M111 Spectral Theory Notes

Based on the 2011 autumn lectures by Prof L Parnovski

M111

SPECTRAL THEORY

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90% exam
10% coursework

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INTRODUCTION

Suppose we have a square matrix, $n \times n$,

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

We find the eigenvalues by setting

$$\det(A - \lambda I) = 0$$

$\Leftrightarrow$ λ is an eigenvalue.

But this is not very convenient for us because we want to deal with infinite-dimensional spaces and there are no determinants here.

We found eigenvectors by doing:

$$\lambda \text{ is an eigenvalue of } A \Leftrightarrow \exists \underline{v} \neq 0 \text{ s.t. } A\underline{v} = \lambda \underline{v}.$$

$\Leftrightarrow$ $(A - \lambda I)$ is not a bijection
$\Leftrightarrow$ $(A - \lambda I)$ is not an injection
$\Leftrightarrow$ $(A - \lambda I)$ is not a surjection

Eigenvalues have different multiplicities – algebraic
– geometric.

Example: $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Eigenvalues are:

$$ch_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = \lambda^2$$

The algebraic multiplicity of $(\lambda = 0)$ is 2.

The collection of all eigenvectors, plus zero, forms a vector space.

ie $\{ \underline{v} : A\underline{v} = \lambda \underline{v} \}$

The geometric multiplicity of λ is $\dim \{ \underline{v} : A\underline{v} = \lambda \underline{v} \}$

Example $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Know $\lambda = 0$

Look at $A\underline{v} = 0$:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \quad \Rightarrow \quad v_2 = 0$$

$\Rightarrow$ any vectors satisfying $A\underline{v} = 0$ is of the form $\begin{pmatrix} v_1 \\ 0 \end{pmatrix}$

ie. the dimension is 1,

ie $\dim \left\{ \begin{pmatrix} v_1 \\ 0 \end{pmatrix} : v_1 \in \mathbb{R} \right\} = 1$.

so the geometric multiplicity is 1.

If the algebraic multiplicity = geometric multiplicity for all geometric multiplicities, then

A is diagonalisable

i.e. $\exists\, U$ s.t. $U^{-1}AU = D$ (diagonal matrix)

We want everything to be diagonalisable.
Are there special cases for when we can look at a matrix and say "yes - diagonalisable"?

Yes! - Symmetric matrices! and we can find an orthogonal U.

Very often we will have huge matrices (e.g. to find eigenvalues of the Millennium Bridge). People use numerical methods here, obviously there are some approximations.

Example:

$$A = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{pmatrix} \qquad 31 \times 31$$

What is the spectrum of A? i.e.
What is the collection of eigenvalues?

$$ch_A(\lambda) = (-\lambda)^{31}$$

A.M. 31
G.M. 1

So the spectrum of A, $\sigma(A) = \{0\}$.

- Consider $2A = \begin{pmatrix} 0 & 2 & & & \\ & 0 & 2 & & \\ & & \ddots & \ddots & \\ & & & 0 & 2 \\ & & & & 0 \end{pmatrix}$ 31×31

$\sigma(2A) = 0.$

- Consider $B = \begin{pmatrix} 0 & 2 & & & \\ & 0 & 2 & & \\ & & \ddots & \ddots & \\ & & & 0 & 2 \\ 2^{-30} & & & & 0 \end{pmatrix}$ 31×31

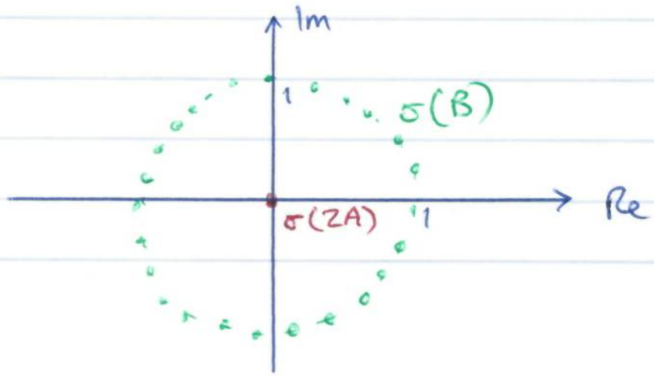
$$ch_B(\lambda) = \det(\lambda I - B)$$

$$= \begin{vmatrix} -\lambda & 2 & & & & \\ & -\lambda & 2 & & & \\ & & -\lambda & 2 & & \\ & & & \ddots & \ddots & \\ & & & & \ddots & 2 \\ 2^{-30} & & & & & -\lambda \end{vmatrix}$$

$$= (-\lambda)^{31} + 2^{-30}\, 2^{30} = -\lambda^{31} + 1$$

$$\Rightarrow \lambda^{31} = 1, \quad \text{roots of unity}$$

so



Extending to infinite dimensions

Defn: A normed space V over a field $\mathbb{F}$ $(=\mathbb{R}$ or $\mathbb{C})$ is a collection of: a linear vector space V and a mapping $\|\,\|: V \to \mathbb{R}_+$ ← including zero

(i) $\|v\| = 0 \iff v = 0 \qquad \forall v \in V$

(ii) $\|v+w\| \leq \|v\| + \|w\| \qquad \forall v, w \in V$

(iii) $\|\lambda v\| = |\lambda| \, \|v\|. \qquad \forall v \in V, \quad \lambda \in \mathbb{F}$

Then $d(v, w) := \|v - w\|$ gives a structure of a metric space.

defn: if every Cauchy sequence converges inside it
eg. $\mathbb{Q}$ not complete $\because \sqrt{2} \notin \mathbb{Q}$.

If this metric space is complete, then we say V is a Banach space.

Defn: H is an inner product space if
(i) H is a vector space
(ii) $\exists$ a mapping $(\cdot, \cdot) : H \times H \to \mathbb{F}$
s.t.
- $(v, v) \geq 0$
- $(v, v) = 0 \iff v = 0$
- $(\lambda v + \mu w, u) = \lambda(v, u) + \mu(w, u)$
- $(v, w) = \overline{(w, v)}$ (complex conj.)

If H is an inner product space, then
$$\|v\| := \sqrt{(v, v)}$$
makes it a proper norm (ie. the norm satisfies all the properties of a norm).

normed + complete

If H with this norm is a complete (Banach) space, then we say H is a <u>Hilbert</u> space.

<u>Examples</u>: (1) Any finite-dimensional normed space is a Banach space.

(2) l_p, $p \in \mathbb{R}, p > 0$

must converge

$$l_p = \left\{ x = (x_1, x_2, x_3, \ldots) : x_j \in \mathbb{F}, \sum_{j=1}^{\infty} |x_j|^p < \infty \right\}$$

with $\|x\| = \|x\|_p = \left(\sum_{j=1}^{\infty} |x_j|^p \right)^{1/p}$

If $1 \le p < \infty$, this is a Banach space.

(3) $l_\infty = \{ x = (x_1, x_2, \ldots) : \{x_j\} \text{ is bounded} \}$

with $\|x\| = \|x\|_\infty = \sup_j |x_j|$ is Banach space

(4) $c = \{ x = (x_1, x_2, \ldots) : \lim_{j \to \infty} x_j \to 0 \}$

with $\|x\| = \|x\|_c = \sup_j |x_j| = \max_j |x_j|$ is Banach space

(5) $C[a,b] = \{ f : [a,b] \to \mathbb{F}, f \text{ is cts} \}$

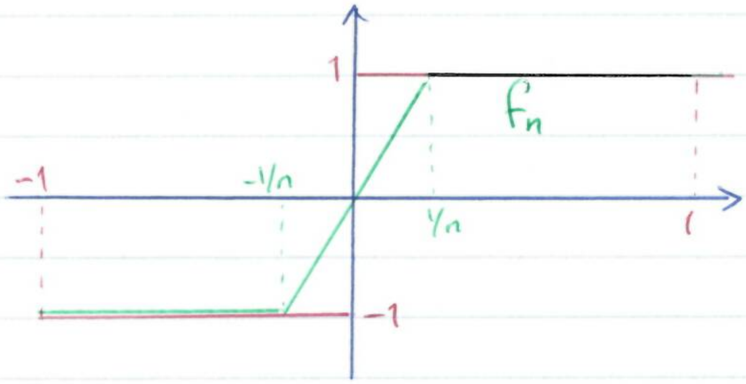
with $\|f\| = \sup_{x \in [a,b]} |f(x)| = \max_{x \in [a,b]} |f(x)|$ is B.s.

(6) $C_p[a,b] = \{ f : [a,b] \to \mathbb{F}, f \text{ is cts} \}$, $1 \le p < \infty$

with $\|f\|_p = \left[\int_a^b |f(x)|^p \, dx \right]^{1/p}$

is normed, but not complete $\Rightarrow$ not B.s.

$C[-1,1]$

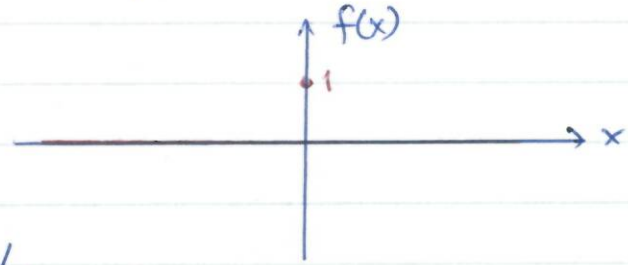


Then $\{f_n\}$ is a Cauchy sequence which has no limit in $C_p[-1,1]$.

If we take the completion of $C_p[a,b]$ and add all limits of all Cauchy sequences (e.g. the step fⁿ drawn above), we will obtain a new function space, $L_p[a,b]$.

$C_p \xrightarrow{\text{complete}} L_p$

Consider this function:



$$f(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}$$

$$\|f\|_p = \left[\int_{-1}^{1} |f(x)|^p \, dx \right]^{1/p} = 0.$$

l_2 is an inner product space with $(x,y) = \sum_{j=1}^{\infty} x_j y_j$

$C_2[a,b]$ or $L_2[a,b]$ are also inner product spaces with

$$(f,g) = \int_a^b f(x)\, g(x)\, dx$$

Operators

If V and W are linear spaces, then a mapping $A: V \to W$ is a <u>linear operator</u> if it preserves/respects the linear structure, ie.

$$A(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 A(v_1) + \lambda_2 A(v_2)$$

<u>Theorem</u>: Let $A: V \to W$ be linear and V, W are normed spaces. Then the FAE:

(1) A is cts.
(2) A is cts. at any given pt, e.g. 0.
(3) A is bounded, ie.
$\exists\, c > 0$ s.t. $A(B_1^V(0)) \subset B_c^W(0)$
(4) $\exists\, c > 0$ s.t. $\|A(v)\| \leq c\|v\|$.

<u>Defn</u>: Whenever A satisfies any of these conditions, we say A is bounded, with norm

$$\|A\| = \sup_{\|v\| \leq 1} \|Av\|_W$$

$Av = A(v)$

$$= \sup_{\|v\|_V \leq 1} \|Av\|$$

$$= \sup_{v \in V} \frac{\|Av\|}{\|v\|}$$

$$= \inf\{c : \|Av\| \leq c\|v\|\}$$

<u>Defn</u>: (handwaving) A is closed if $x_n \to x$ and Ax_n converges $\Rightarrow Ax_n \to Ax$

lin.op from $V \to W$

Banach spaces

<u>Defn</u>: $A \in L(V,W)$ is an unbounded operator if $\exists\, D_A$; a subspace of V and $A : D_A \to W$ s.t. A is not continuous $\leftarrow$ ∵ cts $\Leftrightarrow$ bdd

Usually we will assume that D_A is dense in V, ie. $cl(D_A) = V$.

<u>Example</u>: A acts in $C[0,1]$, $A : f \to f'$.

$$D_A = \{ f \in C[0,1] : f' \in C[0,1] \}.$$

Take $f_n = e^{inx}$.

Then $\|f_n\| = \sup_{x \in [0,1]} |e^{inx}| = 1$.

$$\|Af_n\| = \|f_n'\| = \sup_{x\in[0,1]} |in\, e^{inx}| = n.$$

Thus A is unbounded.

A is the linear operator

<u>Properties of $\|A\|$</u> (the norm)

(1) $\|Av\| \leq \|A\| \|v\|$

(2) $\|\lambda A\| = |\lambda| \|A\|$

(3) $\|A+B\| \leq \|A\| + \|B\|$.

(4) $\|AB\| \leq \|A\| \|B\|$

Let V be a Banach space. We denote by $B(V)$ the collection of all bounded operators $A: V \to V$.

This is a normed space wrt $\|\cdot\|$, and it is bounded.

$\boxed{L2}$

<u>Notation</u>: $L(X, Y)$ linear operators from normed spaces X ~~to~~ Y.

$D_A \subset X$, $A: D_A \to Y$

$B(X,Y)$ bounded operators from X to Y

$B(X,X) \equiv B(X)$

<u>Remark</u>: $B(X)$, X Banach, with the operator norm forms a Banach algebra.

<u>Def^n</u>: Let X be a Banach space. We say that a sequence of elements $x_n \in X$ converges to x <u>strongly</u>, if $\|x - x_n\| \to 0$.

<u>Notation</u>: $x_n \to x$ or $\text{s-}\lim\limits_{n\to\infty} x_n = x$ or $\lim\limits_{n\to\infty} x_n = x$

Consider a linear functional $f: \underset{\text{Banach}}{X} \to \mathbb{F}$. The operator norm of f is called the norm of the functional.

<u>Def^n</u>: $\|f\| = \sup\limits_{\substack{x \in X \\ x \neq 0}} \frac{|f(x)|}{\|x\|}$. This norm is <u>finite</u> if f is bounded (cts).

<u>Def^{n}</u>: A dual space

$$X^* = X' = \{ f : X \to \mathbb{F} \ : \ \|f\| < \infty \}$$

<u>Def^{n}</u>: A sequence $x_n \in X$ converges to x <u>weakly</u> if $\forall f \in X^*$ we have $f(x_n) \to f(x)$

<u>Notation</u>: $w\text{-}\lim x_n = x$ or $x_n \longrightarrow x$

<u>Lemma</u>: Strong convergence $\Rightarrow$ weak convergence.

<u>Proof</u>: $| f(x_n - x) | \leq \|f\| \cdot \|x_n - x\| \to 0$ because $\|Ax\| \leq \|A\| \cdot \|x\|$

$\Rightarrow f(x_n) - x) \to 0$ □

However, if $X = l_p$, $1 \leq p < \infty$, then $X^* \cong l_q$ $(1 < p \leq \infty)$ with $\frac{1}{q} + \frac{1}{p} = 1$.

The natural isometry $I : l_q \to (l_p)^*$ is given by the following formula:

Let $y = (y_1, y_2, \ldots) \in l_q$.
Then $I(y) = f_y$, where, for $x = (x_1, \ldots) \in l_p$,

$$f_y(x) = \sum_{j=1}^{\infty} x_j y_j$$

<u>Example</u>: $X = l_2$. Consider $e_n = (0, 0, \ldots, \overset{n^{th}\text{ place}}{1}, 0, \ldots)$

Then $\| e_n - e_m \| = \sqrt{2} \not\to 0$ as $m, n \to \infty$
so $\{e_n\}$ is not Cauchy.

Claim: $e_n \longrightarrow 0$. Indeed, suppose $f \in (l_2)^*$.

Then $f = f_y$ with $y = (y_1, y_2, \dots) \in l_2$ and $f(e_n) = y_n \longrightarrow 0$, since $\sum y_n^2 < \infty$.

So $f(e_n) \to f(0)$, so $w\text{-}\lim(e_n) = 0$

Def^n: Let $A_n, A \in B(X,Y)$. We say that

(1) A_n converges to A uniformly if $\| A_n - A \| \to 0$.
$\lim A_n = A$

(2) A_n converges to A strongly if $\forall x \in X$, $A_n x \to Ax$ or $\| A_n x - Ax \|_Y \to 0$.
$s\text{-}\lim A_n = A$

(3) A_n converges to A weakly if $\forall x \in X$, $A_n x \rightharpoonup Ax$ (i.e. $\forall f \in Y^*$ $f(A_n x) \to f(Ax)$)
$w\text{-}\lim A_n = A$

Note: $(1) \Rightarrow (2) \Rightarrow (3)$.

Def^n: Let X be a normed space and $x_n, x \in X$. We say that $\sum_{n=1}^{\infty} x_n$ converges to x if

$$S_m \longrightarrow x, \text{ where } S_m = \sum_{n=1}^{m} x_n.$$

The series converges absolutely if $\sum_{n=1}^{\infty} \| x_n \|$ converges.

Thm : X is a Banach space. Then absolute convergence $\Longrightarrow$ convergence.

Proof: Exercise.

Thm : (Corollary from Hahn-Banach Thm).

Let X be a normed space and $x \in X$.
Then $\exists f \in X^*$ s.t. $\|f\| = 1$ and $|f(x)| = \|x\|$
(as large as it can be).

2. SPECTRAL THEORY OF BOUNDED AND GENERAL LINEAR OPERATORS

<u>Defn</u>: Let $A \in B(X,Y)$.

$$\mathrm{Ker}(A) = \{x \in X,\ Ax = 0\} \qquad \text{kernel} \qquad (\Leftrightarrow \text{inj.})$$

$$\mathrm{Ran}(A) = \{Ax : x \in \boldsymbol{X}\} \qquad \text{range} \qquad (\Leftrightarrow \text{surj.})$$

<u>Propn</u>: $\mathrm{Ker}(A)$ is a closed, linear subspace of X.
$\mathrm{Ran}(A)$ is a linear subspace of Y.

<u>Thm</u>: (Banach) (Corollary of open mapping).
Suppose X, Y are Banach, $\mathrm{Ker}\,A = \{0\}$, $\mathrm{Ran}\,A = Y$.

Then $A^{-1} \in B(Y,X)$.

<u>Defn</u>: Let $A : X \to Y$. We say that A_r^{-1} (A_l^{-1}) is a right (left) inverse of A if $AA_r^{-1} = I_Y$ ($A_l^{-1}A = I_X$)

Suppose A_r^{-1} and A_l^{-1} exist. Then

$$A_l^{-1} = A_l^{-1} I_Y = A_l^{-1}(A \cdot A_r^{-1}) = (A_l^{-1}A)A_r^{-1}$$
$$= I A_r^{-1} = A_r^{-1}.$$

<u>Example</u>: $X = l_2$. $\quad x = (x_1, x_2, \ldots)$

$Ax = (x_2, x_3, \ldots) \in l_2$ "left shift"
$Bx = (0, x_1, x_2, \ldots) \in l_2$ "right shift".

$A, B : X \to X$.

$$A \sim \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & \ddots & & \\ & & & \ddots & 1 & \\ & & & & 0 & 1 \\ & & & & & \ddots \end{pmatrix}$$

$$Ax = \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & \ddots & & \\ & & & & 0 & 1 \\ & & & & & \ddots \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ \vdots \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ \vdots \end{pmatrix}$$

$$B \sim \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & 1 & 0 & \\ & & & \ddots & \\ & & & & 0 \\ & & & & 1 & \ddots \end{pmatrix}.$$

$$(AB)\,x = (x_1, x_2, \dots) = x, \qquad \text{ie } AB = I$$
$$(BA)\,x = (0, x_2, \dots) \neq x$$

$$\Longrightarrow \quad A = B_l^{-1}$$
$$B = A_r^{-1}$$

Thm: (1) A, B are invertible, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

(2) If AB is invertible and $AB = BA$, then A and B are invertible

(3) If A and B commute and A is invertible, then A^{-1} commutes with B.

Proof: (2) Let $S = (AB)^{-1}$.

Then $ABS = \underset{\parallel\, SBA}{SAB} = I$

$\Rightarrow BS = A_r^{-1}$

$SB = A_\ell^{-1}$

$\Rightarrow$ A has left & right inverse

$\Rightarrow$ A is invertible.

(3) $A^{-1}B = A^{-1}B\underbrace{AA^{-1}}_{I} = \underbrace{A^{-1}A}_{I}BA^{-1} = BA^{-1}$

so A^{-1} & B commute.

(First Pert. Thm)

Thm: Let X be Banach.

$A \in B(X)$

$\|A\| < 1$.

Then $I - A$ is ~~also~~ invertible and

$$(I-A)^{-1} = I + A + A^2 + \dots = \sum_{n=0}^{\infty} A^n \quad \text{and}$$

$$\|(I-A)^{-1}\| \leq \frac{1}{1-\|A\|}$$

<u>Proof</u>: <u>Claim 1</u>: The series converges absolutely.

Indeed, $\|A^n\| \leq \|A\|^n$.

Thus $$\sum_{n=0}^{\infty} \|A^n\| \leq \sum_{n=0}^{\infty} \|A\|^n$$

$$= \frac{1}{1-\|A\|}$$ Geometric series ($\|A\| \in \mathbb{R}$).

Thus $\sum_{n=0}^{\infty} A^n$ converges absolutely

and since $B(X)$ is Banach,

$\sum_{n=0}^{\infty} A^n = R$ converges.

Then: $$(I-A)R = (I-A)\lim_{m\to\infty} \sum_{n=0}^{m} A^n$$

$$= \lim_{m\to\infty} (I-A)(I+A+A^2+\cdots+A^m)$$

$$= \lim_{m\to\infty} (I-A^{m+1})$$

$\|A^{m+1}\| \leq \|A\|^{m+1} \to 0$

$\Rightarrow A^{m+1} \to 0$

$$= \underline{I}$$

Similarly, $R(I-A) = I$, so $R = (I-A)^{-1}$.

$$\|(I-A)^{-1}\| = \|R\| = \lim_{m\to\infty} \left\| \sum_{n=0}^{m} A^n \right\|$$

$$\leq \lim_{m\to\infty} \sum_{n=0}^{m} \|A^n\|$$

$$= \sum_{n=0}^{\infty} \|A^n\|$$

$$= \frac{1}{1-\|A\|}$$

Thm: (Second Perturbation Thm)

Let X be Banach.
$A \in B(X)$
A is invertible
$\|B\| < \frac{1}{\|A^{-1}\|}$ (and $\therefore$ B is bounded) ie $B \in B(X)$

Then $A+B$ is invertible with

$$(A+B)^{-1} = A^{-1} \sum_{n=0}^{\infty} (-BA^{-1})^n$$
$$= \left[\sum_{n=0}^{\infty} (-A^{-1}B)^n \right] A^{-1}, \quad \text{and}$$

$$\|(A+B)^{-1}\| < \frac{\|A^{-1}\|}{1 - \|B\| \|A^{-1}\|}$$

Proof: $A+B = \underbrace{A}_{\text{invertible}} \underbrace{(I + A^{-1}B)}_{\text{invertible by previous thm}}$
$= (I + BA^{-1}) A$

$$\Rightarrow (A+B)^{-1} = (I + A^{-1}B)^{-1} A^{-1}$$
$$= \left[\sum_{n=0}^{\infty} (-A^{-1}B)^n \right] A^{-1}$$

(Estimate is 'obvious' from previous thm).

The Spectrum

Assume $\mathbb{F} = \mathbb{C}$.

<u>Def^n</u>: Let X be Banach and $A \in B(X)$.

The resolvent set of A,

$$\rho(A) = \{ \lambda \in \mathbb{C} : (A - \lambda I) \text{ is invertible and } (A-\lambda I)^{-1} \in B(X). \}$$

The spectrum of A,

$$\sigma(A) = \mathbb{C} \setminus \rho(A).$$

λ is an eigenvalue of A if $\exists\, x \in X$, $x \neq 0$, s.t. $Ax = \lambda x$.

In this case, x is called an eigenvector.

<u>Thm</u>: λ is an eigenvalue $\Rightarrow$ $\lambda \in \sigma(A)$.

<u>Proof</u>: λ is an eigenvalue $\Rightarrow (A - \lambda I)x = 0$
$\Rightarrow x \in \operatorname{Ker}(A - \lambda I)$
$\Rightarrow (A-\lambda I)$ is not an injection
$\Rightarrow (A - \lambda I)$ is not a bijection
$\Rightarrow (A - \lambda I)$ is not invertible $\square$

<u>Examples</u> (1) If $\dim X < \infty$, then $\sigma(A) = \{\text{eigenvalues of } A\}$

(2) $X = C[0,1]$
$f = f(t) \in C[0,1]$
$(Af)(t) = t f(t)$

$$\|Af\| = \sup_{t\in[0,1]} |t\,f(t)|$$
$$\leq \sup |f(t)|$$
$$= \|f\|$$

Thus A is bounded and $\|A\| \leq 1$

$$[(A-\lambda I)f](t) = (t-\lambda)\,f(t)$$

Thus $[(A-\lambda I)^{-1} f](t) \overset{\text{if it exists}}{=} \dfrac{f(t)}{t-\lambda}$

This is a well-defined operator if $t-\lambda \neq 0$,
ie $\lambda \notin [0,1]$. ($\because t\in[0,1]$).

$\Rightarrow$ $\sigma(A) = [0,1]$
$\rho(A) = \mathbb{C} \setminus [0,1]$.

Eigenvalues: $Af = \lambda f$
$(Af)(t) = (\lambda f)(t)$

$$t\,f(t) = \lambda f(t)$$

$$\Rightarrow (t-\lambda)\,f(t) \equiv 0 \qquad \forall t$$

$$\Rightarrow f \equiv 0$$

$\Rightarrow$ $\nexists$ no eigenvalues for A.

$\boxed{L3}$ **Thm:** Let X be Banach, $A \in \mathcal{B}(X)$.
Then (i) $\sigma(A)$ is closed and
(ii) $\sigma(A) \subset B_c(0, \|A\|)$ Ball (centre, radius)
closed $= \{\lambda \in \mathbb{C} : |\lambda| \leq \|A\|\}$

Proof: (ii) Suppose $|\lambda| > \|A\|$, ie λ not in the ball.
Then we want to show it is not in the spectrum, so it's in the resolvant, ie $A - \lambda I$ is invertible.

$$A - \lambda I = -\lambda(I - \lambda^{-1}A).$$

$$\|\lambda^{-1}A\| = \frac{\|A\|}{|\lambda|} < 1 \Rightarrow (A - \lambda I) \text{ is invertible}$$

(from Pertubation Lemma I)

Thus $\lambda \in \rho(A)$
and $\sigma(A) \subset B(0, \|A\|)$.

(i) Suppose $\lambda_0 \in \rho(A)$.

ie ρ open
Want to show $A - \lambda I$ is invertible for λ close to λ_0. We know $A - \lambda_0 I$ is invertible for definite since $\lambda \in \rho(A)$.

$$A - \lambda I = \overbrace{A - \lambda_0 I}^{\text{'A'}} + \overbrace{(\lambda - \lambda_0) I}^{\text{'B' in Pert. Lem. 2.}}$$

If we have ~~For~~ $\lambda \in \mathbb{C}$ with $|\lambda - \lambda_0| < \frac{1}{\|(A - \lambda_0 I)^{-1}\|}$, "$\|B\| < \frac{1}{\|A^{-1}\|}$"
(which we do for $\lambda \to \lambda_0$)

Perturbation Lemma II implies that $\underset{\text{"A+B"}}{A - \lambda I}$ is invertible, so $\lambda \in \rho(A)$.
$\Rightarrow \rho$ is open $\Rightarrow \sigma$ is closed $\square$

$\sigma(Id) = 1.$

<u>Example</u> (1) $X = \ell_1$ $\quad e_n = (\underbrace{0, 0, \dots, 0, 1}_{n}, 0, \dots)$

$$Ae_n = \lambda_n e_n, \quad \lambda_n \in \mathbb{C}$$

So this is an operator with a diagonal matrix

if $\{\lambda_n\}$ is bounded then A is bounded and $\|A\| = \sup_n |\lambda_n|$

Obviously each $\lambda_n \in \sigma$. Is there anything else in σ?

$\sigma(A)$ is closed so:

<u>Claim</u>: $\sigma(A) = \overline{\{\lambda_n\}_{n=1}^{\infty}}$ ← closure

(Proved in H/W).

(2) $X = \ell_1$

$A(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots)$ left shift

$$\rightarrow \quad A = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 & \ddots \\ & & & & & \ddots \end{pmatrix}$$

$\|A\| = 1.$ $\quad (\because \|A\| \geq 1$ and $\|A\| \leq 1)$

$\Longrightarrow \sigma(A) \subset \underset{\text{closed}}{B_c}(0, \underset{\|A\|}{1})$ (by thm on opposite page, at top)

Eigenvalues of A? Set $Ax = \lambda x$.

$$\underbrace{Ax}_{=(x_2, x_3, \dots)} = \underbrace{\lambda x}_{=(\lambda x_1, \lambda x_2, \dots)}$$

$$\begin{aligned} x_2 &= \lambda x_1 \\ x_3 &= \lambda x_2 = \lambda^2 x_1 \\ &\vdots \\ x_n &= \lambda^{n-1} x_1 \end{aligned}$$

$$\Rightarrow \quad x = x_1(1, \lambda, \lambda^2, \lambda^3, \dots)$$

$$x \in l_1 \iff |\lambda| < 1$$

$\because$ the λ^n have to converge

$\Rightarrow$ the set of eigenvalues of $A = B_o(0,1)$ (open)

and $\sigma(A) \supset \overline{\{\text{eigenvalues}\}} = B_c(0,1)$

Overall, $\sigma(A) = B_c(0,1)$.

Defn: The operator-valued fn

$$\rho(A) \ni \lambda \longrightarrow R(A, \lambda) := (A - \lambda I)^{-1} \in B(X)$$

is called the resolvent of A.

Lemma: (Resolvent eqn)

$$R(A,\lambda) - R(A,\lambda_0) = (\lambda - \lambda_0) R(A,\lambda) R(A,\lambda_0)$$

Proof: $R(A,\lambda) - R(A,\lambda_0) = (A-\lambda I)^{-1} - (A-\lambda_0 I)^{-1}$

$$= (A-\lambda I)^{-1}\left[(A-\lambda_0 I) - (A-\lambda I)\right](A-\lambda_0 I)^{-1}$$

(expand to prove)

$$= (\lambda-\lambda_0)(A-\lambda I)^{-1}(A-\lambda_0 I)^{-1}$$

$$= (\lambda-\lambda_0)\, R(A,\lambda)\, R(A,\lambda_0). \qquad \square$$

Thm: Let Z be a Banach space,
$\Omega \subset \mathbb{C}$ be an open set
$F: \Omega \to Z$ be a Z-valued function.

Then FAE:

(i) $\forall\, \lambda_0 \in \Omega$ $\exists$ a limit $\lim_{\lambda\to\lambda_0} \dfrac{F(\lambda)-F(\lambda_0)}{\lambda-\lambda_0} = F'(\lambda_0)$. $\in Z$

ie $$\lim_{\lambda\to\lambda_0} \left\| \frac{F(\lambda)-F(\lambda_0)}{\lambda-\lambda_0} - F'(\lambda_0) \right\| = 0$$

(ii) $\forall\, \lambda_0 \in \Omega$ has a neighbourhood where

$$F(\lambda) = \sum_{n=0}^{\infty} (\lambda-\lambda_0)^n F_n(\lambda)$$

and the series converges absolutely.

(iii) $\forall\, G \in Z^*$ the complex-valued f^n
$\lambda \mapsto G(F(\lambda))$ is holomorphic in Ω.

(iv) If $Z = \mathcal{B}(X,Y)$ for X, Y Banach, then
(i), (ii), (iii) are also equivalent to:

$\forall x \in X$ and $g \in Y^*$, the complex-valued f^n

$$\lambda \mapsto g\Big[\underbrace{\underbrace{F(\lambda)x}_{\in Z}}_{\in Y}\Big]$$

is holomorphic in Ω.

Def^n: A vector-valued (operator-valued) f^n is called holomorphic (analytic) if it satisfies any of these properties ((i)–(iv))

Thm: The resolvent $R(A,\cdot)$ is analytic in $\rho(A)$ $B(X)$-valued f^{n}s satisfying:

(1) $$\left.\frac{dR(A,\lambda)}{d\lambda}\right|_{\lambda=\lambda_0} = R^2(\lambda,\lambda_0) \quad \forall \lambda_0 \in \rho(A)$$

(by chain rule)

(2) $-\lambda R(A,\lambda) \longrightarrow I$ as $|\lambda| \to \infty$

(3) $$\| R(A,\lambda) \| \geqslant \frac{1}{d(\lambda, \sigma(A))}, \quad \lambda \in \rho(A)$$

and $d(\lambda, K) = \inf_{k \in K} |\lambda - k|$ is the distance from λ to $K \subset \mathbb{C}$

Proof: (1) Let $\lambda_0 \in \rho(A)$.

We know $$R(A,\lambda) - R(A,\lambda_0) = \underbrace{(\lambda-\lambda_0)\overbrace{R(A,\lambda)}^{\text{bounded for } \lambda\to\lambda_0}R(A,\lambda_0)}_{\substack{\to 0 \\ \text{strongly}}}$$

Then $\lim_{\lambda\to\lambda_0} R(A,\lambda) = R(A,\lambda_0)$

Also,

$$\lim_{\lambda\to\lambda_0} \frac{R(A,\lambda)-R(A,\lambda_0)}{\lambda-\lambda_0} \quad \left(= \frac{d}{d\lambda}\left[R(A,\lambda)\right]\right)$$

$$= \lim_{\lambda\to\lambda_0} R(A,\lambda)\,R(A,\lambda_0)$$ by the Resolvent eq^n lemma 2 pgs ago

$$= R^2(A,\lambda)$$

(2) $\| -\lambda R(A,\lambda) - I \|$

$$= \| -\lambda (A-\lambda I)^{-1} - I \|$$

$$= \| (I-\lambda^{-1}A)^{-1} - I \|$$

$$= \left\| \sum_{n=0}^{\infty} (\lambda^{-1}A)^n - I \right\| \quad \Big) (\lambda \text{ is large})$$

$$= \left\| \sum_{n=1}^{\infty} \lambda^{-n} A^n \right\|$$

$$\leq \sum_{n=1}^{\infty} |\lambda|^{-n} \|A\|^n$$

$$= \frac{1}{1-|\lambda|^{-1}\|A\|} \cdot \overbrace{|\lambda|^{-1}\|A\|}^{\because \text{ summing from } 1}$$

$$= \frac{\|A\|}{|\lambda|-\|A\|}$$

$$\longrightarrow 0 \quad \text{as } |\lambda| \to \infty.$$

(3) Suppose $\lambda_0 \in \rho(A)$.

We have proved that if

$$|\lambda - \lambda_0| < \frac{1}{\|R(A, \lambda_0)\|}$$

then $\lambda \in \rho(A)$.

Therefore

$$d(\lambda_0, \sigma(A)) \geqslant \frac{1}{\|R(A, \lambda_0)\|}$$

or

$$\|R(A, \lambda_0)\| \geqslant \frac{1}{d(\lambda_0, \sigma(A))} \qquad \square.$$

<u>Thm</u>: Let $A \in B(X)$. Then $\sigma(A) \neq \emptyset$.

<u>Proof</u>: Suppose $\sigma(A) = \emptyset$.
$\Rightarrow \rho(A) = \mathbb{C}$.

Let $x \in X \setminus \{0\}$ and $g \in X^*$.
Then the function

$$\mathbb{C} \ni \lambda \longmapsto g\big(R(A, \lambda)x\big) =: f(\lambda)$$

is holomorphic complex-valued fn.

s.t. $f(\lambda) \to 0$ as $|\lambda| \to \infty$ (by part 2 of previous thm)
ie it is bounded.

By Liouville's theorem, $f \equiv \text{const} = 0$.
(holomorphic bounded fn is const.)
$\because$ limit is 0.

Therefore by corollary from Hahn-Banach Thm,

$$R(A,\lambda)\,x = 0$$

(otherwise $\exists\, g \in X^*$ s.t. $g(R(A,\lambda)x) \neq 0$)

$$\Rightarrow \quad R(A,\lambda) = 0$$

$$\Rightarrow \quad I = R(A,\lambda)(A - \lambda I) = 0 \qquad \#$$

$$\Rightarrow \quad \sigma(A) \neq \emptyset$$

<u>Thm</u>: Let X be Banach and $A \in B(X)$.
Then $\sigma(A)$ is nonempty, closed set, subset of $B_c(0, \|A\|)$ and the resolvent $(A - \lambda I)^{-1}$ is analytic in $\rho(A)$.

<u>Def^n</u>: The spectral radius of A, $r(A)$,

$$r(A) = \sup_{\lambda \in \sigma(A)} |\lambda|.$$

This is the radius of the smallest (closed) disc centred at 0 and containing $\sigma(A)$.

<u>Property</u>: $r(A) \leq \|A\|$.

<u>Thm</u>: <u>(The Spectral Radius Formula)</u>

$$r(A) = \lim_{n\to\infty} \|A^n\|^{1/n}$$

Proof: We need to prove:

$$(1) \quad r(A) \leq \liminf \|A^n\|^{1/n} \qquad (*)$$

$$(2) \quad r(A) \geq \limsup \|A^n\|^{1/n} \qquad (**)$$

(1) Claim: $\lambda \in \sigma(A) \Rightarrow \lambda^n \in \sigma(A^n)$.

Proof:
$$\underbrace{A^n - \lambda^n I}_{\Rightarrow \text{ non-invertible}} = \underbrace{\underbrace{(A-\lambda I)}_{\text{non-invertible}}(A^{n-1} + \lambda A^{n-2} + \dots + \lambda^{n-1} I)}_{\substack{\text{commutes} \\ \Rightarrow \text{ whole thing non-invertible}}}$$

$$\Rightarrow \lambda^n \in \sigma(A^n). \quad \blacksquare$$

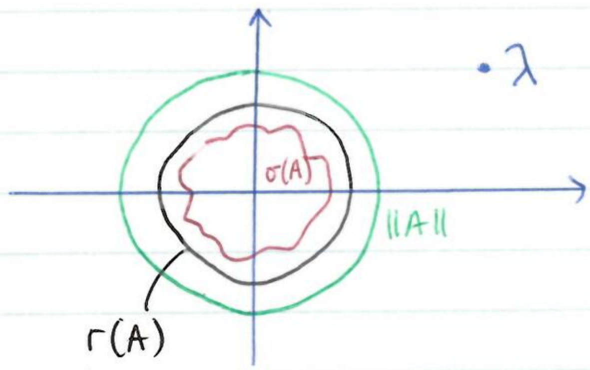
Therefore
$$r(A)^n = \Big[\sup_{\lambda \in \sigma(A)} |\lambda|\Big]^n$$
$$= \sup_{\lambda \in \sigma(A)} |\lambda|^n$$
$$= \sup_{\lambda \in \sigma(A)} |\lambda^n|$$
$$\leq r(A^n)$$
$$\leq \|A^n\|$$

$$\Rightarrow r(A) \leq \|A^n\|^{1/n} \qquad \text{(raising both sides to } 1/n\text{)}$$

$$\Rightarrow r(A) \leq \liminf \|A^n\|^{1/n} \qquad \boxed{1}.$$

[L4]

(2) Suppose
$|\lambda| > \|A\|$



Then $(A-\lambda I)^{-1} = (-\lambda)^{-1}\left(I - \frac{A}{\lambda}\right)^{-1}$

1st Pert. Lemma $= -\sum_{n=0}^{\infty} \lambda^{-n-1} A^n$ $\quad (\dagger)$

Let $x \in X$
$g \in X^*$ and consider

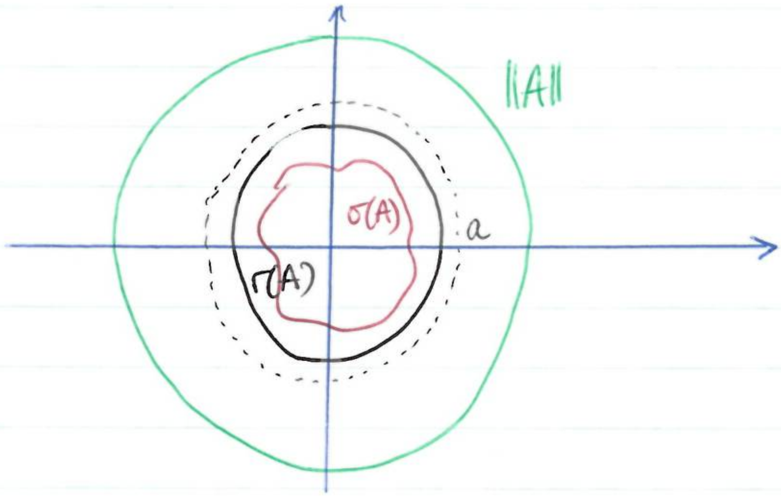
$f(\lambda) := g\left(R(A,\lambda)x\right)$ [g, x fixed so we have analytic f^n of λ] $\quad (\dagger\dagger)$

Then $f(\lambda)$ is analytic in $\rho(A)$, and in particular, in $\{\lambda : |\lambda| > r(A)\}$

We also know that if $|\lambda| > \|A\|$, we have

$$f(\lambda) = -\sum_{n=0}^{\infty} \lambda^{-n-1} g(A^n x) \quad \text{[combining } (\dagger) \text{ and } (\dagger\dagger)]$$

So Laurent's thm implies that (i) this identity holds for $|\lambda| > r(A)$, and (ii) the series converges absolutely



Let $a > r(A)$ and put $\lambda = ae^{i\theta}$, $0 \le \theta \le 2\pi$.

Then $$\lambda^{m+1} f(\lambda) = -\sum_{n=0}^{\infty} \lambda^{m-n} g(A^n x)$$

$\left[\int_0^{2\pi} d\theta :\right]$ $$\int_0^{2\pi} a^{m+1} e^{i(m\;)\theta} f(ae^{i\theta})\, d\theta$$

$$= -\sum_{n=0}^{\infty} g(A^n x) \underbrace{\int_0^{2\pi} a^{m-n} e^{i(m-n)\theta}\, d\theta}_{=0 \text{ unless } m=n}$$

$$= -2\pi g(A^m x)$$

Therefore $$|g(A^m x)| \le \frac{1}{2\pi} a^{m+1} \int_0^{2\pi} |f(ae^{i\theta})|\, d\theta$$ $|e^{i(m+1)\theta}| \le 1$

$$= \frac{1}{2\pi} a^{m+1} \int_0^{2\pi} |g(R(A,\lambda)x)|\, d\theta$$

$$\le \frac{1}{2\pi} a^{m+1} \int_0^{2\pi} \|g\| \cdot \|x\| \cdot \|R(A,\lambda)\|\, d\theta$$

$\left[\text{Let } M(a) = \sup_{|\lambda| = a} \|R(A,\lambda)\|\right]$

$$\le a^{m+1} \|g\| \, \|x\| \, M(a)$$

Choose $g \in X^*$ st. $\|g\| = 1$

and $g(A^m x) = \|A^m x\|$

Then $\|A^m x\| \leq a^{m+1} M(a) \|x\|$

$$\Rightarrow \frac{\|A^m x\|}{\|x\|} \leq a^{m+1} M(a)$$

and
$$\|A^m\| = \sup_{x \neq 0} \frac{\|A^m x\|}{\|x\|} \leq a^{m+1} M(a)$$

$$\Rightarrow \|A^m\|^{1/m} \leq a[aM(a)]^{1/m}$$

$$\Rightarrow \limsup \|A^m\|^{1/m} \leq \limsup a[aM(a)]^{1/m}$$

$$= a$$

Since a is arbitrary n° larger than the spectral radius $(= r(A))$, we can replace a with $r(A)$ and get

$$\Rightarrow r(A) \geqslant \limsup \|A^m\|^{1/m} \qquad \boxed{2}.$$

Let $p(z) = \sum_{n=0}^{N} a_n z^n$ be a polynomial, $a_N \neq 0$ and $A \in B(X)$.

The we can define

$$p(A) = \sum_{n=0}^{N} a_n A^n$$

Theorem: (Spectral Mapping Thm)

p polynomial

$$\sigma(p(A)) = p(\sigma(A)).$$

$$= \{ p(z) : z \in \sigma(A) \}$$

Proof: Let $\mu \in \mathbb{C}$.
Let $\lambda_1, \lambda_2, \dots, \lambda_N \in \mathbb{C}$ be all roots of $p(z) = \mu$.

Then $p(z) - \mu = (z-\lambda_1) \cdot \dots \cdot (z - \lambda_N) \cdot a_N$. by FToA

and $p(A) - \mu I = (A - \lambda_1 I) \cdot \dots \cdot (A - \lambda_N I) \cdot a_N$

$\Rightarrow \mu \notin \sigma(p(A)) \iff p(A) - \mu I$ is invertible

$\iff \forall j = 1, \dots, N, \ (A - \lambda_j I)$ is invertible

$\iff \lambda_j \notin \sigma(A) \quad \forall j = 1, \dots, N$

$\iff \mu \notin p(\sigma(A))$ $\square$

PROJECTIONS

$$V = V_1 \dotplus V_2 = V_1 \oplus V_2$$

direct sum

<u>Def^{n} 1</u>: iff $V = V_1 + V_2 = \{v_1 + v_2 : v_1 \in V_1, v_2 \in V_2\}$ and $V_1 \cap V_2 = \{0\}$.

<u>Def^{n} 2</u>: iff $\forall v \in V \ \exists! \ v_1 \in V_1$ and $v_2 \in V_2$ s.t. $v = v_1 + v_2$

easy exam q? 'check this is a proj' $\Rightarrow$ check $P^2 = P$.

<u>Def^{n}</u>: Let X be a normed space and $P \in B(X)$. P is called a <u>projection</u> if $P^2 = P$ ("idempotent")

<u>Lemma</u>: Let $P \in B(X)$ be a projection.

(i) Then $Q = I - P$ is also a projection,

(ii) $PQ = QP = 0$,

(iii) $Ker(P) = Ran(Q)$

(iv) $Ker(Q) = Ra(P)$.

<u>Proof</u>: (i)
$$\begin{aligned} Q^2 = (I-P)^2 &= I - 2P + P^2 \\ &= I - P. \\ &= Q \end{aligned}$$

(ii)
$$\begin{aligned} PQ &= P(I-P) \\ &= P - P^2 \\ &= 0 \\ &= P - P^2 \\ &= (I-P)P \\ &= QP \end{aligned}$$

(iii) Since $QP = 0$

$\operatorname{Ran} P \subset \operatorname{Ker} Q$

Suppose $x \in \operatorname{Ker} Q$.

Then $$\underset{\displaystyle x - Px}{\underset{\shortparallel}{Qx}} = 0$$

$\Rightarrow x = Px$

$\Rightarrow x \in \operatorname{Ran} P$

$\Rightarrow \operatorname{Ker} Q \subset \operatorname{Ran} P$

$\Rightarrow \operatorname{Ker} Q = \operatorname{Ran} P$

(iv) Similarly $\operatorname{Ker} P = \operatorname{Ran} Q$. $\square$

<u>Lemma</u> Let $P \in B(X)$ be a projection. Then $\operatorname{Ran} P$ is closed and $X = \operatorname{Ker} P \dotplus \operatorname{Ran} P$.

<u>Proof</u>: $\operatorname{Ran} P = \operatorname{Ker} Q$ is closed since $Q \in B(X)$

Since $I = P + \overset{=Q}{(I-P)}$, we have

$$\begin{aligned} \operatorname{Ker} P + \operatorname{Ran} P &= \operatorname{Ran}(I-P) + \operatorname{Ran} P \\ &= X \end{aligned}$$

Let us prove $\operatorname{Ker} P \cap \operatorname{Ran} P = \{0\}$.

Suppose $x \in \operatorname{Ran} P$. $\Rightarrow$ $x = Py$ for $y \in X$.

Then $\underset{\substack{\parallel \\ 0 \text{ since } x \in \operatorname{Ker} P}}{Px} = P^2 y = Py = x \quad \Rightarrow \quad x = 0$. $\square$

<u>Thm</u> Let X be Banach and $P \in \mathcal{B}(X)$ be a projection, $P \neq I$, $P \neq 0$. Then
$$\sigma(P) = \{0, 1\}.$$

<u>Proof</u>: The Spectral Mapping Theorem implies $\sigma(p(A)) = p(\sigma(A))$

$$\begin{aligned} 0 &= \sigma(0) \\ &= \sigma(P^2 - P) \\ &= \{\underbrace{\lambda^2 - \lambda}_{(\lambda)(\lambda-1)} : \lambda \in \sigma(P)\} \end{aligned}$$

$\Rightarrow$ if $\lambda \in \sigma(P)$ then $\lambda^2 - \lambda = 0 \Rightarrow \lambda \in \{0, 1\}$.

$\Rightarrow \sigma(P) \subset \{0, 1\}$.

Suppose $P \neq 0$ and $P \neq I$

$\Downarrow \operatorname{Ran} P \neq \{0\}$ $\quad \Downarrow \operatorname{Ker} P \neq \{0\}$.

$\operatorname{Ker} P \neq 0 \Rightarrow \exists x, x \neq 0, x \in \operatorname{Ker} P$
$\Rightarrow Px = 0$
$\Rightarrow 0$ is an eigenvalue

$\operatorname{Ran} P \neq 0 \Rightarrow \exists y, y \neq 0, y \in \operatorname{Ran} P$
$y \in \operatorname{Ker}(I - P)$
$\Rightarrow (I - P) y = 0$
$\Rightarrow Py = y$
$\Rightarrow 1$ is an eigenvalue

$\Rightarrow \{0,1\} \subset \sigma(P)$

$\Rightarrow \{0,1\} = \sigma(P)$

Compact Operators

Def^n: Let X be a normed space. A set $K \subset X$ is called relatively compact if each sequence $\{x_n\} \subset K$ has a Cauchy subsequence

K is compact if each sequence $\{x_n\} \subset K$ has a convergent (to a point in K) subsequence.

Prop^n: • K is relatively compact $\Rightarrow$ K is bounded
• K is compact $\Rightarrow$ K is closed & bounded

• $K_1 \subset K_2$, K_2 is relatively compact $\Rightarrow$ K_1 is relatively compact.

• $K_1 \subset K_2$, K_2 is compact, K_1 is closed $\Rightarrow$ K_1 is compact.

Thm: (i) Let $\dim X < \infty$.
Then (a) K is rel. compact $\Leftrightarrow$ K is bounded
(b) K is compact $\Leftrightarrow$ K is closed & bounded

(ii) Assume $B_c(0,1)$ is compact.
Then X is finite-dimensional

!

Definition: Let X and Y be normed space and $T : X \to Y$ be a linear operator.

We say T is a <u>compact</u> operator, if it maps bounded sets in X into relatively compact sets in Y.

The set of all such operators is denoted
$$\mathrm{Com}(X, Y).$$

Let $T \in \mathrm{Com}(X, Y)$. Then $T(B_c(0,1))$ is relatively compact and hence bounded. Thus T is bounded,

$\Rightarrow$
$$\mathrm{Com}(X, Y) \subset \mathcal{B}(X, Y).$$

Lemma: $T : X \to Y$ is compact iff $T(B_c(0,1))$ is relatively compact.

Proof: $(\Rightarrow)$ image of arbitrary set is rel. compact, in particular $B_c(0,1)$.

$(\Leftarrow)$ $\forall r > 0$ we have $T(B_c(0,r)) = r\,T(B_c(0,1))$, which is rel. compact. (Find C. sequence in small part, then blow up by r and it's still Cauchy),

Suppose $W \subset X$ is bounded.
Then $W \subset B(0,r)$ for some $r > 0$
and $T(W) \subset T(B_c(0,r))$ is relatively compact. $\square$

Corollary: T is compact iff $\forall \{x_n\} \subset X$, $\|x_n\| \leq 1$, the sequence Tx_n has a Cauchy subsequence.

<u>Thm</u>: Let X, Y, Z be normed:

(i) If $T_1, T_2 \in \mathrm{Com}(X, Y)$ and $\alpha_1, \alpha_2 \in \mathbb{F}$, then $\alpha_1 T_1 + \alpha_2 T_2 \in \mathrm{Com}(X, Y)$.

Com(X,Y) forms a linear space

(ii) $T \in \mathrm{Com}(X, Y)$
$A \in \mathcal{B}(Z, X)$
$B \in \mathcal{B}(Y, Z)$
then $TA \in \mathrm{Com}(Z, Y)$ and $BT \in \mathrm{Com}(X, Z)$.

This space is ideal in the space of compact ~~linear~~ operators

(iii) $T_n \in \mathrm{Com}(X, Y)$
$\|T - T_n\| \to 0$. (ie uniformly)
then $T \in \mathrm{Com}(X, Y)$.

This is a closed ideal.

$\boxed{L5}$ <u>Proof</u>: (i) Let $\{x_n\}$ be a bounded sequence in X, $\|x_n\| \leq 1$.
Since T_1 is compact,
$\exists$ subsequence $\{x_n^{(1)}\} \subset \{x_n\}$
s.t.
$\{T_1 x_n^{(1)}\}$ is Cauchy.

Since T_2 is compact,
$\exists$ subsubsequence $\{x_n^{(2)}\} \subset \{x_n^{(1)}\}$
s.t.
$\{T_2 x_n^{(2)}\}$ is Cauchy

this just uses the def'n of a ~~comp~~ relatively compact set

Then $\{(\alpha_1 T_1 + \alpha_2 T_2)(x_n^{(2)})\}_n$ is Cauchy

$\to \alpha_1 T_1 + \alpha_2 T_2 \in \mathrm{Com}(X, Y)$.

(ii) Let us assume $X = Y = Z$.

- Suppose $\{x_n\}$ is bounded sequence
  $\Rightarrow \{Ax_n\}$ is bounded sequence if A is bdd operator.

Since T is compact, $\{T(Ax_n)\}$ has a Cauchy subsequence.

$\Rightarrow TA \in Com(X, X)$.

- Similarly, if $\{x_n\}$ is bounded and T is compact, $\exists$ subsequence $\{x_n^{(1)}\}$
  s.t.
  $\{Tx_n^{(1)}\}$ is Cauchy.

Since B is bounded,
$\{B(Tx_n^{(1)})\}_{n=1}^{\infty}$ is Cauchy

$\Rightarrow BT \in Com(X, X)$.

(iii) Let $\{x_n\}$ be a bounded sequence, $\|x_n\| \le 1$.

T_1 is compact $\Rightarrow \exists$ subsequence $\{x_n^{(1)}\} \subset \{x_n\}$
s.t.
$\{T_1 x_n^{(1)}\}$ is Cauchy.

T_2 is compact $\Rightarrow \exists$ subsubsequence $\{x_n^{(2)}\} \subset \{x_n^{(1)}\}$
s.t.
$\{T_2 x_n^{(2)}\}$ is Cauchy.

$\vdots$

$\vdots$

T_m is compact $\Rightarrow$ $\exists$ (sub)^{m}sequence $\{x_n^{(m)}\} \subset \{x_n^{(m-1)}\}$ s.t.

$$\{T_m x_n^{(m)}\} \text{ is Cauchy}$$

Take a diagonal subsequence

$$y_n = x_n^{(n)}.$$

Then $\{x_n^{(n)}\}_{n \geq m} \subset \{x_n^{(m)}\}_{n=1}^{\infty}$

Therefore,

$$\{T_m y_n\}_{n=1}^{\infty} \subset \{T_m (x_n^{(m)}\}_{n=1}^{\infty}$$

is Cauchy for each fixed m.

We also have

$\leq \|T - T_\ell\| \cdot \|y_n\| \leq \|T - T_\ell\|$, $\|y_n\| \leq 1$

$$\|Ty_n - Ty_k\| \overset{\triangle}{\leq} \|T_\ell y_n - T_\ell y_k\| + \|Ty_n - T_\ell y_n\| + \|T_\ell y_k - Ty_k\|$$

$\leq \|T - T_\ell\| \cdot \|y_k\| \leq \|T - T_\ell\|$, $\|y_k\| \leq 1$

Let $\varepsilon > 0$ be given.

Since $\|T_\ell - T\| \to 0$, we can find ℓ s.t.

$$\|T_\ell - T\| < \varepsilon/3$$

Since $\{T_\ell y_n\}$ is Cauchy, we can find N s.t. $n, k > N \Rightarrow$

$$\|T_\ell y_n - T_\ell y_k\| < \varepsilon/3$$

$\Rightarrow$ for this choice of N we have

$$\|Ty_n - Ty_k\| \leq 3\left(\frac{\varepsilon}{3}\right) = \varepsilon. \qquad \forall n, k > N.$$

$\square$

<u>Def^n</u>: $T \in \mathcal{B}(X,Y)$ is called a finite rank operator if
$$\dim(\operatorname{Ran} T) < \infty.$$

<u>Claim</u>: T is finite-rank $\Rightarrow$ T is compact.

<u>Proof</u>: $\{x_n\} \subset X$ is bounded $\Rightarrow$ $\{Tx_n\} \subset \operatorname{Ran} T$ is bounded

and $\Rightarrow$ $\operatorname{Ran} T$ is finite-dimensional
$\Rightarrow$ $\{Tx_n\}$ is relatively compact
$\Rightarrow$ T is compact.

<u>Example</u>: $X = C([0,1])$.

Given $f \in C([0,1])$,
$$(Tf)(s) = \int_0^1 k(s,t) f(t)\, dt$$
Here, $k(s,t)$ is an arbitrary smooth f^n on $[0,1] \times [0,1]$.

<u>Claim</u>: T is compact, ie $T \in \operatorname{Com}(X)$.

<u>Proof</u>: Fourier Decomposition Theorem: Given $\varepsilon > 0$, $\exists$ a decomposition $k = k_1 + k_2$, where

Fourier coefficient

$$\begin{cases} k_1 = \displaystyle\sum_{n=-N}^{N} \sum_{m=-N}^{N} a_{nm}\, e^{2\pi i(ns+mt)} \\ \sup |k_2| < \varepsilon \end{cases}$$

Fourier expansion

ucahadø

Then $T = T_1 + T_2$

$$T_1 f(s) = \int_0^1 k_1(s,t)\, f(t)\, dt$$

$$T_2 f(s) = \int_0^1 k_2(s,t)\, f(t)\, dt.$$

Then $\|T_2\| < \varepsilon$

and T_1 has finite rank.

Indeed, $T_1 f(s) = \int_0^1 \sum_n \sum_m e^{2\pi i n s} e^{2\pi i m t} a_{nm} f(t)\, dt$

$$= \sum_{n=-N}^{N} \left[\int_0^1 \sum_{m=-N}^{N} a_{nm} e^{2\pi i m t} f(t)\, dt \right] e^{2\pi i n s}$$

$$\subset \operatorname{span}\{ e^{2\pi i n s} \}_{n=-N}^{N}$$

$\Rightarrow T_1$ has finite rank.

Def$^{\underline{n}}$: $\{e_n\}_{n=1}^{\infty}$ is a Schauder basis if $\forall x$,

$$\exists! \quad x = \sum_{n=1}^{\infty} \alpha_n e_n.$$

Take $A \in B(X)$. $\quad Y \subset X$

$AY \subset Y, \quad \bar{Y} = Y.$

Lemma Let X be normed space and $X_0 \subset X$ be a finite-dimensional subspace.

Then X_0 is closed.

Proof: Let $\{e_1, \dots, e_n\}$ be a basis of X_0.

$\forall x \in X_0 \ \exists! \ x = \sum_{k=1}^{n} \alpha_k e_k$.

Then we can define a different norm on X_0,

$$\|x\|_1 = \sum_{k=1}^{n} |\alpha_k|.$$

Then $\|\cdot\|$ and $\|\cdot\|_1$ are equivalent on X_0.

Suppose $x^1, x^2, x^3, \dots \in X_0$
and $\|x^m - x\| \to 0$ for some $x \in X$.

Want to show $x \in X_0$.

Then $\{x^m\}$ is a Cauchy sequence in $\|\cdot\|$

$\Rightarrow \{x^m\}$ is a Cauchy sequence in $\|\cdot\|_1$

$\Rightarrow x^m = \sum_{k=1}^{\infty} \alpha_k^m e_k$

and all $\{\alpha_1^m\}_{m=1}^{\infty}, \{\alpha_2^m\}_{m=1}^{\infty}, \dots, \{\alpha_n^m\}_{m=1}^{\infty}$

are Cauchy sequences

$\Rightarrow \exists \lim_{m \to \infty} \alpha_k^m = \alpha_k$

Put $y = \sum_{k=1}^{n} \alpha_k e_k \in X_0$.

Then $\|x^m - y\|_1 \to 0$

$\Rightarrow \|x^m - y\| \to 0$

$\Rightarrow y = x$.

$\Rightarrow \lim x^m \in X_0$.

<u>Lemma (almost orthogonality)</u>

Let X be normed and $X_0 \subsetneq X$ be closed subspace.

Then $\forall \varepsilon > 0 \ \exists z \in X$ s.t. $\|z\| = 1$ and $\|z - x\| \geq 1 - \varepsilon$ $\forall x \in X_0$.

<u>Proof</u>: We can assume $\varepsilon < 1$.
Take $x_1 \in X \setminus X_0$.

Since X_0 is closed, there is no sequence of elements in X_0 that converge to $x_1 \notin X_0$, ie

$$d := d(x_1, X_0) = \inf_{x \in X_0} \|x_1 - x\| > 0$$

$$\exists y \in X_0 \text{ s.t. } d \leq \|x_1 - y\| < \frac{d}{1-\varepsilon}$$

Put $z = \dfrac{x_1 - y}{\|x_1 - y\|}$. Then $\|z\| = 1$.

Also, $\|z-x\| = \frac{1}{\|x_1-y\|}\left[\left\|x_1 - \underbrace{\left(y + \|x_1-y\| x\right)}_{\in X_0}\right\|\right] \geqslant \frac{d}{\|x_1-y\|}$

$$\geqslant \frac{1-\varepsilon}{d} \cdot d$$

$$= 1-\varepsilon \qquad \square$$

<u>Corollary</u>: Let X be a normed space s.t. $B(0,1)$ is relatively compact.

Then X is finite-dimensional, ie $\dim X < \infty$

<u>Proof</u>: Choose $x_1 \in X$, $\|x_1\| = 1$.

~~Then~~ $X_1 = \text{span}\{x_1\}$.

Apply the Almost Orthogonality Lemma with $\varepsilon = \frac{1}{2}$.

$\exists\, x_2 \in X$ s.t. $\|x_2\| = 1$ and $\|x_2 - x_1\| > \frac{1}{2}$.

Put $X_2 = \text{span}(x_1, x_2)$.

Apply lemma:

$\exists\, x_3 \in X$ s.t. $\|x_3\| = 1$ and $\|x_3 - x_1\| \geqslant \frac{1}{2}$.

$\|x_3 - x_2\| > \frac{1}{2}$

$\vdots$

Thus we have constructed a bounded sequence $\{x_j\}$, $\|x_j\| = 1$ s.t. $\|x_m - x_n\| > \frac{1}{2}$ $(m \neq n)$

$\Rightarrow$ there is no Cauchy subsequence. $\square$

Corollary If the identity operator I_X is compact, then $\dim X < \infty$.

Theorem: Suppose $T \in Com(X,Y)$, and at least one space out of X and Y is infinite-dimensional. Then T is not invertible. (with bounded inverse)

Proof: Suppose $T^{-1} \in \mathcal{B}(Y,X)$.

Then $TT^{-1} = I_Y$ and $T^{-1}T = I_X$ are compact and $\dim X < \infty$ and $\dim Y < \infty$. $\square$

Corollary Suppose $T \in Com(X,X)$ and $\dim X = \infty$.

Then $0 \in \sigma(T)$.

L6 Thm: If $\lambda \neq 0$ is an eigenvalue of $T^{\in Com(X,X)}$ and $X_\lambda = \{x : Tx = \lambda x\}$ is the eigenspace, then X_λ is finite-dimensional, ie λ has finite multiplicity

restricted to X_λ

Proof: $T|_{X_\lambda} = \lambda I|_{X_\lambda}$ is a compact operator

$\Rightarrow I|_{X_\lambda} = \frac{1}{\lambda} T|_{X_\lambda}$ is compact

$\Rightarrow X_\lambda$ must have finite dimension, since I is invertible. $\square$

<u>Lemma</u>: Suppose $T \in Com(X)$ and $\lambda \neq 0$ is not an eigenvalue. Then $\exists\, c > 0$ s.t.

$$\|(T-\lambda I)x\| \geq c\|x\| \qquad \forall x \in X.$$

<u>Proof</u>: Suppose not. Then for each c (say $c = \frac{1}{k}$, $k \in \mathbb{N}$), $\exists\, x_k$ s.t.

$$\underbrace{\|\underbrace{(T-\lambda I)x_k}_{\neq 0 \because \lambda \text{ not e'val}}\|}_{>0} < \frac{1}{k}\|x_k\|.$$

$$\Rightarrow \|x_k\| > 0.$$

So define $z_k = \frac{x_k}{\|x_k\|}$ s.t. $\|z_k\| = 1$

and $\|(T-\lambda I)z_k\| < \frac{1}{k}$.

Since T is compact and X is Banach, $\exists$ a subsequence z_{k_j} $(j \to \infty)$ s.t.

$$\lim_{j\to\infty} Tz_{k_j} = z.$$

Then $\lambda z_{k_j} = Tz_{k_j} - [(T-\lambda I)z_{k_j}]$

$$\Rightarrow z_{k_j} = \frac{1}{\lambda}\Big[\underbrace{Tz_{k_j}}_{\downarrow\, z} - \underbrace{(T-\lambda I)z_{k_j}}_{\downarrow\, 0}\Big]$$

$$\Rightarrow \lim_{j\to\infty} z_{k_j} = \frac{1}{\lambda}z.$$

Since $\|z_{k_j}\| = 1$ $\forall j$, $z \neq 0$.

Also $\quad Tz = T\left[\lim_{j\to\infty}(\lambda z_{k_j})\right]$

$$= \lambda \lim_{j\to\infty}(T z_{k_j})$$

$$= \lambda z$$

$\Rightarrow \lambda$ is an eigenvalue with corresponding eigenvector z.

$\#$ $\square$.

<u>Lemma</u> Suppose X is Banach and $B \in \mathcal{B}(X)$ satisfies
$\|Bx\| \geq c\|x\|$ for some $c > 0$
for all $x \in X$.

Then $\operatorname{Ker} B = \{0\}$
& $\operatorname{Ran} B$ is closed.

<u>Proof</u>: If $x \neq 0$ then $\|Bx\| \geq c\|x\| > 0$
$\Rightarrow Bx \neq 0$
$\Rightarrow \operatorname{Ker} B = \{0\}$.

Suppose $\exists$ a sequence $y_n \in \operatorname{Ran} B$ s.t. $y_n \to y$ and
$\exists$ $x_n \in X$ s.t. $Bx_n = y_n$.

Suppose $m \neq n$, then $\|x_n - x_m\| \leq \frac{1}{c}\|B(x_n - x_m)\|$

$$= \frac{1}{c}\|y_n - y_m\|$$

$$\longrightarrow 0 \quad \text{as } m,n\to\infty.$$

$\Rightarrow (x_n)$ is a Cauchy sequence
$\Rightarrow (x_n)$ is convergent (since X is Banach).

say $x_n \to x$ as $n \to \infty$.

Therefore
$$Bx = B \lim_{n\to\infty} x_n$$
$$= \lim_{n\to\infty} Bx_n$$
$$= \lim_{n\to\infty} y_n$$
$$= y$$

$\Rightarrow y \in \operatorname{Ran} B$. $\square$

<u>Corollary</u>: Let B be as in the lemma.
Then $\operatorname{Ker} B^n = \{0\}$
and $\operatorname{Ran}(B^n)$ is closed $\forall n \in \mathbb{N}$.

<u>Proof</u>: $\|B^n x\| = \|B(B^{n-1}x)\| \geq c\|B^{n-1}x\|$
$\geq c^2 \|B^{n-2}x\|$
$\geq c^n \|x\|$

and now we apply the lemma. $\square$

<u>Theorem</u>: Let X be Banach, $T \in \operatorname{Com}(X)$,
$\lambda \neq 0$ is not an eigenvalue of T.

Then $\lambda \notin \sigma(T)$.

Proof: We know that $\|(T-\lambda I)x\| \geqslant c\|x\|$.

Therefore $\operatorname{Ker}(T-\lambda I)^n = \{0\}$
and $X_n := \operatorname{Ran}(T-\lambda I)^n$ is closed.

Let $X_0 = X$.

Moreover,
$$\begin{aligned} X_{n+1} &= (T-\lambda I)^{n+1} X \\ &= (T-\lambda I)^n (T-\lambda I) X \\ &= (T-\lambda I)^n X_1 \\ &\subseteq (T-\lambda I)^n X_0 = X_n \end{aligned}$$

$\Rightarrow X_{n+1} \subseteq X_n$.

Claim 1: $\exists\, n$ s.t. $X_{n+1} = X_n$.

Proof of Claim 1: Suppose $X_{n+1} \neq X_n \ \forall n$.

Use the almost orthogonality lemma with $\varepsilon = \frac{1}{2}$ to find
$$x_n \in X_n \setminus X_{n+1}$$
$$\|x_n\| = 1$$
and
$$\|x_n - z\| \geqslant \frac{1}{2} \qquad \forall z \in X_{n+1}.$$

Then for $m > n$,
$$Tx_m - Tx_n = \underbrace{(T-\lambda I)\underbrace{(x_m - x_n)}_{\in X_n}}_{\in X_{n+1}} + \lambda \underbrace{(x_m - x_n)}_{\substack{\in X_m \\ \subset X_{n+1}}}$$

$= \lambda(z - x_n)$

where $z = x_m + \frac{1}{\lambda}(T - \lambda I)(x_m - x_n)$
$\in X_{n+1}$.

$\Rightarrow \|Tx_m - Tx_n\| = |\lambda| \, \|z - x_n\| > \frac{1}{2}|\lambda|$

$\Rightarrow$ there is no Cauchy subsequence of $\{Tx_n\}$
$\Rightarrow$ T is not compact. #

$\Rightarrow \exists\, n$ s.t. $X_n = X_{n+1}$ $\boxed{1}$.

<u>Claim 2:</u> Let $k = \min\{n : X_n = X_{n+1}\}$.
$k = 0$.

<u>Proof of Claim 2:</u> Suppose $k \neq 0$.

Then $X_{k-1} \supsetneq X_k = X_{k+1}$.

Take $z \in X_{k-1} \setminus X_k$.

Then $(T - \lambda I)z \in (T - \lambda I)X_{k-1}$
$= X_k$
$= X_{k+1}$
$= (T - \lambda I)X_k$.

$\Rightarrow \exists\, y \in X_k$ s.t. $(T - \lambda I)y = (T - \lambda I)z$

$\Rightarrow (T - \lambda I)(\underset{\in X_k}{y} - \underset{\notin X_k}{z}) = 0$

$y - z \neq 0$ as $y \in X_k$ and $z \notin X_k$.

$\Rightarrow y - z$ is an eigenvector
$\Rightarrow \lambda$ is an eigenvalue $\cancel{\#}$

$\Rightarrow k = 0$. $\boxed{2}$

and $X = X_0 = X_1$
$= (T - \lambda I) X_0$
$= (T - \lambda I) X$
$= \mathrm{Ran}(T - \lambda I)$

$\Rightarrow \mathrm{Ran}(T - \lambda I) = X$.

Since $\mathrm{Ker}(T - \lambda I) = \{0\}$,

$(T - \lambda I)$ is invertible (and bounded by the inverse mapping thm). $\square$.

<u>Thm:</u> If X is a vector space, $A : X \to X$ is linear and $\{x_n\}$ are eigenvectors of A corresponding to different eigenvalues.

Then they are linearly independent, ie
$$\sum_{n=1}^{m} a_n x_n = 0 \Rightarrow a_n = 0 \quad \forall n.$$

<u>Thm</u>: X is Banach, $T \in \mathrm{Com}(X)$.
Then $\sigma(T)$ is at most countable and has at most one accumulation point, $\lambda = 0$.

<u>Proof</u>: It is enough to prove for $\delta > 0$,

$$\sigma(T) \cap \{\lambda \in \mathbb{C} : |\lambda| > \delta\}$$

is finite.

$$\sigma(T) = \{0\} \cup \Big[\underbrace{\bigcup_{n=1}^{\infty} \left(\sigma(T) \cap \{\lambda : |\lambda| > \tfrac{1}{n}\} \right)}_{\text{countable union of finite sets} \Rightarrow \text{countable.}} \Big]$$

Suppose not, ie $\exists\, \delta > 0$ s.t.

$$\sigma(T) \cap \{\lambda \in \mathbb{C} : |\lambda| > \tfrac{1}{n}\}$$

is not finite, ie there are infinitely many differential eigenvalues $\lambda_1, \lambda_2, \dots$ with $|\lambda_j| > \delta$.

Let $x_j \neq 0$ satisfy $Tx_j = \lambda_j x_j$.

Denote $X_n = \operatorname{span}\{x_1, x_2, \dots, x_n\}$.

$\rightarrow \dim X_n = n$.

Moreover, $X_n \subsetneq X_{n+1}$.

Suppose $z \in X_n$, $z = \sum_{j=1}^{n} a_j x_j$.

Then $Tz = \sum_{j=1}^{n} a_j Tx_j$

$$= \sum_{j=1}^{n} a_j \lambda_j x_j \quad \in X_n.$$

$$(T - \lambda_n I) z = \sum_{j=1}^{n-1} (\lambda_j - \lambda_n) a_j x_j \quad \in X_{n-1}$$

$\Rightarrow$ $TX_n \subset X_n$
$(T-\lambda_n) X_n \subset X_{n-1}$.

Let us use the almost orthogonality lemma to find $y_n \in X_n$ s.t.

$$\|y_n\| = 1$$
$$\text{and } \|y_n - x\| > \tfrac{1}{2} \qquad \forall x \in X_{n-1}.$$

Then for $n > m$ we have

$$Ty_n - Ty_m = \lambda_n y_n + \underbrace{(T - \lambda_n I) y_n - Ty_m}_{\in X_{n-1}}$$

as $y_m \in X_m \subset X_{n-1}$ as $m < n$
$Ty_m \in TX_m \subset X_m \subset X_{n-1}$
and $y_n \in X_n$
$(T - \lambda_n I) y_n \in (T - \lambda_n I) X_n \subset X_{n-1}$.

Therefore $Ty_n - Ty_m = \lambda(y_n - z)$

where $z = -\frac{1}{\lambda_n}(T - \lambda_n I) y_n + \frac{1}{\lambda_n} Ty_m \quad \in X_{n-1}$

$$\|Ty_n - Ty_m\| = |\lambda_n| \, \|y_n - z\|$$
$$> \frac{|\lambda_n|}{2}$$
$$> \frac{\delta}{2}.$$

and so $\{Ty_n\}$ has no Cauchy subsequence

$\Rightarrow$ T is not compact. $\#$

This contradiction shows that $\forall \delta > 0$,

$\sigma(T) \cap \{\lambda ; |\lambda| > \delta\}$ is finite. $\square$.

<u>Def^n</u>: Let $T \in B(X, Y)$. Then we can define the <u>adjoint</u> operation

$$T^* : Y^* \to X^* \qquad \text{by}$$

$$(T^* f) x = f(Tx) \qquad f \in Y^* \quad x \in X.$$

and $\|T^*\| = \|T\|$.

<u>Theorem</u>: $T \in Com(X, Y) \Leftrightarrow T^* \in Com(Y^*, X^*)$.

HILBERT SPACES

Def^n: An inner (scalar) product space is a vector space H together with a map $(\cdot,\cdot) : H \times H \to \mathbb{F}$

s.t.

$$(\lambda x + \mu y, z) = \lambda (x,z) + \mu (y,z)$$
$$(x,y) = \overline{(y,x)}$$
$$(x,x) \geq 0$$
$$(x,x) = 0 \Leftrightarrow x = 0 \qquad \forall x,y,z \in H$$
$$\lambda, \mu \in \mathbb{F}.$$

Example: ℓ^2 : $\quad (x,y) = \sum_{j=1}^{\infty} x_i \overline{y_j}$

$L_2[0,1]$: $\quad (f,g) = \int_0^1 f(t)\overline{g(t)}\,dt$

Thm (Cauchy-Schwarz): $\quad |(x,y)| \leq \|x\|\,\|y\|$

where $\|x\| = \sqrt{(x,x)}$.

Def^n: A collection $\{x_\alpha\}_{\alpha \in A}$ is called <u>orthogonal</u> if

$(x_\alpha, x_\beta) = 0 \quad \forall \alpha \neq \beta.$ $\quad$ [written $x_\alpha \perp x_\beta$]

Thm (Pythagoras): If $x_1, \ldots, x_n$ are orthogonal then

$$\|x_1 + x_2 + \cdots + x_n\|^2 = \|x_1\|^2 + \cdots + \|x_n\|^2$$

Thm (Polarization identity):

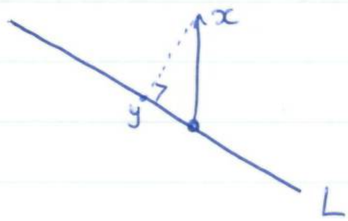
$\mathbb{F}=\mathbb{R}$: $\quad 4(x,y) = \|x+y\|^2 - \|x-y\|^2 \qquad \forall x,y \in H$

$\mathbb{F}=\mathbb{C}$: $\quad 4(x,y) = \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2. \qquad \forall x,y \in H$

Thm (Parallelogram Law):

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

L7 Thm: Let L be a closed subspace of a Hilbert space H. Let $x \in H$. Let $d := d(x,L) = \inf\{d(x,z), z \in L\}$

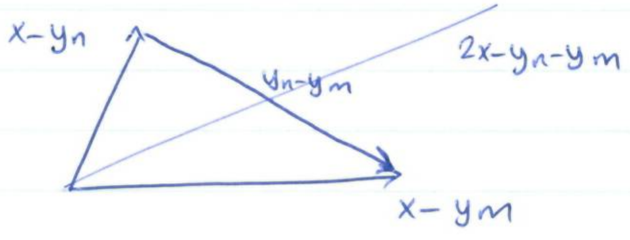


Then $\exists!\, y \in L$ s.t. $d(x,y) = d$ and $(x-y, z) = 0 \quad \forall z \in L$

$y =$ "orthogonal projection"

Proof: Step 1 (existence): $\exists\, y_n \in L$ s.t. $d(x,y_n) = \|x-y_n\| \xrightarrow{n\to\infty} d$.

We apply parallelogram rule to $(x-y_n)$ and $(x-y_m)$



$$0 \leq \|y_n - y_m\|^2 = 2\|x-y_n\|^2 + 2\|x-y_m\|^2 - \|2x-y_n-y_m\|^2$$

$$= 2\|x-y_n\|^2 + 2\|x-y_m\|^2 - 4\|x - \underbrace{\frac{y_n+y_m}{2}}_{\in L}\|^2$$

$$\leq 2\underbrace{\|x-y_n\|^2}_{\downarrow n\to\infty \; d} + 2\underbrace{\|x-y_m\|^2}_{\downarrow m\to\infty \; d} - 4d^2$$

$$\xrightarrow{n,m\to\infty} 2d^2 + 2d^2 - 4d^2$$

$$= 0$$

$\Rightarrow \|y_n - y_m\|^2 \to 0$

$\Rightarrow \|y_n - y_m\| \to 0 \quad \Rightarrow$ we have Cauchy sequence

and since $L \subset H$, a complete space,

(y_n) converges to, say, y.

$$d(x,y) = \lim_{n\to\infty} d(x, y_n) = d.$$

<u>Step 2</u> (orthogonality): Let $z \in L$, $\|z\| = 1$.

Consider $w := y + \lambda z$, $\lambda \in \mathbb{F}$.
$w \in L$.

Then $d^2 \leq d(x,w)^2 = \|x-w\|^2 \quad \forall \lambda$

$$= (x-w, x-w)$$

$$= (x-y-\lambda z, x-y-\lambda z).$$

$$= \|x-y\|^2 - \bar{\lambda}(x-y, z) - \lambda(z, x-y) + |\lambda|^2 + \|z\|^2$$

[Choose λ s.t. $\lambda = (x-y, z)$]

$$= \|x-y\|^2 - |\lambda|^2 - |\lambda|^2 + |\lambda|^2$$

$$= \|x-y\|^2 - |\lambda|^2$$

$$= d^2 - \lambda^2$$

$\Rightarrow \lambda = 0$

Thus $(x-y, z) = 0 \quad \forall z \in L$. ---- $(*)$

<u>Step 3</u> (uniqueness)

Suppose $y_1 \in L$ s.t. $\|x - y_1\| = d$.

Then Step 2 shows that $(x-y_1, z) = 0 \quad \forall z \in L$ .... $(\dagger)$

$(*)-(\dagger)$: $(y_1 - y, z) = 0 \quad \forall z \in L$

and, taking $z = y_1 - y \in L$, we obtain

$\|y_1 - y\|^2 = 0 \rightarrow y_1 = y$ so y is unique.

$\square$

<u>Def^n</u>: Suppose M be a subset of H.
Then the orthogonal complement of M,

$$M^\perp = \{ x \in H : (x, y) = 0 \quad \forall y \in M \}$$

<u>Proposition</u>: (1) $M^\perp$ is a closed subspace of H

$x_n \xrightarrow{(x_n, y) = 0} x$. Cts $\Rightarrow$ $(x, y) = 0 \Rightarrow x \in M^\perp$.

(2) $M_1 \subseteq M_2 \Longrightarrow M_1^\perp \supseteq M_2^\perp$

(3) $M^{\perp\perp} \supseteq M$

(4) Let M be dense in H (ie $\bar{M} = H$).
Then $M^\perp = \{0\}$

(5) $M^{\perp\perp} = \overline{\text{span}(M)}$

[Hint: use ~~lemma~~ thm proved at beginning of lecture to prove "⊆"
$M^{\perp\perp} \supset \overline{\text{span} M}$ is "trivial".
Can't use (1)–(4)]

<u>Theorem</u>: Suppose M is a closed linear subspace of H.
Then $H = M \oplus M^\perp$

orthogonal sum

<u>Proof</u>: Let $x \in H$.

We use the theorem at the beginning of this lecture (about orthogonal projection).

$\exists\, y \in M$ s.t. $(x - y, z) = 0 \quad \forall z \in M$.

Therefore $x - y \in M^\perp$ and

$$x = \underset{\underset{M}{\cap}}{y} + \underset{\underset{M^\perp}{\cap}}{(x-y)} .$$

Let us now prove that $M \cap M^\perp = \{0\}$.

Suppose $w \in M \cap M^\perp$. Then

$$\|w\|^2 = (\underset{\underset{M}{\cap}}{w}, \underset{\underset{M^\perp}{\cap}}{w}) = 0 \quad \Rightarrow \quad w = 0 . \qquad \square$$

<u>Def^n</u> A set $\{e_\alpha\}_{\alpha \in A}$ is called <u>orthonormal</u> if

$$(e_\alpha, e_\beta) = \delta_{\alpha\beta}$$

<u>Prop^n</u>: $\{e_\alpha\}$ is orthonormal $\Longrightarrow$ it is linearly independent.

<u>Thm</u>: Let $\{e_n\}$ be an orthonormal system, and $x \in H$.

Then the series $y := \sum_{n=1}^{\infty} (x, e_n) e_n$ is convergent

and $(x-y, e_n) = 0 \quad \forall n$

$$\|y\|^2 = \sum_{n=1}^{\infty} |(x, e_n)|^2 \leq \|x\|^2$$

(Bessel's inequality)

<u>Def^n</u>: $c_n = (x, e_n)$ is the <u>Fourier coefficient</u>

and $\sum_{n=1}^{\infty} c_n e_n$ is the <u>Fourier series</u>.

Thm: Let $\{e_n\}$ be an orthonormal system.
Then the FAE:

(1) $\forall x \in H$ we have $x = \sum_{n=1}^{\infty} (x, e_n) e_n$

(2) $\sum_{n=1}^{\infty} |(x, e_n)|^2 = \|x\|^2$ (Parseval's identity)

(3) $(x, e_n) = 0 \; \forall n \Rightarrow x = 0$

(4) $\text{span}\{e_n\}$ is dense in H.

Proof: (1) $\Rightarrow$ (2) Let $y = \sum_{n=1}^{\infty} (x, e_n) e_n$.

Then $\|y\|^2 = \sum_{n=1}^{\infty} |(x, e_n)|^2$.

If $y = x$, this gives (2).

(2) $\Rightarrow$ (3) Trivial

(3) $\Rightarrow$ (4) Let $x \in H$. Let $y = \sum_{n=1}^{\infty} (x, e_n) e_n \in \overline{\text{span}\{e_n\}}$

Also know $(x-y, e_n) = 0 \; \forall n$.
$\Rightarrow x = y$
$\Rightarrow x \in \overline{\text{span}\{e_n\}}$.

(4) $\Rightarrow$ (1) Let $x \in H$. Let $y = \sum_{n=1}^{\infty} (x, e_n) e_n$

Want to show $x = y$.

Also know $(x-y, e_n) = 0 \; \forall n$
Thus $(x-y, z) = 0 \quad \forall z \in \overline{\text{span}\{e_n\}} = H$

Take $z = x - y$.
Then $(x-y, x-y) = 0$, so $x = y$. $\square$

<u>Defn</u>: An orthonormal system $\{e_n\}$ is called <u>complete</u> if it satisfies any of the conditions (1) – (4).

<u>Examples</u>:
- $H = \ell^2$
  $e_n = (0, 0, \dots, 0, \overset{n^{th}\text{ place}}{1}, 0, \dots, 0, \dots)$ "natural basis"
  $\{e_n\}$ complete orthonormal system.
- $H = L_2[0,1]$
  $e_n = e^{i2\pi n t} = e_n(t)$ $\quad n \in \mathbb{Z}$

<u>Theorem</u>: (Riesz representation theorem)

Let $f : H \to \mathbb{F}$ be a bounded linear functional in a Hilbert space.

Then $\exists!\ z \in H$ s.t. $f(x) = (x, z)$ $\quad (\forall x \in H)$.
$\|f\| = \|z\|$.

<u>Proof</u>: (existence)

$f = 0 \Rightarrow$ take $z = 0$.

So assume $f \neq 0$.

Hence $\operatorname{Ker} f \neq H$

$\operatorname{Ker} f$ is closed linear subspace $\because$ f bounded.

$\Rightarrow (\operatorname{Ker} f)^{\perp} \neq \{0\}$.

Take $y \in (\operatorname{Ker} f)^{\perp}$, $y \neq 0$.

Then $\forall x \in H$ we have

$$f\left[y \cdot f(x) - x \cdot f(y)\right] = f(y)f(x) - f(x)f(y) = 0$$

$\Rightarrow y f(x) - x f(y) \in \operatorname{Ker} f$

Thus $(y f(x) - x f(y), y) = 0$.

or $f(x)\|y\|^2 - f(y)(x, y) = 0$

Thus, $f(x) = \dfrac{f(y)}{\|y\|^2}(x, y)$

$= (x, z)$

where $z = \dfrac{y\,\overline{f(y)}}{\|y\|^2}$

Keep in mind that x, y are vectors and $f(x), f(y)$ are scalars

Spectral theory in Hilbert spaces

Thm: Let $A \in \mathcal{B}(H)$.
Then $\exists!\ A^* \in \mathcal{B}(H)$ s.t.

$$(Ax, y) = (x, A^*y) \qquad \forall x, y \in H.$$

Proof: Let $y \in H$ be fixed. Then

$$x \longmapsto (Ax, y) = f(x)$$

is a linear functional

Moreover,
$$\begin{aligned} |f(x)| &= |(Ax, y)| \\ &\leq \|Ax\| \|y\| \qquad (C\text{-}S) \\ &\leq (\|A\| \|y\|) \|x\| \end{aligned}$$

$$\Rightarrow \quad \|f\| \leq \|A\| \|y\|$$
ie f is a bounded functional.

Apply Riesz representation thm $\Rightarrow$

$\exists!\ z \in H$ s.t.

$$(Ax, y) = f(x) = (x, z)$$

and $\|f\| = \|z\| \Rightarrow \|z\| \leq \|A\| \|y\|$.

I define the adjoint operator A^* as

$$A^* : y \longmapsto z$$

Then A^* is linear and we have

$$(Ax, y) = (x, A^*y).$$

We also have

$$\|z\| = \|A^*y\| \leq \|A\| \|y\|, \quad \text{and}$$

$$\frac{\|A^*y\|}{\|y\|} \leq \|A\|.$$

Therefore

$$\|A^*\| = \sup_{y \neq 0} \frac{\|A^*y\|}{\|y\|} \leq \|A\|$$

$$\Rightarrow A^* \in \mathcal{B}(H). \qquad \square.$$

<u>Def$^{\underline{n}}$</u>: A^* is called the adjoint operator to A

<u>Thm</u>: (1) $[A_1\alpha_1 + A_2\alpha_2]^* = \overline{\alpha_1} A_1^* + \overline{\alpha_2} A_2^*$

(2) $[AB]^* = B^* A^*$

(3) $A^{**} = A$

(4) $\|A^*\| = \|A\|$

(5) $\|A^*A\| = \|AA^*\| = \|A\|^2$

(6) If A is invertible, A^* is invertible and $(A^*)^{-1} = (A^{-1})^*$.

<u>Proof</u> : (1) $((\alpha_1 A_1 + \alpha_2 A_2)x, y) = \alpha_1 (A_1 x, y) + \alpha_2 (A_2 x, y)$

$$= \alpha_1 (x, A_1^* y) + \alpha_2 (x, A_2^* y)$$
$$= (x, \overline{\alpha_1} A_1^* y) + (x, \alpha_2 A_2^* y)$$
$$= (x, (\overline{\alpha_1} A_1^* + \overline{\alpha_2} A_2^*) y)$$

(2) $(ABx, y) = (Bx, A^+ y)$
$$= (x, B^* A^* y)$$

(3) $(Ax, y) = (x, A^* y)$

$$= \overline{(A^* y, x)}$$

$$= \overline{(y, A^{**} x)}$$

$$= (A^{**} x, y) \qquad \forall x, y \in \mathcal{H}$$

$\Rightarrow (Ax - A^{**} x, y) = 0$.

Taking $y = Ax - A^{**} x$, we see that

$Ax - A^{**} x = 0 \ \forall x \Rightarrow A = A^{**}$.

(4) We have seen

$$\|A^*\| \leq \|A\|.$$

Also, $\|A\| = \|A^{**}\| \leq \|A^*\|$

$\Rightarrow \|A^*\| = \|A\|.$

(5) $\|A^*A\| \leq \|A^*\|\,\|A\| = \|A\|^2$

On the other hand,

$$\begin{aligned}
\|A\|^2 &= \sup_{\|x\|=1} \|Ax\|^2 \\
&= \sup_{\|x\|=1} (Ax, Ax) \\
&= \sup_{\|x\|=1} (x, A^*Ax) \\
\text{(C-S)} &\leq \sup_{\|x\|=1} \left[\cancelto{1}{\|x\|}\ \|A^*Ax\|\right] \\
&= \|A^*A\|
\end{aligned}$$

$\Rightarrow \|A^*A\| = \|AA^*\|.$

(6)

$AA^{-1} = I$	$A^{-1}A = I$
$\Rightarrow (AA^{-1})^* = I^* = I$	$(A^{-1}A)^* = I^* = I$
$\Rightarrow (A^{-1})^*A^* = I$	$A^*(A^{-1})^* = I$

$\Rightarrow$ A^* has left and right inverse and

$$(A^*)^{-1} = (A^{-1})^*.$$

C^*-algebras

Thm: $A \in \mathcal{B}(H)$. Then

$$\begin{aligned} Ker(A^*) &= (Ran\,A)^\perp \quad \text{and} \\ Ker(A) &= (Ran\,A^*)^\perp \end{aligned}$$

Proof: Let $y \in Ker(A^*)$

$\iff A^* y = 0$

$\iff (x, A^* y) = 0 \quad \forall x \in H$

$\iff (Ax, y) = 0 \quad \forall x \in H$

$\iff y \in (Ran\,A)^\perp$

Replace A^* for A to get the other way round. $\square$

L8 $[A^{\perp\perp} = \overline{span A}]$

$$\Rightarrow \quad (Ker A^*)^\perp = (Ran\,A)^{\perp\perp} = \overline{Ran\,A}$$

and $(Ker A)^\perp = \overline{Ran\,A^*}$.

Defn: Let $A \in \mathcal{B}(H)$

(1) A is normal if $AA^* = A^*A$

(2) A is self-adjoint if $A^* = A$, ie

$$(Ax, y) = (x, Ay)$$

(3) Let $U \in \mathcal{B}(H_1, H_2)$.

$$U \text{ is unitary if } \quad U^* U = I_{H_1}$$
$$U U^* = I_{H_2}$$
$$\text{ie} \quad U^* = U^{-1}.$$

Consider the case of unbounded $A : D_A \overset{\subset H}{\longrightarrow} H$, $\quad \overline{D_A} = H$

e.g. A is differential operator

Consider for a fixed $y \in H$,

does z exist? Maybe only for certain y

$$f(x) = (Ax, y) \overset{?}{=} (x, z) \qquad \forall x \in D_A$$

Suppose $(Ax, y) = (x, z)$ (test uniqueness)
$$= (x, z')$$

Substract: $(x, z - z') = 0 \qquad \forall x \in D_A$

$$\Rightarrow z - z' \in D_A^{\perp}$$
$$= \{0\} \quad \because \overline{D_A} = H \Rightarrow D_A^{\perp} = \{0\}.$$

$$\Rightarrow z = z'$$

<u>Def^n</u>: The domain of A^*, ~~$D_{A^*} =$~~

$$D_{A^*} = \{ y \in H : \exists\, z = z(y) \in H$$
$$\text{s.t. } (Ax, y) = (x, z(y)) \ \forall x \in D_A \}$$

Then we define
$$A^* y = z(y).$$

$$\Rightarrow (Ax, y) = (x, A^* y) \quad \forall x \in D_A \quad \forall y \in D_{A^*}$$

D_{A^*} always contains 0

$\underline{\text{Def}^n}$: (1) A is <u>self-adjoint</u> if $A^* = A$,
in particular $D_{A^*} = D_A$.

(1) $\Rightarrow$ (2)

(2) A is symmetric if
$$(Ax, y) = (x, Ay) \qquad \forall x, y \in D_A$$

This means that if $y \in D_A \Rightarrow y \in D_{A^*}$ and $Ay = A^*y$.

In other words, A^* is the extension of A.
$$A \subset A^*$$ means $D_A \subset D_{A^*}$
and if we look at the set where they are both defined, $A = A^*$.

<u>Examples</u>

1. $H = L_2[0,1]$
$Af = f'$
$D_A = \{ f \in L_2[0,1] : f' \in L_2[0,1] \}$ "Sobolev space"
$= H^1_2[0,1]$ Russian notation
$= W^{1,2}[0,1]$ British notation

differentiate 1 times; L_2

$$(Af, g) = \int_0^1 f'(t) \overline{g(t)}\, dt$$

[parts] $$= f(1)\overline{g(1)} - f(0)\overline{g(0)} - \int_0^1 f(t)\overline{g'(t)}\, dt$$

$$= (f, h)$$ by a theorem which we haven't seen

We need the boundary (1st 2) terms to disappear, then we have $h = -g'(t)$.

Change approach:

~~ta~~ Let $Af = if'$
$D_A = H^1[0,1]$

$$(Af, g) = i\int_0^1 f'(t)\overline{g(t)}\,dt$$

$$= if(1)\overline{g(1)} - if(0)\overline{g(0)} + \int_0^1 f(t)\overline{ig'(t)}$$

$$= (f, h) \quad \text{if } h(t) = ig'(t),$$

$$\text{or } A^* g = ig'$$

We can have the equality so long as $\overline{g(1)} = \overline{g(0)} = 0$.

$\Rightarrow D_{A^*} = \{ g \in H^1[0,1] : g(0) = 0,\ g(1) = 0 \}$.

2. $H = L_2[0,1]$
$Af = f'$
$D_A = \{ f \in L_2[0,1] : f' \in L_2[0,1],\ f(0) = f(1) = 0 \}$

$$(Af, g) = \int_0^1 f'(t)\overline{g(t)}\,dt$$

$$= -\int_0^1 f(t)\overline{g'(t)}\,dt$$ parts with first terms killed

$\Rightarrow D_{A^*} = \{ g \in H^1[0,1] \} \supset D_A$

$\rightarrow A^* \supsetneq A$

A is symmetric but not self-adjoint.

3. $H = L_2[0,1]$

$Af = f'$

$D_A = \{f \in L_2[0,1] : f' \in L_2[0,1],\ f(0) = f(1)\}$

$D_{A^*} = \{g \in H^1[0,1] : g(1) = g(0)\}$

$A = A^*$

4. $H = L_2[0,1]$

$Af = f'$

$D_A = \{f \in L_2[0,1] : f' \in L_2[0,1],\ f(0) = \alpha f(1),\ |\alpha| = 1\}$

$D_{A^*} = \{g \in H^1[0,1] : \alpha g(1) = g(0)\}$.

$A = A^*$.

5. $H = L_2[0,1]$

$Af = -f''$ (minus for convenience later) historical reasons!

$D_A = \{f \in L_2[0,1];\ f'' \in L_2[0,1],\ f' \in L_2[0,1]\}$ follows from $f'' \in L_2$

$= H^2[0,1]$

$$(Af, g) = -\int_0^1 f''(t)\overline{g(t)}\,dt$$

$$= -f'(1)\overline{g(1)} + f'(0)\overline{g(0)} + \int_0^1 f'(t)\overline{g'(t)}\,dt$$

$$= -f'(1)\overline{g(1)} + f'(0)\overline{g(0)} + f(1)\overline{g'(1)} - f(0)\overline{g'(0)} - \int_0^1 f(t)\overline{g''(t)}\,dt$$

$$\stackrel{?}{=} (f, A^* g)$$

$$A^* g = g''$$

$$D_{A^*} = \{ g \in H^2[0,1] : g(0) = g(1) = g'(0) = g'(1) = 0 \}$$

6. $H = L_2[0,1]$

$Af = f''$

$$D_A = \left\{ f \in L_2 : f'' \in L_2, f' \in L_2 : \begin{array}{l} f(0) = f(1) \\ f'(0) = f'(1) \end{array} \right\}$$ periodic boundary conditions

7. $H = L_2$

$Af = f''$

$D_A = \{ f \in L_2 : f'' \in L_2, f' \in L_2 \quad f(0) = f(1) = 0 \}$ Dirichlet b.c.s.

$D_{A^*} = \{ g \in H_2 \quad g(0) = g(1) = 0 \}$

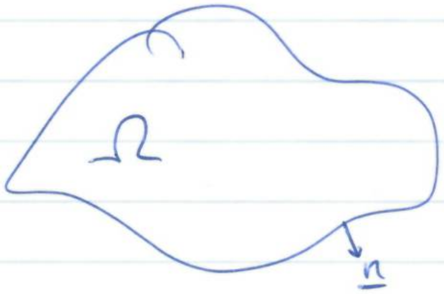
8. $H = L_2$

$Af = f''$

$D_A = \{ f \in L_2 : f'' \in L_2, f' \in L_2 \quad f'(0) = f'(1) = 0 \}$ Neuman b.c.s.

$D_{A^*} = \{ g \in H_2 \quad g'(0) = g'(0) = 0 \}$

9. Take a domain $\Omega \subset \mathbb{R}^d$



$$H = L_2(\Omega)$$

$$Af = -\nabla^2 f = -\sum_{j=1}^{d} \frac{\partial^2 f}{\partial x_j^2}$$

$$D_A = H^2(\Omega) = \left\{ f \in L_2(\Omega) : \frac{\partial f}{\partial x_j} \in L_2(\Omega),\ \frac{\partial^2 f}{\partial x_j \partial x_k} \in L_2(\Omega) \right\}$$

$$(Af, g) = -\int_\Omega \nabla^2 f \, \bar{g}$$

$$= -\int_{\partial\Omega} \frac{\partial f}{\partial \underline{n}} \bar{g} + \int_\Omega \nabla f \, \overline{\nabla g}$$

$$= -\int_{\partial\Omega} \frac{\partial f}{\partial \underline{n}} \bar{g} + \int_{\partial\Omega} f \frac{\partial \bar{g}}{\partial \underline{n}} - \int_\Omega f \, \overline{\nabla^2 g}$$

$$= (f, A^* g)$$

if $A^* g = -\nabla^2 g$

and the first 2 terms are killed.

$$D_{A^*} = \{ g \in H^2(\Omega) : g|_{\partial\Omega} \equiv \frac{\partial g}{\partial \underline{n}}\Big|_{\partial\Omega} \equiv 0 .$$

10. $D_A = \{ f \in L_2(\Omega),\ f|_{\partial\Omega} \equiv 0 \}$

$D_{A^*} = \{ g \in H^2(\Omega),\ g|_{\partial\Omega} = 0 \}$

Dirichlet bcs

11. $D_A = \{ f \in L_2(\Omega),\ \frac{\partial f}{\partial \underline{n}}\Big|_{\partial\Omega} \equiv 0 \}$

$D_{A^*} = \{ g \in H^2(\Omega),\ \frac{\partial g}{\partial \underline{n}}\Big|_{\partial\Omega} = 0 \}$

Neumann bcs

12. $D_A = \{ f \in L_2(\Omega),\ f|_{\partial\Omega} = \lambda \frac{\partial f}{\partial \underline{n}}\Big|_{\partial\Omega}$

$\lambda \in \mathbb{R}$, λ can be a f^n?

and $g|_{\partial\Omega} = \lambda \frac{\partial g}{\partial \underline{n}}\Big|_{\partial\Omega} \}$

$D_{A^*} = \{ g \in H^2(\Omega) \}$.

<u>Thm</u>: Let $A \in \mathcal{B}(H)$.
A is normal $\iff \forall x \in H \quad \|Ax\| = \|A^*x\|$

<u>Proof</u>: $(\Longrightarrow)$ $\|Ax\|^2 = (Ax, Ax)$
$= (A^*Ax, x)$

$\|A^*x\|^2 = (A^*x, A^*x)$
$= (AA^*x, x)$

$\because$ normal.

$(\Longleftarrow)$ $(Ax, Ay) = (A^*x, A^*y)$ by polarisation identity

$\mathbb{F} = \mathbb{R}$ $\|$

$\frac{1}{4}\Big[(A(x+y), A(x+y)) - (A(x-y), A(x-y))\Big]$

$\frac{1}{4}\Big[(A^*(x+y), A^*(x+y)) - (A^*(x-y), A^*(x-y))\Big]$

$\Rightarrow \quad (A^*Ax, y) = (AA^*x, y)$

$\Rightarrow \quad ((A^*A - AA^*)x, y) = 0 \qquad \forall x, y$

$\Rightarrow (A^*A - AA^*)x \in H^{\perp}$ $\forall y \in H$

$\Rightarrow (A^*A - AA^*)x = 0 \qquad \forall x.$

$\Rightarrow \quad A^*A = AA^* \qquad \square$

<u>Thm</u> Let $A \in \mathcal{B}(H)$ be normal. Then

(i) $(\mathrm{Ran}\, A^*)^{\perp} = \mathrm{Ker}\, A$ — we know true for all operators

$(\mathrm{Ran}\, A)^{\perp} = \mathrm{Ker}\, A^*$ ← new!

(ii) $Ax = \lambda x \Rightarrow A^*x = \bar{\lambda}x \qquad (\lambda \in \mathbb{C})$

(iii) $Ax = \lambda x, \ Ay = \mu y, \ \lambda \neq \mu \Rightarrow (x, y) = 0.$

<u>Proof</u> (i) A is normal $\Rightarrow \|Ax\| = \|A^*x\|$ by previous thm.

Suppose $x \in \mathrm{Ker}\, A$.
$\Leftrightarrow x \in \mathrm{Ker}\, A^*$ $\Bigg\} \Rightarrow \mathrm{Ker}\, A = \mathrm{Ker}\, A^*$

(ii) A is normal $\Rightarrow A - \lambda I$ is normal

$Ax = \lambda x \Rightarrow x \in \mathrm{Ker}(A - \lambda I)$
$\Rightarrow x \in \mathrm{Ker}(A - \lambda I)^*$
$= \mathrm{Ker}(A^* - \bar{\lambda} I)$

$\Rightarrow A^*x = \bar{\lambda} I.$

(iii) $$\begin{aligned}\lambda(x,y) &= (Ax,y)\\ &= (x, A^*y)\\ &= (x, \bar{\mu}y)\\ &= (\mu x, y)\\ &= \mu(x,y)\end{aligned}$$

Hence if $\lambda \neq \mu$, $(x,y) = 0$.

Thm $A \in \mathcal{B}(H)$

(i) $A = A^*$, $B = B^*$ $\Rightarrow$ $(A+B)$ is self-adjoint

(ii) $A = A^*$, $\lambda \in \mathbb{R}$ $\Rightarrow$ λA is self-adjoint

(iii) $A = A^*$, $B = B^*$ $\Rightarrow$ (AB) is self adjoint iff $AB = BA$.

(iv) $A_n = A_n^*$, $\|A_n - A\| \to 0$ $\Rightarrow$ A is self-adjoint

Proof: exercise. $\square$

Defn: $A \in \mathcal{B}(H)$.
Then the bilinear form associated with A,

$$f_A(x,y) = (Ax, y)$$

The quadratic form of A

$$q(x) = q_A(x) = (Ax, x)$$

Thm: Let $A \in \mathcal{B}(H)$, $\mathbb{F} = \mathbb{C}$.

Then $A = A^* \iff (Ax, x) \in \mathbb{R} \quad \forall x \in H$.

Proof: $(\Rightarrow)$ $A = A^* \Rightarrow (Ax, x) = (x, Ax)$

$$= \overline{(Ax, x)}$$

$$\Rightarrow (Ax, x) \in \mathbb{R}.$$

$(\Leftarrow)$ $(Ax, x) \in \mathbb{R} \Rightarrow (Ax, x) = \overline{(x, Ax)}$

$$= (x, Ax) \quad \because \text{real}$$

$$= (A^*x, x)$$

Need to show $(Ax, y) = (A^*x, y)$.

Use polarisation identity for operators.

$$4(Ax, y) = (A(x+y), x+y) - (A(x-y), x-y) + i(A(x+iy), x+iy) - i(A(x-iy), x-iy)$$

$$4(A^*x, y) = (A^*(x+y), x+y) - (A^*(x-y), x-y) + i(A^*(x+iy), x+iy) - i(A^*(x-iy), x-iy)$$

Terms ~~by~~ have equality by what we just showed $((Ax, x) = (A^*x, x))$

Thus $(Ax - A^*x, y) = 0$

$\Rightarrow Ax = A^*x \quad \forall x$

$\Rightarrow A = A^*$. $\square$

Thm: If $A = A^*$, Then all eigenvalues of A are real. and

Proof: Let $Ax = \lambda x$

$$\Rightarrow \quad q_A(x) = (Ax, x) = \lambda(x,x) = \lambda \|x\|^2 \in \mathbb{R}$$

$$\Rightarrow \quad \lambda \in \mathbb{R}. \quad \square$$

Thm: Let $P \in \mathcal{B}(H)$ be a projection. Then FAE:

(i) P is self-adjoint $\quad (P^* = P)$

(ii) P is normal $\quad PP^* = P^*P$

(iii) $(\operatorname{Ker} P)^\perp = \operatorname{Ran} P$

(iv) $(Px, x) = \|Px\|^2 \quad \forall x \in H$.

Proof: (cheating! Assume $\mathbb{F} = \mathbb{C}$).

(i) $\Rightarrow$ (ii) Trivial

(ii) $\Rightarrow$ (iii) We have proved that for normal operators, $(\operatorname{Ker} P) = (\operatorname{Ran} P)^\perp$.

$$\rightarrow \quad (\operatorname{Ker} P)^\perp = (\operatorname{Ran} P)^{\perp\perp} = \overline{\operatorname{Ran} P}$$

$$= \overline{\operatorname{Ker}(I-P)} = \operatorname{Ker}(I-P) = \operatorname{Ran} P.$$

(iii) $\Rightarrow$ (iv) Suppose $x, y \in H$.

$$\text{Then } (\underset{\in \operatorname{Ran} P}{Px}, \underset{\in \operatorname{Ker} P}{(I-P)y}) \underset{\because\ (\operatorname{Ker} P)^\perp = \operatorname{Ran} P}{=} 0$$

$$\begin{aligned}
\text{Therefore } (Px, y) &= (Px, Py + (I-P)y) \\
&= (Px, Py) \qquad \because (Px, (I-P)y) = 0. \\
&= (Px + (I-P)x, Py) \\
&= (x, Py)
\end{aligned}$$

So P is self-adjoint

$$\Rightarrow \quad (Px, x) = (Px, Px) = \|Px\|^2$$

(iv) $\Rightarrow$ (i) $\quad q_P(x) = \|Px\|^2 \in \mathbb{R}$ cheating.

$\Rightarrow$ P is self-adjoint. $\square$

<u>Def^{n}</u>: If P is a projection satisfying any of these properties then P is called an <u>orthogonal projection</u>.

Thm: Suppose $A \in \mathcal{B}(H)$ and $\exists\, c > 0$ s.t.

$$|(Ax, x)| \geq c\|x\|^2.$$

Then A is invertible and $\|A^{-1}\| \leq c^{-1}$.

Recall
$\|Bx\| \geq c\|x\|$
$\Downarrow$
$\operatorname{Ker} B = \{0\}$
$\operatorname{Ran} B$ is closed

Proof: $|(Ax, x)| \geq c\|x\|^2$

$\|Ax\|\|x\| \geq |(Ax,x)|$ by Cauchy-Schwarz

$\Rightarrow \|Ax\| \geq c\|x\|$ (cancelling $\|x\|$) $\quad (*)$

$\Rightarrow \operatorname{Ker} A = \{0\}$
$\operatorname{Ran} A$ is closed by green in top-right corner.

Now will show $\operatorname{Ran} A$ is dense in H,
$\Leftrightarrow (\operatorname{Ran} A)^{\perp} = \{0\}$.

Suppose $x \in (\operatorname{Ran} A)^{\perp}$.

$\Rightarrow (Ax, x) = 0 \Rightarrow \overset{(\text{RHS})}{\|x\| = 0}$
$\Rightarrow x = 0$.

$\Rightarrow \overline{\operatorname{Ran} A} = H$.
$\Rightarrow \operatorname{Ran} A = H \quad \because \operatorname{Ran} A$ is closed.

$\operatorname{Ker} A = \{0\}$, $\operatorname{Ran} A = H \Rightarrow$ <u>A is invertible</u>.

$\|A^{-1}\| \leq c^{-1}$ is "an easy exercise" follows from $(*)$ $\square$

<u>Def</u> : $A \in \mathcal{B}(H)$.
Then the set

$$\mathrm{Num}(A) = \{ (Ax, x) : x \in H, \ \|x\| = 1 \}$$

is called the <u>numerical range</u> of A.

Since $|(Ax,x)| \leq \|Ax\| \, \|x\| \leq \|A\| \, \|x\|^2 = \|A\|$

so $\mathrm{Num}A \subset B_c(0, \|A\|)$ closed disc

<u>Thm</u>: $\mathrm{Num}A$ is a convex set, ie
if $a, b \in \mathrm{Num}A$, interval $[a,b] \subset \mathrm{Num}A$.

<u>Thm</u>: $\sigma(A) \subset \overline{\mathrm{Num}A}$.

<u>Proof</u>: Suppose $\lambda \notin \overline{\mathrm{Num}A}$, ie $\exists \, d > 0$ s.t.
[9] $|\lambda - \mu| \geq d \quad \forall \mu \in \mathrm{Num}A$.

Suppose $\|x\| = 1$. Then

$$|((A\lambda - I)x, x)| = |(Ax,x) - \lambda \underbrace{(x,x)}_{= \|x\|^2 = 1}| =$$

$$= |\underbrace{(Ax,x)}_{\in \mathrm{Num}A} - \lambda|$$

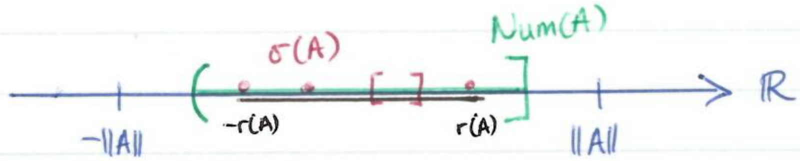
$$\geq d$$

$$= d\|x\|^2$$

This inequality holds $\forall x$ (you just scale both sides by multiplying by the relevant constant), so we can apply the previous lemma and obtain that $(A - \lambda I)$ is invertible, so $\lambda \in \rho(A)$. $\square$

Spectrum of self-adjoint operators

We know $\sigma(A) \subset \mathbb{R}$ when $A = A^*$ since $\overline{\text{Num} A} \subset \mathbb{R}$ and $\sigma(A) \subset \overline{\text{Num} A}$.



Claim: $\|A\| = r(A) = \sup\limits_{\|x\|=1} |(Ax, x)|$.

Thm: If $A = A^*$ then $\|A\| = \sup\limits_{\|x\|=1} |(Ax, x)|$

Proof: $\|A\| = \sup\limits_{\|x\|=1} \|Ax\|$

$$= \sup_{\substack{\|x\|=1 \\ \|y\|=1}} |(Ax, y)|$$

$$\geq \sup_{\|x\|=1} |(Ax, x)| = c$$ → and so $|(Az, z)| \leq c\|z\|^2$ ← (**)

We have (*)

$$(A(x+y), x+y) - (A(x-y), x-y)$$
$$= 2(Ax, y) + 2(Ay, x)$$

Polarization identity?

$$= 2\left[(Ax,y) + \overline{(Ax,y)}\right]$$

$$= 4\,\mathrm{Re}\left[(Ax,y)\right]$$

Therefore

this come from line (*) and observation (**)

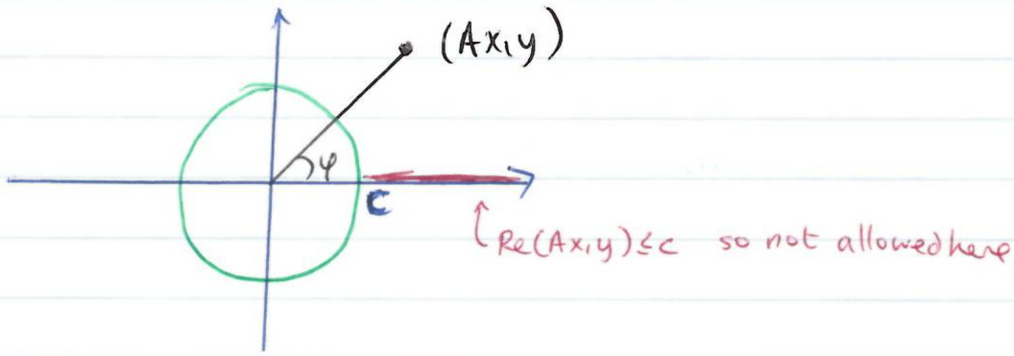
$$4\,\mathrm{Re}(Ax,y) \le c\left(\|x+y\|^2 + \|x-y\|^2\right)$$

$$= 2c\left(\|x\|^2 + \|y\|^2\right) \quad \text{Parallelogram law}$$

$$= 4c \quad \text{if } \|x\| = \|y\| = 1.$$

$$\Rightarrow \mathrm{Re}(Ax,y) \le c \quad \forall \|x\| = \|y\| = 1.$$

But really want to prove that $|(Ax,y)| \le c$
Why is this implied from what we just showed?



Suppose that for some x, y, we have
$|(Ax,y)| > c$, ie
$|(Ax,y)| = e^{i\varphi} r$, $r > c$

Let $y_1 = e^{i\varphi} y$, which satisfies

$$\|y_1\| = \|e^{i\varphi}\|\,\|y\| = \|y\| = 1$$

and $\quad (Ax, y_1) = (Ax, e^{i\varphi} y)$
$$= e^{-i\varphi}(Ax, y)$$
$$= r$$

$\Rightarrow \quad Re(Ax, y_1) = r > c$, which contradicts our results.

$$\Rightarrow \sup_{\substack{\|x\|=1 \\ \|y\|=1}} |(Ax,y)| \leq c$$

$$\Rightarrow \quad \|A\| = \sup_{\substack{\|x\|=1 \\ \|y\|=1}} |(Ax,y)| = c \qquad \square$$

<u>Thm</u>: Let B be self-adjoint and denote

$$M = \sup_{\|x\|=1} (Bx, x)$$

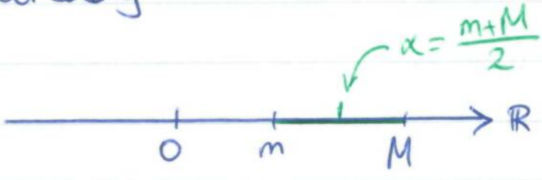
$$m = \inf_{\|x\|=1} (Bx, x)$$

Then (1) $\sigma(B) \subset \overline{Num B} = [m, M] \subset \mathbb{R}$

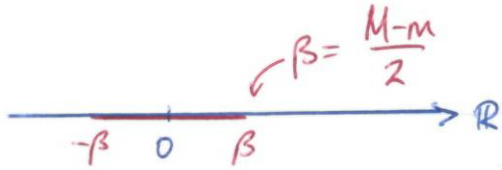
(2) $m \in \sigma(B)$ and $M \in \sigma(B)$.

<u>Proof</u>: (1) Proved already

(2)



Consider $A := B - \alpha I$ to shift interval to be symmetric about zero.



Then $A = A^*$,

$$(Ax, x) = (Bx, x) - \alpha$$

$$\Rightarrow \inf_{\|x\|=1} (Ax, x) = \inf_{\|x\|=1} \left[(Bx, x) - \alpha \right] = -\beta$$

$$\Rightarrow \sup_{\|x\|=1} (Ax, x) = \sup_{\|x\|=1} \left[(Bx, x) - \alpha \right] = \beta$$

The previous theorem implies that

$$\|A\| = \beta.$$

We know that $\exists\ x_n \in H$ s.t. $\|x_n\| = 1$ and $(Ax_n, x_n) \to \beta$ as $n \to \infty$.

Then
$$\|(A - \beta I)x_n\|^2 = \left((A - \beta I)x_n, (A - \beta I)x_n \right)$$
$$= \underbrace{\|Ax_n\|^2}_{\leq \beta^2} + \beta^2 - 2\beta(Ax_n, x_n)$$
$$\leq 2\beta^2 - 2\beta \overbrace{(Ax_n, x_n)}^{\to \beta \text{ as } n \to \infty.}$$
$$\xrightarrow{n \to \infty} 0$$

This implies $A - \beta I$ has no bounded inverse.

Indeed, if $(A - \beta I)^{-1}$ existed and was bounded, then

$$(A-\beta I)^{-1} : y_n \longmapsto \frac{x_n}{\|(A-\beta I)x_n\|} \nearrow \infty$$

where $y_n := \dfrac{(A-\beta I)x_n}{\|(A-\beta I)x_n\|}$

so $\|(A-\beta I)^{-1}\| = \infty$

Therefore $\beta \in \sigma(A)$ (and $B = A + \alpha I$ satisfies $M = \alpha + \beta \in \sigma(B)$).

Similarly $-\beta \in \sigma(A)$ (and $m = -\beta + \alpha \in \sigma(B)$).. $\square$

<u>Corollary</u>: $A = A^*$. Then $\exists\, \lambda \in \mathbb{R}$ s.t.

$|\lambda| = \|A\|$ and $\lambda \in \sigma(A)$; so $r(A) = \|A\|$.

In particular, if $r(A) = 0$. Then $A = 0$.

<u>Thm</u>: Suppose A is normal. Then
$$\|A\| = r(A) = \sup_{\|x\|=1} |(Ax, x)| .$$

Thm: $U \in \mathcal{B}(H)$. Then the FAE:

(1) U is unitary
(2) $\operatorname{Ran} U = H$ and $(Ux, Uy) = (x, y)$
(3) $\operatorname{Ran} U = H$ and $\|Ux\| = \|x\|$

Proof: (1) ⇒ (2): U is unitary $\Rightarrow UU^* = I$

$\Rightarrow \mathrm{Ran}(U) = H$

Also $(Ux, Uy) = (x, U^*Uy) = (x,y)$

(2) ⇒ (3): trivial

(3) ⇒ (2): follows from polarization identity

(2) ⇒ (1): We have $(U^*Ux, y) = (Ux, Uy)$

$= (x,y)$

$\Rightarrow U^*Ux = x \quad \forall x$

$\Rightarrow U^*U = I$

Since $\mathrm{Ker}\, U = \{0\}$ and $\mathrm{Ran}\, U = H$, U^{-1} exists.

Thus $U^* = U_l^{-1} = U^{-1}$

$\Rightarrow$ U is unitary. □

Thm: Suppose U is unitary. Then

$$\sigma(U) \subset \{\lambda : |\lambda| < 1\}$$

Proof: $\|U\| = 1 \Rightarrow \sigma(U) \subset B_c(0,1)$.

Suppose that $|\lambda| < 1$. Then

$$U - \lambda I = U(I - \lambda U^{-1})$$

and $\|U^{-1}\| = 1$ since $U^{-1} = U^*$ and $\|A\| = \|A^*\|$ $\forall A$

$\Rightarrow$ $U - \lambda I$ is invertible since $\|\lambda U^*\| = |\lambda| < 1$ and U^{-1} exists.

Therefore $\lambda \in \rho(U)$. $\square$

Thm: (Hilbert-Schmidt)

Suppose $T \in \mathrm{Com}(H)$, $T = T^*$.

Then there is a finite or countable orthonormal set $\{e_n\}_{n=1}^N$ ($N \in \mathbb{N}$ or $N = \infty$ or $N = 0$) of eigenvectors of T,

$$Te_n = \lambda_n e_n.$$

$\forall x \in H$ $\exists!$ decomposition $x = \sum_{n=1}^N c_n e_n + y$

where $y \in \operatorname{Ker} T$.

s.t. $Tx = \sum_{n=1}^N c_n \lambda_n e_n$.

Moreover, $\lambda_n \neq 0$ and $\sigma(T) \setminus \{0\} = \bigcup_{n=1}^N \lambda_n \subset \mathbb{R}$, $|\lambda_{n+1}| \leq |\lambda_n|$.

If $N = \infty$, then $\lim_{n \to \infty} \lambda_n = 0$.

<u>Proof</u>: We know that $\sigma(T) \setminus \{0\} = \bigcup_{n=1}^{N} \mu_n$,

where $\mu_n \neq \mu_m$ for $n \neq m$. Each μ_n has finite multiplicity and $\mu_n \in \mathbb{R}$.

We can assume that $|\mu_{n+1}| \leq |\mu_n|$.

Let N_n be an eigenspace corresponding to μ_n.

We take an orthonormal basis of each of N_K and arrange these vectors in a sequence $\{e_n\}_{n=1}^{N}$.

(we put elements of the basis of N_K before N_{K+1}).

For each e_n we have $Te_n = \lambda_n e_n$ where $\lambda_n = \mu_K$.

Let $L = \operatorname{span}\{e_n\}_{n=1}^{N} \subset \operatorname{Ran} T$.

Then $\operatorname{Ker} T = (\operatorname{Ran} T)^{\perp} \subset L^{\perp}$

<u>Claim</u>: $\operatorname{Ker} T \supset L^{\perp}$

<u>Proof</u>: Suppose $y \in L^{\perp}$. Then

$$\begin{aligned}
(Ty, e_n) &= (y, Te_n) \\
&= (y, \lambda_n e_n) \\
&= \lambda_n (y, e_n) && \because \overline{\lambda_n} = \lambda_n \\
&= 0 && \because y \in L^{\perp}.
\end{aligned}$$

So $Ty \in L^{\perp}$.

Thus $L^\perp$ is an invariant subspace of T.

$L^\perp$ is closed so it is a Hilbert space.

Consider $T_1 = T|_{L^\perp}$. Then
$T_1 \in \mathrm{Com}(L^\perp)$, $T_1{}^* = T^*$.

Moreover, T_1 has no nonzero eigenvalues (since all eigenvectors of T with nonzero eigenvalues belong to $N_K \subset L$).

Since $T_1 \in \mathrm{Com}(L^\perp)$,
$$\sigma(T_1)\setminus\{0\} = \{\text{eigenvalues}\} = \emptyset.$$

$\Rightarrow \sigma(T_1) = \{0\}$
$\Rightarrow r(T_1) = 0$
$\Rightarrow T_1 \equiv 0$.

But $T_1 = T|_{L^\perp} = 0$
$\Rightarrow L^\perp \subset \mathrm{Ker}\, T$ $\square$

$\Rightarrow L^\perp = \mathrm{Ker}\, T$.
$\Rightarrow H = \overline{L} \oplus L^\perp$
$= \overline{L} \oplus \mathrm{Ker}\, T$.

$\Rightarrow \forall x \, \exists! \; x = \sum_{n=1}^{N} c_n e_n + y$ where $y \in \mathrm{Ker}\, T$ $\square$

L10 Hilbert-Schmidt operators

Let $T \in \mathcal{B}(H)$ and $\{e_n\}_{n=1}^{\infty}$ and $\{f_m\}_{m=1}^{\infty}$ are two complete orthonormal sets.

Then
$$\sum_{n=1}^{\infty} \|Te_n\|^2 = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} |(Te_n, f_m)|^2$$
$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |(e_n, T^* f_m)|^2$$
$$= \sum_{m=1}^{\infty} \|T^* f_m\|^2.$$

Thus $\sum_{n=1}^{\infty} \|Te_n\|^2$ does not depend on the choice of $\{e_n\}$

Defn: If $\sum_{n=1}^{\infty} \|Te_n\|^2$ is finite, then we say that T is a Hilbert-Schmidt operator, and the H-S norm is given by
$$\|T\|_{HS} = \left[\sum_{n=1}^{\infty} \|Te_n\|^2\right]^{1/2}$$

Remark: $\|T\|_{HS}^2 = \sum_{n=1}^{\infty} \|Te_n\|^2$
$$\geq \sup_{\|e_1\|=1} \|Te_1\|^2$$
$$= \|T\|^2.$$

Theorem: If T is Hilbert-Schmidt, then $T \in Com(H)$.

Proof: Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis.

For all $x = \sum_{n=1} a_n e_n$,

$$Tx = T\sum a_n e_n = \sum a_n Te_n.$$

Define $T_N : x \longmapsto \underbrace{\sum_{n=1}^{N} a_n Te_n}_{\in \operatorname{span}(Te_1, Te_2, \dots, Te_N)}$.

Then
$$\|(T-T_N)x\| = \Big\| \sum_{n=N+1}^{\infty} a_n Te_n \Big\|$$
$$\leq \sum_{n=N+1}^{\infty} |a_n| \, \|Te_n\|$$
$$\overset{\text{c.s.}}{\leq} \Big[\sum_{n=N+1}^{\infty} |a_n|^2\Big]^{1/2} \Big[\sum_{n=N+1}^{\infty} \|Te_n\|^2\Big]^{1/2}$$
$$\leq \Big[\sum_{n=1}^{\infty} |a_n|^2\Big]^{1/2} \Big[\sum_{n=N+1}^{\infty} \|Te_n\|^2\Big]^{1/2}$$
$$= \|x\| \Big[\sum_{n=N+1}^{\infty} \|Te_n\|^2\Big]^{1/2}$$

Therefore, $\|T-T_N\| \leq \Big[\sum_{n=N+1}^{\infty} \|Te_n\|^2\Big]^{1/2}$

$\to 0$ as $n \to \infty$ $\because$ tail of convergent sum

Since T_N are finite-rank operators, T is compact. $\square$

<u>Example</u> $(Tf)(t) = \int_0^1 k(t,s) f(s) \, ds$

$H = L_2([0,1])$

<u>Claim</u>: T is H-S and $\|T\|_{HS}$ can be computed as

$$\|T\|_{HS} = \int_0^1\!\!\int_0^1 |k(t,s)|^2 \, ds \, dt$$

Proof: Denote $k_t(s) := k(t,s)$.

Then $Te_n(t) = \int_0^1 k(t,s)\, e_n(s)\, ds$

$= (k_t, e_n)$

where $\{e_n\}_{n=1}^{\infty}$ is a real orthonormal basis in $L_2[0,1]$.

Therefore $\|Te_n\|^2 = \int_0^1 |Te_n(t)|^2\, dt$

$= \int_0^1 |(k_t, e_n)|^2\, dt$

and $\sum_{n=1}^{\infty} \|Te_n\|^2 = \int_0^1 \sum_{n=1}^{\infty} |(k_t, e_n)|^2\, dt$

$//$

$\|T\|_{HS}^2$ $= \int_0^1 \|k_t\|^2\, dt$

$= \int_0^1 \left[\int_0^1 |k(t,s)|^2\, ds \right] dt$ $\square$.

Let $T \in Com(H)$.

Def$\underline{^n}$: The eigenvalues of T^*T are called the S-numbers of T.

$$S_1(T) \geq S_2(T) \geq \cdots .$$

$$S_n \xrightarrow{n\to\infty} 0.$$

Def$\underline{^n}$: (Schatten-von Neumann)

$$T \in \sigma_p \Leftrightarrow (S_1, S_2, \ldots) \in l_p \qquad 1 \leq p \leq \infty$$

$$\|T\|_p = \left[\sum_{j=1}^{\infty} S_j^p\right]^{1/p}.$$

<u>Exercise</u>: $\|T\|_2 = \|T\|_{HS}$

<u>Def 1</u>: σ_1 is called the trace class operators.

Let $T \in \sigma_1$. Then

$$\begin{aligned} tr(T) &= \sum_{j=1}^{\infty} \lambda_j(T) \\ &= \sum_{n=1}^{\infty} (Te_n, e_n) \qquad \leftarrow \text{Lidski's theorem} \end{aligned}$$

where $\{e_n\}$ is an orthonormal basis.