M111 Spectral Theory Notes

Based on the 2012 autumn lectures by Prof L Parnovski

The Author has made every effort to copy down all the content on the board during lectures. The Author accepts no responsibility what so ever for mistakes on the notes nor changes to the syllabus for the current year. The Author highly recommends that reader attends all lectures, making their own notes and to use this document as a reference only

M 111 Spectral theory

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Functional ~~and~~ books:

(1) Kolmogorov and Fomin

(2) B. Bollobas

(3) Kreyszeig

Func.

I for bounded operators, rigorous.

II

Почему нужно $\dim V < \infty$ для ⊘
$A - \lambda I$ is not bijection => bijec

1/10/2012

$A : V \to V$, $\dim V = n < +\infty$
think of A as $n \times n$ matrix.
λ is an eigenvalue of A, if
$\exists v \neq 0 \in V$, s.t. $Av = \lambda v$
=> v is called an eigenvector

$\Leftrightarrow (A - \lambda I) v = 0$
$\Updownarrow$
this is not injection
$(A - \lambda I)$
$\Updownarrow$ if $\dim V < +\infty$ ⊘

~~det (A - λI) ≠ 0~~
$\Updownarrow$
$(A - \lambda I)$ is not a bijection
$\Updownarrow$
$Ch_A(\lambda) = \det (A - \lambda I) = 0$
is characteristic polynomial of A

! linear Algebra разобр.

$$Ch_A(\lambda) = \det(A - \lambda I) = (\lambda_1 - \lambda)^{a_1} (\lambda_2 - \lambda)^{a_2} \cdots (\lambda_k - \lambda)^{a_k}$$
$$a_1 + a_2 + \ldots + a_k = n \qquad a_j \geq 1$$

$\sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ spectrum ~~of~~ of A
a_j is called the algebraic multiplicity of λ_j.
$V_{\lambda_j} = \{ v, Av = \lambda_j \cdot v \}$ is a vector space
V_{λ_j} is linear subspace of V
$\dim V_{\lambda_j}$ is called the geometric multiplicity.

Consider $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$$Ch_A(\lambda) = \det \begin{pmatrix} -\lambda & 1 \\ 0 & -\lambda \end{pmatrix} = \lambda^2 = (\lambda - 0)^2$$

$\lambda_1 = 0$ is an eigenvalue with alg. m. = 2
$V_{\lambda_1} = \{ v, \ Av = 0 \}$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

V

$$\begin{pmatrix} v_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

=> $\dim(V_{\lambda_1}) = 1$ = geom. m.
alg. m $\geq$ geom. m. if matr. is diag. => alg. m = geom. m

mean conjugate

if $A^* \overset{def}{=} \left(\overline{A^T}\right) = A$ then A is symmetric, and alg. m. = geom. m.

if

Def-n. A is normal, if $AA^* = A^*A$

Example Consider $A =$

$B =$

$$\overbrace{\begin{pmatrix} 0 & 2 & & & & \\ & 0 & 2 & & 0 & \\ & & 0 & 2 & & \\ & & & \ddots & \ddots & \\ & 0 & & & 0 & 2 \\ 2^{-30} & & & & & 0 \end{pmatrix}}^{31} \Bigg\} 31$$

$Ch_A(\lambda) = (-\lambda)^{31}$

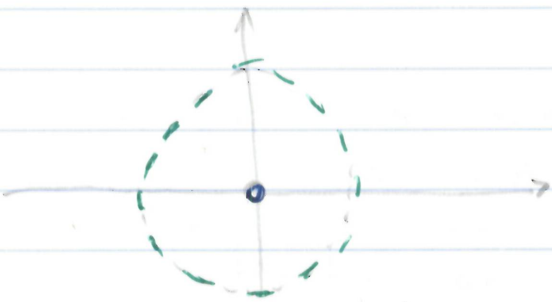
$\sigma(A) = \{0\}$

$\sigma(B)$

$ch_B(\lambda) = (-\lambda)^{31} + 1 = 0$

iff $-\lambda^{31} = -1$

$\lambda^{31} = 1$



• spectrum of A

– spectrum of B $\sigma(B)$

Spaces

vector spaces over field $\mathbb{F} = \mathbb{R}, \mathbb{C}$

Normed space is a vector space with norm i.e.

$$\| \ \| : V \to \mathbb{R}_+ \ , \ \text{s.t.}$$

(1) $\|$

2) $\|\lambda v\| = \lambda \|v\|$

3) $\|v + w\| \leq \|v\| + \|w\|$

the Distance $d(v, w) = \|v - w\|$

if V becomes a complete m. s. with this distance, it is called a Banach Space

an inner product on a vector space $\mathcal{H}$ is $(\cdot, \cdot) : \mathcal{H} \times \mathcal{H} \to \mathbb{F}$, s.t.

(1) $(\lambda x + \mu y, z) = \lambda(x, z) + \mu(y, z)$

(2) $(x, y) = \overline{(y, x)}$

(3) $(x, x) \geq 0 \qquad (x, x) = 0 \iff x = 0$

Then $\|x\| = \sqrt{(x, x)}$, becomes a norm

Remark: $|(x, y)| \leq \|x\| \cdot \|y\|$

Cauchy - Schwarz - Bunjakowski

If $\mathcal{H}$ becomes a Banach space with this norm, then it is called a Hilbert space.

Recall.

Examples [0) If $\dim V < +\infty$ and V is normed, then V is Banach]

1) $\ell^p = \{ x = (x_1, x_2, x_3, \dots), \ x_j \in \mathbb{F},$

$\|x\|_p := \left(\sum_{j=1}^{\infty} |x_j|^p \right)^{1/p} < +\infty \} \qquad 1 \leq p < +\infty$

ℓ^p is a normed space (proved in functional analysis) and Banach space

$\ell^\infty = \{x = (x_1, x_2, \dots), (x_j) \text{ is bounded}\}$

$\|x\|_\infty = \sup |x_j| < +\infty$

Banach.

In ℓ^2 we can introduce the inner product

$$(x,y) = \sum_{j=1}^{\infty} x_j y_j$$

$c_0 = \{x = (x_1, x_2, \dots), \lim_{j\to\infty} x_j = 0\}$

$\|x\|_{c_0} = \sup |x_j|$ is Banach

2) $C[a,b]$ $\quad a, b \in \mathbb{R} \quad a < b$

f

$\|f\|_{C[a,b]} = \sup_{t\in[a,b]} |f(t)| \quad \|\ \|_\infty$

Banach

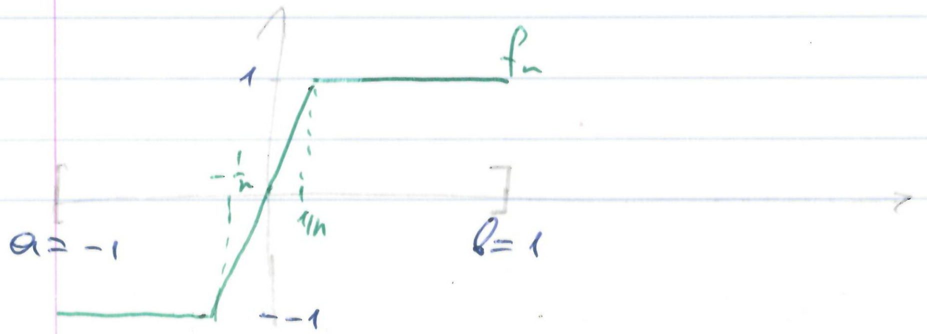
3) $C^{(n)}[a,b] = \{f : [a,b] \to \mathbb{R},\ f, f', \dots, f^{(n)} \in C[a,b]\}$

$\|f\|_{C^{(n)}[a,b]} = \|f\|_{C[a,b]} + \|f'\|_{C[a,b]} + \dots + \|f^{(n)}\|_{C[a,b]}$

4) $C_p[a,b] = \{f : [a,b] \to \mathbb{R},\ f \text{ is continuous}\}$

$\|f\|_p = \left[\int_a^b |f(t)|^p \, dt\right]^{1/p}$

It is not complete (i.e. Banach), and yet is a normed space.



Norming $\|A\| =$
$= \inf\{C>0, \|Ax\| \le C\|x\|\}$
$= \sup_{\|x\| \le 1} \|Ax\|$

3

supremum norm

Then $\|f_n - f_m\|_p \xrightarrow{n,m\to\infty} 0$, thus $\{f_n\}$ is Cauchy seq., but if $\lim_{n\to\infty} f_n = f(x) = \begin{cases} +1 & ; x>0 \\ -1 & ; x<0 \\ 0 & ; x=0 \end{cases}$ is not continuous

expl. on add. page

The space, made out of all limits of all Cauchy seq.-es in $C_p[a,b]$ is denoted $L^p[a,b]$, i.e. $L^p[a,b]$ is completion of $C_p[a,b]$

all my balls are open unless otherwise indicated

Linear ~~operators~~ spaces ~~Operators~~ "vector spaces"

def. Let X and Y be linear space then a mapping $B : X \to Y$ is called ~~linear~~ linear operator if $B(\lambda x + \mu y) = \lambda Bx + \mu By$ $x, y \in X$, $\lambda, \mu \in F$.

def. If X and Y are normed spaces, $A : X \to Y$ is linear bijection and $\|Ax\|_Y = \|x\|_X$, then such mapping is called an isometry

Th-def. Let X and Y be normed spaces and $A : X \to Y$ be linear operator

better by D.N

We say that A is bounded ~~and~~ (continuous) if any of the following conditions are satisfied

w. proof in functional analysis

1) ~~Conti~~ A is continuous
2) A is continuous at 0
3) $\exists C>0$ s.t. $\|Ax\|_Y \le C\|x\|_X$ $\forall x \in X$

exercise

4) $\exists C>0 \; (A B_X(0,1)) \subset B_Y(0,C)$

Then the norm of A

$$\|A\| = \inf\{C>0, \|Ax\| \le C\|x\|\}$$
$$= \sup_{\|x\|\le 1} \|Ax\| = \sup_{\|x\|=1} \|Ax\| = \sup_{x\in X, x\ne 0} \frac{\|Ax\|}{\|x\|}$$

?

$\|?$

A/W

what is compact?
why K is cont. + compact
$\Rightarrow$ bounded

Examples (1) $I : X \to X$

$Ix = x$

(2) $A : C[0,1] \to \mathbb{R}$?

$|Af| = |\int_0^1 f(t)\,dt| \leq C\|f\| = C \sup_{t\in[0,1]} |f(t)|$

$|\int_0^1 f(t)\,dt| \leq \int_0^1 |f(t)|\,dt \leq \int_0^1 \|f\|\,dt = \|f\|$

$\Rightarrow A$ is bounded.

08 October, 2012

Examples:

(1) $A: \quad X \to X \quad , \quad X = C[0,1]$

$Af(t) = \int_0^1 K(t,s)\,f(s)\,ds$, where $K : [0,1]^2 \to \mathbb{F}$ is continuous and is called the (integral) Kernel of A

Recall: [Since K is continuous and $[0,1]^2$ is compact, K is bounded]

say $|K(s,t)| < M$

therefore $|Af(t)| = |\int_0^1 K(t,s)\,f(s)\,ds| \leq M \int_0^1 |f(s)|\,ds \leq$
$\leq M\,\|f\|$

and $\|Af\| = \sup_{t\in[0,1]} |Af(t)| \leq M\,\|f\|$

thus, A is bounded and $\|A\| \leq M$

$\|A\| = \sup \frac{\|Af\|}{\|f\|} \leq M$

(2) $Af(t) = f'(t)$ $\qquad X = C[0,1]$

$D_A = \{ f \in C[0,1],\ f' \in C[0,1] \}$

$A : D_A \to X$

Claim: A is not bounded

"need to find sequence of bounded function but derivatives huge"

How dense was calculated? $\overline{D_A}$ (X) ①

How analogous if $D_A = X \Rightarrow A$ is bounded of ⑤

What is Banach algebra?

take $f_n = e^{inx}$

then $\|f_n\| = 1$

However, $\|Af_n\| = n$ so A is unbounded

We have (topological) $\overline{D_A} = X$ (i.e. so) D_A is not closed ⑥

A subset A of a space X is called dense (in X) if every point x in X either belongs to A or is a limit point of A

Def. the set is Dense (in X) if the closure of set is entire domain X

Remark:

A is (bounded =) continuous, if $x_n \xrightarrow{\text{norm conv.}} x \Rightarrow Ax_n \to Ax$

Def. A is closed, if $\left.x_n \to x,\ Ax_n \to y\right\} \Rightarrow y = Ax$

Th. Closed Graph theorem

$A: X \to Y$ X, Y are Banach

A is closed $\Rightarrow$ A is continuous

~~i.e. if $D_A = X \Rightarrow A$ is bounded operator (in this course)~~ ⑤

+ page **Def** A is unbounded operator from X to Y if $\exists$ a Dense ④ (subspace $D_A \subset X$) and $A: D_A \to Y$

i.e. it is important: $D_A \neq X$

notation: the space of all ~~unbound~~ linear of operators is denoted by $L(X, Y)$

the space of all bounded oper. is den. by $B(X, Y)$

$L(X, X) = L(X)$

$B(X, X) = B(X)$

Th. $A, B \in B(X, Y)$ then $\|A+B\| \le \|A\| + \|B\|$

if $A_1 \in B(Y, Z)$ then $\|A_1 A\| \le \|A_1\| \cdot \|A\|$

if Y is Banach then $B(X, Y)$ is Banach

Remark: the property $\|A_1 A\| \le \|A_1\| \|A\|$ means that $B(X, Y)$ is a Banach algebra.

a special case of $B(X, Y)$ happens when $Y = \mathbb{F}$

$f: X \to \mathbb{F}$ is called a (linear) functional

Remark If $\dim X < +\infty$, $\dim Y < +\infty$ then each linear operator is bounded

The set of all b-d-d linear functionals is called a dual space to X. $X^* = X' \overset{\text{def.}}{=} B(X, \mathbb{F})$

example: $X = l^p = \{ x = (x_1, x_2, \dots), \|x\| = \left(\sum_{i=1}^{\infty} |x_j|^p \right)^{1/p} < +\infty \}$

$X^* \cong l^q$, where $\frac{1}{p} + \frac{1}{q} = 1$

There is a natural isometry between $(l^p)^*$ and l^q.

Suppose, $y \in l^q$, $y = (y_1, y_2, \dots)$, define $f_y : l^p \to \mathbb{F}$ by

$$f_y(x) = \sum_{j=1}^{\infty} x_j y_j, \quad x \in l^p$$

we need to prove the following facts

Then 1) this is well defined

2) $\|y\|_{l^p} = \|f_y\|_{(l^p)^*}$

3) $\forall f \in (l^p)^* \ \exists y \in l^q$ s.t. $f = f_y$

! $1 \le p < +\infty$, $\frac{1}{p} + \frac{1}{q} = 1$

Remark (1) $(l^\infty)^* \not\cong l^1$

(2) $(c_0)^* \cong l^1$

Def. Let X be a Banach space; $x_n, x \in X$

1) we say that x_n converges strongly to x

$s\text{-}\lim_{n\to\infty} x_n = x$, $x_n \to x$, $\lim_{n\to\infty} x_n = x$

if $\|x_n - x\| \xrightarrow{n\to\infty} 0$

2) we say that x_n converges to x weakly

$w\text{-}\lim_{n\to\infty} x_n = x$, $x_n \rightharpoonup x$

if $f \in X^*$ we have $f(x_n) \to f(x)$

? ~~$|f(x_n - x)| \le \|f\| \|x_n - x\|$~~
откуда $\|Ax\|_y \le \|A\| \|x\|_x$
и как $\|A\| = \sup\limits_{x \neq 0} \frac{\|Ax\|}{\|x\|}$ прямо
для

Th $x_n \to x \Rightarrow x_n \rightharpoonup x$

Proof.

Suppose $x_n \to x$ and $f \in X^*$, then $|f(x_n) - f(x)| = |f(x_n - x)| \le$
$\le \|f\| \|x_n - x\| \to 0$ as $n \to \infty$, so $f(x_n) - f(x) \to 0$
i.e. $f(x_n) \to f(x)$ $\square$

Remark: We have used that $|f(x)| \le \|f\| \cdot \|x\|$ $f \in X^*$, $x \in X$, which is corollary of $\|Ax\|_y \le \|A\| \, \|x\|_x$, $A \in B(X, Y)$, $x \in X$, since
$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

Example: $X = c_0$, $e_n = (\underbrace{0, 0, \dots, 0, 1}_{n}, 0, \dots) \in c_0$

Claim: $e_n \rightharpoonup 0$, $e_n \not\to 0$

Proof: $\|e_n - 0\| = \|e_n\| = 1 \not\to 0$, so $e_n \not\to 0$

Suppose that $f \in X^* \cong \ell^1$. Then $\exists y \in \ell^1$ s.t. $f = f_y$, i.e. $f_y(x) = \sum\limits_{i=1}^{\infty} x_i y_i$

Therefore $f_y(e_n) = y_n$, but $y \in \ell^1 \Rightarrow \sum\limits_{n \geq 1} |y_n| < +\infty$
$\Rightarrow y_n \xrightarrow{n \to \infty} 0$.

Thus, $f(e_n) = y_n \to 0 = f(0)$ as $n \to \infty$ and so $w\text{-}\lim\limits_{n \to \infty} e_n = 0$

Def. Let $A_n, A \in B(X, Y)$ we say that (1) A_n converges to A uniformly
$\lim A_n = A$, if $\|A_n - A\| \to 0$ as $n \to \infty$

(2) A_n converges to A strongly
$s\text{-}\lim A_n = A$, if $\forall x \in X$ we have $A_n x \to Ax$ as $n \to \infty$
i.e. $\|A_n x - Ax\| \to 0$

(3) A_n conv. to A weakly
$w\text{-}\lim A_n = A$, if $\forall x \in X$ we have $w\text{-}\lim A_n x = Ax$
i.e. $\forall f \in Y^*$ we have $f(A_n x) \xrightarrow{n \to \infty} f(Ax)$

Claim: $1 \Rightarrow 2 \Rightarrow 3$
$\quad\;\; \not\Leftarrow \quad \not\Leftarrow$

Def. Let X be a normed space and $x_n, x \in X$.
(1) we say that the series $\sum\limits_{n=1}^{\infty} x_n$ converges to x, if the sequence $S_m = \sum\limits_{n=1}^{m} x_n$ of partial sums converges strongly to x.
Note: $\sum\limits_{n=1}^{\infty} x_n = x$

2) We say that $\sum_{n=1}^{\infty} x_n$ converges absolutely, if $\sum_{n=1}^{\infty} \|x_n\|$ converges

Th. X is Banach, $\sum_{n=1}^{\infty} x_n$ converges absolutely $\Rightarrow \sum_{n=1}^{\infty} x_n$ converges

Proof: Put $S_m = \sum_{j=1}^{m} x_j$ then for $m > n$ we have $\|S_m - S_n\| = \|\sum_{j=n+1}^{m} x_j\| \le$
$\le \sum_{j=n+1}^{m} \|x_j\| \le \sum_{j=n+1}^{\infty} \|x_j\| \to 0$ as $n \to \infty$.

Thus, $\|S_m - S_n\| \to 0$ as $n, m \to \infty$ so S_n is Cauchy seq.
so $\exists \lim_{n\to\infty} S_n = x$, so $\sum_{j=1}^{\infty} x_j \to x$

~~chapter 2~~

Def. Let X, Y be a normed spaces and $A \in B(X, Y)$
the Kernel of A, $\operatorname{Ker} A = \{x \in X, Ax = 0\}$
the range of A, $\operatorname{Ran} A = \{y \in Y, \exists x \in X, \text{ with } y = Ax\}$
both $\operatorname{Ker} A$ and Ran are subspaces

Th. $\operatorname{Ker} A$ is closed

Proof: suppose, $x_n \in \operatorname{Ker} A$, $x = \lim x_n$, then $Ax = A \lim_{n\to\infty} x_n \overset{*}{=} \lim_{n\to\infty} Ax_n =$
$= \lim 0 = 0$, so $x \in \operatorname{Ker} A$ □

$$\boxed{A \lim_{n\to\infty} x_n = \lim_{n\to\infty} Ax_n}$$
if A is bounded

(one-to-one)
A is injection $\Leftrightarrow \operatorname{Ker} A = \{0\}$
(onto)
A is surjection $\Leftrightarrow \operatorname{Ran} A = Y$
A is bijection $\Leftrightarrow$ both
If A is bijection, then $\exists A^{-1} : Y \to X$

Th. Banach inverse mapping theorem
X, Y are Banach, $A \in B(X, Y)$, A^{-1} exists then $A^{-1} \in B(Y, X)$
$Y \to X$

Suppose $A : X \to Y$ and A_l^{-1} and A_r^{-1} are operators s.t.
$A_l^{-1} A = I_X \qquad A A_r^{-1} = I_Y$

Claim then $A_l^{-1} = A_r^{-1} = A^{-1}$

Proof: $A_l^{-1} = A_l^{-1} I = A_l^{-1}(AA_r^{-1}) = (A_l^{-1} A) A_r^{-1} = I A_r^{-1} = A_r^{-1}$ □

Ex.: $\dim X < +\infty$, $AB = I \Rightarrow BA = I$

! use in [illegible] when presented

[illegible] A is invertible if $\exists M \in B(X) \ni I = AM$ & $MA = I$

<u>Example</u> $X = \ell^1 \ni x = (x_1, x_2, \dots)$ $\qquad e_n = (\underbrace{0, 0, \dots, 1}_{n}, 0 \dots)$

$Ax = (x_2, x_3, x_4, \dots)$

$Bx = (0, x_1, x_2, x_3, \dots)$

$Ae_1 = 0$, $Ae_{n+1} = e_n$; $Be_n = e_{n+1}$, $n \in \mathbb{N}$

We can think of A and B as the matrices:

$$A = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & 0 \\ & & \ddots & \ddots & \\ & 0 & & 0 & 1 \\ & & & & \ddots \end{pmatrix} \qquad B = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & 0 \\ & 1 & 0 & & \\ & 0 & \ddots & \ddots & \\ & & & 1 & 0 \\ & & & & \ddots \end{pmatrix}$$

$ABx = AB(x_1, x_2, \dots) = A(0, x_1, x_2, \dots) = (x_1, x_2, \dots)$; $AB = I$

$BAx = B(x_2, x_3, \dots) = (0, x_2, x_3, \dots) \Rightarrow BA \neq I$

Th 1) A, B are invertable (i.e. A^{-1} & B^{-1} exists), then AB is invertable and $(AB)^{-1} = B^{-1}A^{-1}$

2) AB is invertable and ~~A and B are invertable~~ $AB = BA$, then A and B are invertable (i.e. $\exists A^{-1}, B^{-1}$)

3) A, B commutative, A^{-1} exists $\Rightarrow$ A^{-1} and B commute.

15/10/2012

Proof (1) $(AB)(B^{-1}A^{-1}) = I = (B^{-1}A^{-1})(AB)$

(2) Denote $(AB)^{-1} = S$ then

$A(BS) = S\overset{BA}{AB} \overset{\text{they commute}}{=} I$

then $BS = A^{-1}_r$ & $SB = A^{-1}_l$

so A is invertable

similarly B

(3) $A^{-1}B = A^{-1}BAA^{-1} = A^{-1}ABA^{-1} = BA^{-1}$

Th 1st perturbation theorem

Suppose X is Banach space

$A \in B(X)$

$\|A\| < 1$

then $(I_X - A)$ is invertable and

$(I_X - A)^{-1} = \sum A^n$

Proof: We have $\|A^n\| \leq \|A\|^n$

Since $\sum_{n=0}^{\infty} \|A\|^n$ conv. $\Rightarrow \sum_{n=0}^{\infty} \|A^n\|$ conv.

$\Rightarrow \sum_{n=0}^{\infty} A^n$ cv. since X is Banach.

Put $R := \sum_{n=0}^{\infty} A^n = \lim_{N\to\infty} \sum_{n=0}^{N} A^n$

then $(I-A)R = (I-A)\lim_{N\to\infty} \sum_{n=0}^{N} A^n \overset{①}{=}$

$= \lim_{N\to\infty} (I-A)\sum_{n=0}^{N} A^n \overset{②}{=} \lim_{n\to\infty} (I - A^{N+1}) = I$

Similarly $R(I-A) = I$,

so $R = (I-A)^{-1}$ □

① as A is bounded $\Rightarrow$ A is continuous $\Rightarrow$ $\Rightarrow$ A respects limit

② geometric prog.

A^0 ...

Remark we also have: $\|(I-A)^{-1}\| \leq \frac{1}{1-\|A\|}$

Proof. $\|(I-A)^{-1}\| = \|\sum_{n=0}^{\infty} A^n\| \overset{③}{\leq} \sum_{n=0}^{\infty} \|A\|^n = \frac{1}{1-\|A\|}$

③ Δ-ineq. works with infinite sum series

Th 2nd perturbation theorem

Let X be Banach;

$A, B \in B(X)$;

A is invertible;

$\|B\| < \frac{1}{\|A^{-1}\|}$;

then $(A+B)$ is invertible

$(A+B)^{-1} = A^{-1}\sum_{n=0}^{\infty}(-BA^{-1})^n$

$= \left[\sum_{n=0}^{\infty}(-A^{-1}B)^n\right] A^{-1}$

and $\|(A+B)^{-1}\| \leq \dfrac{\|A^{-1}\|}{1-\|B\|\,\|A^{-1}\|}$

Proof We have $A+B = A(I-(-A^{-1}B)) =$

$= (I-(-BA^{-1}))A$

Since $\|-A^{-1}B\| \leq \|A^{-1}\|\,\|B\| < 1$,

so we can use the 1st perturbation theorem □

From now on $\mathbb{F} = \mathbb{C}$ (unless specified otherwise)

Def. Let X be Banach and

$A \in B(X)$

2

- the resolvent set of A is
  $\rho(A) := \{\lambda \in \mathbb{C}, \exists (A-\lambda I)^{-1} \in B(X)\}$

bounded is automatic fact

- the spectrum of A is complement of $\rho(A)$:
  $\sigma(A) := \mathbb{C} \setminus \rho(A)$

- a point $\lambda \in \mathbb{C}$ is an eigenvalue of A, if $\exists x \in X \setminus \{0\}$ s.t. $Ax = \lambda x$
  (and then x is called an eigenvector/eigenfunction)

Suppose that $A : D_A \to X$ is unbounded operator

$$(A-\lambda I)^{-1}(A-\lambda I) = I_{D_A}$$
$$(A-\lambda I)(A-\lambda I)^{-1} = I_X$$

the rest 3 definition is the same

Th λ is an eigenvalue $\Rightarrow \lambda \in \sigma(A)$

1 Taking the bounde $\Rightarrow$

Proof: $Ax = \lambda x \Rightarrow (A-\lambda I)x = 0 = (A-\lambda I)0$,
so $(A-\lambda I)$ is not injective $\Rightarrow (A-\lambda I)^{-1}$ doesn't exist.

Examples (1) $\boxed{\text{If } \dim X < +\infty, \text{ then } \sigma(A) = \{\text{eigenvalues of } A\}}$

(2) consider $X = C[0,1]$.
Define $Af(t) = tf(t)$.

I Eigenvalues: $Af = \lambda f$
$tf(t) = \lambda f(t)$.
Therefore, if $t \neq \lambda$ we have $f(t) = 0$,
so $f \equiv 0$

$\exists$ solution but they ($\lambda = \delta$ fn) $\notin C[0,1]$
$\Rightarrow$ no sol-n

II $(A-\lambda I) : f(t) \to (t-\lambda) f(t)$
$(A-\lambda I)^{-1} : f(t) \to \frac{1}{t-\lambda} f(t)$
if $\lambda \notin [0,1]$, this is a well-defined operator,
so $\lambda \in \rho(A)$

Suppose $\lambda \in [0,1]$, then

Compl. Rx: $\lambda \in \mathbb{C}$: $|\lambda^{-1}| \overset{?}{=} |\lambda|^{-1}$

$\mathrm{Ran}(A - \lambda I) \subset \{ g \in C[0,1],\ g(\lambda) = 0 \} \neq X$

So $(A - \lambda I)$ is not surjective so $(A - \lambda I)^{-1}$ doesn't exist.

Thus $\sigma(A) = [0,1]$

Th. Let X be a Banach and $A \in B(X)$

Then (1) $\sigma(A)$ is closed and

(2) $\sigma(A) \subset \{ \lambda \in \mathbb{C},\ |\lambda| \le \|A\| \}$

Proof. Suppose $|\lambda| > \|A\|$

then $(A - \lambda I) = (-\lambda)(I - \lambda^{-1} A)$

is invertible, since $\|\lambda^{-1} A\| = \frac{\|A\|}{|\lambda|} < 1$, by the first perturbation th-m,

so $\lambda \in \rho(A)$.

Thus $\sigma(A) \subset B_c(0, \|A\|)$

Claim ~~Cla~~ Suppose $\lambda_0 \in \rho(A)$, and $|\lambda - \lambda_0| < \frac{1}{\|(A - \lambda_0 I)^{-1}\|}$

then $\lambda \in \rho(A)$

Proof. We have $A - \lambda I = (A - \lambda_0 I) + (\lambda_0 - \lambda) I$

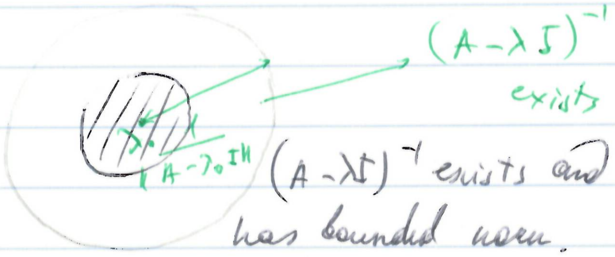
Since $\|(\lambda_0 - \lambda) I\| = |\lambda - \lambda_0| < \frac{1}{\|(A - \lambda_0 I)^{-1}\|}$

we can apply the second perturbation, ~~therefore~~ th. to deduce that

$(A - \lambda I)^{-1}$ exists $\Rightarrow \lambda \in \rho(A)$

Thus, $\rho(A)$ is open and $\sigma(A)$ is closed $\square$

Remark



Def. The operator-valued function
$\rho(A) \ni \lambda \longrightarrow (A-\lambda I)^{-1} =: R(A;\lambda) \in B(X)$
is called the resolvent of A

Th. the 1-st resolvent identity
$R(A,\lambda) - R(A,\lambda_0) = (\lambda-\lambda_0)\, R(A,\lambda) R(A,\lambda_0)$

Proof $R(A,\lambda) - R(A,\lambda_0) = (A-\lambda I)^{-1} - (A-\lambda_0 I)^{-1} =$
$= (A-\lambda I)^{-1}\left[(A-\lambda_0 I) - (A-\lambda I)\right](A-\lambda_0 I)^{-1}$
$= (\lambda-\lambda_0)(A-\lambda I)^{-1}(A-\lambda_0 I)^{-1}$

Th. the 2nd resolvent identity   proof by ourself
$R(A,\lambda) - R(B,\lambda) = R(A,\lambda)(B-A)\cdot R(B,\lambda)$

Th. Let X be normed space and   from functional An.
$x \in X$
then $\exists\, g \in X^*$ s.t. $\|g\| = 1$ and $|g(x)| = \|x\|$ (choosing such x)

Th. Let Z be a complex Banach space
$\Omega \subset \mathbb{C}$ be open
$F : \Omega \to Z$ be a vector-valued function
TFAE   then the following are equiv.

(1) $\forall \lambda_0 \in \Omega$ the derivative exists.
$\frac{dF}{dx}(\lambda) = F'(\lambda_0) := \underset{\lambda\to\lambda_0}{s\text{-}\lim} \frac{F(\lambda) - F(\lambda_0)}{\lambda-\lambda_0} \in Z$,
i.e. $\left\| \frac{F(\lambda)-F(\lambda_0)}{\lambda-\lambda_0} - F'(\lambda_0) \right\| \xrightarrow{\lambda\to\lambda_0} 0$

(2) $\forall \lambda_0 \in \Omega$ has a neighbourhood s.t. there
$F(\lambda) = \sum_{n=0}^{\infty} (\lambda-\lambda_0)^n F_n(\lambda_0)$, $F_n(\lambda_n) \in Z$
and the series converges absolutely.

(3) $\forall\, G \in Z^*$ a complex-valued f-n
$\Omega \ni \lambda \longrightarrow G(F(\lambda)) \in \mathbb{C}$ is analytic in Ω

(4) If $\mathcal{Z} = B(X, Y)$ for Banach X and Y then the equivalent definition is $\forall x \in X$ $\forall g \in Y^*$ a complex-valued f-n $\Omega \ni \lambda \longrightarrow g(F(\lambda)x) \in \mathbb{C}$ is analytic in Ω

1 => 2 straight

3 => 4

Def A vector-valued function (operator-valued) is analytic, if it satisfies any of the above property

f $f(\cdot)$ a function

$f(t)$ function at point t

Th Let X be Banach

$A \in B(X)$

then the $B(X)$-valued function $R(A, \cdot)$ is analytic in $\rho(A)$, and the following hold

$R(A, \lambda) = \frac{1}{A - \lambda}$

(1) $\left.\frac{d\, R(A, \lambda)}{d\lambda}\right|_{\lambda = \lambda_0} = R^2(A^*, \lambda_0)$, $\lambda_0 \in \rho(A)$

(2) $(-\lambda) R(A, \lambda) \underset{|\lambda| \to \infty}{\longrightarrow} I$

(3) $\| R(A, \lambda) \| > \frac{1}{d(\lambda, \sigma(A))}$

$\lambda \in \rho(A)$

Def. $d(\lambda, S) := \inf \{ d(\lambda, s),\ s \in S \}$

$S \subset \mathbb{C}$

$d(T, S) := \inf \{ d(t, s),\ t \in T,\ s \in S \}$

$T \subset \mathbb{C}$

Proof Let $\lambda_0 \in \rho(A)$.

Then $R(A, \lambda)$ exists and has a bounded norm in some neighbourhood of λ_0

Therefore, $R(A, \lambda) - R(A, \lambda_0) = (\lambda - \lambda_0) \underbrace{R(A, \lambda)}_{\to 0,\ \lambda \to \lambda_0} R(A, \lambda_0)$

and $\lim_{\lambda \to \lambda_0} R(A,\lambda) = R(A,\lambda_0)$

Moreover, ~~$R(A,\lambda)$~~

$\lim_{\lambda \to \lambda_0} \frac{R(A,\lambda) - R(A,\lambda_0)}{\lambda - \lambda_0} = \lim_{\lambda \to \lambda_0} R(A,\lambda) R(A,\lambda_0) =$

$= R^2(A,\lambda_0)$

Thus $R(A,\cdot)$ is analytic in $\rho(A)$

(2) $\|(-\lambda) R(A,\lambda) - I\| = \|(-\lambda)(A - \lambda I)^{-1} - I\| =$

$= \|(I - \lambda^{-1}A)^{-1} - I\| \overset{1^{st}\ part\ \Pi\text{-}m}{=\!=} \|\sum_{n=1}^{\infty} (\lambda^{-1}A)^n\| \leq$ + assuming $|\lambda| > \|A\|$

$\leq \sum_{n=1}^{\infty} (|\lambda|^{-1} \|A\|)^n = \frac{|\lambda|^{-1}\|A\|}{1 - |\lambda|^{-1}\|A\|} = \frac{\|A\|}{|\lambda| - \|A\|} \xrightarrow[|\lambda| \to \infty]{} 0$

(3) Let $\lambda_0 \in \rho(A)$, then $\lambda \in B_0\left(\lambda_0, \frac{1}{\|R(A,\lambda_0)\|}\right)$

$\Rightarrow \lambda \in \rho(A)$

therefore, $d(\lambda_0, \sigma(A)) \geq \frac{1}{\|R(A,\lambda_0)\|}$

and $\|R(A,\lambda_0)\| \geq \frac{1}{d(\lambda_0, \sigma(A))}$

21/10/12

Lemma If $A \in B(X)$

then $\sigma(A) \neq \emptyset$

Proof. Suppose, $\rho(A) = \mathbb{C}$

$g(R(A,\lambda)x)$

Take $x \in X$, $x \neq 0$

$g \in X^*$

Put $f(\lambda) := g(R(A,\lambda)x)$

Then $f(\lambda)$ is analytic in $\mathbb{C}$,

$f(\lambda) \to 0$ as $|\lambda| \to \infty$, then

by Liouville's theorem: $f(\lambda) \equiv 0$.

$f: \mathbb{C} \ni \lambda \to g(R(A,\lambda)x)$ is holomorphic complex function

by 2nd part of proof. Then

(i.e. f is bounded)

unnepolko now!

Lemma (corollary from Hahn-Banach): $\exists g \in X^*$ s.t. $g(x) = \|x\|$ $\forall x \in X$

~~implies that $\exists g \in X^*$ s.t. $\|g\| = 1$~~

~~and $g(R(A,0)x) = \|R(A,0)x\|$~~ so $0 \equiv f(\lambda) = g(R(A,\lambda)x) =$

$= \|R(A,\lambda)x\|$

take $\lambda = 0$: $0 = \|R(A,0)x\|$

Thus $R(A, \cancel{\lambda}^{0}) x = 0$, and

$x = A\left(R(A, \cancel{\lambda}^{0}) x\right) = 0$

so $x = 0$ but we assumed $x \neq 0$ #

Thus, $\sigma(A) \neq \emptyset$

<u>Th</u> Let X be Banach

$A \in B(X)$

then $\sigma(A)$ is a non-empty closed subset of $B_{\mathbb{C}}(0, \|A\|)$,

and $R(A, \lambda)$ is analytic in $\rho(A) := \mathbb{C} \setminus \sigma(A)$

$A = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{pmatrix}$

<u>Ex.</u> 1) $X = \ell^2$, $x = (x_1, x_2, \dots)$

$Ax = (\alpha_1 x_1, \alpha_2 x_2, \dots)$ where

$\{\alpha_n\}$ is bounded seq. of compl. numbers

ex:

as $\sigma(A)$ is closed

(*) $\sigma(A) \supset \overline{\{\alpha_j\}_{j=1}^{\infty}}$, is

$$(A - \lambda I)^{-1} = \begin{pmatrix} (\alpha_1 - \lambda)^{-1} & & 0 \\ & (\alpha_2 - \lambda)^{-1} & \\ 0 & & \ddots \end{pmatrix}$$

$(A - \lambda I)^{-1} = \begin{pmatrix} (\alpha_1 - \lambda)^{-1} & & 0 \\ & (\alpha_2 - \lambda)^{-1} & \\ 0 & & \end{pmatrix}$

is bounded, if $\lambda \notin \overline{\{\alpha_j\}}$ i.e. if $\{\lambda \notin \overline{\{\alpha_j\}}\} \subset \rho(0)$

$\therefore \sigma(A) \subset \{\lambda \in \overline{\{\alpha_j\}}\}$ (**)

(*)&(**): so $\sigma(A) = \overline{\{\alpha_j\}_{j=1}^{\infty}}$

2) $X = \ell^2$ $Ax = (x_2, x_3, \dots)$

eigenvalues: $Ax = \lambda x$, or

$(x_2, x_3, x_4, \dots) = (\lambda x_1, \lambda x_2, \lambda x_3, \dots)$

or $x_2 = \lambda x_1$

$x_3 = \lambda x_2 = \lambda^2 x_1$

$x_4 = \lambda x_3 = \lambda^3 x_1$

$\vdots$

$x_n = \lambda^{n-1} x_1$

$x = x_1(1, \lambda, \lambda^2, \dots)$

$\sum_{j=1}^{\infty} |\lambda^j|$ converges so $x \in \ell_1$

iff $|\lambda| < 1$

Thus, $\sigma_{pp}(A) := \{\text{eigenvalues of } A\} = B_o(0,1)$

and $\sigma(A) \supset \{\text{eigenvalues}\} = B_c(0,1) \supset \sigma(A)$ since $\|A\| = 1$

so $\sigma(A) = B_c(0,1)$

pp = (pure) point spectrum

$\|Ax\| \le \|x\|$

$\frac{\|Ax\|}{\|x\|} \le 1$

but take

$y = (0, ?, 1, \dots)$

$Ay =$

$\therefore \|Ay\| = \|y\|$

$\frac{\|Ay\|}{\|y\|} = 1$ is a lower bound

$\Rightarrow \|A\| = 1$

Def The spectral radius

$r(A) := \sup\{|\lambda|,\ \lambda \in \sigma(A)\}$

this is the radius of the smallest disc centred at the origin and containing $\sigma(A)$

we have $r(A) \le \|A\|$

Th $\boxed{r(A) := \lim_{n\to\infty} \|A^n\|^{1/n}}$

Proof I need to prove that

(1) $r(A) \le \liminf_{n\to\infty} \|A^n\|^{1/n}$

(2) $r(A) \ge \limsup_{n\to\infty} \|A^n\|^{1/n}$

step 1 Suppose, $\lambda \in \sigma(A)$ then $\lambda^n \in \sigma(A^n)$, because

~~then~~ $(A^n - \lambda^n I) = (A - \lambda I)(A^{n-1} + \lambda A^{n-2} + \lambda A^{n-3} + \dots + \lambda^{n-1} I)$

since the operators in the RHS commute

and $(A - \lambda I)$ is not invertible,

$(A^n - \lambda^n I)$ is not invertible

and so $\lambda^n \in \sigma(A^n)$.

Therefore,

$r(A)^n = [\sup\{|\lambda|, \lambda \in \sigma(A)\}]^n =$

Analysis I: ①

+ [illegible] lower limit?

$\lambda \notin \rho(A)$

$R(A,\lambda) = ?$

$= \sup\{ |\lambda|^n, \lambda \in \sigma(A) \} \le r(A^n) \le$

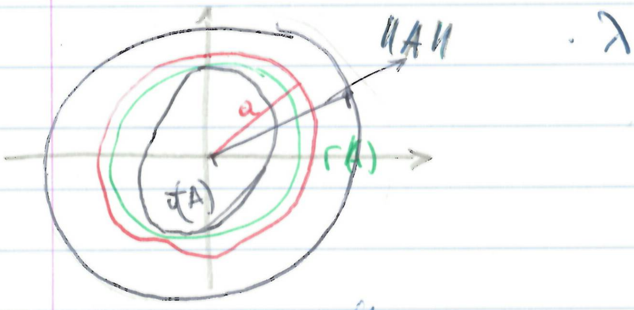
$\le \| A^n \|$, and $r(A) \le \| A^n \|^{1/n}$

① thus, $r(A) \le \liminf_{n \to \infty} \| A^n \|^{1/n}$

if $\lambda \in \sigma(A) \to \lambda^n \in \sigma(A^n)$

$|\lambda|^n = |\lambda^n|$

Step 2



If $\lambda \in \mathbb{C}$ with $|\lambda| > \|A\|$,

then $\lambda \notin \sigma(A)$

and $R(A,\lambda) = (A - \lambda I)^{-1} = (-\lambda)^{-1}(I - \lambda^{-1}A)^{-1} \overset{1^{st}\text{ pert. Thm.}}{=} -\sum_{n=0}^{\infty} \lambda^{-n-1} A^n$

Let $x \in X$, $g \in X^*$ and put

$$f(\lambda) = g\,(R(A,\lambda)x)$$

$f(\lambda)$ analytic for $|\lambda| > r(A)$.

If $|\lambda| > \|A\|$ we have

$$f(\lambda) = -\sum_{n=0}^{\infty} \lambda^{-n-1}\, g(A^n x).$$

② Laurent's theorem implies that this formula is valid for $|\lambda| > r(A)$

wants an add ?

Take $\lambda = a e^{i\theta}$, $a > r(A)$ $\quad \theta \in [0, 2\pi]$

then $(m \in \mathbb{N})$ $\lambda^{m+1} f(\lambda) = -\sum_{n=0}^{\infty} \lambda^{m-n}\, g(A^n x)$

$\int_0^{2\pi} a^{m+1} e^{i(m+1)\theta} f(a e^{i\theta})\, d\theta = -\sum_{n=0}^{\infty} a^{m-n} \int_0^{2\pi} e^{i(m-n)\theta}\, d\theta\; g(A^n x) =$

$= -2\pi\, g(A^m x)$

if $n \ne m$: $\int_0^{2\pi} e^{i(m-n)\theta}\, d\theta = \left. \frac{e^{i(m-n)\theta}}{i(m-n)} \right|_0^{2\pi} = 0$

if $n = m$: $\int_0^{2\pi} e^{i(m-n)\theta} = 2\pi$

Therefore, $|g(A^m x)| \le \frac{1}{2\pi}\, a^{m+1} \int_0^{2\pi} |f(a e^{i\theta})|\, d\theta$

$\int_0^{2\pi} |f(a e^{i\theta})|\, d\theta \le \cancel{a^{m+1}} \int_0^{2\pi} \|g\| \, \|R(A, a e^{i\theta})\| \times \|x\|\, d\theta =$

$f(\lambda) = g(R(A,\lambda)x)$

$= \frac{\|g\|\,\|x\|\, a^{m+1}}{2\pi} M(a)$

where $M(a) = \int_0^{2\pi} \| R(A, a e^{it}) \|\, d\theta$

$$\Rightarrow |g(A^m x)| \le \|g\| \, \|x\| \, a^{m+1} \frac{M(a)}{2\pi}$$

Take $g \in X^*$ s.t. $\|g\| = 1$ and

(3) $g(A^m x) = \|A^m x\|$, (apply corollary from Hahn–Banach) why do we need this

Then $\dfrac{\|A^m x\|}{\|x\|} \le \dfrac{a^{m+1} M(a)}{2\pi}$

Taking $\sup\limits_{x \ne 0}$ in the LHS, we obtain

(4) $\|A^m\| \le a^m \dfrac{a\,M(a)}{2\pi}$, Hence $\limsup \left(\dfrac{a\,M(a)}{2\pi}\right)^{1/m} = 1$

or $\|A^m\|^{1/m} \le a \left(\dfrac{a\,M(a)}{2\pi}\right)^{1/m}$

take lim sup:

(1) $\limsup \|A^m\|^{1/m} \le a$

Since a was arbitrary number s.t.

$a > r(A)$, this implies $\limsup \|A^m\|^{1/m} \le r(A)$ □

th **The spectral mapping theorem**

Let $p(\xi) = \sum_{n=0}^{N} a_n \xi^n$ be a polynomial with $a_N \ne 0$

We put $p(A) = \sum_{n=0}^{N} a_n A^n$

then $\sigma(p(A)) = p(\sigma(A)) = \{p(\xi), \xi \in \sigma(A)\}$

Proof Take any $\mu \in \mathbb{C}$; ~~$p(\xi) = \mu$ (including multipli~~

Denote $\lambda_1, \lambda_2, \dots, \lambda_N$ the sol-ns of $p(\xi) = \mu$

(including multiplicities)

Then $p(\xi) - \mu = a_N (\xi - \lambda_1)(\xi - \lambda_2) \dots (\xi - \lambda_N)$

and $p(A) - \mu I = a_N (A - \lambda_1 I)(A - \lambda_2 I) \dots (A - \lambda_N I)$

and the terms in the RHS commute

Then $\mu \in \sigma(p(A)) \iff$

$p(A) - \mu I$ is invertible

$\iff$ all terms $(A - \lambda_j I)$ are invertible, $j = 1, \dots, N$ as they commute

$\iff$ all $\lambda_j \in \sigma(A)$, $j = 1, \dots, N$

$\iff \mu \in p(\sigma(A))$

Assume it is (bounded) (closed)

Projection operators

Def Let X be a normed space
an operator $P \in B(X)$ is called
a **projection**, if $P^2 = P$

$X = V_1 \oplus V_2$
$x = v_1 + v_2$
$Px = v_1$

Algebraic proof
$\Rightarrow$ it is not important that P is bounded!

Lemma Let $P \in B(X)$ be a projection
then $Q := I - P$ is a projection

$$\boxed{\begin{array}{l} QP = PQ = 0 \text{ and} \\ \operatorname{Ker} P = \operatorname{Ran} Q \\ \operatorname{Ran} P = \operatorname{Ker} Q \end{array}}$$

Q dual projection to P

Proof $Q^2 = (I-P)^2 = I - 2P + P^2 = I - P = Q$
$QP = (I-P)P = P - P^2 = P - P = 0$
$PQ = P(I-P) = P - P^2 = 0$
Since $QP = 0$ $\operatorname{Ran} P \subset \operatorname{Ker} Q$
Suppose that $x \in \operatorname{Ker} Q$.
then $\left.\begin{array}{l} Qx = 0 \\ (I-P)x = 0 \end{array}\right\}$ or $x = Px$ so $x \in \operatorname{Ran} P$
$\therefore$ $\operatorname{Ker} Q \subset \operatorname{Ran} P$
$\therefore$ $\operatorname{Ran} P = \operatorname{Ker} Q$
To prove $\operatorname{Ker} P = \operatorname{Ran} Q$ we replace P and Q.

$X = V_1 \oplus R_2$
$V_1 + V_2 = X$
$V_1 \cap V_2 = \{0\}$

Lemma If $P \in B(X)$ is a projection
then $\operatorname{Ran} P$ is closed
and $X = \operatorname{Ker} P \oplus \operatorname{Ran} P$ — direct sum

Proof $\operatorname{Ran} P = \operatorname{Ker} Q$ is closed since Q is bounded
Suppose, $x \in X$, then $x = \underbrace{Px}_{\in \operatorname{Ran} P} + \underbrace{(I-P)x}_{\in \operatorname{Ran} Q = \operatorname{Ker} P}$,

so $X = \operatorname{Ran} P + \operatorname{Ker} P$
Suppose, $x \in \operatorname{Ran} P \cap \operatorname{Ker} P$
then $\exists y \in X$ s.t. $x = Py$ since $x \in \operatorname{Ker} P$,
we have $Px = 0$
$$\overset{\parallel}{P(Py)} = P^2 y = Py = x \quad \text{so } x = 0$$

th Let P is non-trivial projection
$(i.e., P \neq I, P \neq 0)$
then $\sigma(P) := \{0, 1\}$

Proof: $P^2 - P = 0 \Rightarrow$

spectral mapping th

$0 = \sigma(0) = \sigma(P^2 - P) = \{\lambda^2 - \lambda, \lambda \in \sigma(P)\}$

Thus, $\lambda \in \sigma(P) \Rightarrow \lambda(\lambda - 1) = 0$
$\Rightarrow \lambda = 0$ or $\lambda = 1$
and therefore $\sigma(P) \subset \{0, 1\}$
We also know that $\operatorname{Ran} P \neq \{0\}$ as otherwise $P = 0$ #
and $\operatorname{Ran}(I - P) \neq \{0\}$
Therefore, $\operatorname{Ker}(I - P) \neq \{0\}$ and
$\operatorname{Ker}(P) \neq 0$
Thus P is not invertible and
$I - P$ is also not invertible
therefore $0 \in \sigma(P)$ and
$1 \in \sigma(P)$
Thus $\sigma(P) = \{0, 1\}$. $\blacksquare$

Def Let X be a normed space
A set $K \subset X$ is called relatively compact
if each sequence in K has a Cauchy subsequence
A set $K \subset X$ is called compact
if each sequence in K has a subsequence which converges to an element of K.

actualy it is sequent. compact
but in metric space
comp. $\Leftrightarrow$ seq com

Prop rel. comp. $\Rightarrow$ bounded
comp. $\Rightarrow$ closed and bounded

$\left.\begin{array}{l} K_1 \subset K \\ K \text{ rel. comp.} \end{array}\right\} \Rightarrow K_1$ is rel. com.

$\left.\begin{array}{l} K_1 \subset K \\ K \text{ compact} \\ K_1 \text{ closed} \end{array}\right\} \Rightarrow K_1$ is comp.

Prop. $\dim X < +\infty$ then
K is rel. comp. $\Leftrightarrow$ K is b-dd
K is comp. $\Leftrightarrow$ K is closed, ~~bo~~ b-dd

Example $X = l_1$. then $B_c(0,1)$ is b-dd, but not rel. compact since
$x_n = e_n = (0, \dots, 0, 1, 0 \dots)$
then $d(x_n, x_m) = 2$ $\quad n \neq m$
so there is no Cauchy ~~seq~~ subseq.

$\Leftrightarrow$ thi

Lemma Let X be a finite-dim. vector space,
$\|\cdot\|_1$ and $\|\cdot\|_2$ are two norms on X
then these norms are equivalent.
(i.e. $\exists c_1, c_2 > 0$ s.t. $\forall x \in X$
$c_1 \|x\|_1 \leq \|x\|_2 \leq c_2 \|x\|_1$)

prop. definition of equivalence of norms

Corollary Let X be normed space and
$X_0 \subset X$ be a finite-dimensional subspace of X
then X_0 is closed

Proof: Let $e_1, \dots, e_n$ be a basis of X_0
then $\forall x \in X_0$ it has a form
$x = \sum_{j=1}^{n} c_j e_j$ and
we put $\|x\|' := \max_{1 \le j \le n} |c_j|$ — the norm in X
Suppose, $x_k \in X_0$, converges to $x \in X$ in $\| \ \|$
$x_k = \sum_{j=1}^{n} c_j^k e_j$,
$\Rightarrow x_k$ is a Cauchy sequence in $\| \ \|$
$\Rightarrow x_k$ is Cauchy seq. in $\| \ \|'$ — as $\| \ \|$ & $\| \ \|'$ are equivalent on X_0 (due to $\dim X_0 < \infty$)
$\Rightarrow \forall j = 1, \dots, n$ the seq. $\{c_j^k\}_{k=1}^{\infty}$ is Cauchy
$\Rightarrow \forall j = 1, \dots, n$ there is a limit $\lim_{k\to\infty} c_j^k =: c_j$
Let us put $x = \sum_{j=1}^{n} c_j e_j \in X_0 \Rightarrow$
$\Rightarrow x_k$ converges to x in $\| \ \|'$
$\Rightarrow x_k$ converges to x in $\| \ \|$
$\Rightarrow x = \lim_{k\to\infty} x_k \in X_0$
□ So X_0 is closed

Def Let X and Y be normed spaces
A linear operator $T: X \to Y$ is compact operator
if it maps bounded sets of X into
relatively compact sets of Y
The set of all compact operators from
X to Y is denoted by $Com(X, Y)$
$Com(X) := Com(X, X)$

Prop $Com(X, Y) \subset B(X, Y)$

Lemma Suppose $T(B_c(0,1))$ is relatively compact $T \in L(X, Y)$
then T is compact op

Proof: (a) $T(B_c(0,r)) = T(r B_c(0,1)) \overset{T \text{ is lin.}}{=} r T(B_c(0,1))$
is relatively compact

(d) If $W \subset X$ is bounded,
then $W \subset B_c(0,r)$ for some $r>0$
□ therefore $T(W) \subset T(B_c(0,r))$ is relat. comp.

th (a) let $T_1, T_2 \in Com(X,Y)$
$\alpha_1, \alpha_2 \in \mathbb{F}$
then $\alpha_1 T_1 + \alpha_2 T \in Com(X,Y)$

(b) let $T \in Com(X)$
$A \in B(X)$
then $TA, AT \in Com(X)$

(c) let $T_n \in Com(X)$
$\|T_n - T\| \to 0$
then $T \in Com(X)$

! also applicable for $Com(X,Y)$, just make sure $Com(X,Y)$ $Com(Y,Z)$

$\Rightarrow$ compact operators form an ideal in the algebra of bounded operators.
i.e. this ideal is closed

Proof (a) Suppose $\{x_n\}$ is bounded seq. in X
Since T_1 is compact,
$\exists$ a subseq. $\{x_n^{(1)}\} \subset \{x_n\}$ s.t.
$T_1 x_n^{(1)}$ is Cauchy
Since T_2 is compact,
$\exists$ a subseq. $\{x_n^{(2)}\} \subset \{x_n^{1}\}$ s.t.
$T_2 x_n^{(2)}$ is Cauchy
then $\{T_1 x_n^{(2)}\}$ is also Cauchy
therefore $\{(\alpha_1 T_1 + \alpha_2 T_2) x_n^{(2)}\}$ is Cauchy

$\{x_n$ is Cauchy
y_n is — " —
$\Rightarrow x_n + y_n$ is Cauchy
due to Δ ineq!
one of the def-n of bounded operators

(b) Suppose $\{x_n\}$ is bounded
then $\{Ax_n\}$ is also bounded,
and $\{(TA)x_n\}$ has a Cauchy subseq.,
So TA is compact

Suppose $\{x_n\}$ is bounded
Then $\exists$ subseq. $\{x_n^{(1)}\} \subset \{x_n\}$ s.t.
$\{T x_n^{(1)}\}$ is a Cauchy seq.
and, since A is bounded,
$\{A T x_n^{(1)}\}$ is also Cauchy
thus, $AT \in Com(X)$

(c) let $\{x_n\}$ be bounded

T_1 is compact $\Rightarrow \{x_n^{(1)}\} \subset \{x_n\}$ s.t.
$\{T_1 x_n^{(1)}\}$ is Cauchy
T_2 is compact $\Rightarrow \{x_n^{2}\} \subset \{x_n^{(1)}\}$ s.t.
$\{T_2 x_n^{(2)}\}$ is Cauchy

.....

T_m is compact $\Rightarrow \{x_n^{(m)}\} \subset \{x_n^{(m-1)}\}$ s.t.
$\{T_m x_n^m\}$ is Cauchy
Consider the Diagonal sequence
$$y_n = x_n^{(n)}$$
Obviously y_n is a subseq. of $\{x_n\}$.
Note that $\forall m$, the seq. $\{y_m, y_{m+1}, y_{m+2}, \dots\} \subset \{x_n^{(m)}\}_{n=1}^{\infty}$
Since $\{T_m x_n^{(m)}\}$ is Cauchy
$\Rightarrow \{T_m y_n\}_{n=1}^{\infty}$ is Cauchy $\forall m$
Let us prove that $\{T y_n\}$ is Cauchy seq.
wlog we assume that $\|y_n\| \le 1$
we have
$$\|T y_n - T y_m\| \le \underbrace{\|T y_n - T_k y_m\|}_{\le\|T - T_k\|} + \|T_k y_n - T_k y_m\| + \underbrace{\|T_k y_n - T y_m\|}_{\le\|T - T_k\|} \le$$
$$\le 2\|T - T_k\| + \|T_k y_n - T_k y_m\| < \varepsilon \quad \text{if}$$
Given $\varepsilon > 0$ choose k s.t.
$$\|T - T_k\| < \frac{\varepsilon}{3}$$
Since $\{T_k y_n\}_{n=1}^{\infty}$ is Cauchy:
$\exists N$ s.t. $\|T_k y_n - T_k y_m\| < \frac{\varepsilon}{3}$ $\boxed{\text{for } n, m > N}$

□ Thus, $\{T y_n\}$ is Cauchy.

Def An operator $T \in B(X, Y)$ is called
finite-rank operator
if $\dim(\operatorname{Ran} T) < +\infty$

Prop T is finite-rank operator $\Rightarrow$ T is compact operator

(image of bounded set of bounded map is bounded set & each bounded subset of finite dim subspace is compact (proved))

Ex Consider $T : X \to X$, $X = C[0,1]$
given by $Tf(t) = \int_0^1 K(t,s) f(s)\, ds$
$K \in C[0,1]^2$

Prop: $T \in \mathrm{Com}(X)$

? ← Weierstrass-th. implies that $\forall n$ $\exists$ a pol-l p s.t.

$\| K(t,s) - P_n(t,s) \| < \frac{1}{n}$

P is pol-l $\Leftrightarrow$ $P_n(t,s) = \sum_{k,l=0}^{N(n)} a_{kl}^{(n)} t^k s^l$

Denote $T_n f(t) = \int_0^1 P_n(t,s) f(s)\, ds$

Claim 1: $\| T - T_n \| < \frac{1}{n}$

Claim 2: T_n is finite rank

Pr. of claim 1: $|(T-T_n)f(t)| = |\int_0^1 (K-P_n)(s,t) f(s)\, ds| \leq$

$\leq \frac{1}{n} \| f \|$

$\Rightarrow \| (T-T_n) f \| \leq \frac{\| f \|}{n}$

$\| T - T_n \| = \sup \frac{\| (T-T_n) f \|}{\| f \|} \leq \frac{1}{n}$ ↗ Def.

Pr. of claim 2: $T_n f(t) = \left[\int_0^1 P_n(t,s) f(s)\, ds \right] =$

$= \int_0^1 \sum_{k=0}^{N} \sum_{l=0}^{N} a_{kl} t^k s^l f(s)\, ds =$

$= \sum_{k=0}^{N} t^k \left[\int_0^1 \sum_{l=0}^{N} s^l f(s)\, ds \subset \mathrm{span}(1, t, t^2, \dots, t^N) \right.$

Since $\dim(\mathrm{span}(1, t, \dots, t^N)) < +\infty$,

$\dim(\mathrm{Ran}\, T_n) < +\infty$

Therefore T is compact.

next lect: pr. properties of comp.

12/11/12

Lemma almost orthogonality

Let 1. X be a normed space and

2. $X_0 \subsetneq X$ be any closed subspace of X

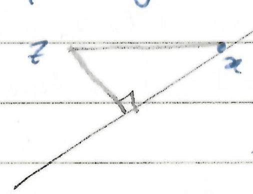
3. $\varepsilon > 0$ $\varepsilon \in (0,1)$

Then $\exists z \in X \setminus X_0$ s.t.

$\| z \| = 1$ and

$\forall x \in X_0$ $\| z - x \| \geq 1 - \varepsilon$

Pr. of orthogonal vector



$\| z \| = 1$

$\Rightarrow \| z - x \| \geq 1$

Proof: Since $X_0 \neq X$ $\exists x_1 \in X \setminus X_0$

Since X_0 is closed ~~the distance~~

$d := d(X_1, X_0) = \inf_{x \in X_0} d(x_1, x) > 0$

There exists $y \in X_0$ s.t.

$d \leq d(x_1, y) \leq \frac{d}{1-\varepsilon}$

Put $z := \frac{x_1 - y}{\|x_1 - y\|}$

then $\|z\| = 1$

Suppose $x \in X_0$

then $\|z - x\| = \frac{1}{\|x_1 - y\|} \left\| \left[x_1 - \overbrace{(y + x\|x_1 - y\|)}^{\in X_0} \right] \right\|$

$\geq \frac{1-\varepsilon}{d} d = 1 - \varepsilon$

! Proved for $\forall\, \varepsilon$ but in the course 'll be using $\varepsilon = \frac{1}{2}$ only.

Th. Let X be a normed space s.t. $B_c(0,1)$ is relatively compact

then $\dim X < +\infty$

Remark If $\dim X < +\infty$ then $B_c(0,1)$ is compact

<u>Proof</u>:

Suppose $\dim X = +\infty$

Choose $x_1 \in X$ s.t. $\|x_1\| = 1$

Put $X_1 = \operatorname{span}\{x_1\}$

then $X_1 \neq X$ is closed.

Apply Lemma with $\varepsilon = \frac{1}{2}$

we find x_2 s.t. $\|x_2\| = 1$ and $\|x_2 - x_1\| \geq \frac{1}{2}$

Put $X_2 = \operatorname{span}\{x_1, x_2\}$

Apply Lemma with $\varepsilon = \frac{1}{2}$

we find x_3 s.t. $\|x_3\| = 1$ and $\|x_3 - x_1\| \geq \frac{1}{2}$

$\|x_3 - x_2\| \geq \frac{1}{2}$

...

Thus we construct a sequence $\{x_n\}$ s.t.

$\|x_n\| = 1$ (so $\{x_n\} \subset B_c(0,1)$)

and $\|x_n - x_m\| \geq \frac{1}{2}$ for $n \neq m$

so there is no Cauchy subsequence. $\square$

Corollary $I_X \in Com(X)$ iff $\dim V < +\infty$

$\dim X < \infty$
$\forall A \in L(X)$
$\Rightarrow A$ is bounded
moreover and compact

Th Let $T \in Com(X, Y)$ and
$\dim X = +\infty$ or $\dim Y = +\infty$
then ~~T is not invertible~~ has no bounded inverse

Proof: suppose T is invertible: $T^{-1} \in B(Y, X)$
i.e. $T^{-1} T = I_X$ and
$T T^{-1} = I_Y$.
then I_X and I_Y are compact
so $\dim X < \infty$ and $\dim Y < +\infty$ # $\square$

Corollary let $\dim X = +\infty$ and
$T \in Com(X)$
then $0 \in \sigma(T)$

i.e. T has no inverse

Th let $T \in Com(X)$ and
λ be its eigenvalue
then λ has finite multiplicity
i.e. $\dim X_\lambda < +\infty$

Recall: $X_\lambda := \{x,\ Tx = \lambda x\} \cup \{0\}$

comp. op = comp. op in inf. dim.

Proof $T|_{X_\lambda} = \lambda I_{X_\lambda}$ is not compact
if $\dim X_\lambda = +\infty$ $\square$

Lemma Let X be Banach

$T \in Com(X)$

Suppose $\lambda \neq 0$ is not an eigenvalue of T

then $\|(T-\lambda I)x\| \geq^{*} c\|x\| \quad \exists c>0 \quad \forall x \in X$

X is Banach from now on

Proof

Suppose not.

Then $\exists$ x_k s.t. $\|(T-\lambda I)x_k\| \leq \frac{1}{k}\|x_k\|$

also assume $x_k \neq 0$ (otherwise * works)

(for each k $\exists$ x_k ...)

Put $z_k = \frac{x_k}{\|x_k\|}$.

then $\|z_k\| = 1$

and $\|(T-\lambda I)z_k\| < \frac{1}{k}$, i.e

$(T-\lambda I)z_k \longrightarrow 0$ as $k \to \infty$

Since T is compact,

$\exists$ a subsequence z_{k_j} s.t. $\{Tz_{k_j}\}_{j=1}^{\infty}$ is Cauchy

and, since X is Banach

$\exists \lim_{j\to\infty} Tz_k = z$

then $z_{k_j} = \frac{\lambda z_{k_j}}{\lambda} = \frac{\overset{0}{\uparrow}\!\!-(T-\lambda I)z_{k_j} + \overset{z}{\uparrow}\!\! Tz_{k_j}}{\lambda}$

$\longrightarrow \frac{z}{\lambda}$ as $j \to \infty$

Since $\|z_{k_j}\| = 1$, we have $z \neq 0$

Moreover, $(T-\lambda I)z = (T-\lambda I)\lim(\lambda z_{k_j}) =$

$= \lambda \lim_{j\to\infty}(T-\lambda I)z_{k_j} = 0$

So $Tz = \lambda z$, and λ is an eigenvalue #

$\square$

Lemma Let X be Banach and

Suppose $A \in B(X)$ satisfies (for $c>0$)

$$\|Ax\| \geq c\|x\| \quad \forall x \in X$$

Then $\ker A = \{0\}$ and

$\operatorname{Ran} A$ is closed

Proof I $x \in \ker A \to Ax = 0 \Rightarrow x = 0$

II Suppose $y \in \overline{\operatorname{Ran} A}$

$y_n \in \operatorname{Ran} A$ s.t. $y_n \to y$

$\therefore \exists x_n$ s.t. $y_n = Ax_n$, i.e.

$Ax_n \to y$

then $\|x_n - x_m\| \leq \frac{1}{c}\|Ax_n - Ax_m\| \to 0$ as $n,m \to \infty$, so $\{x_n\}$ is Cauchy

Since X is Banach, $\{x_n\}$ is converging seq.

$\therefore \exists \lim_{n\to\infty} x_n =: x$

Now apply $Ax = A \lim_{n\to\infty} x_n \overset{A \text{ is bounded}}{=} \lim_{n\to\infty} Ax_n = y$,

so $y \in \operatorname{Ran} A$ and $\operatorname{Ran} A$ is closed. $\square$

Corollary Suppose X is Banach and

$$\|Ax\| \geq c\|x\|$$

then $\ker(A^n) = \{0\}$ and

$\operatorname{Ran}(A^n)$ is closed

Proof We have $\|A^n x\| \geq c\|A^{n-1}x\| \geq c^2\|A^{n-2}x\| \geq \dots$

$\dots \geq c^n\|x\|$

So A^n satisfies all the conditions of the theorem (previous) $\square$

Th Let X be Banach

$T \in Com(X)$

Suppose $\lambda \neq 0$ is not an eigenvalue of T

then $\lambda \notin \sigma(T)$

Proof: Put $A = T - \lambda I$

Denote $X_0 = X$ and $X_n = Ran(A^n) = A^n X$

$X_{n+1} \subset X_n$

then $X_{n+1} = A^{n+1} X = A(A^n X) = A X_n$

or $= A^n X_1 \subset A^n X = X_n$

therefore,

$X = X_0 \supset X_1 \supset X_2 \supset X_3 \supset \dots$

I Let us prove that for some n we have

$X_{n+1} = X_n$

Suppose not $X_{n+1} \neq X_n$ $\forall n$

Since X_{n+1} is closed apply almost orthogonality l:

$\exists x_n \in X_n$ s.t. $\|x_n\| = 1$ and

$\|x_n - z\| \geq \frac{1}{2}$ $\forall z \in X_{n+1}$ $\Rightarrow x_n \notin X_{n+1}$

Suppose, $m > n$ then

$T x_m - T x_n = \underbrace{A x_m}_{\in X_{n+1}} - \overbrace{A x_n}^{\in X_{n+1}} + \lambda(\overbrace{x_m}^{\in X_{n+1}} - \overbrace{x_n}^{\in X_n}) =$

$= \lambda(z - x_n)$,

where $z = x_m + \frac{A x_m - A x_n}{\lambda} \in X_{n+1}$

therefore $\|T x_m - T x_n\| = |\lambda| \|z - x_n\| \geq$

$\geq \frac{|\lambda|}{2}$

So $\{T x_n\}$ has no Cauchy subseq.

This proves Claim 1

therefore, $X = X_0 \underset{\neq}{\supset} X_1 \underset{\neq}{\supset} X_2 \dots \underset{\neq}{\supset} X_k = X_{k+1}$

Denote by k the smallest index s.t.

$$X_k = X_{k+1}$$

II $k = 0$

Suppose not. Then $X_{k-1} \neq X_k$,

so $\exists\, z \in X_{k-1}$ & $z \notin X_k$

then $Az \in X_k = X_{k+1} = AX_k$,

therefore $\exists\, y \in X_k$ s.t. $Az = Ay$,

or $A(z-y) = 0$

$$(T-\lambda I)(z-y) = 0$$

$$T(z-y) = \lambda(z-y)$$

Since $z \in X_k$, $y \in X_k$

$(z-y) \neq 0$ is an eigenvector

corresponding to λ, i.e.

λ is an eigenvalue #

thus $k = 0$ and $X = X_1 = \mathrm{Ran}(T-\lambda I)$

So $\mathrm{Ker}(T-\lambda I) = \{0\}$

$\mathrm{Ran}(T-\lambda I) = X$,

so $(T-\lambda I)^{-1}$ exists

Since $\|(T-\lambda I)x\| \geq c\|x\|$

put $x = (T-\lambda I)^{-1}y$, Then y is any vector

$$\|y\| \geq \|(T-\lambda I)^{-1}y\|\, c$$

i.e. $\|(T-\lambda I)^{-1}y\| \leq c^{-1}\|y\|$,

so $\|(T-\lambda I)^{-1}\| \leq c^{-1}$

thus $\lambda \in \rho(T)$

Def Suppose X is a linear space and

$\{x_\alpha\}_{\alpha \in J}$

then we say that this set is linear independent,

if $\sum_{j=1}^{N} c_{\alpha_j} x_{\alpha_j} = 0 \Rightarrow c_{\alpha_j} = 0$

[1] several opera' need to check their spectra {not a projection}!

in EXAM: check.

Th Let X be a linear space and

$A : X \to X$ be a lin operators

Suppose $\{x_n\}$ are eigenvectors of A correspond to different eigenvalues λ_n

then $\{x_n\}$ is linearly independent set

Th Let X is Banach and

$T \in Com(X)$

then $\sigma(T)$ is at most countable and the only possible accumulation point of $\sigma(T)$ is 0

Proof: we will prove that for each positive δ $(\delta > 0)$

$S_\delta = \sigma(T) \cap \{\lambda \in \mathbb{C}, |\lambda| > \delta\}$

is a finite set

$\sigma(T) \subset \bigcup_{n \in \mathbb{N}} S_{\frac{1}{n}} \cup \{0\}$

Suppose that for some $\delta > 0$ $\in \sigma(T)$, there are points $\lambda_1, \lambda_2, \dots$

$|\lambda_j| > \delta$, $\lambda_n \neq \lambda_m$ for $n \neq m$.

Let $x_j \neq 0$ be eigenvectors

$T x_j = \lambda_j x_j$

Put $X_n = \operatorname{span}\{x_1, x_2, \dots, x_n\}$

Since $\{x_j\}$ are lin. ind.,
we have $\dim X_n = n$

$$X_1 \subsetneq X_2 \subsetneq X_3 \subsetneq \dots$$

$$X_n = \left\{ \sum_{j=1}^{n} \alpha_j x_j ,\ \alpha_j \in \mathbb{C} \right\}$$

$$T x_j = \lambda_j x_j$$

$$T X_n \subset X_n \quad \text{though } T X_n = X_n$$

$$(T - \lambda_n I) X_n \subset X_{n-1}$$

Use almost orthogonality lemma $(\varepsilon = \frac{1}{2})$
to show that $\exists\, y_n \in X_n$ s.t.

$$\| y_n \| = 1$$

$$\| y_n - x \| \geq \frac{1}{2},\quad \forall x \in X_{n-1}$$

Let $n > m$. then

$$\| T y_n - T y_m \| = \| \lambda_n y_n + [\underbrace{(T - \lambda_n I) y_n}_{\in X_{n-1}} + \underbrace{T y_m}_{\in}] \| =$$

$$= \lambda_n (y_n - z),\ \text{where}$$

$$z = -\frac{1}{\lambda_n} [(T - \lambda_n I) y_n + T y_m] \in X_{n-1}$$

Therefore $\| T y_n - T y_m \| = |\lambda_n| \, \| y_n - z \| \geq \frac{\delta}{2}$

$\{ T y_n \}_{n=1}^{\infty}$ has no ~~sub~~ Cauchy Subseq.

Let X be Banach and

$T \in B(X)$

$Y \subset X$ is invariant, if $TY \subset Y$

want: Y is closed
otherwise is trivial

Hillbert Spaces

$(\cdot,\cdot) : \mathcal{H} \times \mathcal{H} \to \mathbb{F}$

norm on all $x, y \in \mathcal{H}$!

$(x,y) = \overline{(y,x)}$

Examples: l_2, $L_2[a,b]$

$(f,g) = \int_a^b f(t)\,\overline{g(t)}\,dt$ inner product of L_2

$(x,y) = \sum_{j=1}^{\infty} x_j \overline{y_j}$

$|(x,y)| \le \|x\|\,\|y\|$, where $\|x\| = \sqrt{(x,x)}$

system = set

Def. A ~~system~~ set $\{x_j\}_{j\in J}$ is called orthogonal system if $(x_j, x_k) = 0$, $\forall j,k \in J$, $j \neq k$ $\Rightarrow x_j \perp x_k$

Th $\boxed{x_1, \dots, x_n \text{ orthogonal system} \Rightarrow \|x_1 + \dots + x_n\|^2 = \sum_{j=1}^{n} \|x_j\|^2}$ $= \|x_1\|^2 + \|x_2\|^2 + \dots + \|x_n\|^2$

Th (polarization identity)

If $\mathbb{F} = \mathbb{R}$, then $4(x,y) = \|x+y\|^2 - \|x-y\|^2$ (**)

If $\mathbb{F} = \mathbb{C}$, then $4(x,y) = \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x+iy\|^2$

(Parallelogram law)

Th $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$ (*)

if $x,y \in$ Banach Sp. H then * is true iff H is Hilbert space with (**) inner prod.

1. (Orthogonality)

Let L be a closed subspace of H

then for any $x \in H$

$\exists$ a unique point $y \in L$ s.t.

$$\|x-y\| = d(x,L) := \inf\{\|x-z\|, z \in L\}$$

We also have

$$(x-y, z) = 0 \quad \forall z \in L$$

Proof: step 1

Put $d = d(x, L)$

then $\exists\, y_n \in L$ s.t. $\|x-y_n\| \to d$ as $n \to \infty$

Use parallelogram law to $x-y_n$ & $x-y_m$

$$0 \le \|y_n - y_m\|^2 = -\|2x - y_n - y_m\|^2 + 2\|x-y_n\|^2 + 2\|x-y_m\|^2 =$$

$$= 2\|x-y_n\|^2 + 2\|x-y_m\|^2 - 4\Big\|x - \overbrace{\frac{y_n+y_m}{2}}^{\in L}\Big\|^2 \le$$

$$\ge 4d^2$$

$$\le 2\|x-y_n\|^2 + 2\|x-y_m\|^2 - 4d^2 \xrightarrow[n,m\to\infty]{} 4d^2 - 4d^2 = 0$$

$$\downarrow 2d^2 \qquad \downarrow 2d^2$$

Thus $\|y_n - y_m\| \to 0$ as $n, m \to \infty$ and there is a limit

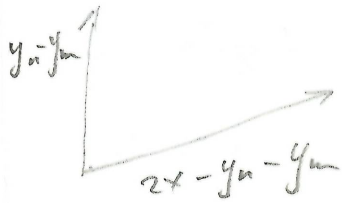
$$y = \lim_{n\to\infty} y_n \in L$$

Also $\|x-y\| = \lim \|x-y_n\| = d$

y is called orthogonal projection

in Banach space this doesn't work general

> i.e. you can find e.g. s.t. $\nexists$ limit s.t. $\|x-y_n\| \to d$



$y_n - y_m + 2x - y_n - y_m =$

$= 2x - y_m - y_m$

> ?

y_n conv. as $y_n \in H$ & H is Banach

Step 2

Let $z \in L$, we can assume that $\|z\| = 1$

Put $w = y + \lambda z \in L$ $(\lambda \in \mathbb{R})$

Then $d^2 \le \|x - w\|^2 = (x - y - \lambda z, x - y - \lambda z)$ need to know properties of $(\,,\,)$

$= \|x-y\|^2 - \lambda (z, x-y) - \bar{\lambda}(x-y, z) + |\lambda|^2 \overset{2}{=}$

st $\lambda = (x-y, z)$

$= \|x - y\|^2 - |\lambda|^2 = d^2 - |\lambda|^2$

Thus, $\lambda = 0 = (x-y, z)$ [1]

<u>step 3</u> uniqueness

Suppose y_1 is another point s.t.

$y_1 \in L$ and

$\|y_1 - x\| = d$.

Then step 2 implies $(\overset{x-y_1}{\cancel{y_1 - x}}, z) = 0 \ \forall z \in L$ [2]

Then $(y - y_1, z) \ \forall z \in L$ $(*)$ $(*) = [2] - [1]$ ...;

Take $z = y - y_1$

$\|y - y_1\|^2 = (y - y_1, y - y_1) = 0$ # $\square$

<u>Def</u> Let $M \subset \mathcal{H}$ be any set

the orthogonal complement is

$$\boxed{M^\perp = \{x \in \mathcal{H}, (x, y) = 0 \ \forall y \in M\}}$$

<u>Th.</u> (1) $M^\perp$ is a closed linear subspace of $\mathcal{H}$

(2) $M_1 \subset M_2 \Rightarrow M_2^\perp \subseteq M_1^\perp$

(3) $M^{\perp\perp} \supset M$

(4) $M^{\perp\perp} = \overline{\text{Span}(M)}$ (H/W 6)

need to orthog. R to prove (4)

(5) Suppose M is dense in $\mathcal{H}$

then $M^{\perp} = \{0\}$

M is dense in $\mathcal{H}$ $(\overline{M} = \mathcal{H})$

Proof: exercise

Th. Let M be a closed linear subspace $\mathcal{H}$

Then $\mathcal{H} = M \oplus M^{\perp}$

Proof: Let $x \in \mathcal{H}$

Use orthogonality theorem to find $y \in M$

with $\|x-y\| = d(x, M)$.

Then $(x-y, z) = 0 \quad \forall z \in M$

so $x - y \in M^{\perp}$

$$x = \underbrace{y}_{\in M} + \overbrace{x - y}^{\in M^{\perp}}$$

Now let us prove that

$M \cap M^{\perp} = \{0\}$.

Let $x \in M \cap M^{\perp}$

then $\underset{\|x\|^2}{\underset{\shortparallel}{(x, x)}} = 0$, so $x = 0$.

$\square$

Def. A set $\{e_j\}_{j\in J}$ is called orthonormal system

if $(e_j, e_k) = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}$

Th $\boxed{\text{Any orthonormal system} \to \text{linearly independent}}$

Proof: obvious

Def. let $x \in \mathcal{H}$

we call (x, e_j) the Fourier coefficient of x w.r.t. the system $\{e_j\}$

& $\sum_{j=1}^{\infty} (x, e_j) e_j$ is Fourier series

Th (Bessel inequality)

$$\|x\|^2 \geq \sum_{j \in J} |(x, e_j)|^2$$

there possibly are uncountably many zeroes and countably many non-zeroes.

Th let $\{e_n\}_{n \in \mathbb{N}}$ be an orthonormal system, $c_n \in \mathbb{F}$ satisfies $\sum |c_n|^2 < +\infty$

then $\exists\, x \in \mathcal{H}$ s.t. $c_n = (x, e_n)$

We can choose

Moreover $x = \sum_{n=1}^{\infty} c_n e_n$ strong convergence

Th-Def. let $\{e_n\}$ be an orthonormal system

T.F.A.E

(1) $\forall x \in \mathcal{H}$: $x = \sum_{n=1}^{\infty} (x, e_n) e_n$

(2) $\forall x \in \mathcal{H}$: $\|x\|^2 = \sum_{n=1}^{\infty} |(x, e_n)|^2$ Parseval's identity

(3) $(x, e_n) = 0 \ \forall n \Rightarrow x = 0$

(4) $\overline{\text{Span}\{e_n\}} = \mathcal{H}$

Such systems are called complete orthonormal systems.

Proof: in functional analysis.

26/11/2012

$\ell_2 \quad \{e_n\}_{n=1}^{\infty}$

$e_n = (0, \ldots, 0, 1, 0, \ldots)$

$L_2(0, 2\pi)$

$$e_n = \frac{e^{int}}{\sqrt{2\pi}}$$

$\{e_n\}_{n=-\infty}^{+\infty}$

Th. (Reisz representation th-m)

Let $f : \mathcal{H} \to \mathbb{F}$ be a bounded linear functional

then $\exists!\, z \in \mathcal{H}$

s.t. $f(x) = (x, z)$ $(\forall x \in \mathcal{H})$

Moreover, $\|f\|_{\mathcal{H}^*} = \|z\|_{\mathcal{H}}$

H/W: uniqueness & rest

Proof: $(\exists)$ if $f = 0$, then $z = 0$

Suppose, $f \neq 0$,

then $\operatorname{Ker} f \neq \mathcal{H}$ is closed

therefore $(\operatorname{Ker} f)^\perp \neq \{0\}$

Take $y \in (\operatorname{Ker} f)^\perp \quad y \neq 0$

Then $\forall x \in \mathcal{H}$

$f(x)\, y - f(y)\, x \in \operatorname{Ker} f$

and, therefore,

$$\underbrace{(f(x)y - f(y)x, y)}_{\substack{\shortparallel \\ f(x)\|y\|^2 = (x, \overline{f(y)}\, y)}} = 0$$

thus, $f(x) = (x, z)$, where z

$$z = \frac{\overline{f(y)}\, y}{\|y\|^2}. \quad \square$$

Th. Let $A \in B(\mathcal{H}_1, \mathcal{H}_2)$

Then $\exists!$ operator $A^* \in B(\mathcal{H}_2, \mathcal{H}_1)$

s.t. $\boxed{(Ax, y) = (x, A^* y) \quad x \in \mathcal{H}_1 \quad y \in \mathcal{H}_2}$

Moreover, $\boxed{\|A^*\| \le \|A\|}$

Def: A^* is called adjoint operator of A

Proof: Let $y \in \mathcal{H}_2$.

Then $x \longmapsto (Ax, y) =: f(x)$ is a lin. functional

Moreover, $|f(x)| = |(Ax, y)| \le \|A\| \|x\| \|y\|$, Cauchy-Schwartz

so f is bounded and

$$\|f\| \le \|A\| \|y\|$$

therefore, $\exists!$ $z \in \mathcal{H}_1$ s.t.

$$\begin{aligned} f(x) &= (x, z) \\ &\shortparallel \\ (Ax&, y) \end{aligned}$$

Moreover, $\|z\| = \|f\| \leq \|\ast\| \|y\|$

$$A(x, y) = (x, Ay)$$

we define $A^* y := z$

then A^* is linear

$\mathcal{H}/\mathcal{W}$: check

Moreover, $\frac{\|A^* y\|}{\|y\|} \leq \|A\|$,

thus $\|A^*\| \leqslant \|A\|$. $\square$

Th. Properties of Adjoint operator

see in the same space s.t product makes sense.

(i) $(\alpha_1 A_1 + \alpha_2 A_2)^* = \bar{\alpha}_1 A_1^* + \bar{\alpha}_2 A_2^*$

$A_1, A_2 \in B(\mathcal{H})$

$\alpha_1, \alpha_2 \in \mathbb{F}$

(ii) $(AB)^* = B^* A^*$

(iii) $A^{**} = A$

(iv) $\|A^*\| = \|A\|$

(v) $\|A^* A\| = \|AA^*\| = \|A\|^2$

(vi) If A^{-1} exists

then $(A^*)^{-1}$ exists

and $(A^*)^{-1} := (A^{-1})^*$

Proof:

(ii) $(ABx, y) = (Bx, A^*y) = (x, B^*A^*y) =$

$= (x, (AB)^*y)$

- C^* algebra
- algebra with involution

Therefore $(x, \underbrace{B^*A^*y - (AB)^*y}_{y}) = 0$

taking $x =$,

we see that $B^*A^*y = (AB)^*y$ $\forall y$

(iv) we proved $\|A^*\| \leq \|A\|$

Therefore $A = \|A^{**}\| \leq A^*$

(v) We have: $\|A^*A\| \leq \|A^*\| \|A\| = \|A\|^2$

On the other hand

$\|A\|^2 = \sup_{\|x\|=1} \|Ax\|^2 = \sup_{\|x\|=1} (Ax, A^{**}x) =$

$= \sup_{\|x\|=1} |(A^*Ax, x)| \leq \sup_{\|x\|=1} \|A^*Ax\| = \|A^*A\|$

just to be on safe side

(vi) $A^{-1}A = I = AA^{-1}$

Therefore, $(A^{-1}A)^* = I^* = I = (AA^{-1})^*$

$(A^{-1}A)^* = A^*(A^{-1})^*$, $(AA^{-1})^* = (A^{-1})^*A^*$

$\Rightarrow (A^*)^{-1} = (A^{-1})^*$ $\square$

If $A \in B(\mathcal{H})$

then $\mathrm{Ker}(A^*) = (\mathrm{Ran}\, A)^\perp$

$\mathrm{Ker}\,(A) = (\mathrm{Ran}\,(A^*))^\perp$

Proof: $y \in \ker(A^*) \Leftrightarrow A^* y = 0$

$\Leftrightarrow x(A^* y) = 0 \quad \forall x \in \mathcal{H}$

$\Leftrightarrow (Ax, y) = 0, \quad \forall x \in \mathcal{H}$

$\Leftrightarrow y \in (\operatorname{Ran} A)^\perp$

Corollary
$$\boxed{\begin{aligned}(\ker A^*)^\perp &= \overline{\operatorname{Ran} A}\\ (\ker A)^\perp &= \overline{\operatorname{Ran} A^*}\end{aligned}}$$

Def. Let $A \in B(\mathcal{H})$

(i) A is self-adjoint (symmetric),

if $\boxed{A^* = A}$

i.e. $(Ax, y) = (x, Ay) \quad \forall x, y$

(ii) A is normal if $\boxed{AA^* = A^* A}$

(iv) U is unitary, if $\boxed{U^{-1} = U^*}$

i.e. $UU^* = U^* U = I$

Def. Let $A : D_A \to \mathcal{H}$ be an unbounded operator, with domain $D_A \subset \mathcal{H}$ and $\overline{D_A} = \mathcal{H}$

for given $y \in \mathcal{H}$, consider the functional $f : D_A \to \mathbb{F}$

$$\underset{\in D_A}{x} \longmapsto (Ax, y) =: f(x)$$

Suppose that for a given y this

functional has a form $f(x) = (x, z)$ where z is unique

We need $\bar{D}_A = \mathcal{H}$ to guarantee

then we say that $\boxed{y \in D_{A^*} \text{ and } A^* y = z}$

if $f(x)$ can not be expression as (x, z) we say that $\boxed{y \in D_{A^*}}$

$$(Ax, y) = (x, A^* y)$$

just for some values of y, e.g. $y = 0$.

Def. An unbounded operator is called a self-adjoint operator if

$$\boxed{\begin{array}{l} (1)\ D_{A^*} = D_A, \text{ and} \\ (2)\ A^* = A \end{array}}$$

Def A is symmetric, if

$$\boxed{(Ax, y) = (x, Ay), \quad \forall x, y \in D_A}$$

this means that A^* is extension of A $(A^* \supset A)$ meaning: that

$$D_{A^*} \supset D_A \quad \text{and}$$

$$\boxed{A^*|_{D_A} = A|_{D_A}}$$

Examples $L_2[0,1]$

$$Af = if'$$

$$D_A = \{f \in L_2, f' \in L_2\} = \mathcal{H}^1[0,1] = W^{1,2}$$

= called Sobolev space

$$(Af,g) = i\int_0^1 f'\bar{g} = if\bar{g}\Big|_0^1 + \int_0^1 f\overline{(ig')} \overset{?}{=}$$

$$\overset{?}{=} (f, A^* g)$$

$$= \int_0^1 f \overline{A^* g}$$

Using some strange formula that implies $if\bar{g}\big|_0^1 = 0$

Thus, $A^* g = ig'$

$D_{A^*} = \{ g \in L_2,\ g' \in L_2,\ g(0) = 0,\ g(1) = 0 \}$

$Af = if'$ $\quad D_A = \{ f \in K;\ f(0) = 0,\ f(1) = 0 \}$

$$(Af,g) = i\int^1 f' g = i[f(1)\overline{g(1)} - f(0)\overline{g(0)}] +$$

$$+ i\int_0^1 f\overline{(ig')}$$

$$=$$

Thus $A^* g = ig'$ (the same formula)

$D_{A^*} = \{ g \in L_2,\ g' \in L_2 \}$

$D_A = \{ f \in K;\ f(1) = f(0) \}$

Thus $D_{A^*} = \{ g \in L_2,\ g' \in L_2,\ g(1) = g(0) \}$

self-adjoint ex.

$\Rightarrow$ A is self-adjoint.

Example $L_2[0,1]$

$$Af = -f'' \quad \left(\text{or } Af = -f'' + q(t) f(t) \right)$$

Shrödinger operator

interesting operator

just to have +ve spectrum

2nd Sobolev space $\mathcal{H}^2$

$$D_A = \{ f \in L_2,\ f'' \in L_2,\ f' \in L_2 \} = \mathcal{H}^2$$

$$(Af, g) = -\int_0^1 f'' \bar{g} = -f' \bar{g} \Big|_0^1 + \int f' \bar{g}' =$$

$$= \left[-f'(1)\bar{g}(1) \Big|_0^1 + f'(0)\bar{g}(0) + f(1)\bar{g}'(1) - f(0)\bar{g}'(0) \right]$$

$$-\int f \bar{g}''$$

$$D_{A^*} = \{ g \in \mathcal{H}^2,\ g(1) = g(0) = g'(1) = g'(0) = 0 \}$$

Self-adjoint boundary conditions:

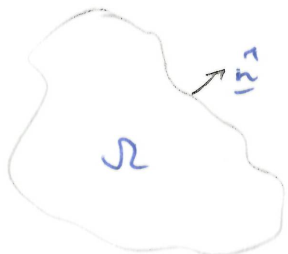
(1) $f(1) = \beta f(0)$, $f'(1) = \beta f'(0)$, $|\beta| = 1$ — Periodic b.c.

(2) $f'(1) = \alpha f(1)$, $f'(0) = \alpha f(0)$, $\alpha \in \mathbb{R}$ — Robin b.c.

(3) $f(0) = f(1) = 0$ — Dirichlet b.c.

(4) $f'(0) = f'(1) = 0$ — Neumann b.c.

Example



$\Omega \subset \mathbb{R}^d$

$\partial\Omega$ is smooth $\rightarrow$ for simplicity means: a well-defined outside looking vector

$\mathcal{H} = L_2(\Omega)$

$$Af = -\Delta f := -\sum_{j=1}^{d} \frac{\partial^2 f}{\partial x_j^2}$$

$$D_A = \{ f \in L_2,\ \partial_j f \in L_2,\ \partial_{jk} f \in L_2 \} = H^2(\Omega)$$

$\partial_j = \partial_{x_j}$

Stokes form = = integr. by parts in Diff. G.

$$(Af, g) = -\int_\Omega \Delta f\, g = \int_{\partial\Omega} \partial_{\vec{n}} f \cdot g + \int_\Omega \nabla f \nabla g =$$

$$= -\int_{\partial\Omega} \partial_{\vec{n}} f \cdot g + \int_{\partial\Omega} f \cdot \partial_{\vec{n}} g - \int_\Omega f \cdot \Delta g$$

$$A^* g = -\Delta g$$

$$D_{A^*} = \{ g \in H^2(\Omega),\ g|_{\partial\Omega} \equiv 0,\ \partial_{\vec{n}} g|_{\partial\Omega} = 0 \}$$

self-adjoint B.C.

(1) $f|_{\partial\Omega} = 0$ — Dirichlet b.c.

(2) $\partial_{\vec{n}} f|_{\partial\Omega} = 0$ — Neumann b.c.

(3) $\partial\Omega = \partial_1\Omega \cup \partial_2\Omega$ — Mixed B.C.

$f|_{\partial_1\Omega} = 0$

$\partial_{\vec{n}} f|_{\partial_2\Omega} = 0$

(4) $\partial_{\vec{n}} f|_{\partial\Omega} = \alpha f|_{\partial\Omega}$ (Robin b.c.)

$\alpha = \alpha(x) \in \mathbb{R}$

Th. $A \in B(\mathcal{H})$

A is normal iff. $\boxed{\begin{array}{l} \forall x \in \mathcal{H} \\ \|Ax\| = \|A^* x\| \end{array}}$

Proof:

$\Rightarrow$ $\|Ax\|^2 = (Ax, Ax) = (x, A^*Ax)$ if A is normal

$\|A^*x\|^2 = (A^*x, A^*x) = (x, AA^*x)$

~~if A~~

$\Leftarrow$ We know that $(Ax, Ax) = (A^*x, A^*x)$

$(Ax, Ay) = (A^*x, A^*y)$

Use polarization identity:

$\forall x, y \in \mathcal{H}$ $(Ax, Ay) = (A^*x, A^*y)$

Therefore, $(A^*Ax - AA^*x, y) = 0$

and $A^*Ax - AA^*x = 0$,

so $A^*A = AA^*$ $\square$

Th. Suppose $A \in B(\mathcal{H})$ be ~~bounded~~ normal

then (1) $\boxed{\begin{array}{l} (\operatorname{Ran} A^*)^\perp = \ker A \\ (\operatorname{Ran} A)^\perp = \ker A^* \end{array}}$

(2) $\boxed{Ax = \lambda x \Rightarrow A^*x = \bar{\lambda} x}$

(3) Eigenvectors corresponding to different eigenvalues of A are orthogonal to each other

Proof: ① $\|Ax\| = \|A^* x\|$, $\Rightarrow$

$x \in \operatorname{Ker} A \iff x \in \operatorname{Ker} A^*$

LHS $= 0$ iff RHS $= 0$

② $Ax = \alpha x \iff$ ~~$A^* x = \bar{\alpha} x$~~

$\iff x \in \operatorname{Ker}(A - \alpha I)$

$\iff x \in \operatorname{Ker}(A - \alpha I)^* = \operatorname{Ker}(A^* - \bar{\alpha} I)$

$\iff A^* x = \bar{\alpha} x$

③ Suppose, $Ax = \alpha x$ and

$Ay = \beta y$, $\alpha \neq \beta$

then

$\alpha(x, y) = (Ax, y) = (x, A^* y) = (x, \bar{\beta} y)$

$= \beta (x, y)$

as $\alpha \neq \beta \Rightarrow (x, y) = 0$ $\square$

Th Self-adjoint operator properties

Suppose $A \in B(H)$, $A = A^*$

(1) $\boxed{\lambda \in \mathbb{R} \Rightarrow \lambda A \text{ is a self-adj.}}$

2) $\boxed{A_1 + A_2 \text{ is a self-adj-op}}$

(3) If ~~Also~~ $A_1 A_2 = A_2 A_1$ (commute)

then $\boxed{A_1 A_2 \text{ is s.-a.}}$

$(A_1 A_2)^* = A_2^* A_1^* =$

$A_2 A_1 = A_1 A_2$

(4) $\|A_n - A\| \xrightarrow{n \to \infty} 0$, A_n are s. a.

then $\boxed{A \text{ is self. adj.}}$

Proof: Exercise!

Def. Let $A \in \mathcal{B}(H)$
the quadratic form of A

$$\boxed{\begin{aligned} q_A(x) &:= (Ax, x) \\ q_A &: \mathcal{H} \to \mathbb{F} \end{aligned}}$$

If A is s-a then

$$(Ax, x) = (x, Ax) = \overline{(Ax, x)}$$

so $(Ax, x) \in \mathbb{R}$

Thm Let $\mathbb{F} = \mathbb{C}$
then $\boxed{A = A^* \Leftrightarrow q_A(x) \in \mathbb{R}, \ \forall x \in \mathcal{H}}$ $\mathcal{H}$

Proof: ($\Rightarrow$) have already proven
($\Leftarrow$) $(Ax, x) = \overline{(Ax, x)} = (x, Ax)$

We need to prove:

$$4(Ax, y) \overset{?}{=} 4(x, Ay)$$

But $4(Ax, y) = (A(x+y), A(x+y)) -$
$- (A(x-y), A(x-y)) + i(A(x+iy), A(x+iy))$
$- i(A(x-iy), A(x-iy))$

polarization identity for operator

03/12/12

Corollary If $A = A^*$
then all its eigenvalues are real, and eigenfunctions corresponding to different eigenvalues are orthogonal to each other.

Proof Suppose, $Ax = \lambda x$.
Then $\lambda = \frac{(Ax, x)}{\|x\|^2} \in \mathbb{R}$ □

The 2nd st.: (orthogonality one) has been proved for normal A. & self-adj are normal

Thm Suppose, $P \in \mathcal{B}(H)$ is a projection
TFAT:
(1) P is s. a.
(2) P is normal
(3) $\operatorname{Ran} P = (\ker P)^\perp$
(4) $(Px, x) = \|Px\|^2, \ \forall x \in H$

Proof: (i) $\Rightarrow$ (ii) obvious
(ii) $\Rightarrow$ (iii) for normal operators we have:
$$\ker P = (\operatorname{Ran} P)^\perp$$
$$\therefore (\ker P)^\perp = \overline{\operatorname{Ran} P} = \operatorname{Ran} P$$
since P is a projection

(iii) $\Rightarrow$ (i)

Recall: $\operatorname{Ran} P = \ker(I-P)$
and $\ker P = \operatorname{Ran}(I-P)$

$\Rightarrow \operatorname{Ran} P = (\ker P)^\perp = \operatorname{Ran}(I-P)^\perp$

what P:

Therefore,
$$(Px, (I-P)y) = 0 = ((I-P)x, Py) \quad \forall x, y \in H$$
Thus, $(Px, y) = (Px, y) + \underbrace{\cancel{(I-P)x}}_{=0}\ (Px, (I-P)y) = (Px, Py)$
$$= (Px, Py) + \underbrace{((I-P)x, Py)}_{=0} = (x, Py)$$
so P is symmetric (= self adjoint)

(iii) $\Rightarrow$ (iv)

We have: $(Px, (I-P)x) = 0$, therefore

$$(Px, x) = (Px, P) - \underbrace{(Px, (I-P)x)}_{=0} =$$

$$= (Px, Px) = \|Px\|^2$$

iv => i

Suppose, $\mathbb{F} = \mathbb{C}$.
then, $(Px, x) = \|Px\|^2 \in \mathbb{R}$, so P is ≥ 0.

suppose, $\mathbb{F} = \mathbb{R}$, ... $\square$

! exercis on exam !

Def. If P is a projection and any (all) of these conditions are satisfied then P is called orthogonal projection

Th Suppose $A \in \mathcal{B}(H)$, and $c > 0$, $c \in \mathbb{R}$
s.t. $|(Ax, x)| \geq c\|x\|^2$
then
- A is invertible, and
- $\|A\|^{-1} \leq \frac{1}{c}$

$(0, x) = (0 \cdot x, x) = 0 \cdot (x, x) = 0$!

Proof: we have:

$$c\|x\|^2 \leq |(Ax, x)| \overset{\text{c. sh.}}{\leq} \|Ax\| \|x\|$$

So $\|Ax\| \geq c\|x\|$ * and, by Theorem ...,
Ker $A = \{0\}$ and Ran A is closed

Suppose, $x \in (\mathrm{Ran}(A))^\perp$
Then $(Ax, x) = 0$ so $x = 0$
thus $(\mathrm{Ran}\, A)^\perp = \{0\}$, and

$$\overline{\mathrm{Ran}\, A} = \{0\}^\perp = H$$

$\overline{\mathrm{Ran}\, A} = \mathrm{Ran}\, A$ thus A is invertible

The ineq.

$$\|A^{-1}\| \le \frac{1}{c}$$ follows from *

□

Def Let $A \in B(H)$. the following set is called the numerical range of A

$$\text{Num}\, A := \{(Ax, x),\ \|x\| = 1\} =$$
$$= \left\{ \frac{(Ax, x)}{\|x\|^2}\quad x \in H \setminus \{0\} \right\}$$

if $x_1, x_2 \in A$

$\Rightarrow$  $\in A$

Remark
$$\boxed{\begin{array}{l} 1.\ \text{Num}\, A \text{ is convex} \\ 2.\ \text{Num}\, A \subset B_c(0, \|A\|) \end{array}}$$

Th $\boxed{\sigma(A) \subset \overline{\text{Num}\, A}}$

Num A is not always closed.

Proof: Suppose $\lambda \notin \overline{\text{Num}\, A}$

$d(\lambda x, \overline{\text{Num}\, A})$

$\Rightarrow \exists\ d = d(\lambda, \overline{\text{Num}\, A}) > 0$

s.t. $\forall x \in H \setminus \{0\}$ satisfies

$$\left| \frac{(Ax, x)}{\|x\|^2} - \lambda \right| \ge d$$

Therefore $|(Ax, x) - \lambda(x, x)| \ge d\|x\|^2$

$$|((A - \lambda I)x, x)| \ge d\|x\|^2$$

$\therefore$ by prev. Th. $A - \lambda I$ is inv,

so $\lambda \in \rho(A)$

Therefore $\sigma(A) \subset \overline{\text{Num}\, A}$ □

Th
$$\boxed{\begin{array}{l} \text{Let } A = A^* \\ \text{then } \|A\| = \sup\limits_{\|x\|=1} |(Ax, x)| \end{array}}$$

Proof: Put $c := \sup\limits_{\|x\| = 1} |(Ax, x)|$

Then $c \le \sup_{\|x\|=1} \|Ax\|\cdot\|x\| = \|A\|$
C.-S.

$c := \sup |(Ax,x)| = \sup_{x\neq 0} \frac{|(Ax,x)|}{\|x\|^2}$

Let us prove that $\|A\| \overset{?}{\le} c$, or

$\|Ax\| \le c$ for each $\|x\| = 1$

We have:

$(A(x+y), x+y) - (A(x-y), x-y) =$

$= 2(Ay, x) + 2(Ax,y) =$

$= 2[(Ax,y) + \overline{(Ax,y)}] = 4\,\mathrm{Re}[(Ax,y)]$

$\frac{|(Ax,x)|}{\|x\|^2} \le C$

Therefore,

$4\,\mathrm{Re}[(Ax,y)] \le c\|x+y\|^2 + c\|x-y\|^2 =$
if $\|x\| = 1 = \|y\|$

$\overset{\text{Par. law}}{=} c(2\|x\|^2 + 2\|y\|^2) = 4c$

Suppose, $Ax = 0$, Then $\|Ax\| \le c$.

suppose, $Ax \neq 0$ i.e. $\|Ax\| > 0$

then we put $y = \frac{Ax}{\|Ax\|}$

and get: $\mathrm{Re}(Ax,y) \le c$

$\frac{\|Ax\|^2}{\|Ax\|} = \|Ax\|$

so $\|Ax\| \le c$ if $\|x\| = 1$

$\therefore \|A\| = c$ Q.E.D. $\square$

Th. Let $A = A^*$

Put $m = \inf_{\|x\|=1} (Ax,x)$

$= \inf_{x\neq 0} \frac{(Ax,x)}{\|x\|^2}$

$M = \sup_{\|x\|=1} (Ax,x)$

Therefore

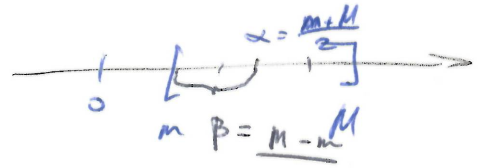
$\boxed{\begin{array}{l}(1)\ \sigma(A) \subset [m, M] \\ (2)\ m, M \in \sigma(A)\end{array}}$

Proof. (1) We have proved that

$\overline{\mathrm{Num}\, A} = [m, M]$ and

therefore $\sigma(A) \subset [m, M]$

(2)



$0 \in [m, M]$
or $0 \notin [m, M]$
it doesn't matter

Put $\alpha = \frac{m+M}{2}$

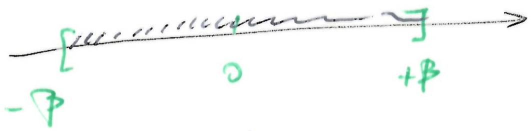
$\beta = \frac{M-m}{2}$

Denote $B = A - \alpha I$, (B is s. adj.)

as A is s.a.
αI is s.a.

then, $\inf_{\|x\|=1} (Bx, x) = m - \alpha = \frac{m-M}{2} = -\beta$ $B = B^*$

and $\sup_{\|x\|=1} (Bx, x) = M - \alpha = \beta$



then $\sup_{\|x\|=1} |(Bx, x)| = \beta$ and

the prev. theorem implies $\|B\| = \beta$

There exists a sequence $x_n \in \mathcal{H}$, s.t.

$\|x_n\| = 1$ and $(Bx_n, x_n) \to \beta$

then $\|(B - \beta I)x_n\|^2 = ((B - \beta I)x_n, (B - \beta I)x_n) =$

$= \|Bx_n\|^2 + \beta^2 \|x_n\|^2 - 2\beta(Bx_n, x) \leq$ $\quad \|x_n\|^2 = 1$

$\leq 2\beta^2 - 2\beta \underbrace{(Bx_n, x_n)}_{\to \beta} \underset{n\to\infty}{\to} 0$

$B = A - \alpha I$

Claim: therefore, $(B - \beta I)$ cannot have bounded inverse.

Suppose not; $(B-\beta I)^{-1} = R$ is bounded

then $1 = \|x_n\| = \|R(B-\beta I)x_n\| \leq \|R\| \|(B-\beta I)x_n\|$
$\to 0$ #

Thus $(B-\beta I) = A - (\alpha+\beta)I =$

$= A - MI$ is not invertible

so $M \in \sigma(A)$

Similarly, $m \in \sigma(A)$ $\square$

Corollary Let $A = A^*$
then $\exists \lambda \in \sigma(A)$ s.t.
$|\lambda| = \|A\|$

In particular, $r(A) = \|A\|$

Corollary If $A = A^*$ and
$\sigma(A) = \{0\}$
then $A = 0$

Th Suppose A is normal
then $\|A\| = r(A) = \sup_{\|x\| = 1} |(Ax, x)|$

Th (Hilbert – Schmidt)

Suppose, $A = A^*$ is compact then $\exists$ an orthonormal set $\{e_n\}_{n=1}^N$ s.t. $N \in \mathbb{N} \cup \{+\infty\}$

$$A e_n = \lambda_n e_n \quad \lambda_n \neq 0 \quad \text{and}$$

$\forall x \in \mathcal{H}$ has a decomposition $\quad ! \, c_n = (x, e_n)$

$$x = \sum_{n=1}^N c_n e_n + y \quad \text{s.t.}$$

$$y \in \operatorname{Ker} A \quad \text{and}$$

$$A x = \sum_{n=1}^N c_n \lambda_n e_n$$

Moreover, $\sigma(A) \setminus \{0\} = \bigcup_{n=1}^N \lambda_n \subset \mathbb{R}$

$|\lambda_{n+1}| \leq |\lambda_n|$ and

$\lim_{n \to \infty} \lambda_n = 0$ if $N = +\infty$

Proof: A is compact, so $\sigma(A) \setminus \{0\}$ is at most countable $\quad$ = finite or infinite set

and consists of points $\mu_1, \mu_2, \mu_3, \dots, \mu_N$ $\quad$ can be finit / infinit

$\lim_{n \to \infty} \mu_n = 0$, $\quad$ if μ_n are infinit

$|\mu_1| \geq |\mu_2| \geq |\mu_3| \geq \dots$ $\quad$ 2 different μ_k -> are orthogonal as $A = A^*$

Since $A = A^*$, $\mu_j \in \mathbb{R}$

Denote by N_k the eigenspace corresponding to μ_k, $\dim N_k < +\infty$

Take an orthonormal basis of each N_k and write the resulting collection of vectors in a sequence $\{e_n\}_{n=1}^N$

(so that elements of N_k come before elements of N_{k+1})

Then $Ae_n = \lambda_n e_n$ for $\lambda_n \in \{\mu_n\}$.

Denote $L = \overline{\operatorname{span}\{e_n\}_{n=1}^{N}}$

<u>claim</u> $\operatorname{Ker} A = L^\perp$

Proof: $\operatorname{Ker} A \subset L^\perp$, since each 2 eigenfunctions corresponding to different eigenvalues are orthogonal.

$$\operatorname{Ker} A \overset{?}{\supset} L^\perp.$$

Suppose, $y \in L^\perp$, then

$$(Ay, e_k) = (y, Ae_k) = \lambda_k (y, e_k) = 0,$$

So $Ay \in L^\perp$, thus $L^\perp$ is invariant under A.

Consider $B = A\big|_{L^\perp}$ $\Rightarrow B$ is compact, B is self-adj.

as A is compact
B has no allowed to have non-zero eigenvalue.

B has no non-zero eigenvalue (since if $Bx = \lambda x$, $\Rightarrow Ax = \lambda x$, and by construction $x \in L$)

Therefore, $\sigma(B)$ has no non-zero point.

Since B is compact, any non-zero point in the spectrum is an eigenvalue so $\sigma(B) = \{0\}$, and $r(B) = 0$, and

$B = 0$. ■

Therefore, $L^\perp = \operatorname{Ker} A$

and $H = L \oplus \operatorname{Ker} A$ □