}$ are our eigenvalues.

Corresponding eigenfunctions are $X_k(x) = \sin\left(\frac{k\pi x}{L}\right)$. We then move on to the y-equation: $Y'' - \lambda Y = 0 \Rightarrow$

$Y_k'' - \frac{k^2\pi^2}{L^2} Y_k = 0;\ \ Y_k(y) = C_k \cosh\left(\frac{k\pi}{L}y\right) + D_k \sinh\left(\frac{k\pi}{L}y\right)$. Use other boundary conditions $Y_k(0) = 0 \Rightarrow C_k = 0.\ \forall k.$

Then $Y_k(y) = D_k \sinh\left(\frac{k\pi y}{L}\right)$. We can now write the general solution: $u(x,y) = \sum_{k=1}^{\infty} X_k(x)Y_k(y) = \sum_{k=1}^{\infty} D_k \sin\left(\frac{k\pi x}{L}\right)\sinh\left(\frac{k\pi y}{L}\right)$.

At the top boundary, $u(x,h) = f(x)$. Write as $\sum_{k=1}^{\infty} E_k X_k(x) = f(x)$ where $E_k = D_k \sinh\left(\frac{k\pi y}{L}\right)\Big|_{y=h} = D_k \sinh\left(\frac{k\pi h}{L}\right)$.

Take inner product with X_j, then $\left\langle \sum_{k=1}^{\infty} E_k X_k, X_j \right\rangle = E_j \langle X_j, X_j \rangle = \langle f, X_j \rangle \Rightarrow E_j = \frac{\langle f, X_j\rangle}{\langle X_j, X_j\rangle} = \frac{\int_0^L f(x)\sin\left(\frac{j\pi x}{L}\right)dx}{\int_0^L \sin^2\left(\frac{j\pi x}{L}\right)dx}.$

$\int_0^L \sin^2\left(\frac{j\pi x}{L}\right) = \frac{L}{2} \Rightarrow E_j = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{j\pi x}{L}\right)dx$ //

If $f(x) = \frac{4x(L-x)}{L^2}$, find $u(x,y)$.

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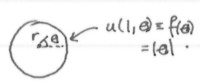
Soln. By application of the formula we have found, $E_k = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{k\pi x}{L}\right)dx = \begin{cases} 32/\pi^3k^3 & k \text{ odd} \\ 0 & k \text{ even.} \end{cases}$

Write $k = 2m+1$, then $u(x,y) = \frac{32}{\pi^3}\sum_{m=0}^{\infty} \frac{\sin\left(\frac{(2m+1)\pi x}{L}\right)\sinh\left(\frac{(2m+1)\pi y}{L}\right)}{(2m+1)^3 \sinh\left(\frac{(2m+1)\pi h}{L}\right)}$

Ex. Laplace's equation on a circle, with $-\pi < \theta \le \pi$, $0 \le r \le 1$, $u(1,\theta) = f(\theta) = |\theta|$.



Soln. $\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$. Separate variables $u(r,\theta) = R(r)T(\theta)$. Then $\nabla^2 u = \frac{T}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{R}{r^2}\frac{d^2T}{d\theta^2} = 0.$

Divide through by $\frac{RT}{r^2}$: $\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = -\frac{1}{T}\frac{d^2T}{d\theta^2} = \lambda$. (separation constant).

Notice the absence of regular/singular S-L boundary conditions, instead consider periodic problem: $\frac{d^2T}{d\theta^2} = -\lambda T$ takes periodic boundary conditions

$T(-\pi) = T(\pi),\ T'(-\pi) = T'(\pi).\ \ T'' + \lambda T = 0 \Rightarrow T(\theta) = \begin{cases} A\cos(\sqrt{\lambda}\theta) + B\sin(\sqrt{\lambda}\theta) & \lambda > 0 \\ A\theta + B & \lambda = 0 \\ \text{trivial solutions} & \text{otherwise.} \end{cases}$ For $\lambda = 0$, $T(-\pi) = T(\pi) \Rightarrow A = 0$ but B arbitrary.

Eigenfunction: $T_0(\theta) = 1$. Then for $\lambda > 0$, then functions are periodic on $[-\pi,\pi]$ if $\lambda = k^2$. We have two eigenfunctions $T_{k_1}(\theta) = \cos k\theta$, $T_{k_2}(\theta) = \sin k\theta$.

Then for R-equation: $\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = \lambda \Rightarrow r^2\frac{d^2R_k}{dr^2} + r\frac{dR_k}{dr} - k^2R_k = 0$, which is an Euler-type equation. We try $R_k(r) = r^p$

Then $p(p-1)r^p + pr^p - k^2r^p = 0 \Rightarrow p^2 - k^2 = 0 \Rightarrow p = \pm k$. $R_k(r) = \frac{C_k}{r^k} + D_k r^k$ for $k \ge 1$. If $\lambda = 0$, $\frac{d}{dr}\left(r\frac{dR_0}{dr}\right) = 0 \Rightarrow R_0 = C_0 \log r + D_0$.

Solution must be finite (by physical limitations) at $r = 0 \Rightarrow C_k = 0.\ \forall k$. Then $R_k(r) = D_k r^k$.

General solution is: $u(r,\theta) = R_0(r)T_0(\theta) + \sum_{k=1}^{\infty} R_k(r)\overbrace{\left(a_k T_{k_1}(\theta) + b_k T_{k_2}(\theta)\right)}^{\text{lin. comb}} = \frac{a_0}{2} + \sum_{k=1}^{\infty} r^k(a_k\cos k\theta + b_k\sin k\theta)$ $\quad D_k = 1,\ D_0 = \frac{a_0}{2}$ WLOG. D_k can be absorbed into a_k, b_k.

Use boundary condition at $r = 1$: $u(1,\theta) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k\cos k\theta + b_k\sin k\theta = f(\theta) = |\theta|$ $\quad -\pi < \theta \le \pi$.

Get constants a_0, a_k, b_k by forming Fourier integrals: $\begin{Bmatrix} a_k \\ b_k \end{Bmatrix} = \frac{1}{\pi}\int_{-\pi}^{\pi}|\theta|\begin{Bmatrix}\cos k\theta \\ \sin k\theta\end{Bmatrix}d\theta$. Since $|\theta|$ is even, $\sin k\theta$ is odd $\Rightarrow b_k = 0$.

$a_k = \frac{2}{\pi}\int_0^{\pi}\theta\cos(k\theta)\,d\theta = \frac{2}{\pi}\left(\left[\theta\cdot\frac{1}{k}\sin(k\theta)\right]_0^{\pi} - \frac{1}{k}\int_0^{\pi}\sin(k\theta)\right) = +\frac{2}{k\pi}\left[\frac{1}{k}\cos(k\theta)\right]_0^{\pi} = \frac{2}{k^2\pi}\left[(-1)^k - 1\right] = \begin{cases} -\frac{4}{\pi k^2} & k \text{ odd} \\ 0 & k \text{ even.} \end{cases}$

$a_0 = \frac{2}{\pi}\int_0^{\pi}\theta\,d\theta = \pi$. Hence, $u(1,\theta) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\cos((2m+1)\theta)}{(2m+1)^2}r^{2m+1}$ //.

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Ex. Heat equation on the "radiating" rod. Heat equation: $u_t = u_{xx}$, unit: thermal diffusivity.

$0 < x < 1,\ t > 0$. Boundary conditions: $u(0,t) = 0$ (fixed temperature on LHS of rod), $\frac{\partial u}{\partial x}(1,t) = -\frac{1}{\alpha}u(1,t)$ (radiation condition).

Initial condition: $u(x,0) = 1$ (uniform initial temperature).

Soln. Separate variables: $u(x,t) = X(x)T(t)$. Insert into heat equation: $X\frac{dT}{dt} = T\frac{d^2X}{dx^2}$, divide by XT — $\frac{1}{T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} = -\lambda$ (separation constant).

Attempt to solve $\frac{1}{X}\frac{d^2X}{dx^2} = -\lambda \Rightarrow X'' + \lambda X = 0$. $X(0) = 0,\ X'(1) = -\frac{1}{\alpha}X(1) \Rightarrow \alpha X'(1) + X(1) = 0$. This is a Sturm-Liouville problem (c.f. Ex3, §2)

$X(x) = \begin{cases} A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x & \lambda > 0 \\ \text{trivial solutions} & \text{otherwise.} \end{cases}$ $X(0) = 0 \Rightarrow A = 0$. $X(x) = B\sin(\sqrt{\lambda}x)$, $X'(x) = B\sqrt{\lambda}\cos(\sqrt{\lambda}x)$. $\alpha X'(1) + X(1) = B\left[\alpha\sqrt{\lambda}\cos(\sqrt{\lambda}) + \sin(\sqrt{\lambda})\right] = 0.$

$\alpha\sqrt{\lambda}\cos\sqrt{\lambda} + \sin\sqrt{\lambda} = 0$. $q = \sqrt{\lambda}$ must be root of $\alpha q + \tan q = 0$. Draw graphs of $-\alpha q$, $\tan q$ $(q > 0)$ to demonstrate that there are infinitely many solutions $\{q_k\} \Rightarrow$ eigenvalues are $\lambda_k = q_k^2$ //, eigenfunctions are $X_k(x) = \sin(q_k x)$.

Now examine T-equation for eigenvalues $\lambda_k = q_k^2$. $\frac{dT_k}{dt} + q_k^2 T_k = 0 \Rightarrow T_k(t) = A_k e^{-q_k^2 t}$. Hence, general solution is $u(x,t) = \sum_{k=1}^{\infty} X_k(x)T_k(t)$

$\Rightarrow u(x,t) = \sum_{k=1}^{\infty} A_k \sin(q_k x)e^{-q_k^2 t}$. To get specific solution, use initial conditions — by Sturm-Liouville theory, we can obtain $\{A_k\}$.

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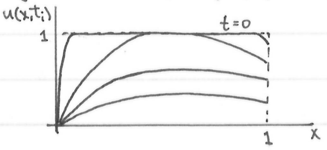
$u(x,0) = \sum_{k=1}^{\infty} A_k \sin(q_k x) = 1 = \sum_{k=1}^{\infty} X_k(x)$. Take inner product with X_j, which yields $\langle \sum_{k=1}^{\infty} A_k X_k, X_j \rangle = \langle 1, X_j \rangle \Rightarrow \sum_{k=1}^{\infty} A_k \langle X_k, X_j \rangle = A_j \langle X_j, X_j \rangle = \langle 1, X_j \rangle$.

Thus, $A_j = \frac{\langle 1, X_j \rangle}{\langle X_j, X_j \rangle} = \frac{\int_0^1 \sin q_j x \, dx}{\int_0^1 \sin^2 q_j x \, dx} = \frac{4(1-\cos q_j)}{2q_j - \sin(2q_j)}$. Hence, solution is given by:

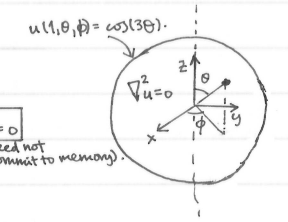
$u(x,t) = \sum_{k=1}^{\infty} \frac{4(1-\cos q_k)}{2q_k - \sin(2q_k)} \sin(q_k x)\, e^{-q_k^2 t}$ //

At late times, $t \to \infty \Rightarrow u \sim \sin(q_1 x)$ //, as $e^{-q_2^2 t} \ll e^{-q_1^2 t}$ etc.



Ex Laplace's equation in a sphere (of unit radius)

Find the steady temperature distribution inside the sphere, given the applied surface temperature $u(1,\theta,\phi) = \cos(3\theta)$.



soln. Use Laplace's equation in spherical polar coordinates (r,θ,ϕ): $\boxed{\nabla^2 u \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} = 0}$ (need not commit to memory).

Here, $0 \le r \le 1$, $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$. We observe that boundary condition does not depend on ϕ, so we seek axisymmetric ϕ-independent solutions. Then $\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial \theta}\right) = 0$.

Write $u(r,\theta) = R(r)\,T(\theta)$, then $0 = \frac{T}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{R}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right) = 0$. Divide by $\frac{RT}{r^2}$: $\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = -\frac{1}{T\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right) = \lambda$ (separation constant).

Consider T-equation: $\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right) + \lambda T = 0$. Use $z = \cos\theta$ and define $T(\theta) = w(z(\theta))$. Then apply chain rule:

$\frac{dT}{d\theta} = \frac{dw}{dz}\cdot\frac{dz}{d\theta} = \frac{dw}{dz}(-\sin\theta) \Rightarrow \sin\theta\frac{dT}{d\theta} = -\sin^2\theta\frac{dw}{dz} = -(1-z^2)\frac{dw}{dz}$. $\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right) = -\frac{d}{dz}\left[(1-z^2)\frac{dw}{dz}\right]\cdot\frac{dz}{d\theta} = -\frac{d}{dz}\left[(1-z^2)\frac{dw}{dz}\right](-\sin\theta)$.

Thus, $\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right) = \frac{d}{dz}\left((1-z^2)\frac{dw}{dz}\right) \Rightarrow \frac{d}{dz}\left[(1-z^2)\frac{dw}{dz}\right] + \lambda w = 0$, which is Legendre's equation.

At $z=\pm 1$, $\theta = 0, \pi$. Then at poles, $T(\theta)$ is finite $\Rightarrow w(\pm 1)$ is finite. Hence, equation and boundary conditions yield a singular Sturm-Liouville system.

General solution is $\lambda = \nu(\nu+1) \Rightarrow w(z) = A P_\nu(z) + B Q_\nu(z)$. This is singular at $z=\pm 1$, except when $\nu \in \mathbb{N}$. Then we get Legendre polynomials.

Eigenvalues: $\lambda_k = k(k+1)$, eigenfunctions $w_k(z) = P_k(z) \Rightarrow T_k(\theta) = P_k(\cos\theta)$ by back substitution.

Then we consider R-equation. $\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \lambda R = 0 \Rightarrow$ insert $\lambda_k = k(k+1)$. $r^2\frac{d^2R_k}{dr^2} + 2r\frac{dR_k}{dr} - k(k+1)R_k = 0$ (Euler equation).

Try $R_k(r) = r^p$. Then $r^p[p(p-1) + 2p - k(k+1)] = 0 \Rightarrow p(p+1) - k(k+1) = 0 \Rightarrow p = k$ or $-(k+1)$. $R_k(r) = A_k r^k + \frac{B_k}{r^{k+1}}$ $(k \ge 0)$.

Since temperature is continuous, solution is finite (non-singular) at origin $r=0 \Rightarrow B_k = 0 \Rightarrow R_k(r) = A_k r^k$.

Then, general solution is $u(r,\theta) = \sum_{k=0}^{\infty} T_k(\theta) R_k(r) = \sum_{k=0}^{\infty} A_k r^k P_k(\cos\theta)$. Apply boundary conditions at $r=1$ to find $\{A_k\}$.

By de Moivre's theorem, $\cos 3\theta = \mathrm{Re}(\cos\theta + i\sin\theta)^3 = 4\cos^3\theta - 3\cos\theta = \sum_{k=0}^{\infty} A_k P_k(\cos\theta)$.

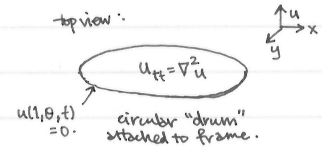
Using Rodrigues's formula, $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$, $P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$. Letting $x = \cos\theta$, we clearly require only A_1, A_3 to be non-zero.

Thus, $A_1\cos\theta + A_3\left(\frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta\right) = \left(A_1 - \frac{3}{2}A_3\right)\cos\theta + \frac{5}{2}A_3\cos^3\theta$. By comparing coefficients, $A_3 = \frac{8}{5}$, $A_1 = -\frac{3}{5}$.

Solution is thus $u(r,\theta) = \frac{8}{5}r^3 P_3(\cos\theta) - \frac{3}{5}r P_1(\cos\theta)$ //.

Ex Waves on a circular membrane (such as a drum).

Domain: $0 \le r \le 1$, $0 \le \theta \le 2\pi$, $t > 0$. Initial conditions: $u(r,\theta,0) = f(r)$ (initial displacement), $u_t(r,\theta,0) = g(r)$ (initial velocity).



Wave equation, with u as the vertical displacement

soln. We observe that f, g depend only on radius $\Rightarrow$ axisymmetric setup. Hence, we seek solutions that are θ-independent.

Separate to get $u(r,t) = R(r)T(t)$. Recall that $\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \Rightarrow \nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$.

Wave equation yields: $R\frac{d^2T}{dt^2} = \frac{T}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) \Rightarrow \frac{1}{T}\frac{d^2T}{dt^2} = \frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = -\lambda$. We cannot use T-equation, not Sturm-Liouville.

R-equation is $\frac{d}{dr}\left(r\frac{dR}{dr}\right) + r\lambda R = 0 \Rightarrow$ Bessel's equation of index 0. This is a Sturm-Liouville equation, with $w(r) = r$, $p(r) = r$.

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Boundary conditions are $R(1) = 0$, $R(0)$ finite. Substitute $R(r) = Y(q(r))$ with $q(r) = \sqrt{\lambda}r$. Equation then becomes:

$q\frac{d^2Y}{dq^2} + \frac{dY}{dq} + qY = 0$, the "standard" Bessel's equation. General solution is $Y(q) = AJ_0(q) + BY_0(q)$. or $R(r) = A J_0(\sqrt{\lambda}r) + BY_0(\sqrt{\lambda}r)$.

$Y(0)$ is finite $\Rightarrow B = 0$. $R(1) = 0 \Rightarrow AJ_0(\sqrt{\lambda}) = 0 \Rightarrow \lambda_k = j_{0k}^2$ are eigenvalues, $J_0(j_{0k}x) = y_k$ are eigenfunctions. where j_{0k}, $k \ge 1$ are Bessel zeros.

Now look at T-equation: $\frac{d^2T}{dt^2} + \lambda T = \frac{d^2T_k}{dt^2} + j_{0k}^2 T = 0$ (SHM): $T_k(t) = a_k\cos(j_{0k}t) + b_k\sin(j_{0k}t) = \mathrm{Re}\left[A_k e^{ij_{0k}t}\right]$ with $A_k = a_k - ib_k$.

Then general solution is $u(r,t) = \sum_{k=1}^{\infty} R_k(r)T_k(t) = \mathrm{Re}\left[\sum_{k=1}^{\infty} A_k e^{ij_{0k}t} J_0(j_{0k}r)\right]$. We now use initial conditions to find $\{A_k\}$.

At $t=0$, $u(r,0) = \mathrm{Re}\left[\sum_{k=1}^{\infty} A_k J_0(j_{0k}r)\right] = \sum_{k=1}^{\infty} a_k R_k(r) = f(r)$. Use inner product with R_j. $a_j\langle R_j, R_j\rangle_w = \langle f, R_j\rangle_w$. Then

$a_j = \frac{\int_0^1 r f(r) J_0(j_{0k}r)\,dr}{\int_0^1 r\,(J_0(j_{0k}r))^2\,dr} = \frac{2}{(J_1(j_{0k}))^2}\int_0^1 r f(r) J_0(j_{0k}r)\,dr$. We can also use the other condition $- u_t(r,0) = g(r)$ in order to show that

$b_j = \frac{2}{j_{0k}(J_1(j_{0k}))^2}\int_0^1 r g(r) J_0(j_{0k}r)\,dr$. This gives us our final solution. //

Comments: • Each "normal mode" (solution for one value of k) has angular frequency $\omega_k = j_{0k}$ $k \ge 1$.

Compare waves on a (unit length) string, $\omega_k = k\pi$. $k \ge 1$. These are all integer multiples of the fundamental mode ω_1. However, j_{02}, j_{03}, j_{04} etc. certainly are not integer multiples of j_{01} — which is why drums are unable to produce notes like stringed instruments.

• Note that unlike stringed solution, membrane solution is not periodic in time (Bessel functions are not in phase).

Chapter 4
INTEGRAL TRANSFORMS.

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So far, we have been working in bounded domains. Can we extend our analysis to infinite or semi-infinite domains?

To do that, we need to extend the idea of Fourier series to $\mathbb{R}$ (the real line).

<u>Fourier Transforms</u>.

In this course, we will provide only a sketch proof of Fourier transform theory.

Start with Fourier series on $[-L,L]$: $f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right)$, where $\begin{Bmatrix} a_m \\ b_m \end{Bmatrix} = \frac{1}{L}\int_{-L}^{L} f(x) \begin{Bmatrix} \cos\left(\frac{m\pi x}{L}\right) \\ \sin\left(\frac{m\pi x}{L}\right)\end{Bmatrix} dx$.

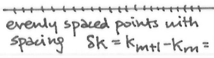
We can also express this in complex form: $\boxed{f(x) = \sum_{m=-\infty}^{\infty} A_m e^{\frac{im\pi x}{L}}, \text{ where } A_m = \frac{a_m - ib_m}{2} = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{\frac{-im\pi x}{L}} dx,\ A_{-m} = A_m^*}$ the complex conjugate.

To see that this is an equivalent representation, $f(x) = A_0 + \sum_{m=1}^{\infty} A_m e^{i\frac{m\pi x}{L}} + A_{-m} e^{i\frac{m\pi x}{L}} = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(\frac{a_m - ib_m}{2}\right)(\cos(\,)+i\sin(\,)) + \left(\frac{a_m+ib_m}{2}\right)(\cos(\,)+i\sin(\,))$.

Combining our formulae above, replacing variable of integration to t, we get $f(x) = \sum_{m=-\infty}^{\infty}\left(\frac{1}{2L}\int_{-L}^{L} f(t)\, e^{-\frac{im\pi t}{L}} dt\right) e^{\frac{im\pi x}{L}}$.

Now, we take the limit $L \to \infty$ to generalise the Fourier series to the real line $\mathbb{R}$. To do this, we introduce some definitions.

First: Define the set of points $\{k_m : k_m = \frac{m\pi}{L},\ m \in \mathbb{Z}\}$. Then spacing $\delta k = \frac{\pi}{L}$.


$\{k_m\}$.
evenly spaced points with spacing $\delta k = k_{m+1} - k_m = \frac{\pi}{L}$.

Notice that the $\{k_m\}$ become <u>dense</u> in $\mathbb{R}$ in the limit $L \to \infty$.

Then we now have $f(x) = \sum_{m=-\infty}^{\infty} \frac{1}{2\pi}\cdot\frac{\pi}{L}\left(\int_{-L}^{L} f(t)\, e^{-i\frac{m\pi}{L}t} dt\right) e^{i\frac{m\pi}{L}t} = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} \delta k \left(\int_{-L}^{L} f(t)\, e^{-ik_m t} dt\right) e^{ik_m x}$

Define $g(k_m) = \left(\int_{-L}^{L} f(t)\, e^{-ik_m t}\right) e^{ik_m x}$. Recall the (loose) definition of the Riemann integral on $\mathbb{R}$: $\lim_{\delta k \to 0}\sum_{m=-\infty}^{\infty} g(k_m)\,\delta k = \int_{-\infty}^{\infty} g(k)\,dk$ for $\delta k = k_{m+1} - k_m$

So in the limit $L \to \infty$ (i.e. $\delta k \to 0$):

- limits on inner integral go to $\pm\infty$
- outer sum can be replaced by an integral.

As such, the result is $\boxed{f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(t)\, e^{-ikt} dt\right) e^{ikx} dk}$. This is known as the <u>Fourier integral formula</u>.

We can split this formula up by defining $\boxed{\hat{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{-ikt} dt}$ (forward transform) then the formula becomes $\boxed{f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(k)\, e^{ikx} dk}$ (inverse transform).

The $\hat{f}(k)$ here take the role of the Fourier transforms.

Remarks: in the forward transform, we often replace $t \mapsto x$.

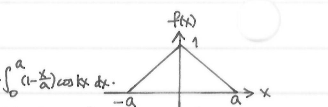
For what class of functions $f(x)$ does the Fourier transform $\hat{f}(k)$ exist?

Answer: If $f(x) \in L^1(\mathbb{R})$, then $\hat{f}(k)$ exists for all $k \in \mathbb{R}$.

Proof - $f(x) \in L^1(\mathbb{R}) \Rightarrow \int_{-\infty}^{\infty} |f|\,dx < \infty$ by definition. $\hat{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-ikx} dx$. $|\hat{f}(k)| \le \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} |f(x)\, e^{-ikx}|\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|f|\,dx < \infty$, q.e.d.

We look at some examples:

Ex $f(x) = \begin{cases} 1 - \frac{|x|}{a} & |x| < a \\ 0 & |x| \ge a \end{cases}$. Find its Fourier transform.

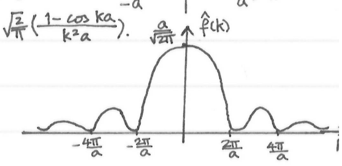


Soln. $\hat{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-ikx} dx = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a}\underbrace{\left(1 - \frac{|x|}{a}\right)(\cos kx}_{\text{even}} - i\underbrace{\sin kx}_{\text{odd}})\,dx = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a}\left(1 - \frac{|x|}{a}\right)\cos(kx)\,dx = \sqrt{\frac{2}{\pi}}\int_0^a \left(1 - \frac{x}{a}\right)\cos kx\,dx$.

Then $\hat{f}(k) = \sqrt{\frac{2}{\pi}}\left(\left[\left(1 - \frac{x}{a}\right)\frac{\sin kx}{k}\right]_0^a + \int_0^a \frac{1}{a}\frac{\sin kx}{k}dx\right) = \sqrt{\frac{2}{\pi}}\int_0^a \frac{1}{a}\frac{\sin kx}{k}dx = \sqrt{\frac{2}{\pi}}\left[-\frac{\cos kx}{k^2 a}\right]_0^a = \sqrt{\frac{2}{\pi}}\left(\frac{1 - \cos ka}{k^2 a}\right)$.

Since $1 - \cos(ka) = 2\sin^2\left(\frac{ka}{2}\right)$, $\hat{f}(k) = \frac{a}{\sqrt{2\pi}}\left(\frac{\sin(ak/2)}{(ak/2)}\right)^2$

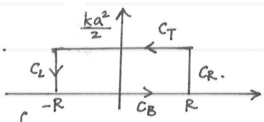
We plot what it looks like on the right:



Note: $f(x)$ scales horizontally $\sim a$, $\hat{f}(k)$ scales horizontally as $\sim \frac{1}{a}$.

Ex Consider the function $f(x) = e^{-\frac{x^2}{a^2}}$ (Gaussian distribution). Find its Fourier transform.

Soln. $\hat{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{a^2} - ikx} dx$. Consider the contour C, as shown on the right; and evaluate $\oint_C e^{-\frac{z^2}{a^2}} dz$.



We know that $\oint_C e^{-\frac{z^2}{a^2}} dz = 0$, by Cauchy's Theorem since e^{-z^2/a^2} is analytic in region enclosing C.

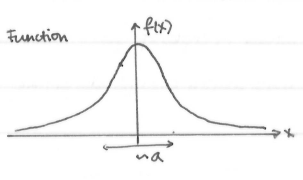
$\oint_C = \int_{C_B} + \int_{C_R} + \int_{C_T} + \int_{C_L}$. By our understanding, we know that $\lim_{R\to\infty}\int_{C_L}, \int_{C_R} = 0$. Then $\lim_{R\to\infty}\int_{C_T} = -\lim_{R\to\infty}\int_{C_B}$

or, $\lim_{R\to\infty}\int_{(-C_T)} = \lim_{R\to\infty}\int_{C_B}$. $\lim_{R\to\infty}\int_{C_B} e^{-\frac{z^2}{a^2}} dz = \int_{-\infty}^{\infty} e^{-\frac{x^2}{a^2}} dx = \sqrt{\pi}\,a$. $\because I = \int_{-\infty}^{\infty} e^{-\frac{x^2}{a^2}} dx \Rightarrow I^2 = \iint_{\mathbb{R}^2} e^{-\frac{x^2-y^2}{a^2}} dx\,dy$.

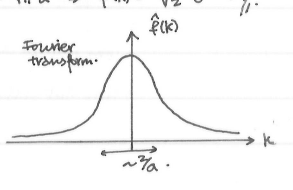
$I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-\frac{r^2}{a^2}} r\,dr\,d\theta = \pi a^2 \Rightarrow I = \sqrt{\pi}\,a$. Then $\lim_{R\to\infty}\int_{(-C_T)} e^{-\frac{z^2}{a^2}} dz = \int_{-\infty}^{\infty} e^{-(x + \frac{ika^2}{2})^2/a^2} dx = \sqrt{\pi}\,a$.

$\int_{-\infty}^{\infty} e^{-\frac{x^2}{a^2} - ikx + \frac{k^2a^2}{4}} dx = \sqrt{2\pi}\, e^{\frac{k^2a^2}{4}}\hat{f}(k)$. Then $\sqrt{2\pi}\, e^{\frac{k^2a^2}{4}}\hat{f}(k) = \sqrt{\pi}\,a \Rightarrow \hat{f}(k) = \frac{a}{\sqrt{2}}\, e^{-\frac{k^2a^2}{4}}$.

Function



Fourier transform.



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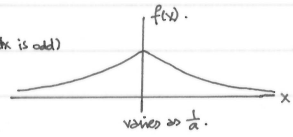
Note: the Fourier transform of a Gaussian curve is also a Gaussian.

this is the logic behind the Heisenberg's uncertainty Principle.

If we chose $a=\sqrt{2}$, $f(x)=e^{-\frac{x^2}{2}}$, $\hat{f}(k)=e^{-\frac{k^2}{2}}$ $\Rightarrow$ the function is its own transform.

$\boxed{Ex}$ consider the function $f(x)=e^{-a|x|}$ $(a>0)$. Find its Fourier transform.



soln. $\hat{f}(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a|x|-ikx}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a|x|}\cos kx\,dx$ $\quad(\because\ e^{-ikx}=\cos kx - i\sin kx$ and $\int f(x)\sin(kx)dx$ is odd)

$=\sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-ax}\cos kx\,dx = \sqrt{\frac{2}{\pi}}\,\mathrm{Re}\left\{\int_0^{\infty} e^{-ax-ikx}dx\right\} = \sqrt{\frac{2}{\pi}}\,\mathrm{Re}\left[-\frac{1}{a+ik}e^{-ax-ikx}\right]_0^{\infty}$

$=\sqrt{\frac{2}{\pi}}\,\mathrm{Re}\left(\frac{1}{a+ik}\right)=\sqrt{\frac{2}{\pi}}\,\mathrm{Re}\,\frac{a-ik}{a^2+k^2}$. then $\hat{f}(k)=\sqrt{\frac{2}{\pi}}\frac{a}{a^2+k^2}$

Note: this decays with order $\sim a$.

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Med Sci G40/ Drayton B20.

Recall that our transforms are Forward transform: $\hat{f}(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}$, inverse transform: $f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(k)e^{ikx}dk$.

Notation: We will use $\mathcal{F}[f](k)$ as an alternative to $\hat{f}(k)$.

Properties of Fourier Transforms:

P① Transform of a derivative: (since $f\in L^1$, $f(x)\to 0$ as $x\to\pm\infty$.)

$\mathcal{F}[f'(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f'(x)e^{-ikx}dx = \frac{1}{\sqrt{2\pi}}\left[f(x)e^{-ikx}\right]_{-\infty}^{\infty}+\frac{1}{\sqrt{2\pi}}(ik)\int_{-\infty}^{\infty} f(x)e^{-ikx}dx$. Hence, $\mathcal{F}[f']=(ik)\hat{f}(k)$.

For instance: we know that $\frac{d}{dx}e^{-x^2/2}=-xe^{-x^2/2}$. $\mathcal{F}[e^{-x^2/2}]=e^{-k^2/2}$. Then $\mathcal{F}[\frac{d}{dx}e^{-x^2/2}]=ik\,e^{-k^2/2}$ $\Rightarrow$ $\mathcal{F}[xe^{-x^2/2}]=-ik\,e^{-k^2/2}$ $\because$ transform is linear.

P② Derivative of transform:

$\frac{d}{dk}\hat{f}(k)=\frac{d}{dk}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx\right)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{d}{dk}f(x)e^{-ikx}dx=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} -ix f(x)e^{-ikx}dx = \mathcal{F}[-ixf(x)]$. then $\mathcal{F}[xf(x)]=i\frac{d\hat{f}}{dk}$

Returning to previous example, $\mathcal{F}[xe^{-x^2/2}]=i\frac{d}{dk}(\mathcal{F}[e^{-x^2/2}])=i\frac{d}{dk}e^{-k^2/2}=-ik\,e^{-k^2/2}$.

we can extend these results inductively: i.e. $\mathcal{F}[f^{(n)}(x)]=(ik)^n\hat{f}(k)$

shift formulae.

P③ consider $\mathcal{F}[f(x-c)]$, $c\in\mathbb{R}$. $\mathcal{F}[f(x-c)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-c)e^{-ikx}dx$. Taking $u=x-c$, $du=dx$, $\mathcal{F}[f(u)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)e^{-ik(u+c)}du$

then $\mathcal{F}[f(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}\cdot e^{-ikc}dx = e^{-ikc}\hat{f}(k)$. Hence, $\mathcal{F}[f(x-c)]=e^{-ikc}\mathcal{F}[f(x)]$.

For instance: $\mathcal{F}[e^{-\frac{(x-c)^2}{2}}]=e^{-ikc}\mathcal{F}[e^{-\frac{x^2}{2}}]=e^{-ikc}e^{-\frac{k^2}{2}}$

P④ Consider $\mathcal{F}[e^{-icx}f(x)]$, $c\in\mathbb{R}$. $\mathcal{F}[e^{-icx}f(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-icx}e^{-ikx}dx=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i(k+c)x}dx=\hat{f}(k+c)$. i.e. transform evaluated at $k+c$.

For example: $\mathcal{F}[e^{-icx-\frac{x^2}{2}}]=e^{-\frac{(k+c)^2}{2}}$

Convolution function:

P⑤ $\boxed{\text{Definition}}$ the convolution of two functions $f(x),g(x)\in L^1(\mathbb{R})$ is the function $(f*g)(x)=\int_{-\infty}^{\infty}f(x-y)g(y)dy$.

Does $f*g=g*f$? $f*g=\int_{-\infty}^{\infty}f(x-y)g(y)dy \overset{(u=x-y)}{=} \int_{\infty}^{-\infty}f(u)g(x-u)(-du)=\int_{-\infty}^{\infty}f(y)g(x-y)dy=g*f$. convolution is commutative.

claim: $f,g\in L^1(\mathbb{R})\Rightarrow f*g\in L^1(\mathbb{R})$.

If $\int_{-\infty}^{\infty}|(f*g)(x)|dx$ exists, then $f*g\in L^1(\mathbb{R})$. $\int_{-\infty}^{\infty}|(f*g)(x)|dx=\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty}f(x-y)g(y)dy\right|dx\le\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f(x-y)||g(y)|dy\,dx$.

Let $u=x-y$. then $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f(u)||g(y)|dy\,du=\int_{-\infty}^{\infty}|f(u)|du\cdot\int_{-\infty}^{\infty}|g(y)|dy=\int_{-\infty}^{\infty}|f(x)|dx\int_{-\infty}^{\infty}|g(x)|dx<\infty$, as $f,g\in L^1$.

Convolution theorem: what is $\mathcal{F}[f*g]$? $\mathcal{F}[f*g]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}f(x-y)g(y)dy\right)e^{-ikx}dx=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x-y)g(y)e^{-ikx}dx\,dy$

LHS $=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}g(y)\left(\int_{-\infty}^{\infty}f(x-y)e^{-ikx}dx\right)dy$. change variables in inner integral. write $u=x-y$, $x=u+y$, $dx=du$. then

$\mathcal{F}[f*g]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}g(y)\left(\int_{-\infty}^{\infty}f(u)e^{-ik(u+y)}du\right)dy=\sqrt{2\pi}\cdot\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}g(y)e^{-iky}dy\right)\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(u)e^{-iku}du\right)=\sqrt{2\pi}\,\hat{f}(k)\,\hat{g}(k)$.

Hence, the convolution theorem states that $\boxed{\mathcal{F}[f*g]=\sqrt{2\pi}\,\hat{f}(k)\hat{g}(k)}$ (we also write $\widehat{f*g}$).

$\boxed{Ex}$ Given that $\mathcal{F}[\frac{1}{x^2+a^2}]=\sqrt{\frac{\pi}{2}}\frac{e^{-a|k|}}{a}$, find $\mathcal{F}[\int_{-\infty}^{\infty}\frac{e^{-\frac{(x-y)^2}{2}}}{y^2+a^2}dy]$.

soln. Let $\int_{-\infty}^{\infty}\frac{e^{-\frac{(x-y)^2}{2}}}{y^2+a^2}dy=f*g$. then $f(x)=e^{-\frac{x^2}{2}}$, $g(y)=\frac{1}{y^2+a^2}$. then $\mathcal{F}[\int_{-\infty}^{\infty}\frac{e^{-(x-y)^2/2}}{y^2+a^2}]=\sqrt{2\pi}\,\mathcal{F}[f(k)]\,\mathcal{F}[g(k)]$

$\sqrt{2\pi}\,\hat{f}(k)\,\hat{g}(k)=\frac{\pi}{a}e^{-k^2/2-a|k|}$.

Applications of Fourier Transform theory:

A① Solve an integral equation, such as find $f(x)$ if $\frac{1}{2}\int_{-\infty}^{\infty}e^{-|x-u|}f(u)du=e^{-x^2/2}$.

$\int_{-\infty}^{\infty}e^{-|x-u|}f(u)du=g*f$ for $g(x)=e^{-|x|}$. Take Fourier transforms of both sides: $\frac{1}{2}(g*f)=h$ where $h=e^{-x^2/2}$. By convolution theorem,

$\frac{1}{2}\sqrt{2\pi}\,\hat{g}(k)\hat{f}(k)=\mathcal{F}[h]=\hat{h}(k)$. $\hat{g}(k)=\sqrt{\frac{2}{\pi}}\cdot\frac{1}{k^2+1}$, $\hat{h}(k)=\mathcal{F}[e^{-x^2/2}]=e^{-k^2/2}$. $\Rightarrow$ $\frac{1}{2}\cdot\sqrt{2\pi}\cdot\sqrt{\frac{2}{\pi}}\frac{1}{k^2+1}\hat{f}(k)=e^{-k^2/2}$ $\Rightarrow$ $\hat{f}(k)=(k^2+1)e^{-k^2/2}$.

since $\hat{f}(k) = (k^2+1)e^{-k^2/2}$, and we recall that $\mathcal{F}[p''(x)] = (ik)^2\hat{p}(k) = -k^2\hat{p}(k)$, $\hat{f}(k) = k^2 e^{-k^2/2} + e^{-k^2/2} \Rightarrow f(x) = (-\frac{d^2}{dx^2}+1)e^{-x^2/2}$.

$\therefore f(x) = (2-x^2)e^{-x^2/2}$.

F⑥. (Parseval's theorem).

Consider the inverse transform for the convolution: $(f*g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{2\pi}\,\hat{f}(k)\hat{g}(k)e^{ikx}dk$. This is true $\forall x \in \mathbb{R}$.

Set $x=0$: then we get $\int_{-\infty}^{\infty} f(-y)g(y)\,dy = \int_{-\infty}^{\infty}\hat{f}(k)\hat{g}(k)\,dk$. $-(\dagger)$ Define a new function $h(y) = f^*(-y)$ (complex conjugate).

Then $\hat{h}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f^*(-y)e^{-iky}dy = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{-\infty} f^*(u)e^{iku}(-du) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f^*(u)e^{iku}du = \frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{\infty} f(u)e^{-iku}du\right)^* = \hat{f}^*(k)$.

Thus, $\hat{f}(k) = \hat{h}^*(k)$, $h^*(y) = f(-y)$. Then substitute into $(\dagger)$: $\int_{-\infty}^{\infty} h^*(y)g(y)dy = \int_{-\infty}^{\infty}\hat{h}^*(k)\hat{g}(k)\,dk$ - $(\ddagger)$. This is true for all h, so we can choose

$h(y) = g(y)$. This changes $(\ddagger)$ to $\boxed{\int_{-\infty}^{\infty}|g(y)|^2\,dy = \int_{-\infty}^{\infty}|\hat{g}(k)|^2\,dk}$. This is Parseval's theorem.

This gives us a method to simply calculate complicated integrals by relating them to their transforms:

A② Use Parseval's theorem with $g(x) = e^{-a|x|}$ to obtain $\int_{-\infty}^{\infty}\frac{1}{(x^2+a^2)^2}dx$:

We know that $g(x) = e^{-a|x|} \Rightarrow \hat{g}(k) = \sqrt{\frac{2}{\pi}}\frac{a}{k^2+a^2} \Rightarrow |\hat{g}(k)|^2 = \frac{2}{\pi}\cdot a^2\frac{1}{(k^2+a^2)^2}$. $\int_{-\infty}^{\infty}|g(x)|^2dx = \int_{-\infty}^{\infty}e^{-2a|x|}dx = 2\int_0^{\infty}e^{-2ax}dx$

$\int_{-\infty}^{\infty}|g(x)|^2dx = -\frac{1}{a}\left[e^{-2ax}\right]_0^{\infty} = \frac{1}{a}$. From Parseval, $\int|g|^2dx = \frac{1}{a} = \int_{-\infty}^{\infty}|\hat{g}(k)|^2dk = \frac{2}{\pi}\int_{-\infty}^{\infty}\frac{a^2}{(a^2+k^2)^2}dk$.

Hence, $\int_{-\infty}^{\infty}\frac{1}{(x^2+a^2)^2}dx = \frac{\pi}{2a^3}$ //.

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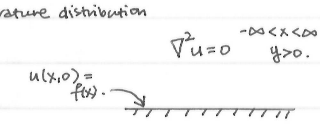
Note: We can check this result using contour integration.

Recall that for any $h(x), g(x)$, $(\ddagger)$: $\int_{-\infty}^{\infty}h^*(x)g(x)\,dx = \int_{-\infty}^{\infty}\hat{h}^*(k)\hat{g}(k)\,dk$. Choose $g(x) = e^{-a|x|}$, $h(x) = e^{-b|x|}$. Then $\hat{g}(k) = \sqrt{\frac{2}{\pi}}\frac{a}{a^2+k^2}$, $\hat{h}(k) = \sqrt{\frac{2}{\pi}}\frac{b}{b^2+k^2}$.

Then $\int_{-\infty}^{\infty}h^*(x)g(x)\,dx = 2\int_0^{\infty}e^{-(a+b)x}dx = \frac{2}{a+b}$. From $(\ddagger)$, $\frac{2}{a+b} = \int_{-\infty}^{\infty}\hat{h}^*(k)\hat{g}(k)\,dk = \frac{2}{\pi}\int_{-\infty}^{\infty}\frac{ab}{(k^2+a^2)(k^2+b^2)}dk$. Then $\int_{-\infty}^{\infty}\frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{ab(a+b)}$ //.

A③ PDEs: Laplace's equation on the half-plane.

Physical interpretation — Find the steady temperature distribution $u(x,y)$ in a semi-infinite conducting plate, when a temperature distribution $u(x,0) = f(x)$ is applied to its boundary. Assume $f(x) \in L^1$. Physical considerations require $u(x,y) \to 0$ as $y \to \infty$.



We take Fourier Transforms in x-direction, since y only goes from 0 to ∞. $\nabla^2 u = u_{xx} + u_{yy} = 0$.

$\mathcal{F}[u_{yy}] = \frac{\partial^2}{\partial y^2}\Big(\underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}u(x,y)e^{-ikx}dx}_{\text{function of } y, k}\Big) = \frac{\partial^2}{\partial y^2}\hat{u}(k,y) = \frac{\partial^2\hat{u}}{\partial y^2}(k,y)$. $\mathcal{F}[u_{xx}] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial^2 u}{\partial x^2}e^{-ikx}dx = (ik)^2\hat{u}(k,y) = -k^2\hat{u}(k,y)$.

As such, our equation is transformed into the form $\frac{\partial^2\hat{u}}{\partial y^2} - k^2\hat{u} = 0$. General solution is $\hat{u}(k,y) = A(k)e^{-ky} + B(k)e^{ky}$ $-(\circledast)$, where A, B are arbitrary functions.

Now use boundary conditions to find $A(k), B(k)$. Note that $(\circledast)$ is equivalent to $\hat{u}(k,y) = \tilde{A}(k)e^{-|k|y} + \tilde{B}(k)e^{|k|y}$, by setting $\tilde{A}(k) = \begin{cases} A(k) & k>0 \\ B(k), & k\le 0\end{cases}$ etc.

Since $u \to 0$ as $y \to \infty$, $\hat{u} \to 0$ as $y \to \infty \Rightarrow \tilde{B}(k) = 0$ as $|k|y \to \infty$ as $y \to \infty$. $\therefore \hat{u}(k,y) = \tilde{A}(k)e^{-|k|y}$. Now use boundary condition at $y=0$.

$u(x,0) = f(x)$. Take Fourier transform: $\hat{u}(k,0) = \hat{f}(k)$. $\hat{u}(k,0) = \tilde{A}(k)e^{-|k|\cdot 0} = \tilde{A}(k)$. Hence, $\tilde{A}(k) = \hat{f}(k)$. As such, $\hat{u}(k,y) = \hat{f}(k)e^{-|k|y}$.

Notice that $\hat{u}(k,y) = \sqrt{2\pi}\,\hat{f}(k)\hat{g}(k,y)$ for $\hat{g}(k) = \frac{e^{-|k|y}}{\sqrt{2\pi}}$. We know that $\mathcal{F}[f*g] = \sqrt{2\pi}\,\hat{f}\hat{g}$, $f*g = g*f$.

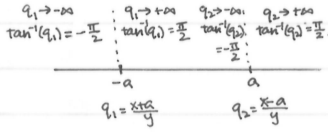
Hence, $u(x,y) = \int_{-\infty}^{\infty}f(x-t)g(t,y)\,dt = f*g$. [Note: use t as integration variable to avoid confusion with y. To complete solution, need to find $g(x,y)$. $\hat{g}(k,y) = \frac{e^{-|k|y}}{\sqrt{2\pi}}$. Use inverse formula: $g(x,y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{g}(k,y)e^{ikx}dk = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{-|k|y}}{\sqrt{2\pi}}\cos(kx)\,dk = \frac{1}{\pi}\int_0^{\infty}e^{-ky}\cos kx\,dk$.

$g(x,y) = \frac{1}{\pi}\mathrm{Re}\left[\int_0^{\infty}e^{-k(y+ix)}dk\right] = \frac{1}{\pi}\mathrm{Re}\left[-\frac{e^{-k(y+ix)}}{y+ix}\right]_0^{\infty} = \frac{1}{\pi}\mathrm{Re}\left[\frac{1}{y+ix}\right] = \frac{1}{\pi}\mathrm{Re}\frac{y-ix}{y^2+x^2} = \frac{1}{\pi}\cdot\frac{y}{x^2+y^2}$. Hence, our final solution is.

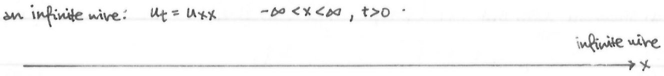
$u(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(x-t)}{t^2+y^2}dt$ //. Choose $f(x) = \begin{cases}1 & |x|\le a \\ 0 & |x|>a.\end{cases} \Rightarrow f(x-t) = \begin{cases}1 & |x-t|\le a \\ 0 & |x-t|>a.\end{cases}$ $|x-t|\le a \Rightarrow -a\le t-x\le a \Rightarrow x-a\le t\le x+a$.

then $u(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(x-t)}{t^2+y^2}dt = \frac{y}{\pi}\int_{x-a}^{x+a}\frac{1}{t^2+y^2}dt = \frac{y}{\pi}\left[\frac{1}{y}\arctan\left(\frac{t}{y}\right)\right]_{x-a}^{x+a} = \frac{1}{\pi}\left[\arctan\left(\frac{x+a}{y}\right) - \arctan\left(\frac{x-a}{y}\right)\right]$ //

explanation for diagram.



A④ Heat equation in an infinite wire: $u_t = u_{xx}$ $\quad -\infty < x < \infty$, $t > 0$.



$u \to 0$ as $x \to \pm\infty$. Also $u(x,0) = f(x)$.

Take the transform of this equation: $u_t = u_{xx} \Rightarrow \mathcal{F}[u_t] = \hat{u}_t$, $\mathcal{F}[u_{xx}] = -k^2\hat{u}$ (as above). Then $\hat{u}_t = -k^2\hat{u}$. We then integrate to get:

$\hat{u}(k,t) = A(k)e^{-k^2t}$. Apply initial condition: $u(x,0) = f(x) \Rightarrow \hat{u}(k,0) = \hat{f}(k)$. $\Rightarrow A(k) = \hat{f}(k)$ where $t=0$. Then $\hat{u}(k,t) = \hat{f}(k)e^{-k^2t}$.

Introduce $\hat{g}(k,t) = \frac{e^{-k^2t}}{\sqrt{2\pi}}$, then $\hat{u}(k,t) = \sqrt{2\pi}\,\hat{f}(k)\hat{g}(k,t)$. $\underbrace{\int_{-\infty}^{\infty}f*g\,dy}_{u(x,t)=} = \int_{-\infty}^{\infty}f(x-y)g(y,t)\,dy$, with y as the dummy variable.

We need to find $g(x,t)$: We know that $\mathcal{F}[e^{-x^2/a^2}] = \sqrt{\frac{a}{2}}\,e^{-k^2a^2/4}$. Choose $a^2 = 4t$, then $\mathcal{F}[e^{-\frac{x^2}{4t}}] = \sqrt{2t}\,e^{-k^2t} \Rightarrow \mathcal{F}\left[\frac{1}{2\sqrt{\pi t}}e^{-\frac{x^2}{4t}}\right] = \frac{e^{-k^2t}}{\sqrt{2\pi}} = \hat{g}(k,t)$

Hence, $g(x,t) = \frac{1}{2\sqrt{\pi t}}e^{-x^2/4t}$. Thus, general solution is $u(x,t) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty}f(x-y)e^{-\frac{y^2}{4t}}dy$ //.

Note: this integral has a smoothing effect on $f(x)$ — scale of smoothing $\sim\sqrt{4t}$. Also, the function $g(x,t) = \frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}$ is known as the heat kernel in 1D.

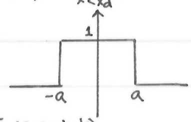
This converts singular distributions into Gaussians.

Generalised Inversion Formula. (cross-refer to handout)

Our inversion formula $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} dk$ is valid for continuous $f(x)$. If $f(x)$ is discontinuous, say at $x = x_d$, then our formula is modified to:

$\frac{1}{2}[f(x_d^+) + f(x_d^-)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} dk$, where $f(x)$ is continuous, $f(x) = f(x^+) = f(x^-) \Rightarrow$ recover original. At $x = x_d$, $f(x^+) = \lim_{\substack{x \to x_d \\ x > x_d}} f(x)$, $f(x^-) = \lim_{\substack{x \to x_d \\ x < x_d}} f(x)$.

Consider $f(x) = \begin{cases} 1 & |x| \le a \\ 0 & |x| > a \end{cases}$. Then $f(a) = 1$, $f(a^+) = 0$, $f(a^-) = 1$. Then $\frac{1}{2}[f(a^+) + f(a^-)] = \frac{1}{2}$.



This formula gives another approach to computing complicated integrals.

Consider $f(x) = \begin{cases} 1 & |x| \le a \\ 0 & |x| > a \end{cases}$. Then $\hat{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) e^{-ikx} dx = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{-ikx} dx = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} \cos(kx)\, dx = \frac{1}{\sqrt{2\pi}}\left[\frac{\sin(kx)}{k}\right]_{-a}^{a} = \sqrt{\frac{2}{\pi}}\frac{\sin ka}{k}$ (even in k).

Use generalised inversion formula: $\frac{1}{2}[f(x^+) + f(x^-)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} dk = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} \frac{\sin ka \cos kx}{k} dk$.

Hence, $\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin ka \cos kx}{x} dk = \frac{1}{2}[f(x^+) + f(x^-)] = \begin{cases} 1 & |x| < a \\ 1/2 & |x| = a \\ 0 & |x| > a \end{cases}$. Replace $k \mapsto x$, $x \mapsto b$, then $\int_{-\infty}^{\infty} \frac{\sin xa \cos xb}{x} dx = \pi \cdot \begin{cases} 1 & |b| < a \\ 1/2 & |b| = a \\ 0 & |b| > a \end{cases}$ //.

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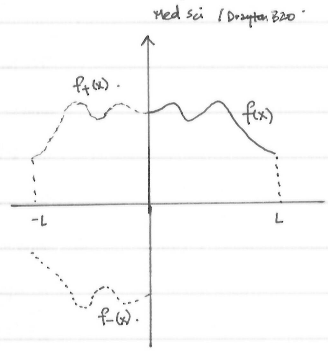
Fourier Sine and Cosine Transforms.



$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi x}{L}\right)$, $f(x) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi x}{L}\right)$ give Fourier cosine/sine series of $f(x)$ on $0 \le x \le L$.

We can extend $f(x)$ to $f_+(x)$, where $f_+(x) = \begin{cases} f(x) & 0 \le x \le L \\ f(-x) & -L < x < 0 \end{cases}$. This is the even extension of $f(x)$.

Alternatively, we can also define the odd extension of x, $f_-(x) = \begin{cases} f(x) & 0 \le x \le L \\ -f(-x) & -L < x < 0 \end{cases}$, which is the odd extension of $f(x)$.

We can also adopt this idea with transforms: allows us to take transforms of functions defined on the half-line, $0 \le x < \infty$.

Take the Fourier Transform of $f_+(x)$: $\hat{f}_+(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_+(x) e^{-ikx} dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_+(x)(\cos kx - i\sin kx)\, dx$

$\hat{f}_+(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_+(x)\cos(kx)\, dx = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f_+(x)\cos(kx)\, dx = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos(kx)\, dx$.

This allows us to define the cosine transform: $\boxed{\mathcal{F}_c[f](k) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos(kx)\, dx}$

To get the inversion formula for $\hat{f}_+(k)$, we see that $f_+(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}_+(k) e^{ikx} dk$. We know that $\hat{f}_+(k)$ is even, so $f_+(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} \hat{f}_+(k)\cos(kx)\, dx$

then $f_+(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} \mathcal{F}[f](k)\cos(kx)\, dk$: for $x \ge 0$, $\boxed{f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} \mathcal{F}_c[f](k)\cos(kx)\, dk}$ Hence, forward and inverse transforms are symmetric.

For the Fourier sine transforms, we work with the odd extension. Then $\hat{f}_-(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_-(x) e^{-ikx} dx = \sqrt{\frac{2}{\pi}}\int_0^{\infty} -i f_-(x)\sin(kx)\, dx$ since product of two odds is even.

We define the sine transform: $\boxed{\mathcal{F}_s[f](k) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin(kx)\, dx}$, whilst $\hat{f}_-(k) = -i\mathcal{F}_s[f](k)$. Inverse transform is $f_-(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}_-(k) e^{ikx} dx$. We know that $\hat{f}_-(k)$ is odd in k, and $f_-(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} \hat{f}_-(k) \cdot i\sin(kx)\, dk = \sqrt{\frac{2}{\pi}}\int_0^{\infty} \mathcal{F}_s[f](k)\sin(kx)\, dk$. Then set $x > 0$: $\boxed{f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} \mathcal{F}_s[f](k)\sin(kx)\, dk}$.

Transforms of Derivatives:

Recall that $\mathcal{F}_c[f'](k) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f'(x)\cos(kx)\, dx = \sqrt{\frac{2}{\pi}}\underbrace{[f(x)\cos(kx)]_0^{\infty}}_{} + \sqrt{\frac{2}{\pi}}\int_0^{\infty} k f(x)\sin(kx)\, dx = k\mathcal{F}_s[f] - \underbrace{\sqrt{\frac{2}{\pi}} f(0_+)}_{\text{boundary term}}$

$f(x) \to 0$ as $x \to \infty$ $\because f \in L^1[(0,\infty)]$. $\lim_{x>0,\, x\to 0} f(x)$.

$\mathcal{F}_s[f'] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f'(x)\sin(kx)\, dx = \sqrt{\frac{2}{\pi}}[f(x)\sin(kx)]_0^{\infty} - k\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos(kx)\, dx = -k\mathcal{F}_c[f]$. (no boundary term).

Then $\mathcal{F}_c[f''] = k\mathcal{F}_s[f'] - \sqrt{\frac{2}{\pi}} f'(0_+) = -k^2\mathcal{F}_c[f] - \sqrt{\frac{2}{\pi}} f'(0_+)$.

Laplace Transforms.

Fourier sine and cosine transforms can be used with function $f(x)$ defined on the half-line $(0 \le x < \infty)$. It is often easier to work with the Laplace Transform:

$\boxed{\mathcal{L}[f(t)] = \bar{f}(s) = \int_0^{\infty} f(t) e^{-st} dt}$, $s \in \mathbb{C}$. However, their inverses are more complicated.

Existence: $\mathcal{L}[f]$ exists for a much wider class of functions than those in $L^1[(0,\infty)]$. (Fourier sine and cosine transforms)

If $f(t) \sim e^{\beta t}$ as $t \to \infty$, $\beta \in \mathbb{R}$, $\beta > 0$, then the integral $\bar{f}(s)$ will exist if $\int_0^{\infty} e^{(\beta - s)t} dt = \left[\frac{e^{(\beta-s)t}}{\beta - s}\right]_0^{\infty}$ exists i.e. $Re(s) > \beta$.

Ex Find $\mathcal{L}[1]$, $\mathcal{L}[t^m]$, $\mathcal{L}[e^{-at}]$ (and conditions for its existence). $m \in \mathbb{Z}$. $a \in \mathbb{R}$

soln. $\mathcal{L}[1] = \int_0^{\infty} e^{-st} dt = \frac{1}{s}$, with $Re(s) > 0$ // $\mathcal{L}[t^m] = \int_0^{\infty} t^m e^{-st} dt$. Note that $\left(\frac{d}{ds}\right)^m \int_0^{\infty} e^{-st} dt = \int_0^{\infty} (-1)^m t^m e^{-st} dt$

Thus, $\mathcal{L}[t^m] = \left(-\frac{d}{ds}\right)^m \left[\int_0^{\infty} e^{-st} dt\right] = (-1)^m \cdot \frac{m!(-1)^m}{s^{m+1}} = \frac{m!}{s^{m+1}}$. [Alternatively recall $\mathcal{L}[t^\alpha] = \int_0^{\infty} t^\alpha e^{-st} dt$, α const, $\alpha \in \mathbb{R}$.]

set $q = st$, then $\frac{dq}{s} = dt \Rightarrow \mathcal{L}[t^\alpha] = \int_0^{\infty} \left(\frac{q}{s}\right)^\alpha e^{-q} \frac{dq}{s} = \frac{1}{s^{\alpha+1}}\int_0^{\infty} q^\alpha e^{-q} dq = \frac{1}{s^{\alpha+1}}\Gamma(\alpha+1)$. Since $\alpha = m \in \mathbb{Z}$, $\Gamma(m+1) = m!$], with $Re(s) > 0$ //.

$\mathcal{L}[e^{-at}] = \int_0^{\infty} e^{-(a+s)t} = \left[\frac{e^{-(a+s)t}}{-(a+s)}\right]_0^{\infty} = \frac{1}{s+a}$, with $Re(s) > -a$ //.

Then, we consider $\mathcal{L}[\cos wt]$, $\mathcal{L}[\sin wt]$. To do this, take $\mathcal{L}[e^{iwt}]$, then $\mathcal{L}[\cos wt] = Re\, \mathcal{L}[e^{iwt}]$, $\mathcal{L}[\sin wt] = Im\, \mathcal{L} e^{iwt}$.

$\mathcal{L}[e^{iwt}] = \int_0^{\infty} e^{iwt} e^{-st} dt = \int_0^{\infty} e^{t(iw-s)} dt = \left[\frac{e^{(iw-s)t}}{iw-s}\right]_0^{\infty} = \frac{1}{s-iw}$. This is valid when $|e^{(iw-s)t}| = |e^{-st}| < 1 \Rightarrow Re(s) > 0$.

Then $\frac{1}{s-iw} = \frac{s+iw}{s^2+w^2}$. Since s^2, w^2 are always real, $\mathcal{L}[\cos wt] = \frac{s}{s^2+w^2}$, $\mathcal{L}[\sin wt] = \frac{w}{s^2+w^2}$ //.

Properties of Laplace Transform: Given $f(t)$ s.t. $\mathcal{L}[f(t)] = \bar{f}(s)$

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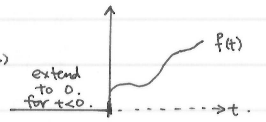
① Linearity.

② Shifting. We have two shifting results:

- $\mathcal{L}[e^{-\alpha t}f(t)] = \int_0^\infty f(t)\, e^{-\alpha t} e^{-st} dt = \int_0^\infty f(t)\, e^{-(s+\alpha)t} dt = \bar{f}(s+\alpha)$. [First shift result] $\boxed{\mathcal{L}[e^{-\alpha t} f(t)] = \bar{f}(s+\alpha)}$
  $\alpha > 0$.
- $\mathcal{L}[f(t-\alpha)] = \int_0^\infty f(t-\alpha) e^{-st} dt \overset{u=t+\alpha}{=} \int_{-\alpha}^\infty f(u)\, e^{-s(u+\alpha)} du = e^{-s\alpha}\int_0^\infty f(u) e^{-su} du$ (with assumption that $f(t-\alpha) \equiv 0$ for $t<\alpha$.)



Hence, this yields the following: [Second shift result] $\boxed{\mathcal{L}[f(t-\alpha)] = e^{-s\alpha}\mathcal{L}[f]}$.

Note that $\mathcal{L}[e^{\beta t}] = \mathcal{L}[1 \cdot e^{\beta t}]$, so by first shift result, $\mathcal{L}[e^{\beta t}] = \frac{1}{r}\big|_{r=s-\beta} = \frac{1}{s-\beta}$ indeed.

③ Derivative of Transform:

$\mathcal{L}[t^n f(t)] = \int_0^\infty f(t)\, t^n e^{-st} dt$. Note that $t^n e^{-st} = (\frac{d}{ds})^n e^{-st} \cdot (-1)^n = (-\frac{d}{ds})^n e^{-st}$. Then $\mathcal{L}[t^n f(t)] = \int_0^\infty f(t) (-\underline{\frac{d}{ds}})^n e^{-st} \underline{dt} = (-\frac{d}{ds})^n \int_0^\infty f(t)\, e^{-st} dt$ [different variables.]

Hence, $(-\frac{d}{ds})^n \mathcal{L}[f(t)](s) = \mathcal{L}[t^n f(t)] \Rightarrow \mathcal{L}[t^n f(t)] = (-1)^n \bar{f}^{(n)}(s)$.

$\boxed{\text{Ex}}$ Find $\mathcal{L}[t \sin wt]$.

soln. $\mathcal{L}[t \sin wt] = (-1)^1 \bar{f}'(s) = -\mathcal{L}[\sin wt]' = -\frac{d}{ds}\left(\frac{w}{s^2+w^2}\right) = -\frac{2ws}{(s^2+w^2)^2}$ //.

Transform of Derivative:

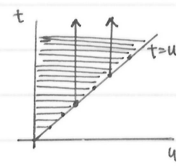
Consider $\mathcal{L}[f'(t)] = \int_0^\infty f'(t) e^{-st} dt = [f(t) e^{-st}]_0^\infty - \int_0^\infty f(t)(-s)e^{-st} dt = -f(0) + s\int_0^\infty f(t) e^{-st} dt = -f(0) + s\mathcal{L}[f(t)]$. Hence $\boxed{\mathcal{L}[f'(t)] = s\bar{f}(s) - f(0)}$.

For $\mathcal{L}[f''(t)]$, define $g(t) = f'(t) \Rightarrow \mathcal{L}[g'(t)] = s\bar{g}(s) - g(0) = s\bar{f'}(s) - f'(0) = s(s\bar{f}(s) - f(0)) - f'(0) = s^2\bar{f}(s) - sf(0) - f'(0)$.

Thus inductively, $\mathcal{L}[f^{(n)}(t)] = s^n \bar{f}(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0)$ //.

④ Convolution Theorem:

integrate $u \le t < \infty$
$0 \le u < \infty$.



For Laplace Transforms, we define $(f*g)(t) = \int_0^t f(t-u)\, g(u)\, du = \int_{-\infty}^\infty f(t-u) g(u) du$ as f, g are non-zero only for non-negative arguments.

$\mathcal{L}[f*g] = \int_0^\infty e^{-st}\{\int_0^t f(t-u) g(u) du\} dt = \int_0^\infty\int_u^\infty e^{-st} f(t-u) g(u)\, dt\, du$. Let $v = t-u$. $dv = dt \Rightarrow \mathcal{L}[f*g] = \int_0^\infty\int_0^\infty e^{-s(v+u)} f(v) g(u)\, dv\, du$

$\Rightarrow \mathcal{L}[f*g] = \int_0^\infty e^{-sv} f(v) dv \int_0^\infty e^{-su} g(u) du = \mathcal{L}[f]\mathcal{L}[g]$ Hence $\mathcal{L}[f*g] = \mathcal{L}[f]\mathcal{L}[g]$.

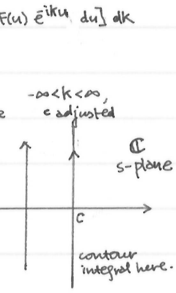
<u>Inversion:</u>

We use several tricks to obtain the inverse Laplace transform of a function.

1. Partial Functions: Consider $\bar{x}(s) = \frac{e^{-s\pi}}{s^2(1+s^2)}$, we want to obtain $x(t)$. $\bar{x}(s) = e^{-s\pi}\left(\frac{1}{s^2} - \frac{1}{1+s^2}\right) = \frac{e^{-s\pi}}{s^2} - \frac{e^{-s\pi}}{1+s^2}$. We know that $\mathcal{L}[t] = \frac{1}{s^2}$, $\mathcal{L}[\sin t] = \frac{1}{s^2+1}$.
   $\mathcal{L}[f(t-\alpha)] = e^{-\alpha s}\bar{f}(s)$ where $f(t-\alpha) = 0$ if $t<\alpha$. Then $\bar{x}(s) = e^{-s\pi}\mathcal{L}[t] - e^{-s\pi}\mathcal{L}[\sin t] \Rightarrow x(s) = \begin{cases} (t-\pi) - \sin(t-\pi) & t>\pi \\ 0 & t<\pi \end{cases}$.
2. Convolution: Consider $\bar{x}(s) = \frac{1}{(s+1)(s+2)} = \left(\frac{1}{s+1}\right)\left(\frac{1}{s+2}\right)$. We know that $\mathcal{L}[e^{-t}] = \frac{1}{s+1}$. If $f(t) = e^{-t}$, $g(t) = e^{-2t}$, $\mathcal{L}[g] = \frac{1}{s+2}$.
   Hence, $\bar{x}(s) = \mathcal{L}[f]\mathcal{L}[g] = \mathcal{L}[f*g] \Rightarrow x(t) = f*g = \int_0^t f(t-u) g(u) du = \int_0^t e^{-(t-u)} e^{-2u} du = e^{-t}\int_0^t e^{-u} du = e^{-t}[-\frac{e^{-u}}{u}]_0^t = e^{-t}(1-e^{-t})$ //.
   The technique depends on the problem; in this case using PF would have been easier.
3. Consider $\bar{x}(s) = \frac{1}{(s-1)^5} = (\frac{d}{ds})^4 \frac{1}{s-1} \cdot \frac{1}{4!} = (\frac{d}{ds})^4 \mathcal{L}[e^t]\frac{1}{4!} = \frac{1}{4!}(-\frac{d}{ds})^4 \mathcal{L}[e^t] = \frac{1}{4!}\mathcal{L}[t^4 e^t]$. By linearity, $x(s) = \frac{1}{4!}t^4 e^t$ //
   Alternatively: let $f(t) = \frac{t^4}{24} \Rightarrow \bar{f}(s) = \frac{1}{s^5}$. i.e. $\frac{1}{(s-1)^5} = \bar{f}(s-1) \overset{\text{shift}}{=} \mathcal{L}[e^t f(t)]$. Hence, $\mathcal{L}^{-1}[\bar{f}(s-a)] = e^{at}f(t) = e^t f(t) = \frac{e^t t^4}{24}$ //. indeed.

Or, if all else fails, we use the general method of contour integration:

4. Contour integration. Recall the Fourier integral theorem; which combines both the forward and inverse Fourier transforms: $F(t) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{ikt}\left[\int_{-\infty}^\infty F(u)\, e^{-iku} du\right] dk$
   Take $F(u) = e^{-cu}f(u)$ with c s.t. $\int_{-\infty}^\infty f(u) e^{-cu} e^{-iku} du$ exists. Then $e^{-ct}f(t) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{ikt}\int_{-\infty}^\infty f(u)\, e^{-(c+ik)u} du\, dk$. Rearranging,
   $f(t) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{(c+ik)t}\int_0^\infty f(u)\, e^{-(c+ik)u} du\, dk$, assuming as usual that $f(u) = 0\ \forall u<0$. Let $s = c+ik$, then $dk = \frac{ds}{i}$. Then we have
   $-\infty < k < \infty$, c adjusted
   $f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\int_0^\infty f(u)\, e^{-su} du\, ds = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\bar{f}(s) ds$. Hence, we have that $\boxed{f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\bar{f}(s)\, ds}$.
   This is called the <u>Bromwich Inversion Integral</u>, which is evaluated over the <u>Bromwich Inversion Contour</u>.



We restrict c to take values $c \ge 0$. What considerations do we have for selecting c? c is defined as the <u>convergence factor</u>.

Since we are examining the Fourier Transform of $F(t) = \begin{cases} e^{-ct}f(t) & t \ge 0 \\ 0 & t<0 \end{cases}$, we require $F(t) \in L^1(\mathbb{R})$ for the transform to be valid i.e. $\int_{-\infty}^\infty |F(t)|\, dt$ exists.

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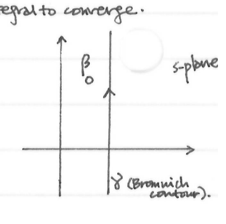
Suppose that $f(t) \sim e^{\beta t}$ (for $\beta \in \mathbb{C}$) as $t \to \infty$. Recall $\bar{f}(s)$ exists for $Re(s) > Re(\beta)$ (multiplying by e^{-st} will make this an exponential decay function).

Then analogously, $\int_{-\infty}^\infty |F(t)|\, dt = \int_0^\infty |F(t)|\, dt$ will exist $\sim \int_0^\infty |e^{\beta t} e^{-ct}|\, dt$ exists. Then $c > Re(\beta)$ is a requirement, for the integral to converge.

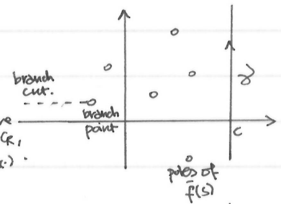
There is an interpretation of this result in terms of the poles of $\bar{f}(s)$ in the complex s-plane: Recall that $\mathcal{L}[e^{\beta t}] = \frac{1}{s-\beta}$

Hence, the analytic continuation of the Laplace transform $\bar{f}(s)$ has a pole at $s=\beta$. For this specific example,

Our Bromwich integral γ must lie to the right of $s=\beta$, as $c > Re(\beta)$. This is true in general $\Rightarrow$ c must be chosen s.t. γ lies to the right of all poles (+ branch points) of $\bar{f}(s)$ in the s-plane.



If there are no branch cuts, and there are only a finite number of poles, we can enclose our poles and evaluate $\int_\gamma$ using the residue theorem on a semicircular contour.

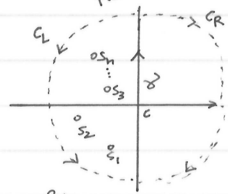


Should we enclose the contour on the left or the right? We consider the "rotated version" of Jordan's lemma: provided $|\bar{f}(s)| \to 0$ everywhere on C_L, C_R,

$\int_{C_R} \bar{f}(s)\, e^{st}\, ds \to 0$ for $t<0$, $\int_{C_L} \bar{f}(s)\, e^{st}\, ds \to 0$ for $t>0$ (recall in the orientation we require $a>0$ in $\int f(x)\, e^{iax}\, dx$.)

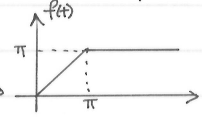
$\Rightarrow$ we close contour to the right for $t<0$. Since it has no poles, $f(t) = \frac{1}{2\pi i}\int_\gamma \bar{f}(s)\, e^{st}\, ds = 0$ for $t<0$; AND

when we close to the left for $t>0$, then by residue theorem, $f(t) = \sum_j \operatorname{Res}\{\bar{f}(s)e^{st},\ s = s_j\}$. where s_j are poles of $\bar{f}(s)$.



Ex Using contour integration, without and with solve the ODE $\ddot{X} + X = f(t)$ subject to initial conditions $X(0) = \dot{X}(0) = 1$ for $f(t) = \begin{cases} t & 0<t<\pi \\ \pi & t>\pi \end{cases}$

Remark: this is a forced oscillator.



Soln. Take Laplace Transforms of whole equation. $\mathcal{L}[\dot{x}] = s\bar{x} - x(0)$, $\mathcal{L}[\ddot{x}] = s^2\bar{x} - s\,x(0) - \dot{x}(0) = s^2\bar{x} - s - 1$

Hence, $\mathcal{L}[\ddot{x}+x] = s^2\bar{x} - s - 1 + \bar{x} = \bar{f} \Rightarrow \bar{x} = \frac{\bar{f}}{s^2+1} + \frac{s+1}{s^2+1}$ with $\frac{\bar{f}}{s^2+1}$ as the forcing and $\frac{s+1}{s^2+1}$ is from initial conditions

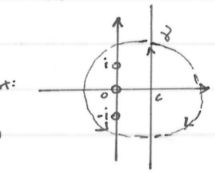
Let $\bar{x}(s) = \bar{x}_1(s) + \bar{x}_2(s)$ as above. $\bar{x}_2(s) = \frac{s}{s^2+1} + \frac{1}{s^2+1} \Rightarrow x_2(t) = \cos t + \sin t$ (by inspection). For $\bar{x}_1(s)$, note that

$\bar{x}_1(s) = \bar{f}(s)\bar{g}(s)$ is the transform of a convolution for $\bar{g}(s) = \frac{1}{s^2+1} \Rightarrow g(t) = \sin t$. Hence, $\bar{x}_1(s) = \bar{f}(s)\bar{g}(s) \Rightarrow x_1(t) = f * g = \int_0^t f(u)\, g(t-u)\, du$.

Hence, $x_1(t) = \int_0^t f(u)\sin(t-u)\, du$, for general $f(u)$. Substitute $f(u) = \begin{cases} u & 0<u<\pi \\ \pi & u>\pi \end{cases}$. and solve (left as an exercise).

Slightly more generally, we can use the inversion formula: $\bar{x}(s) = \frac{\bar{f}(s)}{s^2+1}$. Find $\bar{f}(s) = \int_0^\pi t e^{-st}\, dt + \pi\int_\pi^\infty e^{-st}\, dt = (-\frac{d}{ds})(\int_0^\pi e^{-st}\, dt) + \pi\int_\pi^\infty e^{-st}\, dt$.

$\bar{f}(s) = (-\frac{d}{ds})\left[\frac{-e^{-st}}{s}\right]_0^\pi + \pi\left[-\frac{e^{-st}}{s}\right]_\pi^\infty = -\frac{d}{ds}\left(\frac{1}{s} - \frac{e^{-s\pi}}{s}\right) + \frac{\pi e^{-s\pi}}{s} = \frac{1-e^{-s\pi}}{s^2} - \frac{\pi e^{-s\pi}}{s} + \frac{\pi e^{-s\pi}}{s}$. Hence, $\bar{x}_1(s) = \frac{1-e^{-s\pi}}{s^2(s^2+1)}$.



We use the Bromwich inversion formula to get $x_1(t)$: Write $\bar{x}_1(s) = \bar{x}_{1a}(s) + \bar{x}_{1b}(s) = \frac{1}{s^2(s^2+1)} + \frac{-e^{-s\pi}}{s^2(s^2+1)}$. We find $x_{1a}(t)$ first:

$x_{1a}(t) = \frac{1}{2\pi i}\int_\gamma \frac{e^{st}}{s^2(s^2+1)}\, ds$. Singularities at $0, \pm i \Rightarrow c>0$. By closing to right, $x_{1a}(t) = 0$ for $t<0$, and by closing to left,

$x_{1a}(t) = \begin{cases} 0 & t<0 \\ \sum \operatorname{Res}\{\frac{e^{-st}}{s^2(s^2+1)};\ s = \pm i, 0\} & t>0 \end{cases}$. $\operatorname{Res}\{s=i\} = \left[\frac{(s-i)e^{st}}{s^2(s+i)(s-i)}\right]_{s=i}$ by simple pole formula: $\operatorname{Res}\{s=i\} = \frac{e^{it}}{-2i}$.

Similarly, $\operatorname{Res}\{s=-i\} = \frac{e^{-it}}{2i}$. Then $\operatorname{Res}\{s=i\} + \operatorname{Res}\{s=-i\} = \frac{e^{-it} - e^{it}}{2i} = -\sin t$. For $s=0$: we could use the double pole formula $\operatorname{Res}\{s=0\} = \frac{1}{1!}\frac{d}{ds}\left[\frac{s^2 e^{st}}{s^2(s^2+1)}\right]_{s=0}$

but it might prove easier to use the Laurent series: $\frac{e^{st}}{s^2(s^2+1)} = \frac{1}{s^2}(1 + st + \frac{s^2t^2}{2!} + \dots)(\sum_{n=0}^\infty (-s^2)^n)$ $\because \sum_{n=0}^\infty x^n = \frac{1}{1-x}$ for $|x|<1$

$\frac{e^{st}}{s^2(s^2+1)} = \frac{1}{s^2}(1 + st + \frac{s^2t^2}{2!} + \dots)(1 - s^2 + \dots) \Rightarrow$ coefficient of $\frac{1}{s} = \operatorname{Res}\{s=0\} = t \Rightarrow x_{1a}(t) = t - \sin t$. Then we consider $x_{1b}(t)$. We see that

$x_{1b}(t) = \frac{1}{2\pi i}\int_\gamma \frac{-e^{st}e^{-s\pi}}{s^2(s^2+1)}\, ds = -\frac{1}{2\pi i}\int_\gamma \frac{e^{s(t-\pi)}}{s^2(s^2+1)}\, ds$. Here, $t-\pi$ takes the role of t in the formula of $x_{1a}(t)$. Hence, $t-\pi<0$ when closed to right,

$t-\pi>0$ when closed to left. Hence, $x_{1b}(t) = -x_{1a}(t-\pi)$ (for $t-\pi>0$) $= \begin{cases} -x_{1a}(t-\pi) & t-\pi>0 \\ 0 & t-\pi<0 \end{cases} = \begin{cases} \sin(t-\pi) - (t-\pi) & t>\pi \\ 0 & t<\pi \end{cases} = \begin{cases} (\pi-t) - \sin(t) & t>\pi \\ 0 & t<\pi \end{cases}$.

So final answer is $X(t) = \begin{cases} 0 & t<0 \\ x_{1a}(t) + x_2(t) & 0<t<\pi \\ x_{1a}(t) + x_{1b}(t) + x_2(t) & t>\pi \end{cases} = \begin{cases} 0 & t<0 \\ t + \cos t & 0<t<\pi \\ \pi + \cos t - \sin t & t>\pi \end{cases}$ //

END OF SYLLABUS.