# 7402 Mathematical Methods 4 Notes

Based on the 2011-2012 lectures by Dr G Esler

The Author has made every effort to copy down all the content on the board during lectures. The Author accepts no responsibility what so ever for mistakes on the notes or changes to the syllabus for the current year. The Author highly recommends that reader attends all lectures, making his/her own notes and to use this document as a reference only.

Tuesday $10^{th}$ January

1. Series solution of ODEs and special functions.

$2^{nd}$ - order, homogeneous linear ODE.
Consider :

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0 \quad \cdots\cdots (*)$$

The general solution of $(*)$ is a linear combination of 2 linearly independent solutions $y_1(x)$ and $y_2(x)$.

$$y = Ay_1(x) + By_2(x).$$

Note $y_1(x)$ and $y_2(x)$ not uniquely defined.

e.g. $y'' - y = 0$ ; we could use $y_1 = e^x$, $y_2 = e^{-x}$

or

$\tilde{y}_1 = \cosh x$, $\tilde{y}_2 = \sinh x$.

We have seen two special cases:

(i) $p(x) = a \quad q(x) = b \quad a, b$ const.

Solution : If $m_1$ and $m_2$ are the roots of the $m^2 + am + b = 0$.

$$y(x) = \begin{cases} Ae^{m_1 x} + Be^{m_2 x} & m_1, m_2 \text{ real roots} \\ (Ax + B)e^{m_1 x} & m_1 = m_2 \text{ double roots.} \\ e^{m_r x}(A\cos m_i x + B\sin m_i x) & m_{1,2} = m_r \pm i m_i \text{ complex conjugate roots.} \end{cases}$$

(ii) $p(x) = a/x \quad q(x) = b/x^2 \qquad a, b \in \mathbb{R}$

(Euler-type)

Use substitution $Y(t) = y(e^t) = y(x) \qquad x = e^t$

Chain rule: $dY/dt = x\, dy/dx \qquad d^2Y/dt^2 = x^2\, d^2y/dx^2 + x\, dy/dx$

Leads to: $\dfrac{d^2Y}{dt^2} + (a-1)\dfrac{dY}{dt} + bY = 0$

and gives:

$$y(x) = \begin{cases} Ax^{m_1} + Bx^{m_2} & m_1, m_2 \text{ real roots} \\ (A\log x + B)\,x^{m_1} & m_1 \text{ double root} \\ x^{m_r}\left(A\cos(m_i \log x) + B\sin(m_i \log x)\right) & m_{1,2} = m_r \pm m_i i \end{cases}$$

Quick method: Use $y(x) = x^m$ in equation:

Example: $x^2y'' - 2xy' + 2y = 0.$

$y = x^m \quad y' = m(x^m) \quad y'' = m(m-1)x^m$

$\Rightarrow m(m-1)x^m - 2mx^m + 2x^m = 0$

$m^2 - 3m + 2 = 0$

$m_1 = 2 \quad m_2 = 1 \qquad \Rightarrow y(x) = Ax^2 + Bx$

What if $p(x)$ and $q(x)$ have a more general form?

e.g. $p(x) = \dfrac{p_1(x)}{p_2(x)} \qquad q(x) = \dfrac{Q_1(x)}{Q_2(x)}$

for $p_1, p_2, Q_1, Q_2$ polynomials.

"Naive" power series method.

Try a power series solution ; $y(x) = \sum_{k=0}^{\infty} a_k x^k$

$\{a_k\}$ undetermined coefficients.

Example : $y'' - y = 0$ $\qquad\qquad ' \equiv d/dx$

Differentiate : $\quad y'(x) = \sum_{k=0}^{\infty} a_k \cdot k \cdot x^{k-1}$

$$y''(x) = \sum_{k=0}^{\infty} a_k \cdot k \cdot (k-1)\, a^{k-1}$$

Insert into equation :

$$\sum_{k=0}^{\infty} a_k\, k(k-1) x^{k-2} - \sum_{k=0}^{\infty} a_k x^k = 0$$

Re-index first term :

$$\sum_{\substack{k=0 \\ k=2}}^{\infty} a_k\, k(k-1) x^{k-2} = \sum_{\substack{k=-2 \\ k=0}}^{\infty} a_{k+2} (k+2)(k+1) x^k$$

Eqn : $\quad \sum_{k=0}^{\infty} \left\{ a_{k+2}(k+2)(k+1) - a_k \right\} x^k = 0$

Power series in $x$ = zero everywhere
$\Rightarrow$ all coefficients must be zero.

Deduce that : $a_{k+2} = \dfrac{a_k}{(k+2)(k+1)}$ - recurrance relation.
$(k \geq 0)$

$\Rightarrow$ Even coefficients can be found in terms of $a_0$
$\Rightarrow$ Odd " " " " " " " $a_1$

$a_0$ and $a_1$ are undetermined ... they will take the role of the arbitrary const. in the soln.

Even coefficients: $a_2 = a_0/2\cdot1$ $\quad a_4 = a_2/4.3 = a_0/4.3.2.1$

Try $a_{2k} = a_0/(2k)!$ $\quad$ check: $a_{2k+2} = \dfrac{a_{2k}}{(2k+2)(2k+1)} = \dfrac{a_0}{(2k+2)(2k+1)(2k)!}$ ✓

Odd coefficients: $a_3 = a_1/3.2$ $\quad a_5 = a_3/5.4 = a_1/5.4.3.2$

Try $a_{2k+1} = a_1/(2k+1)!$

Solution: $$y(x) = \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_{2k} x^{2k} + \sum_{k=0}^{\infty} a_{2k+1} x^{2k+1}$$

$$= a_0 \sum_{k=0}^{\infty} x^{2k}/(2k)! + a_1 \sum_{k=0}^{\infty} x^{2k+1}/(2k)!$$

$$= a_0 \cosh x + a_1 \sinh x.$$

The Frobenius Method.

The Naive power series method above is not sufficiently general:
- We do not know in general where the power series should begin. Could be a -ve or non-integer power.
- We know nothing about the nature (or existence) of solutions.
- ~~We~~ May find it helpful to extend our thinking to $\mathbb{C}$ (complex plane)
- to use power series results.

To address these points:

Try to solve $\quad w''(z) + p(z)w'(z) + q(z)w(z) = 0 \qquad (\dagger)$

with the ~~ant~~ ansatz: $\quad w(z) = \sum_{k=0}^{\infty} a_k z^{k+c} \qquad a_0 \neq 0 \quad \dots a_{-2}, a_{-1} = 0$ by asump.

Power series starts (by construction) at $z^c$.

$c$ is a const. to be found.

Example 2.

Solve $zw'' + \frac{1}{2}w' + \frac{1}{4}w = 0$

$$w' = \sum_{k=0}^{\infty} a_k (k+c) z^{k+c-1} \qquad w'' = \sum_{k=0}^{\infty} a_k (k+c)(k+c-1) z^{k+c-2}$$

Insert in equation :

$$\sum_{k=0}^{\infty} a_k (k+c)(k+c-1) z^{k+c-1} + \frac{1}{2}\sum_{k=0}^{\infty} a_k (k+c) z^{k+c-1} + \frac{1}{4}\sum_{k=0}^{\infty} a_k z^{k+c} = 0$$

Convention is to reindex downwards :

$$\sum_{k=0}^{\infty} a_k z^{k+c} = \sum_{k=0}^{\infty} a_{k-1} z^{k-1+c}$$

Equation is :

$$\sum_{k=0}^{\infty} \left\{ a_k (k+c)(k+c-1) + \frac{1}{2} a_k (k+c) + \frac{1}{4} a_{k-1} \right\} z^{k+c-1} = 0$$

Set coeff. to zero : $a_k (k+c)(k+c-1) + \frac{1}{2} a_k (k+c) + \frac{1}{4} a_{k-1} = 0$

$$\Rightarrow a_k (k+c)(k+c-\tfrac{1}{2}) + \tfrac{1}{4} a_{k-1} = 0 \qquad (k \geq 0)$$

$[k=0]$ $\quad \underset{(\neq 0)}{a_0}(c)(c-\frac{1}{2}) = 0$

$\Rightarrow c(c-\frac{1}{2}) = 0$ $\quad$ Indicial equ.

$[k \geq 1]$ $\quad a_k = \dfrac{-a_{k-1}}{4(k+c)(k+c-\frac{1}{2})}$ $\quad$ Recurrance relation.

Two linearly independent solutions from $c=0$ and $c=1/2$.

[$c=0$]

$$a_k = \frac{-a_{k-1}}{2k(2k-1)}$$ (Set $a_0$ to be 1 without loss of generality)

$$a_1 = \frac{-1}{2\cdot 1} \qquad a_2 = \frac{-a_1}{4\cdot 3} = \frac{a_0}{4.3.2.1} = \frac{1}{4.3.2.1}$$

Try $a_k = \frac{(-1)^k}{(2k)!}$. First solution: $w_1(z) = \sum_{k=0}^{\infty} {(-1)^k}/{(2k)!}\, z^k$.

[$c=1/2$]

$$a_k = \frac{-a_{k-1}}{(2k+1)(2k)}$$ ($a_0 = 1$ w.l.o.g)

$$a_1 = \frac{-1}{3\cdot 2} \qquad a_2 = \frac{-a_1}{5\cdot 4} = \frac{1}{5.4.3.2}.$$

Try $a_k = \frac{(-1)^k}{(2k+1)!}$. $W_2(z) = \sum_{k=0}^{\infty} {(-1)^k}/{(2k+1)!}\, z^{k+1/2}$.

General solution: $w(z) = Aw_1 + Bw_2$

$$= A\cos(z^{1/2}) + B\sin(z^{1/2}).$$

Frobenius Method Theory

Under what conditions (in $p(z)$, $q(z)$) does this method work?

Definition.

A point $z = z_0$ is said to be an ordinary point of $(\dagger)$ if both $p(z)$ and $q(z)$ are analytic at $z = z_0$.

Definition

A point $z = z_0$ is said to be a regular singular point of $(\dagger)$ if both $(z-z_0)p(z)$ and $(z-z_0)^2 q(z)$ are analytic there.

i.e. $p(z)$ has at worst a simple pole at $z = z_0$
$q(z)$ " " " " pole of order 2.

In example 2 $\quad z_0 = 0 \quad p(z) = 1/2z \quad q(z) = 1/4z$

$$\left.\begin{aligned} zp(z) &= 1/2 \\ z^2 q(z) &= 1/4\, z \end{aligned}\right\} \text{both analytic}$$

$\Rightarrow$ $z = 0$ is a regular sing. point. but not an ordinary point in Ex 2.

Thursday $12^{th}$ January 2012

Frobenius Method.

Essential singularities: $z = z_0$ is neither an ordinary point or regular singular point.

Rule of thumb.

If $z=z_0$ is an O.P use Naive power series method

If $z=z_0$ is an R.S.P use Frobenius.

If $z=z_0$ is an essential singularity .... Neither!

Theorem

The general solution of $(\dagger)$ is obtainable by the Frobenius method in the form of a power series about $z=z_0$, provided that $z_0$ is a R.S.P (or O.P) of $(\dagger)$

(by Fuch)

Corollary

Further, when $z_0$ is an ordinary point the solutions will be analytic at $z_0$ and have radius of convergence at least as great as the minimum of $p(z)$ and $q(z)$.

Note: We can set $z_0=0$ w.l.o.g by making the transform $\tilde{z} = z - z_0$ in $(\dagger)$.

Recall

The Radius of Convergence $R$ of a complex power series $\sum_{k=0}^{\infty} a_k (z-z_0)^k$ is a real number for which the series;

DIVERGES for all $|z-z_0| > R$

CONVERGES " " $|z-z_0| < R$.

Use D'Alembert Ratio test to get R.

$$R = \lim_{k \to 0} \left| \frac{a_k}{a_{k+1}} \right|$$

Theory (Partial proof of theorem) Assume $z_0 = 0$.

If $z=0$ is an R.S.P of (†).... then $zp(z)$ and $z^2q(z)$ have Taylor series.

$$zp(z) = \sum_{k=0}^{\infty} p_k z^k \qquad z^2q(z) = \sum_{k=0}^{\infty} q_k z^k$$

Insert in (†)

$$\underbrace{\sum_{k=0}^{\infty} a_k (k+c)(k+c-1) z^{k+c-2}}_{w''} + \overbrace{\left(\sum_{k=0}^{\infty} p_k z^k\right)}^{zp(z)} \overbrace{\left(\sum_{k=0}^{\infty} a_k (k+c) z^{k+c-2}\right)}^{z^{-1}w'}$$

$$+ \underbrace{\left(\sum_{k=0}^{\infty} q_k z^k\right)}_{z^2q} \underbrace{\left(\sum_{k=0}^{\infty} a_k z^{k+c-2}\right)}_{z^{-2}w} = 0$$

Use formula for multiplying power series:

$$\left(\sum_{k=0}^{\infty} f_k z^k\right)\left(\sum_{k=0}^{\infty} g_k z^k\right) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} f_{k-j}\, g_j\right) z^k$$

Apply to (†):

$$\sum_{k=0}^{\infty} \left\{ a_k (k+c)(k+c-1) + \sum_{j=0}^{k} p_{k-j} a_j (j+c) + q_{k-j} a_j \right\} z^{k+c-2} = 0$$

Set coefficients to zero:

$$a_k\left[(k+c)(k+c-1) + p_0(k+c) + q_0\right] + \sum_{j=0}^{k-1} a_j \left(p_{k-j}(j+c) + q_{k-j}\right) = 0$$

$k \geq 0$

k term moved to left.

$k=0$ $\quad a_0\left(c(c-1)+p_0c+q_0\right)=0 \qquad a_0\neq 0$

$F(c)\equiv c^2+(p_0-1)c+q_0=0$ — INDICIAL EQUATION. (I.E)

$k\geq 1$ $\quad a_k = -\dfrac{\sum_{j=0}^{k-1} a_j\left(p_{k-j}(j+c)+q_{k-j}\right)}{F(k+c)}$ - RECURRENCE RELATION.

Gives $a_k$ in terms of previous coeff. $\{a_0, a_1, \dots, a_{k-1}\}$

We consider 3 possibilities

Case 1 : I.E has two distinct real roots which do not differ by an integer ($c_1-c_2 \notin \mathbb{Z}$) E.g. example 2 $c(c-\tfrac{1}{2})=0$

Case 2 : I.E has a double root (i.e. $F(c)=(c-c_1)^2$)

Case 3 : (Not on syllabus, see handout)

I.E has 2 distinct roots. $F(c)=(c-c_1)(c-c_2)$ and $c_1-c_2\in\mathbb{Z}$.

R.R fails if $F(k+c)=0$

If $c_1<c_2$ and $c_2-c_1=m>0$ integer

Solution for $c=c_1$ will fail because $F(c_1+m)=F(c_2)=0$

($c=c_2$ still works).

Tuesday 17th January 2012

Frobenius Method - the three cases.

To ~~solve~~ apply the method more generally, it turns out to be useful to consider the Frobenius Ansatz:

$$W(z,c) = \sum_{k=0}^{\infty} a_k(c)\, z^{k+c} \quad \text{as a function of } \underline{\text{two}} \text{ variables } z, c.$$

Here the $\{a_k(c)\}$ satisfy the recurrance relation but $c \in \mathbb{R}$ is allowed to vary freely.

Insert $W(z,c)$ into $(\dagger)$

$$\left( \frac{d^2}{dz^2} + p(z)\frac{d}{dz} + q(z) \right) W(z,c)$$

$$= \sum_{k=0}^{\infty} \left[ a_k (k+c)(k+c-1) + p_0 (k+c) + q_0 + \sum_{j=0}^{k-1} a_j \left(p_{k-j}(j+c) + q_{k-j}\right) \right] z^{k+c-2}$$

$$= a_0^{=1} \underbrace{\left( c^2 + (p_0 - 1)c + q_0 \right)}_{F(c)} z^{c-2}$$

(Recurrance relation ensures all terms vanish for $k \geq 1$).

Case I.

We can make $W(z,c)$ a solution of $(\dagger)$ by setting $c = c_1$ and $c = c_2$. (roots of $F(c)=0$) Since $c_1 > c_2$ and $c_2 - c_1 \notin \mathbb{Z}$ both satisfy that they are well-behaved.

E.g. in example 2: $c_1 = 1/2 \quad c_2 = 0$

two solutions were $w_1(z) = W(z,0)$

$w_2(z) = W(z, 1/2)$.

In general

$$w_1(z) = W(z, c_1) = \sum_{k=0}^{\infty} a_k(c_1) z^{k+c_1}.$$

$$w_2(z) = W(z, c_2) = \sum_{k=0}^{\infty} a_k(c_2) z^{k+c_2}.$$

Case II

In this cas we have :

$$\left( \frac{d^2}{dz^2} + p(z)\frac{d}{dz} + q(z) \right) W(z,c) = F(c) z^{c-2}$$

$$= (c-c_1)^2 z^{c-2} \quad - c_1 \text{ double root}$$

First solution : $w_1(z) = W(z, c_1)$ as before

Differentiate with respect to c !!

$$\left( \frac{d^2}{dz^2} + p(z)\frac{d}{dz} + q(z) \right) \frac{\partial W}{\partial c} = 2(c-c_1) z^{c-2} + (c-c_1)^2 \log z \cdot z^{c-2}$$

$$\left\{ \text{where } \frac{\partial}{\partial c} z^{c-2} = \frac{\partial}{\partial c} e^{(\log z)(c-2)} = \log z \cdot e^{(\log z)(c-2)} = z^{c-2} \log z . \right\}$$

Evaluate at $c = c_1$

$$\left( \frac{d^2}{dz^2} + p(z)\frac{d}{dz} + q(z) \right) \frac{\partial W}{\partial c}(z, c_1) = 0$$

$\Rightarrow$ possible second solution is

$$w_2(z) = \frac{\partial W}{\partial c}(z, c_1) = \left[ \frac{\partial}{\partial c} \left( \sum_{k=0}^{\infty} a_k(c) z^{k+c} \right) \right]_{c=c_1}$$

$$= \left( \sum_{k=1}^{\infty} \partial/\partial c \, a_k \, z^{k+c} + \sum_{k=0}^{\infty} a_k(c) z^{k+c} \cdot \log z \right) \Big|_{c=c_1}$$

$$= \sum_{k=1}^{\infty} \frac{\partial a_k}{\partial c}(c) \, z^{k+c_1} + \sum_{k=0}^{\infty} a_k(c) \, z^{k+c_1} \log z$$

$\frac{\partial a_0}{\partial c} = 0$ as $a_0 \equiv 1$

$$= \sum_{k=1}^{\infty} \frac{\partial a_k}{\partial c}(c) \, z^{k+c_1} + w_1(z) \log z.$$

Know that $w_2(z)$ is linearly independent of $w_1$ .... due to $\log z$ terms.

$\Rightarrow w_2(z)$ is second solution.

Example 3 (Bessel's equation, index zero).

$$zw'' + w' + zw = 0.$$

Insert $w = \sum_{k=0}^{\infty} a_k z^{k+c}$

$$\sum_{k=0}^{\infty} a_k (k+c)(k+c-1) z^{k+c-1} + \sum_{k=0}^{\infty} a_k (k+c) z^{k+c-1} + \sum_{k=0}^{\infty} a_k z^{k+c+1} = 0$$

Re-index final term:

$$\sum_{k=0}^{\infty} \left[ a_k (k+c)^2 + a_{k-2} \right] z^{k+c-1} = 0.$$

Set coefficients to zero.

$K=0$ $\quad \cancel{a_0}^{1} C^2 = 0$ INDICIAL EQUATION $\quad C = 0, 0$ ; double root

$K \geqslant 1$ $\quad a_K = \dfrac{-a_{K-2}}{(K+C)^2}$ RECURRENCE RELATION

Generates <u>even</u> coefficients only : $\{a_0, a_2, a_4, \dots\}$ non zero
$\{a_1, a_3, \dots\}$ zero.

Even coefficients given by $\{a_{2K}\}$.

$$\text{R.R}: \quad a_{2K} = \frac{-a_{2K-2}}{(2K+C)^2} = \frac{-a_{2K-2}}{2^2(K+C/2)^2}$$

Substitute $b_K = a_{2K}$ ;

$$b_K = \frac{-b_{K-1}}{2^2(K+C/2)^2} \qquad \text{Set } b_0 = 1 \text{ w.l.o.g.}$$

Try ;

$$b_K(C) = \frac{(-1)^K}{2^{2K}(K+C/2)^2(K-1+C/2)^2 \dots \dots (2+C/2)^2(1+C/2)^2}$$

Confirm by checking with recurrance relation.

$$W(z,C) = \sum_{K=0}^{\infty} b_K(C) z^{2K+C}$$ (Recall even terms only in the series).

First solution , insert $C=0$

$$W_1(z) = \sum_{K=0}^{\infty} b_K(0) z^{2K} = \sum^{\infty} {(-1)^K}/{2^{2K}(K!)^2} \; z^{2K}$$

$$= J_0(z)$$

Bessel function of first kind of index zero

Second solution

$$W_2(z) = \frac{\partial W}{\partial c}(z,0)$$

$$\frac{\partial W}{\partial c}(z,c) = \sum_{k=1}^{\infty} \frac{\partial b_k}{\partial c}(c)\, z^{2k+c} + \sum_{k=0}^{\infty} b_k(c)\, z^{2k+c} \log z$$

Need $\frac{\partial b_k(c)}{\partial c}$ .... ? Use logarithmic differentiation

$$\log(b_k(c)) = \log(-1)^k - \log 2^{2k} - 2\sum_{j=1}^{k} \log(j + c/2)$$

Recall $\log\left(\prod_{j=1}^{k} c_j^{-2} = -2\sum_{j=1}^{k} \log c_j \text{ etc}\right).$

Differentiate w.r.t c.

$$\frac{\partial}{\partial c}\log(b_k(c)) = \frac{1}{b_k(c)}\frac{\partial b_k}{\partial c}(c) = -\sum_{j=1}^{k} \frac{1}{j + c/2}$$

Evaluate at $c = 0$

$$\frac{\partial b_k}{\partial c}(0) = -b_k(0)\left(\sum_{j=1}^{k} \frac{1}{j}\right) = -\frac{(-1)^k S_k}{(2^{2k})(k!)^2} \qquad (k \geq 1)$$

$$W_2(z) = \frac{\partial W}{\partial c}(z,0) = -\sum_{k=1}^{\infty} \frac{(-1)^k S_k}{2^{2k}(k!)^2} z^{2k} + W_1(z)\log z$$

second solution.

Linear combination of $W_1(z)$ and $W_2(z)$ gives $Y_0(z)$
(Bessel func. of second kind, index zero).

Gamma Function.

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt$$

$$\Gamma(1) = \int_0^\infty e^{-t} \, dt = \left[ -e^{-t} \right]_0^\infty = 1$$

$$\Gamma(x) = \left[ -t^{x-1} e^{-t} \right]_0^\infty + (x-1) \int_0^\infty t^{x-2} e^{-t} \, dt$$

$$= (x-1)\Gamma(x-1) \qquad (x \neq 1)$$

$n \in \mathbb{N}$ $\quad \Gamma(n) = (n-1)\Gamma(n-1)$

$$= (n-1)(n-2)\Gamma(n-2)$$

$$= (n-1)(n-2) \dots 1 \, \Gamma(1) = (n-1)!$$

$$(k+c)(k+c-1)(k+c-2)(k+c-3) \dots (1+c)$$

$$\Gamma(2+c) = (1+c)\Gamma(1+c)$$

$$\Gamma(3+c) = (2+c)(1+c)\Gamma(1+c)$$

$$\vdots$$

$$\Gamma(k+c+1) = (k+c)(k-1+c) \dots (1+c)\Gamma(1+c)$$

$$\Rightarrow (k+c)(k-1+c) \dots (1+c) = \frac{\Gamma(k+c+1)}{\Gamma(1+c)}$$

$\Gamma(x+1) = x!$ for $x$ non-integer.

19th January 2012

## Special functions

- Defined primarily in terms of their complex power series.

- Can be found using Frobenius method and allow us to write down solutions for lots of new ODEs.

### Example. Bessel functions

Solutions of Bessel's equation

$$\boxed{z^2 w'' + z w' + (z^2 + \nu^2) w = 0}$$

In full generality $\nu \in \mathbb{C}$, but we will be concered with $\nu \in \mathbb{R}$.

$\nu$ is the index of the equation. (Note $\nu = 0$ recovers Ex. 3.)

Important R.S.P ; $z = 0$ (why? $p(z) = 1/z$, $q(z) = \frac{z^2 - \nu^2}{z^2}$, $zp$ and $z^2 q$ analytic).

Solution is :

$$w(z) = \begin{cases} AJ_\nu(z) + BJ_{\nu}(z) & \nu \notin \mathbb{Z} \\ AJ_0(z) + BY_0(z) & \nu = 0 \\ AJ_m(z) + BY_m(z) & \nu = m \end{cases}$$

~~$J_\nu(m)$~~ $J_\nu(z)$ Bessel function of the first kind, index $\nu$

$Y_m(z)$ " " " " second kind, index $m \geq 0$ (integer).

Properties : 1) As $x \to \infty$, $J_n(x) \sim x^{-1/2} \sin(x + \varepsilon)$

(Exercise ; write $w(z) = f(z)/z^{1/2}$, differentiate repeatedly, show B.E becomes $f''(z) + \left(1 + \frac{1/4 - \nu^2}{z^2}\right) f(z) = 0$.

When $z = x \to \infty \Rightarrow f'' + f = 0 \Rightarrow f \sim A\sin(x+\varepsilon).$)

Hence $J_n \sim \sin(x+\varepsilon)/x^{1/2} \Rightarrow$ infinetly many zeros.

Label : The zeros of $J_m(x)$ ($x$ real) as $\{j_{m1}, j_{m2}, j_{m3}, \dots\}$.

(2) (from power series)

$J_\nu(z) \sim z^\nu$ as $z \to 0$
$Y_0(z) \sim \log z$ " "
$Y_m(z) \sim z^{-m}$ " "

Generating Function

The Bessel functions have a generating function.

$$G(x,t) = \exp\left(\tfrac{x}{2}(t - 1/t)\right) = \sum_{n=-\infty}^{\infty} J_n(x) t^n.$$

Differentiate w.r.t $t$.

← replace with power series.

$$\partial G/\partial t\,(x,t) = \tfrac{x}{2}(1 + 1/t^2)\exp\left(\tfrac{x}{2}(t - 1/t)\right) = \sum_{n=-\infty}^{\infty} J_n(x)\, n t^{n-1}$$

reindex down 1 ↘ reindex up 1 ↙

$$\Rightarrow \tfrac{x}{2}\sum_{n=-\infty}^{\infty} J_n(x) t^n + \tfrac{x}{2}\sum_{n=-\infty}^{\infty} J_n(x) t^{n-2} = \sum_{n=-\infty}^{\infty} J_n(x)\, n t^{n-1}$$

$$\sum_{n=-\infty}^{\infty}\left[\frac{x}{2} J_{n-1}(x) + \frac{x}{2} J_{n+1}(x) - n J_n(x)\right] t^{n-1} = 0$$

single power series in $t^{n-1}$, all coeff. must $= 0$.

$\Rightarrow x\left(J_{n-1}(x) + J_{n+1}(x)\right) = 2n J_n(x)$. ← Recurance relation.

Differentiate w.r.t $x$ to prove: Differentiation identity $J_m'(x) = \tfrac{1}{2}\left(J_{m-1}(x) - J_{m+1}(x)\right)$

$J_{-1}(x) = -J_1(x)$ hence $J_0'(x) = -J_1(x)$.

24th January 2012.

Special Functions.

Have met Bessel's equation :

$$z^2w'' + zw' + (z^2 - \nu^2)w = 0$$

Arises in applied maths/physics applications, especially where there is Axisymmetry.

Legendre's equation

$$(1-z^2)w'' - 2w'z + \nu(\nu+1)w = 0$$

tends to arise in problems with spherical symmetry

$z = 0$ is an ordinary point

$p(z) = {-2w}/{1-z^2}$

$q(z) = {\nu(\nu+1)}/{1-z^2}$

both analytic at $z = 0$.

Use naive power series method to look for solutions.

$$w = \sum_{k=0}^{\infty} a_k z^k$$

( don't enforce $a_0 \neq 0$ in naive method compared to frobenius ).

$$\Rightarrow \sum_{k=0}^{\infty} a_k k(k-1) z^{k-2} - \sum_{k=0}^{\infty} a_k k(k-1) z^k - \sum_{k=0}^{\infty} 2a_k k z^k + \sum_{k=0}^{\infty} \nu(\nu+1) a_k z^k = 0$$

Re-index first term upwards (in frobenius re-index downwards, doesn't matter here wether reindex up or down).

$$\Rightarrow \sum_{k=0}^{\infty} \Big[ a_{k+2}(k+1)(k+2) + a_k\big(\nu(\nu+1) - k(k+1)\big) \Big] z^k = 0$$

Set coefficients to zero

roots $k = \nu, -\nu-1$

$$a_{k+2} = a_k \left\{ \frac{k(k+1) - \nu(\nu+1)}{(k+1)(k+2)} \right\}$$

$$= a_k \left\{ \frac{(k-\nu)(k+\nu+1)}{(k+1)(k+2)} \right\}$$

Will generate two seperate series of coefficients :

$\{a_0, a_2, a_4, \ldots\}$ and $\{a_1, a_3, a_5, \ldots\}$ note: Not setting $a_0$.
even odd

Even $\{a_{2k}\}$

$$a_{2k+2} = a_{2k} \left( \frac{(2k-\nu)(2k+\nu+1)}{(2k+1)(2k+2)} \right)$$

Write $b_k = a_{2k}$

$$b_{k+1} = b_k \left( \frac{(2k-\nu)(2k+\nu+1)}{(2k+1)(2k+2)} \right)$$

First solution is given by

$$W_1(z) = \sum_{k=0}^{\infty} b_k z^{2k} \quad \text{with } \{b_k\} \text{ given by}$$

Odd $\{a_{2k+1}\}$

$$W_2(z) = \sum_{k=0}^{\infty} \tilde{b}_k z^{2k+1} \quad \text{with } \tilde{b}_k = a_{2k+1} \text{ (exercise to find R.R.)}$$

The tabulated Legendre Functions

$P_\nu(z)$ and $Q_\nu(z)$ are linear combinations of $w_1$ and $w_2$.

In general, their power series have radius of convergence equal to 1.

Use D'Alembert's ratio test on $w_1(z)$

$$\lim_{k\to\infty}\left|\frac{b_{k+1}\, z^{2k+2}}{b_k\, z^k}\right| \quad \begin{matrix} >1 & \text{diverges} \\ <1 & \text{converges.} \end{matrix}$$

Diverges if : $|z|^2 > \lim_{k\to\infty} \left|{}^{b_k}/_{b_{k+1}}\right|$

$$= \lim_{k\to 0}\left|\frac{(2k+1)(2k+2)}{(2k-\nu)(2k+\nu+1)}\right| = 1$$

$\Rightarrow$ Radius of convergence $= 1$

$\boxtimes$

Also (almost always) diverge at $z = \pm 1$.

- unless in applications ....!

Is there ever a situation where $P_\nu$ or $Q_\nu$ converge at $z = \pm 1$?

Legendre polynomials.

Recall R.R for $w_1(z)$

$$b_{k+1} = b_k\left(\frac{(2k-\nu)(2k+\nu+1)}{(2k+1)(2k+2)}\right) \qquad k \geq 0.$$

If one of the $\{b_k\}$ is zero for some $k = \cancel{m}\ m+1$ integer all subsequent ones will be zero.

We will have $W_1(z) = \sum_{k=0}^{m} b_k z^{2k}$ (polynomial order $2m$).

This happens ~~when~~ if $\nu = 2m$.

Then when $k=m$,

$$b_{m+1} = b_m \left( \frac{(2m-2m)(2m+\nu+1)}{(2m+1)(2m+2)} \right) = 0$$

Leaves polynomial order $2m$

$$W_1(z) = \sum_{k=0}^{m} b_k z^{2k} \quad \propto \quad P_{2m}(z)$$

where $P_m(z)$ is the Legendre polynomial of index $m$.

$W_1(z)$ generates even L.Ps $\quad P_{2m}(z)$
$W_2(z)$ generates odd L.Ps $\quad P_{2m+1}(z)$.

Rodrigue's Formula.

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left( (x^2-1)^n \right).$$

degree n

ensures $P_n(1)=1$

differentiate n times.

degree 2n

Proof of validity;

Define $h(x) = (x^2-1)^n$

Methods 4

3

$$y(x) = \underset{\frac{d^n h}{dx^n}}{h^{(n)}}(x)$$ will be a solution of Legendre's equation if it satisfies:

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

$\nu = n$ integer

$h'(x) = 2nx(x^2-1)^{n-1}$

$(x^2-1)h'(x) = 2nx(x-1)^n$

$\Rightarrow (x^2-1)h'(x) = 2nx\,h(x)$

$(1-x^2)h'(x) + 2nx\,h(x) = 0 \quad \ldots. (*)$.

Now differentiate $(*)$ $n+1$ times.

Using Leibnitz rule: $(fg)^{(n+1)} = \sum_{k=0}^{n+1} \binom{n+1}{k} f^{(k)} g^{(n+1-k)}$

$$\overset{1}{\binom{n+1}{0}}(1-x^2)h^{(n+2)}(x) + \overset{n+1}{\binom{n+1}{1}}(-2x)h^{(n+1)}(x) + \overset{\frac{n(n+1)}{2}}{\binom{n+2}{2}}(-2)h^{(n)}(x)$$

$\left\{\begin{array}{l}\text{with } f = 1-x^2 \text{ and } g = h' \\ f^{(3)} = f^{(4)} = \ldots = 0\end{array}\right\}$

$$+ \overset{1}{\binom{n+1}{0}} 2nxh^{(n+1)} + \overset{n+1.}{\binom{n+1}{1}} 2nh^{(n)} = 0$$

$\left\{\begin{array}{l}\text{with } f = 2nx \text{ and } g = h \\ f^{(2)} = f^{(3)} = \ldots = 0\end{array}\right\}$

$$= (1-x^2)h^{(n+2)}(x) - 2xh^{(n+1)}(x) + (n)(n+1)h^{(n)}(x) = 0.$$

proves $h^{(n)}(x)$ satisfies L.E ∎

Can use to calculate L.P's

e.g. $P_2(x) = \frac{1}{2^2 2!}\frac{d^2}{dx^2}(x^2-1)^2$

$$\frac{d^2}{dx^2}(x^4-2x^2-1)$$

$$= 12x^2-4$$

$$= \frac{1}{4\cdot 2}\cdot(12x^2-4)$$

$$= \frac{1}{2}(3x^2-1)$$

Generating function.

Like Bessel functions,

$$G(x,t) = \frac{1}{(1-2xt+t^2)^{1/2}} = \sum_{m=0}^{\infty} t^m P_m(x) \qquad \cdots\cdots (X)$$

Differentiation w.r.t $x$ (exercise) gives differentiation identity

$$P'_{m+1}(x) - P_{m-1}(x) = (2m+1)P_m(x) \qquad m \geq 1.$$

Differentiation w.r.t

$$\frac{\partial G}{\partial t} = \frac{x-t}{(1-2xt+t^2)^{3/2}} = \sum_{m=0}^{\infty} m t^{m-1} P_m(x).$$

Sub. (X)

$$(x-t)\sum_{m=0}^{\infty} t^m P_m(x) = (1-2xt+t^2)\sum_{m=0}^{\infty} m t^{m-1} P_m(x)$$

$$\sum_{m=0}^{\infty}\left[x P_m(x) - P_{m-1}(x) - (m+1)P_{m+1} + 2mx P_m - (m-1)P_{m-1}\right]t^m = 0$$

Set coeff. to zero

Methods 4

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$x(2m+1)P_m(x) - (m+1)P_{m+1}(x) - mP_{m-1}(x) = 0$ Bonnet's Recursion formula.

Allows calculation of $P_{m+1}(x)$ given $P_m(x)$ and $P_{m-1}(x)$

$m=0; \quad P_1(x) = xP_0(x)$

$$P_{m+1}(x) = \frac{1}{m+1}\left((2m+1)x\, P_m(x) - m P_{m-1}(x)\right) \quad m \geq 1.$$

26th January 2012

## §2 Orthogonality and Generalised Fourier Series

Recall the definition of the Fourier Series of a function $f(x)$ defined on $(-\pi, \pi]$

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx \, dx$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx \, dx$$

New idea :

Consider the set $V$ of functions defined on $(-\pi, \pi]$ (i.e. $f(x) \in V$)
The Fourier Series formula suggests that we can think of $V$ as a infinite dimensional vector space with the functions $\{\Psi_j(x)\}$ with

$$\Psi_1(x) = 1 \qquad \Psi_{2j}(x) = \cos jx \qquad \Psi_{3j}(x) = \sin jx \qquad j \geqslant 1$$

acting as a complete basis for $V$.

A natural inner product can be associated with $V$,

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x) g(x) \, dx \qquad \text{for } f, g \in V$$

Exercise; Confirm that $\langle \Psi_j, \Psi_k \rangle = 0 \qquad j \neq k$

Also $\langle \Psi_1, \Psi_1 \rangle = 2\pi$

$\langle \Psi_j, \Psi_j \rangle = \pi$ for $j \geqslant 2$

The basis $\{\psi_j(x)\}$ is therefore orthogonal (but not orthonormal as $\langle \psi_j, \psi_j \rangle \neq 1$)

An alternative way of writing the Fourier Series formula is:

$$\boxed{f(x) = \sum_{k=1}^{\infty} \frac{\langle f, \psi_k \rangle}{\langle \psi_k, \psi_k \rangle} \psi_k(x)}$$

Compare this with: $\underset{\sim}{a} = \sum_{j=1}^{n} (a.\underset{\sim}{e}_j)\underset{\sim}{e}_j$

for any $\underset{\sim}{a} \in \mathbb{R}^n$ where $\{e_1, e_2, \dots, e_n\}$ is an orthonormal basis.

Care must be taken with the choice of vector space V. A suitable choice is $L^2(-\pi, \pi)$

Def $L^2(a,b)$ is the set of functions $f: [a,b] \to \mathbb{C}$ for which $\int_a^b |f(x)|^2 dx$ exists.

$L^2(a,b)$ is a Hibert Space under inner products

$$\langle f, g \rangle_w = \int_a^b w(x) f(x) \bar{g}(x) \, dx$$

Here $w(x)$ is a continuous function satisfying $w(x) > 0$ on $[a,b]$ (otherwise arbitrary)

## Eigenvalue problems.

Example 1.

Consider the following eigenvalue problem on $(0, \pi)$

$$\mathcal{L}y = -\lambda y \qquad y(0) = 0 \quad y(\pi) = 0$$

where $\mathcal{L} \equiv d^2/dx^2$ is a linear differential operator.

$$\frac{d^2}{dx^2} y + \lambda y = 0$$

$$y(x) = \begin{cases} A\cosh\sqrt{-\lambda}\,x + B\sinh\sqrt{-\lambda}\,x & \lambda < 0 \\ Cx + D & \lambda = 0 \\ E\cos\sqrt{\lambda}\,x + F\sin\sqrt{\lambda}\,x & \lambda > 0 \end{cases}$$

Only solutions satisfying boundary conditions $(y(0)=0,\ y(\pi)=0)$ are for

Eigenvalues $\lambda_k = k^2$

Eigenvectors $y_k(x) = \sin kx$

$y \sim F\sin\sqrt{\lambda}\,x$ (satisfies bc)

$$\sin\sqrt{\lambda}\,\pi = 0 \quad \Rightarrow \quad \sqrt{\lambda} = k$$
$$\lambda_k = k^2 \,/\!/.$$

31st January 2012

## Differential equations as eigenvalue problems

A differential equation, involving an undetermined coefficient $\lambda$, can form an eigenvalue problem under suitable boundary conditions.

Example 1

$$\mathcal{L} \equiv d^2/dx^2 \qquad \mathcal{L}y = -\lambda y \qquad y(0) = y(\pi) = 0$$

Resulted in : eigenvalues $\lambda_k = k^2$ eigenvectors $y_k(x) = \sin kx$.

Example 2

$$\mathcal{L} \equiv d^2/dx^2 \qquad \mathcal{L}y = -\lambda y \qquad y(0) = y'(\ell) = 0$$

$$\frac{d^2}{dx^2}y + \lambda y = 0$$

$$y(x) = \begin{cases} A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x & \lambda > 0 \\ Cx + D & \lambda = 0 \\ E\cosh\sqrt{-\lambda}x + F\sinh\sqrt{-\lambda}x & \lambda < 0 \end{cases}$$

Don't give eigenvalues.

$$y(0) = 0 \Rightarrow A = 0$$

$$y(x) = B\sin\sqrt{\lambda}x$$

$$dy/dx = B\sqrt{\lambda}\cos\sqrt{\lambda}x \xrightarrow{x=\ell} B\sqrt{\lambda}\cos\sqrt{\lambda}\ell = 0$$

$$\sqrt{\lambda}\ell = \frac{(2k+1)\pi}{2} \quad k\text{-integer}$$

$$\lambda_k = \frac{(2k+1)^2\pi^2}{4l^2} \qquad \text{eigenvalues}$$

$$y_k(x) = \sin\left(\frac{(2k+1)\pi x}{2l}\right) \qquad \text{eigenfunctions}$$

## Analogy with linear algebra

Linear operator ($d^2/dx^2$ here) and b.c together take the role of the matrix in these problems.

Orthogonality: Both examples 1 and 2 generate an orthogonal basis, with respect to inner product

Example 1 $\qquad \langle f, g\rangle_1 = \int_0^{\pi} f(x)\bar{g}(x)\,dx$

$$\langle y_j, y_k\rangle = \int_0^{\pi} \sin jx \sin kx\,dx \qquad \text{when } j \neq k$$

Defn. Let $V$ be an inner product space, with $\langle .,. \rangle : V \times V \to \mathbb{C}$, its inner product. If $A$ is a linear operator defined on $V$, then its Adjoint $A'$ is a linear operator satisfying $\langle \underline{x}, A'\underline{y}\rangle = \langle A\underline{x}, \underline{y}\rangle$ for all $\underline{x}, \underline{y} \in V$.

Defn. If $A = A'$ then $A$ is said to be self-adjoint

Consider the inner product in Ex 1.

$$\langle \mathcal{L}f, g\rangle_1 = \int_0^{\pi} \frac{d^2f}{dx^2}\bar{g}\,dx = \left[\frac{df}{dx}\bar{g}\right]_0^{\pi} - \int_0^{\pi} \frac{df}{dx}\frac{d\bar{g}}{dx}\,dx$$

$$= \left[ \frac{df}{dx}\bar{g} - f\frac{d\bar{g}}{dx} \right]_0^{\pi} + \int_0^{\pi} f \frac{d^2\bar{g}}{dx} dx$$

$$\Rightarrow \langle \mathcal{L}f, \bar{g} \rangle_1 = \langle f, \mathcal{L}\bar{g} \rangle + \left[ {}^{df}/_{dx}\, \bar{g} - f\, {}^{d\bar{g}}/_{dx} \right]_0^{\pi}$$

$\mathcal{L}$ is self-adjoint under the defn. only if we can apply some b.c to $f, \bar{g}$ to make the boundary term vanish.

The b.c in examples 1 and 2 both achieve this.

If this holds, $\mathcal{L}$ is said to be self-adjoint under boundary conditions.

The Strum-Liouville Differential Operator.

Intrested in linear operators that are self-adjount under b.c

What is the most general second-order operator for which this holds?

Consider $$\boxed{\mathcal{L} \equiv \frac{1}{w(x)} \left[ \frac{d}{dx}\left(p(x)\frac{d}{dx}\right) \; r(x) \right]}$$ (Strum-Liouville operator).

acting on functions in $L^2(a,b)$

together with b.c $$\alpha_1 y(a) + \beta_1 y'(a) = 0$$ $$\alpha_2 y(b) + \beta_2 y'(b) = 0$$ Strum-Liouville b.c

$w(x)$ is the same weight function that appears in the inner product ($w(x) > 0$ on $(a,b)$)

$$\langle f, g \rangle_w = \int_a^b w(x) f(x) \bar{g}(x)\, dx$$

$p(x)$ is real, differentiable and $p(x) > 0$ on $[a,b]$
$r(x)$ is real and continuous on $(a,b)$

## Lagrange's Identity

$$(\mathcal{L}f)g - f(\mathcal{L}g) = \frac{1}{w}\left[\underbrace{(pf')'}_{u'v=(vu)'-uv'}+rf\right]g - \frac{1}{w}\left[(pg')'+rg\right]f$$

$$= \frac{1}{w}\left[(pf'g)' + pf'g' + rfg - (pg'f)' + pf'g' + rgf\right]$$

$\Rightarrow$
$$\boxed{(\mathcal{L}f)g - f(\mathcal{L}g) = \frac{1}{w}\frac{d}{dx}\left(p(f'g - fg')\right)}$$

Next consider the inner products

$$\langle \mathcal{L}f, g\rangle_w - \langle f, \mathcal{L}g\rangle_w = \int_a^b w(x)\left((\mathcal{L}f)\bar{g} - f(\mathcal{L}\bar{g})\right)dx$$

$$= \int_a^b \frac{d}{dx}\left(p(f'\bar{g} - f\bar{g}')\right)dx$$

$$\langle \mathcal{L}f, g\rangle_w - \langle f, \mathcal{L}g\rangle_w = \left[p(f'\bar{g} - f\bar{g}')\right]_a^b$$

If we now apply the Strum-Liouville b.c both $f$ and $g$, the b.c term disappears and $\langle \mathcal{L}f, g\rangle_w = \langle f, \mathcal{L}g\rangle_w$

$\Rightarrow \mathcal{L}$ is self-adjoint under b.c ∎

Boundary terms disappear .... Assume $\alpha$, $\beta$ non-zero, and consider only $x=a$

Apply $\alpha f(a) + \beta f'(a) = 0$
$\alpha g(a) + \beta g'(a) = 0$ to $p(a)\left(f'(a)\bar{g}(a) - f(a)\bar{g}'(a)\right)$

$$\Rightarrow p(a)\left(\left(-\tfrac{\alpha_1}{\beta_1} f(a)\right)\bar{g}(a) + \left(\tfrac{\alpha_1}{\beta_1}\bar{g}(a)\right)f(a)\right) = 0$$

Eigenvalue problems involving self-adjoint operators have many properties. For the S-L eigenvalue problem

$$w\mathcal{L}y = -\lambda w y$$

$$\Rightarrow \quad \frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + (r(x) + \lambda w(x))y = 0$$

with $\alpha_1 y(a) + \beta_1 y'(a) = 0$
$\alpha_2 y(b) + \beta_2 y'(b) = 0$

has the following properties:

(1) The eigenvalues $\{\lambda_k\}$ are real and form an infinite unbounded set
$\lambda_1 < \lambda_2 < \dots \quad \Rightarrow \lambda_k \to \infty$

(2) The eigenvector (eigenfunctions) are orthogonal under the inner product
$\langle \cdot, \cdot \rangle_w$ i.e. $\langle y_j, y_k \rangle_w = 0 \quad (j \neq k)$

(3) Eigenvectors associated with a particular eigenvalue are unique up to a multiplicative const.

(4) The set of functions $\{y_k(x)\}$ form a complete orthogonal basis for $L^2(a,b)$ in the sense that $f(x)$ in $L^2(a,b)$ can be expressed as a generalised power series.

$$f(x) = \sum_{k=1}^{\infty} \frac{\langle f, y_k \rangle_w}{\langle y_k, y_k \rangle_w} y_k(x)$$

with equality almost everywhere.

(1) Eigenvalues real :

Follows from $\mathcal{L}$ self adjoint. Let $y_k(x)$ be the eigenvector associated with $\lambda_k$

(i) $\langle \mathcal{L}y_k, y_k \rangle_w = \langle -\lambda_k y_k, y_k \rangle_w = -\lambda_k \langle y_k, y_k \rangle_w$

(ii) $\langle \mathcal{L}y_k, y_k \rangle_w = \langle y_k, \mathcal{L}y_k \rangle = \langle y_k, -\lambda_k y_k \rangle = -\bar{\lambda}_k \langle y_k, y_k \rangle$

$\Rightarrow \lambda_k = \bar{\lambda}_k \quad \Rightarrow$ real.

(2) Eigenvectors orthogonal

Let $y_j(x), y_k(x)$ be the eigenvectors for $\lambda_j, \lambda_k$ respectively

(i) $\langle \mathcal{L}y_j, y_k \rangle_w = \langle -\lambda_j y_j, y_k \rangle_w = -\lambda_j \langle y_j, y_k \rangle_w$

(ii) $\langle \mathcal{L}y_j, y_k \rangle_w = \langle y_j, \mathcal{L}y_k \rangle_w = \langle y_j, -\lambda_k y_k \rangle_w = -\lambda_k \langle y_j, y_k \rangle_w$

Equating (i) and (ii)

$$-\lambda_j \langle y_j, y_k \rangle_w = -\lambda_k \langle y_j, y_k \rangle_w$$

$$(\lambda_k - \lambda_j) \langle y_j, y_k \rangle_w = 0$$

$$\Rightarrow \langle y_j, y_k \rangle_w = 0$$

$$\Rightarrow \int_a^b w(x)\, y_j(x)\, \bar{y}_k(x)\, dx = 0$$

(3) See lecture notes

## (4)(b) Generalised Fourier Series

If the $\{y_k(x)\}$ are complete then we can expand any function $f(x) \in L^2(a,b)$ as a linear combination.

$$f(x) = \sum_{k=1}^{\infty} a_k y_k(x) \qquad \{a_k\} \text{ - undetermined coeff.}$$

Take inner product with $y_j(x)$

$$\langle f, y_j \rangle_w = \langle \sum_{k=1}^{\infty} a_k y_k, y_j \rangle_w$$

$$= \sum a_k \langle y_k, y_j \rangle_w$$

using orthogonality property.

$$= a_j \langle y_j, y_j \rangle_w$$

$$\Rightarrow a_j = \frac{\langle f, y_j \rangle_w}{\langle y_j, y_j \rangle_w} \quad \text{and so} \quad f(x) = \sum_{k=1}^{\infty} \frac{\langle f, y_k \rangle_w}{\langle y_k, y_k \rangle_w} y_k(x)$$

1

2nd February 2012

A generalised Fourier Series.

Ex 3.

Consider the eigenvalue problem on $0 \leq x \leq 1$, $\alpha \in \mathbb{R}$, $\alpha > 0$

$y'' + \lambda y = 0$ $\qquad$ $y(0) = 0$, $\quad y(1) + \alpha y'(1) = 0$ $\qquad$ SL eqn with $p(x) = 1$, $r(x) = 0$, $w(x) = 1$

Use it to expand $f(x) = 1$ in a generalised Fourier Series.

Get (again)

$$y(x) = \begin{cases} A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x & \lambda > 0 \\ - \\ - \end{cases}$$

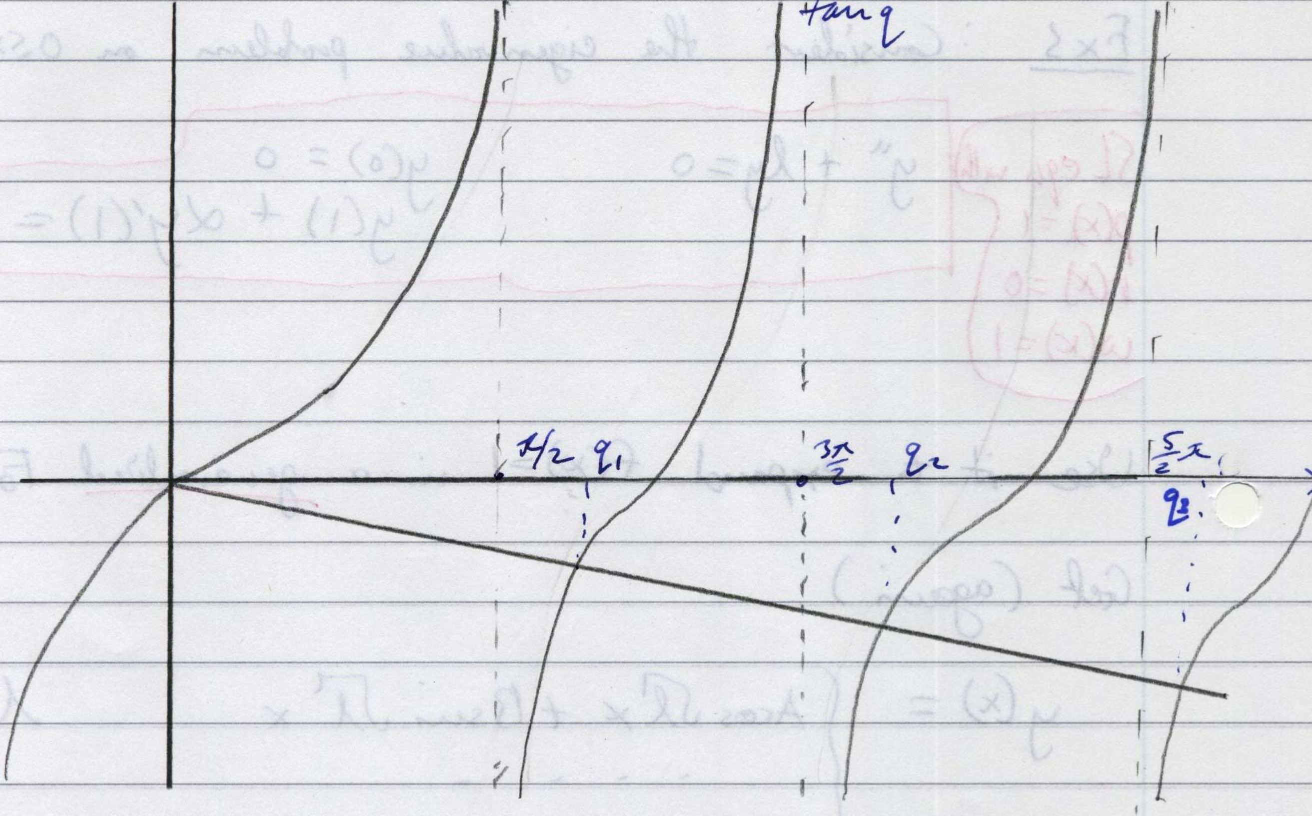
$y(0) = 0 \Rightarrow A = 0$

$y(1) + \alpha y'(1) = 0 \Rightarrow B\sin\sqrt{\lambda} + \alpha\sqrt{\lambda}\, B\cos\sqrt{\lambda} = 0$

$$\tan\sqrt{\lambda} + \alpha\sqrt{\lambda} = 0. \quad \cdots\cdots (*)$$

Eigenvalues will be roots of $(*)$, i.e. roots of $\tan q + \alpha q = 0$.

Need to consider $q > 0$.



tan q
π/2
q1
3π/2
q2
5π/2
q3

There are infinitely many roots (see graph) $\{q_K,\ K=1,2,3,\dots\}$.

Eigenvalues are $\lambda_K = q_K^2$

Eigenvectors $y(x) = \sin q_K x$

$$f(x) = 1 = \sum \frac{\langle f, y_K\rangle_1}{\langle y_K, y_K\rangle_1} y_K(x) \qquad \text{G.F.S}$$

$$\langle y_K, y_K\rangle_1 = \int_0^1 \overbrace{\sin^2 q_K x}^{y_K(x)^2}\, dx$$

$$= \int_0^1 \tfrac{1}{2}(1-\cos 2q_K x)\, dx$$

$$= \left[\tfrac{1}{2}x - \tfrac{1}{4q_K}\sin 2q_K x\right]_0^1$$

$$= \tfrac{1}{2} - \tfrac{1}{4q_K}\sin 2q_K$$

$$\langle f, y_K\rangle_1 = \int_0^1 \sin q_K(x)\, dx = \tfrac{1}{q_K}(1-\cos q_K)$$

Hence the G.F.S is

$$1 = \sum_{K=1}^{\infty} \frac{4(1-\cos q_K)}{2q_K - \sin 2q_K} \sin q_K x \quad \text{at every } x \in (0,1)$$

Connection to Seperation of Variables Problems.

S.O.V problems often have the form

$$P(x)y''(x) + Q(x)y'(x) + R(x)y(x) + \underset{\text{seperation const.}}{\lambda} y(x) = 0$$

with S.L boundary conditions: $\alpha_1 y(a) + \beta_1 y'(a) = 0$
$\alpha_2 y(b) + \beta_2 y'(b) = 0$

Can † be converted to Strum-Liouville form .... ?

Use integrating factor (just as in 1st order ODE)

$$p(x) = \exp\left(\int^x \frac{Q(t)}{P(t)}\,dt\right)$$

$$\frac{d}{dx}\left(p(x)y'(x)\right) = p(x)y''(x) + \underbrace{p(x)\frac{Q(x)}{P(x)}}_{\frac{dp}{dx}}y'$$

$$= \frac{p(x)}{P(x)}\left(P(x)y'' + Q(x)y'\right)$$

Use to rewrite †

$$\frac{P(x)}{P(x)}\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + R(x)y(x) = -\lambda y(x)$$

or $$\frac{1}{w(x)}\left[\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + r(x)y(x)\right] = -\lambda y(x)$$

for $w(x) = \frac{p(x)}{P(x)}$ $\quad r(x) = \frac{p(x)R(x)}{P(x)}$

Conclusion: † can always be converted to S.L form provided that $P(x)>0$ on the interval $[a,b]$.

$p(x)>0$ satisfied automatically

(or $<0$ i.e. no roots $\Downarrow$ would create R.S.P).

Ex 4.

$$x^2y'' - xy' + \lambda y = 0 \qquad \begin{array}{l} y(1) = 0 \\ y(e^{\pi}) = 0 \end{array}$$

Solve using $Y(t) = y(x) \qquad x = e^t$

$$\frac{dY}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = x\frac{dy}{dx}$$

$$\frac{d^2Y}{dt^2} = \frac{d}{dt}\left(x\frac{dy}{dx}\right) = x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx}$$

Equation becomes; $\dfrac{d^2Y}{dt^2} - 2\dfrac{dY}{dt} + \lambda Y = 0.$

Aux. eq.
$m^2 - 2m + \lambda$
$(m-1)^2 - 1 + \lambda = 0$
$\lambda > 1 \quad m = 1 \pm i\sqrt{\lambda - 1}$
(real roots otherwise)

Solution is :

$$Y(t) = e^t\left(A\cos\sqrt{\lambda-1}\,t + B\sin\sqrt{\lambda-1}\,t\right)$$

$$y(x) = x\left(A\cos\left(\sqrt{\lambda-1}\log x\right) + B\sin\left(\sqrt{\lambda-1}\log x\right)\right)$$

Check S-L type :

$$\text{IF}: \quad \exp\left(\int^x \frac{-t}{t^2}\,dt\right) = \exp(-\log x) = \frac{1}{x}$$

S.L: $\underset{\uparrow\, w(x)}{x^3}\dfrac{d}{dx}\left(\underset{\uparrow\, p(x)}{\dfrac{1}{x}}\dfrac{dy}{dx}\right) + \lambda y = 0$

$w(x) = 1/x^3 \quad p(x) = 1/x \quad r(x) = 0$

Apply b.c. $y(1) = 0 \Rightarrow A = 0$

$y(e^{\pi}) = 0 \Rightarrow B\sin(\sqrt{\lambda - 1}\,\pi) = 0$

$\sqrt{\lambda - 1} = k$ integer

$\lambda_k = 1 + k^2$ eigenvalues

$y_k(x) = \sin(k \log x)$ eigenvectors.

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Example 4

$$x^2y'' - xy' + \lambda y = 0 \qquad \begin{array}{l} y(1) = 0 \\ y(e^\pi) = 0 \end{array}$$

SL form

$$x^3 \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dx}\right) + \lambda y = 0$$

$$\mathcal{L}y + \lambda y = 0 \qquad \mathcal{L} \equiv x^3 \, {}^{d}\!/\!{}_{dx} \left( {}^{1}\!/\!{}_{x} \, {}^{d}\!/\!{}_{dx} \right)$$

General Solution

$$y(x) = Ax\cos\left(\sqrt{\lambda - 1}\log x\right) + Bx\sin\left(\sqrt{\lambda - 1}\log x\right)$$

Apply boundary conditions

$$y(1) = 0 \Rightarrow A = 0$$

$$y(e^\pi) = 0 \Rightarrow \sqrt{\lambda - 1} = k$$

Eigenvalues $\lambda_k = k^2 + 1$

Eigenfunctions $y_k(x) = x\sin(k\log x)$

Orthogonality $(j \neq k)$ $\langle y_j, y_k \rangle_w = 0 = \displaystyle\int_1^{e^\pi} \frac{x\sin(j\log x)\, x\sin(k\log x)}{x^3}\, dx$

$$= \int_1^{e^\pi} \frac{\sin(j\log x)\sin(k\log x)}{x}\, dx$$

Write $\theta = \log x$

$d\theta = 1/x \, dx$

$$= \int_0^{\pi} \sin j\theta \sin k\theta \, d\theta = \begin{cases} 0 & j \neq k \\ \pi/2 & j = k \end{cases}$$

Generalised Fourier Series

$f(x) \in L^2(1, e^\pi)$

$$f(x) = \sum_{k=1}^{\infty} \frac{\langle f, y_k \rangle_w}{\langle y_k, y_k \rangle_w} y_k(x)$$

$$= \sum_{k=1}^{\infty} a_k \, x \sin(k \log x)$$

where

$$a_k = \underbrace{\frac{2}{\pi}}_{\frac{1}{\langle y_k, y_k \rangle_w}} \underbrace{\int_1^{e^\pi} \frac{f(x)\, x \sin(k \log x)}{x^2}\, dx}_{\langle f, y_k \rangle_w}$$

Singular SL systems.

The SL operator satisfies

$$\langle \mathcal{L}f, g\rangle_w - \langle f, \mathcal{L}g\rangle_w = \Big[ p(\underbrace{f\bar{g}' - f'\bar{g}}) \Big]_a^b$$

Previously. choose S.L boundary conditions to force $=0$.

However, we could have $p(a)$ or $p(b) = 0$ instead.

Such a problem is known as a Singular Strum-Liouville problem.

( Previously had $p(x) > 0$ on $a \leq x \leq b$)

Now need $p(x) > 0$ on $a < x < b$ and $p(a) = 0$ and/or $p(b) = 0$

$p(a) = 0$ makes $x = a$ a REGULAR SINGULAR POINT of the SL equation.

SL boundary condition is REPLACED at $x = a$ (and/or $x = b$) with the condition $y(a)$ is finite (avoids singular solution there).

Example 5

Consider the singular eigenvalue problem

Legendre $\qquad (1-x^2)y'' - 2xy' + \lambda y = 0 \qquad y(\pm 1)$ finite

$\lambda \to \nu(\nu+1)$

S.L form $\qquad \dfrac{d}{dx}\left[ (1-x^2)\dfrac{dy}{dx}\right] + \lambda y = 0$

$p(x) = 1-x^2$
$r(x) = 0$
$w(x) = 1$

Clearly $p(\pm 1) = 0 \Rightarrow$ it is singular at $x = \pm 1$

(Recall: Legendre's equation has R.S.P at $x = \pm 1$)

Writing $\lambda = \nu(\nu+1)$, L.E has the general solution

$$y(x) = A P_\nu(x) + B Q_\nu(x) \qquad \text{Legendre functions}$$

Key fact: In general, $P_\nu(x)$ and $Q_\nu(x)$ are singular at $x = \pm 1$

Except when $\nu = k$ integer ..... then $P_k(x)$ is a Legendre polynomial

Eigenvalues $\lambda_k = k(k+1)$

Eigenfunctions $y_k(x) = P_k(x)$

The generalised Fourier Series in this case is the Fourier Legendre Series
$f(x) \in L^2(-1,1)$ can be written ....

$$f(x) = \sum_{k=1}^{\infty} a_k P_k(x)$$

$$a_k = \frac{2k+1}{2} \int_{1}^{-1} f(x) P_k(x)\, dx$$

$$\langle P_k, P_k \rangle_w^{-1} \qquad\qquad \langle f, P_k \rangle$$

Orthogonality:

$$\int_{-1}^{1} P_j(x) P_k(x)\, dx = 0 \quad j \neq k$$

Lemma

If $q(x)$ is a polynomial of degree $k-1$ or less then

$$\int_{-1}^{1} q(x) P_k(x)\, dx = 0$$

Proof

The polynomial $q(x)$ can be written as a linear combination of the first $k-1$ L.Ps

$$q(x) = \sum_{j=0}^{k-1} q_j P_j(x)$$

Then ...

$$\int_{-1}^{1} q P_k \, dx = \sum_{j=0}^{k-1} q_j \int_{-1}^{1} P_j(x) P_k(x)\, dx$$

$$= 0 \text{ by orthogonality.}$$

Allows us to calculate

$$\int_{-1}^{1} (P_k(x))^2 dx = \frac{1}{2k+1} \int_{-1}^{1} P_k(x) \left( P'_{k+1}(x) - \overbrace{P'_{k-1}(x)}^{0 \text{ by lemma}} \right)$$

$$= \frac{1}{2k+1} \left( \left[ P_k(x) P_{k+1}(x) \right]_{-1}^{1} - \underbrace{\int_{-1}^{1} P_k'(x) P_{k+1}(x)\, dx}_{=0 \text{ by lemma}} \right)$$

$$= \frac{2}{2k+1} \quad \text{as } P_k(1) = 1, \quad P_k(-1) = (-1)^k$$

Example 6.

Use the eigenvalue problem.

$y(0)$ finite

$y(1) = 0$

integer $q^2$

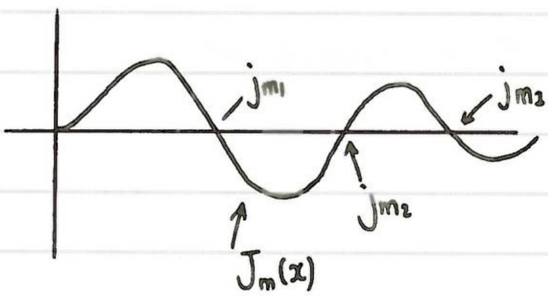
$$x^2 y'' + xy' + (-m^2 + \lambda x^2) y = 0 \quad \cdot\cdot (*)$$

to expand $f(x) \in L^2(0,1)$ in a .

Fourier - Bessel Series .

$$f(x) = \sum_{k=1}^{\infty} a_k J_m (j_{mk} x)$$

$$a_k = \frac{2}{(J_{m+1}(j_{mk}))^2} \int_0^1 x f(x) J_m (j_{mk} x) \, dx$$




System is singular at $x=0$ (Q2b, Sheet 3) as $p(0)=0$ but regular at $x=1$.

Write $Y(q) = y(x(q))$ $\qquad x(q) = q/\sqrt{\lambda}$

$$\frac{dY}{dq} = \frac{dy}{dx} \cdot \frac{1}{\sqrt{\lambda}}$$

$$q \frac{dY}{dq} = x \frac{dy}{dx}$$

$$\frac{d^2Y}{dq^2} = \frac{d^2y}{dx^2} \cdot \frac{1}{\lambda} \qquad q^2 \frac{d^2Y}{dq} = x^2 \frac{dy}{dx}$$

Substituting in (*)

$$q^2 \frac{d^2Y}{dq^2} + q \frac{dY}{dq} + (-m^2 + q^2) Y = 0 \qquad \text{(form met in lectures)}$$

General Solution

$$Y(q) = A J_m(q) + B Y_m(q)$$

$$y(x) = Y(q(x)) = A J_m(\sqrt{\lambda} x) + B Y_m(\sqrt{\lambda} x)$$

$y(0)$ finite $\Rightarrow$ $B = 0$ since $Y_m(x) \to -\infty$ as $x \to 0$

$y(1) = 0$ $\quad A J_m(\sqrt{\lambda}) = 0$ $\Rightarrow$ $\sqrt{\lambda} = j_{mk}$ $\leftarrow$ a zero of $J_m(x)$

$$\lambda_k = j_{mk}^2 \qquad y_k(x) = J_m(j_{mk} x)$$

Periodic Strum-Liouville Systems.

A different type of S-L system emerges under PERIODIC B.C on $a \leq x \leq b$

$$y(a) = y(b) \quad , \quad y'(a) = y'(b)$$

Example 7 (Exercise)

Show that Standard Fourier Series emerge from the periodic S-L problem

$$y'' + \lambda y = 0 \qquad \begin{aligned} y(-\pi) &= y(\pi) \\ y'(-\pi) &= y'(\pi) \end{aligned}$$

Periodic S-L systems exhibit DEGENERACY, more than one eigenfunction per eigenvalue.

$$\lambda_k = k^2 \Rightarrow \left.\begin{matrix} \cos kx \\ \sin kx \end{matrix}\right\} \text{ 2 functions.}$$

(regular and singular systems are non-degenerate).

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1.

## §3. Seperation of variables Revisited.

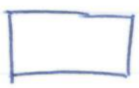
Idea: Use what we have learned about special functions §1 and S-L eigenvalue problems §2 to solve P.D.Es. in new geometries.

E.g. $\nabla^2 u = 0$ Laplace

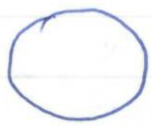
$u_t = \nabla^2 u$ Diffusion (heat) eq.

$u_{tt} = \nabla^2 u$ Wave equ.

in



Rectangle.



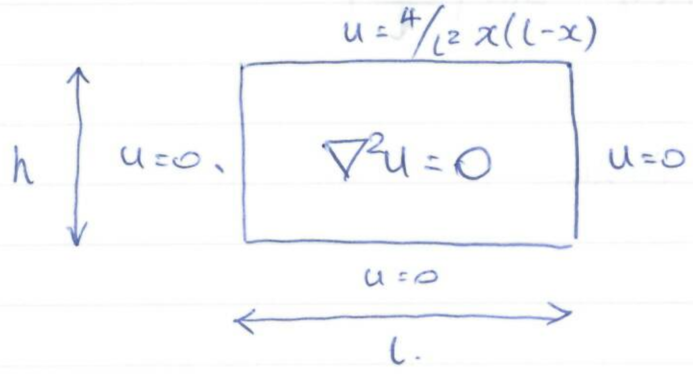
CIRCLE

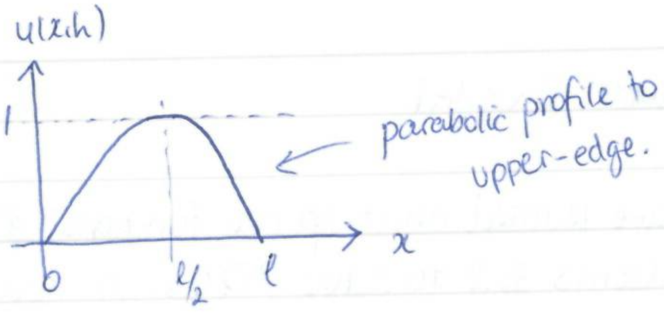
Sphere.

### Example 1.

Find the steady temperature distribution $u(x,y)$ in a rectangular sheet of metal, when the sides are held at temperature

$$u(0,y) = u(l,y) = 0$$
$$u(x,0) = 0$$
$$u(x,h) = \frac{4}{l^2} x(l-x)$$






Use seperation of variables method.

Try $u(x,y) = X(x)Y(y)$

$$\nabla^2 u = u_{xx} + u_{yy} = Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0$$

$$\Rightarrow \underbrace{\frac{1}{Y}\frac{d^2Y}{dy^2}}_{\text{func. of } y} = \underbrace{-\frac{1}{X}\frac{d^2X}{dx^2}}_{\text{func. of } x} = \lambda$$ must be a const.

Look at X equation.... because homogeneous B.C.s.
i.e. S-L boundary conditions.

$$\frac{d^2X}{dx^2} + \lambda X = 0 \qquad \begin{aligned} X(0) &= 0 \leftarrow \text{left side.} \\ X(l) &= 0 \leftarrow \text{right side.} \end{aligned}$$

An S-L eigenvalue problem (see §2, Ex 1).

$$\lambda_k = \frac{k^2\pi^2}{l^2} \qquad X_k(x) = \sin\left(\frac{k\pi x}{l}\right)$$

Next Solve y-equation for $\lambda = \lambda_k$.

$$\frac{d^2Y_k}{dy^2} - \lambda_k Y_k = 0.$$

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$$Y_k(y) = C_k \cosh\left(\overset{=\sqrt{\lambda_k}}{k\pi y/\ell}\right) + D_k \sinh\left(\overset{=\sqrt{\lambda_k}.}{k\pi y/\ell}\right)$$

$$Y_k(0) = 0 \quad \Rightarrow \quad C_k = 0$$

Can now write down <u>General Solution</u>

$$u(x,y) = \sum_{k=1}^{\infty} X_k(x) Y_k(y)$$

$$= \sum_{k=1}^{\infty} D_k \sin\left(\frac{k\pi x}{l}\right) \sinh\left(\frac{k\pi y}{l}\right)$$

Now use the top b.c $(y=h)$.

$$u(x,h) = \sum_{k=1}^{\infty} D_k \sinh\left(k\pi h/\ell\right) \sin\left(\pi k x/\ell\right) = 4/\ell^2 \; x(\ell - x).$$

$$= \sum_{k=1}^{\infty} E_k X_k(x) = 4/\ell^2 \, f(x)$$

$f(x) = x(\ell - x)$
$E_k = D_k \sinh\left(k\pi h/\ell\right).$

Take inner product with $X_j(x)$

$$= \sum_{k=1}^{\infty} E_k \langle X_k, X_j \rangle_1 = 4/\ell^2 \langle f, X_j \rangle_1$$

$$= E_j \langle X_j, X_j \rangle_1 = 4/\ell^2 \langle f, X_j \rangle_1$$

All other terms in sum $= 0$
due to orthogonality
$\langle X_k, X_j \rangle \; (j \neq k) = 0$

$$E_j = \frac{4}{\ell^2} \frac{\langle f, X_j \rangle_1}{\langle X_j, X_j \rangle_1}$$

weight function $= 1$ because

$$\frac{d^2 X}{dx^2} + \lambda X = 0$$

$$\mathcal{L} X = -\lambda X$$

$$\mathcal{L} = \frac{d^2}{dx^2}$$

$w = 1 \quad p = 1 \quad r = 0.$

$$\langle X_j, X_j \rangle = \int_0^\ell \sin^2(j\pi x/\ell)\, dx = \ell/2$$

$$\langle f, X_j \rangle = \int_0^\ell x(\ell - x) \sin\left(\frac{j\pi x}{\ell}\right) dx.$$

$$= \left[ -\frac{l}{j\pi} \cancel{x(l-x)\cos\left(\frac{j\pi x}{\ell}\right)} \right]_0^\ell + \frac{L}{j\pi} \int_0^L (l-2x)\cos\left(\frac{j\pi x}{\ell}\right) dx$$

$$= \left[ \frac{\ell^2}{j^2\pi^2} (\ell - 2x) \sin\left(\frac{j\pi x}{\ell}\right) \right]_0^\ell + \frac{2\ell^2}{j^2\pi^2} \int_0^L \sin\left(\frac{j\pi x}{\ell}\right) dx.$$

$$= \frac{2\ell^3}{j^3\pi^3} \left[ -\cos\left(\frac{j\pi x}{l}\right) \right]_0^l = \frac{4\ell}{j^3\pi^3} \to j\text{-odd}, \quad 0 \ j\text{-even}.$$

$$= -\frac{2\ell^3}{j^2\pi^3} \left( (-1)^j - 1 \right) = \begin{cases} \frac{4\ell^3}{j^3\pi^3} & j \text{ odd} \\ 0 & j \text{ even} \end{cases}$$

$$D_j = \frac{E_j}{\sinh\left(\frac{j\pi h}{l}\right)} = \begin{cases} \frac{4}{\ell^2} \cdot \frac{2}{\ell} \cdot \frac{4\ell^3}{j^3\pi^3} \cdot \frac{1}{\sinh(j\pi h/\ell)} = \frac{32}{\sinh(j\pi h/\ell)\, j^3\pi^3} & \text{odd.} \\ 0 & \text{even} \end{cases}$$

Write $j = 2m+1$ for $m = 0, 1, 2, \ldots$ to pick odd terms.

$$u(x,y) = \sum_{m=0}^\infty D_{2m+1} X_{2m+1}(x) Y_{2m+1}(y)$$

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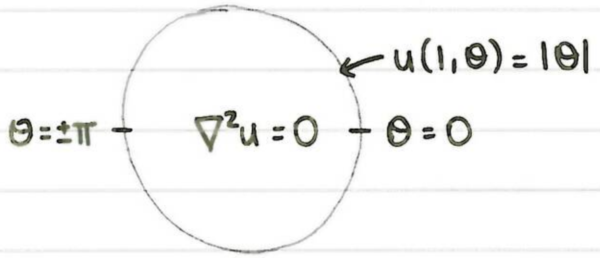
$$= \frac{32}{\pi^3} \sum_{m=0}^{\infty} \frac{\sin\left(\frac{(2m+1)\pi x}{l}\right) \sinh\left(\frac{(2m+1)\pi y}{l}\right)}{(2m+1)^3 \sinh\left(\frac{(2m+1)\pi h}{l}\right)}.$$

Final solution for $u(x,y)$ //.

Tuesday 21st February 2012

## §3. Example 2

Find the steady temperature distribution $u(r,\theta)$ in a circular disk ($0 \le r \le 1$, $-\pi \le \theta \le \pi$) when the boundary is held at temperature
$u(1,\theta) = |\theta| \qquad (-\pi < \theta \le \pi)$




$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

(careful application of chain rule, see handout)

Seek a solution $u(r,\theta) = R(r)T(\theta)$

get : $\frac{T}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{R}{r^2}\frac{d^2T}{d\theta^2} = 0$

Dividing by $RT/r^2$

$$\underbrace{\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right)}_{\text{function of } r} = \underbrace{-\frac{1}{T}\frac{d^2T}{d\theta^2}}_{\text{function of } \theta} = \lambda \leftarrow \text{seperation constant.}$$

Look at T-equation : $\frac{d^2T}{d\theta^2} + \lambda T = 0$

$T(\theta)$ must be periodic in $\theta$ i.e.

$T(\pi) = T(-\pi)$ T continuous

$\frac{dT}{d\theta}(\pi) = \frac{dT}{d\theta}(-\pi)$ $\nabla T$ continuous - physically necessary

Above are b.cs for a periodic S.L system.

In fact the problem in $T(\theta)$ is identical to §2 Ex 7.

General solution :

$T(\theta) = A\cos\sqrt{\lambda}\,\theta + B\sin\sqrt{\lambda}\,\theta \qquad \lambda>0$

Require $\sqrt{\lambda}=k$ integer for periodicity

Eigenvalues $\lambda_k = k^2$
$(k \geq 0)$

Eigenvectors $T^a_k(\theta) = \cos k\theta \qquad T_0(\theta) = 1$
$T^b_k(\theta) = \sin k\theta$

R-equation : $\underbrace{r\frac{d}{dr}\left(r\frac{dR_k}{dr}\right)}_{r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr}} - k^2R_k = 0$

with $\lambda = k^2$

Try $R_k(r) = r^p \qquad (p(p-1)+p-k^2)r^p = 0 \quad : \quad p = \pm k$ for $k \geq 1$.

$K=0 \qquad \frac{d}{dr}\left(r\frac{dR_0}{dr}\right) = 0$

$$r\frac{dR_0}{dr} = B_0$$

$$\frac{dR_0}{dr} = \frac{B_0}{r} \qquad \Rightarrow \quad R_0(r) = B_0 \log r + A_0.$$

General solution.

$$u(r,\theta) = R_0(r)T_0(\theta) + \sum_{K=1}^{\infty} R_K(r)\left(C_K T_K^a(\theta) + D_K T_K^b(\theta)\right)$$

$$= A_0 + B_0 \log r + \sum_{K=1}^{\infty} \underbrace{\left(A_K r^K + B_K/r^K\right)}_{R_K(r)}\left(C_K \cos K\theta + D_K \sin K\theta\right)$$

Physical considerations give b.c $u(0,\theta)$ finite

$\Rightarrow B_K = 0 \quad K \geqslant 0$

Set $A_K = 1$ w.l.o.g

$A_0 = C_0/2$ (re-labelling)

Leaves $u(r,\theta) = \frac{C_0}{2} + \sum_{K=1}^{\infty} r^K\left(C_K \cos K\theta + D_K \sin K\theta\right)$

Insert b.c at $r=1$

$$\frac{C_0}{2} + \sum_{K=1}^{\infty} C_K \cos K\theta + D_K \sin K\theta = |\theta| \qquad (-\pi < \theta \leq \pi)$$

Standard fourier series

$$C_k = \frac{1}{\pi}\int_{-\pi}^{\pi} |\theta| \cos k\theta \, d\theta$$

$$D_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{|\theta| \sin k\theta}_{0} \, d\theta$$

odd function

$$C_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |\theta| \, d\theta = \frac{2}{\pi}\int_{0}^{\pi} \theta \, d\theta = \pi$$

$$C_k = \frac{2}{\pi}\int_{0}^{\pi} \theta \cos k\theta \, d\theta = \frac{2}{\pi}\left[\theta \frac{\sin k\theta}{k}\right]_0^{\pi} - \frac{2}{\pi}\int_0^{\pi} \frac{\sin k\theta}{k} \, d\theta$$

$$= \frac{2}{k^2\pi}\underbrace{\Big[\cos k\theta\Big]_0^{\pi}}_{(-1)^k - 1} = \begin{cases} -4/k^2\pi & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$$

$\therefore$ Final solution given by

$$u(r,\theta) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{j=0}^{\infty} \underbrace{r^{2j+1}}_{R_k(r)} \frac{\underbrace{\cos(2j+1)\theta}_{T_k^{a}(\theta)}}{(2j+1)^2}$$

Write $k = 2j+1$ $\quad j = 0, 1, \ldots$

picks out odd terms.

Example 3

Find the unsteady temperature distribution $u(x,t)$ in a uniform rod $0<x<1$ of unit thermal diffusivity, subject to the boundary conditions.

$u(0,t) = 0$ (held at zero temp)

$\dfrac{\partial u}{\partial x}(1,t) = -\dfrac{1}{\alpha}u(1,t)$ (Radiates to space at a rate proportional to )


To find $u(x,t)$ seek a seperable solution

$$u(x,t) = X(x)T(t)$$

when the initial temperature is $u(x,0) = 1$.

$$X\frac{dT}{dt} = T\frac{d^2X}{dx^2} \qquad \frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{T}\frac{dT}{dt} = -\lambda$$ seperation const.

X equation

$$\frac{d^2X}{dx^2} + \lambda X = 0 \qquad X(0) = 0 \qquad x=0$$

radiation condition. $\quad \alpha\dfrac{dX}{dx}(1) + X(1) = 0 \qquad x=1$

Exactly the same S-L problem as §2 Ex 3.

Eigenvalues $\lambda_K = q_K^2$

Eigenvectors $X_K(x) = \sin(q_K x)$

Where the $\{q_K\}$ were the roots of $\boxed{\tan q + \alpha q = 0}$ ($\infty$ many)

T-equation
with $\lambda = q_K^2$

$$\frac{dT_K}{dt} + q_K^2 T_K = 0$$

$$T_K(t) = A_K e^{-q_K^2 t}$$

Construct general solution

$$u(x,t) = \sum_{K=1}^{\infty} X_K(x) T_K(t) = \sum_{K=1}^{\infty} A_K \sin(q_K x) e^{-q_K^2 t}$$

Use initial condition $(t=0)$

$$u(x,0) = \sum_{K=1}^{\infty} A_K \sin(q_K x) = 1$$

to find $\{A_K\}$ take inner product with $X_j$

$$\sum_{K=1}^{\infty} A_K \underbrace{\langle X_K, X_j \rangle}_{=0 \text{ unless } j=K} = \langle 1, X_j \rangle$$

$$A_j = \frac{\langle 1, X_j \rangle}{\langle X_j, X_j \rangle} = \frac{4(1-\cos q_j)}{2q_j - \sin 2q_j}$$ (from §2, Ex 3)

Solution is :

$$u(x,t) = \sum_{K=1}^{\infty} \frac{4(1-\cos q_j)}{2q_j - \sin 2q_j} \sin(q_K x) e^{-q_K^2 t}$$

At large times $t \gg 1$, the $k=1$ term in the sum dominates

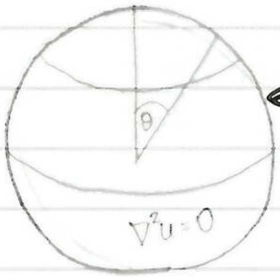
$$u(x,t) \simeq A_1 \sin q_1 x \, e^{-q_1 t}$$

(all other terms decay more rapidly as $q_k^2 > q_1^2$ for $k>1$).

Example 4

Find the steady temperature distribution inside a sphere ($0<r<1$, $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$) when an axisymmetric temperature distribution $u(1,\phi,\theta) = f(\theta)$ is applied to its surface.

Uniform conducting




$u(1,\phi,\theta) = f(\theta)$

axisymmetric
(independent of $\phi$)

Expect solution to be $\phi$ independent. Seek a solution of Laplace's equation of the form $u = u(r,\theta)$. In spherical polars

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \cancel{\frac{1}{r^2 \sin\theta}\frac{\partial^2 u}{\partial \phi^2}}^{\,0} = 0$$

Try $u(r,\theta) = R(r)T(\theta)$

get $$\frac{T}{r^2}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \frac{R}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{dT}{d\theta}\right) = 0$$

Divide by $\frac{RT}{r^2}$ $$\frac{1}{R}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) = \frac{-1}{T\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{dT}{d\theta}\right) = \lambda.$$

Consider T-equation

$$\frac{d}{d\theta}\left(\sin\theta \frac{dT}{d\theta}\right) + \lambda T \sin\theta = 0$$

Make a transformation of variables $z = \cos\theta$ and write $T(\theta) = w(z(\theta))$

Use chain rule : $\frac{dT}{d\theta} = \frac{dw}{dz}\frac{dz}{d\theta} = -\sin\theta \frac{dw}{dz}$

$$\frac{d}{d\theta}\left(\sin\theta \frac{dT}{d\theta}\right) = \frac{dz}{d\theta}\frac{d}{dz}\left(-\sin^2\theta \frac{dw}{dz}\right)$$

$$= \sin\theta \frac{d}{dz}\left(-\underbrace{(1-z^2)}_{1-\cos^2\theta}\frac{dw}{dz}\right)$$

Equation is now

$$\sin\theta \frac{d}{dz}\left((1-z^2)\frac{dw}{dz}\right) + \lambda w \sin\theta = 0$$

Write $\lambda = \nu(\nu+1)$

$$(1-z^2)w'' - 2zw' + \nu(\nu+1)w = 0$$

Legendre's equation.

Thursday 23rd February

SL-form

$$\frac{d}{dz}\Big( \underbrace{(1-z^2)}_{p(z)=1-z^2}\frac{dw}{dz}\Big) + \nu(\nu+1)w = 0$$

$p(\pm 1) = 0$

The points $z=\pm 1$ corresponds to the poles of our sphere, $\theta = 0, \pi$ resp.

$T(\theta)$ must be finite at the poles $\theta = 0, \pi$

$\Rightarrow$ $w(z)$ must be finite at $z = \pm 1$

The condition $w(\pm 1)$ finite gives us a singular SL system.

Seen already: §2, Ex 5.

Solutions of L.E are singular at $z=\pm 1$ unless $\nu = k$ (integer) when we obtain Legendre polynomials.

$\Rightarrow$ Eigenvalues $\lambda_k = k(k+1)$

Eigenvectors $w_k(z) = P_k(z)$

$T_k(\theta) = P_k(\cos\theta)$ (Legendre polynomial evaluated at $\cos\theta$).

R-equation

$$\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) - \underbrace{(k)(k+1)}_{\lambda_k}R_k = 0$$

$$r^2\frac{d^2R}{dr} + 2r\frac{dR}{dr} - k(k+1)R_k = 0$$

Try $R_k(r) = r^p$

$$\left(p(p-1) + 2p - k(k+1)\right) r^p = 0$$

$$p(p+1) - k(k+1) = 0$$

$$p = k, \; -(k+1)$$

So $\quad R_k(r) = A_k r^k + \dfrac{B_k}{r^{k+1}}$

General Solution

$$u(r,\theta) = \sum_{k=0}^{\infty} T_k(\theta) R_k(\theta)$$

$$= \sum_{k=0}^{\infty} \left(A_k r^k + B_k/r^{k+1}\right) P_k(\cos\theta)$$

$u(0,\theta)$ finite $\Rightarrow$ $R(0)$ finite $\Rightarrow$ $B_k = 0$ for all $k$.

Left with $\quad u(r,\theta) = \sum_{k=0}^{\infty} A_k r^k P_k(\cos\theta)$

$$u(1,\theta) = \sum_{k=0}^{\infty} A_k P_k(\cos\theta) = f(\theta)$$

or $\quad = \sum_{k=0}^{\infty} A_k P_k(z) = F(z) \qquad$ where $F(\cos\theta) = f(\theta)$

or $F(z) = f(\cos^{-1}(z))$

Now use orthogonality :

$$\sum_{k=0}^{\infty} A_k \underbrace{\langle P_k, P_j \rangle}_{\substack{=0 \text{ for} \\ j \neq k}} = \langle f, P_j \rangle$$

$$A_j = \frac{\langle F, P_j \rangle}{\langle P_j, P_j \rangle} = \frac{\int_{-1}^{1} F(z) P_j(z)\, dz}{\int_{-1}^{1} P_j(z)^2\, dz}$$

$$= \frac{2}{2j+1} \int_{-1}^{1} F(z) P_j(z)\, dz$$

(Fourier-Legendre Series §2), completes solution.

Exercise Show that if $f(\theta) = \begin{cases} 1 & 0 \leq \theta \leq \pi/2 \\ 0 & \pi/2 < \theta \leq \pi. \end{cases}$

then the solution is

$$u(r, \theta) = 1/2 + 1/2 \sum_{m=0}^{\infty} {}^{4m+3}/_{2m+1} \; {}^{(-1)^m (2m)!}/_{2^{2m} (m!)^2} \; r^{2m+1} P_{2m+1}(\cos\theta).$$

Hint use :

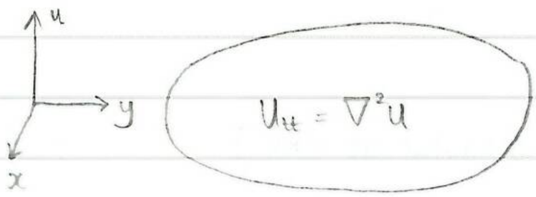
§2 Q2c $\qquad \int_x^1 P_n(t)\, dt = 1/_{2n+1} \left( P_{n-1}(x) - P_{n+1}(x) \right)$

§2 Q2d $\qquad P_{2n}(0) = {}^{(-1)^n (2n)!}/_{4^n (n!)^2} \qquad P_{2n+1}(0) = 0$

28th February 2012

Example 5

Waves on a circular membrane




$u(1,\theta,t) = 0$

Membrane $0 \le r \le 1$

$0 \le \theta \le 2\pi$

Find the time dependent displacement field $u(r,t)$ if the membrane initially has : $u(r,\theta,0) = f(r)$ initial vertical displacement

$u(r,\theta,0) = g(r)$ initial vertical velocity

Recall in polar coordinates

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}}$$

Solutions will be $\theta$-independent.

Seperate variables $u(r,t) = R(r)T(t)$

Insert in wave equation :

$$R\frac{d^2T}{dt^2} = \frac{T}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right)$$

Divide by RT : $\frac{1}{T}\frac{d^2T}{dt^2} = \frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = -\lambda$ ← seperation const.

R-equation : $$\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \lambda r R = 0$$

$rR'' + R'$ "like" Bessel's equation index zero.

Outer b.c $R(1) = 0$

$R(0) = \text{finite}$

An S-L system (regular at $r=1$, singular at $r=0$)

Exactly §2, Ex 6, $m = 0$

We found that :

Eigenvalues: $\lambda_k = j_{0k}^2$

Eigenfunctions : $R_k(r) = J_0(j_{0k} r)$

T-equation : $$\frac{d^2T}{dt^2} + \lambda T = 0$$

substitute $\lambda = \lambda_k$

$$\frac{d^2T_k}{dt^2} + j_{0k}^2 T_k = 0$$

has solution

$$T_k(t) = A_k \cos(j_{0k} t) + B_k \sin(j_{0k} t)$$

General solution is therefore :

$$u(r,t) = \sum_{k=1}^{\infty} R_k(r) T_k(t) = \sum_{k=1}^{\infty} J_0(j_{0k} r)\left(A_k \cos(j_{0k} t) + B_k \sin(j_{0k} t)\right).$$

Notice that the angular frequencies of the T-equation are given by

$$\omega_k = j_{0k} \qquad\qquad k \geq 1$$

(Differs from string when $\underset{\uparrow \text{ "note"}}{\omega_k} \sim k\omega_1$ frequencies integer multiples of fundemental

Next use initial data to get $\{A_k\}$, $\{B_k\}$

$$u(r,0) = \sum_{k=1}^{\infty} A_k \underbrace{J_0(j_{0k} r)}_{R_k(r)} = f(r)$$

$$\underbrace{\sum_{k=1}^{\infty} A_k \langle R_k, R_j \rangle_w}_{A_j \langle R_j, R_j \rangle_w.} = \langle f, R_j \rangle_w$$

Recall $w(x) = x$
$w(r) = r$

$$A_j = \frac{\langle f, R_j \rangle_w}{\langle R_j, R_j \rangle_w} = \frac{2}{J_1(j_{0j})^2} \int_0^1 r\, f(r) J_0(j_{0j} r)\, dr$$

(S3, Q2)

$$u_t(r,0) = \sum_{k=1}^{\infty} j_{0k} B_k J_0(j_{0k} r) = g(r)$$

leads to formula for $\{B_j\}$ (exercise).

Completes solution ✓

Exercise Let $f(r) = 0$, $g(r) = \begin{cases} 0 & \delta \leq r \leq 1 \\ 1 & 0 \leq r \leq \delta \end{cases}$

$$\left( \text{Use } \frac{d}{dx}\left(x J_1(\alpha x)\right) = \alpha x J_0(\alpha x) \right)$$

Show that $u(r,t) = \sum_{k=1}^{\infty} \frac{2\delta J_0(j_{0k}\delta)}{j_{0k}^2 J_1(j_{0k})^2} J_0(j_{0k}r)\sin(j_{0k}t)$

## §4. Integral Transforms

### Fourier transforms

Key idea : Extend the idea of Fourier Series defined for $f(x) \in L'[-L,L]$ to the entire real line $\mathbb{R}$

set of functions defined on $-L \le x \le L$ for which $\int_{-L}^{L} |f(x)|\,dx$ exists.

### Sketch derivation

Recall the Fourier Series for $f(x)$ defined on $-L \le x \le L$

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi x}{L}\right) + b_k \sin\left(\frac{k\pi x}{L}\right)$$

with
$$\begin{Bmatrix} a_k \\ b_k \end{Bmatrix} = \frac{1}{L}\int_{-L}^{L} f(x) \begin{Bmatrix} \cos(k\pi x/L) \\ \sin(k\pi x/L) \end{Bmatrix} dx$$

Complex form : An equivalent formula is

$$f(x) = \frac{1}{2}\sum_{k=-\infty}^{\infty} c_k e^{ik\pi x/L} \qquad c_k = a_k - ib_k, \quad c_{-k} = a_k + ib_k$$

$$c_k = \frac{1}{L}\int_{\cancel{-\infty}\,-L}^{\cancel{\infty}\,L} f(x)\, \underbrace{e^{-ik\pi x/L}}_{\cos\left(\frac{k\pi x}{L}\right) - i\sin\left(\frac{k\pi x}{L}\right)} dx$$

Expand
$$f(x) = \frac{a_0}{2} + \frac{1}{2}\sum_{k=1}^{\infty} (a_k - ib_k)\left(\cos\frac{k\pi x}{L} + i\sin\frac{k\pi x}{L}\right) \quad \text{+ve terms}$$
$$+ (a_k + ib_k)\left(\cos\frac{k\pi x}{L} - i\sin\frac{k\pi x}{L}\right) \quad \text{-ve terms}$$

Complex form allows us to write, defining $q_k = k\pi/L$

$$f(x) = \frac{1}{2}\sum_{k=-\infty}^{\infty}\frac{1}{L}\left(\int_{-L}^{L} f(x) e^{-iq_k x}\, dx\right) e^{iq_k x}$$

The set of real numbers $\{q_k\}$ are equally spaced along the real line, with ~~integral~~ interval $\delta q = \pi/L$

Use $\delta q = \pi/L$ to write:

$$f(x) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\left(\int_{-L}^{L} f(\bar{x}) e^{-iq_k \bar{x}}\, d\bar{x}\right) e^{iq_k x}\, dq$$

Now consider the limit $L \to \infty$, $\delta q \to 0$ (simultaneous as $\delta q = \pi/L$)

(1) Limits on inner integral $\to \infty$

(2) The set of numbers $\{q_k\}$ become dense on the real line (in $q$)

(3) If we consider the expression for $f(x)$ in the form

$$f(x) = \sum_{k=-\infty}^{\infty} g(q_k)\,\delta q$$

then taking the limit $\delta q \to 0$ and using the (non-rigorous) Riemann integral

$$\lim_{\delta q \to 0}\sum_{k=-\infty}^{\infty} g(q_k)\,\delta q = \int_{-\infty}^{\infty} g(q_k)\, dq$$

Applying (1), (2), (3):

$$\boxed{f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(\tilde{x}) e^{-iq\tilde{x}}\, d\tilde{x}\right) e^{iqx}\, dq}$$

Fourier Integral formula.

Can split formula to obtain :

Forward transform :

$$\hat{f}(q) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-iqx}\, dx$$

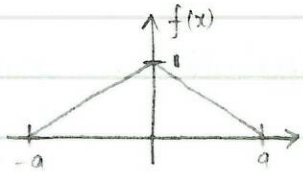
Inverse transform

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(q)\, e^{iqx}\, dq$$

Non unique. Check for signs, factors of $\sqrt{2\pi}$.

Example 1.

$$f(x) = \begin{cases} 1 - |x|/a & |x| \leq a \\ 0 & |x| > a \end{cases}$$




Fourier transform is

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$$

$\cos kx + i \sin kx$

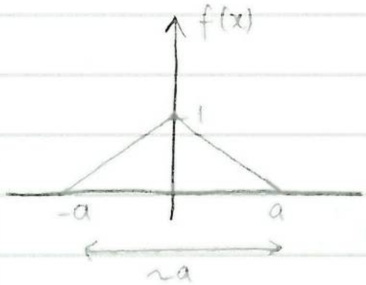
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cos kx \, dx \qquad (f \text{ even})$$

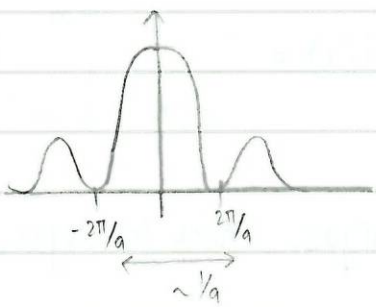
$$= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \cos kx \, dx = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} (1 - x/a) \cos kx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left( \left[ (1 - \tfrac{x}{a}) \frac{\sin kx}{k} \right]_0^a + \int_0^a \frac{1}{ak} \sin kx \, dx \right)$$

$$= -\sqrt{\frac{2}{\pi}} \left[ \frac{\cos kx}{ak^2} \right]_0^a = \sqrt{\frac{2}{\pi}} \left( \frac{1 - \cos ka}{ak^2} \right)$$

$$= \frac{a}{\sqrt{2\pi}} \left( \frac{\sin (ak/2)}{(ak/2)} \right)^2$$




Scale of transform ~ (scale of function)$^{-1}$

Applications : - Solutions of integeral equations
(e.g. find $f(a)$ if $\int_{-\infty}^{\infty} f(x)h(y-x)\,dx = g(y)$ for given $g, h$)
- evaluation of some "difficult" integrals.
e.g. $\int_{-\infty}^{\infty} \frac{\cos kx \sin ak}{k} \, dk$

- solving PDEs especially on infinite or semi-infinite domains.

$u \to 0$ as $x^2+y^2 \to \infty$ $\quad u = f(x) \quad \nabla^2 u = 0$.

1st March 2012

# Fourier Transforms

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} \, dx$$ Forward transforms

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} \, dx$$ Inverse transforms.

$\hat{f}(k)$ is the fourier transform of the function $f(x)$

Notation : We will also write

$$F[f(x)] \text{ for } \hat{f}(k)$$

Ex1 :
$$f(x) = \begin{cases} 1 - |x|/a & |x| \leq a \\ 0 & |x| > a \end{cases}$$

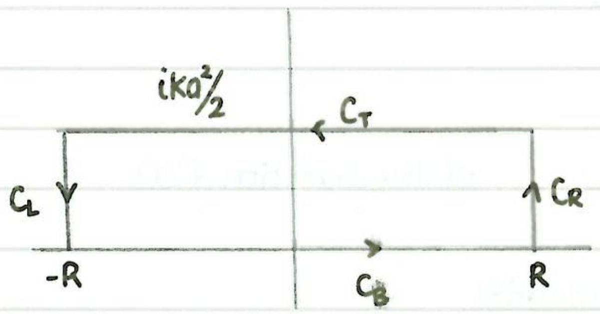
$$\hat{f}(k) = \frac{a}{\sqrt{2\pi}} \left( \frac{\sin(ak/2)}{(ak/2)} \right)^2$$

Ex 2

$$f(x) = e^{-x^2/a^2}$$

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/a^2} e^{-ikx}\, dx$$

Consider the following countour in $\mathbb{C}$




$C = C_L + C_B + C_R + C_T$

Consider then $\displaystyle\oint_C e^{-z^2/a^2}\, dz = 0$ (by cauchy)

$$\lim_{R\to\infty} \int_{C_B} = \int_{-\infty}^{\infty} e^{-x^2/a^2}\, dx \qquad t = x/a$$

$$= a \int_{-\infty}^{\infty} e^{-t^2}\, dt = a\sqrt{\pi}$$

$$\lim_{R\to\infty} \int_{C_T} = - \int_{-\infty}^{\infty} e^{-\frac{(x+ika^2/2)^2}{a^2}}\, dx$$

direction of $C_T$ from R to L

Use parametrisation $z(x) = x + \frac{ika^2}{2}$, $-\infty < x < \infty$.

$$= -e^{\frac{k^2a^2}{4}} \int_{-\infty}^{\infty} e^{-x^2/a^2 - ikx}\, dx$$

$$= -\sqrt{2\pi}\, e^{k^2a^2/4} \hat{f}(k)$$

Need to show that $\lim_{R\to\infty} \int_{C_L} \cdot \int_{C_R} = 0$

(Exercise) Use parametrisation $z(y) = \pm R + iy,\ 0 \le y \le ka^2/2$

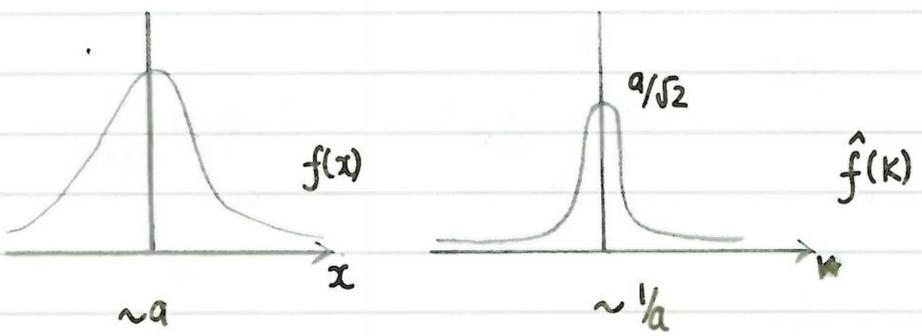
$$\lim_{R\to\infty} \oint_C = \lim_{R\to\infty}\int_{C_B} + \lim_{R\to\infty}\int_{C_T} + \lim_{R\to\infty}\int_{C_L} + \lim_{R\to\infty}\int_{C_R} = 0$$

$$= \sqrt{\pi}\, a - \sqrt{2\pi}\, e^{k^2a^2/4} \hat{f}(k)$$

$$\hat{f}(k) = \sqrt{a^2/2}\, e^{-k^2a^2/4} = a/\sqrt{2}\, e^{-k^2a^2/4}$$

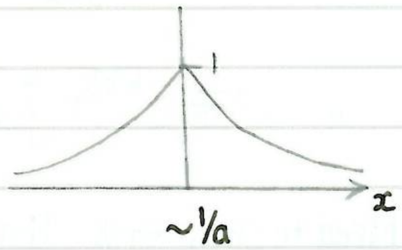
If $a = \sqrt{2}$

$f(x) = e^{-x^2/2}$ $\qquad \hat{f}(k) = e^{-k^2/2}$



Ex 3

$$f(x) = e^{-a|x|} \qquad a>0$$




$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} e^{-ikx}\, dx$$

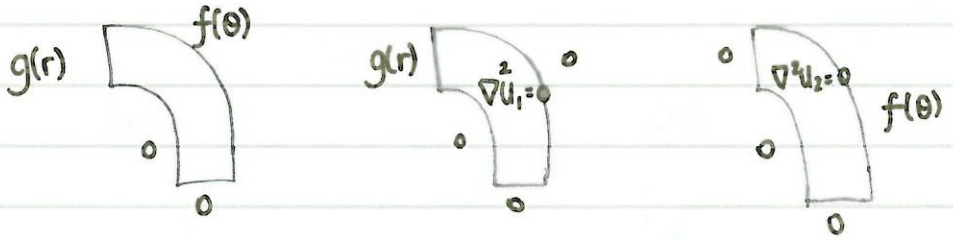
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} \cos kx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{-ax} \cos kx\, dx$$

$$= \mathrm{Re}\left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ax-ikx}\, dx \right\}$$

$$= \frac{2}{\sqrt{2\pi}} \mathrm{Re}\left\{ -\left[ \frac{e^{-(a+ik)x}}{a+ik} \right]_0^{\infty} \right\}$$

$$= \sqrt{\frac{2}{\pi}} \mathrm{Re}\left\{ \frac{1}{a+ik} \right\} = \sqrt{\frac{2}{\pi}} \mathrm{Re}\left\{ \frac{a+ik}{a^2+k^2} \right\} = \sqrt{\frac{2}{\pi}} \frac{a^2}{a^2+k^2}.$$




$U = U_1 + U_2$

Linearity : $\nabla^2(U_1 + U_2) = \nabla^2 U_1 + \nabla^2 U_2 = 0$

Existence

A sufficient condition on $f(x)$ for its transform $\hat{f}(k)$ to exist is that $f(x) \in L^1(\mathbb{R})$

Here $L^1(\mathbb{R})$ is the function space of functions for which $\int_{-\infty}^{\infty} |f(x)|\, dx$ exists.

$$|\hat{f}(k)| = \frac{1}{\sqrt{2\pi}} \left| \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \right|$$

$$\leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)| \qquad \text{as } |e^{-ikx}| = 1$$

$$< \infty \qquad \text{since } f \in L^1(\mathbb{R})$$

Lemma (Riemann - ~~Lesb~~ Lebesgue)

If $f(x) \in L^1(\mathbb{R})$ then $\hat{f}(k) \to 0$ as $|k| \to \infty$

- F. Ts decay as $k \to \pm\infty$. (see printed notes).

Fourier Transform Properties.

$$F[f'(x)] = 1/\sqrt{2\pi} \int_{-\infty}^{\infty} f'(x) e^{-ikx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left( \left[ f(x) e^{-ikx} \right]_{-\infty}^{\infty} + ik \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx \right)$$

Boundary terms vanish as $f(x) \to 0$ as $x \to \pm\infty$ is a necessary condition for $f \in L^1$

Hence $\mathcal{F}[f'] = ik\hat{f}$

Higher derivatives : $\mathcal{F}[f^{(n)}] = (ik)^n\hat{f}$

6th March 2012

Fourier Transforms.

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} \, dx$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{-ikx} \, dk$$

Write $F[f(x)]$ for $\hat{f}(k)$

(1) Transform of a derivative

$$F[f'(x)] = ik\hat{f}(k)$$

$$F[f^{(n)}(x)] = (ik)^n \hat{f}(k)$$ transform of $n^{th}$ derivative.

Example

$f(x) = e^{-x^2/2}$ $\qquad$ $\hat{f}(k) = e^{-k^2/2}$

$$F[f'(x)] = F[-xe^{-x^2/2}] = ik\hat{f}(k) = ike^{-k^2/2}$$

or $F[xe^{-x^2/2}] = -ike^{-k^2/2}$

(2) Derivative of transform

$$\frac{d\hat{f}}{dk} = \frac{d}{dk}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx\right)$$

$$= \frac{1}{\sqrt{2\pi}}\frac{d}{dk}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx\right)$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\frac{\partial}{\partial k}\left(e^{-ikx}\right)dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} (-ixf(x))e^{-ikx}\,dx$$

$$= F[-ixf(x)] = -i\,F[xf(x)]$$

or $F[xf(x)] = i\, d\hat{f}/dk$

Use $f(x) = e^{-x^2/2}$

$$F\left[xe^{-x^2/2}\right] = i\, d/dk\; e^{-k^2/2}$$

$$= ike^{-k^2/2} \quad \text{(as before)}$$

3. Shift formulae

(i) Let $g(x) = f(x+c)$ where $\hat{f}(k)$ is known ($c$ const.)

$$\hat{g}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x+c)e^{-ikx}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\,e^{-ik(t-c)}$$

$$= e^{ikc}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{-ikt}\, dt\right) = e^{ikc}\hat{f}(k)$$

Example

$f(x) = e^{-x^2/2}$ $\qquad$ $g(x) = e^{-(x+c)^2/2}$

$$F\left[e^{-(x+c)^2/2}\right] = e^{ikc}e^{-k^2/2}$$

(ii) Consider the transform 'shifted'

$$\hat{f}(k+c) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-i(k+c)x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \left(f(x)\, e^{-icx}\right) e^{-ikx}\, dx$$

$$= F\left[f(x)\, e^{-icx}\right]$$

Example

$$F\left[e^{-x^2/2}e^{-icx}\right] = e^{-(k+c)^2/2} \quad \left(= \hat{f}(k+c)\right)$$

(4) Convolution Theorem

For $f, g \in L^1(\mathbb{R})$ define their convolution to be the function

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)\, g(y)\, dy$$

(i) $f * g \in L^1$

Proof.

We need to show that $\int_{-\infty}^{\infty} |(f*g)(x)|\, dx$ exists.

$$\int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} f(x-y)g(y)\, dy \right| dx$$

$$\leq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x-y)g(y)| dy\, dx$$

$$\leq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x-y)|\, |g(y)|\ dy\, dx$$

Write $u = x-y$ $\quad du = dx$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(u)||g(y)|\, du\, dy$$

$$= \left(\int_{-\infty}^{\infty} |f(u)|\, dy\right)\left(\int_{-\infty}^{\infty} |g(y)|\, dy\right) = C \quad \text{since } f, g \in L^1$$

(ii) $f*g = g*f$ (convolution operator commutes)

$$(f*g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\, dy \qquad \begin{aligned} t &= x-y \\ -dt &= dy \end{aligned}$$

$$= \int_{-\infty}^{\infty} f(t)g(x-t)\ (-dt)$$

$$= \int_{-\infty}^{\infty} f(t)g(x-t)\,dt$$

$$= g*f \qquad (t, y \text{ dummy variables of integration})$$

(iii) Convolution Theorem

$$\widehat{f*g} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} f(x-y)g(y)\,dy \right) e^{-ikx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x-y)g(y)e^{-ikx}\,dx\,dy$$

Change variables in inner integral (y fixed)

Write $t = x-y$, $dt = dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)g(y)e^{-ik(t+y)}\,dt\,dy$$

$$= \frac{1}{\sqrt{2\pi}} \left( \int_{-\infty}^{\infty} f(t)e^{-ikt}\,dt \right) \left( \int_{-\infty}^{\infty} g(y)e^{-iky}\,dy \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left( \sqrt{2\pi}\,\hat{f} \right) \left( \sqrt{2\pi}\,\hat{g} \right)$$

$$\therefore \boxed{\widehat{f*g} = \sqrt{2\pi}\,\hat{f}\hat{g}} \quad \left( = \widehat{g*f} \text{ by (ii)} \right)$$

Convolution Theorem.

(5) Parseval's Theorem.

Convolution theorem : $\widehat{f*g} = \sqrt{2\pi}\,\hat{f}\hat{g}$.

Take inverse transform
$$f*g = \frac{\sqrt{2\pi}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k)\hat{g}(k)e^{ikx}\,dk$$

$$\int_{-\infty}^{\infty} f(x-y)g(y)\,dy = \int_{-\infty}^{\infty} \hat{f}(k)\hat{g}(k)\,e^{ikx}\,dk$$

Two functions of $x$, equal everywhere, must be equal to $x=0$

$$\int_{-\infty}^{\infty} f(-y)g(y)\,dy = \int_{-\infty}^{\infty} \hat{f}(k)\hat{g}(k)\,dk \qquad (\dagger)$$

Introduce a new function

$$h(y) = f(-y)^* \qquad * \text{ complex conjugate.}$$

Find its F.T

$$\hat{h}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(-y)^* e^{-iky}\,dy$$

Use C.O.V $x=-y$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)^* e^{ikx}\,(-dx)$$

$$= \frac{1}{\sqrt{2\pi}} \left( \int_{-\infty}^{\infty} f(x)e^{-ikx} \right)^* = \hat{f}(k)^*$$

Also $\hat{f}(k) = \hat{h}(k)^*$

Substitute into (†)

$$\int_{-\infty}^{\infty} h(y)^* g(y)\, dy = \int_{-\infty}^{\infty} \hat{h}(k)^* \hat{g}(k)\, dk \qquad (‡)$$

Free to choose any $f(y)$, hence any $h(y)$

Choose $h(y) = g(y)$

$$\boxed{\int_{-\infty}^{\infty} |g(y)|^2 dy = \int_{-\infty}^{\infty} |\hat{g}(k)|^2 dk}$$

Applications

(1) Solution of Integral Equation.

Example: Find $f(x)$

$$e^{-x^2/2} = \underbrace{\frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-u|} f(u)\, du}$$

$$g = h * f$$

for $g(x) = e^{-x^2/2}$ , $h(x) = \frac{1}{2}e^{-|x|}$

Take F.T

$$\hat{g}(k) = \sqrt{2\pi}\, \hat{h}\, \hat{f}$$

$$\hat{g}(k) = F\left[e^{-x^2/2}\right] = e^{-k^2/2}$$

From Ex 3. $F[e^{-a|x|}] = \sqrt{2/\pi}\; a/(a^2+k^2)$

$$F[\tfrac{1}{2}e^{-|x|}] = 1/\sqrt{2\pi}\; 1/1+k^2$$

$$e^{-k^2/2} = 1/\sqrt{2\pi} \cdot \sqrt{2\pi}\; 1/1+k^2\; \hat{f}(k)$$

$$\hat{f}(k) = (1+k^2)e^{-k^2/2}$$

Recall "transform of derivative" formula

$$F[g^{(n)}(x)] = (ik)^n \hat{g}$$

$$F[g''(x)] = -k^2\hat{g} \qquad \text{choose } \hat{g}(k) = e^{-k^2/2}$$

$$F[d^2/dx^2\, e^{-x^2/2}] = -k^2 e^{-k^2/2}$$

Using this on $\hat{f}(k) = (1+k^2)\,e^{-k^2/2}$

$$\Rightarrow f(x) = (1 - d^2/dx^2)\,e^{-x^2/2}$$

$$= e^{-x^2/2}(2-x^2)$$

2 Solution of 'difficult' integrals using Parseval.

Consider $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^2}\,dx$

Choose $g(y) = e^{-a|y|}$ $\qquad \hat{g}(k) = \sqrt{2/\pi}\; a/a^2+k^2$

5

Parseval gives

$$\int_{-\infty}^{\infty} \underbrace{\left(e^{-a|y|}\right)}_{|g(y)|^2} dy = \frac{2a^2}{\pi} \int_{-\infty}^{\infty} \underbrace{\frac{1}{(a^2+k^2)^2}}_{|\hat{g}(k)|^2} dk$$

$$2\int_{0}^{\infty} e^{-2ay} \, dy = \frac{2a^2}{\pi} I$$

$$2\left[-\frac{1}{2a} e^{-2ay}\right]_0^{\infty} = \frac{1}{a} = \frac{2a^2}{\pi} I$$

Hence ~~or~~ $I = \pi/2a^3$

Parseval's Theorem

$$\int_{-\infty}^{\infty} |g(y)|^2 \, dy = \int_{-\infty}^{\infty} |\hat{g}(k)|^2 \, dk$$

More generally:

$$\int_{-\infty}^{\infty} h(y)^* g(y) \, dy = \int_{-\infty}^{\infty} \hat{h}(k)^* \hat{g}(k) \, dk \qquad (\ddagger)$$

for $g, h \in L^1 \cap L^2$

Application

$$I = \int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^2} \, dx = \frac{\pi}{2a^3} .$$

Can also use $(\ddagger)$: Choose $h(y) = e^{-a|y|}$ $\quad g(y) = e^{-b|y|}$

$$\hat{h}(k) = \sqrt{\frac{2}{\pi}} \frac{a}{a^2+k^2} \qquad \hat{g}(k) = \sqrt{\frac{2}{\pi}} \frac{b}{b^2+k^2} \qquad (a, b > 0 \text{ real})$$

Insert in $(\ddagger)$

$$\frac{2}{\pi} \int_{-\infty}^{\infty} \frac{ab}{(a^2+k^2)(b^2+k^2)} \, dk = \int_{-\infty}^{\infty} e^{-(a+b)|y|} \, dy$$

$$= 2 \int_{0}^{\infty} e^{-(a+b)y} \, dy = \frac{2}{a+b}$$

Allows us to evaluate

$$J = \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$$

$$\frac{2ab}{\pi} J = \frac{2}{a+b}$$

$$J = \pi / ab(a+b)$$

Generalised Inversion Formula

The 'sketch' dervation of the Fourier Integral formula seen in lectures is valid only for $f(x) \in L^1 \cap C^0$

($f$ continuous and $\int_{-\infty}^{\infty} |f| \, dx$ exists)

($C^n$ - set of $f(x)$ with $f, f', \dots f^{(n)}$ continuous).

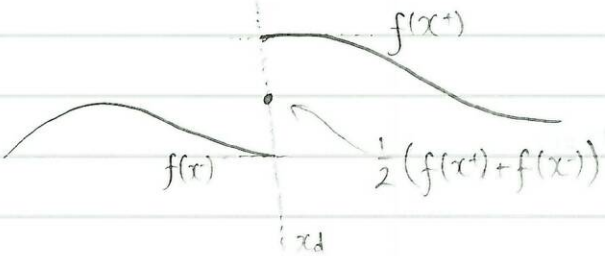
The generalised inversion formula, valid also at points of discontinuity for discontinuous $f(x)$ is

$$\frac{1}{2}\left(f(x^+) + f(x^-)\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} \, dk$$

At points where $f(x)$ is continuous $f(x^+) = f(x^-) = f(x)$

At points $x = x_d$ (say) where $f(x)$ is discontinuous

$$f(x^+) = \lim_{\substack{x \to x_d \\ x > x_d}} f(x) \qquad f(x^-) = \lim_{\substack{x \to x_d \\ x < x_d}} f(x)$$




(3) Application : Find the value of a 'difficult' integral using the Generalised inversion formula

Choose $f(x) = \begin{cases} 1 & |x|<a \\ 0 & |x| \geq a \end{cases}$

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{ikx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \cos kx\, dx = \frac{1}{\sqrt{2\pi}} \left[ \frac{\sin kx}{k} \right]_{-a}^{a}$$

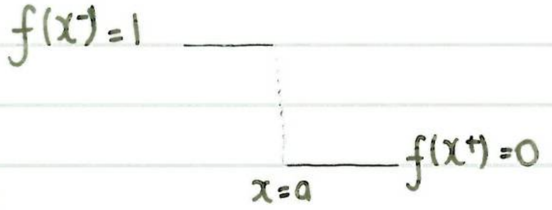
$$= \sqrt{\frac{2}{\pi}}\, \frac{\sin ka}{k}$$

Use Gen. inversion formula

$$\frac{1}{2}\left(f(x^+) + f(x^-)\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}}\, \frac{\sin kn}{k}\, e^{ikx}$$

even in $k$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin ka \cos kx}{k}\, dk = \begin{cases} 1 & |x|<a \\ 0 & |x|>a \\ \tfrac{1}{2} & x = \pm a \end{cases}$$




or

$$\int_{0}^{\infty} \frac{\sin ka \cos kx}{k}\, dk = \frac{\pi}{2}\begin{cases} 1 & |x|<a \\ 0 & |x|>a \\ \tfrac{1}{2} & x = \pm a \end{cases}$$

(4) Application Laplace's equation in the half plane.


Find the steady temperature distribution $u(x,y)$ in a semi-infinite plate $(-\infty<x<\infty,\ 0\leq y<\infty)$ when its lower boundary is held at temperature $u(x,0)=f(x)$ ($f(x)\in L^1 \Rightarrow$ boundary distribution is sufficiently "localised").

Take F.T in $x$

$\hat{u}(k,0) = \hat{f}(k)$ boundary condition

$F[\nabla^2 u] = F[u_{xx}] + F[u_{yy}]$ Fourier transform is linear

$$F[u_{yy}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial y^2} e^{-ikx}\, dx$$

$$= \frac{\partial^2}{\partial y^2} \left( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u e^{-ikx} dx \right) = \frac{\partial^2 \hat{u}}{\partial y^2}(k,y)$$

$$F[u_{xx}] = F[\partial^2 u/\partial x^2] = (ik)^2 \hat{u}(k,y) = -k^2 \hat{u}(k,y)$$

$\nabla^2 u = 0$ gives $\dfrac{\partial^2 \hat{u}}{\partial y^2} - k^2 \hat{u} = 0$

Laplace's Equation in the ½ plane.

$$\nabla^2 u = 0 \qquad u \to 0 \text{ as } x^2+y^2 \to \infty.$$


Find the steady temp. $u(x,y)$

~~The~~ Take the $x$-transforms:

$$\nabla^2 u = 0 \quad \Rightarrow \quad \frac{\partial^2 \hat{u}}{\partial y^2} \overbrace{- k^2 \hat{u}}^{\mathcal{F}[u_{xx}]} = 0 \quad (1) \qquad \hat{u} = \hat{u}(k,y)$$

$$u(x,0) = f(x) \quad \Rightarrow \quad \hat{u}(k,0) = \hat{f}(k) \quad (2)$$

$$u \to 0 \text{ as } y \to \infty \quad \Rightarrow \quad \hat{u}(k,y) \to 0 \text{ as } y \to \infty \quad (3)$$

Integrate (1)

$$\hat{u}(k,y) = \underline{A(k)}e^{-ky} + \underline{B(k)}e^{ky}$$

Arbitrary functions

equivalently $= C(k)e^{-|k|y} + D(k)e^{|k|y}$

$$C(k) = \begin{cases} A(k) & k \geq 0 \\ B(k) & k < 0 \end{cases} \qquad D(k) = \begin{cases} B(k) & k \geq 0 \\ A(k) & k < 0 \end{cases}$$

Now b.c (3) implies $D(k) = 0$

Now $\hat{u}(k,y) = C(k)e^{-|k|y}$

Use (2)

$$\hat{u}(k,0) = C(k) = \hat{f}(k)$$

$$\hat{u}(k,y) = \hat{f}(k)e^{-|k|y}$$

$$= \sqrt{2\pi}\,\hat{f}(k)\hat{g}(k,y) \qquad \text{For } \hat{g}(k,y) = \frac{e^{-|k|y}}{\sqrt{2\pi}}$$

$(= f*g)$.

Hence, using the convolution theorem :

$$u(x,y) = \underbrace{\int_{-\infty}^{\infty} f(x-t)g(t,y)\,dt}_{f*g}$$

Use t (not y!) to avoid confusion.

Next find $g(x,y)$

$$g(x,y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{g}(k,y)e^{ikx}\,dk$$

$$= \frac{1}{\sqrt{2\pi}^{\,2}}\int_{-\infty}^{\infty} e^{-|k|y}\,e^{ikx}\,dk$$

$$= \frac{1}{\pi}\int_{0}^{\infty} e^{-ky}\cos kx\,dk$$

since $\hat{g}(k,y)$ is even in k
see EX 3.

$$= \frac{1}{\pi}\,\mathrm{Re}\left\{\int_{0}^{\infty} e^{-ky+ikx}\,dk\right\}$$

$$= \frac{1}{\pi} \operatorname{Re}\left[ \frac{e^{-k(y-ix)}}{ix-y} \right]_0^{\infty} = \frac{1}{\pi} \frac{y}{x^2+y^2}$$

limits in $k$

Ex $$f(x) = \begin{cases} 1 & |x| \leq a \\ 0 & |x| > a \end{cases}$$

$$f(x-t) = \begin{cases} 1 & |x-t| \leq a \\ 0 & |x-t| > a \end{cases}$$

$x-a \leq t \leq x+a$

$-a \leq t-x \leq a$

So $$u(x,y) = \frac{y}{\pi} \int_{x-a}^{x+a} \frac{dt}{t^2+y^2}$$

$f(x-t)=1$ between those limits in $t$ and zero elsewhere.

$$= \frac{y}{\pi} \left[ \frac{1}{y} \tan^{-1}\left(\frac{t}{y}\right) \right]_{x-a}^{x+a}$$

$$= \frac{1}{\pi} \left( \tan^{-1}\left(\frac{x+a}{y}\right) - \tan^{-1}\left(\frac{x-a}{y}\right) \right)$$

Consistency check : Does $u(x,0) = f(x)$? (as it has to be!)

Make the substitution $t = sy$

$$u(x,y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(x-sy)}{y^2(s^2+1)} (y\, ds)$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(x-sy)}{s^2+1} ds$$

$$\lim_{y\to 0} u(x,0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{s^2+1}\,ds = f(x)$$

## Application 5 : Heat equation on the real line

Find the time-dependent temperature distribution $u(x,t)$ in an ~~ins~~ insulated metal rod $(-\infty < x < \infty)$ when its initial temperature is $u(x,0) = f(x)$


$u(x,t)$ satisfies : $\quad u_t = u_{xx}$ (heat equation) in $-\infty < x < \infty$

$\qquad u(x,0) = f(x)$ (initial condition) $t \geq 0$

Take F.T in $x$ : $\quad \hat{u}_t = \underbrace{-k^2\hat{u}}_{F[u_{xx}]}$

$\qquad \hat{u}(k,0) = \hat{f}(k)$

Integrating $\quad$ arb. fn.

$\hat{u}(k,t) = A(k)e^{-k^2t}$

Evaluate at $t=0$

$\hat{u}(k,0) = A(k) = \hat{f}(k)$

$$\hat{u}(k,t) = \hat{f}(k)e^{-k^2t}$$

$$= \sqrt{2\pi}\,\hat{f}(k)\hat{g}(k,t) = \widehat{f*g}$$

Take inverse T

$$u(x,t) = f*g = \int_{-\infty}^{\infty} f(x-q)g(q,t)\,dq$$

choose $q$ as variable of integration.

Need to find $g(x,t)$

Recall (Ex 2) that

$$F\left[e^{-x^2/a^2}\right] = a/\sqrt{2}\; e^{-a^2k^2/4}$$

Choose $a = \sqrt{4t}$

$$F\left[e^{-x^2/4t}\right] = \sqrt{2t}\, e^{-k^2 t}$$

Divide by $\sqrt{4\pi t}$

$$F\left[\frac{1}{\sqrt{4\pi t}}\, e^{-x^2/4t}\right] = \hat{g}(k,t)$$

or $g(x,t) = \dfrac{1}{\sqrt{4\pi t}}\, e^{-x^2/4t}$ Heat kernel

Physical property: Spatial length scale increases in proportion to $\sqrt{t}$
All solutions of heat equation eventually spread with length $\sim \sqrt{t}$.

General Solution

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} f(x-q)\, e^{-q^2/4t}\, dq$$

Exercise

Find the particular solution (on Moodle) for $f(x) = \begin{cases} 1 & |x| \le a \\ 0 & |x| > a \end{cases}$

Fourier Sine and Cosine Transforms.

There are defined for functions $f(x)$ defined on the half-line $(0 \le x < \infty)$

Fourier transforms of odd and even functions.

Let $f(x)$ be even :

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) (\cos kx - \cancel{i \sin kx}) \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \cos kx \, dx$$

Real.

Let $f(x)$ be odd :

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) (\cancel{\cos kx} - i \sin kx) \, dx$$

$$= -i \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \sin kx \, dx$$

imaginary

Let $f^{+}(x)$ and $f^{-}(x)$ be the odd and even extensions of $f(x)$ respectively.

Define the COSINE and SINE transforms of $f(x)$ as follows :

$$F_c[f](k) = \sqrt{2/\pi}\int_0^\infty f(x)\cos kx\,dx = \hat{f}^+(k)$$ F.T of even extension

$$F_s[f](k) = \sqrt{2/\pi}\int_0^\infty f(x)\sin kx\,dx \quad i\hat{f}^-(k)$$ ← F.T of odd extension.

Consider inverse formulae for odd and even extensions

$$f^+(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}^+(k)\, e^{ikx}\,dk$$

↖even in k

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}^+(k)\cos kx\,dk$$

For $x \geq 0$

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_c[f](k)\cos kx\,dk$$

Cosine transform is exactly symmetric.

$$f^-(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}^-(k)\, e^{ikx}\,dk$$

$$= i\sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}^-(k)\sin kx\,dk$$

For $x \geq 0$

$$f(x) = i\sqrt{\frac{2}{\pi}}\int_0^\infty \left(-i\, F_s[f](k)\right)\sin kx\,dk$$

( Assume $f, f' \to 0$ at $\infty$ $\quad f(0^+) = \lim_{x\to 0} f(x)$

$f'(0^+)$ $\lim_{x\to 0} f'(x)$ exists.

Transforms of derivatives

$$F_c[f'] = \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x)\cos kx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \Big[ f(x)\cos kx \Big]_0^\infty + \sqrt{\frac{2}{\pi}} \int_0^\infty k f(x) \sin kx \, dx$$

$$= -\sqrt{\frac{2}{\pi}} f(0^+) + k F_s[f]$$

$$F_s[f'] = \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x)\sin kx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \Big[ f(x)\sin kx \Big]_0^\infty - \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) k\cos kx \, dx$$

$$= -k F_c[f]$$

Second derivatives

$$F_c[f''] = -\sqrt{\frac{2}{\pi}} f'(0^+) + k F_s[f'] = -\sqrt{\frac{2}{\pi}} f(0^+) - k^2 F_c[f]$$

$$F_s[f''] = -k F_c[f'] = -k^2 F_s[f] + k\sqrt{2/\pi}\, f(0^+)$$

## Properties of Laplace transform

(1) Shift results

(a) $\mathcal{L}\left[e^{-\alpha t} f(t)\right] = \int_0^\infty f(t) e^{-(s+\alpha)t}\, dt = \bar{f}(s+\alpha)$

(b) $\mathcal{L}\left[f(t-\alpha)\right] = \int_0^\infty f(t-\alpha) e^{-st}\, dt$

Valid for real $\alpha > 0$
$f(t) = 0$ for $t < 0$
$f(t-\alpha) = 0$ for $t < \alpha$

$$= \int_{-\alpha}^{\infty} f(u) e^{-(u+\alpha)s}\, du$$

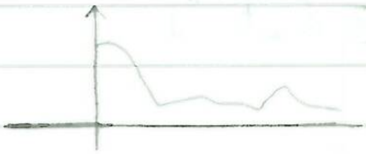
$$= e^{-\alpha s} \bar{f}(s)$$

15th March 2012.

## §4b. Laplace transforms

Defined for functions $f(t)$ defined on the HALF-LINE $(0 \le t < \infty)$

Although we extend $f(t)$ so that $f(t)=0$, $t<0$.



The Laplace transform is defined to be

$$\mathcal{L}[f](s) = \bar{f}(s) = \int_0^\infty f(t) e^{-st}\, dt$$

Examples

(1) $$\mathcal{L}[e^{\beta t}] = \int_0^\infty e^{(\beta - s)t}\, dt = \left[ -\frac{e^{(\beta-s)t}}{s-\beta} \right]_0^\infty = \frac{1}{s-\beta}$$

for $\mathrm{Re}(s) > \beta$ required for convergence as $t \to \infty$ ($\beta$ real).

Laplace transforms exist (at least somwhere in the complex $s$-plane) for a much wider class of functions than F.Ts

e.g. $f(t) = e^{\beta t} \not\in L^1(0,\infty)$ for $\beta > 0$.

(2) $$\mathcal{L}[1] = \int_0^\infty e^{-st}\, dt = \frac{1}{s}$$

(3) $\mathcal{L}[t^n] = \int_0^\infty t^n e^{-st}\,dt$

(n-integer)
$n \geq 1$

$$= \cancelto{0}{\left[-t^n \frac{e^{-st}}{s}\right]_0^\infty} + \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}\,dt$$

$$= \frac{n}{s}\mathcal{L}[t^{n-1}] = \frac{n}{s}\cdot\frac{n-1}{s}\mathcal{L}[t^{n-2}]$$

$$= \frac{n}{s}\cdot\frac{n-1}{s}\cdots\cdot\frac{2}{s}\overbrace{\mathcal{L}[t^0]}^{1/s} = \frac{n!}{s^{n+1}}$$

(4) $\mathcal{L}[t^\alpha] = \int_0^\infty t^\alpha e^{-st}\,dt$ $\qquad u = st$

($\alpha$ - non-integer)

$$= \int_0^\infty \left(\frac{u}{s}\right)^\alpha e^{-u}\,\frac{du}{s}$$

$$= \frac{1}{s^{\alpha+1}}\int_0^\infty u^\alpha e^{-u}\,du = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$$

(5)

$$\mathcal{L}\begin{bmatrix}\cos\omega t\\ \sin\omega t\end{bmatrix} = \begin{Bmatrix}\mathrm{Re}\\ \mathrm{Im}\end{Bmatrix}\mathcal{L}[e^{i\omega t}]$$

$$\mathcal{L}[e^{i\omega t}] = \int_0^\infty e^{(i\omega - s)t}\,dt = \left[\frac{e^{(i\omega-s)t}}{i\omega - s}\right]_0^\infty = \frac{1}{s-\omega i} = \frac{s+\omega i}{s^2+\omega^2}$$

Hence $\mathcal{L}[\cos\omega t] = s/s^2+\omega^2$
$\mathcal{L}[\sin\omega t] = \omega/s^2+\omega^2$

20/3/12

# Laplace Transforms - Properties

**Shift results**

$$\mathcal{L}[e^{-\alpha t} f(t)] = \bar{f}(s+\alpha)$$

$$\mathcal{L}[\underbrace{f(t-\alpha)}] = e^{-\alpha s}\bar{f}(s)$$

$(\alpha > 0 \quad f(t-\alpha) = 0 \text{ for } t < \alpha)$

**Derivative of Transform**

$$\left(-\frac{d}{ds}\right)\bar{f}(s) = -\frac{d}{ds}\int_0^\infty f(t)e^{-st}\,dt$$

$$= \int_0^\infty -f(t)\frac{d}{ds}(e^{-st})\,dt$$

$$= \int_0^\infty t f(t) e^{-st}\,dt = \mathcal{L}[t f(t)]$$

Apply many times $\mathcal{L}[t^n f(t)] = \left(\frac{-d}{ds}\right)^n \bar{f}(s)$

$$= (-1)^n \bar{f}^{(n)}(s)$$

e.g.

$$\mathcal{L}[t \sin\omega t] = \left(\frac{-d}{ds}\right)\underbrace{\left(\frac{\omega}{s^2+\omega^2}\right)}_{\mathcal{L}[\sin \omega t]} = \frac{2s\omega}{(s^2+\omega^2)^2}$$

$\boxed{\text{Transform of Derivative}}$ $\mathcal{L}[f'(t)] = \int_0^\infty f'(t)e^{-st}\,dt.$

$$= \left[f(t)e^{-st}\right]_0^\infty + \int_0^\infty sf(t)e^{-st}\,dt \quad \left(\text{Using parts}\right)$$

$$= \underbrace{-f(0_+)}_{\text{Boundary term}} + s\bar{f}(s)$$

<u>Similarly</u> $\quad \mathcal{L}[f''(t)] = s\mathcal{L}[f'(t)] - f'(0_+)$

$$= s^2\bar{f}(s) - sf(0_+) - f'(0_+)$$

<u>Apply n times</u>

$$\mathcal{L}[f^{(n)}(t)] = s^n\bar{f}(s) - \underbrace{s^{n-1}f(0_+) - s^{n-2}f'(0_+) - \dots - f^{(n-1)}(0_+)}_{n \text{ boundary terms.}}$$

<u>Convolution</u>:

$$(f*g)(t) = \int_{\substack{-\infty\\0}}^{\substack{\infty\\t}} \overbrace{f(t-u)}^{=0 \text{ for } u>t}\,\underbrace{g(u)}_{=0 \text{ for } u<0}\,du$$

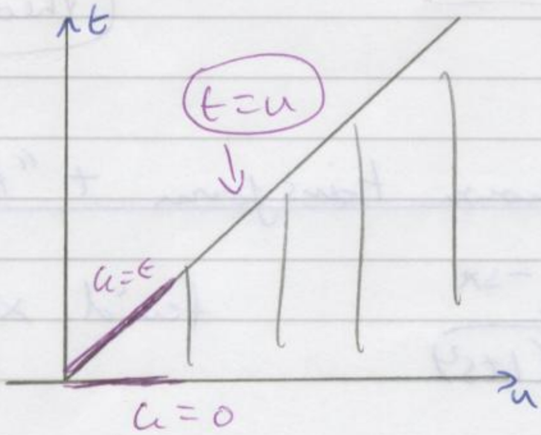
Consistent with F.T. defn Why?

Recall that (by convention) $\begin{cases} f(t) = 0 & t < 0 \\ g(t) = 0 & t < 0 . \end{cases}$

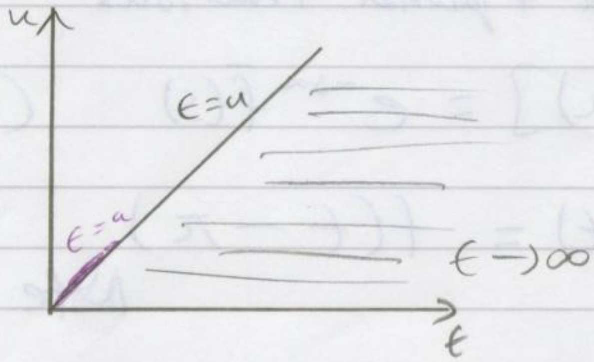
$\boxed{\text{Convolution Theorem}}$ $\quad \mathcal{L}[f * g] = \int_0^\infty \left( \int_0^t f(t-u)\, g(u)\, du \right) \cdots$

$\cdots e^{-st} dt$

$$= \int_0^\infty \int_0^t f(t-u)\, g(u)\, e^{-st} \, du\, dt$$




Change order of integration ... take care with limits




$$= \int_0^\infty \int_u^\infty f(t-u)\, g(u)\, e^{-st} \, dt\, du$$

Now substitute $v = t - u$ in inner integral

$$\mathcal{L}[f*g] = \int_0^\infty \left( \int_0^\infty f(v) g(u) e^{\underbrace{-s(u+v)}_{}} dv \right) du$$

$v=0$ when $t=u$. $\quad t = u+v$

$$= \left( \int_0^\infty f(v) e^{-sv} dv \right) \left( \int_0^\infty g(u) e^{-su} du \right)$$

$$\boxed{\mathcal{L}[f*g] = \bar{f}(s)\,\bar{g}(s)}$$ Convolution theorem

<u>Inversion</u> – <u>Using known transform + "tricks"</u>

<u>Ex 1</u>: $\bar{x}(s) = \dfrac{e^{-s\pi}}{s^2(1+s^2)}$ find $x(t)$.

– Try "shift result" + partial fractions

Recall $\mathcal{L}[f(t-\alpha)] = e^{-s\alpha}\bar{f}(s) \quad (\alpha > 0)$

Using this $x(t) = f(t-\pi)$

~~Note that $x(t)=0$~~

where $\bar{f}(s) = \dfrac{1}{s^2(1+s^2)}$ Note that $x(t)=0$ for $t < \pi$ (not 0)

$$= \frac{1}{s^2} - \frac{1}{s^2+1} \quad \left(\text{partial fraction}\right)$$

$= \mathcal{L}[t] - \mathcal{L}[\sin(t)]$ $\left(\begin{array}{l}\text{from} \\ \text{examples}\end{array}\right)$

Hence $f(t) = t - \sin t \quad (t > 0)$

and $x(t) = f(t - \pi) = \begin{cases} (t-\pi) - \sin(t-\pi) & t > \pi \\ 0 & t < \pi \end{cases}$

<u>Ex</u>: $\bar{x}(s) = \dfrac{1}{(s+1)(s+2)}$

- this time use convolution (could use partial fracs)

$\bar{x} = \bar{f}(s)\bar{g}(s) \qquad \bar{f}(s) = \dfrac{1}{s+1} \qquad f(t) = e^{-t}$

$\Downarrow$ use convolution theorem $\qquad \Rightarrow$

$\Rightarrow x(t) = (f * g)(t) \qquad \bar{g}(s) = \dfrac{1}{s+2} \qquad g(t) = e^{-2t}$

$$= \int_0^t f(t-u)g(u)\, du \quad \left(\begin{array}{l}\text{from} \\ \text{exs}\end{array}\right)$$

$$= \int_0^t \overbrace{e^{-(t-u)}}^{f(t-u)} \overbrace{e^{-2u}}^{g(u)}\, du$$

$$= e^{-t} \int_0^t e^{-u}\, du$$

$$= e^{-t} - e^{-2t}$$

Ex 3: $\bar{x}(s) = \dfrac{1}{(s-1)^5}$

- Use derivate of transform formula:

$$\bar{x}(s) = \frac{1}{4!}\left(-\frac{d}{ds}\right)^4 \frac{1}{(s-1)}$$

$$\boxed{\frac{d^n}{ds^n}\frac{1}{s-1} = (-1)^n \frac{n!}{(s-1)^{n+1}}}$$

know that

$$\mathcal{L}[t^n f(t)] = \left(\frac{-d}{ds}\right)^n \bar{f}(s)$$

Hence $x(t) = \dfrac{1}{4!} t^4 f(t)$

where $\bar{f}(s) = \dfrac{1}{s-1} \Rightarrow f(t) = e^t$

therefore $x(t) = \dfrac{1}{4!} t^4 e^t = \dfrac{t^4 e^t}{24}$

<u>Inversion of the Laplace transform</u>

Recall the FOURIER INTEGRAL RESULT

$$F(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\underbrace{\left(\int_{-\infty}^{\infty} F(u) e^{-iku}\,du\right)}_{\sqrt{2\pi}\,\hat{F}(k)} e^{ikt}\,dk$$

relates $F(t) \in L^1(\mathbb{R})$ to integrals involving itself:

Write:

$$F(t) = \begin{cases} f(t)e^{-ct} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

where $c > 0$ is a constant known as a CONVERGENCE FACTOR. It needs to be large enough so that $F \in L^1$

Substituting

$$f(t)e^{-ct} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_0^{\infty} f(u)e^{-cu}e^{-iku}\,du\right)e^{+ikt}\,dk$$

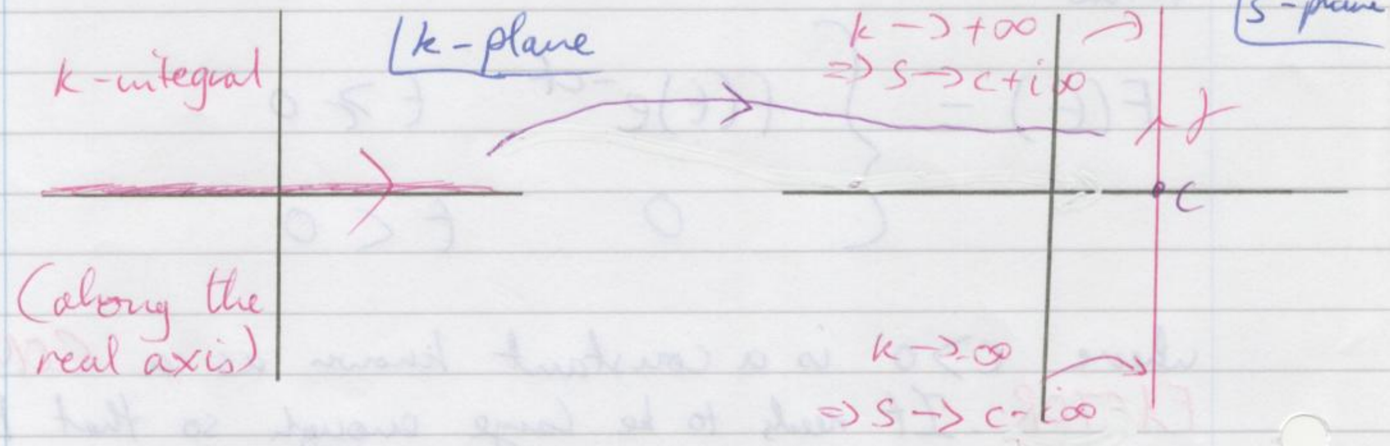
Next, make the COMPLEX substitution $s = c + ik$. This has no effect on the inner integral (re-labelling a constant) but changes the PATH of the complex plane.

$$f(t)\cancel{e^{-ct}} = \frac{1}{2\pi}\int_{\gamma}\underbrace{\left(\int_0^{\infty} f(u)e^{-su}\,du\right)}_{\bar{f}(s)}\underbrace{e^{(s-\cancel{c})t}}_{e^{ikt}}\underbrace{\left(\frac{ds}{i}\right)}_{dk}$$

$$\boxed{f(t) = \frac{1}{2\pi i}\int_{\gamma}\bar{f}(s)e^{st}\,ds}$$

BROMWICH INVERSION FORMULA

What is the path (in the complex plane) repersented by $\gamma$?




- $\gamma$ is a vertical line in the complex s-plane given by $Re(s) = c$

- known as the BROMWICH CONTOUR

- sometimes write $\int_{c-i\infty}^{c+i\infty}$ in place of $\int_\gamma$

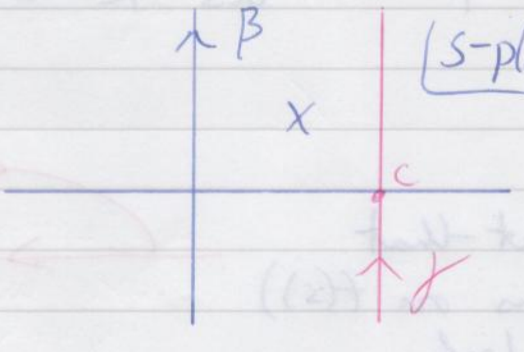
What constrains the choice of convergence factor $c$?

- Essentially, we need $\bar{f}(s) = \int_0^\infty f(u) e^{-st}\, du$. for all $s$ on $\gamma$

- Suppose $f(u) = e^{\beta u}$ for complex $\beta$ and with $Re\, \beta > 0$.

The integral for $\bar{f}(s)$ converges only for $Re(s) > Re(\beta)$

Then $\bar{f}(s) = \dfrac{1}{s-\beta}$




Hence $c > \mathrm{Re}(\beta)$ for the integral to converage and give $\bar{f}(s) = \dfrac{1}{s-\beta}$ everywhere on $\gamma$.

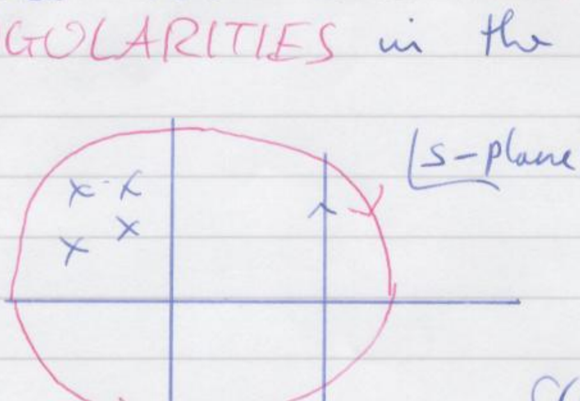
More generally, for the Laplace transform to exist for all $s$ on $\gamma$, $c$ must be CHOSEN large enough so that all of the SINGULARITIES (and branch pts) of $\bar{f}(s)$ lie to the left of $\gamma$.

REMARK: $\bar{f}(s)$ exist over (nearly) all of the s-plane, but is only equal to the integral

$$\int_0^\infty f(u)e^{-su}\,du \quad \text{for} \quad \mathrm{Re}(s) > \mathrm{Re}\begin{pmatrix}\text{right most}\\ \text{singularity}\end{pmatrix}$$

this is called ANALYTIC CONTINUATION

Suppose $\bar{f}(s)$ has a FINITE NUMBER of ISOLATED SINGULARITIES in the complex s-plane.




$$f(t) = \frac{1}{2\pi i}\int_\gamma \bar{f}(s)e^{st}\,ds$$

If we can close on the right $f(t) = 0$. We can do this if $t < 0$

Jordan's Lemma gives the conditions from:

$$\left| \int_{C_R} f(z) e^{iaz} \, dz \right| \to 0 \quad \text{as } R \to \infty$$

Requires $a \geq 0$


If $t \geq 0$, it turns out that (under "mild" conditions on $f(s)$) we can close to the left.

Hence

$$f(t) = \begin{cases} \sum_j \operatorname{Res}\left(\bar{f}(s) e^{st} ; s = s_j\right) & t > 0 \\ 0 & t < 0 \end{cases}$$

from Cauchy's residue theorem.

22nd March 2012.

Application : Solution of ODE.

Forced oscillator

$$\underbrace{\ddot{x}+x}_{\text{equation of SHM}} = \underset{\text{forcing}}{f(t)} \qquad x(0)=1, \quad \dot{x}(0)=1$$

$$f(t) = \begin{cases} t & 0<t<\pi \\ \pi & t \geq \pi \end{cases}$$

Use L.T.

Recall 'transform of derivatives'

$$\mathcal{L}[\ddot{x}] = s^2\bar{x} - sx(0_+) - \dot{x}(0_+).$$

$$= s^2\bar{x} - s + 1 \qquad \text{from b.c at } t=0$$

Equation is

$$(s^2+1)\bar{x} - (s+1) = \bar{f}$$

$$\mathcal{L}[f] = \int_0^{\pi} t e^{-st} \, dt + \int_{\pi}^{\infty} \pi e^{-st} \, dt$$

$$= \left[ -t\frac{e^{-st}}{s} \right]_0^{\pi} + \int_0^{\pi} \frac{e^{-st}}{s} \, dt + \int_{\pi}^{\infty} \pi e^{-st} \, dt$$

$$= -\frac{\pi e^{-s\pi}}{s} + \left[-\frac{e^{-st}}{s^2}\right]_0^{\pi} + \left[-\frac{\pi}{s}e^{-st}\right]_{\pi}^{\infty}$$

$$= -\frac{\pi e^{-s\pi}}{s} + \frac{(1-e^{-s\pi})}{s^2} + \frac{\pi}{s}e^{-st} = \frac{1-e^{-s\pi}}{s^2}$$

$$\therefore \quad \bar{X}(s^2+1)-(s+1) = \frac{1-e^{-s\pi}}{s^2}$$

$$\bar{X} = \frac{s^3+s^2+1}{s^2(s^2+1)} - \frac{e^{-s\pi}}{s^2(s^2+1)}$$ L.T of $x(t)$

Now use Bromwich contour method to obtain $x(t)$.

$$x(t) = \frac{1}{2\pi i}\int_{\gamma} \bar{X}(s)e^{st}\,ds$$

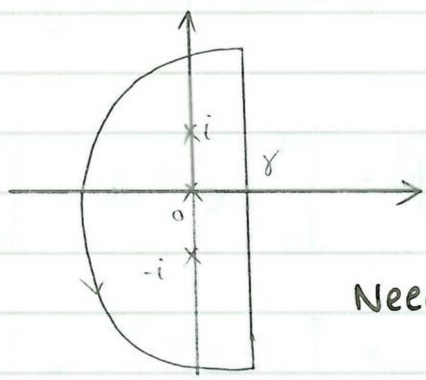
Split $x(t) = X_1(t) + X_2(t)$

$$\bar{X}_1(s) = \frac{s^3+s^2+1}{s^2(s^2+1)}$$

$$\bar{X}_2(s) = \frac{e^{-s\pi}}{s^2(s^2+1)}$$

$$X_1(t) = \frac{1}{2\pi i}\int_{\gamma} \frac{s^3+s^2+1}{s^2(s^2+1)}e^{st}\,ds$$

Can close to the left when $t>0$




Need $c>0$

$$x_1(t) = \sum_j \text{Res}\left\{\frac{s^3+s^2+1}{s^2(s^2+1)}e^{st} \; ; \; s=s_j\right\}$$

$s_j = \{-i, 0, i\}$

$s=0$ Pole of order 2
$s \pm i$ simple pole (order 1)

For $s=0$,
use Laurent expansion

$$\overbrace{(s+1-1/s^2)(1-s^2+s^4-s^6+\dots)}^{1/s^2+1 \text{ geo. progression}}\overbrace{(1+st+s^2t^2/2!+\dots)}^{e^{st}}$$

$$= 1/s^2 + t/s + O(1)$$

$\Rightarrow \text{Res}\{s=0\} = t$

For $s \pm i$, use simple pole formula.

$$\text{Res}\{f(z),\ z=z_j\} = [(z-z_j)f(z)]_{z=z_j}$$

$$\text{Res}\{s=\pm i\} = \left[\frac{(s\mp i)e^{st}(s^3+s^2+1)}{s^2(s^2+1)}\right]_{s=\pm i}$$

$$= \left[\frac{e^{st}(s^3+s^2+1)}{s^2(s\pm i)}\right]_{s=\pm i} \qquad \left(\frac{s-i}{s^2+1} = \frac{1}{s+i} \quad \frac{s+i}{s^2+1} = \frac{1}{s-i}\right)$$

$$= \frac{e^{\pm it}(\overset{(-1)(\pm i)}{\pm i^3}-1+1)}{(-1)(\pm 2i)} = \frac{e^{\pm it}}{2}$$

Hence $\text{Res}(s=i) + \text{Res}(s=-i) = \frac{e^{it}}{2} + \frac{e^{-it}}{2} = \cos t$

and $x_1(t) = \sum \text{Res}(s=s_j) = \begin{cases} t+\cos t & t>0 \\ 0 & t<0 \end{cases}$

$$X_2(t) = \frac{1}{2\pi i}\int_\gamma \frac{e^{s(t-\pi)}}{s^2(s^2+1)}\, ds.$$

Here : Can close to the left when $t-\pi > 0$

right when $t-\pi < 0$.

$\Rightarrow X_2(t)$ does not "switch on" until $t=\pi$.

$$X_2(t) = \begin{cases} \sum_j \text{Res}\left\{ \frac{e^{s(t-\pi)}}{(s^2+1)s^2} ; \quad s=s_j \right\} & t>\pi \\ 0 & t<\pi \end{cases}$$

Res $s=0$

Use Laurent

$$\frac{1}{s^2}(1-s^2+s^4-\dots)\left(1+(t+\pi)s+\frac{(t+\pi)^2}{2!}s^2+\dots\right)$$

$$\Rightarrow \text{Res}\{s=0\} = t-\pi$$

Res $s=\pm i$

(use simple pole formula)

$$\text{Res}\{s=\pm i\} = \left[\frac{(s\mp i)e^{s(t-\pi)}}{s^2(s^2+1)}\right]_{s=\pm i}$$

$$= \left[\frac{e^{s(t-\pi)}}{s^2(s^2+1)}\right]_{s=\pm i} = \frac{(e^{\pm it})(\overset{=-1}{e^{\pm\pi i}})}{(-1)(\pm 2i)} = \frac{e^{\pm it}}{\pm 2i}$$

Hence $\text{Res}(s=i)+\text{Res}(s=-i) = e^{it}/2i - e^{-it}/2i = \sin t$

$$X_2(t) = \text{Res}\{s=s_j\} = \begin{cases} \sin t + t + \pi & t>\pi \\ 0 & t<\pi \end{cases}$$

$$X = X_1(t) - X_2(t) = \begin{cases} t+\cos t & 0<t<\pi \\ \pi+\cos t - \sin t & t>\pi. \end{cases}$$