MATH0072 Riemannian Geometry Notes

Based on the 2018 autumn lectures by Dr J Lotay

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04-10-18

Riemannian Geometry - Felix Schulze room 607

HW weekly, due Fri
Office hour Thursday 12:45 - 1:45

Do Carmo
Boothby

Notes online - print them! Use both!

3 basic examples in 2d:
- flat plane $\mathbb{R}^2$ (flat)
- sphere S^2 (+vely curved)  $z = x^2 + y^2$
- hyperbolic space $\mathbb{H}^2$ (-vely curved)  $z = x^2 - y^2$

Objects:
- geodesics (shortest paths between points)
  - $\nwarrow$ length minimisers $\inf\{\text{length of all paths } a \rightsquigarrow b\}$
- curvature (area of small triangles $\leftarrow$ geodesic sides)
  - fat (bigger than in flat space / sum of angles $> \pi$)
    $\cong$ positive curvature 
  - thin (smaller than in flat space / sum of angles $< \pi$)
    $\cong$ negative curvature 

Idea:
local geometry $\Rightarrow$ restrictions on global geometry

Let M be an n-dim manifold, K its curvature,
Here are some example statements:
- If $K \leq 0$ then M is essentially $\mathbb{R}^n$ topologically
  (Cartan-Hadamard thm)

• If $K \geq \delta > 0$ then M has finite diameter (and is therefore compact) and there are only finitely many distinct closed loops (Bonnet-Myers thm). (i.e. fundamental group is finitely generated).

• If $\frac{1}{4} < K \leq 1$ then M is essentially the n-dim sphere S^n topologically (sphere thm)

up to a quotient i.e. universal cover

§1 Manifolds (def^{n}s & examples)

First fake def^n

A manifold is a natural notion of a smooth object.

Basic examples:

[dimension = how many coords needed to describe each point]

• $\mathbb{R}^2$ is a 2-dim manifold.

• $\mathbb{R}^n$ is an n-dim manifold.

• The upper half plane $\mathbb{H}^2 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 > 0\}$ is a 2-dim manifold.

• Similarly $\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^2 \mid x_n > 0\}$ is an n-dim manifold.

• The unit disc $B^2 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < 1\}$ is a 2-dim manifold

• Similarly $B^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum_i x_i^2 < 1\}$ is an n-dim manifold.

• The n-dim sphere $S^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_i x_i^2 = 1\}$ is an n-dim manifold

• The torus $\{(2+\cos\theta)\cos\phi, (2+\cos\theta)\sin\phi, \sin\theta) \in \mathbb{R}^3 \mid \theta, \phi \in \mathbb{R}\}$ is a 2-dim manifold

• The n-dim torus $T^n = \{(\cos\theta_1, \sin\theta_1, \cos\theta_2, \sin\theta_2, \dots, \cos\theta_n, \sin\theta_n) \mid \theta_1, \dots, \theta_n \in \mathbb{R}\}$

product of n circles.

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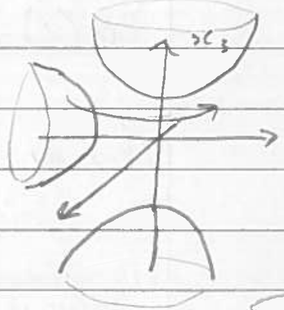
§1.2 some non-examples

<u>2nd fake definition</u>
An n-dim manifold is something which locally "looks like" $\mathbb{R}^n$.

<u>non-examples of manifolds</u>

- A cube
- The closed disc in $\mathbb{R}^2$
  $\{x \in \mathbb{R}^2 : |x| \leq 1\}$
- The hyperboloid of one sheet
  $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 - x_3^2 = 1\}$
  and the hyperboloid of two sheets
  $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 - x_3^2 = -1\}$
  are 1-dim manifolds, but
  $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 - x_3^2 = 0\}$ is a cone
  and not a manifold since it is not smooth at 0.



§1.3 More advanced examples of mfds.

<u>Ex</u>

$M_n(\mathbb{R})$ real $n \times n$ matrices

$GL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid det(A) \neq 0\}$

$SL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid det(A) = 1\}$

Note $det(AB) = det(A)det(B)$, I in both, so both groups.

<u>Claim</u>

$GL_n(\mathbb{R})$ is a n^2-dim manifold.

$SL_n(\mathbb{R})$ is a (n^2-1)-dim manifold.

Ex

Let I be the identity matrix in $M_n(R)$

Then $O(n) = \{A \in M_n(\mathbb{R}) \mid A^TA = I\}$

$SO(n) = \{A \in M_n(\mathbb{R}) \mid A^TA = I, \det(A) = 1\}$

Again note both of these sets are groups since I is in both and $(AB)^T(AB) = B^TA^TAB = B^TIB = I$

Then $O(n)$ and $SO(n)$ are $\frac{1}{2}n(n-1)$ dim manifolds

Ex

$SU(2) = \left\{\begin{pmatrix} a & b \\ -\bar{b} & \bar{a}\end{pmatrix} : a, b \in \mathbb{C},\ |a|^2 + |b|^2 = 1\right\}$

This is a group and a 3-dim manifold.

In general, we have the special unitary group

$SU(n) = \{A \in M_n(\mathbb{C}) \mid \bar{A}^TA = I, \det(A) = 1\}$ ← dim n^2-1

$U(n) = \{A \in M_n(\mathbb{C}) \mid \bar{A}^TA = I\}$ ← dim n^2

Remark

These examples given in terms of matrices are all examples which are groups. In fact these are almost the def^n of a Lie group

$:\approx$ group with manifold structure s.t. group structure is compatible with the manifold

A bit more interesting:

Ex

Let $\mathbb{R}P^n$ be the space of straight lines in $\mathbb{R}^{n+1}$ through 0.

Then $\mathbb{R}P^n$ is the real projective n-space and it is an n-dim manifold.

Equivalently we can say $\mathbb{R}P^n$ is the quotient of $\mathbb{R}^{n+1} \setminus \{0\}$ by the equivalence relation $x \sim y$ if $x = \lambda y$ for some $\lambda \in \mathbb{R}$.

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Hence one can denote points in $\mathbb{RP}^n$ by equivalence classes $[x]$.

Ex

We have $\mathbb{C}^n$ is a $2n$-dim mfd.

We consider $\mathbb{CP}^n$ the set of complex lines in $\mathbb{C}^{n+1}$ through 0. This is a $2n$-dim mfd called the complex projective n-space.

More explicitly, $\mathbb{CP}^n$ is the quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ by the equivalence relation $z \sim w$ if $z = \lambda w$ for some $\lambda \in \mathbb{C}$. We denote points in $\mathbb{CP}^n$ by equivalence classes $[z]$.

§1.4 Constructing mfds, regular values.

Let $F: \mathbb{R}^n \to \mathbb{R}^m$, written $F(x) = (F_1(x), \dots, F_n(x))$, then $dF_p = \left(\frac{\partial F_i}{\partial x_j}\right)_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$

Thm 1.1 (Regular value thm)

Let $F: \mathbb{R}^{n+m} \to \mathbb{R}^m$ be a smooth map, and suppose that $\forall p \in F^{-1}(c)$, where $F^{-1}(c) = \{p \in \mathbb{R}^{n+m} \mid F(p) = c\} \neq \emptyset$ ~~with~~ the derivative $dF_p: \mathbb{R}^{n+m} \to \mathbb{R}^m$ is surjective $\forall p \in F^{-1}(c)$ i.e. c is a regular value of F.

Then $F^{-1}(c)$ is an n-dim manifold.

Examples

Let $F: \mathbb{R}^{n+1} \to \mathbb{R}$ be $F(x_1, \dots, x_{n+1}) = \sum_{i=1}^{n+1} x_i^2$

Note $dF_x = (2x_1, \dots, 2x_{n+1})$

If $x \in F^{-1}(1)$ then $dF_x \neq 0$, but if $x \in F^{-1}(0)$ then $dF_x = 0$.

ie. 1 is a regular value and 0 is not.
So we see that $F^{-1}(1) = S^n$ is an $(n+1)-1 = n$ dim mfd
but $F^{-1}(0) = \{0\}$ is not an n-dim mfd.

Ex

Let $F: \mathbb{R}^{2n} \to \mathbb{R}^n$, $F(x_1, \dots, x_{2n}) = (x_1^2 + x_2^2 - 1, \dots, x_{2n-1}^2 + x_{2n}^2 - 1)$
This is smooth and for $x \in F^{-1}(0)$

$$dF_x = \begin{pmatrix} 2x_1 & 2x_2 & & 0 & & & \\ 0 & 0 & 2x_3 & 2x_4 & & 0 & \\ & \vdots & & & \vdots & & \\ & & 0 & & & 2x_{2n-1} & 2x_{2n} \end{pmatrix}$$

which has rank n since all rows are linearly independent as $(x_{2i-1}, x_{2i}) \neq (0,0)$ $\forall i$.
So the map is surjective. $F^{-1}(0) = T^n \subseteq \mathbb{R}^{2n}$ is an n-dim mfd by Thm 1.1.

Example

$F(x,y) = x^3 - y^3$, $F: \mathbb{R}^2 \to \mathbb{R}$
$dF_{(x,y)} = (3x^2 \quad -3y^2)$?
So 0 is not a regular value because $dF_{(0,0)} = 0$
However $F^{-1}(0) = \{(x,y) \in \mathbb{R}^2 \mid x^3 = y^3\} = \{(x,y) \in \mathbb{R}^2 \mid x = y\}$
is a 1-dim mfd.

Example

$F: M_n(\mathbb{R}) \to M_n(\mathbb{R})$ by $F(A) = A^T A - I$
Note that $(F(A))^T = F(A)$, so $F(A)$ maps into $Sym_n(\mathbb{R})$, the symmetric $(n \times n)$-matrices.
? Note that $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ and $Sym_n(\mathbb{R}) \cong \mathbb{R}^{\frac{1}{2}n(n-1)}$
Clearly F smooth (since polynomial).
We want to compute its derivative, which is the part of $F(A+B) - F(A)$ which is linear in B

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So we see

$$F(A+B) - F(A) = (A+B)^T(A+B) - A^TA$$
$$= \underbrace{B^TA + A^TB}_{\text{linear in } B} + \underbrace{B^TB}_{\text{quadratic in } B}$$

(all norms equiv.)

Hence $\dfrac{|F(A+B) - F(A) - (B^TA + A^TB)|}{|B|} = \dfrac{|B^TB|}{|B|} \to 0$

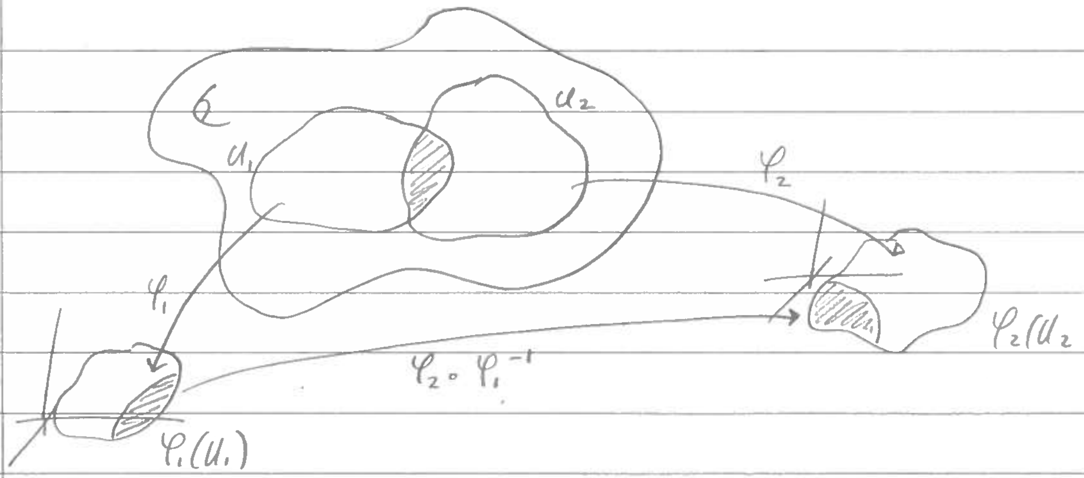
as $|B| \to 0$, so $dF_A(B) = B^TA + A^TB$.

If $C \in Sym_n(\mathbb{R})$ and $A \in F^{-1}(0)$ then
$dF_A(\frac{1}{2}AC) = C$.
So Thm 1.1 $\Rightarrow$ $O(n) = F^{-1}(0)$ is an $n^2 - \frac{1}{2}n(n+1) = \frac{1}{2}n(n-1)$ dimensional mfd.

§1.5 The formal definition

Key points
- "abstract objects" e.g. the torus in $\mathbb{R}^3$ we know from geometry is "the same" as $T^2 = S^1 \times S^1 \subseteq \mathbb{R}^4$
- smooth geometric objects: sphere OK ✓, cone not OK ✗
- objects on which we can measure how quantities change from point to point, i.e. we can differentiate.

<u>Def 1.2</u>
An n-dim mfd is a (seperable / & second countable) metric space M s.t. there exists a family
$\mathcal{A} = \{(U_i, \varphi_i) : i \in I\}$ where
- $U_i \subset M$ is open and $\bigcup_{i \in I} U_i = M$
- $\varphi_i : U_i \to \mathbb{R}^n$ are continuous bijections onto open sets $\varphi_i(U_i)$ with continuous inverse (i.e. homeomorphisms)
- Whenever $U_i \cap U_j \neq \emptyset$, the transition map $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ is a smooth (infinitely diff.) bijection with smooth inverse (i.e. a diffeomorphism).



The family A is called an atlas and the pairs (U_i, φ_i) are called coordinate charts.

Remark

- Seperable means there is a countable dense subset.
e.g. for $\mathbb{R}^n$ take $\mathbb{Q}^n$ [since $\overline{\mathbb{Q}^n} = \mathbb{R}^n$]
- Second countable means there is a collection of open sets which form a basis for all open sets.
e.g. for $\mathbb{R}^n$ take balls $B_r(x)$ with $x \in \mathbb{Q}^n$, $r \in \mathbb{Q}^+$.

Ex

$\mathbb{R}^n$ is a manifold.
Take $U_1 = \mathbb{R}^n$ and $\varphi_1 = \mathrm{Id}$.
Same works for any open $\Omega \subset \mathbb{R}^n$.

Ex

In fact, any open subset U of a manifold M is a manifold of the same dimension - take the atlas $\{(U_i \cap U, \varphi_i|_U) : i \in I\}$ if $\{(U_i, \varphi_i) : i \in I\}$ is an atlas for M.

$\Rightarrow GL_n(\mathbb{R})$, $M_n(\mathbb{R})$, $GL_n^+(\mathbb{R})$ are n^2-dim mfds.

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Ex

Consider S^n.

Let $N = (0, \dots, 0, 1)$ and $S = (0, \dots, 0, -1)$ be the "North" and "South" poles.

- Let $U_N = S^n \setminus \{N\}$, $U_S = S^n \setminus \{S\}$.
- Let $\varphi_N : U_N \to \mathbb{R}^n$, $\varphi_N(x) = \dfrac{(x_1, \dots, x_n)}{1 - x_{n+1}}$

and $\varphi_S : U_S \to \mathbb{R}^n$, $\varphi_S(x) = \dfrac{(x_1, \dots, x_n)}{1 + x_{n+1}}$

(These are stereographic projections)

We have explicit inverses

$$\varphi_N^{-1}(y) = \left(\frac{2y_1}{1+|y|^2}, \dots, \frac{2y_n}{1+|y|^2}, \frac{|y|^2 - 1}{1+|y|^2} \right)$$

$$\varphi_S^{-1}(y) = \left(\frac{2y_1}{1+|y|^2}, \dots, \frac{2y_n}{1+|y|^2}, \frac{1 - |y|^2}{1+|y|^2} \right)$$

So φ_N, φ_S are clearly homeomorphisms.

- $U_N \cap U_S = S^n \setminus \{N, S\}$, $\varphi_N(U_N \cap U_S) = \mathbb{R}^n \setminus \{0\}$

and $\varphi_S \circ \varphi_N^{-1} : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$

is $\varphi_S \circ \varphi_N^{-1}(y) = \dfrac{y}{|y|}$

which is a diffeo because it is smooth if $y \neq 0$ and it is its own inverse.

$\therefore$ by Def^n 1.2, S^n is an n-dim mfd.

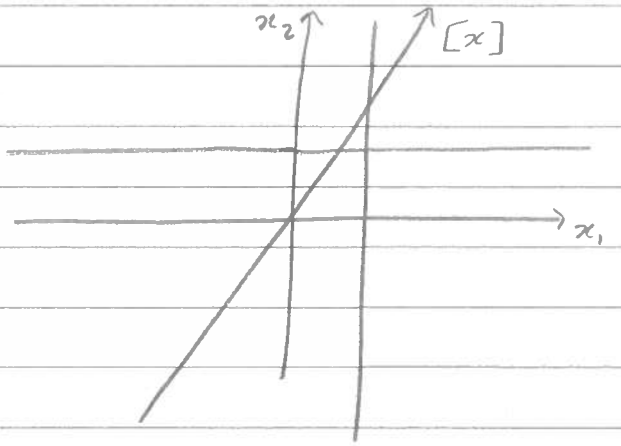
Ex

For $\mathbb{RP}^n$ we have the following atlas

- For $i = 1, \dots, n+1$ let $U_i = \{[x_1, \dots, x_{n+1}] \in \mathbb{RP}^n : x_i \neq 0\}$
- We define $\varphi_i : U_i \to \mathbb{R}^n$ by

$$\varphi_i([x]) = \left(\frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_{n+1}}{x_i} \right)$$

Then the conditions of Def^n 1.2 are satisfied for $\{(U_i, \varphi_i) : i = 1, \dots, n+1\}$ and $\mathbb{RP}^n$ is an n-dim mfd.



$\mathbb{RP}^2$

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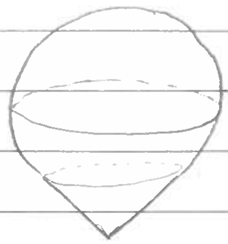
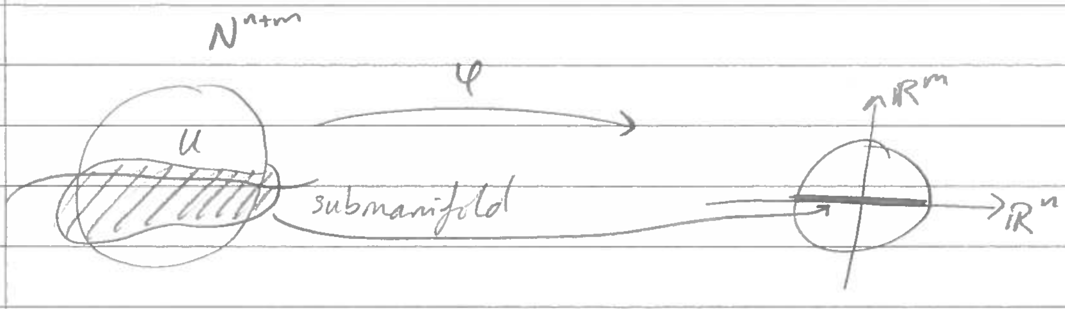
<u>Example</u>

Take $S^2 \subset \mathbb{R}^3$ and add a cone tip: (not a smooth submanifold)



This is a smooth manifold since we can take the transition functions to overlap away from the cone point.



Recap: Atlas $A = \{(U_i, \varphi_i),\ i \in I\}$ st. $\bigcup_{i \in I} U_i = M$ with smooth transition maps.

Two atlases are <u>equivalent</u> ($A_1 = \{(U_i, \varphi_i), i \in I\}$, $A_2 = \{(V_j, \psi_j), j \in J\}$) if all transition maps $\varphi_i \circ \psi_j^{-1}$ and $\psi_j \circ \varphi_i^{-1}$ are smooth.

Such an equivalence class is called a smooth structure.

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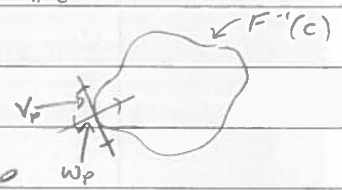
Thm 1.1 (Regular Value Thm)
Let $F : \mathbb{R}^{n+m} \to \mathbb{R}^m$ be smooth and suppose that $\forall p \in F^{-1}(c) \neq \emptyset$ the derivative $dF_p : \mathbb{R}^{n+m} \to \mathbb{R}^m$ is surjective. Then $F^{-1}(c)$ is a smooth n-dim. (sub)manifold of $\mathbb{R}^{n+m}$.

Proof

level set

- By the Implicit Function Thm, $\forall p \in F^{-1}(c)$ there exists a splitting $\mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m = \text{Ker}\, dF_p \times \mathbb{R}^m$ such that if $p = (a, b)$ wrt this splitting, then there exist open sets $a \in V_p \subseteq \mathbb{R}^n$, $b \in W_p \subseteq \mathbb{R}^m$ and a smooth map $G_p : V_p \to W_p$ with $G_p(a) = b$ s.t.



$F^{-1}(c) \cap (V_p \times W_p) = \{(a, G_p(q)) : q \in V_p\}$
[writing as graph over tangent space].

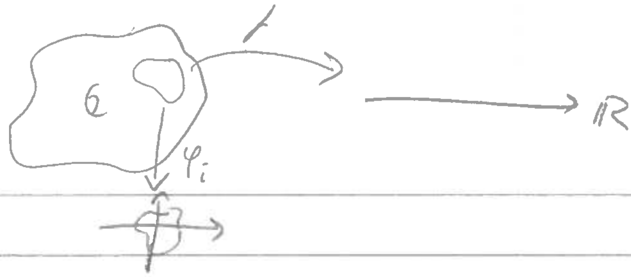
Let $U_p = F^{-1}(c) \cap (V_p \times W_p)$ which is (relatively) open and $\bigcup_{p \in F^{-1}(c)} U_p = F^{-1}(c)$ since $p \in U_p$.

- $\forall p \in F^{-1}(c)$ let $\varphi_p : U_p \to V_p \subseteq \mathbb{R}^n$ be given by $\varphi_p((q, G_p(q)) = q$.
Then $\varphi_p^{-1}(q) = (q, G_p(q))$ is it is a homeomorphism.
- Claim: the transition maps $\varphi_p \cdot \varphi_{p'}^{-1}$ are smooth. [Exercise]

□

Prop^n 1.3 (proof in notes)
A surface in $\mathbb{R}^3$ is a 2-dim manifold.

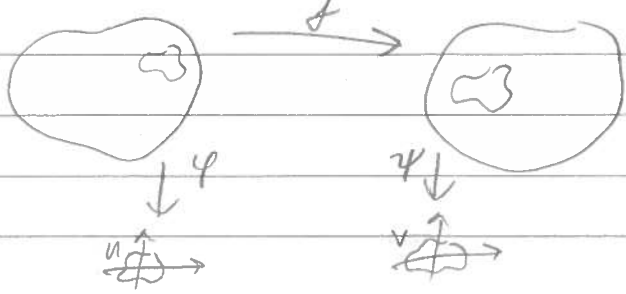
Prop^n 1.4 (proof in notes)
Let M be an n-dim (sub)manifold of $\mathbb{R}^{n+m}$ as defined in Multivariable Analysis. Then M is a smooth n-dim manifold.



§1.6 Smooth maps

Def^n

Let M, N be mfds of dim n, m resp. A map $f: M \to N$ is smooth at $p \in M$ if for some coordinate charts (U, φ) at p and (V, ψ) at $f(p)$ with $f(U) \subseteq V$, the maps
$\psi \circ f \circ \varphi^{-1} : \varphi(U) \subseteq \mathbb{R}^m \longrightarrow \psi(V) \subseteq \mathbb{R}^n$ is smooth at $\varphi(p)$.



Remark:
This makes sense because of def^n 1.2.
If we take $(U, \varphi), (U', \varphi')$ around p and $(V, \psi), (V', \psi')$ around $f(p)$ with $f(U) \subseteq V$, $f(U') \subseteq V'$ then $\psi' \circ f \circ (\varphi')^{-1} = (\psi' \circ \psi^{-1}) \circ (\psi \circ f \circ \varphi^{-1}) \circ (\varphi \circ (\varphi')^{-1})$.
The LHS is smooth iff $\psi \circ f \circ \varphi^{-1}$ is smooth since the transition maps are smooth.

Examples

1). The maps φ in the atlas of M are smooth from M to $\mathbb{R}^n$, since taking (U, φ) as a coordinate we get $\varphi \circ \varphi^{-1} = id_{\mathbb{R}^n} : \varphi(U) \to \varphi(U)$ which is smooth.
Similarly the inverses $\varphi^{-1} : \varphi(U) \to U$ are smooth.

2). The identity map $id: M \to M$ is smooth, take coordinates (U, φ) on both sides, then

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$\varphi \circ id_M \circ \varphi^{-1} = id_{\varphi(U)}$ is smooth.

necessary since smoothness on M and in $\mathbb{R}^n$ are not necessarily equiv.

3). If $M \subset \mathbb{R}^n$ is a submanifold, then the restriction of any smooth map on $\mathbb{R}^n$ to M is again a smooth map.

If $N \subseteq \mathbb{R}^m$ is a smooth submanifold and the map $f: \mathbb{R}^n \to \mathbb{R}^m$ is smooth s.t. $f(M) \subseteq N$, then the restriction $f: M \to N$ is smooth.

This is very helpful:
i.e. take $f: \mathbb{R}^4 \to \mathbb{R}^3$,
$f(x_0, x_1, x_2, x_3) = (x_0^2 + x_1^2 - x_2^2 - x_3^2,\ 2x_0x_3 - 2x_1x_2,\ 2x_1x_3 - 2x_0x_2)$
which is smooth and $f(S^3) \subset S^2$.
The restriction $f: S^3 \to S^2$ is smooth.

4). Take G to be any of the groups of matrices we discussed, the multiplication map $m: G \times G \to G$ given by $m(A,B) = AB$ and the inversion map $i: G \to G$ given by $i(A) = A^{-1}$ are smooth.
$\left[A^{-1} = \frac{1}{\det A} A^{\#} \right]$

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Recall: $f: M \to N$ smooth $\Leftrightarrow$ $p \in V_p \subset M$, $f(p) \in U_{f(p)} \subset N$, (V_p, φ), $(U_{f(p)}, \psi)$ the map $\psi \circ f \circ \varphi^{-1}$ is smooth.

<u>Def^n 1.6</u>
A map $f: M \to N$ is a <u>diffeomorphism</u> if it is a smooth bijection with smooth inverse.
Then the manifolds are called diffeomorphic.

A map $f: M \to N$ is a <u>local diffeo at p</u> if $\exists$ open $U \ni p$, open $V \ni f(p)$ s.t. $f: U \to V$ is a diffeo.
We say f is a local diffeo if it is a local diffeo $\forall p \in M$.

e.g. $\bigcirc \longrightarrow \circledcirc$ (wrapping circle twice around itself is local but not global diffeo).

<u>Examples</u>

1). $id: M \to M$ a diffeo
f, g diffeo $\Rightarrow f \circ g$ and f^{-1} is diffeo
Hence the diffeomorphisms form a group $Diff(M)$.

2). the maps $\varphi: U \to \mathbb{R}^n$ in the atlas of M are local diffeos since the maps $\varphi: U \to \varphi(U)$ are diffeo by def 1.2

3). Any matrix $A \in M_n(\mathbb{R})$ defines a linear map on $\mathbb{R}^n$ given by $x \mapsto Ax$, since it is linear it is smooth. Moreover this map is invertible iff $\det A \neq 0$, with inverse $x \mapsto A^{-1}x \Leftrightarrow A \in GL_n(\mathbb{R})$.
So the group of linear diffeos of $\mathbb{R}^n$ is $GL_n(\mathbb{R})$.

4). The map $f: (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ given by $f(x) = \tan x$ is smooth with inverse $f^{-1} = \tan^{-1}: \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$, so f is a diffeo.

5). The left and right multiplication maps L_A, R_A are diffeos on the group G.

6). The map $f: \mathbb{R} \setminus \{0\} \to \mathbb{R}_{>0}$ given by $f(x) = x^2$ is a local diffeo. It is smooth, surjective, but not injective since $f(-x) = f(x)$.

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§1.7 Quotient constructions

Def^n 1.7

A group G acts on a manifold M by diffeos if there is a homeomorphism $G \to \mathrm{Diff}(M)$,
i.e. $\forall g \in G$ there exists f_g diffeo on M s.t.
- $f_e = id_M$
- $f_g \circ f_h = f_{gh} \quad \forall g, h \in G.$

Let G be a discrete group (i.e. a finite group or $\mathbb{Z}^n$ or some other countable group) acting by diffeos on M. We say that G acts freely and properly discontinuously if
- $\forall p \in M \ \exists$ open $V \ni p$ with $V \cap f_g(V) = \emptyset \quad \forall g \neq e$.
- $\forall p, q \in M$ with $p \neq f_g(q) \ \forall g \in G$, $\exists$ open $V \ni p$ and open $W \ni q$ with $V \cap f_g(W) = \emptyset \quad \forall g \in G$

Thm 1.8

Let M be an n-dim manifold and G a discrete group acting freely and properly discontinuously on M by diffeos. Define an equivalence relation $\sim$ on M by $p \sim q \Leftrightarrow q = f_g(p)$ for some $g \in G$. Then the quotient space $M/_\sim = M/_G$ is an n-dim manifold.

Proof
- Let $\{(V_i, \psi_i) : i \in I\}$ be an atlas of M s.t. $V_i \cap f_g(V_i) = \emptyset \ \forall g \in G$.

  Let $\pi : M \to M/_G$ be the projection map, which is an open map.

  Then $U_i = \pi(V_i)$ is open, $\bigcup_{i \in I} U_i = M/_G$
- Since $\pi_i := \pi|_{V_i} : V_i \to U_i$ is a homeomorphism, so

  [open & injective $\Rightarrow$ inverse is continuous]

we can define $\varphi_i = \psi_i \circ \pi_i^{-1} : U_i \to \psi_i(V_i) \subseteq \mathbb{R}^n$
which is a homeo.

- If $U_i \cap U_j \neq \emptyset$, then

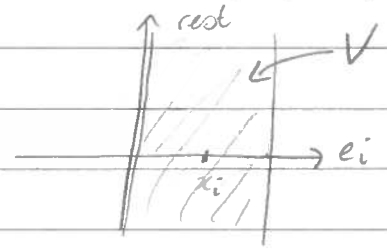
$$\begin{aligned}\varphi_i(U_i \cap U_j) &= \psi_i \circ \pi^{-1}(U_i \cap U_j)\\ &= \psi_i(V_i \cap \pi^{-1}(U_j))\\ &= \psi_i(V_i \cap \bigcup_{g\in G} f_g(V_j))\end{aligned}$$

which is a disjoint union of open sets and clearly $\varphi_j \circ \varphi_i^{-1}$ is a homeo, so it suffices to show that $\varphi_j \circ \varphi_i^{-1}$ is smooth.

Let $p \in \varphi_i(U_i \cap U_j)$. Then $\exists!\, g \in G$ s.t.
$p \in W = \psi(V_i \cap f_g(V_j))$.
Then $\psi_i^{-1}(W) = V_i \cap f_g(V_j)$ and $\varphi_j \circ \varphi_i^{-1}|_W = \psi_j \circ \pi_j^{-1} \circ \pi_i \circ \psi_i^{-1}|_W$.
It is enough to show that $\pi_j^{-1} \circ \pi_i$ is smooth on $V = V_i \cap f_g(V_j)$.
If $q \in V$ and $q' = \pi_j^{-1} \circ \pi_i(q) \in V_j$
$\Rightarrow \pi_j(q') = \pi_i(q) \Rightarrow \exists\, g_q \in G$ s.t. $f_{g_q}(q') = q$
$\Rightarrow q \in f_{g_q}(V_j) \cap f_g(V_j) \Rightarrow g_q = g$ and hence
$\pi_j^{-1} \circ \pi_i|_V = f_{g^{-1}}|_V$ which is smooth. $\square$

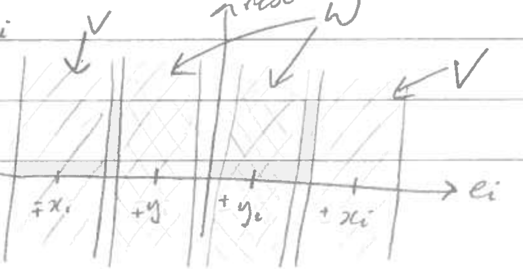
Examples

1). Let $\mathbb{Z}_2 = \{1, -1\}$ act on $\mathbb{R}^n$ via $f_1 = id$, $f_{-1} = -id$.
Clearly $-id$ is a diffeo, but the action is not free since 0 is fixed.
Claim: $\mathbb{Z}_2$ acts freely and prop. discont. on $\mathbb{R}^n \setminus \{0\}$.
Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n \setminus \{0\}$ and $x_i \neq i$



We have $V \cap -V = \emptyset$

If $x, y \in \mathbb{R}^n \setminus \{0\}$ with $y \neq x$, $y \neq -x$, $y_i \neq x_i$, $y_i \neq -x_i$



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V, W open, $V \cap W = \emptyset$, $W \cap -V = \emptyset$
$\Rightarrow \mathbb{Z}_2$ acts freely and prop. discont. on $\mathbb{R}^n \setminus \{0\}$.

Hence it acts freely and prop discont. on any submanifold $M \subset \mathbb{R}^n \setminus \{0\}$ s.t. $-M = M$.

Examples

(a) $0 \notin S^n$ and $-S^n = S^n$ so $S^n / \mathbb{Z}_2 = \mathbb{RP}^n$

(b) 0 is not in the cylinder $C = \{(x,y,z) : x^2+y^2=1, -1 < z < 1\}$ and $-C = C$, hence $C/\mathbb{Z}_2$ is a 2-dim mfd called the Möbius band.

? poss also Klein?

(c) Similarly $\mathbb{Z}_2$ acts freely and prop. discont. on $T^2 \subset \mathbb{R}^3$ and hence $T^2/\mathbb{Z}_2$ is a 2-dim manifold called the Klein bottle.

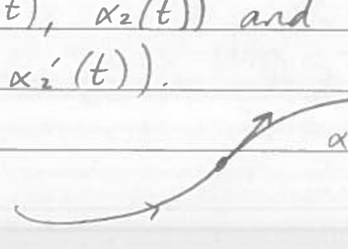
Example

Let $a = (a_1, \dots, a_n) \in \mathbb{Z}^n$, define $f_a : \mathbb{R}^n \to \mathbb{R}$ by $f_a(x_1, \dots, x_n) = (x_1 + a_1, \dots, x_n + a_n)$.
This gives a homeo from $\mathbb{Z}^n \to \mathrm{Diff}(\mathbb{R}^n)$ by $a \mapsto f_a$ which is free and prop. discont.
Then $\mathbb{R}^n / \mathbb{Z}^n$ is an n-dim mfd, called the n-dim torus T^n.

§2 Tangent vectors

§2.1 Tangent vectors and regular values

Let $\alpha : \mathbb{R} \to \mathbb{R}^2$ (or $\mathbb{R}^n$) be a curve.
The tangent vector to α can be calculated via $\alpha(t) = (\alpha_1(t), \alpha_2(t))$ and then taking derivative $\alpha'(t) = (\alpha_1'(t), \alpha_2'(t))$.



$\alpha(t) = \alpha(t_0) + \alpha'(t_0)(t - t_0) + o((t-t_0)^2)$

Assume $M^2 \subset \mathbb{R}^3$ is a 2-dim surface, $p \in M$. Then a tangent vector at p is given by a curve $\alpha : (-\varepsilon, \varepsilon) \to M$, $\alpha(0) = p$ and we take the tangent vector as $\alpha'(0)$.

Let $M^n \subset \mathbb{R}^{n+m}$ be an n-dim (sub)manifold.

<u>Prop^n</u>

Let $F : \mathbb{R}^{n+m} \to \mathbb{R}^m$ be smooth, c a regular value of F st. $M = F^{-1}(c) \neq \emptyset$.
Then $\forall\, p \in M$, $T_pM \cong \operatorname{Ker} dF_p$.

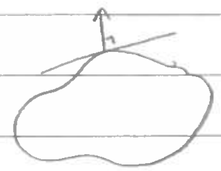
<u>Proof</u>

Let $p \in M = F^{-1}(c)$ and α a curve through p.
Then $F(\alpha(t)) = c$ since $\alpha(t) \in M = F^{-1}(c)$.
Differentiating gives $\frac{d}{dt} F(\alpha(t)) = 0$.

Chain rule at $t = 0$:
$dF_{\alpha(0)}(\alpha'(0)) = dF_p(\alpha'(0)) = 0$
$\Rightarrow T_pM \subset \operatorname{Ker} dF_p$.
Note $\dim T_pM = n$ and $\dim \operatorname{Ker} dF_p = n+m-m = n$
$\Rightarrow T_pM = \operatorname{Ker} dF_p$.

$\square$

$f : \mathbb{R}^n \to \mathbb{R}$, $df = (\partial_1 f, \ldots, \partial_n f)$



<u>Examples</u>

(i) $S^n = F^{-1}(0)$, $F(x) = \sum_{i=1}^{n+1} x_i^2 - 1$
$dF_x = (2x_1, \ldots, 2x_{n+1})$
So $\operatorname{Ker} dF_x = \{ y \in \mathbb{R}^{n+1} : \langle y, x \rangle = 0 \} = \langle x \rangle^\perp$

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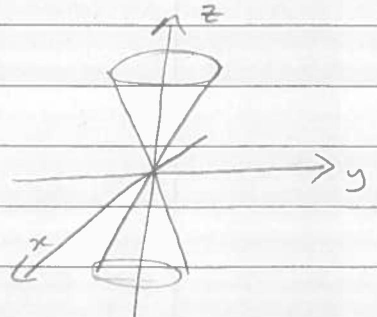
2). Let $f: \mathbb{R}^n \to \mathbb{R}^m$ be smooth and let
$F(x,y) = f(x) - y$. (Level set at 0 is graph of f)
$dF_{(x,y)} = (df_x \;\; -I)$
$\Rightarrow F^{-1}(0) = \text{graph}(f) = \{(x, f(x)) : x \in \mathbb{R}^n\}$
is an n-dim manifold.
We also get $\operatorname{Ker} dF_{(x,y)} = \{(u,v) \in \mathbb{R}^{n+m} : df_x(u) = v\}$
$= \text{graph}(df_x)$.
$\Rightarrow T_{(x,f(x))}\text{Graph}(f) \cong \text{Graph}(df_x) \subseteq \mathbb{R}^{n+m}$.

3). Consider $SL(n,\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det A = 1\}$
is $F^{-1}(0)$ where $F: M_n(\mathbb{R}) \to \mathbb{R}$ is given by
$F(A) = \det A - 1$.
Assuming $\det A \neq 0$, we can write

$$F(A+B) - F(A) = \det(A+B) - \det(A)$$
$$= \det(A(I + A^{-1}B)) - \det(A)$$
$$= \det(A)(\det(I + A^{-1}B)) - \det(A)$$

By expanding:
$$\det(I + A^{-1}B) = 1 + tr(A^{-1}B) + o(|B|^2) \qquad [A \in SL_n(\mathbb{R})]$$
$\Rightarrow dF_A(B) = tr(A^{-1}B)$
For $A \in SL_n(\mathbb{R})$ this is non zero since
$dF_A(A) = tr(I) = n$
Thus by Thm 1.1 we have that $SL_n(\mathbb{R})$ is an n^2-1 dim mfd.
Moreover $T_A SL_n(\mathbb{R}) = \{B \in M_n(\mathbb{R}) \mid tr(A^{-1}B) = 0\}$
$\Rightarrow T_I SL_n(\mathbb{R}) = \{B \in M_n(\mathbb{R}) \mid tr(B) = 0\} = sl_n(\mathbb{R})$
(Lie algebra is tangent space)

Example
Let $C = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2\}$
then the curves $(t,0,t)$, $(0,t,t)$ are in C and have tangent vectors $(1,0,1)$ and $(0,1,1)$ at zero.



However $(1,-1,0) = (1,0,1) - (0,1,1) \in \mathrm{Span}\{(1,0,1),(0,1,1)\}$ is not in the tangent space to C at zero.
So C can't be a 2-dim submanifold of $\mathbb{R}^3$.

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§2.2 Tangent vectors as differential operators

Assume $\alpha: \mathbb{R} \to \mathbb{R}^2$ through $p \in \mathbb{R}^2$, and $f: \mathbb{R}^2 \to \mathbb{R}$ smooth. $\Rightarrow f \circ \alpha : \mathbb{R} \to \mathbb{R}$ smooth.
Differentiating at 0, $(\alpha(0) = p)$, we get
$$(f \circ \alpha)'(0) = \frac{\partial f}{\partial x_1}(p)\, \alpha_1'(0) + \frac{\partial f}{\partial x_2}(p)\, \alpha_2'(0)$$

We get a map $f \mapsto (f \circ \alpha)'(0)$ from $C^\infty(\mathbb{R}^2) \to \mathbb{R}$ given by $f \longmapsto \left(\alpha_1'(0) \frac{\partial}{\partial x_1}\Big|_p + \alpha_2'(0) \frac{\partial}{\partial x_2}\Big|_p\right) f$

which is a differential operator acting on functions.

Can think of $\left\{ \frac{\partial}{\partial x_1}\Big|_p, \frac{\partial}{\partial x_2}\Big|_p \right\}$ as a basis of a 2-dim vector space and then identify with $\alpha'(0) = (\alpha_1'(0), \alpha_2'(0))$.

This works also on a mfd M:
Let $\alpha: \mathbb{R} \to M$ a curve st. $\alpha(0) = p$,
$f: M \to \mathbb{R}$ a smooth function.
Let (U, φ) be coordinates around p, and
$\varphi \circ \alpha(t) = (a_1(t), \dots, a_n(t)) \in \varphi(U) \subset \mathbb{R}^n$.
$$(f \circ \alpha)'(0) = \frac{d}{dt}(f \circ \alpha)(t)\Big|_{t=0} = \frac{d}{dt}\left(f \circ \varphi^{-1} \circ \varphi \circ \alpha\right)(t)\Big|_{t=0}$$
$$= \frac{d}{dt}\Big[(f \circ \varphi^{-1})(a_1(t), \dots, a_n(t))\Big]\Big|_{t=0}$$
$$= \sum_{j=1}^n a_j'(0) \frac{\partial (f \circ \varphi^{-1})}{\partial x_j}\Big|_{\varphi(p)} = \left(\sum_{j=1}^n a_j'(0) \frac{\partial}{\partial x_j}\Big|_{\varphi(p)}\right)(f \circ \varphi^{-1})$$

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Using $\left\{ \frac{\partial}{\partial x_j}\Big|_{\varphi(p)} \right\}$ as a basis we can identify the tangent vector to $\varphi \circ \alpha$ in $\mathbb{R}^n$ at $\varphi(p)$ with the differential operator

$$\sum_{j=1}^{n} a_j'(0) \frac{\partial}{\partial x_j}\Big|_{\varphi(p)} \quad \text{acting on functions } f \circ \varphi^{-1}$$

Note: $\frac{\partial}{\partial x_j}\Big|_{\varphi(p)}$ is the tangent vector to the curve

$t \longmapsto \varphi^{-1}((0, \ldots, t, \ldots 0) + \varphi(p))$

Def^n

Let $\alpha : (-\varepsilon, \varepsilon) \to M$, $\alpha(0) = p$, be a smooth curve. Let $U \ni p \in M$ be open and $f : U \subseteq M \to \mathbb{R}$ be smooth. Then $f \circ \alpha : (-\varepsilon, \varepsilon) \to \mathbb{R}$ is smooth and we call $\alpha'(0) : f \longmapsto (f \circ \alpha)'(0) \in \mathbb{R}$ the tangent vector to α at 0.

We say that X is a tangent vector to M at p if there exists a curve $\alpha : (-\varepsilon, \varepsilon) \to M$, smooth, $\alpha(0) = p$, s.t. $\alpha'(0) = X$.

More precisely, we identify X with the equivalence class of curves $[\alpha]$, where $\alpha \sim \beta$ if $\alpha, \beta : (-\varepsilon, \varepsilon) \to M$ both smooth, $\alpha(0) = p = \beta(0)$, and $(f \circ \alpha)'(0) = (f \circ \beta)'(0)$ for all $f : U \ni p \to \mathbb{R}$, f smooth, U open.

Def^n

Let T_pM denote the set of tangent vectors to M at p. Clearly this is a real vector space. We call this the tangent space of M at p.

Prop^n

$\dim(T_pM) = n$

Proof

Follows from the observation that given p and a chart (U, φ) around p, we can identify T_pM with linear combinations of $\left\{ \frac{\partial}{\partial x_i}\Big|_{\varphi(p)} \right\}$ □

2.3 Differential

Def^n

Let $f: M \to N$ be smooth. Let $X = \alpha'(0) \in T_pM$.
? Then $f \circ \alpha$ is a smooth curve in through $f(p)$.
We define the differential of f at p, which is a linear map $df_p : T_pM \to T_{f(p)}N$ by
$df_p(X) = (f \circ \alpha)'(0)$.

Assume $X = \alpha'(0) = \beta'(0)$. Then we need to show
$(f \circ \alpha)'(0) = (f \circ \beta)'(0)$

$\Leftrightarrow$ let h be any smooth function around $f(p) \in N$, $[h : V \subseteq N \to \mathbb{R}]$
so $(h \circ f \circ \alpha)'(0) = (h \circ f \circ \beta)'(0)$. $(*)$
But $h \circ f$ is a smooth function around $p \in M$, so since $\alpha'(0) = \beta'(0)$ we have that $(*)$ holds.

Take α a curve through $p \in M$ and a chart (U, φ) around p, then we have a curve $a = \varphi \circ \alpha$ in Euclidean space.
The curve $f \circ \alpha$ defines $b = \psi \circ f \circ \alpha$ in Euclidean space where (V, ψ) is a chart around $f(p) \in N$.
Then the relationship between the tangent vectors of the curves a and b at 0 is

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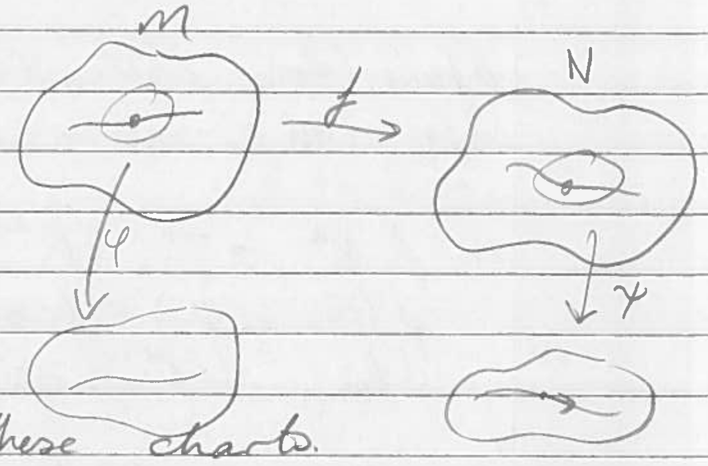
$$b'(0) = (\psi \circ f \circ \alpha)'(0)$$
$$= (\psi \circ f \circ \varphi^{-1} \circ a)'(0)$$
$$= d(\psi \circ f \circ \varphi^{-1})_{\varphi(p)}(a'(0))$$



Hence the differential df_p may be viewed as $d(\psi \circ f \circ \varphi^{-1})_{\varphi(p)}$ given these charts.

Remark: In particular, if $M \subseteq \mathbb{R}^m$ and $N \subseteq \mathbb{R}^n$ are submanifolds and $f: \mathbb{R}^m \to \mathbb{R}^n$ is smooth and $f(M) \to N$, then $df_p\ T_pM \to T_{f(p)}N$ is just the restriction of the linear map $df_p: \mathbb{R}^m \to \mathbb{R}^n$.

Example

Let $f: \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^2 \setminus \{0\}$ given by

$$f(r, \theta) = (r\cos\theta, r\sin\theta)$$

$$df_{(r,\theta)} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

If we let $\partial_r, \partial_\theta$ denote diff. w.r.t. r, θ and ∂_1, ∂_2 diff. w.r.t. x_1, x_2 on $\mathbb{R}^2$, then

$$df_{(r,\theta)}(\partial_r) = \cos\theta\, \partial_1 + \sin\theta\, \partial_2$$
$$df_{(r,\theta)}(\partial_\theta) = -r\sin\theta\, \partial_1 + r\cos\theta\, \partial_2$$

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$$f: M \to N, \quad \underset{\in T_pM}{X = \alpha'(0)}, \quad \alpha: (-\varepsilon, \varepsilon) \to M, \quad \alpha(0) = p$$

$$df_p(X) = (f \circ \alpha)'(0) \in T_{f(p)}N$$

Examples

1) $f: \mathbb{R}^2 \to S^2 \subset \mathbb{R}^3, \quad f(\theta, \varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$

$$df_{(\theta,\varphi)} = \begin{pmatrix} \cos\theta\cos\varphi & -\sin\theta\sin\varphi \\ \cos\theta\sin\varphi & \sin\theta\cos\varphi \\ -\sin\theta & 0 \end{pmatrix}$$

$df_{(\theta,\varphi)}(\partial_\theta) = \cos\theta\cos\varphi\,\partial_1 + \cos\theta\sin\varphi\,\partial_2 - \sin\theta\,\partial_3$

$df_{(\ \ )}(\partial_\varphi) = -\sin\theta\sin\varphi\,\partial_1 + \sin\theta\cos\varphi\,\partial_2$

? θ,φ or $0,1$?

2). $f: \mathbb{R}^n \to T^n \subset \mathbb{R}^{2n}$

$f(\theta_1, \dots, \theta_n) = (\cos\theta_1, \sin\theta_1, \dots, \cos\theta_n, \sin\theta_n)$

$df_{(\theta_1,\dots,\theta_n)}(\partial\theta_j) = -\sin\theta_j\,\partial_{2j-1} + \cos\theta_j\,\partial_{2j}$

3). $f: S^2 \to \mathbb{RP}^2$ given by $f(x) = [x]$ at $(0,0,1) \in \mathcal{U}_3$

Let $X \in T_{(0,0,1)}S^2$, $f(0,0,1) = [(0,0,1)] \in \mathcal{U}_3$

where $\mathcal{U}_3 = \{[y_1, y_2, y_3] \in \mathbb{RP}^2 : y_3 \neq 0\}$.

Calculate $df_{(0,0,1)}(X)$.

Recall $\varphi_S(0,0,1) = (0,0,0)$

and for $(x_1, x_2) \in \mathbb{R}^2$ s.t. $|x|^2 < 1$

$$\varphi_3 \circ f \circ \varphi_S^{-1}(x_1, x_2) = \varphi_3\left[\left(\frac{2x_1}{1+|x|^2}, \frac{2x_2}{1+|x|^2}, \frac{1-|x|^2}{1+|x|^2}\right)\right]$$

$$= \left(\frac{2x_1}{1-|x|^2}, \frac{2x_2}{1-|x|^2}\right)$$

$$\Rightarrow d(\varphi_3 \circ f \circ \varphi_S^{-1})\Big|_{(0,0)} = \frac{2}{1-|x|^2}\begin{pmatrix} 1+x_1^2-x_2^2 & 2x_1x_2 \\ 2x_1x_2 & 1-x_1^2+x_2^2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

<u>Prop^n</u>

(a) The identity map $id: M \to M$ satisfies

$did_p = id: T_pM \to T_pM$.

(b) If $f: P \to N$ and $g: M \to P$ are smooth maps then $f \circ g: M \to N$ satisfies the chain rule

$d(f\circ g)_p = df_{g(p)} \circ dg_p$

<u>Proof</u>

Exercise. [For b, use $(f\circ g)(\alpha(t)) = f(g(\alpha(t)))$]

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Example

$f: M \to N$ diffeo.

$\Rightarrow f^{-1} \circ f = id_M$, $f \circ f^{-1} = id_N$

So by the chain rule,

$d(f^{-1})_{f(p)} \circ df_p = id$, $df_p \circ d(f^{-1})_{f(p)} = id$ $\quad \forall p \in M$.

So $df_p : T_pM \to T_{f(p)}N$ is invertible with inverse $(df_p)^{-1} = d(f^{-1})_{f(p)}$.

§2.4 Local diffeo

Prop^n 2.7

A smooth map $f: M \to N$ is a local diffeo at p iff $df_p : T_pM \to T_{f(p)}N$ is an isomorphism.

Proof

[$\Rightarrow$] Assume f is a local diffeo. at p

$\Rightarrow \exists\, U \ni p$ open in M and $V \ni f(p)$ open in N

s.t. $f: U \to V$ is a diffeo.

By the previous example, df_p is an isomorphism.

[$\Leftarrow$] Assume df_p is an isomorphism.

Let (U, φ), (V, ψ) be charts around p and $f(p)$.

Then $d\varphi^{-1}_{\varphi(p)} : \mathbb{R}^n \to T_pM$, $d\psi_{f(p)} : T_{f(p)}N \to \mathbb{R}^n$

are isomorphisms since φ^{-1} and ψ are local diffeos.

$\Rightarrow d(\psi \circ f \circ \varphi^{-1})_{\varphi(p)} : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism

by the chain rule.

By the inverse function thm $\psi \circ f \circ \varphi^{-1}$ is a local diffeo. $\square$

Example

The map $f: \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^2 \setminus \{0\}$ satisfied

$df_{(r,\theta)} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$ which has full rank (two non vanishing vectors which are $\perp$)

$\Rightarrow f$ is a local diffeo.

Example

$f : \mathbb{R}^2 \to S^2$ as before

$$df_{(\theta,\varphi)} = \begin{pmatrix} \cos\theta\cos\varphi & -\sin\theta\sin\varphi \\ \cos\theta\sin\varphi & \sin\theta\cos\varphi \\ -\sin\theta & 0 \end{pmatrix}$$

has full rank if $\sin\theta \neq 0$.

$\Rightarrow f$ is a local diffeo but not global.

Example

$f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(\theta,\varphi) = ((2+\cos\theta)\cos\varphi, (2+\cos\theta)\sin\varphi, \sin\theta)$

has $$df_{(\theta,\varphi)} = \begin{pmatrix} -\sin\theta\cos\varphi & -(2+\cos\theta)\sin\varphi \\ -\sin\theta\sin\varphi & (2+\cos\theta)\cos\varphi \\ \cos\theta & 0 \end{pmatrix}$$

which always has full rank, 2

$\Rightarrow f$ is a local diffeo.

§2.5 Regular values

Thm 2.8 (Regular value Thm)

Let M be a manifold of dim $m+n$ and N an manifold of dim m. Suppose $f : M \to N$ is smooth and let $c \in N$ s.t. $F^{-1}(c) \neq \emptyset$ and $df_p : T_pM \to T_{f(p)}N$ is surjective $\forall p \in F^{-1}(c)$. Then $F^{-1}(c)$ is an n-dim submanifold of M and $T_p(F^{-1}(c)) = \operatorname{Ker} dF_p$ $\forall p \in F^{-1}(c)$.

Proof

Go to charts. $\square$

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Examples

1). $F: S^n \to \mathbb{R}$, $F(x_1, \dots, x_{n+1}) = x_{n+1}$.
Then as a map from $\mathbb{R}^{n+1} \to \mathbb{R}$ we have
$dF_x = (0, \dots, 0, 1)$ which is non-zero on
$T_x S^n$ except at points where $x = (0, \dots, 0, \pm 1)$ i.e.
when $F(x) = \pm 1$
$\Rightarrow F^{-1}(c)$, $|c| < 1$ is an $(n-1)$-dim submanifold of S^n.

§ Immersions, embeddings, submersions

Def^n
A smooth map $f: M \to N$ is an immersion if
$df_p : T_p M \to T_{f(p)} N$ is injective $\forall p \in M$
(i.e. we need $\dim M \leq \dim N$).
An injective immersion is called an embedding.
If $f: M \to N$ is an embedding, then $f(M)$ is a manifold and $f: M \to f(M)$ is a diffeomorphism.

A smooth map $f: M \to N$ is called a submersion
if $df_p : T_p M \to T_{f(p)} N$ is surjective $\forall p \in M$
and $\dim N \leq \dim M$.

Example
$f: \mathbb{R} \to \mathbb{R}^2$, $f(\theta) = (\cos\theta, \sin\theta)$
$\Rightarrow df_\theta(\partial_\theta) = -\sin\theta\, \partial_1 + \cos\theta\, \partial_2$
which is non-zero $\forall \theta \in \mathbb{R}$, so df_θ is injective
$\forall \theta \in \mathbb{R} \Rightarrow f$ is an immersion.
No embedding since $f(\theta + 2\pi) = f(\theta)$.
Define a free and properly discont. action of $\mathbb{Z}$ on
$\mathbb{R}$ by $f_n(\theta) = \theta + 2\pi n$ $\forall n \in \mathbb{Z}$ $\forall \theta \in \mathbb{R}$
Then the map $F: \mathbb{R}/\mathbb{Z} \to \mathbb{R}^2$ given by
$F([\theta]) = f(\theta)$ is well-defined and is injective.

Let $\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ be the projection map.
Then $f = F \circ \pi \Rightarrow df = dF \circ d\pi$
Since $d\pi$ is injective and df is injective we get dF is injective so F is an immersion
Thus F is an embedding and thus a diffeo,
ie. $\mathbb{R}/\mathbb{Z} \cong S^1$.

Example
- Let $C = \{(\cos\theta, \sin\theta, t) \in \mathbb{R}^3 \mid \theta, t \in \mathbb{R}^2\}$
Let $f : S^1 \to C$ be given by
$f(\cos\theta, \sin\theta) = (\cos\theta, \sin\theta, 0)$
f is an immersion. Since f is injective it is an embedding.
- Now let $g : C \to S^1$ be given by
$g(\cos\theta, \sin\theta, t) = (\cos\theta, \sin\theta)$.
Then $T_{(\cos\theta, \sin\theta, t)} C = \text{span}\{-\sin\theta\,\partial_1 + \cos\partial_2, \partial_3\}$
and $dg_{(\cos\theta, \sin\theta, t)}(-\sin\theta\,\partial_1 + \cos\theta\,\partial_2) = -\sin\theta\,\partial_1 + \cos\theta\,\partial_2$
$dg_{(\cos\theta, \sin\theta, t)}(\partial_3) = 0. \Rightarrow g$ is a submersion.

§3 Vector field

Fake definition
A vector field is a choice of tangent vector that varies smoothly.

Recall: on $\mathbb{R}^n$ a vector field has the form
$X = \sum_{i=1}^n a_i(x) \frac{\partial}{\partial x_i} = \sum_{i=1}^n a_i \partial_i$ where $a_i : \mathbb{R}^n \to \mathbb{R}$ are smooth

§3.1 Tangent bundle

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Def^n

The tangent bundle TM of M is given by
$TM = \bigcup_{p \in M} T_pM$.

Thm

The tangent bundle TM is a $2n$-dim manifold s.t.
- there exists a smooth surjective map $\pi : TM \to M$ s.t.
- $\pi^{-1}(p) = T_pM$ which is a vector space $\forall p \in M$.
- $\forall p \in M$, $\exists$ open set $U \ni p$ and a diff $\Psi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ s.t. $\Psi : \pi^{-1}(q) \to \{q\} \times \mathbb{R}^n$ is an isomorphism $\forall q \in U$.

Proof

Let $\{(U_i, \varphi_i) \mid i \in I\}$ be an atlas of M and let $\pi : TM \to M$ be the natural projection, $\pi(X) = p \ \forall X \in T_pM$.
- Let $V_i = \pi^{-1}(U_i)$ which we define to be open and clearly $\bigcup_{i \in I} V_i = TM$.
- Let $\Psi_i : V_i \longrightarrow \mathbb{R}^n \times \mathbb{R}^n$ be given by $\Psi_i(p, X) = (\varphi_i(p), d(\varphi_i)_p(X))$ so that $\Psi_i : V_i \to U_i \times \mathbb{R}^n$ is a homeomorphism. It is clearly a bijection and continuous with continuous inverse because the same is true of φ_i and $(d\varphi_i)_p$ (this is an iso. by prop^n 2.7).

If $V_i \cap V_j \neq \emptyset$ then
$$\Psi_j \circ \Psi_i^{-1} = (\varphi_j \circ \varphi_i^{-1}(q),\ d(\varphi_j)_{\varphi_i^{-1}(q)} \circ d(\varphi_i)^{-1}_q(u))$$
$$= (\varphi_j \circ \varphi_i^{-1}(q),\ d(\varphi_j \circ \varphi_i^{-1})_q(u))\ ?$$
$\Rightarrow TM$ is a smooth $2n$-dim manifold. $\square$

Examples

1). $T_p\mathbb{R}^n = \mathbb{R}^n \ \forall p \in \mathbb{R} \Rightarrow T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$

2). Clearly points in TS^1 are given by $p \in (\cos\theta, \sin\theta)$ and $q = \lambda(-\sin\theta, \cos\theta)$ since q must be orthogonal to p. for some $\theta, \lambda \in \mathbb{R}$.

so there is a diffeo $f: S^1 \times \mathbb{R} \to TS^1$
given by $f: (\theta, \lambda) \mapsto \lambda(-\sin\theta, \cos\theta)$.
$\Rightarrow TS^1 \cong S^1 \times \mathbb{R}$. (note: not always a product like this).

25-10-18 Recall: Tangent bundle $TM = \bigcup_{p \in M} T_pM$
2n-dim mfd $\longrightarrow$

§ 3.2 Def^n of a vector field

Def^n 3.3 (a vector field is a section!)
A vector field X on a smooth mfd M is a smooth map $X: M \to TM$ s.t. $X(p) \in T_pM \quad \forall p \in M$
We denote the set of vector fields by $\Gamma(M)$.

Examples
1). on $\mathbb{R}^n$ we have the standard vector fields $\partial_i := \frac{\partial}{\partial x_i}$

2). $M \subset \mathbb{R}^n$ and X is a vector field on M, then (exercise) X is the restriction of a vector field $\tilde{X}$ on $\mathbb{R}^n$ s.t. $\tilde{X}(p) \in T_pM \quad \forall p \in M$
Take for example $S^1 \subseteq \mathbb{R}^2$ then $(x_1, x_2) = \cos\theta, \sin\theta \in S^1$
we have $T_{(x_1,x_2)} S^1 = T_{(\cos\theta, \sin\theta)} S^1 \cong \{\lambda(-\sin\theta, \cos\theta) : \lambda \in \mathbb{R}\}$
$= \{\lambda(-x_2, x_1) : \lambda \in \mathbb{R}\}$
$\Rightarrow$ the vector field $X = -x_2 \partial_1 + x_1 \partial_2$ on $\mathbb{R}^2$ restricts to a vector field on S^1.

3). Define on $\mathbb{R}^3$ $E_1 = x_3 \partial_2 - x_2 \partial_3$, $E_2 = x_1 \partial_3 - x_3 \partial_1$,
$E_3 = x_2 \partial_1 - x_1 \partial_2$.
(These correspond to rotations around the coordinate axes).

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4). Let $f: \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^2 \setminus \{0\}$, $f(r,\theta) = (r\cos\theta, r\sin\theta)$

We saw $df_{(r,\theta)}(\partial_r) = \cos\theta\,\partial_1 + \sin\theta\,\partial_2$

$df_{(r,\theta)}(\partial\theta) = -r\sin\theta\,\partial_1 + r\cos\theta\,\partial_2$

We let $X_r = \cos\theta\,\partial_1 + \sin\theta_2 = (x_1\partial_1 + x_2\partial_2)/r$

$X_\theta = -r\sin\theta\,\partial_1 + r\cos\theta\,\partial_2 = -x_2\partial_1 + x_1\partial_2$

5). $f: \mathbb{R}^2 \to S^2 \subseteq \mathbb{R}^3$, $f(\theta,\phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

$df_{(\theta,\phi)}(\partial_\theta) = \cos\theta\cos\phi\,\partial_1 + \cos\theta\sin\phi\,\partial_2 - \sin\theta\,\partial_3 =: X_\theta$ 

$df_{(\theta,\phi)}(\partial_\phi) = -\sin\theta\sin\phi\,\partial_1 + \sin\theta\cos\phi\,\partial_2 =: X_\phi$ 

are vector fields on $S^2 \setminus \{N, S\}$

6). $f: \mathbb{R}^n \to T^n \subseteq \mathbb{R}^{2n}$, $f(\theta_1, \dots \theta_n) = (\cos\theta_1, \sin\theta_1, \dots, \cos\theta_n, \sin\theta_n)$

$df_{(\theta_1,\dots,\theta_n)}(\partial_{\theta_j}) = -\sin\theta_j\,\partial_{2j-1} + \cos\theta_j\,\partial_{2j} =: X_j$

§3.3 Parallelisable manifolds

Def^n 3.4

The tangent bundle TM of M is trivial if there exists a diffeomorphism $\psi: TM \to M \times \mathbb{R}^n$ s.t. $\psi: \pi^{-1}(p) \to p \times \mathbb{R}^n$ is an isomorphism $\forall p \in M$ (i.e. a bundle isomorphism between TM and $M \times \mathbb{R}^n$)
If TM is trivial, we call M parallelisable.

Examples

1). $\mathbb{R}^n$ is trivially parallelisable

2). S^1 is parallelisable by our previous example however (see problem sheet 3): S^3 is parallelisable, but S^{2n} is not, but also S^5 is not.

3). All the matrix groups G we have seen are parallelisable (exercise).

$T_eG \xrightarrow{l_g} T_gG$ ← left multiplication by g give n nowhere vanishing L.I. vector fields.

Prop^n 3.5
An n-dim manifold is parallelisable iff it has n linearly independent vector fields (at each point).

Proof
Problem sheet 3.
$\square$

Examples
1). For a 1-dim mfd we have parallelisable
$\iff \exists$ a nowhere vanishing vector field.
$\Rightarrow S^1$ is parallelisable.

2). On S^n we can think of a vector field as a map
$X : S^n \to \mathbb{R}^{n+1}$ s.t. $X(p) \in T_pS^n = \langle p \rangle^\perp$
Recall: Hairy Ball Thm: every vector field on S^{2n} has at least one point where it vanishes.
Thus by prop^n 3.5, S^{2n} is not parallelisable.

3). T^n is parallelisable since the vector fields
$X_j = -\sin\theta_j \, \partial_{2j-1} + \cos\theta_j \, \partial_{2j}$ on T^n are linearly independent $\forall j = 1, \dots, n$.

§3.4 Push forward

Def^n 3.6
Let $f : M \to N$ be a diffeo. Then we define the push forward $f_* : \Gamma(TM) \to \Gamma(TN)$ by
$f_*(X)(f(p)) = df_p(X(p)) \quad \forall p \in M$.
This defines a vector field since f is a diffeo.

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Examples

1). For $f: \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^2 \setminus \{0\}$, $f(r,\theta) = (r\cos\theta, r\sin\theta)$
we have $X_r = f_*(\partial_r)$, $X_\theta = f_*(\partial_\theta)$.

2). $f: \mathbb{R}^2 \to S^2 \in \mathbb{R}^3$, $f(\theta,\phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$
$X_\theta = f_*(\partial_\theta)$, $X_\phi = f_*(\partial_\phi)$.

3). $f: \mathbb{R}^n \to T^n \in \mathbb{R}^{2n}$, $f(\theta_1, \dots, \theta_n) = (\cos\theta_1, \sin\theta_1, \dots, \cos\theta_n, \sin\theta_n)$
then $X_j = f_*(\partial_i)\ \forall_j$.

Local correspondence of charts

Let (U,φ) be a chart on M and $X \in \Gamma(TM)$.
Then we can map $X|_U \to \varphi_*(X)$.
Explicitly $\varphi_*(X) = \sum_{i=1}^n a_i \partial_i$, $a_i : \varphi(U) \to \mathbb{R}$ smooth, $i=1,\dots,n$.

$\varphi^{-1} : \varphi(U) \to U$, $(\varphi^{-1})_* \left(\sum_{i=1}^n a_i \partial_i\right) = X|_U$

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§3.5

Assume $X, Y \in \Gamma(\mathbb{R}^n)$ given by $X = \sum a_i \partial_i$,
$Y = \sum b_j \partial_j$. Then the operator $X \circ Y$ is given by
$$X \circ Y = \sum_i a_i \partial_i \left(\sum_j b_j \partial_j\right) = \sum_{i,j} \left(a_i b_j \partial^2_{ij} + a_i(\partial_i b_j)\partial_j\right)$$

This is not anymore a linear comb. of ∂_i's so
it is not a vectorfield on $\mathbb{R}^n$.
However $X \circ Y - Y \circ X = \sum_{i,j} \left(a_i(\partial_i b_j) - b_i(\partial_i a_j)\right)\partial_j$
which is a vector field on $\mathbb{R}^n$.

Def^n

Given $X, Y \in \Gamma(TM)$ we define the Lie-bracket of X, Y to
be $[X,Y] = X \circ Y - Y \circ X$, i.e. if f is a smooth function on

M, then $[X,Y](f) = X(Y(f)) - Y(X(f))$.

<u>Prop^n</u>

Let $g: M \to N$ be a diffeo. Then
$g_*[X,Y] = [g_*X, g_*Y]$.

[Note: as a function on N
$g_*(X)(f) = (X(f \circ g)) \circ g^{-1}$]

<u>Proof</u>

Let $f: N \to \mathbb{R}$ be smooth.

$$\begin{aligned} g_*[X,Y](f) &= [X,Y](f\circ g) = X(Y(f\circ g)) - Y(X(f\circ g)) \\ &= X(Y(f\circ g)\circ g^{-1}\circ g) - Y(X(f\circ g)\circ g^{-1}\circ g) \\ &= (g_*X)(g_*Y(f)) - (g_*Y)(g_*X(f)) \\ &= [g_*X, g_*Y](f). \end{aligned}$$

□

<u>Claim:</u> $[X,Y] \in \Gamma(TM)$.

<u>Proof</u>

Let (U, φ) be a chart on M, then

$$\begin{aligned} [X,Y]|_U &= (\varphi^{-1})_*(\varphi_*([X,Y])) \\ &= (\varphi^{-1})_*(\underbrace{[\varphi_*X, \varphi_*Y]}_{\in\, \Gamma(T\mathbb{R}^n|_{\varphi(U)})}) \in \Gamma(TM|_U) \end{aligned}$$

[This uses fact that on $\mathbb{R}^n$ the second partial derivatives commute]

□

<u>Remark</u>

$[Y,X] = -[X,Y]$, so $[X,X] = 0$

<u>Examples</u>

1). ∂_i, ∂_j standard vector fields on $\mathbb{R}^n$

$$[\partial_i, \partial_j] = \partial_i(\partial_j) - \partial_j(\partial_i) = \frac{\partial^2}{\partial x_i \partial x_j} - \frac{\partial^2}{\partial x_j \partial x_i} = 0$$

2). $E_1 = x_3\partial_2 - x_2\partial_{x_1}$, $E_2 = x_1\partial_3 - x_3\partial_1$, $E_3 = x_2\partial_1 - x_1\partial_2$

vector fields on $\mathbb{R}^3$.

$$\begin{aligned} [E_1, E_2] &= (x_3\partial_2 - x_2\partial_3)(x_1\partial_3 - x_3\partial_1) - (x_1\partial_3 - x_3\partial_1)(x_3\partial_2 - x_2\partial_3) \\ &= (x_2\partial_1 - x_1\partial_2) = E_3 \end{aligned}$$

Similarly $[E_2, E_3] = E_1$, $[E_3, E_1] = E_2$.

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3). Let $X = x_1 \partial_1 + x_2 \partial_2$, $Y = -x_2 \partial_1 + x_1 \partial_2$ on $\mathbb{R}^2$

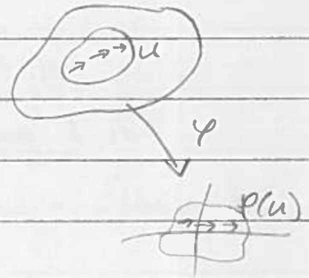
then $[X,Y] = (x_1\partial_1 + x_2\partial_2)(-x_2\partial_1 + x_1\partial_2) - (-x_2\partial_1 + x_1\partial_2)(x_1\partial_1 + x_2\partial_2)$

$$= \cancel{-x_1x_2\partial^2_{1,1}} + \cancel{x_1^2\partial^2_{1,2}} \cancel{- x_2^2\partial^2_{2,1}} + \cancel{x_2x_1\partial^2_{2,2}} + \cancel{x_1\partial_2} - x_2\partial_1$$
$$- \left(\cancel{-x_2x_1\partial^2_{1,1}} \cancel{- x_2^2\partial^2_{1,2}} + \cancel{x_1^2\partial^2_{2,1}} + \cancel{x_1x_2\partial^2_{2,2}} + \cancel{x_1\partial_2} - x_2\partial_1\right)$$
$$= 0$$

4). Clearly the vector fields $X_j = -\sin\theta_j \partial_{2j-1} + \cos\theta_j \partial_{2j}$ on $T \subset \mathbb{R}^{2n}$ satisfy $[X_i, X_j] = 0$.

5) Let (U, φ) be a chart on M. If ∂_i are the standard vector fields on $\mathbb{R}^n$, then $X_i = (\varphi^{-1})_* \partial_i$ are vector fields on U and $[X_i, X_j] = [(\varphi^{-1})_*\partial_i, (\varphi^{-1})_*\partial_j]$

$$= (\varphi^{-1})_* \underbrace{[\partial_i, \partial_j]}_{=0} = 0$$



6). Let $f: \mathbb{R}^2 \to S^2$ be the map given by

$f(\theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

then $[\partial_\theta, \partial_\phi] = 0$

$\Rightarrow [f_*(\partial_\theta), f_*(\partial_\phi)] = 0$.

7). If $\varphi_s^{-1}: \mathbb{R}^3 \to S^3 \setminus \{S\}$, then $Y_i = (\varphi_s^{-1})E_i$ (where E_i are defined in example 2) satisfy $[Y_1, Y_2] = Y_3$.

Prop^n

The Lie bracket satisfies the Jacobi identity, i.e. for $X, Y, Z \in \Gamma(TM)$ we have

$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

Proof
Use a local chart and compute on $\mathbb{R}^n$. $\square$

§4 Riemannian mfds : definitions and examples

$c:[0,1]\to\mathbb{R}^n$, $l(c)=\int_0^1 |c'(t)|\,dt$

§4.1 · Def^{n}s

Fake def^n

A Riemannian metric g on M is a smooth choice of positive definite inner product on each tangent space i.e. $\forall\, p\in M$ we have a symmetric, bilinear map $g_p : T_pM\ T_pM \to \mathbb{R}$

Def^n 4.1

A Riemannian manifold (M,g) is a manifold M with a Riemannian metric g.

What are Riemannian metrics?

An inner product on $\mathbb{R}^n$ corresponds to a symmetric (positive definite) matrix $A\in \mathrm{Mat}_{n,n}(\mathbb{R})$.
For example, if $\langle\cdot,\cdot\rangle$ is an inner product on $\mathbb{R}^n$ then there is a symmetric matrix A s.t.
$\forall\, x,y\in\mathbb{R}^n$, $\langle x,y\rangle = x^TAy$

Examples

1). On $\mathbb{R}^n$, the standard metric g_0 is given by

$$g_0\left(\sum_{i=1}^n a_i\partial_i,\ \sum_{j=1}^n b_j\partial_j\right)=\sum_i a_ib_i$$

$$\iff g_0(\partial_i,\partial_j)=\delta_{ij}$$

Now on $\mathbb{R}^2\setminus\{0\}$ take $r=\sqrt{x_1^2+x_2^2}$, $X=\dfrac{x_1\partial_1+x_2\partial_2}{r}$
$Y=-x_2\partial_1+x_1\partial_2$

$\Rightarrow g_0(X,X)=1,\ g_0(X,Y)=0,\ g_0(Y,Y)=x_2^2+x_1^2=r^2$

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Hence g_0 w.r.t. the basis vector fields X, Y on $\mathbb{R}^2 \setminus \{0\}$ is $\begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$

2). Let $M \subseteq \mathbb{R}^n$ be a submanifold. We can define a metric on M by $g_p(X, Y) = g_0(X, Y)$ if $X, Y \in T_pM \subseteq T_p\mathbb{R}^n$.
We call this the induced metric.
[This is the first Fundamental Form when on surfaces]

3). In particular S^n has Riem. metric induced from the Euclidean metric on $\mathbb{R}^{n+1}$.
We take $X_1 = \cos\theta\cos\phi\,\partial_1 + \cos\theta\sin\phi\,\partial_2 - \sin\theta\,\partial_3$
$X_2 = -\sin\theta\sin\phi\,\partial_1 + \sin\theta\cos\phi\,\partial_2$

on $S^n \setminus \{N, S\}$, then we have
$g(X_1, X_1) = 1$, $g(X_2, X_2) = \sin^2\theta$, $g(X_1, X_2) = 0$
We can identify g w.r.t. X_1, X_2 with $\begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$

In contrast, if we take the vector fields
$X_1 = -x_1\partial_0 + x_0\partial_1 - x_3\partial_2 + x_2\partial_3$
$X_2 = -x_2\partial_0 + x_3\partial_1 + x_0\partial_2 - x_1\partial_3$
$X_3 = -x_3\partial_0 - x_2\partial_1 + x_1\partial_2 + x_0\partial_3$
which are all elements of $\Gamma(TS^3)$ then w.r.t. the induced metric we have $g(X_i, X_j) = \delta_{ij}$.

4). Let $f: \mathbb{R}^2 \to \mathbb{R}^3$ be given by
$f(\theta, \phi) = ((2+\cos\theta)\cos\phi, (2+\cos\theta)\sin\phi, \sin\theta)$
Then $X_1 = f_*(\partial_\theta) = -\sin\theta\cos\phi\,\partial_1 - \sin\theta\sin\phi\,\partial_2 + \cos\theta\,\partial_3$
$X_2 = f_*(\partial_\phi) = -(2+\cos\theta)\sin\phi\,\partial_1 + (2+\cos\theta)\cos\phi\,\partial_2$
are vector fields on T^2. Then
$g(X_1, X_1) = 1$, $g(X_1, X_2) = 0$, $g(X_2, X_2) = (2+\cos\theta)^2$
So we can identify g with the matrix $\begin{pmatrix} 1 & 0 \\ 0 & (2+\cos\theta)^2 \end{pmatrix}$

4.3 Pull back and local metrics

Assume $f: M \to N$ a diffeo, $X, Y \in \Gamma(TM)$ and a metric h on N, then h acts on the vector fields $f_*(X)$, $f_*(Y)$

$\Rightarrow$ we get a metric on M defined by

$$g(X, Y) = h(f_*(X), f_*(Y))$$

Def^n 4.2

Let $f: M \to N$ be smooth and let h be a Riemannian metric on N.

We define the pull back f^*h of h by

$$(f^*h)_p(X, Y) = h_{f(p)}(df_p(X), df_p(Y)).$$

If X, Y are vector fields on M then

$$(f^*h)(X, Y) = h(f_*(X), f_*(Y))$$

Prop^n 4.3

Let $f: M \to N$ be an immersion (so df_p is injective $\forall\, p \in M$) and let h be a Riemannian metric on N. Then $g = f^*h$ is a Riemannian metric on M.

Proof

Let $p \in M$, $X, Y \in T_pM$. Since h is symmetric, bilinear and smooth $\Rightarrow$ g is symmetric, bilinear and smooth.

Positive definite: $g_p(X, X) = h_{f(p)}(df_p(X), df_p(X)) \geq 0$ and $g_p(X, X) = 0$ only if $df_p(X) = 0$.

Since df_p is injective we have $df_p(X) = 0 \Leftrightarrow X = 0$

$\Rightarrow$ g is pos. definite. $\square$

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Recall

Def^n

$f: M \to N$ smooth, h a Riem. metric on N.
Define the pullback f^*h of h by f via
$$(f^*h)_p(X, Y) = h_{f(p)}(df_p(X), df_p(Y))$$

$\forall p \in M$, $\forall X, Y \in T_pM$.

Prop^n 4.3

Let $f: M \to N$ be an immersion, and h a Riem. metric on N, then $g = f^*h$ is a Riem. metric on M

Assume (U, φ) is a chart on (M, g), then $\varphi^{-1}: \varphi(U) \to U \subseteq M$ is a diffeomorphism
$\Rightarrow (\varphi^{-1})^*g$ is a Riemannian metric on $\varphi(U) \subseteq \mathbb{R}^n$.
So we can write it in terms of a symmetric matrix on $\mathbb{R}^n$, in particular
$$(\varphi^{-1})^*g(\partial_i, \partial_j) = g((\varphi^{-1})_* \partial_i, (\varphi^{-1})_* \partial_j) = g(X_i, X_j)$$

Alternatively, we can also write the Euclidean metric on $\mathbb{R}^n$ as $g_0 = dx_1^2 + dx_2^2$
with the rule $dx_i dx_j(\partial_k, \partial_l) = dx_i dx_j(\partial_l, \partial_k) = \begin{cases} 1 & i=k,\ j=l \text{ or } i=l,\ j=k \\ 0 & \text{otherwise} \end{cases}$

So any metric on $\mathbb{R}^n$ can be written as
$$g = \sum_{i,j} g_{ij} dx_i dx_j$$
where g_{ij} is a positive, symmetric matrix of functions.

We see if we write $(\varphi^{-1})^*g = \sum_{i,j} g_{ij} dx_i dx_j$, $g(X_i, X_j) = g_{ij}$

Rmk

We will use the notation g_{ij} frequently in the rest of the course for the functions $g(X_i, X_j)$ where $\{X_1, \ldots, X_n\}$ is a coordinate frame field in the chart (U, φ).

Example

Let $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^2 \setminus \{0\}$, $f(r,\theta) = (r\cos\theta, r\sin\theta)$
then $X_1 = f_*(\partial_r)$, $X_2 = f_*(\partial_\theta)$

$$\Rightarrow f^*g_0(\partial_r, \partial_r) = g_0(f_*(\partial_r), f_*(\partial_r)) = 1$$
$$f^*g_0(\partial_r, \partial_\theta) = g_0(f_*(\partial_r), f_*(\partial_\theta)) = 0$$
$$f^*g_0(\partial_\theta, \partial_\theta) = g_0(f_*(\partial_\theta), f_*(\partial_\theta)) = r^2$$

So $f^*g_0 = dr^2 + r^2 d\theta^2$

Example

$f(\theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ defines local coordinates on S^2 so the standard induced metric on S^2 in (θ, ϕ) coords is given by

$$f^*g_0(\partial_\theta, \partial_\theta) = g_0(f_*\partial_\theta, f_*\partial_\theta) = 1$$
$$f^*g_0(\partial_\theta, \partial_\phi) = 0 \quad , \quad f^*g_0(\partial_\phi, \partial_\phi) = \sin^2\theta.$$
$$\Rightarrow f^*g = d\theta^2 + \sin^2\theta\, d\phi^2.$$

§5 The Levi-Civita Connection

Assume (M, g) a Riemannian manifold.

§5.1 Fundamental Thm of Riemannian geometry

Thm 5.1

There exists a unique map $\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ denoted by $\nabla(X, Y) \mapsto \nabla_X Y$ s.t. if $X, Y, Z \in \Gamma(TM)$ and a, b are smooth fns on M, then

(i) $\nabla_{aX+bY} Z = a\nabla_X Z + b\nabla_Y Z$, (ii) $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$

(iii) $\nabla_X(aY) = a\nabla_X Y + X(a)Y$

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(iv) $X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$

(v) $\nabla_X Y - \nabla_Y X = [X, Y]$

We call $\nabla_X Y$ the covarient derivative of Y w.r.t. X and ∇ the Levi-Civita Connection of g.

Proof

Assume ∇ exists and satisfies (i)-(v) then

$$\text{(iv)} \Rightarrow \quad X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$
$$Y(g(Z,X)) = g(\nabla_Y Z, X) + g(Z, \nabla_Y X).$$
$$Z(g(X,Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y).$$

We deduce from (v)

$$X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y))$$
$$= 2g(\nabla_X Y, Z) + g(X, [Y,Z]) - g(Y, [Z,X]) - g(Z, [X,Y])$$
$$\Rightarrow g(\nabla_X Y, Z) = \frac{1}{2}\Big(X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) - g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y])\Big)$$

$(*) \rightarrow$

So $\nabla_X Y$ is uniquely defined by g if it exists using $(*)$ (called Kostel formula).

We can now define $\nabla_X Y$ using $(*)$ and prove that it satisfies (i) - (v).

For (i), if W is another vector field on U, we can calculate

$$g(\nabla_{aX+bY} Z, W) = \frac{1}{2}\Big((aX+bY)g(Z,W) + Z(g(W,aX+bY)) - W(g(aX+bY,Z)) - g(aX+bY,[Z,W]) + g(Z,[W,aX+bY]) + g(W,[aX+bY,Z])\Big)$$
$$= g(a\nabla_X Z + b\nabla_Y Z, W) + \cancel{Z(a)g(W,X)} + \cancel{Z(b)g(W,Y)}$$
$$- \cancel{W(a)g(X,Z)} - \cancel{W(b)g(Y,Z)} + \cancel{W(a)g(Z,X)} + \cancel{W(b)g(Z,Y)}$$
$$- \cancel{Z(a)g(W,X)} - \cancel{Z(b)g(W,Y)} \quad \text{as required.}$$

(ii) - (v) see notes online. $\square$

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Levi-Civita connection

(M, g) Riem. manifold

$X, Y, Z \in \Gamma(TM)$, $a, b, c \in C^\infty(M)$

$\nabla_{aX+bY} Z = a\nabla_X Z + b\nabla_Y Z$

$\nabla_X (Y+Z) = \nabla_X Y + \nabla_X Z$

$\nabla_X (aY) = a\nabla_X Y + X(a) Y$

$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$

$\nabla_X Y - \nabla_Y X = [X, Y]$

Koszul formula →

$$g(\nabla_X Y, Z) = \frac{1}{2}\Big(X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) - g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]) \Big)$$

Example

On $\mathbb{R}^n$: $[\partial_i, \partial_j] = 0$, $g_0(\partial_i, \partial_j) = \delta_{ij}$

$\Rightarrow \nabla_{\partial_i} \partial_j \equiv 0$

(since $g_0(\nabla_{\partial_i} \partial_j, \partial_k) \equiv 0$)

Example

$T^n \subseteq \mathbb{R}^{2n}$, there we have $X_i = -\sin\theta_i \partial_{2i-1} + \cos\theta_i \partial_{2i}$

which satisfy $g(X_i, X_j) = \delta_{ij}$, $[X_i, X_j] = 0$

$\Rightarrow \nabla_{X_i} X_j = 0$.

Example

On S^2 let $f(\theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

and let $X_1 = f_* \partial_\theta$, $X_2 = f_* \partial_\phi$

Then $[X_1, X_2] = 0$.

Also $g(X_1, X_1) = 1$, $g(X_1, X_2) = 0$, $g(X_2, X_2) = \sin^2\theta$.

Assume $h = h(\theta, \phi)$ is a function on S^2, then we have $X_1(h) = \dfrac{\partial h}{\partial \theta}$, $X_2(h) = \dfrac{\partial h}{\partial \phi}$

$g(\nabla_{X_1} X_1, X_1) = \frac{1}{2} X_1(g(X_1, X_1)) = 0$.

$g(\nabla_{X_1} X_1, X_2) = \frac{1}{2}\big(2X_1(g(X_1, X_2)) - X_2(g(X_1, X_1))\big) = 0$

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Since X_1, X_2 are a basis for $TS^2 \setminus \{N, S\}$ we have $\nabla_{X_1} X_1 \equiv 0$

$$g(\nabla_{X_2} X_2, X_1) = \frac{1}{2}\left(2X_2(g(X_1, X_1)) - X_1(g(X_2, X_2))\right)$$
$$= -\frac{1}{2}\frac{\partial}{\partial\theta}(\sin^2\theta) = -\sin\theta\cos\theta.$$

$$g(\nabla_{X_2} X_2, X_2) = \frac{1}{2}\left(X_2 g(X_2, X_2)\right) = \frac{1}{2}\frac{\partial}{\partial\phi}\sin^2\theta = 0.$$

$$\Rightarrow \nabla_{X_2} X_2 = -\sin\theta\cos\theta\, X_1$$

$$g(\nabla_{X_1} X_2, X_1) = \frac{1}{2}\left(X_1(g(X_2, X_1)) + X_2(g(X_1, X_1)) - X_1(g(X_1, X_2))\right)$$
$$= 0$$
$$g(\nabla_{X_1} X_2, X_2) = \frac{1}{2}\left(X_1(g(X_2, X_2)) + X_2(g(X_1, X_2)) - X_2(g(X_1, X_2))\right)$$
$$= \frac{1}{2}\frac{\partial}{\partial\theta}(\sin^2\theta) = \sin\theta\cos\theta$$

Since $g(X_2, X_2) = \sin^2\theta$, $[X_1, X_2] = 0$

$$\nabla_{X_1} X_2 = \nabla_{X_2} X_1 = \frac{\sin\theta\cos\theta}{\sin^2\theta} X_2 = \cot X_2$$

<u>Example</u>

On S^3 we had the vector fields

$E_1 = -x_1\partial_0 + x_0\partial_1 - x_3\partial_2 + x_2\partial_3$

$E_2 = -x_2\partial_0 + x_3\partial_1 + x_0\partial_2 - x_1\partial_3$

$E_3 = -x_3\partial_0 - x_2\partial_1 + x_1\partial_2 + x_0\partial_3$

Let g be the induced metric, then $g(E_i, E_j) = \delta_{ij}$

$[E_1, E_2] = -2E_3$, $[E_i, E_j] = -2\varepsilon_{ijk}E_k$

Then

$$g(\nabla_{E_i} E_j, E_k) = \frac{1}{2}\left[-g(E_i, [E_j, E_k]) + g(E_j[E_k, E_i]) + g(E_k, [E_i, E_j])\right]$$

$$= \frac{1}{2}\left(2\varepsilon_{ijk} - 2\varepsilon_{kij} - 2\varepsilon_{kij}\right) = -\varepsilon_{ijk}$$

$$\Rightarrow \nabla_{E_1} E_2 = -\nabla_{E_2} E_1 = -E_3, \quad \nabla_{E_2} E_3 = -\nabla_{E_3} E_2 = -E_1,$$

$$\nabla_{E_3} E_1 = -\nabla_{E_1} E_3 = -E_2, \quad \nabla_{E_1} E_1 = \nabla_{E_2} E_2 = \nabla_{E_3} E_3 = 0.$$

Motivation

$X \in \Gamma(TM)$, (U, φ) has vector fields $X_i = (\varphi^{-1})_* \partial_i$

$X|_U = \sum_i a_i X_i$ where $a_i \in C^\infty(U)$

$\nabla_Y X = \nabla_Y (\sum a_i X_i) = \sum (\nabla_Y X_i) a_i + Y(a_i) X_i$

So if we write $Y = \sum_j b_j X_j$

$\Rightarrow \nabla_Y X = \sum_{i,j} \left(b_j a_i \nabla_{X_j} X_i + b_j X_j(a_i) X_i \right)$

§5.2 Christoffel symbols

Def^n 5.2

(U, φ) coordinates on (M, g), $X_i = (\varphi^{-1})_* \partial_i \in \Gamma(TM)$

Since $\{X_i\}_{1 \le i \le n}$ form a basis for $\Gamma(TM)$

we can define functions $\Gamma_{ij}^{\ k}$ on U by

$\nabla_{X_i} X_j = \sum_k \Gamma_{ij}^{\ k} X_k$

which are the christoffel symbols of ∇ in the chart (U, φ).

Examples

(i) On $\mathbb{R}^n$ $\quad \nabla_{\partial_i} \partial_j \equiv 0 \Rightarrow \Gamma_{ij}^{\ k} = 0$

(ii) For S^2, $\nabla_{X_1} X_1 \equiv 0$

$\Rightarrow \Gamma_{11}^{\ 1} = \Gamma_{11}^{\ 2} = 0.$

$\nabla_{X_2} X_2 = -\sin\theta \cos\theta X_1 \Rightarrow \Gamma_{22}^{\ 1} = -\sin\theta\cos\theta,\ \Gamma_{22}^{\ 2} = 0$

$\nabla_{X_1} X_2 = \nabla_{X_2} X_1 = \cot\theta X_2$

$\Rightarrow \Gamma_{12}^{\ 1} = \Gamma_{21}^{\ 1} = 0, \quad \Gamma_{12}^{\ 2} = \Gamma_{21}^{\ 2} = \cot\theta.$

Prop^n 5.3

Let (U, φ) be coords on (M, g) and X_i the coord. vector fields on U. Let g be given by $g_{ij} = g(X_i, X_j)$ on U, then $\Gamma_{ij}^{\ k} = \Gamma_{ji}^{\ k}$ and $g^{ij} = (g^{-1})_{ij}$ and let $\partial_k g_{ij} = X_k(g_{ij})$.

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Then $\Gamma_{ij}^{k} = \frac{1}{2}\sum_{l=1}^{k} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$.

Note: depends on coords!

Proof

$\nabla_{X_i} X_j - \nabla_{X_j} X_i = [X_i, X_j] = 0 \Rightarrow \Gamma_{ij}^{k} = \Gamma_{ji}^{k}$

Now $g(\nabla_{X_i} X_j, X_l) = \sum_{m=1}^{k} g(\Gamma_{ij}^{m} X_m, X_l) = \sum_{m=1}^{k} \Gamma_{ij}^{m} g_{ml}$

Koszul formula

$\overset{\downarrow}{=} \frac{1}{2}\Big[X_i(g(X_j, X_l)) + X_j(g(X_i, X_l)) - X_l(g(X_i, X_j))\Big]$

$= \frac{1}{2}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$

Finally, $\Gamma_{ij}^{k} = \sum_{l,m=1}^{k} \Gamma_{ij}^{m} \underbrace{g_{ml} g^{kl}}_{\delta_{mk}}$. $\square$

note symmetric matrix so order of indices doesn't matter

Example

Take the usual coordinate frame on $T^n \subseteq \mathbb{R}^{2n}$

ie, $X_i = f_* \partial_i$ where $f(x_1, \dots, x_n) = (\cos x_1, \sin x_1, \dots, \cos x_n, \sin x_n)$

$\Rightarrow g_{ij} = g(X_i, X_j) = \delta_{ij} \Rightarrow \Gamma_{ij}^{k} \equiv 0$.

Example

For S^2 take $f(\theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\phi)$

so $X_1 = f_* \partial_\theta$, $X_2 = f_* \partial_\phi$

$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$, $g^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1/\sin^2\theta \end{pmatrix}$

Note: if either i or j is 1 then g_{ij} is constant and $\partial_2 = \partial_\phi$ of anything is zero.

$\Rightarrow \Gamma_{12}^{1} = \frac{1}{2}\sum_{l=1}^{2} g^{1l}(\partial_1 g_{2l} + \partial_2 g_{1l} - \partial_l g_{12})$

$= \frac{1}{2} g^{11}(\partial_1 g_{12} + \partial_1 g_{11} - \partial_1 g_{12}) = 0$.

$\Gamma_{12}^{2} = \frac{1}{2}\sum_{l=1}^{2} g^{2l}(\partial_1 g_{2l} + \partial_2 g_{1l} - \partial_l g_{12})$

$= \frac{1}{2} g^{22}\partial_1 g_{22} = \frac{1}{2\sin^2\theta}\frac{\partial}{\partial\theta}(\sin^2\theta) = \frac{2\sin\theta\cos\theta}{2\sin^2\theta} = \cot\theta$.

§6 Geodesics

Def^n 6.1 Let (M, g) be a Riem. manifold. Assume $\alpha: I \to M$ is a curve and let $f = f(\alpha(t))$ be a function along the curve α, then

$$\alpha'(t) = (f \circ \alpha)'(t) = \frac{d}{dt}(f(\alpha(t)))$$

A curve γ is called a geodesic if $\nabla_{\gamma'}\gamma' = 0$

[Remark
$\nabla_X Y|_p$ is well defined if we know $X(p)$ and Y along a curve α s.t. $\alpha(0) = p$, $\alpha'(0) = X(p)$.]

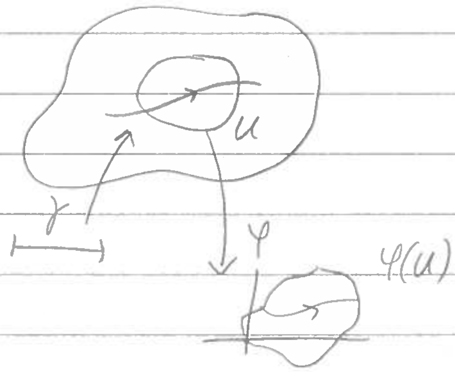
Since

$$\frac{d}{dt} g(\gamma', \gamma') = \gamma' g(\gamma', \gamma')$$
$$= 2g(\nabla_{\gamma'}\gamma', \gamma') = 0$$

$\Rightarrow |\gamma'| = \sqrt{g(\gamma', \gamma')} = \text{const.}$

We say γ is is normalised if $|\gamma'| = 1$.

Let (U, φ) be a coordinate chart and we write $\varphi \circ \gamma = (x_1, \dots, x_n)$



$$\Rightarrow (\varphi \circ \gamma)' = \sum_{i=1}^{n} x_i' \partial_i = \varphi_*(\gamma')$$

$$\Rightarrow \gamma' = (\varphi^{-1})_* \left(\sum_{i=1}^{n} x_i' \partial_i\right)$$

$$= \sum_{i=1}^{n} x_i' \underbrace{(\varphi^{-1})_* \partial_i}_{X_i|_{\gamma(t)}}$$

Thus
$$\nabla_{\gamma'}\gamma' = \sum_{i=1}^{n} \nabla_{\gamma'}(x_i' X_i)$$

$$= \sum_{i=1}^{n} \left(\gamma'(x_i') X_i + x_i' \nabla_{\gamma'} X_i\right)$$

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$$\Rightarrow \nabla_{\gamma'}\gamma' = \sum_{i=1}^{n}\left(x_i'' X_i + x_i' \sum_j x_j' \nabla_{X_j} X_i\right)$$

$$= \sum_{i=1}^{n} x_i'' X_i + \sum_{i,j=1}^{n} x_i' x_j' \Gamma_{ij}^{k} X_k$$

$$= \sum_{k=1}^{n}\left(x_k'' + \sum_{i,j=1}^{n} \Gamma_{ij}^{k} x_i' x_j'\right) X_k$$

Prop^n 6.2
Let (U, φ) be a coordinate chart on (M, g) and let γ be a curve in U.
If we write $\varphi \circ \gamma = (x_1, \dots, x_n)$ then γ is a geodesic iff (in Einstein summation convention)

$$x_k'' + \Gamma_{ij}^{k} x_i' x_j' = 0 \quad \forall k.$$

These are called the geodesic equations.

Examples
1). For $\mathbb{R}^n$ $\Gamma_{ij}^{k} \equiv 0$ so the geodesic equations $\Rightarrow x_k'' = 0$
$\Rightarrow x_k(t) = a_k t + b_k$
γ normalised $\Leftrightarrow \left(\sum a_k^2\right)^{1/2} = 1$.

2). Let $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^2 \setminus \{0\}$ be given by
$f(r, \theta) = (r\cos\theta, r\sin\theta)$
Then $f^* g_0 = g$ is given by

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}, \quad (g^{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix}$$

So we get $(X_1 = \partial_r, \ X_2 = \partial_\theta)$
$\Gamma_{11}^{1} = 0$, $\Gamma_{22}^{1} = -r$, $\Gamma_{12}^{1} = 0$, $\Gamma_{11}^{2} = 0$
$\Gamma_{22}^{2} = 0$, $\Gamma_{12}^{2} = 1/r$
So $\nabla_{X_1} X_1 = 0$, $\nabla_{X_1} X_2 = \nabla_{X_2} X_1 = \frac{1}{r} X_2$, $\nabla_{X_2} X_2 = -r X_1$
$\Rightarrow (x_1 = r, \ x_2 = \theta)$
Geodesic eq^{n}s: $r'' - r(\theta')^2 = 0$, $\theta'' + \frac{2}{r} r'\theta' = 0$.
So $\theta' = 0$, $r'' = 0$ gives a solution corresponding to a ray through the origin.

3). On the standard torus $T^n \subseteq \mathbb{R}^{2n}$ we saw
$\Gamma_{ij}^k \equiv 0$ so w.r.t. the map $f = \varphi^{-1} = (\cos\theta_1, \sin\theta_1, \dots, \cos\theta_n, \sin\theta_n)$
$\Rightarrow \theta_i(t) = a_i t + b_i$
$\Rightarrow \gamma(t) = (\cos(a_1 t + b_1), \sin(a_1 t + b_1), \dots, \cos(a_n t + b_n), \sin(a_n t + b_n))$

4). For S^2 we take a normalised geodesic
$\gamma(t) = (\sin\theta(t)\cos\phi(t), \sin\theta(t)\sin\phi(t), \cos\theta(t))$
Here $f(\theta, \phi)$ as usual.
$X_1 = f_* \partial_\theta, \quad X_2 = f_* \partial_\phi$
$\Gamma_{11}^1 = \Gamma_{12}^1 = 0, \quad \Gamma_{22}^1 = -\sin\theta\cos\theta$
$\Gamma_{11}^2 = \Gamma_{22}^2 = 0, \quad \Gamma_{12}^2 = \cot\theta$
so the geodesic eq^{n}s are:
$\theta'' - \sin\theta\cos\theta(\phi')^2 = 0$
$\phi'' + 2\cot\theta\, \theta'\phi' = 0.$
$|\gamma'|^2 = (\theta')^2 + \sin^2\theta(\phi')^2 = 1$
We see that $\phi' = 0$ and $\theta'' = 0$
gives a solution which is
$\gamma(t) = (\sin(t + \theta_0)\cos\phi_0, \sin(t + \theta_0)\sin\phi_0, \cos(t + \theta_0))$

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Example: Hyperbolic plane

$H^2 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 > 0\}$

$$g = \frac{dx_1^2 + dx_2^2}{x_2^2}$$

$$\Gamma_{11}^1 = \Gamma_{22}^1 = 0, \quad \Gamma_{11}^2 = -\Gamma_{22}^2 = \frac{1}{x_2}, \quad \Gamma_{12}^1 = -\frac{1}{x_2}, \quad \Gamma_{12}^2 = 0$$

$$x_1'' - \frac{2}{x_2} x_1' x_2' = 0$$

$$x_2'' + \frac{1}{x_2}\left((x_1')^2 - (x_2')^2\right) = 0$$

Note $x_1 = \text{const}$, $x_2 = e^t$ is a solution.

Prop^n

Let (U, φ) be a chart on (M, g) and let $L = \frac{1}{2} g_{ij} x_i' x_j'$ (use sum. conv.). Then γ given by $\varphi \circ \gamma = (x_1, \ldots, x_n)$ is a geodesic iff $\forall k$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial x_k'}\right) - \frac{\partial L}{\partial x_k} = 0$$

Proof

$$\frac{d}{dt} g_{ij} = \gamma'(g(X_i, X_j)) = g(\nabla_{\gamma'} X_i, X_j) + g(X_i, \nabla_{\gamma'} X_j)$$
$$= x_l' g(\nabla_{X_l} X_i, X_j) + x_l' g(X_i, \nabla_{X_l} X_j)$$

$$= x_l' \Gamma_{li}^m g(X_m, X_j) + x_l' \Gamma_{lj}^m g(X_i, X_m)$$
$$= x_l' \Gamma_{li}^m g_{mj} + x_l' \Gamma_{li}^m g_{im}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial x_k'}\right) = \frac{d}{dt}(g_{ik} x_i') = g_{ik} x_i'' + x_i' x_l' (\Gamma_{li}^m g_{mj} + \Gamma_{lj}^m g_{im})$$

$X_k = \frac{\partial}{\partial x_k}$

$$\frac{\partial L}{\partial x_k} = \frac{1}{2} X_k(g_{ij}) x_i' x_j' = \frac{1}{2} X_k(g(X_i, X_j)) x_i' x_j'$$

$$= \frac{1}{2}\left(g(\nabla_{X_k} X_i, X_j) + g(X_i, \nabla_{X_k} X_j)\right) x_i' x_j'$$

$$= \frac{1}{2}\left(\Gamma_{ki}^l g_{lj} + \Gamma_{kj}^l g_{il}\right) x_i' x_j'$$

$$\Rightarrow \frac{\partial L}{\partial x_k} = \Gamma_{ki}^{l} g_{lj} x_i' x_j'$$

Multiply these eqns by g^{ka} and notice that $g^{ka} g_{ki} = \delta_{ia}$

Then $\frac{d}{dt}\left(\frac{\partial L}{\partial x_k'}\right) - \frac{\partial L}{\partial x_k} = 0$

$$\Longleftrightarrow x_a'' + \left(\Gamma_{li}^{a} + \cancel{g^{ka}\Gamma_{lk}^{m} g_{im}} - \cancel{g^{ka}\Gamma_{ki}^{m} g_{ml}}\right) x_i' x_l'$$

$$= x_a'' + \Gamma_{li}^{a} x_i' x_l' = 0$$

$\square$

Example Polar coords. (on Plane)

$g = dr^2 + r^2 d\theta^2$

$L = \frac{1}{2}\left((r')^2 + r^2(\theta')^2\right)$

$\frac{\partial L}{\partial r'} = r' \qquad \frac{\partial L}{\partial r} = r(\theta')^2 \qquad ①$

$\frac{\partial L}{\partial \theta'} = r^2\theta' \qquad \frac{\partial L}{\partial \theta} = 0 \qquad ②$

$① \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial r'}\right) - \frac{\partial L}{\partial r} = r'' - r(\theta')^2 = 0$

$② \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \theta'}\right) - \frac{\partial L}{\partial \theta} = (r^2\theta')' = r^2\theta'' + 2rr'\theta' = 0$

These two are equiv. to $\varphi\circ\gamma = (r, \theta)$ being a geodesic.

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Def^n

A smooth map $f:(M,g)\to(N,h)$ between Riemannian manifolds is an isometry if f is a diffeomorphism and $g=f^*h$.

Example

$f:\mathbb{R}^n\to\mathbb{R}^n$, $f(x)=Ax$, $A=(a_{ij})\in M_n(\mathbb{R})$.

When is this an isometry, when $g_0=\delta_{ij}$?

$$\begin{aligned}f^*g_0(\partial_i,\partial_j) &= g_0(f_*\partial_i, f_*\partial_j)\\ &= g_0(A\partial_i, A\partial_j)\\ &= g_0\left(\sum_{k=1}^n a_{ki}\partial_k, \sum_{k=1}^n a_{kj}\partial_l\right)\\ &= \sum_{k=1}^n a_{ki}a_{kj}\end{aligned}$$

Thus $f^*g_0=g_0 \iff \sum_{k=1}^n a_{ki}a_{kj} = \delta_{ij} \iff A^TA=I$

Example

Let $z=x_1+ix_2$, consider (H^2,g) (upper half plane)

$g=\dfrac{dz\,d\bar{z}}{|\mathrm{Im}\,z|^2}$.

If $f:H^2\longrightarrow H^2$, $f^*dz=d(f(z))=f'(z)dz$

$f^*(d\bar{z})=\overline{f'(z)}d\bar{z}$

$$f^*g=\frac{|f'(z)|^2dz\,d\bar{z}}{|\mathrm{Im}(z)|^2}$$

f is an isometry if it is a diffeo and

$|f'(z)|^2|\mathrm{Im}(z)|^2=|\mathrm{Im}\,f(z)|^2$

Set $f(z)=\dfrac{az+b}{cz+d}$ where $a,b,c,d\in\mathbb{R}$ s.t. $ad-bc=1$

i.e. f can be identified with a matrix $\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL(2,\mathbb{R})$.

$$\begin{aligned}f(z)=f(x_1+ix_2) &= \frac{ax_1+iax_2+b}{cx_1+icx_2+d}\\ &= \frac{(acx_1^2+acx_2^2+bd)+i(ad-bc)x_2}{|cz+d|^2}\end{aligned}$$

$$\Rightarrow f(z) = \frac{(ac|z|^2 + bd) + i \operatorname{Im} z}{|cz+d|^2}$$

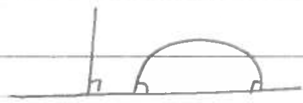
$$f'(z) = \frac{ad-bc}{(cz+d)^2} = (cz+d)^2$$

f sends H^2 to H^2 and has inverse

$$f^{-1}(z) = \frac{dz-b}{-cz+a}$$

$\Rightarrow f$ a diffeo and f is an isometry since

$$\underbrace{|f'(z)|^2 |\operatorname{Im} z|^2}_{= \frac{|\operatorname{Im} z|^2}{|cz+d|^4}} = \underbrace{|\operatorname{Im} f(z)|^2}_{= \frac{|\operatorname{Im} z|^2}{|cz+d|^4}}$$



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Thm

Let $p \in M$. Then $\exists$ an open set $\mathcal{U} \ni p$, $\varepsilon > 0$, a smooth map $\Gamma : (-2, 2) \times V \longrightarrow M$ where $V = \{(q, X) : q \in \mathcal{U},\ X \in B_\varepsilon(0) \subseteq T_q M\} \subset M$ such that $\gamma_{(q,X)}(t) = \Gamma(t, q, X)$ is the unique geodesic in M with $\gamma_{(q,X)}(0) = q$, $\gamma_{(q,X)}'(0) = X$

Proof

See notes.

§7 Curvature

Prop 7.1 (/Defn)

For vector fields X, Y, Z on M we define

$$R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

which is a vector field on M. Then $R(\cdot,\cdot)$ is bilinear in its arguments, $R(X,Y)$ is a linear operator and $R(X,Y)Z(p) \in T_pM$ depends only on $X(p)$, $Y(p)$, $Z(p) \in T_pM$.

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$R(X,Y)$ - operator sending vector fields to vector fields is the <u>Riemann curvature operator</u>.

<u>Proof</u>

Properties of ∇ and $[\cdot,\cdot]$

$$\Rightarrow \begin{cases} R(X_1+X_2,Y) = R(X_1,Y)+R(X_2,Y) \\ R(X,Y_1+Y_2) = R(X,Y_1)+R(X,Y_2) \\ R(X,Y)(Z_1+Z_2) = R(X,Y)Z_1 + R(X,Y)Z_2 \end{cases}$$

Let $f: M \to \mathbb{R}$ be smooth

NTS: $R(fX,Y)Z = fR(X,Y)Z$ etc with fX, fY, fZ resp.

$$\begin{aligned}[fX,Y] &= (fX)Y - Y(fX) \\ &= f(XY-YX) - Y(f)X\end{aligned}$$

$$\begin{aligned}R(fX,Y)Z &= \nabla_{fX}\nabla_Y Z - \nabla_Y(f\nabla_X Z) - \nabla_{f[X,Y]-Y(f)X} Z \\ &= f\nabla_X\nabla_Y Z - (f\nabla_Y\nabla_X Z + \cancel{Y(f)\nabla_X Z}) \\ &\quad - (f\nabla_{[X,Y]}Z - \cancel{Y(f)\nabla_X Z}) \\ &= fR(X,Y)Z\end{aligned}$$

Since $R(X,Y)Z = -R(Y,X)Z$

$\Rightarrow R(X,fY)Z = fR(X,Y)Z$

Left to show $R(X,Y)(fZ) = fR(X,Y)Z$.

$$\begin{aligned}R(X,Y)(fZ) &= \nabla_X\nabla_Y(fZ) - \nabla_Y\nabla_X(fZ) - \nabla_{[X,Y]}(fZ) \\ &= \nabla_X(Y(f)Z + f\nabla_Y Z) - \nabla_Y(X(f)Z + f\nabla_X Z) \\ &\quad - ([X,Y](f)Z + f\nabla_{[X,Y]}Z) \\ &= X(Y(f))Z + \cancel{Y(f)\nabla_X Z} + \cancel{X(f)\nabla_Y Z} + f\nabla_X\nabla_Y Z \\ &\quad - (Y(X(f))Z + \cancel{X(f)\nabla_Y Z} + \cancel{Y(f)\nabla_X Z} + f\nabla_Y\nabla_X Z) \\ &\quad - ([X,Y](f)Z + f\nabla_{[X,Y]}Z) \\ &= \cancel{[X,Y](f)Z} - \cancel{[X,Y](f)Z} + fR(X,Y)Z \\ &= fR(X,Y)Z\end{aligned}$$

$\square$

<u>Example</u>

$M = (\mathbb{R}^n, \underbrace{g_0}_{\text{Euc. metric}})$, ∂_i standard orthonormal vector fields,

$[\partial_i, \partial_j] = 0$, $\nabla_{\partial_i}\partial_j = 0 \Rightarrow R(\partial_i,\partial_j)\partial_k = 0$

Since R is linear, $R \equiv 0$ on $\mathbb{R}^n$

Def^n 7.2
We define R by $R(X,Y,Z,W) := g(R(X,Y)Z, W)$ for vector fields X, Y, Z, W on M.
Well defined and at $p \in M$ depends only on $g(p)$ and the values of $X(p), Y(p), Z(p), W(p) \in T_pM$.
R is called the Riemann curvature tensor.

Remark
If X_i are coord. v.f.s then let $R(X_i, X_j, X_k, X_l) =: R_{ijkl}$
If we take geodesic normal coordinates at p so that $g_{ij}(p) = \delta_{ij}$, $\Gamma^k_{ij}(p) = 0$,

⚠ $g_{ij} = \delta_{ij} - \frac{1}{4} R_{ijkl} x_k x_l + O(|x|^3)$.

If $R \equiv 0$ then a Riemannian manifold is said to be flat. E.g. $(\mathbb{R}^n, g_0)$ is flat.
some tori are flat!

Example
On S^2, if we let $X_1 = f_* \partial_\theta$, $X_2 = f_* \partial_\phi$ for $f(\theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, we have $[X_1, X_2] = 0$ and $\nabla_{X_1} X_1 = 0$, $\nabla_{X_2} X_2 = -\sin\theta\cos\phi\, X_1$,
$\nabla_{X_2} X_1 = \nabla_{X_1} X_2 = \cot\theta\, X_2$.

$$\begin{aligned} R(X_1, X_2) X_1 &= \nabla_{X_1} \nabla_{X_2} X_1 - \nabla_{X_2} \nabla_{X_1} X_1 \\ &= \nabla_{X_1} (\cot\theta\, X_2) = -\frac{1}{\sin^2\theta} X_2 + \cot\theta\, \nabla_{X_1} X_2 \\ &= -X_2 \end{aligned}$$

Similarly $R(X_1, X_2) X_2 = \sin^2\theta\, X_1$
So $R(X_1, X_2, X_1, X_1) = 0$, $R(X_1, X_2, X_1, X_2) = -g(X_2, X_2) = -\sin^2\theta$
$R(X_1, X_2, X_2, X_1) = \sin^2\theta$, $R(X_1, X_2, X_2, X_2) = 0$

Suppose instead we considered the orthonormal basis $E_1 = X_1$, $E_2 = X_2 / \sin\theta$, by linearity
$R(E_1, E_2, E_2, E_1) = 1$

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Prop^n

Let X, Y, Z, W be vector fields on M

(a) $R(Y, X, Z, W) = -R(X, Y, Z, W)$

(b) $R(X, Y, W, Z) = -R(X, Y, Z, W)$

(c) $R(Z, W, X, Y) = R(X, Y, Z, W)$

(d) (Bianchi identity) $R(X, Y, Z, W) + R(Y, Z, X, W) + R(Z, X, Y, W) = 0.$

Proof

$R(X, Y)Z = -R(Y, X)Z \Rightarrow$ (a)

Claim (b) $\Leftarrow R(X, Y, Z, Z) = 0$

$$0 = R(X, Y, Z+W, Z+W)$$
$$= R(X, Y, Z, W) + R(X, Y, W, Z)$$

$$g(\nabla_X \nabla_Y Z, Z) = X(g(\nabla_Y Z, Z)) - g(\nabla_Y Z, \nabla_X Z).$$
$$= \tfrac{1}{2} X(Y(g(Z, Z))) - g(\nabla_Y Z, \nabla_X Z)$$

$$g(\nabla_{[X,Y]} Z, Z) = \tfrac{1}{2}[X, Y](g(Z, Z))$$

$$g(R(X, Y)Z, Z) = \tfrac{1}{2} X(Y(g(Z, Z))) - \tfrac{1}{2} Y(X(g(Z, Z))) - \tfrac{1}{2}[X, Y](g(Z, Z))$$
$$= 0 \quad \text{so (b) is true.}$$

(d) $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y$

$$= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$
$$+ \nabla_Y \nabla_Z X - \nabla_Z \nabla_Y X - \nabla_{[Y,Z]} X$$
$$+ \nabla_Z \nabla_X Y - \nabla_X \nabla_Z Y - \nabla_{[Z,X]} Y$$
$$= \nabla_X [Y, Z] + \nabla_Y [Z, X] + \nabla_Z [X, Y]$$
$$- \nabla_{[Y,Z]} X - \nabla_{[Z,X]} Y - \nabla_{[X,Y]} Z$$
$$= [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

(c) exercise. □

$$R(X_i, X_j) X_k = R_{ijk}^{\;\;\;t} X_t$$
$$R_{ijkl} = g(R(X_i, X_j) X_k, X_l) = g(R_{ijk}^{\;\;\;t} X_t, X_l)$$
$$= R_{ijk}^{\;\;\;t} g_{tl}$$

Furthermore writing $\partial_l \Gamma_{ij}^{\;k} = X_l(\Gamma_{ij}^{\;k})$ we have that

$$R_{ijk}^{\;\;\;l} = \partial_i \Gamma_{jk}^{\;l} - \partial_j \Gamma_{ik}^{\;l} + \Gamma_{im}^{\;l} \Gamma_{jk}^{\;m} - \Gamma_{jm}^{\;l} \Gamma_{ik}^{\;m} \quad \text{(note sum over } m\text{)}$$

Def^n

Let $\sigma = \text{span}\{X, Y\} \subset T_pM$ be a 2-plane.
The *sectional curvature* of σ is given by

$$K(\sigma) = K(X,Y) = \frac{R(X,Y,Y,X)}{g(X,X)g(Y,Y) - g(X,Y)^2}$$

This is well defined.

Take any other basis of the form
$\{aX+bY, cX+dY\}$ s.t. $ad-bc \neq 0$ then
$R(aX+bY, cX+dY, cX+dY, aX+bY) = (ad-bc)^2 R(X,Y,Y,X)$
Similarly $g(aX+bY, aX+bY)g(cX+dY, cX+dY) - g(aX+bY, cX+dY)^2$
$= (ad-bc)[g(X,X)g(Y,Y) - g(X,Y)^2]$

Prop^n

Let $\bar{R}$ be s.t. it has properties a, b, c, d from previous prop^n. Suppose that $\forall \sigma \in \text{Span}\{X,Y\} \subset T_pM$ we have

$$\bar{K}(\sigma) := \frac{\bar{R}(X,Y,Y,X)}{g(X,X)g(Y,Y) - g(X,Y)^2} = K(\sigma).$$

Then $R = \bar{R}$

Proof

Since $K = \bar{K}$, we have $\forall X, Y \in T_pM$
$R(X,Y,Y,X) = \bar{R}(X,Y,Y,X)$
$R(X+Z, Y, Y, X+Z) = \bar{R}(X+Z, Y, Y, X+Z) \quad \forall X, Y, Z \in T_pM$
$LHS = R(X,Y,Y,X) + R(Z,Y,Y,Z) + 2R(X,Y,Y,Z)$
$[R(Z,Y,Y,X) = R(Y,X,Z,Y) = -R(X,Y,Z,Y) = +R(X,Y,Y,Z)]$
$RHS = \bar{R}(X,Y,Y,X) + \bar{R}(Z,Y,Y,Z) + 2\bar{R}(X,Y,Y,Z)$
$\Rightarrow R(X,Y,Y,Z) = \bar{R}(X,Y,Y,Z).$
$R(X, Y+W, Y+W, Z) = \bar{R}(X, Y+W, Y+W, Z)$
$\Rightarrow R(X,Y,W,Z) + R(X,W,Y,Z) = \bar{R}(X,Y,W,Z) + \bar{R}(X,W,Y,Z)$
$R(X,Y,Z,W) - \bar{R}(X,Y,Z,W) = R(X,W,Y,Z) - \bar{R}(X,W,Y,Z)$
$= R(Y,Z,X,W) - \bar{R}(Y,Z,X,W) \quad (*)$

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Cyclic permutations leave (*) invariant:

$$(R-\bar{R})(X,Y,Z,W) = (R-\bar{R})(Y,Z,X,W)$$
$$= (R-\bar{R})(Z,X,Y,W)$$

Bianchi identity

$$\Rightarrow 2(R-\bar{R})(X,Y,Z,W) = (R-\bar{R})(Y,Z,X,W) + (R-\bar{R})(Z,X,Y,W)$$
$$= -(R-\bar{R})(X,Y,Z,W)$$
$$\Rightarrow 3(R(X,Y,Z,W) - \bar{R}(X,Y,Z,W)) = 0$$

□

Example S^2.

$\sigma = T_p S^2$

$$K(\sigma) = \frac{R(X_1, X_2, X_2, X_1)}{\underset{=1}{g(X_1,X_1)}\,\underset{=\sin^2\theta}{g(X_2,X_2)} - \underset{=0}{g(X_1,X_2)^2}}$$

?

$$= \frac{\sin^2\theta}{\sin^2\theta} = 1$$

* See notes for examples! *

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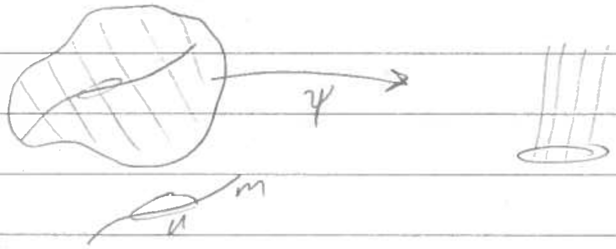
§8 Vector fields revisited

§8.1 Vector bundles

Def^n

A manifold E is a vector bundle over M if
- $\exists$ a smooth surjective map $\pi : E \to M$ s.t.
- $\pi^{-1}(p)$ is a vector space $\forall p \in M$
- $\forall p \in M$ $\exists$ an open set $U \ni p$ and a diffeomorphism $\pi^{-1}(U) \to U \times \mathbb{R}^m$ s.t. $\psi : \pi^{-1}(q) \to \{q\} \times \mathbb{R}^m$ is an isomorphism $\forall q \in U$.



Note: m is the same $\forall p \in M$ and is called the rank of E.
Clearly M is n-dimensional if E is $n+m$ dim
We call E the total space and M the base space.

Examples
1) M manifold, then $M \times \mathbb{R}^m$ is the trivial bundle
Simplest example $S^1 \times \mathbb{R} \cong \{(\cos\theta, \sin\theta, z) : \theta, z \in \mathbb{R}\}$

2) The tangent bundle is a vector bundle of rank n over an n-dim manifold M.

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Def^n

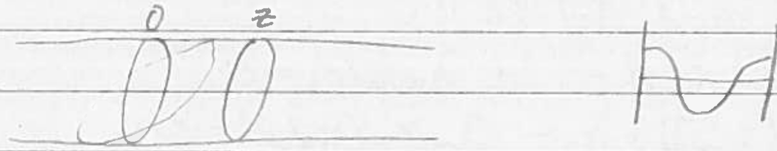
Let E be a vector bundle over M.
A section of E is a smooth map $s: M \to E$ such that $(\pi \circ s)(p) = p$.
We denote the set of sections of E by $\Gamma(E)$, which is naturally a vector space since $s(p) \in \pi^{-1}(p)$, which is a vector space $\forall p \in M$.

Example : A section of TM is a vector field.

Remark: The graph of a section $\{(p, s(p)) : p \in M)$ is clearly diffeomorphic to M using the projection π.

Example

Consider the cylinder $C = S^1 \times \mathbb{R}$, then we have the obvious sections $s: S^1 \to C$ given by $s(\cos\theta, \sin\theta) = (\cos\theta, \sin\theta, z)$ $\forall z \in \mathbb{R}$.
However we also have more interesting sections such as $s(\cos\theta, \sin\theta) = (\cos\theta, \sin\theta, \cos\theta)$



Example

Let $S^2 T_p^* M = \{$symmetric bilinear maps $g_p : T_pM \times T_pM \to \mathbb{R}\}$

Then $S^2 T_p^* M = \bigcup_{p \in M} S^2 T_p^* M$ is a vector bundle of rank $\frac{1}{2}n(n+1)$ over an n-dim manifold M

Def^n

A Riemannian metric is a section of $S^2 T^* M$, i.e. $g \in \Gamma(S^2 T^* M)$ which is positive definite $\forall p \in M$.

Def^n 8.3

A vector bundle of rank m over M is trivial if there exists a diffeo $\Psi: E \to M \times \mathbb{R}^m$ s.t. $\Psi: \pi^{-1}(p) \to \{p\} \times \mathbb{R}^m$ is an isomorphism $\forall p \in M$ (i.e. a bundle isomorphism between E and the trivial bundle $M \times \mathbb{R}^m$).

Prop^n 8.4

A vector bundle of rank m is trivial iff it has m linearly independent sections.

Proof

Assume E is trivial, i.e. $\exists$ a diffeo $\chi: E \to M \times \mathbb{R}^m$ s.t. $\chi(p): \pi^{-1}(p) \to \{p\} \times \mathbb{R}^m$ is an isomorphism $\forall p \in M$.

Then define sections $s_i : M \to E$ $\forall i = 1, \dots, m$ by $s_i(p) = \chi^{-1}(p)(e_i)$.

Clearly $(\pi \circ s_i)(p) = p$ $\forall p \in M$ so there are sections and s_i are smooth since χ (and thus χ^{-1}) are smooth, so $s_i \in \Gamma(E)$.

Moreover $\lambda_1 s_1(p) + \dots + \lambda_m s_m(p) = 0$

$\Rightarrow \lambda_1 \underbrace{\chi_p(s_1(p))}_{=e_1} + \dots + \lambda_m \underbrace{\chi_p(s_m(p))}_{=e_m} = 0$

$\Rightarrow \lambda_1 = \dots = \lambda_m = 0$

So the sections s_i are linearly independent $\forall p \in M$.

Suppose instead that we have linearly independent $s_i \in \Gamma(E)$ for $i = 1, \dots, m$.

Since $\{e_i\}$ form a basis of $\mathbb{R}^m$ we can define $\chi: M \times \mathbb{R}^m \to E$ via $\chi(p, \lambda_1 e_1 + \dots + \lambda_m e_m) = (p, \underbrace{\lambda_1 s_1(p) + \dots + \lambda_m s_m(p)}_{\in \pi^{-1}(p)})$

Clearly $\chi: \{p\} \times \mathbb{R}^m \to \pi^{-1}(p)$ is a well-defined isomorphism and $\pi \circ \chi(p, x) = p$ so χ is a bijection.

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Clearly χ is smooth and its inverse is smooth so this gives the required bundle isomorphism. $\square$

§8.2 Integral curves

Given a curve $\alpha : (-\varepsilon, \varepsilon) \to M$ we can define $\alpha'(t) \in T_{\alpha(t)}M$ by $\alpha'(t) = \alpha_t'(0)$ where $\alpha_t(s) = \alpha(s+t)$.

The map $t \mapsto \alpha'(t)$ from $(-\varepsilon, \varepsilon)$ into TM is smooth, so defines a vector field α' along α.

Let $X \in \Gamma(TM)$ and $p \in M$.
Then there exists a unique curve $\alpha_p : (-\varepsilon, \varepsilon) \to M$ s.t. $\alpha(0) = p$ and $\alpha_p'(t) = X(\alpha_p(t))$.

Why? Take a chart (U, φ) around p so we can write $\varphi \circ \alpha_p(t) = (x_1(t), \ldots, x_n(t))$

? $\boxed{x \text{ or } X}$

$\varphi_*(x) = \sum_{i=1}^n a_i \partial_i$ then we have

$$\varphi_*(\alpha_p'(t)) = (\varphi \circ \alpha_p)'(t) = \sum_{i=1}^n x_i'(t)\, \partial_i$$

and $\varphi_*(X)(\alpha_p(t)) = \sum_{i=1}^n a_i((x_1(t), \ldots, x_n(t))\partial_i$

so we have the ODE

$$x_i'(t) = a_i(x_1(t), \ldots, x_n(t))$$

with the initial condition $(x_1, \ldots, x_n)(0) = \varphi(p)$.

These curves are called integral curves of X

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Recall

$X \subset \Gamma(TM)$

Integral curve through p

$\alpha : (-\varepsilon, \varepsilon) \to M$ s.t. $\alpha(0) = p$, $\alpha'(t) = X(\alpha(t))$.

Examples

$X_i = \partial_i$ on $\mathbb{R}^n$

$\alpha_p(t) = (x_1(t), \dots, x_n(t))$

$\alpha_p'(t) = (x_1'(t), \dots, x_n'(t))$

$= (0, \dots, 1, \dots, 0)$

↖ ith position

$x_j = c_j \qquad j \neq i$

$x_i(t) = c_i + t$

Example

Let $X = x_1 \partial_2 - x_2 \partial_1$ on $\mathbb{R}^3$

and let $(a_1, a_2, a_3) \in \mathbb{R}^3$

The integral curve $\alpha(t) = (x_1(t), x_2(t), x_3(t))$

of X through x satisfies

$x_1' \partial_1 + x_2' \partial_2 + x_3' \partial_3 = x_1(t) \partial_2 - x_2(t) \partial_1$

$\Rightarrow x_1'(t) = -x_2(t), \quad x_2'(t) = x_1(t), \quad x_3'(t) = 0.$

$\Rightarrow x_1''(t) = -x_1(t) \quad \Rightarrow x_1(t) = A\cos(t) + B\sin(t)$

$\Rightarrow x_2(t) = A\sin(t) - B\sin(t)$

$$\Rightarrow \begin{cases} x_1(t) = a_1 \cos t - a_2 \sin t \\ x_2(t) = a_2 \cos t - a_1 \sin t \\ x_3(t) = a_3 \end{cases}$$

Note: X restricts to a vector field on the cylinder

$C = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 = 1\}$

Why? - check $X \in \operatorname{Ker} D(x_1^2 + x_2^2 - 1)$

$\Rightarrow$ Integral curves of X starting on C stay on C.

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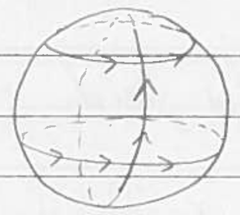
Ex.



§8.3 Flow

Observe the map $(t,q) \longmapsto \alpha_q(t)$ from $(-\varepsilon,\varepsilon)\times V$ into M is smooth by theory of ODEs.

Def^n 8.6

Let $X \in \Gamma(TM)$ and $p \in M$.
Let $V \ni p$ be open st. we have integral curves $\alpha_p : (-\varepsilon,\varepsilon) \to M$ of X through q $\forall q \in V$.
We define the flow of X on V as the family of smooth maps

$$\{\phi_t^X : V \to M \mid t \in (-\varepsilon,\varepsilon)\}, \ \phi_t^X(q) = \alpha_q(t).$$

Note $\phi_0(q) = Id_V$.

Example

$X = \partial_i$ on $\mathbb{R}^n$
$\Rightarrow \alpha_q(t) = q + te_i$
$\Rightarrow \phi_p^{\partial_i}(t)$ is just translation by t in e_i-direction.

Example

Recall the vector field X on the cylinder

$$C = \{(\cos\theta, \sin\theta, z) \in \mathbb{R}^3 \mid \theta, z \in \mathbb{R}\}$$

The flow of X on C is

$$\begin{aligned}\phi_t^X(\cos\theta, \sin\theta, z) &= (\cos\theta\sin t - \sin\theta\cos t, \sin\theta\cos t + \cos\theta\sin t, z) \\ &= (\cos(\theta+t), \sin(\theta+t), z)\end{aligned}$$

Now consider instead the vector field

$Y = x_1 \partial_2 - x_2 \partial_1 + \partial_3$

which restricts again to a vector field on C.

The integral curves satisfy

$x_1'(t) = -x_2(t), \quad x_2'(t) = x_1(t), \quad x_3'(t) = 1$

so the flow of Y is

$\phi_t^Y(\cos\theta, \sin\theta, z) = (\cos(\theta+t), \sin(\theta+t), z+t)$

i.e. screw motion. 

Prop^n 8.7

Let $p \in M$ and let $\{\phi_t^X : V \to M : t \in (-\varepsilon, \varepsilon)\}$ be the flow of $X \in \Gamma(TM)$ on $V \ni p$.

Then $\phi_t^X \circ \phi_{t'}^X = \phi_{t+t'}^X$ if both sides are well defined and ϕ_t^X is a local diffeo around p.

Proof

Note $\phi_t^X \circ \phi_{t'}^X(q) = \alpha_{\alpha_q(t')}(t)$

and $\phi_{t+t'}^X(q) = \alpha_q(t+t')$

Note $\frac{d}{dt} \alpha_q(t+t') = X(\alpha_q(t+t'))$

$\frac{d}{dt} \alpha_{\alpha_q(t')}(t) = X(\alpha_{\alpha_q(t')}(t))$

and $\alpha_q(t+t')\big|_{t=0} = \alpha_q(t'), \quad \alpha_{\alpha_q(t')}(t)\big|_{t=0} = \alpha_q(t')$

so by uniqueness of solutions to ODEs, we have that $\alpha_q(t+t') = \alpha_{\alpha_q(t')}(t) \quad \forall t \in (-\varepsilon, \varepsilon)$.

Note $\phi_{-t}^X \circ \phi_t^X = \phi_0^X = id \Rightarrow \phi_t^X$ is a local diffeo.

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§8.4 Lie derivative

Let $X, Y \in \Gamma(TM)$, $p \in M$ and consider ϕ_t^X the flow of X around p.

We can see how Y changes around p wrt X.

First look at Y along an integral curve $\alpha_p(t)$ of X, i.e.

$$Y(\phi_t^X(p)) \in T_{\phi_t^X(p)}M$$

Recall $\phi_{-t}^X \circ \phi_t^X = id \Rightarrow \phi_{-t}^X(\phi_t^X(p)) = p$

$$\Rightarrow d(\phi_{-t}^X)_{\phi_t^X(p)} : T_{\phi_t^X(p)}M \to T_pM$$

$$\Rightarrow (\phi_{-t}^X)_*(Y(\phi_t^X(p))) \in T_pM$$

Def^n 8.8

Given $X, Y \in \Gamma(TM)$ we define the Lie derivative of Y wrt. X by

$$\mathcal{L}_X(Y(p)) = \lim_{t\to 0} \frac{(\phi_{-t}^X)_*(Y(\phi_t^X(p))) - Y(p)}{t}$$

where $\{\phi_t X ; t \in (-\varepsilon, \varepsilon)\}$ is the flow of X near p.

Example

Let $Y = \sum b_j \partial_j$ be a vector field on $\mathbb{R}^n$.

We know $\phi_t^{\partial_i}(p) = p + te_i$ so $(\phi_t^{\partial_i})_* = id$.

Hence $\mathcal{L}_{\partial_i} Y(p) = \lim_{t\to 0} \frac{(\phi_{-t}^{\partial_i})_*(Y(\phi_t^{\partial_i}(p))) - Y(p)}{t}$

$$= \lim_{t\to 0} \frac{Y(p+te_i) - Y(p)}{t} = \sum_{j=1}^n \partial_i b_j \partial_j$$

$\Rightarrow \mathcal{L}_{\partial_i} \partial_j = 0$

and if $X = x_1\partial_2 - x_2\partial_1$ then

$\mathcal{L}_{\partial_1} X = \partial_2$, $\mathcal{L}_{\partial_2} X = \partial_1$, $\mathcal{L}_{\partial_3} X = 0$

Prop^n

$\mathcal{L}_X Y = [X, Y]$

So we have the properties

$\mathcal{L}_X Y = -\mathcal{L}_Y X$

$\mathcal{L}_X (Y+Z) = \mathcal{L}_X Y + \mathcal{L}_X Z$

$\mathcal{L}_X (fY) = f\mathcal{L}_X Y + X(f) Y$

Note: we don't get a connection since

$\mathcal{L}_{fX} Y = f\mathcal{L}_X Y - Y(f) X$

Example

$X = \sum a_j \partial_j$ on $\mathbb{R}^n$

$$\mathcal{L}_X \partial_i = (-\mathcal{L}_{\partial_i} X) = \sum_{j=1}^{n} a_j \mathcal{L}_{\partial_i} \partial_j - \sum_{j=1}^{n} \partial_i a_j \partial_j$$

$$= -\sum_{j=1}^{n} \partial_i a_j \partial_j$$

For example, let $X = x_1 \partial_2 - x_2 \partial_1$

$\mathcal{L}_X \partial_1 = -\partial_2$, $\mathcal{L}_X \partial_2 = -\partial_1$, $\mathcal{L}_X \partial_3 = 0$.

Def 8.10

Let $X \in \Gamma(TM)$ and let g be a Riemannian metric. Then $\mathcal{L}_X g(p) = \lim_{t\to 0} \frac{(\phi_t^X)_* g_{\phi_t^X(p)} - g_p}{t}$

where ϕ_t^X is the flow of X around p.

We call vector fields X s.t. $\mathcal{L}_X g \equiv 0$ killing fields

? [Isometries $\leftrightarrow$ killing fields]

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Example

Recall that the flow of ∂_i is a translation on $\mathbb{R}^n$ $\Rightarrow (\phi_t^X)^* g_0 = g_0 \Rightarrow \mathcal{L}_{\partial_i} g_0 \equiv 0$.
Similarly the vector fields $X_1 = x_2\partial_1 - x_2\partial_1$, $X_2 = \ldots$ etc define rotations around the coordinate axes
$\Rightarrow (\phi_t^X)^* g_0 = g_0 \Rightarrow \mathcal{L}_{X_i} g_0 \equiv 0$.

§9 Riemannian metrics revisited

§9.1 Isometries and local isometries

Def 9.1

A smooth map $f : (M, g) \to (N, h)$ between Riemannian manifolds is an isometry if f is a diffeo and $g = f^* h$.

Clearly the isometries of (M, g) form a group, in fact a subgroup of $\mathrm{Diff}(M)$, which we call $\mathrm{Isom}(M)$.

Note

Geodesics and curvature are defined purely by the Riemannian metric, they are invarient under isometries.

Example

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be linear, i.e. $f(x) = Ax$
$\Rightarrow f^* g_0 = g_0$ (i.e. $g_0(Ax, Ay) = g_0(x, y)$) $\Leftrightarrow A \in O(n)$
Note that translations are isometries.
Hence (modulo showing every isometry of $\mathbb{R}^n$ is affine linear) we have $\mathrm{Isom}(\mathbb{R}^n) = O(n) \times \mathbb{R}^n$.

Example

Clearly $\mathrm{Isom}(S^n, g_{\text{standard}}) = O(n+1)$

Example

By problem sheet 5 we have $\mathrm{Isom}(H^n, g)$, where g is the hyperbolic metric, is given by

$$\mathrm{Isom}(H^n, g) = O^+(n+1) = \left\{ A = (a_{ij}) \in M_{n+1}(\mathbb{R}) \;\middle|\; \begin{matrix} A^T G A = G, \\ a_{n+1\,n+1} > 0 \end{matrix} \right\}$$

where $G = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix}$

Example

Recall $SU(n)$ (i.e. $A^*A = 1$, $\det A = 1$)

then $T_A SU(n) = \{ B \in M_n(\mathbb{C}) \mid \overline{A}^T B + \overline{B}^T A = 0,\ tr(\overline{A}^T B) = 0 \}$

$= \{ AX \in M_n(\mathbb{C}) : X + \overline{X}^T = 0,\ tr(X) = 0 \}$

Claim: g given by $g_A(B, C) = -tr(\overline{A}^T B \overline{A}^T C)$

$= -tr(X, Y) = g_A(AX, AY)$

$\forall A \in SU(n)$, $B = AX$, $C = AY \in T_A SU(n)$

is a Riemannian metric.

Note $g_A(B, C) = tr(XY) = tr(YX) = g_A(C, B)$

$tr(X, Y) = tr(\overline{X}, \overline{Y}) = tr(X^T, Y^T) = tr((YX)^T)$

$= tr(YX) = tr(XY)$

It is also positive definite: if we write $x_1, \dots, x_n$ for the columns of X,

$-tr(X^2) = tr(\overline{X}^T X) = \sum_{j=1}^{n} |x_j|^2$

Let $L_C : SU(n) \to SU(n)$ be given by $L_C(A) = CA$.

Claim: L_C is an isometry

$AX, AY \in T_A SU(n)$

$(L_C^* g)(AX, AY) = g_{CA}((L_C)_* AX, (L_C)_* AY)$

$= g_{CA}(CAX, CAY) = -tr(XY) = g_A(AX, AY)$

$\Rightarrow g$ is left invariant.

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Moreover $R_C : SU(n) \to SU(n)$ given by $R_C A = AC$ is an isometry since

$$\begin{aligned}(R_C g)_A (AX, AY) &= g_{AC}(AXC, AYC) \\ &= -tr(\overline{AC}^T AXC \, \overline{AC}^T AYC) \\ &= -tr(\overline{C}^T XYC) = -tr(YX) = -tr(XY) \\ &= g_A(AX, AY)\end{aligned}$$

so g is also right invariant.

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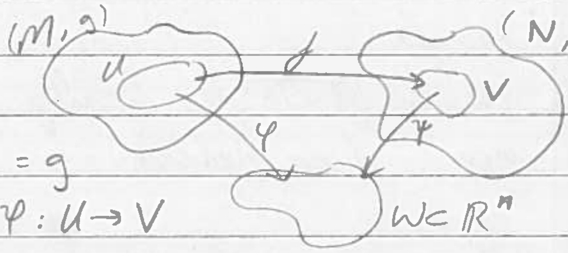
Recall

$f : (M, g) \to (N, h)$ is a local isometry if for $q \in N$ s.t. $f(p) = q$, $\exists\, U$ open, $p \in U$, V open, $q \in V$ s.t. $f : U \to V$ is an isometry (i.e. $f : U \to V$ diffeo and $f^* h = g$).

Assume we have charts (U, φ) on (M, g) and (V, ψ) on (N, h) s.t. $\varphi(U) = \psi(V) = W \subset \mathbb{R}^n$ and $\underset{(*)}{(\varphi^{-1})^* g = (\psi^{-1})^* h}$ on W.



Then
$$\begin{aligned}(\psi^{-1} \circ \varphi)^* h &= \varphi^* \circ (\psi^{-1})^* h \\ &= \varphi^* \circ (\varphi^{-1})^* g = g\end{aligned}$$
so the map $f := \psi^{-1} \circ \varphi : U \to V$ is an isometry.

Note that $(*)$ is equivalent to
$$\begin{aligned}g_{ij} &= g((\varphi^{-1})_* \partial_i, (\varphi^{-1})_* \partial_j) \\ &= (\varphi^{-1})^* g(\partial_i, \partial_j) = (\psi^{-1})^* h(\partial_i, \partial_j) = h_{ij}.\end{aligned}$$

Example

We have minimal surfaces (that is the mean curvature $H = \lambda_1 + \lambda_2 = 0$) in $\mathbb{R}^3$ known as the helicoid
$$M_1 = \{(s\cos t, s\sin t, t) \mid s, t \in \mathbb{R}\}$$

and the catenoid

$$M_2 = \{(\cosh(z)\cos\theta, \cosh(z)\sin\theta, z) \mid z, \theta \in \mathbb{R}\}.$$

Define local coordinates on M_1 by

$$f_1(x_1, x_2) = (\sinh x_1 \cos x_2, \sinh x_1 \sin x_2, x_2)$$

and on M_2 by $f_2(x_1, x_2) = (\cosh x_1 \cos x_2, \cosh x_1 \sin x_2, x_1)$

$$(f_1)_*\partial_1 = \cosh x_1 \cos x_2 \, \partial_1 + \cosh x_1 \sin x_2 \, \partial_2$$
$$(f_1)_*\partial_2 = -\sinh x_1 \sin x_2 \, \partial_1 + \sinh x_1 \cos x_2 \, \partial_2 + \partial_3$$
$$\text{so } (f_1)^* g_0 = \cosh^2 x_1 \, dx_1^2 + (1 + \sinh^2 x_1) dx_2^2$$
$$= \cosh^2 x_1 (dx_1^2 + dx_2^2)$$

$$(f_2)_*\partial_1 = \sinh x_1 \cos x_2 \, \partial_1 + \sinh x_1 \sin x_2 \, \partial_2 + \partial_3$$
$$(f_2)_*\partial_2 = -\cosh x_1 \sin x_2 \, \partial_1 + \cosh x_1 \cos x_2 \, \partial_2$$
$$\text{so } (f_2)^* g_0 = (1 + \sinh^2 x_1) dx_1^2 + \cosh^2 x_1 \, dx_2^2 = \cosh^2 x_1 (dx_1^2 + dx_2^2)$$

$\Rightarrow$ M_1 and M_2 are locally isometric

Example

Pseudosphere is locally isometric to hyperbolic space (see notes).

§ 9.2 Group actions

Example

Given a discrete group G acting freely and properly discontinuously on a manifold M, the quotient map $\pi : M \to M/G$ is a local diffeo. by prop^n 1.9 and hence an immersion by prop^n 2.7.
Thus by prop^n 4.3 if we have a metric h on M/G we get a metric $\pi^* h = g$ on M.

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Thm 9.2
Let G be a discrete group acting freely and properly discontinuously by isometries on a Riem. manifold (M, g), i.e. suppose that the diffeomorphisms f_g on M are isometries $\forall g \in G$. Then there exists a Riem metric h on M/G s.t. $\pi : M \to M/G$ is a local isometry.

Proof
Idea: define h s.t. $\pi^* h = g$ and show it is well defined.
Note $d\pi_p : T_pM \to T_{\pi(p)} M/G$ is an isomorphism $\forall p \in M$ by prop^n 1.9 and π is surjective, so we can define h by
$$h_{\pi(p)}(X, Y) = g_p((d\pi_p)^{-1}X, (d\pi_p)^{-1}Y).$$
Why is this well defined?
Take $q \neq p$ s.t. $\pi(q) = \pi(p)$.
$\Rightarrow q = f_g(p)$ for some $g \in G$
$\Rightarrow \pi(p) = \pi \circ f_g(p)$
So differentiating gives $d\pi_p = d(\pi \circ f_g) = d\pi_{f_p(g)} \circ d(f_g)_p$
$= d\pi_q \circ d(f_g)_p$.
$\Rightarrow d\pi_q = d\pi_p \circ (d(f_g)_p)^{-1}$
$\Rightarrow (d\pi_q)^{-1} = d(f_g)_p \circ (d\pi_p)^{-1}$
$$\begin{aligned}\Rightarrow \quad & g_q((d\pi_q)^{-1}X, (d\pi_p)^{-1}Y) \\ &= g_{f_g(p)}(d(f_g)_p \circ (d\pi_p)^{-1}X, d(f_g)_p \circ (d\pi_p)^{-1}Y) \\ &= (f_g^* g)_p((d\pi_p)^{-1}X, (d\pi_p)^{-1}Y) \\ &= g((d\pi_p)^{-1}X, (d\pi_p)^{-1}Y)\end{aligned}$$
since f_g is an isometry.
Moreover, h is pos. definite since
$$h_{\pi_p}(X, X) = g_p((d\pi_p)^{-1}(X), (d\pi_p)^{-1}(X)) \geq 0$$
and equal to zero iff $X = 0$ since $(d\pi_p)^{-1}$ is an isomorphism. $\square$

Example

Note id and $-id$ are isometries on $\mathbb{R}^{n+1}$, so $\mathbb{R}P^n$, the Möbius band and the Klein bottle obtain Riem. metrics from S^n, the cylinder and $T^2 \subset \mathbb{R}^3$.

Example

We get a flat metric on $\mathbb{R}^n/\mathbb{Z}^n$ (see notes).

Example

By Thm 9.2 the projection $\pi : S^n \to \mathbb{R}P^n = S^n/\mathbb{Z}_2$ is a local isometry.
Therefore, since the condition for being a geodesic is a local one (determined by the Christoffel symbols in a chart) we see that if $\tilde{\gamma}$ is a normalised geodesic in S^n then $\gamma = \pi \circ \tilde{\gamma}$ is a normalised geodesic in $\mathbb{R}P^n$.

Note: $\tilde{\gamma}(t+2\pi) = \tilde{\gamma}(t)$ but $\gamma(t+\pi) = \gamma(t)$ since $\tilde{\gamma}(t+\pi) = -\tilde{\gamma}(\pi)$ and $\pi(\tilde{\gamma}(t)) = \pi(-\tilde{\gamma}(t))$.
By thm 6.5, $\exists!$ geodesic γ in $\mathbb{R}P^n$ through $[p]$ with $\gamma'(0) = X$, $d\pi_p : T_p S^n \to T_{[p]} \mathbb{R}P^n$ is an isomorphism, there exists a unique great circle α through p with $d\pi_p(\alpha'(0)) = X$
$\Rightarrow \pi \circ \alpha$ is a geodesic with $(\pi \circ \alpha)(0) = p$, $(\pi \circ \alpha)'(0) = X$.

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§9.3 Parallel transport

(M, g) a Riemannian manifold

Def^n 9.3

Let α be a curve in M and X a vector field along α (i.e. $X(\alpha(t)) \in T_{\alpha(t)} M$ and $t \mapsto X(\alpha(t))$ is smooth).

We say X is *parallel* (along α) if $\nabla_{\alpha'} X = 0$.

Assume α is contained in coordinate chart (U, φ), write $(\varphi \circ \alpha)(t) = (x_1(t), \ldots, x_n(t))$ and if $X_i = (\varphi^{-1})_* \partial_i$ as usual, then write $X = \sum_{i=1}^{n} a_i X_i$ (think $X_i \circ \alpha$)

Then $\alpha' = \sum_{i=1}^{n} x_i' (\varphi^{-1})_* \partial_i = \sum_{i=1}^{n} x_i' X_i$

Note $\alpha' = \alpha_*(\partial_t)$. Then

$$\nabla_{\alpha'}\left(\sum_i a_i X_i\right) = \sum_{i=1}^{n} \alpha'(a_i) X_i + a_i \nabla_{\alpha'} X_i$$

$$= \sum_i a_i' X_i + a_i \nabla_{\sum_{j=1}^{n} x_j' X_j} X_i$$

$$= \sum_i \left(a_i' X_i + a_i \sum_j x_j' \nabla_{X_j} X_i\right)$$

$$= \sum_i a_i' X_i + a_i \sum_{j,k} x_j' \Gamma_{ij}^{k} X_k$$

$$\Rightarrow \nabla_{\alpha'}\left(\sum_i a_i X_i\right) = \sum_{k=1}^{n} \left(a_k' + \sum_{i,j=1}^{n} \Gamma_{ij}^{k} a_i x_j'\right) X_k \qquad (**)$$

So parallel $\Leftrightarrow$ 1st order ODE (in local coordinates).

Remark

This confirms the geodesic eq^{n}s.

Example

On $\mathbb{R}^n$, $\Gamma_{ij}^k \equiv 0$ $\quad \nabla_{\alpha'} X = \sum_{k=1}^{n} a_k' X_k$

so parallel $\iff$ $a_k' \equiv 0$

Example

Let $X_1 = (f_*(\partial_\theta))$, $X_2 = (f_*(\partial_\phi))$ be the standard vector fields on S^2.

Then $\Gamma_{11}^2 = \Gamma_{11}^1 = 0$

$\Gamma_{22}^1 = -\sin\theta\cos\theta$, $\Gamma_{22}^2 = 0$, $\Gamma_{12}^1 = 0$, $\Gamma_{12}^2 = \cot\theta$.

$\Rightarrow X = a_1 X_1 + a_2 X_2$, then given a curve $\alpha(t) = f(\theta(t), \phi(t))$, we get

$\nabla_{\alpha'} X = (a_1' - (\sin\theta\cos\theta) a_2 \phi') X_1 + (a_2' + \cot\theta (a_1 \phi' + a_2 \theta')) X_2$

So assume $\phi \equiv$ const. along α, i.e. $\theta = t$ and $\phi' \equiv 0$

then $\nabla_{\alpha'} X = a_1' X_1 + (a_2' + \cot(t) a_2) X_2$

$\Rightarrow X_1$ is parallel along α ($\Leftrightarrow \alpha$ geodesic)

but X_2 is not parallel.

Take $\theta \equiv$ const, $\phi = t$

$\Rightarrow \nabla_{\alpha'} X = (a_1' - \sin\theta\cos\theta\, a_2) X_1 + (a_2' + \cot\theta\, a_1) X_2$

$\Rightarrow X_1$ and X_2 are parallel along α iff $\theta \equiv \pi/2$

Example

Discussion on $T^2 \subset \mathbb{R}^3$ - see notes.

Thm 9.4

Let $p, q \in M$ and $\alpha : [0, L] \to M$ be a curve between p and q. Given $X_0 \in T_{\alpha(0)} M = T_p M$ there exists a unique parallel vectorfield X along α s.t. $X(p) = X_0$. The map $\tau_\alpha : T_p M \to T_q M$ given by $\tau_\alpha(X_0) = X(q)$ is an isometry, so an isomorphism s.t.

$$g_p(X_0, Y_0) = g_q(\tau_\alpha(X_0), \tau_\alpha(Y_0))$$

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Proof

Note: it suffices to show this for the case that α is contained in a coordinate chart (U, φ). Otherwise since $[0, L]$ is compact we can cover the image with a finite number of coordinate charts and the result follows by doing it on each sub-interval.

Note: X is parallel iff RHS of $(**)$ $= 0$.
This is a 1st order ODE in n variables $(a_1(t), \dots, a_n(t))$ together with n initial conditions $(a_1(0), \dots, a_n(0)) = X_0$ $\Rightarrow$ solution exists and is unique.

Why is τ_α an isomorphism?
Clearly τ_α is linear (since the sum of two parallel vector fields X, Y is again parallel).

Let $\beta(t) = \alpha(L-t)$ and consider $\tau_\beta : T_qM \to T_pM$.
Thus $\exists!$ vector field Y along β s.t. $Y(q) = X(q)$.

However $\beta'(t) = \alpha'(L-t)$, so $\nabla_{\alpha'} X = 0 \Rightarrow \nabla_{\beta'} X = 0$.
$\Rightarrow X$ is parallel along β.
By uniqueness of Y we have $Y(p) = X_0$.
$\Rightarrow \tau_\beta \circ \tau_\alpha = id_{T_pM}$.

Let $X_0, Y_0 \in T_pM$ and consider X, Y the unique parallel vector fields along α st. $X(p) = X_0$, $Y(p) = Y_0$.
Since $\alpha' = \alpha_*(\partial_t)$ we have

$$\frac{d}{dt} g(X, Y) = \alpha'(g(X,Y)) = g(\underset{\overset{\shortparallel}{0}}{\nabla_{\alpha'} X}, Y) + g(X, \underset{\overset{\shortparallel}{0}}{\nabla_{\alpha'} Y}) = 0$$

$$\begin{aligned}\Rightarrow g_p(X_0, Y_0) &= g_p(X(p), Y(p)) = g(X,Y)(\alpha(0)) \\ &= g(X,Y)(\alpha(L)) = g_q(X(q), Y(q)) = g_q(\tau_\alpha(X_0), \tau_\alpha(Y_0))\end{aligned}$$

$\Rightarrow \tau_\alpha$ is an isometry.

$\square$

Example

Consider S^2 with standard parameterisation $f(\theta, \phi)$. $\alpha(t) = (\sin\theta\cos t, \sin\theta\sin t, \cos\theta)$

$\Rightarrow \alpha' = X_2$, $\quad X = a_1 X_1 + a_2 X_2$

X parallel $\Leftrightarrow$

$\nabla_{\alpha'} X = (a_1' - \sin\theta\cos\theta\, a_2) X_1 + (a_2' + \cot\theta\, a_1) X_2 = 0$

$\Leftrightarrow a_1' = \sin\theta\cos\theta\, a_2$ and $a_2' = -\cot\theta\, a_1$

Differentiating: $a_1'' = -\cos^2\theta\, a_1$

$$\Rightarrow \begin{cases} a_1(t) = a_1(0)\cos(t\cos\theta) + \dfrac{a_2(0)}{\sin\theta}\sin(t\cos\theta) \\ a_2(t) = a_2(0)\cos(t\cos\theta) - \dfrac{a_1(0)}{\sin\theta}\sin(t\cos\theta) \end{cases}$$

So parallel transport along α is the map

$$\tau_\alpha(a_1 X_1 + a_2 X_2) = \left(a_1\cos(t\cos\theta) + \frac{a_2}{\sin\theta}\sin(t\cos\theta)\right) X_1 + \left(\frac{-a_1}{\sin\theta}\sin(t\cos\theta) + a_2\cos(t\cos\theta)\right) X_2$$

§10 Geodesics revisited

Recall

Let $p \in (M, g)$, there is $\varepsilon > 0$ and $U \ni p$ s.t. $\forall q \in U$ and $X \in T_p M$ with $|X| = \sqrt{g(X,X)} < \varepsilon$ there exists a unique geodesic $\gamma_{(q,X)} : (-2, 2) \to M$ s.t. $\gamma_{(q,X)}(0) = q$ and $\gamma'_{(q,X)}(0) = X$.

Let $V = \{(q, X) : q \in U, X \in B_\varepsilon(0) \subset T_q M\}$.

§10.1 Exponential map

Def^n

We define the smooth map $\exp_p : V \to M$ by $\exp_q(q, X) = \gamma_{(q,X)}(1)$. This is called the exponential map.

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We often restrict $\exp_p : B_\varepsilon(0) \subseteq T_pM \to M$ by
$\exp_p(X) = \gamma_{(p,X)}(1)$.

Note $\gamma_{(p,tX)}(1) = \exp_p(tX) = \gamma_{(p,X)}(t)$.

Examples

1). $(\mathbb{R}^n, g)$, $\gamma_{(p,X)}(t) = p + tX$
$\Rightarrow \exp_p(X) = p + X$

2). Consider $T^n \subset \mathbb{R}^{2n}$
If $p = (\cos\theta_1, \sin\theta_1, \dots, \cos\theta_n, \sin\theta_n)$
$X = f_*\left(\sum_{i=1}^n a_i \partial_i\right)$ where $f(x_1, \dots, x_n) = (\cos x_1, \sin x_1, \dots, \cos x_n, \sin x_n)$
then we have
$\gamma_{(p,X)}(t) = (\cos(a_1 t + \theta_1), \sin(a_1 t + \theta_1), \dots, \cos(a_n t + \theta_n), \sin(a_n t + \theta_n))$
$\Rightarrow \exp_p(X) = (\cos(a_1 + \theta_1), \sin(a_1 + \theta_1), \dots, \cos(a_n + \theta_n), \sin(a_n + \theta_n))$.

3). On S^2 we have geodesics for $c \in \mathbb{R}$ given by
$\gamma(t) = (\sin(ct + \theta_0)\cos(\phi_0), \sin(ct + \theta_0)\sin(\phi_0), \cos(ct + \theta_0))$
They all start at $p = \gamma(0) = f(\theta_0, \phi_0)$ and $\gamma'(0) = cX_1$
$\Rightarrow \exp_p(cX_1) = \gamma(1) = (\sin(c + \theta_0)\cos(\phi_0), \sin(c + \theta_0)\cos(\phi_0), \cos(c + \theta_0))$
Note $\exp_p(2\pi X_1) = \exp_p(X_1)$
So the exponential map is not injective.

Example

We saw on the Hyperbolic plane (H^2, g) that
g? the geodesics for $c \in \mathbb{Z}$ are given by
$\gamma(t) = (x_1, e^{ct} x_2)$ $\qquad g = \dfrac{dx^2 + dy^2}{x^2 + y^2}$
$\Rightarrow \gamma(0) = (x_1, x_2) = p$, $\gamma'(0) = c\,\partial_2$
$\Rightarrow \exp_p(c\,\partial_2) = \gamma(1) = (x_1, x_2 e^c)$.

Example
On $SO(n)$ (and many other matrix Lie groups) the exponential map $\exp_I : T_I\, SO(n) \to SO(n)$ is $\exp_I(A) = \exp(A)$

$\left[\overset{\text{metric}}{|A|^2 = tr(A^T A)} \right]$

Thm 10.2
Given $p \in M$ there exists an open set $W \ni p$ and $\delta > 0$ s.t. $\forall q \in W$
$\exp_q : B_\delta(0) \subset T_q M \longrightarrow \exp_q(B_\delta(0)) \supseteq W$
is a diffeomorphism on its image.

Proof
Key: calculate the differential of the exponential map at 0.
We have $\exp_p : T_pM \longrightarrow M$
$\Rightarrow d(\exp_p)_0 : T_0(T_pM) \longrightarrow T_{\exp_p(0)} M = T_pM$
Note T_pM is a vector space, so we can identify $T_0(T_pM) = T_pM$.
If $X \in T_pM$, then

$$d(\exp_p)_0(X) = \frac{d}{dt} \exp_p(tX) \Big|_{t=0}$$

$$= \frac{d}{dt}\Big|_{t=0} \gamma_{(p,X)}(t) = X$$

$\Rightarrow d(\exp_p)_0 = id$

Rest:

Let U, V be as in Thm 6.5. Define $F : V \subset TM \to M \times M$ by $F(q, X) = (q, \exp_q(X))$.
Hence $dF_{(p,0)} : T_{(p,0)} TM \cong T_pM \times T_0(T_pM) \cong T_pM \times T_pM$
$\longrightarrow T_pM \times T_pM$
can be written as $dF_{(p,0)} \begin{pmatrix} I & 0 \\ A & I \end{pmatrix}$ for some matrix A.
?

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$\Rightarrow dF_{(p,0)}$ is an isomorphism
so F is a local diffeo by Prop^n 2.7.

Thus $\exists \varepsilon > \delta > 0$ and open sets $\tilde{U} \subset U$, $\tilde{W} \subset M \times M$ s.t. $(p,p) \in \tilde{W}$ and if $\tilde{V} = \{(q, X)$ s.t. $q \in \tilde{U}$, $X \in B_\delta(0) \subset T_q M\} \subseteq V$ then $F: \tilde{V} \to \tilde{W}$ is a diffeo.

Choose an open set $W \ni p$ s.t. $W \times W \subseteq \tilde{W}$
Then if $q \in W$ we have $W \subseteq \exp_q(B_\varepsilon(0))$ as required. $\square$

06-12-18 ## §10.2 Length and normal nbhds

Def^n 10.3
The length of a piecewise smooth curve $\alpha: [0, l] \to M$ is $L(\alpha) = \int_0^l |\alpha'(t)|\,dt = \int_0^l \sqrt{g(\alpha'(t), \alpha'(t))}\,dt$

Note this is invarient under reparameterisation.

For normalised geodesic $\gamma: [0, l] \to M$, $L(\gamma) = l$ since $|\gamma'(t)| = 1$.

We say a curve α is (length) minimising if $L(\alpha) \leq L(\beta)$ for all piecewise smooth curves $\beta: [0, l'] \to M$ st. $\alpha(0) = \beta(0)$ and $\alpha(l) = \beta(l')$.

Examples See notes.

Def^n 10.4
An open set $U \subseteq M$ with $p \in U$ is called a normal nbhd of p if $\exists$ an open set $V \subseteq T_p M$ s.t. $\exp_p: V \to U$ is a diffeo.

If $\overline{B_\varepsilon(0)} \subseteq V$ we define $B_\varepsilon(p) = \exp_p(B_\varepsilon(0))$ to be the geodesic ball of radius ε centred at p and $\partial B_\varepsilon(p) = S_\varepsilon(p)$ to be the geodesic sphere of radius ε around p.

An open set $W \subseteq M$ is called a totally normal neighbourhood if it is a normal nbhd $\forall p \in W$.

Remark
Thm 10.2 was existence of totally normal nbhds.

We call all geodesics emanating from p radial geodesics.

Note : By Thm 10.2, given a pt. $q \in B_\varepsilon(p)$ (ε sufficiently small) then all radial geodesics from p to q are the unique geodesic connecting p to q in $B_\varepsilon(p)$.

Examples
1). $p \in \mathbb{R}^n$, $X \in T_p\mathbb{R}^n \cong \mathbb{R}^n$, then $\exp_p(X) = p + X$ so $\exp_p$ is defined $\forall X \in T_p\mathbb{R}^n$, so $\exp_p$ is a diffeo between $T_p\mathbb{R}^n$ and $\mathbb{R}^n$.

2). Let N be the north pole in S^n, $\exp_N$ is the map which follows a great circle, and if $X \in T_N S^n$ s.t. $|X| = \pi$, then $\exp_N(X) = S$ (south pole)
Hence $\exp_N : B_\pi(0) \subseteq T_N S^n \to S^n \setminus \{S\}$ is a diffeo, so $S^n \setminus \{S\}$ is a normal nbhd of N.

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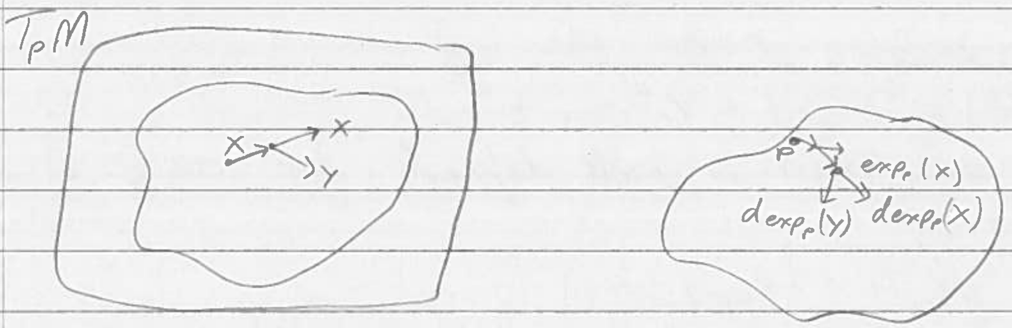
§10.3 Geodesics are locally length minimising

Lemma 10.5 (Gauss lemma)

Let $p \in M$ and $X \in T_pM$ s.t. $\exp_p(X)$ is defined. If $Y \in T_X(T_pM) \cong T_pM$, then

$$g_{\exp_p(X)}(d(\exp_p)_X(X), d(\exp_p)_X(Y)) = g_p(X, Y).$$



Proof

Write $Y = Y^T + Y^\perp$ where $Y^T \in \mathrm{Span}(X)$, $Y^\perp \in \{\mathrm{Span}\, X\}^\perp$

Note the geodesic $\gamma_{(p,X)}$ so that

$\gamma_{(p,X)}(0) = p$, $\exp_p(X) = \gamma_{(p,X)}(1)$

satisfies

$$\gamma_{(p,X)}(t) = \exp_p(tX), \quad \gamma'_{(p,X)}(t) = d(\exp_p)_{tX}(X)$$

$$\Rightarrow \gamma'_{(p,X)}(0) = d(\exp_p)_0(X) = X.$$

also,

$$\gamma'_{(p,X)}(1) = d(\exp_p)_X(X)$$

Moreover, $$|\gamma'_{(p,X)}(t)|^2 = g_{\gamma_{(p,X)}(t)}(\gamma'_{(p,X)}(t), \gamma'_{(p,X)}(t))$$

$$= g_{\exp_p(tX)}(d(\exp_p)_{tX}(X), d(\exp_p)_{tX}(X)).$$

This is constant in t, by def^n of a geodesic.

Thus choosing $t=0$ and $t=1$

$$g_{\exp_p(X)}(d(\exp_p)_X(X), d(\exp_p)_X(X)) = g_p(X, X)$$

Since $Y^T \in \text{Span}(X)$ (i.e. $Y^T = \lambda X$ for some $\lambda \in \mathbb{R}$)
$\Rightarrow g_{\exp_p(X)}(d(\exp_p)_X(X), d(\exp_p)_X(Y^T)) = g_p(X, Y)$

So since $g_p(X, Y^\perp) = 0$ by def^n it is enough to show that
$g_{\exp_p(tX)}(d(\exp_p)_X(X), d(\exp_p)_X(Y^\perp)) = 0$

There exists $\varepsilon > 0$ s.t. if
$X(t) = X\cos t + Y^\perp \sin t$
then $\exp_p(sX(t))$ is well defined for $s \in [0,1]$ and $t \in (-\varepsilon, \varepsilon)$.
Let $f(s,t) = \exp_p(sX(t))$
so $s \longmapsto f(s,t) = \exp_p(sX(t))$ are radial geodesics.

We can differentiate f to get
$\frac{\partial f}{\partial s} = d(\exp_p)_{sX(t)}(X(t))$ and $\frac{\partial f}{\partial t} = d(\exp_p)_{sX(t)}(sX'(t))$.

Since $f(1,0) = \exp_p(X)$ and $X'(0) = Y^\perp$, we see that
$g_{\exp_p(X)}(d(\exp_p)_X(X), d(\exp_p)_X(Y^\perp)) = g_{\exp_p(X)}\left(\frac{\partial f}{\partial s}(1,0), \frac{\partial f}{\partial t}(1,0)\right)$.

Now the covarient derivative along curves where t is constant is $\nabla_{f_* \partial_s} \frac{\partial f}{\partial s} = 0$ (geodesic eq^n)

In a coordinate chart (U, φ) around $f(s_0, t_0)$
s.t. $\varphi \circ f(s,t) = (x_1(s,t), \dots, x_n(s,t))$
we can calculate $\nabla_{f_* \partial_s} \frac{\partial f}{\partial t}$ in the chart (U, φ) as

$$\nabla_{f_* \partial_s} \sum_{j=1}^n \frac{\partial x_j}{\partial t} \partial_j = \sum_{j=1}^n \frac{\partial^2 x_j}{\partial s \partial t} \partial_j + \sum_{j,k=1}^n \frac{\partial x_j}{\partial s} \frac{\partial x_k}{\partial t} \underbrace{\nabla_{\partial_k} \partial_j}_{= \sum_i \Gamma_{kj}^i \partial_i}$$

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which is symmetric in s and t, so
$\nabla_{f_*\partial_s} \frac{\partial f}{\partial t} = \nabla_{f_*\partial_t} \frac{\partial f}{\partial s}$

So we get (as in thm 9.4)

$$\frac{\partial}{\partial s} g\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right) = g\Big(\underbrace{\nabla_{f_*\partial_s} \frac{\partial f}{\partial s}}_{0}, \frac{\partial f}{\partial t}\Big) + g\left(\frac{\partial f}{\partial s}, \nabla_{f_*\partial_s} \frac{\partial f}{\partial t}\right)$$

$$= g\left(\nabla_{f_*\partial_t} \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right) = \frac{1}{2} \frac{d}{dt} \Big|\underbrace{\frac{\partial f}{\partial s}}_{\text{const}}\Big|^2$$

$$= 0$$

$\Rightarrow g\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)(1,0) = g\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)(s,0) \quad \forall s \geq 0$

Now $\frac{\partial f}{\partial t}(s,0) = d(\exp_{sX})(sY^\perp) \to 0$ as $s \to 0$.

$\Rightarrow g\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)(1,0) = g_{\exp_p(x)}\left(d(\exp_p)_x(X), d(\exp_p)_x(Y^\perp)\right) = 0$

$\square$

07-12-18 §10.3 Geodesics locally length minimising

Thm 10.6

Geodesics $\gamma : [0,L] \to M$ in $B_\varepsilon(p)$ with $\gamma(0) = p$ are minimising. Moreover if $\alpha : [0,L] \to M$ is a piecewise smooth curve s.t. $\alpha(0) = \gamma(0)$, $\alpha(L) = \gamma(L)$ and $L(\alpha) = L(\gamma)$ then $\alpha([0,L]) = \gamma([0,L])$.

Proof (non examinable)
Let $\alpha : [0,L] \to M$ be a comparison curve.
If $\gamma(0) = \gamma(L)$ then γ is const. $\Rightarrow L(\gamma) = 0$ ($\Rightarrow \gamma$ minimising)
Wlog. $\gamma(0) \neq \gamma(L)$. Since we are in $B_\varepsilon(p)$ the unique geodesic from $\gamma(0)$ to $\gamma(L)$ is the radial geodesic (thm 6.5)

If $\alpha([0,L]) \notin B_\varepsilon(p)$, choose $T \in [0,L]$ s.t. the first time that $\alpha(T) \in S_\varepsilon(p)$.
Then $L(\alpha) \geq L(\alpha|_{[0,T]})$.

So $\alpha|_{[0,T]} \subseteq \overline{B_\varepsilon(p)}$. Reparameterise s.t. $\alpha|_{[0,T]}$ is defined on $[0,L]$ and call this α.
If we can show $L(\alpha) \geq L(\text{radial geodesic from } \alpha(0) \text{ to } \alpha(L))$ then we're done since we will have shown that the radial geodesic minimises length over all curves connecting p to any other point.
Assume $\alpha([0,L]) \subseteq B_\varepsilon(p)$. W.l.o.g. suppose $\alpha(t) \neq p$, $t > 0$.
Write $\alpha(t) = \exp_p(r(t)X(t))$, $t \in (0,L]$,
$r : (0,L] \to \mathbb{R}^+$ piecewise smooth.
$X(t)$ smooth in T_pM, $|X(t)| = 1$.

Write geodesic $\gamma : [0,L] \to M$ from p to q as
$\gamma(s) = \exp_p\left(\frac{s\, r(L) X(L)}{L}\right)$

Use notation of proof of Gauss Lemma
$\alpha(t) = f(r(t), t)$ s.t. $\alpha'(t) = r'(t)\frac{\partial f}{\partial s}(r(t),t) + \frac{\partial f}{\partial t}(r(t),t)$

By Gauss Lemma,
$$g\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial s}\right)(r(t),t) = g_{\exp_p(r(t)X(t))}\left(d(\exp_p)_{r(t)X(t)}(X(t)), d(\exp_p)_{r(t)X(t)}(X(t))\right)$$
$$= g_p(X(t), X(t)) = 1.$$

$$g\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)(r(t),t) = g_{\exp_p(r(t)X(t))}\left(d(\exp_p)_{r(t)X(t)}(X(t)), d(\exp_p)_{r(t)X(t)}(X'(t))\right)$$
$$= g_p(X(t), X'(t)) = \frac{d}{dt}|X(t)|^2_{g_p} = 0$$

Hence $|\alpha'|^2 = |r'|^2 + \left|\frac{\partial f}{\partial t}\right|^2 \geq |r'|^2$

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So $L(\alpha) = \int_0^L |\alpha'(t)|\,dt \geq \int_0^L |r'(t)|\,dt$

$$\geq \int_0^L r'(t)\,dt = r(L) = L(\gamma)$$

$$= L\left(\frac{r(L)X(L)}{L}\right)$$

So the geodesic γ is minimising.

Now if $L(\alpha) = L(\gamma) \Leftrightarrow \frac{\partial f}{\partial t} = 0 \Leftrightarrow X'(t) = 0$

So $X(t) = X$ const, and $|r'| = r' > 0$

So α is a monotonic reparameterisation of γ

$\left(\gamma(s) = \exp_p\left(\frac{s r(L) X}{L}\right)\right)$ so $\alpha([0,L]) = \gamma([0,L])$

$\square$

<u>Prop^n 10.7</u>

If $\gamma : [0,L] \to M$ is a piecewise smooth curve with $|\gamma'|$ const, and it is locally minimising then γ is a geodesic.

<u>Proof</u>

Let $t \in (0,L)$, W a totally normal nbhd of $\gamma(t)$

Then $\exists \delta > 0$ s.t. $[t-\delta, t+\delta] \subseteq [0,L]$ and $\gamma([t-\delta, t+\delta]) \subseteq W$. Then $\gamma|_{[t-\delta, t+\delta]}$ is piecewise smooth and joins two points

$p = \gamma(t-\delta)$ and $q = \gamma(t+\delta)$ in a geodesic ball centred at p since W is a totally normal nbhd.

By thm 10.6, $L(\gamma|_{[t-\delta,t+\delta]})$ is the length of the radial geodesic $\beta(s) = \exp_p(sX)$ from p to q so $\alpha = \gamma|_{[t-\delta, t+\delta]}$ is a monotonic reparam. of β.

i.e. $\alpha(s) = \exp_p(r(s)X)$, $r(s)$ positive increasing function with $r(0) = 0$.
By proof of thm 10.6,
$|\alpha'|^2 = |r'|^2$, $|r'| = r'$ since r increasing
so r is a multiple of s.
Hence α is a radial geodesic
Therefore γ is a geodesic on $[t-\delta, t+\delta]$ and in particular at t.
t arbitrary, so γ satisfies the geodesic eq^n everywhere and so is a geodesic.

$\square$

See online notes for example about geodesics in $\mathbb{CP}^n$.

§10.4 Completeness

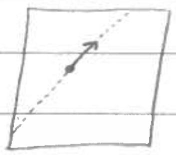
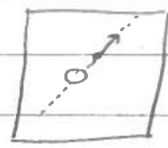
(M, g) a connected Riem. mfd.

Def^n

(M,g) is (geodesically) complete if $\exp_p(X)$ is defined for all $X \in T_pM$ and $\forall p \in M$.

Equivalently, normalised geodesics $\gamma_{(p,X)}(t) = \exp_p(tX)$ are defined $\forall X \in T_pM$, $|X| = 1$, $\forall t \in \mathbb{R}$, $\forall p \in M$.

Example

$(\mathbb{R}^2, g_0)$ complete , $(\mathbb{R}^2 \setminus \{0\}, g_0)$ not complete 

See notes for more examples.

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Prop^n 10.9

If $p,q \in (M,g)$ define $d(p,q) = \inf\{L(\alpha) \mid \alpha$ piecewise smooth curve joining p to $q\}$

Then (M,d) is a metric space.

Proof (non examinable)

The metric balls $B_\varepsilon^d(p)$ in (M,d), ε sufficiently small, are geodesic balls $B_\varepsilon(p)$ in (M,g) by thm 10.6.

Any geodesic ball is an open set in (M,g) by def^n. Moreover given any open set U in (M,g), then $\forall p \in U$ $\exists\, \varepsilon(p) > 0$ s.t. $B_{\varepsilon(p)}(p) \subseteq U$ by existence of normal neighbourhoods, hence U = union of geodesic balls.

So if d defines a metric, d defines same open sets as topology on M.

1). $d(p,p) = 0$ $\forall p \in M$: $\alpha(t) = p$ $\forall t$ $L(\alpha) = 0$ so $d(p,p) = 0$.

2). $d(p,q) = d(q,p)$: $\alpha: [0,L] \to M$ from p to q
$\beta(t) = \alpha(L-t)$, $\beta'(t) = -\alpha'(L-t)$ so $|\beta'(t)| = |\alpha'(L-t)|$
$L(\alpha) = L(\beta)$ so $d(p,q) = d(q,p)$.

3). Δ inequality: $p,q,r \in M$, α, β piecewise smooth curves from p to q, q to r resp.
Take γ from p to r by tracing α then β.
$L(\gamma) = L(\alpha) + L(\beta)$. $d(p,r) \leq L(\alpha) + L(\beta)$ $\forall$ such α, β.
taking inf: $d(p,r) \leq d(p,q) + d(q,r)$.

4). $d(p,q) > 0$ $\forall p \neq q$: $\exists$ open set $U \ni p$, $q \notin U$.
Since $\exp_p$ cts, $\exists \delta > 0$ s.t. $\exp_p(\overline{B_\delta(0)})$ well-defined and contained in U, so $q \notin \exp_p(\overline{B_\delta(0)})$.
Let α be a piecewise smooth curve p to q.
Take β part of α contained in $\exp_p(\overline{B_\delta(0)})$, must meet $S_\delta(p)$. However since geodesics are locally length minimising (thm 10.6), $L(\beta) \geq \delta$,
so $L(\alpha) \geq L(\beta) \geq \delta$. $\Rightarrow d(p,q) \geq \delta > 0$.

So (M, d) is a metric space. $\square$

§10.5 Hopf - Rinow

Key point: if a Riemannian mfd is complete then any two points can be joined by a minimising geodesic.

Theorem 10.10 (Hopf - Rinow thm)
Let (M, g) be a connected Riem. mfd.
Then the following are equivalent.
ⓐ (M, g) is geodesically complete
ⓑ $\exp_p$ is defined on all of T_pM for some $p \in M$
ⓒ closed & bounded subsets of M are compact.
ⓓ (M, d) is complete as a metric space.

Proof (non-examinable
$a \Rightarrow b$: by defⁿ
$b \Rightarrow c$: First show that $\forall q \in M$ $\exists$ geodesic $\gamma : [0, L] \to M$ s.t. $\gamma(0) = p$ and $\gamma(L) = q$
Let $q \in M$, $d(p, q) = L$. Let $\delta > 0$ s.t. $B_\delta(p)$ is a well-defined geodesic ball around p
and let $S_\delta(p) = \partial \overline{B_\delta(p)}$ be usual geodesic sphere.
Consider map $x \mapsto d(q, x)$.
This is cts. for $x \in S_\delta(p)$ so achieves a minimum at some $x_0 \in S_\delta(p)$, write $x_0 = \exp_p(\delta X)$ for some $X \in T_pM$, $|X| = 1$.

Let $\gamma(s) = \exp_p(sX)$ (defined $\forall s \in \mathbb{R}$ by assumption).
Want to show: γ is a geodesic joining p to q.
It points in the right direction.
Let $A = \{ s \in [0, L]$ s.t. $d(\gamma(s), q) = L - s \}$

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$A \neq \phi$ since $0 \in A$, $d(p,q) = L$ $(\gamma(0) = p)$
A closed since d is cts.
If we can show A is open then $A = [0, L]$
so $L \in A$ so $d(\gamma(L), q) = L - L = 0$ so $\gamma(L) = q$
so γ is a geodesic joining p to q.
$L(\gamma) = L|X| = L = d(p,q)$, as desired.

Suppose $s_0 < L$, $s_0 \in A$. Need to show
$s_0 + \delta_0 \in A$ for some $\delta_0 > 0$.
Choose $\delta_0 > 0$ s.t. $B_{\delta_0}(\gamma(s_0))$ is well defined
geodesic ball. Let $y_0 \in S_{\delta_0}(\gamma(s_0))$ be s.t. y_0 is min
of $y \mapsto d(y, q)$.

Since $s_0 \in A$, $L - s_0 = d(\gamma(s_0), q) = \delta_0 + \min_{y \in S_{\delta_0}(\gamma(s_0))} d(y,q) = \delta_0 + d(y_0, q)$

Hence $d(y_0, q) = L - (s_0 + \delta_0)$
If we can show that $y_0 = \gamma(s_0 + \delta_0)$ then
$d(\gamma(s_0 + \delta_0), q) = d(y_0, q) = L - (s_0 + \delta_0)$
so $s_0 + \delta_0 \in A \Rightarrow A$ open.

Now $d(p, y_0) \geq |d(p,q) - d(q, y_0)| = |L - (L - (s_0 + \delta_0))|$
$= s_0 + \delta_0$

However, a curve given by following γ from p
to $\gamma(s_0)$ and then radial geodesic in $B_{\delta_0}(\gamma(s_0))$
from $\gamma(s_0)$ to y_0 has length $L(\alpha) = s_0 + \delta_0$
and piecewise smooth p to y_0 $d(p, y_0) \leq L(\alpha) = s_0 + \delta_0$
$\Rightarrow d(p, y_0) = s_0 + \delta_0$
and minimising $|\alpha'|$ is const. (union of geodesics)
so prop^n 10.7 is a geodesic, so smooth
Uniqueness of geodesics $\Rightarrow \alpha = \gamma$ so $y_0 = \gamma(s_0 + \delta_0)$
as required.

Now if $C \subseteq M$ is closed and bounded then
$C \subseteq B^d_R(p) \subseteq \exp_p(\overline{B_{R'}(0)})$ $R, R' > 0$
(by what we've just shown).

$\overline{B_{R'}(0)}$ compact, $\exp_p$ cts $\Rightarrow \exp_p(\overline{B_{R'}(0)})$ is compact so C is compact, as desired.

<u>$c \Rightarrow d$</u>

Let (p_n) be Cauchy in (M, d).
(p_n) is bounded so $C = \{p_n \mid n \in \mathbb{N}\}$ closed and bdd so compact. So (p_n) has a convergent subsequence so (p_n) converges to $p \in C \subseteq M$.
So (M, d) complete.

<u>$d \Rightarrow a$</u>

Suppose M is not geodesically complete.
Then $\exists$ normalised geodesic γ defined on $[0, s_0)$ but not at s_0. Choose a strictly increasing sequence (s_n) in $[0, s_0)$ with s_n converging to s_0.
(s_n) convergent $\Rightarrow (s_n)$ Cauchy
$\Rightarrow \gamma(s_n)$ is cauchy
$d(\gamma(s_n), \gamma(s_m)) = |s_n - s_m| < \varepsilon \quad \forall n, m \geq N$
(M, d) complete $\Rightarrow \exists p_0 \in M$ s.t. $d(\gamma(s_n), p_0) \to 0$ as $n \to \infty$.

W totally normal nbhd of p_0 $\exists \delta > 0$ s.t.
$\exp_q(B_\delta(0)) \to M$ is a diffeo onto an open set containing W $\forall q \in W$.
Choose N large enough s.t. $n, m > N$.
$\Rightarrow \gamma(s_n) \in W \ \forall n > N$ and $d(\gamma(s_n), \gamma(s_m)) < \delta$
Choose $m, n > N$, $\exists!$ geodesic $\alpha : [0, L] \to W$
s.t. $\alpha(0) = \gamma(s_n)$, $\alpha(L) = \gamma(s_m)$
$\alpha = \gamma$ where they are both defined by uniqueness.

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Since $\exp_{\gamma(s_n)}$ diffeo $B_\delta(0)$,
and its image contains W, α (radial geodesic from $\gamma(s_n)$) extends γ beyond s_0. ※

□

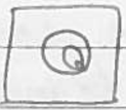
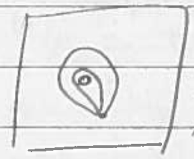
<u>Remark</u>

The minimising geodesic joining p to q is not necessarily unique (e.g. sphere, $p = N$, $q = S$)
The upper half space has property that there is a minimising geodesic joining any two points but is not complete.

(c) tells us any compact manifold must be complete.

<u>§10.6 Cartan-Hadamard thm</u>

Simply connected: every loop in M can be continuously deformed to a point.

$(\mathbb{R}^2, g_0)$  ✓ $(\mathbb{R}^2 \setminus \{0\}, g_0)$  X

<u>thm</u> (Cartan - Hadamard)
Let (M,g) be a simply connected, connected and complete n-dim Riem. mfd. with sectional curvature $K \leq 0$.
Then $\exp_p : T_pM \to M$ is a diffeo. (i.e. $M \cong \mathbb{R}^n$)

S^n connected & simply connected for $n \geq 2$.
$S^n \not\cong \mathbb{R}^n$ so it cannot admit a complete metric with $K \leq 0$.
(cf. Gauss Bonnet for S^2). [areas with $K \leq 0$ but must have $K > 0$ somewhere]

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§11 Differential Forms

§11.1 Review: forms on $\mathbb{R}^n$

On $\mathbb{R}^n$ we have 1-forms: $dx_1, \dots, dx_n$
$\Rightarrow$ any 1-form α can be written as
$\alpha = \sum_{i=1}^{n} \alpha_i dx_i$, $\alpha_i \in C^\infty(\mathbb{R}^n)$

Recall $dx_i(\partial_j) = \delta_{ij}$ $\Rightarrow$ $dx_j\left(\sum_{i=1}^{n} a_i \partial_i\right) = a_j$

k-forms are wedge products of 1-forms
$dx_{i_1} \wedge \dots \wedge dx_{i_k}$ [$\binom{n}{k}$ basis elements]

n-forms on $\mathbb{R}^n$ are multiples of $dx_1 \wedge \dots \wedge dx_n$.

0-form: function $f: \mathbb{R}^n \to \mathbb{R}$ (smooth).

Example
$M^n \subset \mathbb{R}^N$ submanifold, so we can restrict a form on $\mathbb{R}^N$ to M by acting on tangent vectors to M
(writing $M = M^n$, n-dim submfd.)
e.g. $\xi = \dfrac{x_1 dx_2 - x_2 dx_1}{x_1^2 + x_2^2}$ on $\mathbb{R}^2 \setminus \{0\}$

evaluate on $x_1\partial_2 - x_2\partial_1$
$\xi(x_1\partial_2 - x_2\partial_1) = \dfrac{x_1^2 + x_2^2}{x_1^2 + x_2^2} = 1$

whereas $\xi(x_1\partial_1 + x_2\partial_2) = \dfrac{-x_1x_2 + x_2x_1}{x_1^2 + x_2^2} = 0$

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Exterior derivative

Recall $d(a\, dx_{i_1} \wedge \ldots \wedge dx_{i_k}) = \sum_{j=1}^{n} \frac{\partial x}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k}$

extend linearly.

? $i?j?$

On functions: $df = \sum \frac{\partial f}{\partial x} dx$

1-forms: $d\left(\sum_{i=1}^{n} a_i\, dx_i\right) = \sum_{i,j=1}^{n} \frac{\partial a_i}{\partial x_j} dx_j \wedge dx_i$

Note: Ω n-form $\longrightarrow d\Omega = 0$

Properties:

- $d \circ d = 0$
- Assume $f: \mathbb{R}^n \to \mathbb{R}^m$, ω a k-form on $\mathbb{R}^m$
  then $f^*\omega$ is a k-form on $\mathbb{R}^n$,
  $$d f^* \omega = f^*(d\omega)$$

Example

$$\xi = \frac{x_1\, dx_2 - x_2\, dx_1}{x_1^2 + x_2^2} \quad \text{on } \mathbb{R}^2 \setminus \{0\}$$

$$d\xi = \partial_1 \left(\frac{x_1}{x_1^2 + x_2^2}\right) dx_1 \wedge dx_2 - \partial_2 \left(\frac{x_2}{x_1^2 + x_2^2}\right) dx_2 \wedge dx_1$$

$$= \frac{(x_2^2 - x_1^2)(dx_1 \wedge dx_2 + dx_2 \wedge dx_1)}{(x_1^2 + x_2^2)^2} = 0$$

Recall: a form s.t. $d\omega = 0$ is called closed,
a form s.t. $\omega = d\eta$ is call exact [exact $\Rightarrow$ closed]

Forms on manifolds

Def^n 11.1

$\Lambda^k T_p^* M = \{$ alternating k-linear maps $\omega : T_pM \times \dots \times T_pM \to \mathbb{R} \}$

which is a vector space, and let

$$\Lambda^k T^* M = \bigcup_{p \in M} \Lambda^k T_p^* M$$

which is a rank $\binom{n}{k}$ vector bundle over M.

The sections $\Gamma(\Lambda^k T^* M)$ of $\Lambda^k T^* M$ are called k-forms.

Examples

- 0-forms are functions $f : M \to \mathbb{R}$
- $\Lambda^1 T_p^* M = T^*_p M$ and $\Lambda^1 T^* M = T^* M$.
  T^*M is a rank n vector bundle over M, called the cotangent bundle and $T_p^* M$ is called the cotangent space of M at p.

- $\Lambda^n T^* M$ is a rank 1 vector bundle over an n-dim mfd M.

Assume $\omega \in \Gamma(T^*M)$ and $X \in \Gamma(TM)$

Then $\omega(p) \in T^*_p M$ and $X(p) \in T_p M$

$\Rightarrow \omega(p)(X(p)) \in \mathbb{R}$

$\Rightarrow \omega(X) \in C^\infty(M)$

i.e. 1-forms are "dual" to vector fields.

Example

If TM is trivial $\Leftrightarrow$ there are n L.I. vector fields $X_1, \dots, X_n$ on M. Define $\omega_1, \dots, \omega_n$ by $\omega_i(X_j) = \delta_{ij}$.

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Then the ω_i are L.I. (and nowhere vanishing since the X_i are nowhere vanishing)
so prop^n 3.5 $\Rightarrow$ T^*M is trivial as well.

Def^n 11.2

Let g be a Riem. metric on M.
For $X \in T_pM$ define $X^\flat(Y) = g(X,Y)$.
This is well-defined since g is bilinear.
Moreover $X \longmapsto X^\flat$ is injective: $X^\flat = 0$ iff $X = 0$ (since g is non-degenerate).

$\Rightarrow X \longmapsto X^\flat$ is an isomorphism from T_pM to T_p^*M.
The inverse map for $\omega \in T_p^*M$, which we call $\omega^\# \in T_pM$, by $\omega(Y) = g(\omega^\#, Y)$.

§11.3 Pullback and exterior derivative

Def^n 11.3

Let $f: M \to N$ be a map. If $\omega \in \Gamma(\Lambda^k T^*N)$ we can define the pullback $f^*\omega \in \Gamma(\Lambda^k T^*M)$ by $(f^*\omega)(p)(X_1, \dots, X_k) = \omega(f(p))(df_p(X_1), \dots, df_p(X_k))$ $\forall p \in M$, $X_1, \dots, X_k \in T_pM$.

Note: $(f \circ g)^* = g^* \circ f^*$

Example

$$\xi = \frac{x_1 dx_2 - x_2 dx_1}{x_1^2 + x_2^2} \quad \text{on } \mathbb{R}^2 \setminus \{0\}$$

we saw $\xi(X) = 1$ for $X = x_1\partial_2 - x_2\partial_1$
Let $f: \mathbb{R} \to \mathbb{R}^2$ be $f(\theta) = (\cos\theta, \sin\theta)$.
then $f(\mathbb{R}) = S^1$ and $f_*(\partial_\theta) = -\sin\theta\, \partial_1 + \cos\theta\, \partial_2$
which is the restriction of X to S^1.

$\Rightarrow f^*\xi(\partial_\theta) = \xi(f_*\partial_\theta) = \xi(X)\big|_{S^1} = 1$

θ or θ ?

$\Rightarrow f^*\xi = d\theta$, the 1-form dual to $\partial\theta$

Example

Given any chart (U, φ) on M and $\omega \in \Gamma(\Lambda^k(T^*M))$ then we know that $\varphi^{-1} : \varphi(U) \to U$, then $(\varphi^{-1})^*\omega$ is a k-form on $\varphi(U)$.

So we can always locally see a k-form on M as a k-form on $\mathbb{R}^n$.

Def^n 11.4

The exterior derivative $d : \Gamma(\Lambda^k T^*M) \to \Gamma(\Lambda^{k+1} T^*M)$ is defined via requiring that $\omega \in \Lambda^k T^*M$ and (U, φ) is a chart, then

$$d\omega\big|_U = \varphi^*(d((\varphi^{-1})_*\omega))$$

Note: This implies $d \circ d = 0$ and if $f : M \to N$ smooth, then $d(f^*\omega) = f^*(d\omega)$.

§11.4 Lie derivative and Cartan's formula

Def^n 11.5

Given $X \in \Gamma(TM)$ and $\omega \in \Gamma(\Lambda^k T^*M)$, the Lie derivative of ω wrt. X is given by

$$\mathcal{L}_X \omega = \lim_{t \to 0} \frac{(\phi_t^X)^*(\omega(\phi_t^X(p))) - \omega(p)}{t}$$

where $\phi_t^X : t \in (-\varepsilon, \varepsilon)$ is the flow of X near p

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$\Rightarrow \mathcal{L}_X \omega$ is also a k-form on M since $\mathcal{L}_X \omega(p) \in \Lambda^k T_p^* M$ and X, ω are smooth.

Example

$f : M \to \mathbb{R}$ a 0-form

$\mathcal{L}_X f = X(f)$

Prop^n 11.6

Let $X \in \Gamma(TM)$ and $\omega \in \Gamma(\Lambda^k T^* M)$.
We define the inner product of X with ω, called $\iota_X \omega$ to be the k-1 form defined by

$$\iota_X \omega(Y_1, \dots, Y_{k-1}) = \omega(X, Y_1, \dots, Y_k)$$

Then Cartan's formula is

$$\mathcal{L}_X \omega = d(\iota_X \omega) + \iota_X(d\omega).$$

Example

Consider the 1-form $\xi = \dfrac{x_1 dx_2 - x_2 dx_1}{x_1^2 + x_2^2}$ on $\mathbb{R}^2 \setminus \{0\}$.

then $d\xi = 0$ so by Cartan's formula

$$\mathcal{L}_{x_1\partial_1 + x_2\partial_2} \xi = d(\xi(x_1\partial_1 + x_2\partial_2)) = d(0) = 0,$$

and $\mathcal{L}_{x_2\partial_1 - x_1\partial_2} \xi = d(\xi(x_2\partial_1 - x_1\partial_2)) = d(1) = 0.$

Let (M, g) be a Riem. mfd
We can extend the Levi-Civita connection to forms:

If ω is a k-form we can define

$$D_X\omega(Y_1, \dots, Y_k) = X(\omega(Y_1, \dots, Y_k)) - \sum_{j=1}^{k} \omega(Y_1, \dots, Y_{j-1}, D_X Y_j, Y_{j+1}, \dots, Y_k)$$

for $Y_1, \dots, Y_k \in \Gamma(TM)$.

Example

Let $X, Y, Z \in \Gamma(TM)$ we see that

$$\begin{aligned}\nabla_X Y^\flat(Z) &= X(Y^\flat(Z)) - Y^\flat(\nabla_X Z)\\ &= X(g(Y, Z)) - g(Y, \nabla_X Z)\\ &= g(\nabla_X Y, Z)\\ &= (\nabla_X Y)^\flat(Z)\end{aligned}$$

We can extend the def^n of covarient derivatives to metrics (k-tensors) with the same formula.

In particular

$$(\nabla_X g)(Y, Z) = \underbrace{X(g(Y, Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z)}_{=0}$$

Thus g is always parallel.

§12 Orientation and Riem. metrics

§12.1 Partitions of unity

Thm 12.1

Let M be a mfd. with an atlas $\{(U_i, \varphi_i) \mid i \in \underset{\text{index set}}{I}\}$

there exists a smooth family of functions

$\{f_j : M \to \mathbb{R},\ j \in \mathbb{N}\}$ s.t.

- $\forall j \in \mathbb{N}$, $\exists i \in I$ s.t. $\underset{\text{support}}{\operatorname{spt}} f_j = \overline{\{p \in M : f_j(p) \neq 0\}} \subseteq U_i$
- $\forall p \in M$, $\exists$ open set $W \ni p$ s.t. $W \cap \operatorname{spt} f_j \neq \emptyset$ for only finitely many j.
- $f_j(p) \geq 0$ $\forall j \in \mathbb{N}$ and $p \in M$
- $\sum_{j=1}^{\infty} f_j(p) = 1$

We call $\{f_j : j \in \mathbb{N}\}$ a partition of unity, subordinate to the atlas $\{(U_i, \varphi_i) \mid i \in I\}$

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Prop^n 12.2

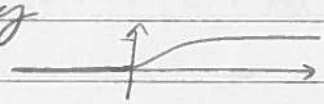
Let $B_r(0)$, $\overline{B_r(0)} \subseteq \mathbb{R}^n$ be the open and closed balls of radius $r>0$ around 0.
For each $r>0$, $\exists$ smooth functions $g_r : \mathbb{R}^n \to \mathbb{R}$ st.
- $g_r \geq 0$
- $g_r \equiv 1$ on $\overline{B_{r/2}(0)}$
- $g_r \equiv 0$ on $\mathbb{R}^n \setminus B_r(0)$

$\Rightarrow$ spt $g_r \subseteq \overline{B_r(0)}$

Proof (non-examinable)

Consider $h : \mathbb{R} \to \mathbb{R}$ given by

$$h(t) = \begin{cases} e^{-1/t} & t \geq 0 \\ 0 & t \leq 0 \end{cases}$$



as $h'(t) = \frac{1}{t^2} e^{-1/t} > 0$ for $t>0$ we see that h is increasing and $0 \leq h < 1$.
Now $h'(t) \to 0$ as $t \to 0$ because

$$t^{-k} e^{-1/t} = t \cdot t^{-k-1} e^{-1/t} \leq (k+1)!\, t \underbrace{\sum_{m=0}^{\infty} \frac{t^{-m}}{m!}}_{e^{1/t}} e^{-1/t} = (k+1)!\, t$$

$\Rightarrow h'(t) \to 0$ as $t \to 0$

Note $h^{(k)}(t) = p_{2k}(\frac{1}{t}) e^{-1/t}$ where $p_{2k}(\frac{1}{t})$ is a polynomial of degree $2k$.
$\Rightarrow h^{(k)}(t) \to 0$ as $t \to 0$ $\Rightarrow h$ is smooth on $\mathbb{R}$.

Consider $h_r : \mathbb{R} \to \mathbb{R}$ given by

$$h_r(t) = \frac{h(r^2 - t^2)}{h(r^2-t^2) + h(t^2 - \frac{1}{4}r^2)}$$

This is well defined since if
$h_r(r^2 - t^2) = 0 \Rightarrow |t| \geq 0$
so $t^2 - \frac{1}{4}r^2 > 0$ and similarly the other way around.
Since the denominator never vanishes, $h_r(t)$ is smooth
Moreover $0 \leq h_r \leq 1$, $h_r(t) = 0$ iff $|t| \geq r$
and $h_r(t) = 1$ iff $h(t^2 - \frac{1}{4}r^2) = 0 \Leftrightarrow |t| \leq r/2$

Then define $g_r : \mathbb{R}^n \to \mathbb{R}$ by $g(x) = h_r(|x|^2)$ $\square$

See notes for proof of existence of partition of unity (non-examinable).

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Def^n 12.3

A mfd M is orientable if $\exists$ an atlas $\{(U_i, \varphi_i) \mid i \in I\}$ s.t. whenever $U_i \cap U_j \neq 0$, the Jacobian $\det(d(\varphi_j \circ \varphi_i^{-1})) > 0$ on $\varphi_i(U_i \cap U_j)$.

Examples

1). The n sphere is orientable

Let's look at S^2 (S^n similarly)

Take atlas $\{(U_N, \varphi_N), (U_S, \varphi_S)\}$ we constructed the take the inversion map $F : \varphi_N \circ \varphi_S^{-1} : y \mapsto \frac{y}{|y|^2}$ on $\mathbb{R}^2 \setminus \{0\}$ so $dF = \frac{1}{|y|^4}\begin{pmatrix} y_2^2 - y_1^2 & -2y_1y_2 \\ -2y_1y_2 & y_1^2 - y_2^2 \end{pmatrix}$

$\Rightarrow \det(dF) = \frac{1}{|y|^8}\left(-(y_1^2 - y_2^2)^2 - 4y_1y_2\right) = -\frac{1}{|y|^8} < 0$

To make it positive everywhere, switch the sign of one of the coordinates for one of the maps, say y_1 in the def^n of φ_N so $\varphi_N(x_1, x_2, x_3) = \frac{-x_1, x_2}{1 - x_2}$

$\Rightarrow F$ becomes $(y_1, y_2) \mapsto \frac{(-y_1, y_2)}{|y|^2}$

This changes the sign of the determinant.

2). T^n is orientable

3). The Möbius band and Klein bottle are not orientable (later today)

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4). All Lie groups are orientable

Thm 12.4

For an n-mfd M, the following are equivalent:

(a) M is orientable

(b) $\exists$ a nowhere vanishing n-form on M (the volume form)

(c) $\Lambda^n T^* M$ is trivial

Proof

(b) $\Leftrightarrow$ (c)

Note $\Lambda^n T^* M$ has rank 1, then we can use prop^n 3.5.

(b) $\Rightarrow$ (a)

Suppose Ω is a nowhere vanishing n-form on M and let $\{(U_i, \varphi_i), i \in I\}$ be an atlas where $\varphi_i(U_i)$ is connected (we can always assume this).

Let $\Omega = dx_1 \wedge \dots \wedge dx_n$ be the standard n-form on $\mathbb{R}^n$. Compare this to $(\varphi_i^{-1})^* \Omega$ which is an n-form on $\varphi_i(U_i)$, so $(\varphi_i^{-1})^* \Omega = \lambda_i \Omega_0$ where λ_i is a non vanishing function on $\varphi_i(U_i)$.

If $\lambda_i < 0$ we change $\varphi_i : p \mapsto (x_1(p), \dots, x_n(p))$ to $\varphi_i : p \mapsto (-x_1(p), \dots, x_n(p))$ so λ_i changes to $-\lambda_i > 0$.

We know from MV Analysis that

$(\varphi_j \circ \varphi_i^{-1})^* \Omega_0 = \det(d(\varphi_j \circ \varphi_i^{-1})) \Omega_0$

Moreover $(\varphi_j \circ \varphi_i^{-1})^* \circ (\varphi_j^{-1})^* \Omega = (\varphi_j^{-1} \circ \varphi_j \circ \varphi_i^{-1})^* \Omega$

$= (\varphi_i^{-1}) \Omega$

$$\det(d(\varphi_j \circ \varphi_i^{-1})) \underset{>0}{\lambda_j} \Omega_0 = \underset{>0}{\lambda_i} \Omega$$

$\Rightarrow \det(d(\varphi_j \circ \varphi_i^{-1})) > 0$

(a) $\Rightarrow$ (b)

Assume M is orientable and let $\{(U_i, \varphi_i), i \in I\}$ be an orientation

By Thm 12.1 we have a partition of unity $\{f_j : M \to \mathbb{R}, j \in \mathbb{N}\}$ subordinate to this atlas.
$\forall j$ choose $i(j)$ s.t. $\operatorname{spt} f_j \subseteq U_{i(j)}$
$\Rightarrow \bigcup_j U_{i(j)} = M \Rightarrow \{(U_{i(j)}, \varphi_{i(j)}), j \in \mathbb{N}\}$ is an atlas

Define $\Omega = \sum_{j=1}^{\infty} f_j \varphi_{i(j)}^* \Omega_0$

where we set $f_j \varphi_{i(j)}^* \Omega_i \equiv 0$ outside of $\operatorname{spt} f_j$.
Ω is nowhere vanishing:
Let $p \in M$ and choose $W \ni p$ open st. $W \cap \operatorname{spt} f_j \neq \emptyset$ for finitely many j, by intersecting with a coordinate chart if necessary, we can ?
$W \subseteq U_k$ for some $k \in \mathbb{N}$.
Then $(\varphi_k^{-1})^*(\Omega) = \sum_{j=1}^{\infty} f_j (\varphi_k^{-1})^* \circ \varphi_{i(j)}^* \Omega_0$

$$= \sum_{j=1}^{\infty} f_j \circ \varphi_k^{-1} (\varphi_{i(j)} \circ \varphi_k^{-1})^* \Omega_0$$

and the sum is actually finite.
Note that for some $j \in \mathbb{N}$, $i(j) = k$ and $f_k \circ \varphi_k^{-1}(p) \neq 0$
so $(\varphi_k^{-1})^*(\Omega) \neq 0$.

□

Examples
1). any parallelisable mfd is orientable
2). On $\mathbb{R}^n$ we have the standard orientation given by Ω. We can use this to say that any basis $\{b_1, \dots, b_n\}$ of $\mathbb{R}^n$ is positively oriented if $\Omega_0(b_1, \dots, b_n) > 0$
Similarly for an oriented manifold we can equip each tangent space with a positively oriented basis.

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Def^n 12.5
We say two orientations Ω and Ω' on M are the same if $\Omega = \lambda \Omega'$ and $\lambda > 0$, $\lambda : M \to \mathbb{R}$.

We say a diffeo $f : M \to N$ is orientation preserving if, given volume forms Ω on M and γ on N,
$$f^* \gamma = \lambda \Omega \qquad \text{for } \lambda : M \to \mathbb{R},\ \lambda > 0.$$

Example
$id : M \to M$ is always orientation preserving.
However $-id : \mathbb{R}^n \to \mathbb{R}^n$
$$(-id)^* \Omega_0 = \det(-id)\, \Omega_0$$
$$= (-1)^n \Omega_0$$
is orientation preserving only if n is even, orientation reversing if n is odd.

Note $-id : S^n \to S^n$ is thus also orientation preserving if n is odd (important that normal vectors $\mapsto$ normal vectors)

Suppose $\mathbb{RP}^n$ is orientable.
Then $\exists$ a volume form Ω on $\mathbb{RP}^n$.
$\pi : S^n \to \mathbb{RP}^n$ is a local diffeo
$\Rightarrow \gamma = \pi^* \Omega$ is an n-form on S^n
and is nowhere vanishing (since π a local diffeo).
So γ is a volume form on S^n.
However $\pi \circ (-id) = \pi \quad \Rightarrow \pi^* = (-id)^* \circ \pi^*$
$\Rightarrow \gamma = \pi^* \Omega = (-id)^* \circ \pi^* \Omega = (-id)^* \gamma = (-1)^{n+1} \gamma$
$\Rightarrow$ contradiction if n is even
See notes: if n is odd, can use the orientation on S^n to construct an orientation on $\mathbb{RP}^n$.

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§12.3 Riemannian metrics

Thm 12.6

Every mfd has a Riem. metric.

Proof

Let M be an n-mfd. with an atlas $\{(U_i, \varphi_i), i \in I\}$ and let g_0 be the standard metric on $\mathbb{R}^n$.
Thm 12.1 $\Rightarrow \exists$ partition of unity $\{f_j, j \in \mathbb{N}\}$ subordinate to the atlas. For each $j \in \mathbb{N}$ $\exists$ $i(j)$ s.t. $\operatorname{spt} f_j \subseteq U_{i(j)}$, take the atlas $\{(\underbrace{U_{i(j)}}_{= U_j}, \underbrace{\varphi_{i(j)}}_{= \varphi_j}), j \in \mathbb{N}\}$

On U_j, since φ_j is a diffeo, $\varphi_j : U_j \to \varphi_j(U_j) \subset \mathbb{R}^n$, we have that $\varphi_j^* g_0$ is a Riem. metric (prop^n 4.3).
write $g_j := \varphi_j^* g_0$.
$\Rightarrow f_j g_j$ is smooth symmetric and bilinear, and we can extend it by 0 to all of M.

Then define $g = \displaystyle\sum_{j=1}^{\infty} f_j g_j \in \Gamma(S^2 T^* M)$
(a finite sum)

Take $p \in M$. Then $g_p(X, X) = \displaystyle\sum_{j=1}^{\infty} f_j(p)(g_j)_p(X, X) \geq 0$
$\forall X \in T_p M$ since $f_j \geq 0$.

Assume $g_p(X, X) = 0$
$\Rightarrow f_j(p) g_j(X, X) = 0 \quad \forall_j$
Since $\sum f_j = 1$ $\exists$ a $j \in \mathbb{N}$ st. $f_j(p) > 0$
$\Rightarrow g_j(p)(X, X) = 0 \Rightarrow X = 0$

$\square$

Example

Assume M orientable, g a metric on M
$\Rightarrow \exists$ a section $\mathrm{vol} \in \Gamma(\Lambda^n T^* M)$ st. $|\mathrm{vol}| = 1$.
This is the so called volume form.

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In a chart (U, φ) one has
$(\varphi^{-1})^* \Omega = \sqrt{\det(g_{ij})}\ \Omega_0$.

§13 Curvature Revisited

§13.1 Ricci and scalar curvature

Def^n 13.1
We define the Ricci curvature tensor $Ric \in \Gamma(S^2 T^* M)$ by
$$Ric(X, Y)(p) = \sum_{i=1}^{n} R(E_i, X, Y, E_i)$$
for all $p \in M$, $X, Y \in T_p M$, where $\{E_1, \dots, E_n\}$ an O.N.-basis for $T_p M$.

Note
$$Ric(Y, X) = \sum_{i=1}^{n} R(E_i, Y, X, E_i) = \sum_{i=1}^{n} R(X, E_i, E_i, Y)$$
$$= \sum_{i=1}^{n} R(E_i, X, Y, E_i) = Ric(X, Y)$$

Why is this a trace?
Recall given X, Y, Z on M we have the map $Z \mapsto R(X, Y)Z$ which sends vector fields to vector fields.
Moreover, at p this depends only on $X(p), Y(p), Z(p)$ so this is a well-defined linear map $T_p M \to T_p M$

The Ricci curvature is then given by
$$Ric(X, Y) = Tr(Z \mapsto R(Z, X)Y)$$

Locally, let (U, φ) be a coordinate chart and $\{X_1, \dots, X_n\}$ the coordinate frame $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$.
Let $\{E_1, \dots, E_n\}$ be an orthonormal frame on U, and write this as $E_k = \sum_{l=1}^{n} a_{lk} X_l$, for

invertible matrix of functions $A = (a_{ij})$

Note $\delta_{kl} = g(E_k, E_l) = g\left(\sum_{i=1}^{n} a_{ik} X_i, \sum_{j=1}^{n} a_{jl} X_j\right)$

$$= \sum_{i,j=1}^{n} a_{ik} a_{jl} g_{ij}$$

which is (in matrix notation) $A^T g A = I$

$\Rightarrow g = (A^T)^{-1} A^{-1} = (AA^T)^{-1}$

$\Rightarrow g^{-1} = AA^T$

$$Ric(X_i, X_j) = \sum_{k=1}^{n} R(X_i, E_k, E_k, X_j)$$

$$= \sum_{k,l,m=1}^{n} R(X_i, a_{lk} X_l, a_{mk} X_m, X_j)$$

$$= \sum_{k,l,m=1}^{n} R_{iml j}\, a_{lk} a_{mk} = \sum_{l,m=1}^{n} R_{imlj}\, g^{lm}$$

Remark

If we take geodesic normal coordinates, i.e. $g_{ij} = \delta_{ij}$ and $\Gamma_{ij}^{k} = 0$ at p, and let Ω be the local Riemannian volume form.

Then $(\varphi^{-1})^* \Omega = \left(1 - \frac{1}{6} \sum_{i,j} Ric(p)_{ij} x_i x_j + O(|x|^3)\right) \Omega_0$

Example

Assume M has dim 2, take $\{E_1, E_2\}$ to be an O.N.-basis of T_pM then

$$K(T_pM) = R(E_1, E_2, E_2, E_1) = \sum_{j=1}^{2} R(E_j, E_2, E_2, E_j)$$

$$= Ric(E_2, E_2) = Ric(E_1, E_1)$$

Remark: In dim 3, the Ricci curvature still determines the curvature R, but the formula is more difficult (not true for dim ≥ 4)

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Remark

Recall that the Ricci tensor is a symmetric $(0,2)$-tensor, as is the Riemannian metric. We say that (M,g) is Einstein if $Ric = \lambda g$ for some constant $\lambda \in \mathbb{R}$.

In particular, in the case $\lambda = 0$, then (M,g) is called Ricci-flat.

$\hookrightarrow$ similar concept to a minimal surface.

Def^n 13.2:

The scalar curvature S of M is a smooth function on M given by

$$S(p) = \sum_{i,j=1}^{n} R(E_i, E_j, E_j, E_i) = \sum_{j=1}^{n} Ric(E_j, E_j)$$

for $p \in M$ and $\{E_1, \dots, E_n\}$ an ON basis of T_pM.

Remark

One can compute that for $p \in M$ the volume of a small geodesic ball $B_\varepsilon(p)$ is given by

? $$vol_g(B_\varepsilon(p)) = \left(1 - \frac{S(p)}{6(n+2)}\varepsilon^2 + O(\varepsilon^4)\right) vol_{Eucl}(B_\varepsilon(0)).$$

Locally, in a coordinate chart (U,φ) we get as before $S = \sum_{i,j=1}^{n} R(E_i, E_j, E_j, E_i)$

$$= \sum_{i,k,j,l} R_{ijkl}\, g^{il} g^{jk} = \sum_{i,j=1}^{n} Ric_{ij}\, g^{ij}$$

§13.2 Constant curvature

Prop^n 13.3

A Riem. mfd (M,g) has constant sectional curvature K iff $\forall$ vector fields X, Y, Z, W on M we have $R(X,Y,Z,W) = K(g(X,W)g(Y,Z) - g(X,Z)g(Y,W))$

Proof

Suppose (M,g) has constant sectional curvature K. Define $\bar{R}(X,Y,Z,W) = K(g(X,W)g(Y,Z) - g(X,Z)g(Y,W))$
Then $R(X,Y,Y,X) = \bar{R}(X,Y,Y,X)$
Since $\bar{R}$ has the same symmetries as R, prop^n 7.5 implies $R = \bar{R}$.
Suppose R is as given, then $K(X,Y) = K$.
$\square$

Prop^n 13.4

If (M,g) has constant sectional curvature K then $Ric = (n-1)Kg$, $S = n(n-1)K$

Proof see notes.

Examples

- $\mathbb{R}^n$ has sectional curvature 0
- S^2 has const. sectional curvature 1, same for $\mathbb{RP}^2$. $\Rightarrow Ric = g$, $S = 2$
- H^2 has const. sectional curvature -1 $\Rightarrow Ric = -g$, $S = -2$.

See notes for some more detail.

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Thm 13.7

Let (M,g) be complete n-dim Riem manifold with const. sectional curvature $K \in \{-1, 0, 1\}$.
Then there exists a discrete group G acting freely and properly discontinuously on S^n, $\mathbb{R}^n$, or H^n by isometries, s.t. (M,g) is isometric to

- S^n/G if $K=1$
- $\mathbb{R}^n/G$ if $K=0$
- H^n/G if $K=-1$.

Prop^n 13.8

Let M be a complete $2n$-dim Riem. mfd with with constant sectional curvature 1. Then M is isometric to S^{2n} or $\mathbb{RP}^{2n}$

Proof

Thm 13.7 says (M,g) is isometric to S^{2n}/G.
Hence $G \subseteq O(2n+1)$ (consider rescaling radius of S^{2n} and applying same isometry).
Let $x \in G$ and f_x the corresponding isometry then $\det f_x = \pm 1$.
If $\det f_x = 1$ then f_x has 1 as an eigenvalue.
So f_x has a fixed point on S^{2n} (corresponding to a unit eigenvector p to the eigenvalue 1).
$\Rightarrow f_x$ does not act freely if f_x is not id.
Assume $\det f_x = -1 \Rightarrow \det(f_x^2) = 1 \Rightarrow f_x^2 = id$
$\Rightarrow f_x = \pm id$.

$\square$

§13.3 Curvature and topology

We say $Ric \geq k$ for $k \in \mathbb{R}$ if $Ric(X,X) \geq k$ for all unit tangent vectors X on M.

Thm 13.9 (Bonnet-Meyers)
Let (M,g) be a complete manifold with $Ric \geq \frac{(n-1)}{r^2} > 0$. Then M is compact and
$$diam(M) \leq \pi r$$
i.e. $dist(p,q) \leq \pi r \ \forall p,q \in M$.

Thm 13.10 (Synge)
Let (M,g) be compact n-dim Riem. mfd with $K > 0$,
ⓐ if M is orientable and n is even, then M is simply connected.
ⓑ if n is odd, then M is orientable

Thm 13.11 (Differentiable Sphere thm)
Let (M,g) be compact, simply connected n-dim Riem. mfd. with $\frac{1}{4} < K \leq 1$. Then M is diffeomorphic to S^n.