M111 Spectral Theory

Based on the 2013 autumn lectures by
Prof L Parnovski.

30/09/13

M111 (Spectral Theory)

Year:	2010–2011
Code:	MATHM111
Value:	Half unit
Term:	1
Structure:	3 hour lectures per week
Assessment:	90% examination, 10% coursework
Normal Pre-requisites:	7102, 3107 (when introduced), 3103 recommended
Lecturer:	Prof L Parnovski

Course Description and Objectives

Spectral theory came to prominence when quantum mechanics was introduced in modern physics. In quantum mechanics classical quantities (position, momentum etc) are represented by operators (bounded, unbounded, self-adjoint etc). The eigenvalues of these operators are the only precise measurements of the quantity. Without requiring knowledge of physics, this course introduces the fundamentals of such operators and the space of their eigenvalues, which is called the spectrum.

Recommended Texts

Recommended books are (i) E. Brian Davies, *Linear Operators and Their Spectra* (Cambridge Studies in Advanced Mathematics), (ii) W. Arveson, *A short course on spectral theory* (Springer Graduate Texts in Mathematics).

Detailed Syllabus

Banach and Hilbert Spaces. Orthogonal projections. Orthonormal bases. Fourier expansion. Riesz representation theorem. Linear operators (bounded and unbounded). Adjoint operator. Symmetric, normal, self-adjoint and compact operators. Resolvent. Spectra of linear operators: classification and properties. Spectral theorem for compact self-adjoint operators. Applications to differential and integral equations. Further topics are chosen from: spectral theorem for bounded self-adjoint operators; unbounded operators and applications; Fredholm operators; extensions of symmetric operators; quadratic forms.

30/09/13

M111
Leonid Parnovski
607 5pm Monday - Office hours.

—/—

$A: \mathbb{R}^n \rightarrow \mathbb{R}^n$

. λ is an eigenvalue if i) $Ax = \lambda x$ for some $x \in \mathbb{R}^n - \{0\}$ (x is an eigenvalue)

ii) $(A - \lambda I)$ is not a bijection.

iii) $\det(A - \lambda I) = 0$.

$$\det(A-\lambda I) = Ch_A(\lambda) = \det\begin{pmatrix} a_{11}-\lambda & a_{12} & \cdots \\ a_{21} & \ddots & \vdots \\ & & a_{nn}-\lambda \end{pmatrix} = (-1)^n(\lambda-\lambda_1)(\lambda-\lambda_2)\ldots(\lambda-\lambda_n)$$

$$= (-1)^n(\lambda-\mu_1)^{\alpha_1}(\lambda-\mu_2)^{\alpha_2}\ldots(\lambda-\mu_k)^{\alpha_k}$$

$\mu_j \neq \mu_l$, $j \neq l$.

The spectrum of A $\sigma(A) = \{\mu_1, \ldots, \mu_k\}$
α_j is an algebraic multiplicity of μ_j.

Def: The eigenspace corresponding to μ_j

is $V_{\mu_j} = \{x,\ Ax = \mu_j x\} \cup \{0\}$.

—/—

The geometric multiplicity of μ_j is $\beta_j := \dim V_{\mu_j}$

$$\boxed{\alpha_j \geq \beta_j}$$

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A - \lambda I = \begin{pmatrix} -\lambda & 1 \\ 0 & -\lambda \end{pmatrix}$$

$$Ch(\lambda) = \lambda^2.$$

$$\sigma(A) = \{0\}$$

$$\alpha = 2$$

$$Ax = 0$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} 0 \cdot x_1 + x_2 &= 0 \\ 0 &= 0 \end{aligned} \quad \Big| \ \beta = 1.$$

$$Ax = 0 \iff x = \begin{pmatrix} t \\ 0 \end{pmatrix}$$

$$\dim(V_0) = 1$$

$$J.N.F(A) = \begin{pmatrix} \begin{matrix} \mu_1 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \mu_1 \end{matrix} & & & \\ & \boxed{\begin{matrix} \mu_1 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \mu_1 \end{matrix}} & & \\ & & \mu_2 & \\ & & & \ddots \end{pmatrix}$$

<u>Th</u>: A is diagonalizable iff $\forall_j \ \alpha_j = \beta_j$

—/—

<u>Th</u>: If A is symmetric, then A is diagonalizable. $A^* = A$.

$A^* = \overline{(A^T)}$

<u>Th</u>: If A is symmetric, $A^* = A \Rightarrow$

$\sigma(A) \subset \mathbb{R}$.

—/—

$T^{-1} A T = D$

<u>Th</u>: $A^* = A$, then for some orthogonal (unitary in complex case) T we have $T^{-1} A T =$ diagonal.

<u>Def</u>: T is orthogonal (unitary) if $T^* = T^{-1}$

—/—

Example:

$$A = \begin{pmatrix} 0 & 2 & & & \\ & \ddots & 2 & & 0 \\ & & \ddots & \ddots & \\ & 0 & & & 2 \\ & & & & 0 \end{pmatrix} \qquad \text{Size: } (31 \times 31)$$

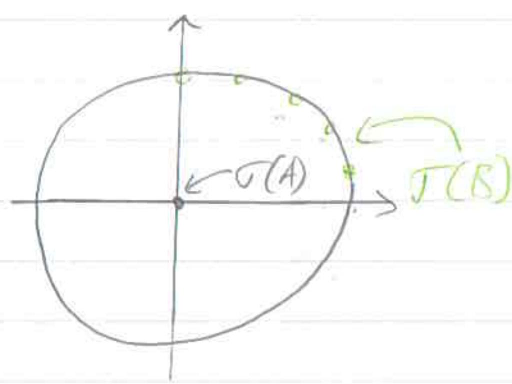
$$ch_A(\lambda) = -\lambda^{31}$$

$$\sigma(A) = \{0\}$$

$$B = \begin{pmatrix} 0 & 2 & & & \\ & \ddots & 2 & & 0 \\ & & \ddots & \ddots & \\ & 0 & & & 2 \\ 2^{-30} & & & & 0 \end{pmatrix} \qquad \text{Size: } (31 \times 31)$$

$$ch_B(\lambda) = -\lambda^{31} + 1$$

$$\sigma(B) = \sqrt[31]{1}$$



If A is symmetric such a problem cannot happen.

I. Introductcon:

$\mathbb{F} = \mathbb{R}, \mathbb{C}$

- a norm on a vector space X
$\|\cdot\| : X \to \mathbb{R}_+ = [0, +\infty)$ s.t.
$\|x\| = 0 \iff x = 0$
$\|\alpha x\| = |\alpha| \, \|x\| \quad \alpha \in \mathbb{F}$
$\|x+y\| \leq \|x\| + \|y\|$

If we define the distance

$$d(x, y) = \|x - y\|,$$

then X becomes a meteric space.

Def: We say that x_n converages to x (strongly), if $d(x_n, x) = \|x - x_n\| \to 0$ as $n \to \infty$.

Def: We say that $\{x_n\}$ is a Cauchy sequence, if $d(x_n, x_m) = \|x_n - x_m\| \to 0$ as $n, m \to \infty$.

—/—

Def: X is complete, if each Cauchy sequence converages.

Def: X is a normed space. X is a Banach space if it is complete.

Examples:

1) $\mathbb{C}^n$ with

$$\|x\| = \left[\sum_{j=1}^{n} |x_j|^2\right]^{1/2}$$

$$x = (x_1, \dots, x_n) \in \mathbb{C}^n$$

is a normed space and it is a Banach space.

2) $\ell_p =. \ell^p = \{ x = (x_1, x_2, \dots), \; x_j \in \mathbb{F},$

$$\sum_{j=1}^{\infty} |x_j|^p < +\infty \} \quad 1 \leq p < +\infty$$

$$\|x\|_p = \left[\sum_{j=1}^{\infty} |x_j|^p\right]^{1/p} \quad 1 \leq p < +\infty$$

is a norm space and Banach space.

$$\ell_\infty = \{ x = (x_1, x_2, \dots) \; x_j \in \mathbb{F},$$
$$\|x\|_\infty = \sup_{1 \leq j < +\infty} |x_j| < +\infty \}$$

Banach.

3) $\mathcal{X} = C[0,1]$, $f \in \mathcal{X}$.

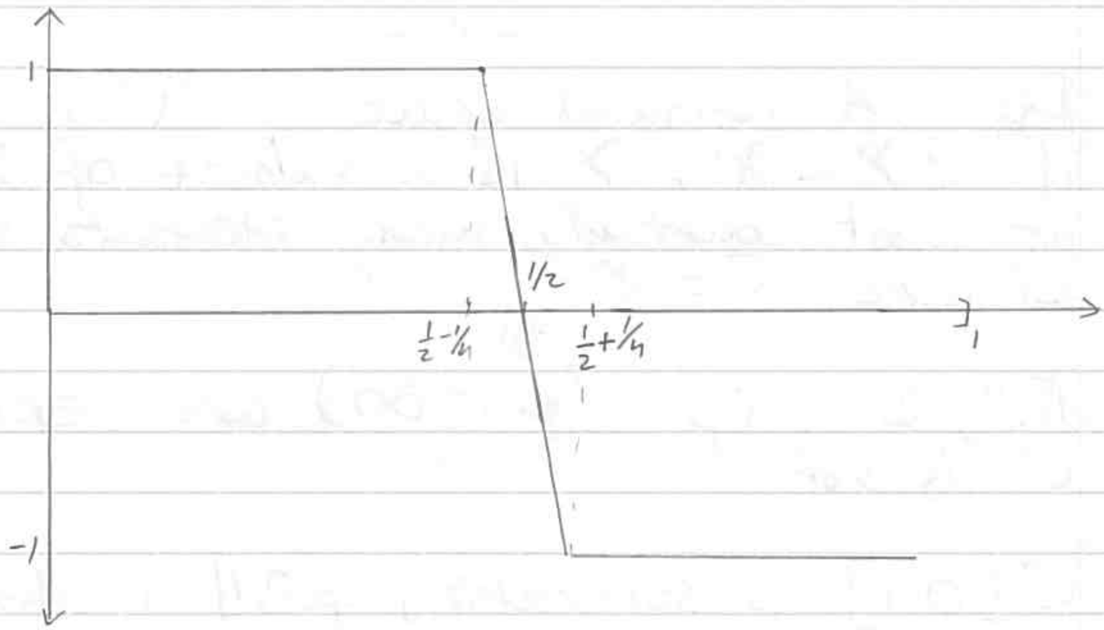
$$\|f\| = \sup_{t \in [0,1]} |f(t)| < +\infty$$

Banach.

4) $\mathcal{X} = C_p[0,1]$ $1 \leq p < +\infty$

$\mathcal{X} = C[0,1]$ as a set.

$f \in \mathcal{X}$, $\|f\|_p = \left[\int_0^1 |f(t)|^p \, dt \right]^{1/p}$.



$$f = \begin{cases} 1, & x < 1/2 \\ 0, & x = 1/2 \\ -1, & x > 1/2 \end{cases}$$

should be a limit of $\{f_n\}$. However, $f \notin C[0,1]$.

C_p, $\|f\|_p = \left[\int_0^1 |f(t)|^p dt\right]^{1/p}$

$L_p[0,1]$ is a set of all continuous functions and all limits of such functions in $\|.\|_p$.

C_p is normed, but incomplete.

L_p is complete.

—/—

<u>Def</u>: A normed space X is separable, if $\exists Y \subset X$, Y is a subset of X, Y has at most countably many elements and the closure $\overline{Y} = X$.

$\mathbb{R}^n$, $\mathbb{C}^n$, ℓ_p $(p < \infty)$ are separable, ℓ^∞ is not.

$C[0,1]$ is separable, $C_p[0,1]$ is also separable

—/—

$$\|f\|_p = \left[\int_0^1 |f(t)|^p dt\right]^{1/p}$$

$$\leq \left[\int_0^1 \|f\|^p dt\right]^{1/p} = \|f\|$$

Suppose, $\mathcal{X}$ and $\mathcal{Y}$ are normed space and

$f: \mathcal{X} \to \mathcal{Y}$

Def: f is called an isometery, if

1) $\|f(x)\|_{\mathcal{Y}} = \|x\|_{\mathcal{X}}$

2) f is surjection.

Def: f is continuous if any of the following holds:

1) If $Z \subset \mathcal{Y}$, Z is open, then $f^{-1}(Z)$ is open.

2) $x_n \to x$ in $\mathcal{X} \Rightarrow f(x_n) \to f(x)$ in $\mathcal{Y}$.

Def Let $\mathcal{X}$ and $\mathcal{Y}$ be vector spaces and

$$A: \mathcal{X} \to \mathcal{Y}$$

A is called linear operator, if

$$A(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 A x_1 + \alpha_2 A x_2$$

$x_1, x_2 \in \mathcal{X}$

$\alpha_1, \alpha_2 \in \mathbb{F}$.

Def - Th: Let $\mathcal{X}$ and Y be normed spaces and $A: \mathcal{X} \to Y$ be linear T.f.a.e

i) A is continuous.

ii) A is continuous at 0.

iii) A is bounded, i.e.

$$B^\circ_{\mathcal{X}}(0,1) = \{x \in \mathcal{X} ; \|x\| < 1\}.$$

$$A(B^\circ_{\mathcal{X}}(0,1)) \subset B^\circ_Y(0,C)$$

for some C.

iv) $\|Ax\| \leq C\|x\|$ for some $C > 0$.

If A satisfies any of these, it is called bounded operator.

Def Let $A: \mathcal{X} \to Y$ be bounded. Then the norm of A

$$\|A\| = \sup_{\substack{\|x\|=1 \\ x \in \mathcal{X}}} \|Ax\|_Y = \sup_{\substack{\|x\| \leq 1 \\ x \in \mathcal{X}}} \|Ax\|_Y$$

$$= \sup_{\substack{x \in \mathcal{X} \\ x \neq 0}} \frac{\|Ax\|}{\|x\|}$$

$$= \inf\{C > 0 \text{ st } \|Ax\|_Y \leq C\|x\|_{\mathcal{X}}\}$$

$$= \inf\{C > 0 \text{ st } \|Ax\|_Y \leq C \text{ for all } x, \|x\| = 1\}.$$

*Def: Suppose A, B are subset of a metric space X, A is dense with respect to B if B is contained in the closure of A, $B \subseteq \bar{A}$.

7/10/13

Office 607 Mathematics.

—/—

Th: (Banach Open mapping theorem) $A: X \to Y$ is linear, bounded, X, Y Banach, $\forall y \in Y$ $A^{-1}y \neq \emptyset$. $\Rightarrow$ A is open, i.e. A (open) is open.

Corollary 1: (Banach image mapping thm) If on top of the above A is bijection, then $A^{-1}: Y \to X$ is continuous.

Corollary 2: (Close graph theorem) If $A: X \to Y$, X, Y Banach; A is closed. Then A is bounded.

A is bounded	A is closed
$x_n \to x$ $\Rightarrow Ax_n \to Ax$	$x_n \to x \in X$ $Ax_n \to y \in Y$ $\Rightarrow Ax = y$.

Def: Let X, Y be Banach. The unbounded operator A is a mapping.

$$A : D_A \to Y$$

where D_A is a dense* linear subspace of X

Notation: $A \in L(X, Y)$ unbounded operators
$A \in B(X, Y)$ bounded operators.

Examples: 1. $X = C[0,1]$, $Y = \mathbb{R}$.

$$(A\ f) = \int_0^1 f(t)\, dt$$

$$\|A\| = \sup_{\substack{f,\ \|f\| \leq 1 \\ \Leftrightarrow \forall t\ |f(t)| \leq 1}} \left| \int_0^1 f(t)\, dt \right| \leq 1, \text{ so } A \text{ is bounded.}$$

However, if $|f(t)| \leq 1$ $\forall t$ then

$$\left| \int_0^1 f(t)\, dt \right| \leq \int_0^1 |f(t)|\, dt \leq \int_0^1 1\, dt = 1$$

on the other hand, if $f(t) = 1$ then $\|f\| = 1$ and $\|Af\| = 1$.

So $\|A\| = 1$

2. $X = C[0,1]$, $Af(t) = f'(t)$

$$D_A = \{ f \in C[0,1],\ f' \in C[0,1] \} := C^{(1)}[0,1]$$

$$A : D_A \to C[0,1]$$

Find $f_n \in C^{(1)}[0,1]$, $\|f_n\| = 1$, $\|f_n'\| \to \infty$ as $n \to \infty$
(then $\sup_{\|f\|=1} \|Af\| = \sup_{\|f\|=1} \|f'\| = +\infty = \|A\|$, so A is unbounded)

Take $f_n(t) = t^n$
$f_n(t) = \sin(nt)$.

$A = \frac{d}{dt}$ is unbounded.

<u>Theorem</u>: Let X and Y be normed. Then $\|\cdot\|$ is a norm in $B(X,Y)$ and $\|AB\| \le \|A\| \|B\|$, $A \in B(X,Y)$, $B \in B(Z, X)$ moreover, if Y is Banach, so is $B(X, Y)$.

<u>Def</u>: $B(X, X) =: B(X)$

<u>Def</u>: Let X be a normed space. The dual space $X^* = X' = B(X, F)$ is a collection of all, bounded linear functionals.

It is a Banach space.

Example: 1) $X = \ell_p$, $1 < p < +\infty$. Then $X^* \cong \ell_q$, where $1/p + 1/q = 1$. This means $\forall f \in X^*$ $\exists!$ ←(unique) $y = (y_1, y_2, \ldots) \in \ell_q$ st

$$\underset{(x_1, x_2, \ldots)}{\underset{\shortparallel}{f(x)}} = \sum_{j=1}^{\infty} x_j y_j$$

In particular, $\|f\|_{X^*} = \|y\|_{\ell_q}$

2) $(\ell_1)^* \cong \ell_\infty$, but $(\ell_\infty)^* \not\cong \ell_1$

—/—

<u>Def</u>: Let X be normed, $x_n, x \in X$

a) we say that x_n converages strongly to x ($x_n \to x$, or $s\text{-}\lim_{n\to\infty} x_n = x$) if $\|x_n - x\| \to 0$ as $n \to \infty$

b) x_n converages weakly to x ($x_n \rightharpoonup x$, or $w\text{-}\lim_{n\to\infty} x_n = x$) if $f(x_n) \to f(x)$

Thm $\forall f \in X^*$ $x_n \to x \Rightarrow x_n \rightharpoonup x$.

Proof: Suppose, $x_n \to x$ and $f \in X^*$. Then
$|f(x_n) - f(x)| = \|f(x_n - x)\|$

$\leq \|f\|_{X^*} \|x_n - x\|_X \to 0$ as $n \to \infty$

so $f(x_n) \to f(x)$.

Excercise: $\|Ax\| \leq \|A\| \|x\|$.

Example: $X = \ell_2$

$x_n = e_n = (\underbrace{0, \dots, 0, 1}_{n}, 0, \dots)$

we have $\|(x_n - x_m)\| = \sqrt{2} \overset{n,m\to 0}{\not\to} 0$ $n \neq m$

So x_n is not Cauchy and is not (strongly) convergent.

Claim: $w\text{-}\lim x_n = 0$.

Proof: Suppose, $f \in \ell_2^* \cong \ell_2$, and $f(x_n)$
$= \sum_{j=1}^{\infty} (x_n)_j y_j = y_n \overset{?}{\to} 0 = f(0)$ as $n \to \infty$ for some $y \in \ell_2$

$(y_1, y_2, \dots)$

$\sum_{j=1}^{\infty} |y_j|^2 < +\infty$

Def: X, Y are Banach, $A_n, A \in B(X,Y)$
i) A_n converages to A. (uniformly)

$\lim A_n = A$ $(A_n \to A)$

if $\|A_n - A\| \to 0$.

ii) A_n converages to A strongly

$(s - \lim\limits_{n\to\infty} A_n = A)$

if $A_n x \to Ax$ $\forall x \in X$ or $\|A_n x - Ax\| \to 0$.

iii) A_n converages to A weakly if $A_n x \rightharpoonup Ax$ $\forall x \in X$, or $\forall f \in Y^*$, $f(A_n x) - f(Ax) \to 0$.

Th.: uniform $\underset{\not\Leftarrow}{\Rightarrow}$ strong $\underset{\not\Leftarrow}{\Rightarrow}$ weak

Def: Let X be normed, $x_n, x \in X$.

i) we say that infinite series $\sum_{n=1}^{\infty} x_n$ converages to x, if the sequence of partial sum $S_n = \sum_{j=1}^{n} x_j$ converages

ii) $\sum_{n=1}^{\infty} x_n$ converages absolutely, if $\sum_{n=1}^{\infty} \|x_n\|$ converages.

Th.: If X is Banach, $\sum x_n$ conv abs $\Rightarrow$ it conv.

Proof: Put $S_n = \sum_{j=1}^{n} x_n$. Then for $m > n$.

$$\| S_n - S_m \| = \| \sum_{j=n+1}^{m} x_j \| \leq \sum_{j=n+1}^{\infty} \|x_j\| \xrightarrow{n\to\infty} 0$$

Thus, $S_n - S_m \to 0$, so $\{S_n\}$ is Cauchy, so it converges.

Operators.

Def: Let $A : X \to Y$ be linear

i) The kernal of A.

$$\text{Ker } A = \{ x \in X, Ax = 0 \}.$$

ii) The range of A

$$\text{Ran } A = \{ y \in Y, \exists x \in X \text{ st } Ax = y \}$$

If A is bounded, then Ker A is closed.

Def: A left (right) inverse to A is an operator A_l^{-1} (A_r^{-1}) st:

$$A_l^{-1} A = I_X \text{ or } A A_r^{-1} = I_Y$$

$A : X \to Y$

$A_l^{-1}, A_r^{-1} : Y \to X$.

Th: If A_l^{-1} exists and A_r^{-1} exist, then $A_l^{-1} = A_r^{-1}$ (and this is the inverse of A).

<u>Proof</u>: $A_l^{-1} = A_l^{-1}(AA_r^{-1}) = (A_l^{-1}A)A_r^{-1}$

$$= A_r^{-1}.$$

<u>Example</u>: $\mathcal{X} = \ell_1$

$$A(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$$

$$B(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$$

Then $AB(x_1, x_2, \ldots) = (x_1, \ldots)$, $\quad AB = I$
$ BA(x_1, x_2, \ldots) = (0, x_2, \ldots)$, $\quad BA \neq I$

Thus, $B = A_r^{-1}$, but $B \neq A_l^{-1}$.

<u>Remark</u>: A can be thought of as a matrix

$$A = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & 1 & \\ & & \ddots & \\ 0 & & 0 & \\ & & & \ddots \end{pmatrix}$$

and $\quad B = \begin{pmatrix} 0 & & & 0 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & 0 & \\ & & & \ddots \end{pmatrix}$

$\dim(\operatorname{Ker} A) = 1$
$\operatorname{Ran} A = \mathcal{X}$.

Note: $Ker(A)$ is not trivial so A is not invertiable.

Lemma: (1) If A and B are invertible then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

(2) If AB is invertiable and A and B commute ($AB = BA$), then A and B are invertible.

(3) If A and B commute, A^{-1} exists then A^{-1} and B commuted.

Proof: (2) AB is invertiable $\Rightarrow \exists s = (AB)^{-1}$ st $ABS = \underset{\overset{\|}{SBA}}{SAB} = I$. Then $BS = A_r^{-1}$ and $SB = A_l^{-1}$

(3): $A^{-1}B = A^{-1}BAA^{-1} = A^{-1}ABA^{-1}$
$= BA^{-1}$.

Lemma (First perturbatcoin lemma) Let X be Banch, $A \in B(X)$, $\|A\| < 1$. Then $I - A$ is invertible,

$$(I-A)^{-1} = \sum_{n=0}^{\infty} A^n.$$ (von Neumann series)

and $\|(I-A)^{-1}\| \leq \frac{1}{1-\|A\|}$

Proof: we have: $\|A^n\| = \|A^{n-1}A\| \leq \|A\|$, $\|A^{n-1}\| \leq \|A\|^n$. Therefore, $\sum_{n=0}^{\infty} \|A^n\| \leq \sum \|A\|^n < +\infty$ and $\sum_{n=0}^{\infty} A^n$ converages absolutely. Therefore since $B(\mathcal{X})$ is Banach, the series $\sum_{n=0}^{\infty} A^n$ converages to $R \in B(\mathcal{X})$. Then

$$(I-A)R = (I-A) \lim_{n\to\infty} \sum_{j=1}^{n} A^j$$

$$= \lim_{n\to\infty} (I-A) \sum_{j=0}^{n} A^j$$

$$= \lim_{n\to\infty} (I - A^{n+1}) = I$$

since $\|A^{n+1}\| \leq \|A\|^{n+1} \to 0$ as $n \to \infty$

Simarlarly, $R(I-A) = I$

14/10/13

$$c_0 = \{ x = (x_1, \dots) : \lim_{n\to\infty} x_n = 0 \}$$

$$\|x\|_{c_0} = \max_{1\le j<+\infty} |x_j|$$

$$c_0^* \cong \ell_1$$

Lemma: Suppose, $\|A\|<1$. Then

$$(I-A)^{-1} = \sum_{n=0}^{\infty} A^n$$

and $\|(I-A)^{-1}\| \leq \dfrac{1}{1-\|A\|}$

The rest of the proof

Proof: We have

$$\|(I-A)^{-1}\| = \|\sum_{n=0}^{\infty} A^n\|$$

$$= \|\lim_{N\to\infty} \sum_{n=0}^{N} A^n\|$$

$$= \lim_{N\to\infty} \|\sum_{n=0}^{N} A^n\|$$

$$\leq \lim_{N\to\infty} \sum_{n=0}^{N} \|A\|^n$$

$$= \lim_{N\to\infty} \frac{1-\|A\|^{N+1}}{1-\|A\|}.$$

$$= \frac{1}{1-\|A\|}$$

Excercise: suppose $A_n \to A$ uniformly, Then $\|A_n\| \to \|A\|$ [Hint use triangle inequality]

Lemma: (The second perturbatcoi lemma) Let X and Y be Banach spaces, $A, B \in B(X, Y)$, is invertible and $\|B\| < 1/\|A^{-1}\|$. Then $(A+B)$ is invertible,

$$(A+B)^{-1} = \left[\sum_{n=0}^{\infty} (-A^{-1}B)^n\right] A^{-1}$$

$$= A^{-1}\left[\sum_{n=0}^{\infty} (-BA^{-1})^n\right]$$

and

$$\|(A+B)^{-1}\| \leq \frac{\|A^{-1}\|}{1 - \|B\|\|A^{-1}\|}$$

Proof: We have:

$$(A+B) = A(I + A^{-1}B)$$

$$= A(I - (-A^{-1}B))$$

Since $\|-A^{-1}B\| \leq \|A^{-1}\|\|B\| < 1$, by the first perturbatcoi lemma $(I + A^{-1}B)$ is invertible. Since A is invertiable, $(A+B)^{-1}$ exist, also $(A+B) = (I + BA^{-1})A$.

Everything follows from the 1-st perturbatcoi lemma.

Def: ($\mathbb{F} = \mathbb{C}$) Let $\mathcal{X}$ be Banach space and $A \in B(\mathcal{X})$. We say that $\lambda \in \mathbb{C}$ is a resolvent point, if $(A - \lambda I)$ is invertiable i.e $(A - \lambda I)^{-1} : \mathcal{X} \to \mathcal{X}$ exists (and is bounded by the Banach inverse mapping theorem).

The collectcoin of all resolvent points is called a resolvent set and is denoted by $\rho(A)$. The specturm of A is the complement of $\rho(A)$. $\sigma(A) = \mathbb{C} \setminus \rho(A)$ i.e. $\sigma(A)$ is a collectcoin of $\lambda \in \mathbb{C}$ st $(A - \lambda I)$ is not invertiable or $(A - \lambda I)^{-1}$ is unbounded.

Def: Let $\mathcal{X}$ be Banach and $A : D_{\mathcal{X}} \to \mathcal{X} \in L(\mathcal{X})$. We say that $\lambda \in \mathbb{C}$ is a resolvent point, if $(A - \lambda I)$ is invertible, $(A - \lambda I)^{-1} : \mathcal{X} \to \mathcal{X}$ exist and is bounded

Def: $\lambda \in \mathbb{C}$ is called an eigenvalue of A, if $\exists x \in \mathcal{X}$, $x \neq 0$ (called an eigenvector) s.t.

$$Ax = \lambda x.$$

Th. λ is an eigenvalue $\Rightarrow \lambda \in \sigma(A)$

Proof: $Ax = \lambda x \Rightarrow x \in \mathrm{Ker}(A - \lambda I)$. If $x \neq 0$, this implies $\mathrm{Ker}(A - \lambda I) \neq \{0\}$, so $(A - \lambda I)$ is not a bijectcoin, so $\lambda \in \sigma(A)$.

$\square$

Example 1) If $\dim \mathcal{X} < +\infty$, then $\sigma(A) = \{$ eigenvalue of $A \}$.

2) $\mathcal{X} = C[0,1]$.

$A : \mathcal{X} \to \mathcal{X}$, $f \in \mathcal{X}$.

$(Af)(t) = t f(t)$.

$A \in B(\mathcal{X})$, $\|A\| = 1$

Eigenvalue:

$(Af)(t) = \lambda f(t)$.

$t f(t) = \lambda f(t)$

$(t - \lambda) f(t) = 0 . \Rightarrow f \equiv 0$.

Therefore, A has no eigenvalues.

$(A - \lambda I) : f(t) \to (t - \lambda) f(t)$.

$$\underset{\displaystyle (t-\lambda) f(t)}{\underset{\shortparallel}{(A - \lambda I) f(t)}} = g(t)$$

$f(t) = \dfrac{g(t)}{t - \lambda}$ - fine for $\lambda \notin [0,1]$.

$\sigma(A) = [0,1]$.

Lemma: X is Banach, $A \in B(X)$. Then $\sigma(X)$ is a compact subset of $\mathbb{C}$ moreover

$$\sigma(A) \subset B_c(0, \|A\|) \leftarrow \text{closed}$$

$$= \{\lambda \in \mathbb{C}, |\lambda| \le \|A\|\}.$$

Proof: Suppose, $\lambda \notin B_c(0, \|A\|)$, i.e. $|\lambda| > \|A\|$. Then

$$(A - \lambda I) = -\frac{1}{\lambda}\left(I - \frac{A}{\lambda}\right).$$

Since: $\left\|\frac{A}{\lambda}\right\| = \frac{\|A\|}{|\lambda|} < 1$

first perturbation lemma implies that $(A - \lambda I)$ is invertible. Thus $\lambda \in \rho(A)$, so $\sigma(A) \subset B_c(0, \|A\|)$

Now, let us prove that $\sigma(A)$ is closed. This follow from...

Claim: Suppose, $\lambda_0 \in \rho(A)$ and

$$|\lambda - \lambda_0| < \frac{1}{\|(A - \lambda_0 I)^{-1}\|}$$

Then $\lambda \in \rho(A)$.

Proof of claim

$$A - \lambda I = (A - \lambda_0 I) + (\lambda_0 - \lambda) I$$

since $\|(\lambda_0 - \lambda) I\| = |\lambda - \lambda_0|$

$$< \frac{1}{\|(A - \lambda_0 I)^{-1}\|},$$

we apply the second perturbation lemma to deduce that $(A - \lambda I)^{-1}$ exist and $\lambda \in \rho(A)$.

Example. $Ax = (x_2, x_3, \ldots)$ $\quad x = (x_1, x_2, \ldots)$

$A : \ell_1 \to \ell_1$

$$\|Ax\| = \sum_{j=2}^{\infty} |x_j| \leq \sum_{j=1}^{\infty} |x_j| = \|x\|,$$

So $\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} \leq 1$

Take $x = (0, 1, 0, 0, \ldots)$. Then $\|Ax\| = \|x\|$.

Thus, $\|A\| = 1$

Thus $\sigma(A) \subset B_c(0, 1)$

Find eigenvalues.

$$\underset{(x_2, x_3, \ldots)}{\underset{\|}{Ax}} = \underset{(\lambda x_1, \lambda x_2, \ldots)}{\underset{\|}{\lambda x}}, \quad x = (x_1, x_2, \ldots)$$

$x_2 = \lambda x_1$ $\qquad$ x_1 is arbitory

$x_3 = \lambda x_2$ $\qquad$ $x_2 = \lambda x_1$

$x_4 = \lambda x_3$ $\qquad$ $x_3 = \lambda^2 x_1$

$\vdots$ $\qquad$ $x_4 = \lambda^3 x_1$

$\vdots$

$x_n = \lambda^{n-1} x_1$

Therefore $x = (x_1, \lambda x_1, \lambda^2 x_1, \dots)$

$= x_1 (1, \lambda, \lambda^2, \dots)$

we have $x \in l_1 \Leftrightarrow \sum_{j=0}^{\infty} \lambda^j$ converges.

$\Leftrightarrow |\lambda| < 1$.

Therefore, each λ s.t. $|\lambda| < 1$ is an eigenvalue.

Therefore:

$B_0(0,1) \subset \sigma(A) \subset B_c(0,1)$

$\overline{B_0(0,1)} \subset \overline{\sigma(A)}$

$\overline{B_0(0,1)} = B_c(0,1)$, $\overline{\sigma(A)} = \sigma(A)$.

so $\sigma(A) = B_c(0,1)$

Def: X is Banach, $A \in B(X)$. The operator-valued functcoin.

$$\rho(A) \ni \lambda \longmapsto R(A;\lambda) = (A - \lambda I)^{-1} \in B(X)$$

is called the <u>resolvent</u> of A.

<u>Lemma</u>: (the resolvent equation).

$$R(A;\lambda) - R(A;\lambda_0)$$
$$= (\lambda-\lambda_0)R(A;\lambda)R(A,\lambda_0)$$

if $\lambda, \lambda_0 \in \rho(A)$.

<u>Proof</u>:

$$R(A;\lambda) - R(A;\lambda_0)$$
$$= (A-\lambda I)^{-1} - (A-\lambda_0 I)^{-1}$$
$$= (A-\lambda I)^{-1}\left[(\cancel{A}-\lambda_0 I) - (\cancel{A}-\lambda I)\right](A-\lambda_0 I)^{-1}$$
$$= (\lambda-\lambda_0)(A-\lambda I)^{-1}(A-\lambda_0 I)^{-1}.$$

<u>Excercise</u>: $R(A;\lambda) - R(B;\lambda)$
$= R(A;\lambda)(B-A)\, R(B;\lambda)$.

—/—

<u>Th</u>: Z is a (complex) Banach space, $\Omega \subset \mathbb{C}$ is open, and $F: \Omega \to Z$ be a vector-valued function. T.f.a.e.

i) $\forall \lambda \in \Omega$ there is a (strong) derivate
$$\frac{dF}{d\lambda}(\lambda_0) = F'(\lambda_0) = \lim_{\lambda\to\lambda_0} \frac{F(\lambda) - F(\lambda_0)}{\lambda-\lambda_0} \in Z$$

i.e. $\left\| \frac{F(\lambda) - F(\lambda_0)}{\lambda - \lambda_0} - F'(\lambda_0) \right\| \xrightarrow{\lambda \to \lambda_0} 0$

(ii) $\forall \lambda_0 \in \Omega$. $F(\lambda)$ can be expressed as a sum of Taylor series in some neighbourhood of λ_0:

$$F(\lambda) = \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n F_n(\lambda),$$

$F_n(\lambda) \in Z$, and the series converages absolutely.

iii) $\forall G \in Z_1^*$ the complex-valued function $G(F(\lambda))$ is analytic in Ω.

If $Z_1 = B(X, Y)$ for some Banach X and Y, then (i)-(iii) are equivalent to (iv) $\forall x \in X$, $g \in Y^*$ is analytic complex-valued function in Ω.

Def: A vector-valued (operator-valued) function is called analytic in Ω if it satisfies i-iii (i-iv) of the theorem above.

Th: The $B(X)$-valued function $R(A, \cdot)$ is analytic in $\rho(A)$ and has the following properties.

$$\left. \frac{dR(A, \lambda)}{d\lambda} \right|_{\lambda = \lambda_0} = R^2(A; \lambda_0) \quad \forall \lambda_0 \in \rho(A)$$

$-\lambda R(A;\lambda) \to I \quad , \quad |\lambda| \to +\infty.$

$$\|R(A;\lambda)\| \geq \frac{1}{d(\lambda,\sigma(A))} \qquad \lambda \in \rho(A).$$

where $d(\lambda,\sigma(A)) = \inf_{\mu\in\sigma(A)} |\lambda-\mu|$.

<u>Proof</u>: $R(A;\lambda) - R(A;\lambda_0)$

$= (\lambda-\lambda_0)R(A;\lambda)R(A;\lambda_0)$.

as $\lambda \to \lambda_0$, the norm of the RHS $\to 0$
so $R(A;\lambda) \xrightarrow{\lambda\to\lambda_0} R(A;\lambda_0)$, now:

$$\lim_{\lambda\to\lambda_0} \frac{R(A;\lambda) - R(A;\lambda_0)}{\lambda-\lambda_0}.$$

$$\lim_{\lambda\to\lambda_0} R(A;\lambda)R(A;\lambda_0) = R^2(A;\lambda_0).$$

21/10/13

Th i) $\left.\dfrac{dR(A;\lambda)}{d\lambda}\right|_{\lambda=\lambda_0} = R^2(A;\lambda)$.

ii) $-\lambda R(A;\lambda) \to I \qquad |\lambda| \to \infty$

$\uparrow$ bounded

iii) $\|R(A;\lambda_0)\| \geq \dfrac{1}{d(\lambda_0, \sigma(A))}$, $\lambda_0 \in \rho(A)$

Proof: ii) $\|-\lambda R(A;\lambda) - I\|$

$= \|-\lambda(A - \lambda I)^{-1} - I\|$

$= \|(I - \lambda^{-1}A)^{-1} - I\|$

$\overset{|\lambda|>\|A\|}{=} \|\sum_{n=1}^{\infty} \lambda^{-n} A^n\| \leq \sum_{n=1}^{\infty} |\lambda|^{-n}\|A\|^n$

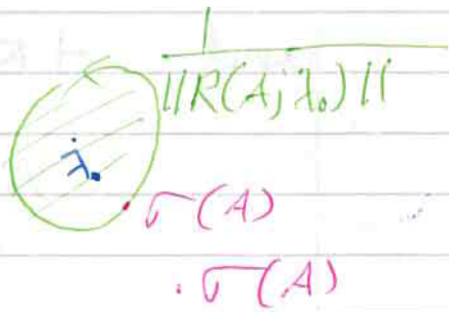
$= -1 + \dfrac{1}{1-|\lambda|^{-1}\|A\|} = \dfrac{|\lambda|^{-1}\|A\|}{1-|\lambda|^{-1}\|A\|} \xrightarrow{|\lambda|\to\infty} 0$

iii) If $\lambda_0 \in \rho(A)$ and $d(\lambda, \lambda_0) = |\lambda - \lambda_0|$.

$< \dfrac{1}{\|R(A;\lambda_0)\|}$ then $\lambda \in \rho(A)$ (have proved already)

Therefore:

$d(\lambda_0, \sigma(A)) \geq \dfrac{1}{\|R(A;\lambda_0)\|}$



and $\|R(A;\lambda_0)\| \geq \dfrac{1}{d(\lambda_0, \sigma(A))}$.

Lemma: $A \in B(X)$ (bounded). Then $\sigma(A) \neq \emptyset$

Proof: Suppose, $\rho(A) = \mathbb{C}$.

Take any $x \in X \setminus \{0\}$, $g \in X^*$. Then

$f(\lambda) := g(R(A;\lambda)x)$

is analytic in $\rho(A) = \mathbb{C}$, $f(\lambda) \to 0$ as $|\lambda| \to \infty$. Liouville's theorem implies $f(\lambda) \equiv 0$.

Claim: Suppose, $z \in X$. Then $\exists g \in X^*$ st $\|g\| = 1$ and $g(z) = \|z\|$ (follows from Hahn–Banach theorem).

This claim implies that $\exists g \in X^*$ st $\|g\| = 1$ and $g(R(A;0)x) = \|R(A;0)x\|$

$= f(0) = 0$.

This, $R(A,0)x = 0$,

and $AR(A;0)x = A0$.

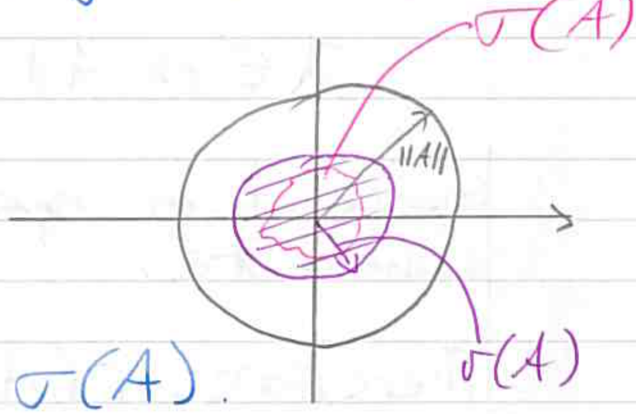
$$\begin{array}{ccc} AR(A;0)x & = & A0 \\ \| & & \| \\ AA^{-1}x & & 0 \\ \| & & \\ x & & \end{array}$$

This contradicts our assumption that $x \neq 0$.

Theorem: Let $\mathcal{X}$ be Banach and $A \in B(\mathcal{X})$. Then $\sigma(A)$ is non-empty, closed bounded set in $\mathbb{C}$.

$$\sigma(A) \subseteq \{\lambda \in \mathbb{C},\ |\lambda| \leq \|A\|\},$$

$(A-\lambda I)^{-1}$ is an analytic function in $\rho(A)$.



Def: The sprectral radius of A is the radius of the smallest disk centred at 0 and containing $\sigma(A)$.

$$r(A) = \overset{(max)}{\sup}\{|\lambda|;\ \lambda \in \sigma(A)\}$$

Obviously, $r(A) \leq \|A\|$

Thm $A \in B(\mathcal{X})$. Then

$$r(A) = \lim_{n\to\infty} \|A^n\|^{1/n}$$

Proof: I will proved that
2) $r(A) \geq \limsup \|A^n\|^{1/n}$ and
1) $r(A) \leq \liminf \|A^n\|^{1/n}$

Proof of (i)

Claim: Suppose, $\lambda \in \sigma(A)$. Then $\lambda^n \in \sigma(A^n)$

Indeed, $(A^n - \lambda^n I)$

$= (A - \lambda)(A^{n-1} + \lambda A^{n-2} + \dots + \lambda^{n-2} A + \lambda^{n-1} I)$

so if $\lambda^n \in \rho(A^n)$, then:

$\lambda \in \rho(A)$.

Since two operators in the RHS commute.

Therefore, $r(A)^n = [\sup\{|\lambda|, \lambda \in \sigma(A)\}]^n$

$= \sup\{|\lambda|^n : \lambda \in \sigma(A)\}$

$= \sup\{|\lambda^n| : \lambda \in \sigma(A)\}$.

$\leq \sup\{|\mu| : \mu \in \sigma(A^n)\}$.

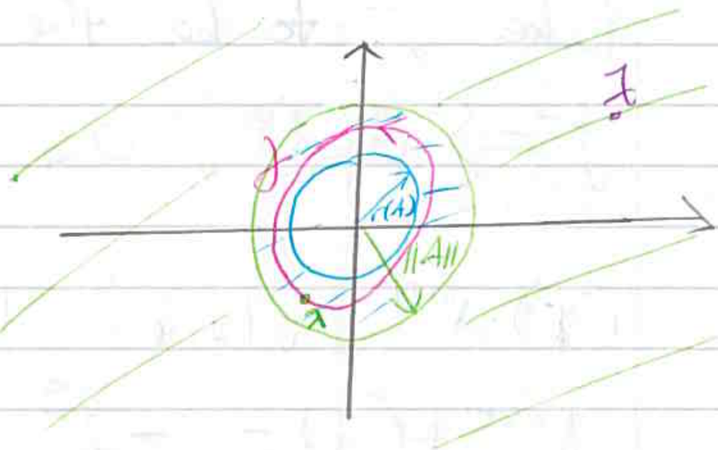
$= r(A^n)$

$\leq \|A^n\|$, so

$r(A) \leq \|A^n\|^{1/n}$ and

$r(A) \leq \liminf \|A^n\|^{1/n}$

<u>Proof of (ii)</u>



Suppose,

$|\lambda| > \|A\|$

Then $\lambda \notin \sigma(A)$
and $R(A;\lambda)$
$= (A - \lambda I)^{-1}$
$= (-\lambda)\left(I - \dfrac{A}{\lambda}\right)^{-1}$

$= -\sum_{n=0}^{\infty} \lambda^{-n-1} A^n$

Now take $x \in X$, $g \in X^*$, and define

$f(\lambda) := g(R(A;\lambda)x)$

Then, for $|\lambda| > \|A\|$ we have

$$f(\lambda) = -\sum_{n=0}^{\infty} \lambda^{-n-1} g(A^n x) \quad (*)$$

Since $f(\lambda)$ is analytic in

$\{\lambda \in \mathbb{C},\ |\lambda| > r(A)\}$.

Laurent's theorem implies that

$$f(\lambda) = -\sum_{n=0}^{\infty} \lambda^{-n-1} g(A^n x)$$

in this region $\{\lambda \in \mathbb{C},\ |\lambda| > r(A)\}$

Take γ to be the contour

$$\gamma = \{ ae^{i\theta},\ 0 \leq \theta < 2\pi \},\ a > r(A).$$

$(*) \cdot \lambda^{m+1}$ (where λ is on the contour)

$$\lambda^{m+1} f(\lambda) = -\sum_{n=0}^{\infty} \lambda^{m-n} g(A^n x)$$

$$\Rightarrow a^{m+1} e^{i(m+1)\theta} f(ae^{i\theta}) = -\sum_{n=0}^{\infty} a^{m-n} e^{i\theta(m-n)} g(A^n x)$$

$$\Rightarrow \int_0^{2\pi} a^{m+1} e^{i(m+1)\theta} f(ae^{i\theta})\, d\theta$$

$$= -\int_0^{2\pi} \sum_{n=0}^{\infty} a^{m-n} e^{i\theta(m-n)} g(A^n x)\, d\theta$$

$$= -2\pi g(A^m x)$$

Therefore

$$|g(A^m x)| = \frac{1}{2\pi} \left| \int_0^{2\pi} a^{m+1} e^{i(m+1)\theta} f(ae^{i\theta})\, d\theta \right|$$

$$\leq \frac{a^{m+1}}{2\pi} \int_0^{2\pi} |g(R(A,\lambda)x)|\, d\theta \leq a^{m+1} M(a) \|g\| \cdot \|x\|$$

where $M(a) = \sup_{0 \leq \theta \leq 2\pi} \| R(A; ae^{i\theta}) \|$

Take $g \in \mathcal{X}^*$ st $\|g\|=1$ and $g(A^m x) = \|A^m x\|$

Then

$$\frac{\|A^m x\|}{\|x\|} \leq a^{m+1} M(a)$$

and $\|A^m\| = \sup \frac{\|A^m x\|}{\|x\|} \leq a^{m+1} M(a)$

so $\|A\|^{1/m} \leq a a^{1/m} M(a)^{1/m}$

and $\limsup_{m\to\infty} \|A^m\|^{1/m} \leq \underbrace{\limsup_{m\to\infty} [a a^{1/m} M(a)^{1/m}]}_{= a}$

Since this inequality holds for all $a > r(A)$, we have $\limsup \|A^m\|^{1/m} \leq r(A)$. $\square$

Example: $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\sigma(A) = \{0\}$

$\underbrace{r(A)}_{= \lim \|A^n\|^{1/n}} = 0$. and $\|A\| > 0$.

$A: e_1 \to 0$
$A: e_2 \to e_1$

Since $A^2 = 0$, $A^n = 0$, so $\lim \|A^n\|^{1/n} = 0$.

Th: (The spectral mapping theorem) Let

$$p(z) = a_N z^N + a_{N-1} z^{N-1} + \dots + a_0 .$$

be a polynomial with $a_N \neq 0$. Let $A \in B(X)$. Then $\sigma(p(A)) = p(\sigma(A))$.

$$= \{ p(z),\ z \in \sigma(A) \}.$$

Proof: Let $\mu \in \mathbb{C}$. Denote by $\lambda_1, \lambda_2, \dots, \lambda_N$ solutions of $p(z) = \mu$ so that

$$p(z) - \mu = a_N (z - \lambda_1)(z - \lambda_2) \dots (z - \lambda_N)$$

Then: $p(A) - \mu I = a_N (A - \lambda_1 I)(A - \lambda_2 I) \dots (A - \lambda_N I)$

Since the operators in the RHS commute, we have

$\mu \notin \sigma(p(A)) \iff p(A) - \mu I$ is invertible.

$\iff \forall j\ (A - \lambda_j I)$ is invertiable

$\iff \forall j\ \ \lambda_j \notin \sigma(A)$

$\iff \mu \notin p(\sigma(A))$

so $p(\sigma(A)) = \sigma(p(A))$

(Comes up in exam) $\square$

Projections

Def: An operator $P \in B(X)$ is called a projection, if $P = P^2$.

Lemma: Let P be a projection. Then $Q := I - P$ is a projection $PQ = QP = 0$ and $\operatorname{Ran} P = \operatorname{Ker} Q$, $\operatorname{Ker} P = \operatorname{Ran} Q$.

Proof: 1) $Q^2 = (I - P)^2$

$$= I - 2P + P^2$$

$$= I - P = Q$$

2) $PQ = P(I - P)$

$$= P - P^2$$

$$= P - P = 0.$$

3) Since $QP = 0$, we have $\operatorname{Ran} P \subset \operatorname{Ker} Q$. On the other hand, let $x \in \operatorname{Ker} Q$

Then $\underset{(I-P)x = x - Px.}{\overset{Qx = 0}{\|}}$

and $x = Px$, so $x \in \operatorname{Ran} P$.

Therefore, $\operatorname{Ran} P = \operatorname{Ker} Q$.

Replacing P and Q, we prove $\operatorname{Ker} P = \operatorname{Ran} Q$.

<u>Def</u>: X is a direct sum of U and W if $U \cap W = \emptyset$, $\underbrace{U + W}_{\{u+w \,:\, u \in U,\, w \in W\}} = X$.

Notation: Can be $X = U \dot{+} W$
$= U \oplus W$

<u>Lemma</u>: Let $P^{\in B(X)}$ be a projection. Then

$$X = \operatorname{Ker} P \oplus \operatorname{Ran} P$$

and $\operatorname{Ran} P$ is closed.

<u>Proof</u>: $\operatorname{Ran} P = \operatorname{Ker}(I - P)$ is closed.

We have: $I = P + (I - P)$

Therefore, $x = \underbrace{Px}_{\in \operatorname{Ran} P} + \underbrace{(I-P)x}_{\in \operatorname{Ran}(I-P)}$ and
$X = (\operatorname{Ran} P) + \underbrace{\operatorname{Ran}(I - P)}_{= \operatorname{Ker} P}$.

Now we need to check that

$$\operatorname{Ker} P \cap \operatorname{Ran} P = \{0\}.$$

Suppose, $x \in \operatorname{Ker} P \cap \operatorname{Ran} P$

Then $x = Py$ for $y \in \mathcal{X}$.

Therefore $\underset{\substack{\shortparallel \; x \in \operatorname{Ker} P \\ 0}}{Px} = P^2 y = Py = x$, so $x = 0$.

Next: Compact sets
Compactness.

28/10/13.

Used in exam, if the operator is ugly.

Th: Suppose $P \in B(X)$ is a projection, $P \neq 0$, $P = I$. Then $\sigma(P) = \{0, 1\}$.

Proof:

$$\{0\} = \sigma(0) = \sigma(P^2 - P):$$

By the spectral mapping theorem.

$$= \{\lambda^2 - \lambda : \lambda \in \sigma(P)\}$$

So $\sigma(P) \subset \{0, 1\}$

Since $P \neq 0$, we have $\underbrace{\operatorname{Ran} P}_{= \operatorname{Ker}(I-P)} \neq \{0\}$,

So $(P - I)$ is not invertiable and $1 \in \sigma(P)$.

Similarly, $P \neq I$, so $\overbrace{\operatorname{Ran}(I - P)}^{= \operatorname{Ker}(P)} = \{0\}$, so P is not invertiable, so $0 \in \sigma(P)$.

Compact operators.

Def: Let X be a normed space. A set $K \subset X$ is relatively compact, if each sequence in K has a Cauchy subsequence.

K is called compact if each sequence has a subsequence converging to an element of K.

Prop: Relatively comp. $\Rightarrow$ bounded, compact $\Rightarrow$ closed and bounded.

A subset of rel. comp. set is rel. comp.

A closed subset of comp set is comp.

Th: If $\dim X < +\infty$, then K is comp $\Leftrightarrow$ K is closed and bounded.

K is rel. comp. $\Leftrightarrow$ K is bounded.

Def: Suppose, $\|\cdot\|$ and $\|\cdot\|$ are norms in a vector space X. They are equivalent, if $\exists\, c_1, c_2 > 0$ st

$$c_1\|x\| \leq \|x\|' \leq c_2\|x\| \quad \forall x \in X.$$

Th: If $\|\cdot\|$ and $\|\cdot\|'$ are two norms on X, and $\dim X < +\infty$, then these norms are equivalent.

Corollary: Any finite dimensional subspace of a normed space X is closed.

Proof: Suppose, $X_0 \subset X$,

$$\dim X_0 = n < +\infty.$$

Let $\{e_1, \dots, e_n\}$ be a basis of X_0, i.e. $\forall x \in X_0 \ \exists!\ x = \sum_{k=1}^{n} c_k e_k$.

unique

Define:

$$\|x\|' := \max_{k=1,\dots,n} |c_k|$$

The theorem implies that the norm in X, restricted to X_0, is equivalent to $\|.\|'$.

Suppose, $\{x_m\}$ is a sequence in X_0 st. $x_m \to x$ in the norm of X. Then $\{x_m\}$ is a Cauchy seq. in the norm $\|.\|$ (the norm in X). Then $\{x_n\}$ is Cauchy seq. in $\|.\|$. Put $x_m = \sum_{k=1}^n c_k^{(m)} e_k$. Then for each $k = 1, \dots, n$, the sequence $\{c_k^{(m)}\}_{m=1}^\infty$ is a Cauchy sequence of numbers.

It must converge say $\lim_{m\to\infty} c_k^{(m)} = c_k$, and put $x' \overset{\in X_0}{=} \sum_{k=1}^n c_k e_k$. Then $x_m \to x'$ in $\|.\|$. Since $\|.\|'$ and $\|.\|$ are equivalent, $x_m \to x'$ in $\|.\|$. Therefore $x = x' \in X_0$

$\square$

<u>Def</u>: Let X and Y be normed spaces. A linear operator, $T: X \to Y$ is called <u>compact</u> (completely continuous) if T maps bounded set in X into relatively compact sets in Y. (i.e. if $\{x_n\}$ is a bounded sequence in X, $\exists$ a subsequence $\{x_{n_k}\}$ st $\{Tx_{n_k}\}$ is Cauchy in Y). The set of all compact operators from X to Y is denoted by <u>$Com(X, Y)$</u>.

Suppose, $T \in Com(X, Y)$. Then $T(B_c(0,1)_X)$ is rel. comp. (in Y) and is therefore bounded. Thus, T is bounded and $Com(X, Y) \subset B(X, Y)$.

<u>Lemma</u>. $T \in B(X, Y)$ is compact iff $T(B_c(0,1)_X)$ is relatively compact in Y.

<u>Proof</u>: $(\Rightarrow)$ obvious.

$(\Leftarrow)$ Suppose $W \subset X$ is bounded. Then $W \subset B_c(0, R)_X = R B_c(0,1)_X$ for some $R > 0$.

Then $TW \subset T(B_c(0,1)_X)$

$$= R \, T(B_c(0,1)_X)$$ relatively comp

is relatively compact $\square$.

<u>Th</u>: Let X be normed space

i) If $T_1, T_2 \in Com(X) = Com(X, X)$ and $\alpha_1, \alpha_2 \in \mathbb{F}$, then $\alpha_1 T_1 + \alpha_2 T_2 \in Com(X)$.

ii) If $T \in Com(X)$ and $A \in B(X)$ then TA, $AT \in Com(X)$

iii) If $T_n \in Com(X)$, $\|T_n - T\| \to 0$ as $n \to \infty$ then $T \in Com(X)$.

<u>Proof</u>: i) Suppose, $x_n \in X$ is a bounded seq. Since T_1 is compact, $\exists$ a subsequence $\{x_n^{(1)}\}$ st $T_1 x_n^{(1)}$ is Cauchy. Since T_2 is comp

$\exists$ a subsequence of $\{x_n^{(1)}\}$ say $\{x_n^{(2)}\}$ s.t. $T_2 x_n^{(2)}$ is Cauchy. Also $T_1 x_n^{(2)}$ is Cauchy

Therefore $\{(\alpha_1 T_1 + \alpha_2 T_2) x_n^{(2)}\}$ is Cauchy. Thus $\alpha_1 T_1 + \alpha_2 T_2 \in Com(\mathcal{X})$.

ii) $T[A(\text{bounded set})] = T(\text{bounded set})$ $=$ relatively comp. set.

Thus, $TA \in Com(\mathcal{X})$. Suppose now than $\{x_n\}$ is bounded sequence. Then $\exists$ a subsequence $\{x_n^{(1)}\}$ s.t. $\{Tx_n^{(1)}\}$ is Cauchy. Then $\{A(Tx^{(1)})\}$ is Cauchy.

Thus, $AT \in Com(\mathcal{X})$

iii) $T_n \in Com(\mathcal{X})$. $\|T_n - T\| \to 0$.

Suppose, $\{x_n\} \subset \mathcal{X}$, $\|x_n\| \le 1$

Since $T_1 \in Com(\mathcal{X})$, $\exists$ a subsequence $\{x_n^{(1)}\}$ st $\{T_1 x_n^{(1)}\}$ is Cauchy.

Since $T_2 \in Com(\mathcal{X})$ $\exists$ a subsequence of $\{x_n^{(1)}\}$, say $\{x_n^{(2)}\}$ st. $\{T_2 x_n^{(2)}\}$ is a Cauchy. ... In this way we construct a sub-sequence $\{x_n^{(m)}\}$ st. $\{T_1 x_n^{(m)}\}, \{T_2 x_n^{(m)}\}, \ldots, \{T_m x_n^{(m)}\}$ are Cauchy.

Now take a diagonal subsequence, $y_n = x_n^{(n)}$. Then $\{y_n\}_{n=N+1}^{\infty}$ is a subsequence of $\{x_n^{(N)}\}$, therefore $\{T_N y_n\} \subset \{T_N x^{(n)}\}$ (except first N terms) therefore, $\{T_N y_n\}_{n=1}^{\infty}$ is Cauchy for each N.

Now I need to prove that $\{Ty_n\}$ is Cauchy.

Indeed; $\|Ty_n - Ty_m\|$

$\leq \|T y_n - T_N y_n\| + \|T_N y_m - Ty_m\| + \|T_N y_m - Ty_m\|$

$\leq \|T - T_N\| \|y_n\| \leq \|T - T_N\|$

$\leq 2\|T - T_N\| + \|T_N y_n - T_N y_m\|$

Let $\varepsilon > 0$. Choose N s.t.

$\|T - T_N\| < \varepsilon/3$ (possible since $\|T - T_N\| \to 0$ as $N \to \infty$).

Then choose M s.t. $\forall n, m \geq M$ we have $\|T_N y_n - T_N y_m\| < \varepsilon/3$. (possible since $\{T_N y_n\}_{n=1}^{\infty}$ is Cauchy). Then

$\|Ty_n - Ty_m\| < \varepsilon$ for all $n, m \geq M$, so $\{Ty_n\}$ is Cauchy so $T \in \mathrm{Com}(\mathcal{X})$.

Def: $T \in B(\mathcal{X}, Y)$ is finite rank operator, if $\operatorname{Ran} T$ is finite-dimensional.

Prop: Finite rank operator is compact. Also, uniform limit of sequence of finite rank operators is compact.

Example: Integral operator. $T: C[0,1] \to C[0,1]$. $\ni f$

$$Tf(t) = \int_0^1 k(t,s) f(s)\, ds$$

$k \in C([0,1]^2)$ is called the integral kernel of T.

Claim: T is compact.

$\forall\, \varepsilon = 1/n > 0\ \exists$ a polynomial

$$p_n(t,s) = \sum_{j,k=1}^{m} c_{jk}^{(n)} t^j s^k.$$

Result from Fun. Analysis from the Stone-Weierstrass theorem

s.t. $\|p_n(t,s) - k(t,s)\|_{C[0,1]} < 1/n$.

Denote

$$T_n f(t) = \int_0^1 p_n(t,s) f(s)\, ds$$

Then $(T_n - T) f(t)$

$$= \int_0^1 [p_n(t,s) - k(t,s)] f(s)\, ds,$$

so $\|T - T_n\| < 1/n$ and $T_n \to T$ as $n \to \infty$.

also.

$$T_n f(t) = \int_0^1 \left[\sum_{j,k=0}^{m} c_{jk}^{(n)} t^j s^k\right] f(s)\, ds$$

$= \sum_{j=0}^{m} \left[\sum_{k=0}^{m} \int_0^1 c_{jk}^{(N)} s^k f(s)\, ds \right] t^j$

$\in \operatorname{Span}\{t^0, t^1, t^2, \dots, t^m\}$

Since $\dim \operatorname{span}\{t^0, \dots, t^m\} = m+1 < +\infty$

T_n is finite rank operator.

—/—

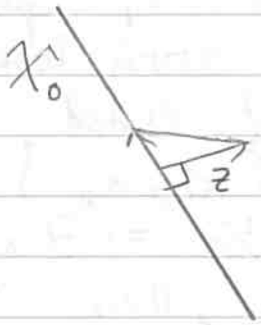
$Y \subset X$ is an invariant subspace for A, if $\forall x \in Y \Rightarrow Ax \in Y$.

11/10/13

$$(Bu)(s) = \int_0^s K(s,t)\, u(t)\, dt$$

$X = C[0,1]$ Volterra.

Lemma: (Almost-orthogonality lemma)
Let X be a normed space and $X_0 \subset X$ be a closed subspace, $X_0 \neq X$. Then $\forall \varepsilon > 0$ there exist $z \in X \setminus X_0$, st $\|z\| = 1$ and $\|z - x\| \geq 1 - \varepsilon, \forall x \in X_0$.



Proof: Take $\varepsilon \in (0,1)$

Let $x_1 \in X$, $x_1 \notin X_0$

Then $d := d(x_1, X_0) > 0$ (since X_0 is closed)
Then $\exists y \in X_0$ st $d \leq \|x_1 - y\| \leq \frac{d}{1-\varepsilon}$

Put: $z := \frac{x_1 - y}{\|x_1 - y\|}$. Then $\|z\| = 1$

Let $x \in X_0$. Then we have
$$\|z - x\| = \left\| \frac{x_1 - y}{\|x_1 - y\|} - x \frac{\|x_1 - y\|}{\|x_1 - y\|} \right\|$$

$$= \frac{\| x_1 - \overbrace{(y + x\|x_1 - y\|)}^{\in X_0} \|}{\|x_1 - y\|} \geqslant \frac{d}{\left(\frac{d}{1-\varepsilon}\right)} = 1 - \varepsilon .$$

<u>Th</u>: Every bounded subset of a normed space X is relativ compact $\Longleftrightarrow$ $\dim X < +\infty$.

<u>Proof</u> ($\Leftarrow$) Know already. (from functional)

($\Rightarrow$) Suppose that $\dim X = +\infty$

Take $x_1 \in X$ with $\|x_1\| = 1$. Put $X_1 = \operatorname{span}\{x_1\}$. Then X_1 is closed. Apply almost orth. lemma with $\varepsilon = 1/2$. $\exists x_2$ st $\|x_2\| = 1$, and $\|x_2 - x_1\| \geqslant 1/2$. Put $X_2 = \operatorname{span}\{x_1, x_2\}$, by almost orth. lemma with $\varepsilon = 1/2 \Rightarrow \exists x_3$ st $\|x_3\| = 1$ and $\|x_3 - x_1\| \geqslant 1/2$, $\|x_3 - x_2\| \geqslant 1/2$ etc. Thus, $\exists$ a sequence $\{x_n\}$ st $\|x_n\| = 1$ and $\|x_n - x_m\| \geqslant 1/2$ for $n \neq m$. This sequence has no Cauchy subsequence.

<u>Corollary</u> 1) I_X is compact iff $\dim X < +\infty$

<u>Corollary</u> 2) Let $T \in \operatorname{Com}(X, Y)$ and $\dim X = +\infty$ or $\dim Y = +\infty$. Then T is not invertible (there is no bounded inverse)

<u>Proof</u>: Suppose, $T^{-1} \in B(X, Y)$. Then $I_Y = TT^{-1}$ and $I_X = T^{-1}T$ are compact, and so $\left.\begin{matrix}\dim X \\ \dim Y\end{matrix}\right\} < +\infty$.

Corollary 3): Suppose, $\dim \mathcal{X} = +\infty$ and $T \in Com(\mathcal{X})$. Then $0 \in \sigma(T)$

Th. Let $T \in Com(\mathcal{X})$, and let $\lambda \neq 0$ be an eigenvalue of T. Then the (geometric) multiplicity of λ is finite.

Proof: Let $M_\lambda = \{x \in \mathcal{X}, Tx = \lambda x\}$ be an eigenspace. Then $T|_{M_\lambda}$ is compact and

$$I|_{M_\lambda} = \frac{1}{\lambda}\left(T|_{M_\lambda}\right)$$

is compact, so $\dim M_\lambda < +\infty$

Lemma: Let $\mathcal{X}$ be Banach, $T \in Com(\mathcal{X})$, $\lambda \neq 0$ is not an eigenvalue of T. Then $\exists\, c > 0$ st

$$\|(T - \lambda I)x\| \geq c\|x\| \quad \forall x \in \mathcal{X}.$$

Proof: Suppose not. Then $\forall c = 1/k$ $\exists x_k \neq 0$ st

$$\|(T - \lambda I)x_k\| < \frac{1}{k}\|x_k\|$$

Then put $z_k = x_k / \|x_k\|$ we have: $\|z_k\| = 1$ and

$$\|(T - \lambda I)z_k\| < \frac{1}{k}$$

Then $(T - \lambda I) z_k \to 0$

Since T is compact, $\exists$ a subsequence $\{z_{k_j}\}$ st $T z_{k_j}$ converages: $T z_{k_j} \to z$. Then

$$z_{k_j} = \frac{1}{\lambda}\Big[\underbrace{T z_{k_j}}_{\to z} - \underbrace{(T - \lambda I) z_{k_j}}_{\to 0}\Big] \to \frac{z}{\lambda} =: z'$$

We know: $\|z'\| = \|\lim_{j\to\infty} z_{k_j}\| = 1$.

Also $(T - \lambda I) z' = (T - \lambda I)(\lim z_{k_j})$

$= \lim \left[(T - \lambda I) z_{k_j}\right] = 0$.

Thus, $z' \neq 0$ is an eigenvalue corresponding to λ.

Lemma: Let X be Banach and $A \in B(X)$ st $\|Ax\| \geq c\|x\|$ $\forall x \in X$ where $c > 0$ is a constant. Then:
i) $\operatorname{Ker} A = \{0\}$.
ii) $\operatorname{Ran} A$ is closed.

Proof: 1) Obvious
2) Suppose, $y_n \in \operatorname{Ran} A$ (so that $y_n = Ax_n$) and $y_n \to y$. Then

$$\|x_n - x_m\| \leq \frac{1}{c}\|A(x_n - x_m)\| = \frac{1}{c}\|y_n - y_m\| \xrightarrow{n,m\to\infty} 0$$

Thus $\{x_n\}$ is Cauchy seq, since $\mathcal{X}$ is Banach, $\exists x = \lim x_n$. Then $Ax = A(\lim x_n) = \lim(Ax_n)$ $= \lim y_n = y$, so $y \in \operatorname{Ran} A$.

Corollary: Suppose, A is as in lemma. Then $\operatorname{Ker}(A^n) = \{0\}$ and $\operatorname{Ran}(A^n)$ is closed.

Proof: $\|A^n x\| = \|A(A^{n-1}x)\| \geq C\|A^{n-1}x\| \geq C^2\|A^{n-2}x\| \geq \dots \geq C^n\|x\|$, so A^n satisfies the assumptions of Lemma.

Th: Let $\mathcal{X}$ be Banach, $T \in \operatorname{Com}(\mathcal{X})$, $\lambda = 0$ is not an eigenvalue. Then $\lambda \notin \sigma(T)$

Proof: Put $\mathcal{X}_0 = \mathcal{X}$ and $\mathcal{X}_n = \operatorname{Ran}(T-\lambda I)^n$. Then $\mathcal{X}_n$ is closed (by the last corollary)

We have:

$$\mathcal{X}_{n+1} = (T-\lambda I)^{n+1}\mathcal{X}$$
$$= (T-\lambda I)\left[(T-\lambda I)^n \mathcal{X}\right]$$
$$= (T-\lambda I)\mathcal{X}_n$$

also $\mathcal{X}_{n+1} = (T-\lambda I)^n[(T-\lambda I)\mathcal{X}] \subseteq \underbrace{(T-\lambda I)^n\mathcal{X}}_{\mathcal{X}_n}$

Therefore: $\mathcal{X}_0 \supseteq \mathcal{X}_1 \supseteq \mathcal{X}_2 \supseteq \mathcal{X}_3 \dots$

could be equal

Claim $\exists\, n$ st $X_{n+1} = X_n$

Proof: Suppose, $\forall n$ $X_{n+1} \subsetneq X_n$

cannot be equal

By the almost orthogonality lemma $\exists\, x_n \in X_n \setminus X_{n+1}$ st $\|x_n\| = 1$ and $\|x_n - z\| \geq 1/2$ $\forall z \in X_{n+1}$.

Suppose, $m \overset{m \geq n+1}{>} n$

$$Tx_m - Tx_n = \frac{\lambda}{\lambda}(T - \lambda I)(x_m - x_n) + \lambda(x_m - x_n)$$
$$= \lambda(z - x_n)$$

where $z = \frac{1}{\lambda}\underbrace{(T - \lambda I)\underbrace{(x_m - x_n)}_{X_n}}_{\in X_{n+1}} + \underbrace{x_m}_{X_m \subset X_{n+1}} \in X_{n+1}$

Therefore,

$$\|Tx_m - Tx_n\| = |\lambda| \|z - x_n\| \geq \frac{|\lambda|}{2}$$

so $\{Tx_n\}$ cannot have a Cauchy subsequence. This contradicts the compactness of T!

Thus, $\exists n \ \ X_{n+1} = X_n$.

Let k be the smallest number st $X_{k+1} = X_k$

Claim 2. $k = 0$.

Proof: Suppose, $k > 0$.

Then $X_{k-1} \supsetneq X_k$. Take $x \in X_{k-1} \setminus X_k$

Then $(T - \lambda I)x \in X_k = X_{k+1}$

$$= (T - \lambda I) X_k,$$

so $\exists y \in X_k$ st

$(T - \lambda I)x = (T - \lambda I)y$ or

$(T - \lambda I)(x - y) = 0$ so

$x - y \neq 0$ is an eigenvalue and λ is an eigenvalue.

Therefore, $k = 0$, so

$X = X_0 = X_1 = \mathrm{Ran}(T - \lambda I)$

Since $\mathrm{Ker}(T - \lambda I) = \{0\}$, $(T - \lambda I)$ is invertible with $\|(T - \lambda I)^{-1}\| \leq 1/c$.

Result from linear algebra only

Lemma: Let A be a linear operator, $\{\lambda_j\}_{j=1}^n$ are eigenvalue and $\{\varphi_j\}$ are eigenvectors assume $\lambda_j \neq \lambda_k$ $(j \neq k)$. Then $\{\varphi_j\}$ are linear independant.

Proof: Suppose $\{\varphi_j\}_{j=1}^n$ is lin dep. Let m be a minimal number st $\{\varphi_j\}_{j=1}^m$ is linearly dep.

Then $\exists c_1, \dots, c_m \neq 0$ st $\sum_{j=1}^m c_j \varphi_j = 0$. Then

$$\varphi_m = -\sum_{j=1}^{m-1} \left(\frac{c_j}{c_m}\right) \varphi_j$$

and $$\underbrace{(A - \lambda_m I)\varphi_m}_{=0} = -\sum_{j=1}^{m-1} \left(\frac{c_j}{c_m}\right)(A - \lambda_m I)\varphi_j$$

$$= -\sum_{j=1}^{m-1} \left(\frac{c_j}{c_m}\right) \underbrace{(\lambda_j - \lambda_m)}_{\neq 0} \varphi_j$$

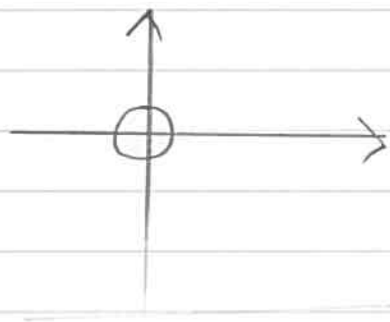
not all are zeros

so $\{\varphi_j\}_{j=1}^{m-1}$ is linearly depenant, which contradicts our assumption.

Th: Lets X be Banach, $T \in Com(X)$. Then $\sigma(T)$ is at most countable and the only possible accumulatcon point is zero.

Proof: This is equivalent to: Let $\delta > 0$. Then $\sigma(T) \cap \{\lambda \in \mathbb{C}, |\lambda| < \delta\}$ is finite.

$$\sigma(T) = \{0\} \cup \bigcup_{n=1}^{\infty} \left(\sigma(T) \cap \{\lambda, |\lambda| > 1/n\} \right)$$



18/11/13

Th: Let X be Banach $T \in Com(X)$. Then $\sigma(T)$ is at most countable and the only accumulation point might be zero.

Proof It is enough to show that for each $\delta > 0$ $\sigma(T) \cap \{\lambda \in \mathbb{C}, |\lambda| < \delta\}$ is finite, assume the opposite $\exists\ \lambda_1, \lambda_2, \ldots \in \sigma(T)$, $|\lambda_j| > \delta$.

$\lambda_j \neq \lambda_k$ for $j \neq k$

Each λ_j is an eigenvalue corresponding to φ_j.

$$T\varphi_j = \lambda_j \varphi_j$$

Let $X_n = \text{span}\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$.

Then $\dim X_n = n$, since $\{\varphi_j\}$ are linearly independent.

Each X_n is closed, and:

$$X_1 \subsetneq X_2 \subsetneq X_3 \subsetneq \ldots$$

$$TX_n \subset X_n$$

$$(T - \lambda_n X)X_n \subset X_{n-1}$$

$$\ni x = \sum_{j=1}^{n} \alpha_j \varphi_j$$

$Tx = \sum_{j=1}^{n} \alpha_j \lambda_j \varphi_j$, $(T - \lambda_n I)x$
$= \sum_{j=1}^{n} (\alpha_j \lambda_j - \alpha_j \lambda_n)\varphi_j$
$= \sum_{j=1}^{n-1} \ldots$

Since X_{n-1} is closed proper subspace of X_n, we can apply almost-orthogonality lemma (with $\varepsilon = 1/2$) and show that $\exists y_n \in X_n$, $\|y_n\| = 1$ and $\|y_n - x\| \geq 1/2$, $\forall x \in X_{n-1}$.

Suppose $n > m$. Then

$$Ty_n - Ty_m = \lambda_n y_n + \underbrace{[(T-\lambda_n I)y_n - Ty_m]}_{X_{n-1}}$$

$$= \lambda\left[y_n + \frac{1}{\lambda_n} z\right], \text{ where } z = \underbrace{\boxed{(T-\lambda_n I)y_n}}_{X_{n-1}} - \underbrace{Ty_m}_{X_m \subset X_{n-1}}$$

Therefore:

$\|Ty_n - Ty_m\|$

$$= |\lambda_n| \left\| y_n + \frac{z}{\lambda_n} \right\| \geq \frac{1}{2}\delta,$$

Recall:

$$TX_n \subset X_n$$
$$(T - \lambda I)X_n \subset X_{n-1}$$

and there is no Cauchy subsequence of $\{Ty_n\}$, which contradicts our assumption that $T \in Com(X)$. □

—/—

Thm: Let X be an infinite-dimensional Banach space and $T \in Com(X)$. Then:

1) $0 \in \sigma(T)$
2) If $\lambda \in \sigma(T)$, $\lambda \neq 0$, then λ is an eigenvalue of finite multiplicity

3) There are at most countably many eigenvalues, with only possible accumulation point at 0.

—/—

$A \in B(X, Y)$ $\qquad f \in Y^*$

$A^* \in B(Y^*, X^*)$ $\qquad x \in X$

$$(A^*f)(x) := f(Ax)$$

Th If $T \in Com(X, Y)$, then $T^* \in Com(Y^*, X^*)$

(No proof)

The geometry of Hilbert spaces.

Def: The inner (scalar) product space is a vector space $\mathcal{H}$ with a map

$$(\cdot, \cdot) = \langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{F}.$$

st:

1) $(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z)$.

2) $(x, y) = \overline{(y, x)}$

3) $(x, x) \geq 0$, $(x, x) = 0 \Leftrightarrow x = 0$.

$$\alpha, \beta \in \mathbb{F}.$$

Then:

$$(x, \alpha y + \beta z) = \bar{\alpha}(x,y) + \bar{\beta}(x,z)$$

$$(x,0) = (0,x) = 0.$$

Def: $\|x\| = \sqrt{(x,x)}$

Th: $\|\cdot\|$ is a proper norm.

Lemma: (Cauchy-Schwarz-Buniakowski inequality)

$$|(x,y)| \leq \|x\| \, \|y\|.$$

Examples:

1) $\mathbb{R}^n$, $\mathbb{C}^n$

$$(x,y) = \sum_{j=1}^{n} x_j \overline{y_j}$$

2) ℓ_2

$$(x,y) = \sum_{j=1}^{\infty} x_j \overline{y_j}$$

3) $C_2[0,1]$ or $L_2[0,1]$

$$(f,g) = \int_0^1 f(t)\overline{g(t)}\, dt$$

$C^{(1)}[0,1] \ni f, g$

$$(f,g)_{C_2^{(1)}[0,1]} = \int_0^1 \left[f(t)\overline{g(t)} + f'(t)\overline{g'(t)}\right] dt$$

<u>Def</u>: A Hilbert space is an inner product space that becomes Banach with the norm $\|x\| = \sqrt{(x,x)}$

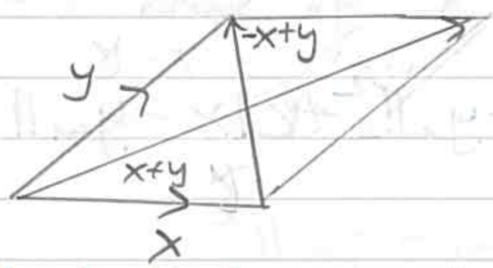
<u>Thm</u>: (polarization identity)
If $\mathbb{F} = \mathbb{R}$, then:

$$4(x,y) = \|x-y\|^2 - \|x-y\|^2$$

If $\mathbb{F} = \mathbb{C}$, then

$$4(x,y) = \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x+iy\|^2.$$

<u>Thm</u>: (parallelogram law)



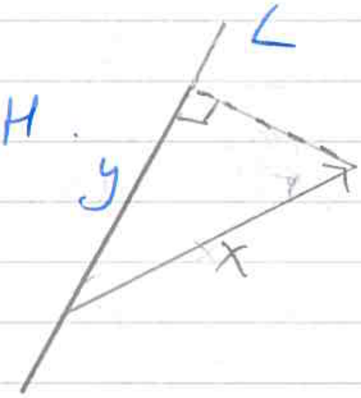
$$2\|x\|^2 + 2\|y\|^2 = \|x+y\|^2 + \|x-y\|^2$$

Let H be an inner product space.

Thm: (Jordan – van Neumann)
If a normed space satisfies the parallelgram law, then it comes from an inner product space.

always linear subspace

Th: Let L be a closed subspace of a Hilbert space H. For each $x \in H$ $\exists!$ st



1) $\|x-y\| = d(x,L)$

$$= \{\|z-x\|, z \in L\}$$

2) $(x-y, z) = 0, \forall z \in L$

Def: Two vectors x_1 and x_2 are orthogonal, $x_1 \perp x_2$, if $(x_1, x_2) = 0$.

Proof: Step 1: Put $d := d(x,L)$. Then $\exists y_n \in L$ st $\|x-y_n\| \to d$.

We use parall. law to $x-y_n$ and $x-y_m$:

$$0 \le \|y_n - y_m\|^2 = 2\|x-y_n\|^2 + 2\|x-y_m\|^2 - \|2x - y_n - y_m\|^2$$

$$= 2\|x-y_n\|^2 + 2\|x-y_m\|^2 - 4\left\|x - \frac{y_n+y_m}{2}\right\|^2$$

$$\le 2\|\underbrace{x-y_n}_{\to 2d^2}\|^2 + 2\|\underbrace{x-y_m}_{\to 2d^2}\| - 4d^2 \xrightarrow{n,m\to\infty} 0$$

Since H is Hilbert, and $\{\varphi_n\}$ is Cauchy seq, it converages: $y_n \to y$

$$\|x-y\| = \lim \|x-y_n\| = d.$$

Step 2: Take $z \in L$. If $z=0$, we have $(x-y, z) = 0$.

If $z \neq 0$, we can assume that $\|z\| = 1$. Then $w := y + \lambda z \in L$ $\forall \lambda \in \mathbb{F}$, and

$$d^2 \leq \|x-w\|^2 = (x-y-\lambda z, x-y-\lambda z)$$

$$= \|x-y\|^2 - \lambda(z, x-y) - \bar{\lambda}(x-y, z) + |\lambda|^2$$

Put $\lambda = (x-y, z)$

$$= \|x-y\|^2 - \lambda\bar{\lambda} - \bar{\lambda}\lambda + |\lambda|^2$$

$= d^2 - |\lambda|^2$, so

$$0 = \lambda = (x-y, z)$$

Step 3: y is unique. Suppose not; $\exists$ y and y_1 st

$$\|x-y\| = \|x-y_1\| = d.$$

Then from step 2 we know:

$(x-y, z) = 0 \quad (\alpha) \qquad \forall z \in L$
$(x-y_1, z) = 0 \quad (\beta)$

$(\alpha) - (\beta) \Rightarrow (y_1 - y, z) = 0.$

Take $z = y_1 - y$

Then $\|y_1 - y\|^2 = 0$, so $y_1 = y$.

Def: Let M be a subset of a Hilbert space H. The orthogonal complement

$$M^\perp = \{x \in H, (x,y) = 0, \forall y \in M\}.$$

Proposition: (Prove them yourself)

1) $M^\perp$ is a closed linear subspace of H.
2) $M_1 \subset M_2 \Rightarrow M_1^\perp \supset M_2^\perp$ (can be equal)
3) $(M^\perp)^\perp \supset M$.
4) $M^{\perp\perp} = \overline{\text{span } M}$
5) If M is dense in $H \Rightarrow M^\perp = \{0\}$.

Proof Excercise (The most differcult is (4))

Def: $\text{Span } M = \{\sum_{j=1}^{n} \alpha_j x_j, \ \alpha_j \in \mathbb{F}, x_j \in M\}$.
from linear algebra

Thm: Suppose M is a closed linear subspace of H. Then $H = M \oplus M^\perp$

Proof: Let $x \in H$. Then, by the previous theorem, $\exists y \in M$ s.t. $d(x,y) = d(x, L)$

and $(x-y) \perp z \quad \forall z \in M$. Then $x-y \in M^\perp$,
and $x = \underbrace{(x-y)}_{\in M^\perp} + \underbrace{y}_{\in M}$

2) Suppose, $x \in \underset{\Downarrow}{M}$ and $\underset{\Downarrow}{M^\perp}$. Then $(x, x) = 0$ so $x = 0$

Therefore, $M \cap M^\perp = \{0\}$.

Complete orthonormal sets

Def: A set $\{x_\alpha\}_{\alpha \in A}$ is called orthonormal if $x_\alpha \perp x_\beta$, $\alpha \neq \beta$.

Th.: (Pythagorous' theorem)
Suppose $\{x_j\}_{j=1}^{n}$ is orthogonal. Then

$$\|x_1 + \dots + x_n\|^2 = \|x_1\|^2 + \dots + \|x_n\|^2$$

Def: A set $\{e_\alpha\}_{\alpha \in A}$ is called orthonormal if:

$$(e_\alpha, e_\beta) = \delta_{\alpha\beta} = \begin{cases} 0 & , \alpha \neq \beta \\ 1 & , \alpha = \beta \end{cases}.$$

Thm: If $\{e_\alpha\}$ is orthonormal, it is linearly independant.

Suppose $\{e_j\}_{j=1}^{\infty}$ is orthonormal, and $x \in H$.

Def: $\alpha_j := (x, e_j)$ are called the Fourier coefficients of x. The series $\sum_{j=1}^{\infty} (x, e_j) e_j$ is called the Fourier series of x.

Th: 1) $\sum_{j=1}^{\infty} |(x, e_j)^2| \leq ||x||^2$

(Bessel's inequality)

2) If H is Hilbert, then the Fourier series converges $y := \sum_{j=1}^{\infty} (x, e_j) e_j$ satisfies.

$$||y||^2 = \sum_{j=1}^{\infty} |(x, e_j)|^2$$

Th-Def: Let $\{e_j\}_{j=1}^{\infty}$ be an orthonormal system in a Hilbert space H. T.f.a.e.

1) $\forall x \in H \quad x = \sum_{j=1}^{\infty} (x, e_j) e_j$

2) $\sum_{j=1}^{\infty} |(x, e_j)|^2 = ||x||^2, \ \forall x \in H$

(Parseval's identity)

3) $(x, e_n) = 0 \quad \forall n \Rightarrow x = 0$.

4) $\overline{\text{span}\{e_n\}} = H$

If any of these properties are satisfied, we say that $\{e_j\}_{j=1}^{\infty}$ is a complete orthonormal system.

Examples:

1) $\mathbb{C}^n$. $e_j = (\underbrace{0, \ldots, 0, 1}_{j}, 0, \ldots)$

Then $\{e_j\}_{j=1}^n$ is complete orthonormal system.

2) ℓ_2 $e_j = (\underbrace{0, \ldots, 0, 1}_{j}, 0, \ldots)$

$\{e_j\}_{j=1}^{\infty}$ is c.o.s - complete orthonormal system.

3) $L_2([0,1], \mathbb{C})$

$\{e^{i2\pi x n}\}_{n=-\infty}^{\infty}$ is c.o.s.

If there exist in H a countable dense set

Th: H is separable iff it contains a finite or countable c.o.s.

Th: (Riesz reperesentation theorem) Let H be Hilbert, and f be a bounded linear functional. Then $\exists ! z$ st

$$f(x) = (x, z) \qquad \forall x \in H.$$

Moreover, $\|f\|_{H^*} = \|z\|_H$

Proof (May be in exams)

1) Uniqueness.
Suppose, $\exists z_1$ and z_2 st $f(x) = (x, z_1) = (x, z_2)$.

Then $(x, z_1) - (x, z_2) = 0$.
$$\begin{array}{c} (x,z_1)-(x,z_2) = 0 \\ \| \\ (x, z_1 - z_2) \end{array}$$

Take $x = z_1 - z_2 \Rightarrow z_1 = z_2$.

2) Existence: If $f = 0$, take $z = 0$. suppose, $f \neq 0$.

Then $\operatorname{Ker} f$ is a closed proper subspace of H.

Therefore: $(\operatorname{Ker} f)^\perp \neq \{0\}$.

Take $y \in (\operatorname{Ker} f)^\perp$, $y \neq 0$.

Claim: $f(x)y - f(y)x \in \operatorname{Ker} f$. $\forall x \in H$

Proof: $f[f(x)y - f(y)x]$
$$= f(x)f(y) - f(y)f(x) = 0.$$

Therefore, $(\underbrace{f(x)y - f(y)x}_{\in \operatorname{Ker} f}, \underbrace{y}_{\in (\operatorname{Ker} f)^\perp}) = 0$.

$= f(x)|y|^2 - f(y)(x, y)$, and.

$$f(x) = \frac{f(y)}{|y|^2}(x,y) = (x,y) \quad \text{where}$$

$$z = \overline{\frac{f(y)}{|y|^2}}$$

3) $\|f\| = \|z\|$ excercise.

25/11/13.

Th. (Riesz)
If f is a bounded linear functional in H, then $\exists!\ z$ st $f(x) = (x, z)$ and $\|f\|_{H^*} = \|z\|_H$.

Th-def: Let $A \in B(H)$, H is Hilbert. There exists a unique operator A^* (called the adjoint of A) st:

$$(Ax, y) = (x, A^*y) \quad \forall x, y \in H.$$

moreover, $A^* \in B(A)$, $\|A^*\| \leq \|A\|$.

Proof: Let us fix $x \in H$. Then $x \mapsto (Ax, y) = f(x)$ defines a linear functional. moveover $|f(x)| < |(Ax, y)| \leq \|A\| \|x\| \|y\|$, so

$$\underbrace{\sup_{x \neq 0} \frac{|f(x)|}{\|x\|}}_{\|f\|} \leq \|A\| \|y\|.$$

apply the Riesz representation th-m:
$\exists!\ z$ st $(Ax, y) = f(x) = (x, z)$, with $\|z\| = \|f\| \leq \|A\| \|y\|$.

Define a mapping:

$$A^* : y \mapsto z.$$

A^* is linear;

$$(Ax, y) = (x, A^* y)$$

and

$$\sup_{y \neq 0} \underbrace{\frac{\|A^* y\|}{\|y\|}}_{\|A^*\|} \quad \text{so } A^* \in B(H)$$

Th: Suppose, $A_1, A_2, A, T \in B(H)$
Then:

(i) $(\alpha_1 A_1 + \alpha_2 A_2)^* = \overline{\alpha_1} A_1^* + \overline{\alpha_2} A_2^*$, $\alpha_1, \alpha_2 \in \mathbb{F}$

(ii) $(AT)^* = T^* A^*$

(iii) $(A^*)^* = A$.

(iv) $\|A^*\| = \|A\|$.

(v) $\|A^* A\| = \|A A^*\| = \|A\|^2$

(vi) If A is invertiable then A^* is also inv and $(A^*)^{-1} = (A^{-1})^*$.

Proof: (i–ii) Do it yourself.

(iii) $(A^{**}x, y) = \overline{(y, A^{**}x)}$

$= \overline{(A^* y, x)} = (x, A^* y)$

$= (Ax, y)$, so:

$(A^{**}x - Ax, y) = 0 \quad \forall x, y.$

Take $y = A^{**}x - Ax$. Then $A^{**}x - Ax = 0$, so $A^{**}x = Ax \quad \forall x$.

(iv) We have seen: $\|A^*\| \leq \|A\|$. Therefore, $\underbrace{\|A^{**}\|}_{= \|A\|} \leq \|A^*\|$, so $\|A^*\| = \|A\|$

(v) $\|A^*A\| \leq \|A^*\| \, \|A\| = \|A\|^2$. On the other hand,

$$\|A\|^2 = \sup_{\|x\|=1} \|Ax\|^2 = \sup_{\|x\|=1} (Ax, Ax)$$

$$= \sup_{\|x\|=1} |(x, A^*Ax)| \leq \sup_{\|x\|=1} \cancel{\|x\|} \, \|A^*Ax\| = \|A^*A\|$$

Thus, $\|A^*A\| = \|A\|^2$. Take A^* instead of A:

$$\underbrace{\|A^{**}A^*\|}_{= \|AA^*\|} = \|A^*\|^2 = \|A\|^2$$

(vi) $AA^{-1} = A^{-1}A = I$.

Take adjoint

$$(A^{-1})^* A^* = A^*(A^{-1})^* = I^* = I.$$

Therefore, $(A^*)^{-1} = (A^{-1})^*$.

<u>Th.</u> Let $A \in B(H)$. Then

$$\operatorname{Ker}(A^*)^{\perp} = (\operatorname{Ran} A)^{\perp\perp}$$

and

$$\operatorname{Ker}(A) = (\operatorname{Ran} A^*)^{\perp}.$$

<u>Proof</u>: Let $x \in H$. Then $x \in \operatorname{Ker}(A^*) \iff$
$A^* x = 0 \iff (y, A^* x) = 0, \ \forall y \in H \iff$
$(Ay, x) = 0, \ \forall y \in H \iff x \in (\operatorname{Ran} A)^{\perp}$

<u>Corollary</u>: $(\operatorname{Ker} A^*)^{\perp} = \overline{(\operatorname{Ran} A)}$

$$(\operatorname{Ker} A)^{\perp} = \overline{\operatorname{Ran} A^*}$$

<u>Def</u>: Let $A \in B(H)$. A is said to be:
i) normal, if $A^* A = A A^*$
ii) self-adjoint (or symmetric) if $A^* = A$,
i.e. $(Ax, y) = (x, Ay)$
iii) $u \in B(H)$ is unitary if $u^* u = u u^* = I$

—/—

Let A be a linear unbounded operator, $A: D_A \to H$.

Let $y \in H$ and consider a linear functional

$$f : \underset{\substack{\cap \\ D_A}}{x} \mapsto f(x) \quad = (Ax, y)$$

$f : D_A \to \mathbb{F}$. Suppose there $\exists z \in H$ st

$(Ax, y) = f(x) = (x, z)$.

Since $\overline{D_A} = H$, such z must be unique.

If such z exists, we say that $y \in D_{A^*}$ and $A^* y = z$.

Def: 1) An unbounded operator A is called symmetric if

$$(Ax, y) = (x, Ay), \quad x, y \in D_A.$$

2) A is called self-adjoint if $A = A^*$ (in particular $D_A = D_{A^*}$).

Remark: A is s-adj $\Rightarrow$ A is symmetric and A is symmetric $\Leftrightarrow$ A^* is an extension of A ($A^* \supset A$)

i.e. $D_{A^*} \supseteq \underset{=H}{D_A}$ and $A^*|_{D_A} = A$.

Examples: 1) $A : L_2[0,1] \to L_2[0,1]$,

$Af = f'$.

Sobolev space

$$D_A = \{f \in L_2[0,1], f' \in L_2[0,1]\} = H^1[0,1] = W^{1,2}[0,1]$$

Different notation – Same thing.

$$(Af, g) = \int_0^1 f'\bar{g}$$

$$= f\cdot\bar{g}\Big|_0^1 - \underbrace{\int_0^1 f\bar{g}'}_{=(f,\,Ag)}$$

Now consider $Af = if'$. (in the same domain then:

$$(Af, g) = \int_0^1 if'\bar{g}$$

$$= if\bar{g}\Big|_0^1 + \underbrace{\int_0^1 f\overline{(ig')}}_{=(f,\,Ag)}$$

$$= if(1)\overline{g(1)} - if(0)\overline{g(0)} + (f, Ag)$$

$\overset{?}{=} (f, A^*g)$ is possible iff $A^*g = Ag$.
and $g(1) = g(0) \overset{!}{=} 0$.

$D_{A^*} = \{g \in H^1[0,1],\ g(0) = g(1) = 0\}$.

$A^*g = ig'$

$D_A = \{f \in H^1[0,1],\ f(0) = f(1) = 0\}$.

$Af = if'$

Then A is symmetric,

$D_{A^*} = H^1[0,1] \supsetneq D_A$, so dispite $A^*f = Af$, A is not self-adjoint.

If $D_A = \{ f \in H^1[0,1], f(0) = f(1) \}$. Then, $(Af = if')$,
in order to have

$$if(1)\overline{g(1)} - if(1)\overline{g(0)} = 0.$$

we should have: $g(0) = g(1)$

Therefore, $D_{A^*} = D_A$, and $A^* f = if'$, so
$A^* = A$.

$H = L_2[0,1]$, $Af = \cancel{\overline{f''}}$.

$D_A = \{ f \in L_2, f'' \in L_2, f' \in L_2 \} = H^2[0,1]$.

$D_A = \{ f \in H^2[0,1] \}$

$$(Af, g) = \int_0^1 f'' \bar{g} = f' \bar{g} \Big|_0^1 - \int_0^1 f' \bar{g}'$$

$$= f'(1)\overline{g(1)} - f'(0)\overline{g(0)} - f(1)\overline{g'(1)}$$

$$+ f(0)\overline{g'(0)} + \int_0^1 f \bar{g}''$$

$$\stackrel{?}{=} (f, A^* g) = \int f \overline{A^* g}$$

$$A^* g = g'', \quad D_{A^*} \subset H^2[0,1].$$

If $H^2[0,1] \Rightarrow D_{A^*} = \{ g \in H^2[0,1], g(0) = g(1) = g'(0) = g'(1) \}$

$D_A = \{ f \in H^2[0,1], f(0) = f(1) = f'(0) = f'(1) = 0 \}$

$\Rightarrow D_{A^*} = \{ g \in H^2[0,1] \}$.

Suppose :

$D_A = \{ f \in H^2[0,1], f(0) = f(1), f'(0) = f'(1) \}$
(periodic)

$\Rightarrow D_{A^*} = \{ g \in H^2[0,1], g'(1) = g'(0), g(1) = g(0) \}$

Suppose :

$D_A = \{ f \in H^2[0,1], f(0) = f'(0), f(1) = f'(1) \}$

$\Rightarrow D_A = \{ g \in H^2[0,1]; g(0) = g'(0) \; g(1) = g'(1) \}$.

Suppose :

$D_A = \{ f \in H^2[0,1], f'(1) = \alpha f(1), f'(0) = \beta f(0) \}$
$\alpha, \beta \in \mathbb{R}$ Robin bc.

$D_{A^*} = \{ g \in H^2[0,1], g'(1) = \alpha g(1), g(0) = \beta g(0) \}$.

Suppose :

$D_A = \{ f \in H^2[0,1]; f'(1) = f'(0) = 0 \}$
Neumann bc.

$\Rightarrow D_{A^*} = \{ g \in H^2[0,1], g'(1) = g'(0) = 0$

Suppose:

$$D_A = \{ f \in H^2[0,1],\ f(1)=f(0)=0 \}$$

Dirichlet bc.

$$\Rightarrow D_{A^*} = \{ g \in H^2[0,1],\ g(1)=g(0)=0 \}$$

—/—

$\Omega \subset \mathbb{R}^d$.

Ω is bounded with smooth boundary.

$\Gamma = \partial\Omega$, $H = L_2(\Omega)$

$$Af = \Delta f = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2} f.$$

$$D_A \subset \left\{ f \in L_2,\ \frac{\partial^2}{\partial x_j^2} f \in L_2,\ \frac{\partial f}{\partial x_i} \in L_2 \right\} = H^2(\Omega).$$

$$(Af, g) = \int_\Omega \Delta f \bar{g}$$

$$= \int_\Gamma \frac{\partial f}{\partial \vec{n}} \cdot \bar{g} - \int_\Omega \nabla f \cdot \overline{\nabla g}$$

$$= \int_\Gamma \left[\frac{\partial f}{\partial \vec{n}} \bar{g} - f \frac{\partial g}{\partial \vec{n}} \right] + \int_\Omega f \overline{\Delta g}$$

$\overset{?}{=} (f, A^* g)$

$= \int_\Omega f \, \overline{A^* g}$.

$D_{A^*} \subset H^2(\Omega)$, $A^* g = \Delta g$.

$D_A \subset H^2(\Omega) \Rightarrow D_{A^*} = \left\{ g \in H^2(\Omega), g|_\Gamma, \frac{\partial g}{\partial n}\Big|_\Gamma = 0 \right\}$

$D_A = \left\{ f \in H^2(\Omega), f|_\Gamma = 0, \frac{\partial f}{\partial \vec{n}}\Big|_\Gamma = 0 \right\}$

Look at $D_A = \left\{ f \in H^2(\Omega) : f|_\Gamma = 0 \right\}$

Dirichlet

$\Rightarrow D_{A^*} = \left\{ g \in H^2(\Omega) : g|_\Gamma = 0 \right\}$

Look at $D_A = \left\{ f \in H^2(\Omega) : \frac{\partial f}{\partial \vec{n}}\Big|_\Gamma = 0 \right\}$

Neumann

$\Rightarrow D_{A^*} = \left\{ g \in H^2(\Omega) : \frac{\partial g}{\partial \vec{n}}\Big|_\Gamma = 0 \right\}$.

Look at $D_A = \left\{ f \in H^2(\Omega), \left(\frac{\partial f}{\partial \vec{n}} - \alpha f \right)\Big|_\Gamma = 0 \right\}$

$\alpha : \Gamma \to \mathbb{R}$ Robin.

$D_{A^*} = \left\{ g \in H^2(\Omega) : \left(\frac{\partial g}{\partial \vec{n}} - \alpha g \right)\Big|_\Gamma = 0 \right\}$.

Th.: Let $A \in B(H)$. Then A is normal $\iff \|Ax\| = \|A^*x\| \quad \forall x \in H$.

Proof: $\Rightarrow$

$$\|Ax\|^2 = (Ax, Ax) = (A^*Ax, x)$$

$$\|A^*x\|^2 = (A^*x, A^*x) = (AA^*x, x)$$

$\Leftarrow$ Suppose $\|Ax\| = \|A^*x\|$

This, using polarization identity, we obtain.

$$(Ax, Ay) = (A^*x, A^*y) \quad \forall x, y.$$

i.e. $(A^*Ax, y) = (AA^*x, y)$

i.e. $(A^*Ax - AA^*x, y) = 0 \quad \forall x, y$, so,

by taking:

$$y = (A^*A - AA^*)x,$$

we obtain that $A^*Ax = AA^*x$.

Thus A is normal.

Corollary: If A is normal then $\operatorname{Ker} A = \operatorname{Ker} A^*$

Th: Let $A \in B(H)$ be normal. Then

a) $(\operatorname{Ran} A^*)^\perp = \operatorname{Ker} A \overset{\text{Just proved.}}{=} \operatorname{Ker} A^*$

holds for arbitrary A $= (\operatorname{Ran} A)^\perp$

b) $Ax = \alpha x \Rightarrow A^* x = \bar{\alpha} x$.

c) Two eigenvalues of A corresponding to different eigenvalues are othogonal to each other.

Proof: a) Proved already.

b) $Ax - \alpha x = 0 \Rightarrow x \in \operatorname{Ker}(A - \alpha I)$

$\Rightarrow x \in \operatorname{Ker}(A^* - \bar{\alpha} I)$

$\Rightarrow A^* x = \bar{\alpha} x$.

c) Suppose, $Ax = \alpha x$ and $Ay = \beta y$, $\alpha \neq \beta$.

Then

$\alpha(x,y) = (\alpha x, y) = (Ax, y) = (x, A^* y)$

$\overset{b)}{=} (x, \bar{\beta} y) = \beta(x,y)$

Since $\alpha \neq \beta$, this implies $(x, y) = 0$.

<u>Th</u>: (i) A is s-a, $\alpha \in \mathbb{R} \Rightarrow \alpha A$ is s-a.

(ii) A_1, A_2 are s-a $\Rightarrow A_1 + A_2$ is s-a.

(iii) A_1, A_2 are s-a.

$\Rightarrow A_1 A_2$ is s-a $\Leftrightarrow A_1$ and A_2 commute.

(iv) If A_n are s-a and $\|A_n - A\| \to 0$ then A is s-a.

<u>Proof</u>: Excercise.

2/12/13

A_1 and A_2
D_{A_1}, D_{A_2}

—/—

Def: Let $A \in B(H)$. The quadratic form of A.
$q_A : H \to \mathbb{F}$, is defined by

$$q_A(x) = (Ax, x)$$

Th: Let $\mathbb{F} = \mathbb{C}$ and $A \in B(H)$. Then A is self-adjoint iff $q_A(x) \in \mathbb{R}$ $\forall x \in H$.

Proof ($\Rightarrow$): $A = A^*$. Then

$$q_A(x) = (Ax, x)$$
$$= (x, Ax)$$
$$= \overline{(Ax, x)} \in \mathbb{R}$$

($\Leftarrow$) We know that $\underset{\mathbb{R}\ni}{q_A(z)} = (Az, z)$
$$= \overline{(Az, z)}$$
$$= (z, Az) \quad \forall z \in H$$

We have...

$$4(Ax, y) = (A(x+y), x+y) - (A(x-y), x-y)$$
$$+ i(A(x+iy), x+iy)$$
$$- i(A(x-iy), x-iy)$$

(Polarization identity for operators)

similarly:

$$4(x, Ay) = (x+y, A(x+y)) -$$
$$(x-y, A(x-y)) + i(x+iy, A(x+iy))$$
$$- i(x-iy, A(x-iy))$$

Thus: $(Ax, y) = (x, Ay)$ $\forall x, y \in H$, so $A = A^*$.

<u>Th</u>: Let $A = A^* \in B(H)$. Then all eigenvalues of A are real, and eigenvectors corresponding to different eigenvalues are orthogonal to each other.

<u>Proof</u>: Suppose, $Ax = \lambda x$. Then

$$(Ax, x) = (\lambda x, x)$$
$$= \lambda \|x\|^2$$

so $$\lambda = \frac{(Ax, x)}{\|x\|^2} \in \mathbb{R}$$

by the last theorem. The second statement we have proved for normal operators.

Th-def: Let $P \in B(H)$ be a projection.
T.f.a.e.
i) P is self-adjoint
ii) P is normal.
iii) $\mathrm{Ran}(P) = (\mathrm{Ker}(P))^{\perp}$
iv) $(Px, x) = \|Px\|^2 \quad \forall x \in H$.

If any of these properties are satisfied, we say that P is an orthogonal projection

Proof: (i) $\Rightarrow$ (ii) obvious.

(ii) $\Rightarrow$ (iii) P is normal. Therefore,

$$\mathrm{Ker}(P) = (\mathrm{Ran}(P))^{\perp}$$

This implies

$$\begin{aligned}(\mathrm{Ker}(P))^{\perp} &= (\mathrm{Ran}(P))^{\perp\perp} \\ &= \overline{\mathrm{Ran}(P)} \\ &\underset{\substack{\uparrow \\ \text{Since } P \text{ is} \\ \text{a projection}}}{=} \mathrm{Ran}(P)\end{aligned}$$

(iii) $\Rightarrow$ (i) Suppose; $(\mathrm{Ker}\, P)^{\perp} = \mathrm{Ran}\, P$.

$$(\underbrace{Px}_{\mathrm{Ran}\,P}, \underbrace{(I-P)y}_{\mathrm{Ker}\,P}) = 0 = ((I-P)x, Py) \quad \forall x, y \in H.$$

Therefore:

$$\begin{aligned}(Px, y) &= (Px, y) - \overset{=0}{(Px, (I-P)y)} \\ &= (Px, Py) \\ &= (Px, Py) + \overset{=0}{((I-P)x, Py)} \\ &= (x, Py)\end{aligned}$$

so P is self-adjoint.

Therefore (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii)

(iii) $\Rightarrow$ (iv). Put $y = x$ in the last identity. Then $(Px, x) = (Px, Px)$ $= \|Px\|^2$.

(iv) $\Rightarrow$ (i). Assume $\mathbb{F} = \mathbb{C}$. $\leftarrow$ cheating!

Then $q_P(x) = (Px, x)$

$$= \|Px\|^2 \in \mathbb{R}. \quad \forall x \in \mathbb{R}.$$

Therefore, P is self-adjoint.

Def: Let $A \in B(H)$. Then the numerical range of A is

$$\text{Num}(A) = \{ q_A(x) \, , \, \|x\| = 1 \}$$

$$= \{ (Ax, x) \, , \, \|x\| = 1 \}$$

$$= \left\{ \frac{(Ax, x)}{\|x\|^2} \, , \, x \in H \setminus \{0\} \right\}$$

—/—

Exam: When finding $\text{Num}(A)$ make sure you remember that $\|x\| = 1$

—/—

Since $|(Ax, x)| \leq \|Ax\| \, \|x\|$

$$\leq \|A\| \cdot \|x\|^2.$$

we can deduce that

$$\text{Num}\, A \subset B_{\mathbb{C}}(0, \|A\|)$$

Th. $\text{Num}(A)$ is convex.

(No proof)

Th: $\sigma(A) \subset \overline{Num(A)}$

Lemma: Suppose, $B \in B(A)$ st
$$|(Bx, x)| \geq c\|x\|^2 \quad \text{for } c > 0.$$
Then B is invertible and $\|B^{-1}\| \leq c^{-1}$

Proof: We have:
$$c\|x\|^2 \leq |(Bx, x)|$$
$$\leq \|Bx\| \cdot \|x\|,$$
so $c\|x\| \leq \|Bx\|$. Then
$$\operatorname{Ker} B = \{0\}$$
and

Ran B is closed (by the theorem proved when we discuss compact operators)

We also have:
$$(\operatorname{Ran} B)^{\perp} = \{0\}$$
so
$$(\operatorname{Ran} B)^{\perp\perp} = H.$$
$$\overset{\shortparallel}{\overline{\operatorname{Ran} B}} = \operatorname{Ran} B$$

Thus B is invertible.

Proof of the theorem:

$$\sigma(A) \overset{?}{\subset} \overline{Num(A)}$$

Suppose $\lambda \notin \overline{Num(A)}$, i.e.

$$d(\lambda, Num(A)) =: d > 0.$$

Suppose, $z \in H$, $\|z\| = 1$

Denote $B = A - \lambda I$

Then $(Bz, z) = ((A - \lambda I)z, z)$

$$= |\underbrace{(Az, z)}_{Num(A)} - \lambda| \geq d.$$

Then $\forall x \in H$ we have

$$|(Bx, x)| \geqslant d\|x\|^2$$

and we can apply the previous lemma to show that $B = (A + \lambda I)$ is invertible and so $\lambda \notin \sigma(A)$. Thus

$$\sigma(A) \subset \overline{Num(A)}$$

Spectrum of self-adjoint operators

Th: Let $A = A^* \in B(H)$. Then

$$\|A\| = \sup_{\|x\|=1} |(Ax, x)|$$

Proof: Denote $c := \sup_{\|x\|=1} |(Ax, x)|$

$$\leq \sup_{\|x\|=1} (\|A\| \cdot \|x\|^2) = \|A\|$$

Let us prove that

$$\|A\| \leq c$$

i.e. $\|Ax\| \overset{?}{\leq} c \quad \forall \|x\| = 1$

We have:

$$|(Ax, x)| \leq c\|x\|^2 \quad \forall x \in H \ (*)$$

We have $\forall x, y \in H$

$$(A(x+y), x+y) - (A(x-y), (x-y))$$

$$= 2(Ax, y) + 2(Ay, x)$$

Using $A = A^*$

$$= 2(Ax, y) + 2\overline{(Ax, y)}$$

$= 4\mathrm{Re}(Ax, y)$

Therefore,

$$4\,\mathrm{Re}(Ax,y) \leq |(A(x+y),(x+y))| + |(A(x-y),(x-y))|$$

Using (*)

$$\leq c\left[\|x+y\|^2 + \|x-y\|^2\right]$$

By parallogram law

$$= 2c\left[\|x\|^2 + \|y\|^2\right]$$

If x, y have length 1 this implies

$\mathrm{Re}(Ax, y) \leq c$. Take:

$y = \frac{Ax}{\|Ax\|}$ (if $x = 0$ we automatically have $\|Ax\| \leq c$)

Then $\underbrace{\mathrm{Re}\,\frac{\|Ax\|^2}{\|Ax\|}}_{\|Ax\|} \leq c$, so $\|A\| \leq c$ and $\|A\| = c$.

<u>Th</u>: Let $A = A^* \in B(H)$ and

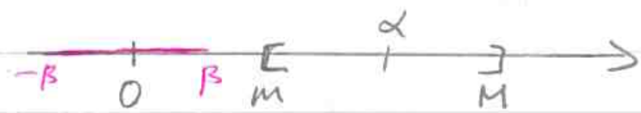
$$m := \inf_{\|x\|=1}(Ax, x), \quad M = \sup_{\|x\|=1}(Ax, x)$$

Then: (i) $\sigma(A) \subset [m, M] \subset \mathbb{R}$
(ii) $m \in \sigma(A)$ and $M \in \sigma(A)$

<u>Proof</u>: (i) Have proved already:

$\overline{\text{Num } A} = [m, M]$

(ii)



Denote $\alpha := \frac{m+M}{2}$, and $\beta = \frac{M-m}{2}$

and $B := A - \alpha I$. Then $B = B^*$,

$$\inf_{\|x\|=1} (Bx, x) = m - \alpha = -\beta$$

$$\sup_{\|x\|=1} (Bx, x) = M - \alpha = \beta$$

The the previous theorem implies:

$$\|B\| = \sup_{\|x\|=1} |(Bx, x)| = \beta$$

I know that $\exists\, x_n$, $\|x_n\| = 1$ st

$$(Bx_n, x_n) \xrightarrow{n\to\infty} \beta$$

Note $\|x\| = 1$

Then $((B - \beta I)x_n, (B - \beta I)x_n)$

$$= \|Bx_n\|^2 + \beta^2 - 2\beta (Bx_n, x_n)$$

$$\dots \leq 2\beta^2 - 2\beta(Bx_n, x_n) \xrightarrow{n\to\infty} 2\beta^2 - 2\beta^2 = 0.$$

Therefore, $\|(B-\beta I)x_n\| \xrightarrow{n\to\infty} 0$

Suppose $(B-\beta I)^{-1} \in B(H)$, $\|(B-\beta I)^{-1}\| = Z_1$. Then:

$$1 = \|x_n\| = \|(B-\beta I)^{-1}(B-\beta I)x_n\|$$

$$\leq Z_1 \|(B-\beta I)x_n\| \xrightarrow{n\to\infty} 0$$

This contradictions shows that $(B-\beta I)$ is not inveritible so $\beta \in \sigma(B)$ and

$$M = \beta + \alpha \in \sigma(A)$$

similarly $-\beta \in \sigma(A)$ and $m = -\beta + \alpha \in \sigma(A)$

Corollary: Let $A^* \in B(H)$. Then $r(A) = \|A\|$ ($\exists \lambda$ with $|\lambda| = \|A\|$ st $\lambda \in \sigma(A)$)
In particular if $r(A) = 0$, then $A = 0$.

Th (Hilbert - Schmidt)
Suppose, $T \in Com(H)$, $T = T^*$. Then there is an orthonormal set of eigenvectors $\{e_n\}_{n=1}^N$ ($N \in \mathbb{N} \cup \{\infty\}$) st $\forall x \in H$ $\exists!$ decompostion

$$x = \sum_{n=1}^N \alpha_n e_n + y. \quad , y \in Ker(T)$$

Then $Tx = \sum_{n=1}^N \alpha_n \lambda_n e_n$, where λ_n is an eigenvalue of T corresponding to e_n.

Moreover, $\sigma(T) \setminus \{0\} = \{\lambda_n\}_{n=1}^{N} \subset \mathbb{R}$. $\lambda_n \xrightarrow{n\to\infty} 0$ (assuming $N = \infty$), $|\lambda_1| \geqslant |\lambda_2| \geqslant |\lambda_3| \geqslant \ldots$

Proof: We know that $\sigma(T) \setminus \{0\}$ consists of at most countably many eigenvalues say $\{\mu_k\} \subset \mathbb{R}$, $\mu_k \neq \mu_j$ for $k \neq j$. We can arrange them so that $|\mu_{n+1}| \leq |\mu_n|$
Let N_k be the eigenspace corresponding to μ_k

$$N_k = \{x \in H, \ Tx = \mu_k x\}.$$

We know: $\dim N_k < +\infty$.

Take an orthonormal basis of each N_k and arrange them into an orthonormal system $\{e_n\}_{n=1}^{N}$, $N \in \mathbb{N} \cup \{\infty\}$.

Let λ_n be an eigenvalue corresponding to e_n.

Then $\lambda_n \xrightarrow{n\to\infty} 0$

Let $L = \operatorname{span}\{e_n\}_{n=1}^{N}$

Claim: $\operatorname{Ker} T = L^{\perp}$

Proof: Suppose, $x \in \operatorname{Ker} T$. Then $(x, e_n) = 0$ so $x \in L^{\perp}$, and $\operatorname{Ker} T \subset L^{\perp}$

Let us prove that $L^\perp \subset \operatorname{Ker} T$. Suppose, $y \in L^\perp$. Then $(y, e_n) = 0$, and (Ty, e_n) $= (y, Te_n) = \lambda_n (y, e_n) = 0$, so $Ty \in L^\perp$

Thus, $TL^\perp \subset L^\perp$, and $L^\perp$ is an invariant subspace of T. Consider $T|_{L^\perp}$ it is a compact operator. It cannot have any non-zero eigenvalues.

Therefore $\sigma(T|_{L^\perp})$ has no non-zero points. Thus:

$$r(T|_{L^\perp}) = 0.$$

$L^\perp = \operatorname{Ker}(T)$

so $T|_{L^\perp} = 0$.

and $L^\perp \subset \operatorname{Ker}(T)$.

Therefore, $H = \overline{L} \oplus L^\perp$, so $\forall x \in H \ \exists!$

$$x = \underbrace{\sum_{n=1}^{N} \alpha_n e_n}_{\in \overline{L}} + \underbrace{y}_{\in L^\perp = \operatorname{Ker} T}$$

and $Tx = \sum_{n=1}^{N} \alpha_n \lambda_n e_n$.

9/12/13

Hibert - Schmidt operators.

$T \in B(H)$. Suppose $\{e_n\}$ and $\{f_m\}$ are orthonormal bases.
and complete.

Consider:

$$\sum_{n=1}^{\infty} \|Te_n\|^2 = \sum_{n=1}^{\infty} \sum_{m=1} |(Te_n, f_m)|^2$$

$$= \sum_{m=1}^{\infty} \underbrace{\sum_{n=1}^{\infty} |(T^* f_m, e_n)|^2}_{= \|T^* f_m\|^2}$$

$$= \sum_{m=1}^{\infty} \|T^* f_m\|^2$$

Therefore, the quantity $\sum_{n=1}^{\infty} \|Te_n\|^2$ does not depend on the choice of the orthonormal basis $\{e_n\}$.

Def: We say that T is a Hilbert-Schmidt operator, if:

$$\|T\|_{HS} := \left[\sum_{n=1}^{\infty} \|Te_n\|^2 \right]^{1/2} < \infty$$

for any (for all) orth. basis $\{e_n\}$.

Lemma 1:

1) $\|\cdot\|_{HS}$ is a proper norm. (excerise)

2) $\|T\|_{HS} \geq \|T\|$.

Proof 2) We have:

$$\|T\|_{HS}^2 = \sum_{n=1}^{\infty} \|Te_n\|^2$$

$$\geq \left[\sup_{\|e_1\|=1} \|Te_1\|\right]^2 = \|T\|^2.$$

Lemma 2. T is $H-S \Rightarrow T \in Com(H)$

Proof: Suppose T is $H-S$ and $\{e_n\}$ is an orthonormal basis.

Suppose, $x \in H$, $x = \sum_{n=1}^{\infty} a_n e_n$, $a_n \in \mathbb{F}$.
We put:

$$T_N x = \sum_{n=1}^{N} a_n Te_n$$

Then:

1) T_N is finite rank. Indeed, $\operatorname{Ran} T_N = \operatorname{span}(Te_1, Te_2, \dots, Te_N)$

2) $\|T - T_N\| \overset{?}{\longrightarrow} 0$

Indeed, if $x = \sum_{n=1}^{\infty} a_n e_n$, then

$$\|(T - T_N)x\| = \left\|\sum_{n=N+1}^{\infty} a_n Te_n\right\|$$

$$\leq \sum_{n=N+1}^{\infty} |a_n| \, \|Te_n\|$$

$$\overset{\text{C-Sch.-B}}{\leq} \underbrace{\left(\sum_{n=1}^{\infty} |a_n|^2\right)^{1/2}}_{= \|x\|} \left(\sum_{n=N+1}^{\infty} \|Te_n\|^2\right)^{1/2}$$

$$= \|x\| \left(\sum_{n=N+1}^{\infty} \|Te_n\|^2\right)^{1/2}$$

Therefore

$$0 \leq \underbrace{\sup_{x \neq 0} \frac{\|(T-T_N)x\|}{\|x\|}}_{= \|T - T_N\|} \leq \left(\sum_{n=N+1}^{\infty} \|Te_n\|^2\right)^{1/2} \xrightarrow{N \to \infty} 0$$

Therefore, $\|T - T_N\| \to 0$.

Thus, T is compact.

<u>Example</u>: $K : L_2[0,1] \to L_2[0,1]$.

$$(Kf)(t) = \int_0^1 k(t, \tau) f(\tau) \, d\tau.$$

$k(t,\tau) \in L_2([0,1]\times[0,1])$.

Lemma: K is Hilbert-Schmidt, with

$$\|K\|_{HS}^2 = \int_0^1\int_0^1 |k(t,\tau)|^2 \, dt\, d\tau.$$

Proof: For $t\in[0,1]$, put $k_t(\tau) = k(t,\tau)$

Let $\{e_n\}$ be an orthonormal basis

Then:

$$(Ke_n)(t) = \int_0^1 k(t,\tau)e_n(\tau)\,d\tau.$$

$$= \int_0^1 k_t(\tau)\overline{\overline{e_n(\tau)}}\,d\tau = (k_t, \overline{e_n})$$

Therefore,

$$\|Ke_n\|^2 = \int_0^1 |(Ke_n)(t)|^2\,dt$$

$$= \int_0^1 |(k_t,\overline{e_n})|^2\,dt,$$

and $\underbrace{\sum_{n=1}^{\infty} \|Ke_n\|^2}_{\|K\|_{HS}^2} = \int_0^1 \sum_{n=1} |(k_t,\overline{e_n})|^2\,dt$

$$\ldots = \int_0^1 \|k_t\|^2 \, dt$$

$$= \int_0^1 dt \int_0^1 |k_t(\tau)|^2 \, d\tau$$

$$= \int_0^1 \int_0^1 |k(t,\tau)|^2 \, dt \, d\tau .$$

—/—

Schatten – von Neumann Classes.

Let $T \in Com(H)$

Then T^*T is a self-adjoint compact operator

$Num(T^*T) \subset [0, +\infty)$. Since $(T^*Tx, x) = (Tx, Tx) \geq 0$.

Denote by s_j^2 the j-th eigenvalue of T^*T (in decreasing order, counting multiplicities).

Def: $\{s_j\}_{j=1}^{\infty}$ are called the s-numbers of T.

Def: Let $1 \leq p < +\infty$. Then we say that T belongs to the class σ_p, if the sequence $\{s_j\}$ belongs to ℓ_p i.e. $\underbrace{\sum_{j=1}^{\infty} s_j^p}_{\|T\|_{\sigma_p}^p} < +\infty$.

<u>Examples:</u> 1) σ_2 = Hilbert-Schmidt operators (excercise).

2) σ_1 (trace class operators, or nuclear operators).

$$T \in \sigma_1 \Rightarrow \sum_{j=1}^{\infty} s_j < +\infty.$$

$$\Downarrow$$

$$\sum_{\ell=1}^{\infty} \lambda_\ell < +\infty$$

$$\parallel$$

$$\mathrm{Tr}(T)$$

$\{e_j\}$ - any orth. basis

$$\sum_{j=1}^{\infty} (Te_j, e_j) = \mathrm{Tr}(T)$$ Lidskii theorem

—/—

(END OF COURSE)

Spectral geometry: (Not for exam)

Ω — *d-dimensional* Riemannian man. on a domain in $\mathbb{R}^d$

Let Δ be a Laplace operator on Ω (say with Dirichlet boundary condition on $\Gamma = \partial\Omega$)

$$\Delta = -\sum_{j=1}^{\infty} \frac{\partial^2}{\partial x_j^2}$$

Then $\sigma(\Delta_D) = \underset{\substack{\vee \\ 0}}{\lambda_1} \leq \lambda_2 \leq \lambda_3, \dots \to +\infty$

with Dirichlet boundary cond.

and $\sigma(\Delta_N) = 0 = \mu_1 \leq \mu_2 \leq \dots \to +\infty$

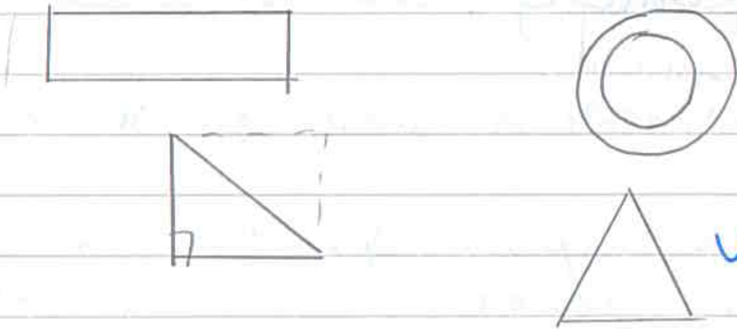
Denote by $\tilde{\Omega}$ the ball of the volume as Ω.

<u>Thm</u>: $\lambda_1(\Omega) \geq \lambda_1(\tilde{\Omega})$

$\mu_2(\Omega) \leq \mu_2(\tilde{\Omega})$

$\lambda_2(\Omega)$

$\mu_3(\Omega)$



Domains that seperation of variables that work.

—/—

$$N(\lambda) = \#\{\lambda_j < \lambda\}$$

λ_n

$$N(\lambda) = C_d \, vol(\Omega) \, \lambda^{d/2} + R(\lambda)$$

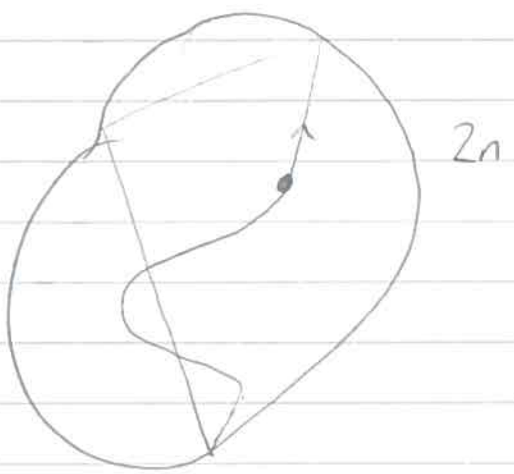
$$R(\lambda) = o(\lambda^{d/2})$$

small "o"

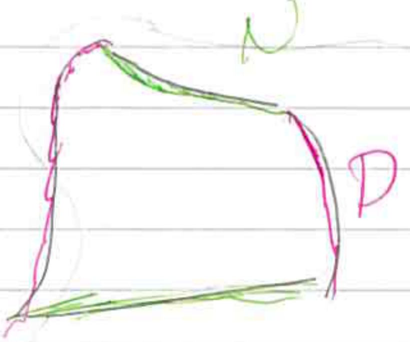
for D condition

$$N(\lambda) = C_d \, vol_d(\Omega) \, \lambda^{d/2} \pm \tilde{C}_d \, vol_{d-1}(\Gamma) \, \lambda^{\frac{d-1}{2}} + o(\lambda^{\frac{d-1}{2}})$$

for N condition

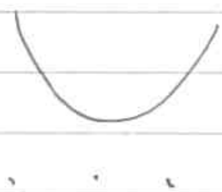


$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$

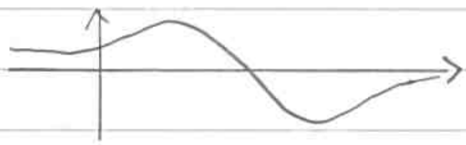


$\mathbb{R}^d$

$[-\Delta + V(x)]u = \lambda u.$



Suppose:



$-\lambda_1 \geq -\lambda_2 \geq -\lambda_3 \ldots \to 0.$

$$\sum_{i=1}^{\infty} \lambda_i \overset{?}{\leq} C_d \left(\int_{\mathbb{R}^d} |V_-|^{p(d)} \right)^{q(d)}$$

$\sin x + \sin(\sqrt{2}x)$ - almost per.