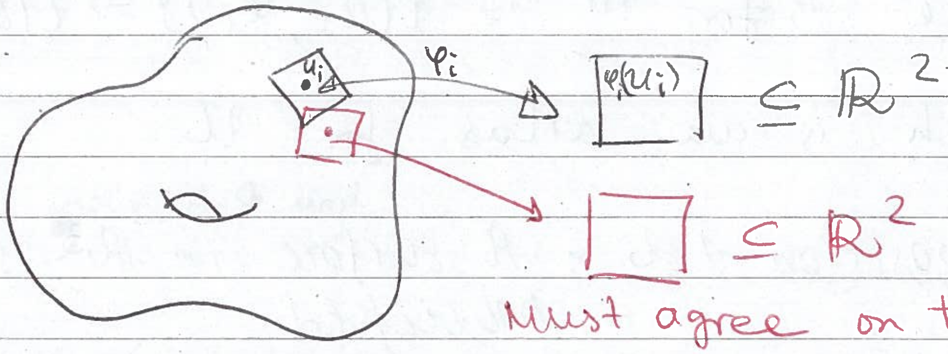
M114 Riemannian Geometry Notes

Based on the 2015 autumn lectures by Dr J Lotay

Riemannian Geometry

Manifolds



Definition 1.1: An n-dimensional manifold M is a separable metric space s.t. $\exists$ $\mathcal{A} = \{(U_i, \varphi_i) : i \in I\}$ with

- $U_i \subset M$ open $\forall i$ & $\bigcup_{i \in I} U_i = M$
- $\varphi_i : U_i \to \varphi_i(U_i) \subseteq \mathbb{R}^n$ continuous bijection with continuous inverse (i.e. homeomorphism)
- whenever $U_i \cap U_j \neq \emptyset$, the map

$$\varphi_j \circ \varphi_i^{-1} : \underset{\underset{\mathbb{R}^n}{\cap}}{\varphi_i(U_i \cap U_j)} \to \underset{\underset{\mathbb{R}^n}{\cap}}{\varphi_j(U_i \cap U_j)}$$

is a smooth bijection with smooth inverse (i.e. diffeomorphism)

This shows that the overlap is the same, and they have the same variation i.e. differen

Then A is called an atlas, (U_i, φ_i) is a chart and $\varphi_j \circ \varphi_i^{-1}$ are transition maps

Example: $\mathbb{R}^n$ is an n-dim manifold
Take $\mathcal{A} = \{(\mathbb{R}^n, id)\}$

Note: If M is an n-dim manifold and $U \subset M$ open then U is an n-dim manifold as well

Take $\mathcal{A}$ for M is $\{(U_i, \varphi_i)\} \Rightarrow \{(U_i \cap U, \varphi_i|_{U_i \cap U})\}$

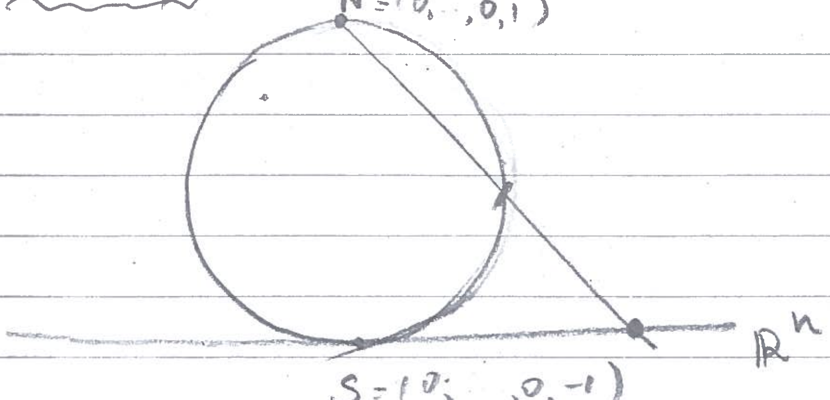
which is an atlas for U

<u>Proposition 1.2</u> : A ~~surface~~ in $\mathbb{R}^3$ is a 2-dim manifold
from Diff geom

<u>Proposition 1.3</u> A submanifold in $\mathbb{R}^n$ is a manifold

<u>Example</u> : Let $S^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1}$ s.t. $\sum_{i=1}^{n+1} x_i^2 = 1\}$

<u>Claim</u>: S^n is an n-dim manifold



Let $U_N = S^n \setminus \{N\}$, $U_S = S^n \setminus \{S\}$

U_N, U_S are open and $U_N \cup U_S = S^n$

· Let $\varphi_N(x_1, \dots, x_{n+1}) = \dfrac{(x_1, \dots, x_n)}{1 - x_{n+1}}$ we divide by $1 - x_{n+1}$ because antipodal points will be mapped to the same thing i.e. not a bijection

$\varphi_S(x_1, \dots, x_{n+1}) = \dfrac{(x_1, \dots, x_n)}{1 + x_{n+1}}$

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$\varphi_N(U_N) = \mathbb{R}^n = \varphi_S(U_S)$

$\varphi_N : U_N \to \mathbb{R}^n$ and $\varphi_S : U_S \to \mathbb{R}^n$ are continuo

$$\varphi_N^{-1}(y_1, \dots, y_n) = \frac{(2y_1, \dots, 2y_n, \sum_{i=1}^n y_i^2 - 1)}{\sum_{i=1}^n y_i^2 + 1} \text{ is}$$

continuous

$$\varphi_S^{-1}(y_1, \dots, y_n) = \frac{(2y_1, \dots, 2y_n, 1 - \sum_{i=1}^n y_i^2)}{\sum_{i=1}^n y_i^2 + 1} \text{ is}$$

also continuous

$\therefore$ φ_N, φ_S are homeomorphisms

$U_N \cap U_S = S^n \setminus \{N, S\}$

$\varphi_N(U_N \cap U_S) = \mathbb{R}^n \setminus \{0\} = \varphi_S(U_N \cap U_S)$

$$\varphi_S \circ \varphi_N^{-1}(y_1, \dots, y_n) = \varphi_S\left(\frac{(2y_1, \dots, 2y_n, \sum y_i^2 - 1)}{\sum y_i^2 + 1}\right)$$

$$= \frac{(y_1, \dots, y_n)}{\sum y_i^2} \text{ is smooth away from } 0$$

and is its own inverse

$\Rightarrow$ the transition map $\varphi_S \circ \varphi_N^{-1} : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ is a diffeomorphism

Example: Let $\mathbb{RP}^n$ denote the set of lines through the origin in $\mathbb{R}^{n+1}$. equivalently it is the set of pairs of antipodal points in S^n. Denote pts in $\mathbb{RP}^n$ by $[x]$ where $x=(x_1,\dots,x^{n+1}) \in \mathbb{R}^{n+1}\setminus\{0\}$

- Let $U_i = \{[x] \in \mathbb{RP}^n : x_i \neq 0\}$ for $i=1,\dots,n+1$

These are open and $\forall [x] \in \mathbb{RP}^n \ \exists i$ s.t. $x_i \neq 0$

$\Rightarrow \bigcup_{i=1}^{n+1} U_i = \mathbb{RP}^n$

- Let $\varphi_i : U_i \to \mathbb{R}^n$ be $\varphi_i([x]) = \underbrace{\frac{(x_1, \dots, \hat{x_i}, \dots, x_{n+1})}{x_i}}_{\in \mathbb{R}}$

$\to$ this is why we divide by $x_i \neq 0$

This is well defined as $\varphi_i([\lambda x]) = \varphi_i([x])$ $\forall \lambda \neq 0$ and since $x_i \neq 0$ this is continuous.

$\varphi_i(U_i) = \mathbb{R}^n$ open and

$\varphi_i^{-1}(y_1, \dots, y_n) = [(y_1, \dots, y_{i-1}, 1, y_i, \dots, y_n)] \in U_i$

is an inclusion so φ_i^{-1} is continuous

So $\varphi_i : U_i \to \mathbb{R}^n$ is a homeomorphism $\forall i$

- $U_i \cap U_j = \{[x] \in \mathbb{RP}^n : x_i \neq 0 \neq x_j\}$

suppose wlog $i > j$

$$\varphi_i(U_i \cap U_j) = \{y \in \mathbb{R}^n : y_j \neq 0\}$$

$$\varphi_j \circ \varphi_i^{-1}(y_1, \dots, y_n) = \varphi_j[(y_1, \dots, y_{i-1}, 1, y_i, \dots, y_n)]$$

$$= \frac{(y_1, \dots, \hat{y_j}, \dots, y_{i-1}, 1, y_i, \dots, y_n)}{y_j}$$

But $y_j \neq 0$ so it is smooth on $\varphi_i(U_i \cap U_j)$ and similarly its inverse is smooth so it is a diffeomorphism

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Example: Let $F : \mathbb{R}^{n+1} \to \mathbb{R}$ be
$$F(x_1, \dots, x_{n+1}) = \sum_{i=1}^{n+1} x_i^2$$

- $F^{-1}(1) = S^n$ - n-dim manifold
- $F^{-1}(0) = \{0\}$ - not n-dim manifold

$$dF_x = (2x_1, \dots, 2x_{n+1}) \neq 0 \iff x \neq 0$$

Recall $F : \mathbb{R}^n \to \mathbb{R}^m$ then $dF = \left(\frac{\partial F^i}{\partial x^j}(x) \right)$

$dF_x : \mathbb{R}^{n+1} \to \mathbb{R}$ surjective iff $x \neq 0$
by the following thm $F^{-1}(1)$ is an n-dim manifold

Theorem 1.4: Let $F : \mathbb{R}^{n+m} \to \mathbb{R}^m$ be a smooth map and suppose $c \in \mathbb{R}^n$ s.t. $F^{-1}(c) \neq \emptyset$ and $dF_p : \mathbb{R}^{n+m} \to \mathbb{R}^m$ is surjective $\forall p \in F^{-1}(c)$ (i.e. the matrix of dF_p has full rank) then $F^{-1}(c)$ is an n-dim manifold.

Proof (non-examinable)

- Implicit Function Thm: $\forall p \in F^{-1}(c)$ $\exists$ splitting
$$\mathbb{R}^{n+m} = \underbrace{\operatorname{Ker}(dF_p)}_{\cong\, \mathbb{R}^n} \times \mathbb{R}^m$$
since dF_p surj $\Rightarrow \operatorname{Im} = \mathbb{R}^m$ by ker-rank $\operatorname{Ker} \cong \mathbb{R}^n$

s.t. if $p = (a, b)$ then $\exists$ open set $\underset{\mathbb{R}^n}{\underset{\cap}{V_p}} \ni a$, $\underset{\mathbb{R}^m}{\underset{\cap}{W_p}} \ni b$

and a smooth map $G_p : V_p \to W_p$ so that
$$F^{-1}(c) \cap (V_p \times W_p) = \{(x, G_p(x)) : x \in V_p\}$$

Let $U_p = F^{-1}(c) \cap (V_p \times W_p)$ and $\varphi_p : U_p \to \varphi_p(U_p) = V_p \subset \mathbb{R}^n$

$\varphi_p : (x, G_p(x)) \longmapsto x$

$\varphi_p^{-1} : x \longrightarrow (x, G_p(x))$

Then $\mathcal{A} = \{ (U_p, \varphi_p) : p \in F(c) \}$ is an atlas ∎

Example : Let $F : \mathbb{R}^{2n} \to \mathbb{R}^n$ be $F(x_1, \dots, x_{2n}) =$
$= (x_1^2 + x_2^2 - 1, \dots, x_{2n-1}^2 + x_{2n}^2 - 1)$

$$F^{-1}(0) = \{ (x_1, \dots, x_{2n}) \in \mathbb{R}^{2n} : x_1^2 + x_2^2 = \dots = x_{2n-1}^2 + x_{2n}^2 = 1 \}$$

$$= \underbrace{S^1 \times S^1 \times \dots \times S^1}_{n \text{ times}} = T^n$$

n-torus in $\mathbb{R}^{2n}$

$$dF_x = \begin{pmatrix} 2x_1 & 2x_2 & 0 & \dots & & 0 \\ 0 & & & & & \\ \vdots & & & & & \\ 0 & \dots & & & 2x_{2n-1} & 2x_{2n} \end{pmatrix}$$

$n \times 2n$ matrix

This is surjective iff $(x_{2i-1}, x_{2i}) \neq (0,0)$ $\forall i$ this is true since $x_{2i-1}^2 + x_{2i}^2 = 1$

Theorem 1.4 $\Rightarrow$ $F^{-1}(0)$ is $2n - n = n$-dim manifold.

Remark : $T_4^2 \subseteq \mathbb{R}^4$ unlike $T_3^2 \subset \mathbb{R}^3$ given by
$\{ ((2 + \cos\theta)\cos\varphi, (2 + \cos\theta)\sin\varphi, \sin\theta) :$
$\varphi \in [0, 2\pi], \theta \in [0, \pi] \}$

Note Even though T_4^2 & T_3^2 are the same manifolds they are not the same Riemannian manifolds, have different curvature

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Example: Let $F : \mathbb{R}^2 \to \mathbb{R}$ be $F(x_1, x_2) = x_1^3 - x_2^3$.

$$dF_x = (3x_1^2 \quad -3x_2^2) = 0 \quad \text{if}$$

$$(x_1, x_2) = (0,0)$$

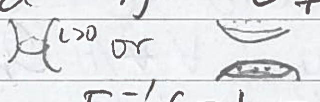
$F^{-1}(0)$ contains $(0,0)$ so dF_x is not surjective for some $x \in F^{-1}(0)$ so the Thm does not apply.

However $F^{-1}(0) = \{(x_1, x_2) \in \mathbb{R}^2 \text{ s.t. } x_1^3 = x_2^3\}$
$= \{(x_1, x_1) \in \mathbb{R}^2 \text{ s.t. } x_1 \in \mathbb{R}\}$
which <u>is</u> a 1-dim manifold!

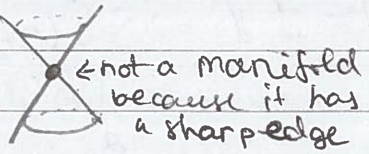
Example: Let $F : \mathbb{R}^3 \to \mathbb{R}$ be $F(x_1, x_2, x_3) =$
$= x_1^2 + x_2^2 - x_3^2$

$$dF_x = (2x_1 \quad 2x_2 \quad -2x_3) \neq 0$$
iff $x \neq 0$

Therefore theorem 1.4 $\Rightarrow$ $F^{-1}(c)$ is a 2-dim manifold if $c \neq 0$ which are hyperbolas



What is $F^{-1}(0) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \text{ s.t. } x_1^2 + x_2^2 = x_3^2\}$
= cone is <u>not</u> a manifold



Example (non-exam): Let $F : M_n(\mathbb{R}) \to Sym_n(\mathbb{R})$
$$F(A) = A^T A - I$$

$$F^{-1}(0) = \{A \in M_n(\mathbb{R}) : A^T A = I\} = O(n) \text{ - orthogonal matrices}$$

Claim: O_n is $\left(n^2 - \frac{1}{2}n(n+1)\right) = \frac{1}{2}n(n-1)$ - dim mani

$$\text{Recall } \frac{|F(A+B) - F(A) - dF_A(B)|}{|B|} \longrightarrow 0 \quad \text{as } |B| \to 0 \quad (*)$$

$$F(A+B) - F(A) = \{(A+B)^T(A+B) - I\} - (A^TA - I) =$$

$$= A^TA + B^TA + A^TB + \underbrace{B^TB}_{\text{non linear in } B \text{ so we discard}} - A^TA =$$

$$=$$

$\Rightarrow dF_A(B) = B^TA + A^TB$ since $\exists$ unique linear map for which $(*)$ is true

Suppose A is an orthogonal matrix $C \in Sym_n(\mathbb{R})$. WTS $dF_A(B) = C$

let $B = \frac{1}{2}(AC)$

$$\Rightarrow dF_A\left(\tfrac{1}{2}AC\right) = \tfrac{1}{2}C^TA^TA + \tfrac{1}{2}A^TA\cdot C =$$

$$= \tfrac{1}{2}C^T + \tfrac{1}{2}C = \tfrac{1}{2}C + \tfrac{1}{2}C = C$$

As $A^TA = id$ and $C^T = C$

$\Rightarrow dF_A$ is surj $\forall\, A \in O(n) = F^{-1}(0)$

$\Rightarrow$ Thm 1.4 $\Rightarrow O_n$ is a manifold

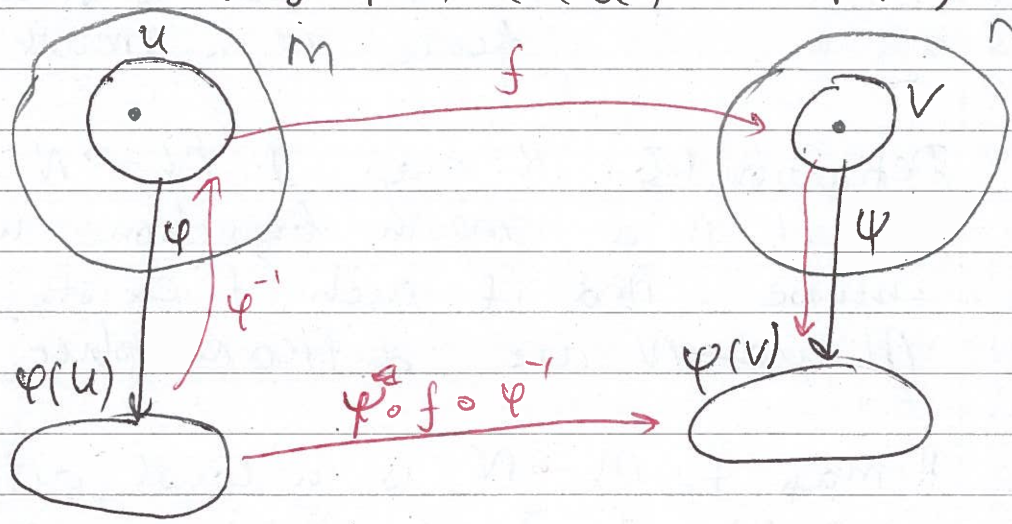
$A \in O_n \Rightarrow \det(A) = \pm 1 \Rightarrow SO(n) = \{A \in O_n : \det = 1\}$

this is open in $O(n)$ so is $\frac{1}{2}n(n-1)$-dim

manifold.

Definition 1.5: Let M, N be manifolds and let $f: M \to N$. Then f is smooth at $p \in M$ if $\exists$ charts (U, φ) at $p \in U$ and (V, ψ) at $f(p)$ s.t. $f(U) \subseteq V$ and $\psi \circ f \circ \varphi^{-1}: \varphi(U) \to \psi(V)$

$\psi \circ f \circ \varphi^{-1}: \varphi(U) \to \psi(V)$ is smooth



We say f is smooth if f is smooth $\forall p \in M$

Q: Is this well-defined?

Suppose we have charts (U, φ), (U', φ') at p
(V, ψ), (V', ψ') at $f(p)$

$$\psi' \circ f \circ (\varphi')^{-1} = \underbrace{(\psi' \circ \psi^{-1})}_{\text{transition map}} \circ (\psi \circ f \circ \varphi^{-1}) \circ \underbrace{(\varphi \circ (\varphi')^{-1})}_{\text{transition map}}$$

$\psi' \circ \psi^{-1}$ and $\varphi \circ (\varphi')^{-1}$ are transition maps and thus smooth. Then $\psi' \circ f \circ (\varphi')^{-1}$ is smooth iff $\psi \circ f \circ \varphi^{-1}$ is smooth. Hence it's well-defined.

Example: (U, φ) is a chart then $\forall\ \varphi: U \to \mathbb{R}^n$ is smooth, $U \subseteq M$

Choose (U, φ) on M and $(\mathbb{R}^n, \mathrm{id})$ on $\mathbb{R}^n$

$id \circ \varphi \circ \varphi^{-1} = id \ : \varphi(U) \to \varphi(U)$ is smooth

Example For any M, $id : M \to M$ is smooth

Take chart (U, φ) on M, then
$\varphi \circ id \circ \varphi^{-1} = id$ is smooth on $\varphi(U)$
hence id is smooth on M

Definition 1.6: A map $f : M \to N$ is a diffeomorphism if it is a smooth bijection with smooth inverse. And if such f exists, we say M and N are diffeomorphic

A map $f : M \to N$ is a local diffeomorphism at $p \in M$ if $\exists$ open $U \ni p$, open $V \ni f(p)$ s.t. $f : U \to V$ is a diffeomorphism

Remark: A local diffeomorphism is a diffeomorphism iff it is a bijection as well.

Example • $id : M \to M$ is a diffeomorphism
• (U, φ) chart, $\varphi : U \to \mathbb{R}^n$ is a local diffeomorphism (it may not be surj.)

Theorem 1.7: Let M be a manifold and let G be a discrete group (i.e. countable) with identity e. We say G acts freely and properly discontinuously (by diffeomorphisms) on M if: $\forall$
$\forall g \in G \ \exists$ diffeo $\phi_g : M \to M$ with
• $\phi_e = id$

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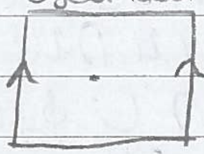
- $\phi_g \circ \phi_h = \phi_{gh} \quad \forall\, g, h \in G$
- $\forall\, p \in M \ \exists$ open $V \ni p$ s.t. $V \cap \phi_g(V) = \emptyset$ $\forall g \neq e$ $\Rightarrow$ free i.e. no fixed pts
- $\forall\, p, q \in M$ s.t. $q \neq \phi_g(p)\ \forall g \in G$ $\exists$ open $V \ni p$, open $W \ni q$ s.t. $V \cap \phi_g(W) = \emptyset \ \forall\, g \in G$ $\Rightarrow$ properly discontinuous

Define $p \sim q \iff q = \phi_g(p)$ for some $g \in G$.
Then if M is an n-dim manifold then $M/\sim\ = M/G$ is an n-dim manifold

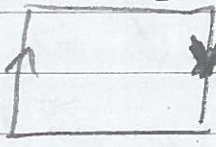
<u>Example</u> : $\mathbb{Z}_2 = \{-1, +1\}$ acts on $\mathbb{R}^n$ by $\phi_1 = id \quad \phi_{-1} = -id$ and acts freely and properly discontinuously on $\mathbb{R}^n \setminus \{0\}$

- $S^n \subseteq \mathbb{R}^{n+1}$; $0 \notin S^n \Rightarrow \underset{\mathbb{R}P^n}{\underset{\parallel}{S^n/\mathbb{Z}_2}}$ is an n-dim manifold
- $C = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 = 1,\ |x_3| < 1\}$ $\quad 0 \notin C \Rightarrow$ $C/\mathbb{Z}_2$ is a 2-dim manifold

Cylinder

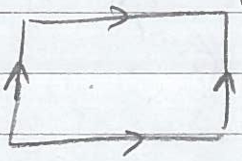


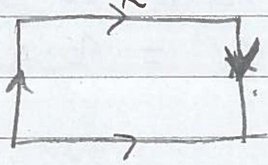
$C/\mathbb{Z}_2 = $ Möb



- $0 \notin T^2 \subseteq \mathbb{R}^3 \Rightarrow \underset{K^2}{\underset{\parallel}{T^2/\mathbb{Z}^2}}$ is 2-dim manifold

T^2



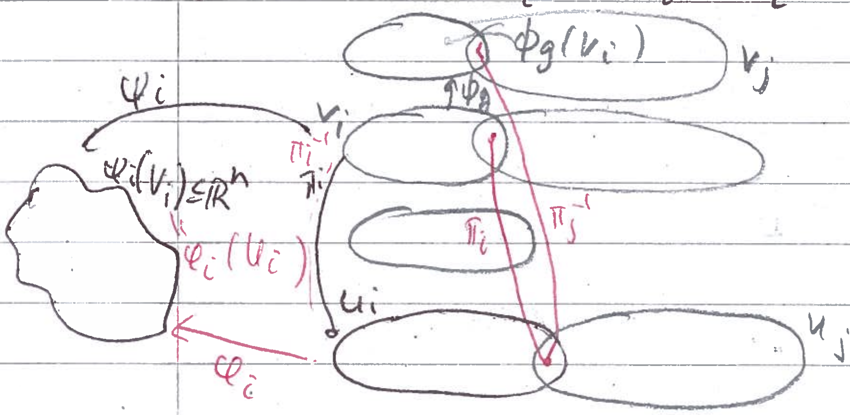


<u>Proof of Thm 1.7</u> :
Let $\{(V_i, \psi_i)\}$ be an atlas for M s.t. $V_i \cap \phi_g(V_i) = \emptyset \quad \forall g \notin id$, possible by the choice of action as free and properly discontinuous.

• Let $\pi : M \to M/G$ be the projection map
π are open maps. $\Rightarrow$ $\pi(V_i) = U_i \subseteq M/G$
is open and $\bigcup_i (U_i) = M/G$

• $\pi_i = \pi|_{V_i} : V_i \to U_i$ is a bijection because

$\pi(p) = \pi(q)$ iff $q = \phi_g(p)$ for some $g \in G$
and we chose $V_i \cap \phi_g(V_i) = \emptyset$ $\forall g \neq e$
$\Rightarrow$ it is injective
In fact π_i is a homeomorphism.
we let $\varphi_i = \psi_i \circ \pi_i^{-1}$



φ_i is a homeomorphism since ψ_i and π_i are
• $U_i \cap U_j \neq \emptyset$ $\Rightarrow$ $\varphi_i(U_i \cap U_j) = \psi_i \circ \pi_i^{-1}(U_i \cap U_j)$
$$= \psi_i\Big(V_i \cap \bigcup_{g \in G} \phi_g(V_j)\Big)$$

$\varphi_j \circ \varphi_i^{-1}$ is a homeomorphism so we just need to show it & its inverse are smooth

$$\varphi_j \circ \varphi_i^{-1} = \underbrace{\psi_j}_{\text{smooth}} \circ \pi_j^{-1} \circ \pi_i \circ \underbrace{\psi_i^{-1}}_{\text{smooth}}$$

Since ψ_i and ψ_j are smooth it is enough to show $\pi_j^{-1} \circ \pi_i$ is smooth.

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Recall : M n-dim manifold, group G $\Rightarrow$
M/G is n-dim manifold

Atlas $\{(V_i, \psi_i)\}$ for M, $V_i \cap \phi_g(V_i) = \emptyset$ $\forall g \neq e$

$\pi : M \to M/G$ then $\{(U_i, \varphi_i)\}$ in M/G
$U_i = \pi(V_i)$, $\pi_i = \pi|_{V_i} : V_i \to U_i$

$\varphi_i = \psi_i \circ \pi_i^{-1} : U_i \to \mathbb{R}^n$

We just need to show $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ is smooth

$\varphi_j \circ \varphi_i^{-1} = \psi_j \circ \pi_j^{-1} \circ \pi_i \circ \psi_i^{-1}$, ψ_j & ψ_i^{-1} smooth since chart maps, so we want to show $\pi_j^{-1} \circ \pi_i$ is smooth.

Look at $\pi_j^{-1} \circ \pi_i$ on $\psi_i^{-1} \circ \varphi_i(U_i \cap U_j) =$
$= \pi_i^{-1}(U_i \cap U_j) =$
$= \bigcup_{g \in G} V_i \cap \phi_g(V_j)$

Let $p \in \bigcup_{g \in G} V_i \cap \phi_g(V_j) \Rightarrow \exists!\, g \in G$ s.t. $p \in V_i \cap \phi_g(V_j)$

Let $q \in V_i \cap \phi_g(V_j)$ and let $q' = \pi_j^{-1} \circ \pi_i(q)$
$\Rightarrow \pi_j(q') = \pi_i(q)$

That means $\exists!\, g_q \in G$ s.t. $q = \phi_{g_q}(q') \Rightarrow$

$q' \in V_j \Rightarrow q \in \phi_{g_q}(V_j)$ but $q \in \phi_{g_q}(V_j) \cap \phi_g(V_j)$

but this is only possible if $g_q = g$
so $\pi_j^{-1} \circ \pi_i = \phi_{g^{-1}}$ on $V_i \cap \phi_g(V_j)$ and

hence is smooth at p.

Proposition 1.8 : If G acts freely and properly discontinuously on M then $\pi : M \to M/G$ is a local diffeomorphism

Proof : Claim : $\pi_i : V_i \to U_i$ is a diffeomorphism

Use (V_i, ψ_i) on M and (U_i, φ_i) on M/G

Then $\varphi_i \circ \pi_i \circ \psi_i^{-1} = \psi_i \circ \pi_i^{-1} \circ \pi_i \circ \psi_i^{-1} = id$ which is smooth, φ_i & ψ_i^{-1} are local diffeomorphism, hence π_i is a diffeo ■

2. Tangent Vectors and the tangent bundle

Think again about tangent vectors even in $\mathbb{R}^n$.

Let α be a curve in $\mathbb{R}^n$ with $\alpha(t) = (a_1(t), \dots, a_n(t))$
Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function
Then $f \circ \alpha : \mathbb{R} \to \mathbb{R}$ is a smooth map

$$\frac{d}{dt}(f \circ \alpha)\Big|_{t=0} = \frac{d}{dt} f(a_1(t), \dots, a_n(t))\Big|_{t=0} =$$

$$= \sum_{i=1}^{n} a_i'(t) \frac{df}{dx_i}(a_1(t), \dots, a_n(t))\Big|_{t=0}$$

Now let $t=0$

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$$= \left(\sum_{i=1}^{n} a_i'(0) \frac{\partial}{\partial x_i}\Big|_{\alpha(0)} \right) f$$

Moral: We can think of tangent vectors as differential operators on functions.

Definition 2.1 : A curve in M (through p) is a smooth map $\alpha : (-\varepsilon, \varepsilon) \longrightarrow M$ (with $\alpha(0) = p$) s.t. $\forall t \in (-\varepsilon, \varepsilon)\ \exists \delta > 0$ and a chart (U, φ) s.t. $\alpha(t-\delta, t+\delta) \subseteq U$ and $\varphi \circ \alpha$ is a curve in $\varphi(U) \subseteq \mathbb{R}^n$.

Definition 2.2 : Let α be a curve in M through p. Let $f : \underset{\text{open}}{U \subseteq M} \longrightarrow \mathbb{R}$ be smooth $p \in U$. Then $f \circ \alpha : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}$ is smooth at 0 so $(f \circ \alpha)'(0) \in \mathbb{R}$. The tangent vector to α at 0 is $\alpha'(0) : f \longrightarrow (f \circ \alpha)'(0)$ for any such f.

If we were in a chart (U, φ) and write $\varphi \circ \alpha(t) = (a_1(t), \dots, a_n(t))$ then

$$(f \circ \alpha)'(0) = (\underbrace{f \circ \varphi^{-1}}_{\text{function on } \mathbb{R}^n} \circ \underbrace{\varphi \circ \alpha}_{\in \mathbb{R}^n})'(0) = \underbrace{\left(\sum_{i=1}^{n} (a_i'(0)) \frac{d}{dx_i}\Big|_{\varphi \circ \alpha(0)} \right)}_{\text{represents } \alpha'(0) \text{ in } (U, \varphi)} (f \circ \varphi^{-1})$$

Definition 2.3 : A tangent vector X at $p \in M$ is given by $X = \alpha'(0)$ for some curve α through p

Definition 2.4 Let $T_pM = \{$ tangent vectors at $p \}$

Proposition T_pM is an n-dim vector space

Proof: In (U, φ) $\left.\frac{\partial}{\partial x_i}\right|_{\varphi(p)}$ is a basis.

Remark: $\left.\frac{\partial}{\partial x_i}\right|_{\varphi(p)} = \alpha_i'(0)$ where

$$\alpha_i(t) = \varphi^{-1}(0, 0, \ldots, 0, \underset{i^{th} \text{ place}}{t}, 0, \ldots, 0)$$

Proposition 2.6: Let $F: \mathbb{R}^{n+m} \to \mathbb{R}^m$ be smooth s.t. $F^{-1}(c) \neq \emptyset$ and $dF_p: \mathbb{R}^{n+m} \to \mathbb{R}^m$ surjective $\forall p \in F^{-1}(c)$
Then $\forall p \in F^{-1}(c)$, $T_p F^{-1}(c) \cong \operatorname{Ker} dF_p$

Proof: $T_p F^{-1}(c)$ and $\operatorname{Ker} dF_p$ are n-dim vector spaces. Let α be a curve in $F^{-1}(c) \Rightarrow F(\alpha(t)) = c$

Then $\frac{d}{dt} F(\alpha(t)) = 0$

$$dF_p\left(\frac{d\alpha}{dt}(0)\right) = 0$$

$$\Rightarrow \frac{d\alpha}{dt}(0) \in \operatorname{Ker} dF_p$$

since $T_p F^{-1}(c)$ & $\operatorname{Ker} dF_p$ are both n-dim v.s. this is enough to show $\cong$ ∎

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Example: $S^n = F^{-1}(1)$ where

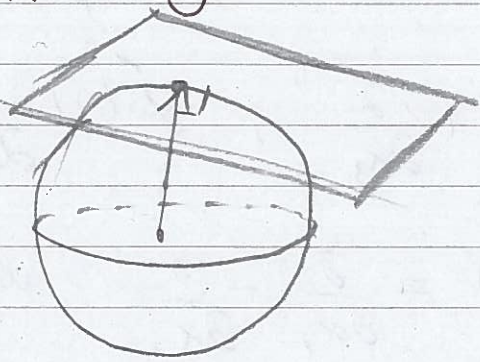
$$F(x_1, \dots, x_{n+1}) = \sum_{i=1}^{n+1} x_i^2$$

$$dF_x = 2(x_1, \dots, x_{n+1})$$

$$dF_x \begin{pmatrix} y_1 \\ \vdots \\ y_{n+1} \end{pmatrix} = 2.\langle x, y \rangle = 2(x_1 y_1 + x_2 y_2 + \dots x_{n+1} y_{n+1})$$

just the dot product

$$\operatorname{Ker} dF_x = \{ y \in \mathbb{R}^{n+1} : \langle x, y \rangle = 0 \} \cong T_x S^n$$



Example: $O_n = \{ A \in M_n(\mathbb{R}) : A^T A = I \}$

$$= F^{-1}(0)$$

$$F(A) = A^T A - I$$

$$dF_A(B) = B^T A + A^T B \Rightarrow$$

$$dF_I(B) = B^T + B$$

$$\Rightarrow \operatorname{Ker} dF_I = \{ B \in M_n(\mathbb{R}) \mid B^T = -B \} = \text{skew symmetric } n \times n \text{ mat}$$

$$\cong T_I(O_n) = o(n) - \text{Lie algebra of } O_n$$

The tangent space at the identity is its Lie algebra

Exercise: Show $SL(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) : \det A = 1\}$ is a manifold, compute its dim and: $T_I SL(n, \mathbb{R})$

Non-example: Let $C = \{(x_1, x_2, x_3) \in \mathbb{R}^3$ s.t. $x_1^2 + x_2^2 = x_3^2\}$

Consider $\alpha(t) = (t, 0, t)$ $\quad \alpha'(0) = (1, 0, 1)$
$\beta(t) = (0, t, t)$ $\quad \beta'(0) = (0, 1, 1)$

i.e. $\alpha'(0) = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}$, $\beta'(0) = \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}$

$\alpha'(0) - \beta'(0) = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}$, which is not tangent!

$\Rightarrow$ The set of tangent vectors at 0 in C is not a vector space so C is not a mani-

Definition 2.7 Let f be a smooth map between manifolds $f : M \to N$. Let $p \in M$ and let $X = \alpha'(0) \in T_p M$. α is a curve through p i.e. $f \circ \alpha$ is a curve through $f(p)$ so we define the differential

$$\boxed{\begin{gathered} df_p : T_p(M) \to T_{f(p)}(N) \quad \text{by} \\ df_p(X) = (f \circ \alpha)'(0) \end{gathered}}$$

Suppose $\alpha'(0) = \beta'(0)$, I want to show $(f \circ \alpha)'(0) = (f \circ \beta)'(0)$

Let h be a smooth real valued function defined near $f(p)$

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Then $(f\circ\beta)'(0)(h) = (h\circ(f\circ\beta))'(0) =$
$= (h\circ f\circ\beta)'(0) =$
$= \beta'(0)(h\circ f) =$
$= \alpha'(0)(h\circ f) =$
$= (f\circ\alpha)'(0)(h)$

$\Rightarrow (f\circ\beta)'(0) = (f\circ\alpha)'(0)$ so df_p is well defined!

Remarks : We can work in charts. Let (U,φ) be a chart at p and (V,ψ) be a chart at $f(p)$. α is a curve through p
$\Rightarrow a = \varphi\circ\alpha$ is a curve in $\varphi(U)$
$f\circ\alpha$ is a curve through $f(p)$
$\Rightarrow b = \psi\circ f\circ\alpha$ is a curve in $\psi(V)$

$$b'(0) = (\psi\circ f\circ\alpha)'(0) = (\psi\circ f\circ\varphi^{-1}\circ\varphi\circ\alpha)'(0) = d(\psi\circ f\circ\varphi^{-1})_{\varphi(p)}(a'(0))$$

$\Rightarrow$ We can view df_p as $d(\psi\circ f\circ\varphi^{-1})_{\varphi(p)}$ w.r.t. these charts

Example : Let $\pi: S^2 \longrightarrow \mathbb{R}P^2$ $\quad p=(0,0,1)\in S^2$

Recall chart $(U_S^{\ni p},\varphi_S)$ on S^2 $\quad \pi(p) = [(0,0,1)]$
chart $(U_3^{\ni\pi(p)},\varphi_3)$ on $\mathbb{R}P^2$

we want to calculate $\varphi_3\circ\pi\circ\varphi_S^{-1}$ near $\varphi_S(0,0,1) = (0,0)$

$$\varphi_3\circ\pi\circ\varphi_S^{-1}(y_1,y_2) = \varphi_3\circ\pi\left(\frac{2y_1,\,2y_2,\,1-y_1^2-y_2^2}{1+y_1^2+y_2^2}\right) =$$
$$= \frac{2(y_1,y_2)}{1-y_1^2-y_2^2} \quad \text{if } y_1^2+y_2^2<1$$

$$d(\varphi_3 \circ \pi \circ \varphi_S^{-1})_{\underset{\varphi_S(p)}{\overset{\parallel}{(0,0)}}} =$$

$$\frac{2}{(1-y_1^2-y_2^2)^2}\begin{pmatrix} 1+y_1^2-y_2^2 & 2y_1y_2 \\ 2y_1y_2 & 1+y_2^2-y_1^2 \end{pmatrix}_{(0,0)} =$$

$$= 2I$$

Proposition 2.8

A smooth map $f: M \to N$ is a local diffeomorphism at $p \in M$ iff $df_p : T_pM \to T_{f(p)}N$ is an isomorphism.

Proof: $id: M \to M$ then $d\,id_p = id \ \forall p \in M$
if $f_1: M_1 \to M_2$, $f_2: M_2 \to M_3$ then

$$d(f_2 \circ f_1)_p = df_{2\,f_1(p)} \circ df_1(p)$$

The chain rule holds since it holds in $\mathbb{R}^n$.

$\Rightarrow$) Let $f: M \to N$ be local diffeo at $p \in M$
$\Rightarrow \exists \underset{\text{open}}{U} \ni p$, open $V \ni f(p)$ s.t. $f: U \to V$
is a diffeomorphism

$f \circ f^{-1}|_V = id = f^{-1} \circ f|_U \Rightarrow d(f \circ f^{-1})_{f(p)} = id = d(f^{-1} \circ f)_p$

$\Rightarrow df_p \circ d(f^{-1})_{f(p)} = id = d(f^{-1})_{f(p)} \circ df_p$

$\Rightarrow df_p$ is an isomorphism and $(df^{-1}_{f(p)})^{-1} = df_p$

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$\Leftarrow$) Suppose df_p is an isomorphism

Let (U, φ) be a chart at p, (V, ψ) be a chart at $f(p)$ with $f(U) \subseteq V$

Then $\Rightarrow$) gives us $d\varphi^{-1}_{\varphi(p)}\ \mathbb{R}^n \to T_p(M)$

and $d\psi_{f(p)}\ T_{f(p)}N \to \mathbb{R}^n$ since df_p an iso

M, N have the same dimension n (say)

are isomorphisms $d(\psi \circ f \circ \varphi^{-1})_{\varphi(p)} =$

$= \underset{\text{iso}}{\underset{\downarrow}{d\psi_{f(p)}}} \circ \underset{\text{iso}}{\underset{\downarrow}{df_p}} \circ \underset{\text{iso}}{\underset{\downarrow}{d\varphi^{-1}_{\varphi(p)}}}$ is an isomorphism.

But this is a map between Euclidean spaces whose derivative is an iso so by the Inverse function Theorem i.e.

$\exists$ open $\underset{U}{\underset{\cap}{\tilde{U}}} \ni p$, $\underset{V}{\underset{\cap}{\tilde{V}}} \ni f(p)$ s.t. $\psi \circ f \circ \varphi^{-1} : \varphi(\tilde{U}) \to \psi(\tilde{V})$ is diffeo.

$\Rightarrow f : \tilde{U} \to \tilde{V}$ is a local diffeomorphism

$\blacksquare$

Definition 2.9 The tangent bundle

$$\boxed{TM = \bigcup_{p \in M} T_p M}$$

Theorem 2.10: TM is a $2n$-dim manifold

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Example: $f: \mathbb{R}^2 \to T^2 \subseteq \mathbb{R}^3$

$$f(\theta, \phi) = ((2+\cos\theta)\cos\phi, (2+\cos\theta)\sin\phi, \sin\theta)$$

$$df_{(\theta,\phi)} = \begin{pmatrix} -\sin\theta\cos\phi & -(2+\cos\theta)\sin\phi \\ -\sin\theta\sin\phi & +(2+\cos\theta)\cos\phi \\ \cos\theta & 0 \end{pmatrix}$$

This has rank 2 $\quad df_{(\theta,\phi)}: T_{(\theta,\phi)}\mathbb{R}^2 \to T_{f(\theta,\phi)}T^2$

is surjective and injective since it does from 2-dim space to 2-dim space
So f is a local diffeomorphism.

It is not a diffeomorphism because it is not 1-1

Recall: $TM = \bigcup_{p \in M} T_pM = \{(p, X): p \in M, X \in T_pM\}$

Theorem 2.10: TM is $2n$-dim manifold.

Proof: Let $\{(U_i, \varphi_i)\}$ be an atlas for M and let $\pi: TM \to M$ be projection.

- Let $V_i = \pi^{-1}(U_i)$, open and $\bigcup_i V_i = TM$
- Let $\psi_i: V_i \to \mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ be

$$\psi_i(p, X) = (\underset{\text{homeomorphism}}{\varphi_i(p)}, \underbrace{d(\varphi_i)_p(X)}_{\text{isomorphism}})$$

$\Rightarrow \psi_i: V_i \to \psi_i(V_i)$ is a homeomorphism

- $\psi_j \circ \psi_i^{-1}(q, Y) = (\varphi_j \circ \varphi_i^{-1}(q), d(\varphi_j)_{\varphi_i^{-1}(q)} \circ d(\varphi_i^{-1})_q(Y))$

$$= \Big(\underbrace{\varphi_j \circ \varphi_i^{-1}(q)}_{\text{diffeomorphism}} \, , \, \underbrace{d(\varphi_j \circ \varphi_i^{-1})_q (v)}_{\text{isomorphism}} \Big)$$

So $\psi_j \circ \psi_i^{-1}$ is a diffeomorphism ∎

Examples:

- $T_p \mathbb{R}^n = \mathbb{R}^n \Rightarrow T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$

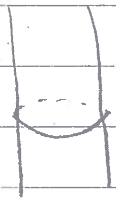
  For every pt in $\mathbb{R}^n$ stick in another pt in $\mathbb{R}^n$!

- $T_p S^1 = \{ q \in \mathbb{R}^2 : \langle p, q \rangle = 0 \}$, $TS^1 = \{ ((\cos\theta, \sin\theta), \lambda(-\sin\theta, \cos\theta)) : \theta, \lambda \in \mathbb{R} \}$

  $p \overset{\shortparallel}{} (\cos\theta, \sin\theta)$

  smth orthogonal to $(\cos\theta, \sin\theta)$

  $TS^1 \cong \underset{\text{for } \theta}{S^1} \times \underset{\text{for } \lambda}{\mathbb{R}}$



  infinite cylinder

- $TS^2 \neq S^2 \times \mathbb{R}^2$ (Hairy dog theorem)

Definition 2.11 Let $f: M \to N$ be smooth. Then the pushforward $f_*: TM \to TN$ is

$$\boxed{f_*(p, X) = (f(p), df_p(X))}$$

Examples:

- $id: M \to M \Rightarrow id_* = id: TM \to TM$
- $(f \circ g)_* = f_* \circ g_*$ (Chain Rule)
- $f: M \to N$ is a diffeomorphism $\Rightarrow$ $f_*: TM \to TN$ is a diffeomorphism s.t. $f_*|_{T_pM}: T_pM \to T_{f(p)}M$ is an isomorphism $\forall p \in M$

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This f_* is called a bundle isomorphism

Definition 2.12: A manifold E is a vector bundle over M if:

- $\exists$ smooth surjective map $\pi: E \to M$ s.t.
- $\pi^{-1}(p)$ is a vector space $\forall p \in M$ and
- $\forall p \in M$ $\exists$ open $U \ni p$ and a diffeomorphism $\psi: \pi^{-1}(U) \to U \times \mathbb{R}^m$ for some m s.t. $\psi|_{\pi^{-1}(q)}: \pi^{-1}(q) \to \{q\} \times \mathbb{R}^m$ is an isomorphism $\forall q \in U$

(namely ψ is a bundle isomorphism)

Remark: The integer m is the same $\forall p \in M$ and it is called the rank of E

$\Rightarrow E$ is an $(m+n)$-dimensional manifold!

Examples

- $M \times \mathbb{R}^m$ is a vector bundle of rank m over M
- TM is a vector bundle of rank n over M

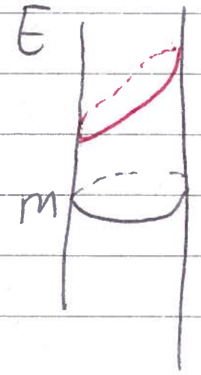
Definition 2.13: A vector bundle E of rank m over M is trivial if $\exists$ a bundle isomorphism $\psi: E \to M \times \mathbb{R}^m$

If TM is trivial we say M is parallelizable

Example: S^1 and S^3 are parallelizable but S^2 is not

Definition 2.14 A section of a vector bundle E over M is a smooth map $s: M \to E$ s.t. $\pi \circ s = id_M$

$$\begin{tikzcd} E \arrow[d, "\pi"'] \\ M \arrow[u, bend right, "s"'] \end{tikzcd}$$



We let $\Gamma(E) = \{\text{sections of } E\}$, which is a vector space (infinite dimensional)

Proposition 2.15: The vector bundle of rank m is trivial iff it has m l.i. sections

Proof: $\Rightarrow$) E is a trivial vector bundle of rank m over $M \Rightarrow \exists$ a bundle isomorphism $\chi : M \times \mathbb{R}^m \to E$.

Let $e_1, \dots, e_m$ be a basis for $\mathbb{R}^m$ and define $s_i : M \to E$ by $s_i(p) = \chi(p, e_i)$

χ is smooth $\Rightarrow s_i$ is smooth and $\pi \circ s_i(p) = p$ because χ is a bundle isomorphism $\forall p \Rightarrow s_i \in \Gamma(E)\ \forall i$ and if $(\lambda_1 s_1 + \dots + \lambda_m s_m)(p) = 0$ for some $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ and $p \in M$ then

$$\lambda_1 \chi(p, e_1) + \dots + \lambda_m \chi(p, e_m) = 0$$

$$\chi(p, \lambda_1 e_1) + \dots + \chi(p, \lambda_m e_m)) = \chi(p, \lambda_1 e_1 + \dots + \lambda_m e_m) = 0$$

iff $\lambda_1 e_1 + \dots + \lambda_m e_m) = 0$

$\Rightarrow \lambda_1 = \lambda_2 = \dots = \lambda_m = 0$ as e_i forms a basis for $\mathbb{R}^m$

$\Leftarrow$) Suppose $s_1, \dots, s_m \in \Gamma(E)$ are l.i

Define $\chi : M \times \mathbb{R}^m \to E$ by

$$\chi(p, \lambda_1 e_1 + \dots + \lambda_m e_m) = \lambda_1 s_1(p) + \dots + \lambda_m s_m(p)$$

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Then s_i smooth $\Rightarrow$ χ is smooth
s_i l.i and sections $\Rightarrow$ χ is a bundle isomorphism ∎

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§ 3 Vector Fields

Definition 3.1 A vector field X on M is a section of TM i.e. $X : M \to TM$ smooth s.t. $X(p) \in T_pM$ $\forall p \in M$

Example: Let e_i be the i^{th} coordinate vector on $\mathbb{R}^n$. Then we define ∂_i vector field on $\mathbb{R}^n$ by $\partial_i(p) = e_i \in T_p\mathbb{R}^n$. For a function $f : \mathbb{R}^n \to \mathbb{R}$ we have

$$\partial_i(f)(p) = \alpha_i'(0)(f) = (f \circ \alpha_i)'(0) =$$

$$= \frac{d}{dt} f(p + te_i)\Big|_{t=0} =$$

Note $e_i = \alpha_i'(0)$, where $\alpha_i(t) = p + te_i$

$$= \frac{\partial f}{\partial x_i}(p)$$

Therefore ∂_i is the differential operator $\frac{\partial}{\partial x_i}$.

In general if X is a vector field, $f : M \to \mathbb{R}$ is a function, then $X(f) : M \to \mathbb{R}$ is another function. And X is a differential operator on f

Definition 3.2: Let $f: M \to N$ be a diffeomorphism, We define the **push forward** $f_* : \Gamma(TM) \to \Gamma(TN)$ by

$$\boxed{\underbrace{f_*(X)}_{X \in \Gamma(TM)}(f(p)) = \underbrace{df_p X(p)}_{\in T_{f(p)}N} \quad \forall p \in M}$$

Example: (U, φ) chart on $M \Rightarrow$

$\varphi : U \to \varphi(U) \subseteq \mathbb{R}^n$ is a diffeo

$\Rightarrow$ if X is a vector field on U

Then $\varphi_*(X)$ is a vector field on $\mathbb{R}^n$

$\varphi_*(X) = \sum_{i=1} x_i \partial_i$, where $x_i : \varphi(U) \to \mathbb{R}$

are smooth functions. And ∂_i form a basis

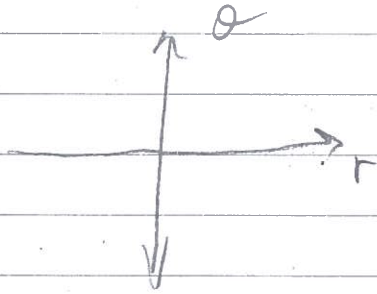
Example $f : \mathbb{R}^+ \times S^1 \to \mathbb{R}^2 \setminus \{0\}$

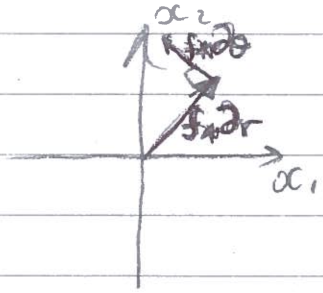
$(r, \theta) \mapsto (r\cos\theta, r\sin\theta)$

This is a diffeomorphism

$$f_* \partial_r = \cos\theta\, \partial_1 + \sin\theta\, \partial_2$$

$$f_* \partial_\theta = -r\sin\theta\, \partial_1 + r\cos\theta\, \partial_2$$





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Example $f: (0,\pi)\times(0,2\pi) \longmapsto S^2$
$(\theta,\phi) \to (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

$$f_* \partial_\theta = \cos\theta\cos\phi\,\partial_1 + \cos\theta\sin\phi\,\partial_2 - \sin\theta\,\partial_3$$

$$f_* \partial_\phi = -\sin\theta\sin\phi\,\partial_1 + \sin\theta\cos\phi\,\partial_2$$

Suppose X and Y are vector fields on $\mathbb{R}^n \Rightarrow X = \sum_i X_i \partial_i$, $Y = \sum_i Y_i \partial_i$

What is $(X \circ Y)(f) = X(Y(f)) =$

$$= X\left(\sum_i Y_i \frac{\partial f}{\partial x_i}\right) = \left(\sum_{i,j} X_j \frac{\partial Y_i}{\partial x_j}\frac{\partial f}{\partial x_i}\right) + \sum_{i,j} X_j Y_i \frac{\partial^2 f}{\partial x_j x_i}$$

In general this can't be a vector field since we have 2^{nd} derivatives

$$(YX)(f) = \sum_{i,j} Y_j \frac{\partial X_i}{\partial x_j}\frac{\partial f}{\partial x_i} + \sum_{i,j} X_j Y_i \frac{\partial^2 f}{\partial x_j \partial x_i}$$

$\Rightarrow XY - YX$ is a vector fields.

Definition 3.3 $X, Y \in \Gamma(TM) \Rightarrow XY - YX = [X,Y]$ is a vectorfield called the Lie bracket of X and Y. If (U,φ) is a chart and $\varphi_* X = \sum X_i \partial_i$ and $\varphi_* Y = \sum Y_i \partial_i$

Then $$\boxed{\varphi_*[X,Y] = \sum_{i,j}\left(X_i \frac{\partial Y_j}{\partial x_i} - Y_i \frac{\partial X_j}{\partial x_i}\right)\partial_j}$$

Remark: $[Y,X] = -[X,Y]$

Example $[\partial_i, \partial_j] = 0$

Example: Let
$$X = x_3 \partial_2 - x_2 \partial_3$$
$$Y = x_1 \partial_3 - x_3 \partial_1$$
$$Z = x_2 \partial_1 - x_1 \partial_2$$

$$[X, Y] = (x_3\partial_2 - x_2\partial_3)(x_1\partial_3 - x_3\partial_1) -$$
$$-(\underbrace{x_1\partial_3}_{=0} - x_3\partial_1)(x_3\partial_2 - x_2\partial_3) =$$
$$= x_3\partial_2(x_1)\partial_3 - x_3\partial_2(x_3)\partial_1 - x_2\partial_3(x_1)\partial_3$$
$$= -x_2\partial_3(-x_3)\partial_1 - x_1\partial_3(x_3)\partial_2 =$$
$$= x_2\partial_1 - x_1\partial_2 = Z$$

Similarly $[Y, Z] = X$ and $[Z, X] = Y$

$\Rightarrow [aX + bY + cZ, a'X + b'Y + c'Z] =$
$= (bc' - b'c)X + (ca' - c'a)Y + (ab' - ba')Z$

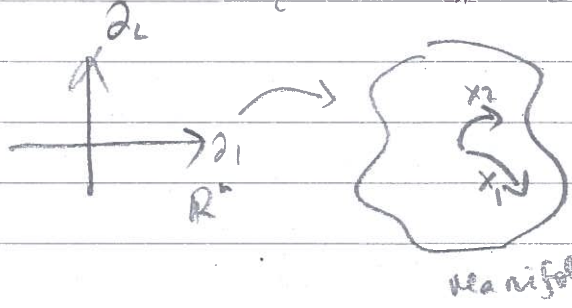
a, b, c constants.

$\Rightarrow$ If we identify $\mathrm{Span}\{X, Y, Z\} \cong \mathbb{R}^3$ then Lie bracket corresponds with the cross product.

Remark: $\varphi_*[X, Y] = [\varphi_* X, \varphi_* Y]$

Proposition 3.4: Let $f: M \to N$ is a diffeomorphism. Then $\forall X, Y \in \Gamma(TM)$, we have
$$f_*[X, Y] = [f_* X, f_* Y]$$

Example: If (U, φ) is a chart, then $X_i = (\varphi^{-1})_* \partial_i$ (the coordinate vector fields on (U, φ)) satisfy
$$[X_i, X_j] = 0$$
By proposition 3.4



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Proof (Prop. 3.4)

Let (U, φ) be a chart on $M \Rightarrow (f(U), \varphi \circ f^{-1} = \psi)$ is a chart on N

Look at $\psi_* \circ f_* [X, Y] = (\varphi \circ f^{-1})_* \circ f_* [X, Y] =$

$= \varphi_* \circ (f^{-1}_*) \circ f_* [X, Y] = \varphi_* \circ (f^{-1} \circ f)_* [X, Y]$

$= \varphi_* [X, Y] = [\varphi_* X, \varphi_* Y] =$

$= [\psi_* \circ f_* X, \psi_* \circ f_* Y] =$

$= \psi_* [f_* X, f_* Y]$

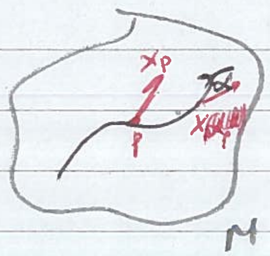
Since ψ_* is a diffeomorphism. This holds $\forall$ charts $(f(U), \psi) \Rightarrow f_*[X, Y] = [f_* X, f_* Y]$

Proposition 3.5 : The Lie bracket satisfies the Jacobi identity

Namely $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

Let $\alpha : (-\varepsilon, \varepsilon) \to M$ be a curve, $\forall t \in (-\varepsilon, \varepsilon)$, $\alpha'(t) \in T_{\alpha(t)} M \Rightarrow \alpha' : t \to \alpha'(t)$ is smooth

α' is a vector field along α. Let $X \in \Gamma(TM)$ and let $p \in M$



Claim: $\exists$ unique curve $\alpha_p : (-\varepsilon, \varepsilon) \to M$ s.t. $\alpha'_p(t) = X(\alpha_p(t))$, $\alpha_p(0) = p$

Suppose $(\mathcal{U}, \varphi)$ is a chart at p. $\varphi \circ \alpha(t) = (a_1(t), \dots, a_n(t))$, $\varphi_* X = \sum_i X_i \partial_i$

$\Rightarrow (\varphi \circ \alpha)' = \varphi_* \circ \alpha'$
By Chain Rule

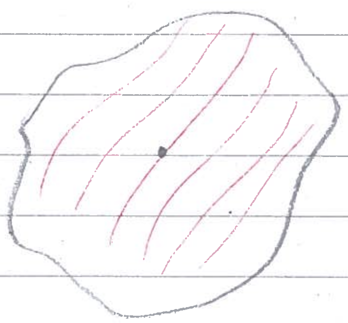
$\varphi_* \alpha'(t) = \sum_{i=1}^{n} a_i' \partial_i$ and $\varphi_* X(\alpha(t)) = \sum_{i=1}^{n} X_i(a_1(t), \dots, a_n(t)) \partial_i$

$\therefore \alpha' = X(\alpha(t)) \iff a_i'(t) = X_i(a_1(t), \dots, a_n(t))$ $\forall i$ and $a_1(0), \dots, a_n(0) = \varphi(p)$

We have a system of n linear ODEs with n initial conditions. So there exist a unique solution.

Definition 3.6 Let $X \in \Gamma(TM)$ and $p \in M \Rightarrow \exists$ open $V \ni p$ s.t. $\forall q \in V$, $\exists!$ curve $\alpha_q : (-\varepsilon, \varepsilon) \longrightarrow M$, $\alpha_q'(t) = X(\alpha_q(t))$ and $\alpha_q(0) = q$

These curves are called the integral curves of X.



Example: ∂_i on $\mathbb{R}^n \Rightarrow (a_1', \dots, a_n') = e_i = (0, \dots, 0, 1, 0, \dots, 0)$
ith place

$a_j' = 0 \quad \forall j \neq i$
$a_i' = 1$

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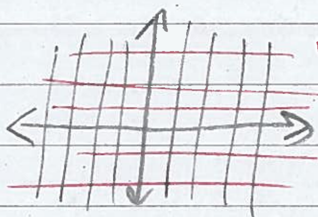
$\Rightarrow$ the integral curve through 0 is $(a_1, \dots, a_n) = (0, 0, \dots 0, \underset{i^{th} \text{ place}}{t}, 0, \dots 0)$

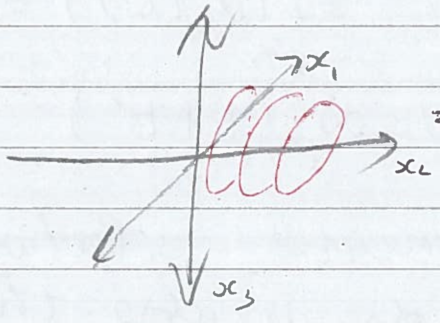


Example: $X = x_3 \partial_2 - x_2 \partial_3$ in $\mathbb{R}^3$.
The integral curves are

$$\begin{cases} a_1' = 0 \\ a_2' = a_3 \\ a_3' = -a_2 \end{cases} \Rightarrow (a_1(t), a_2(t), a_3(t)) = $$
$$= ($$

$$a_2'' = -a_2$$

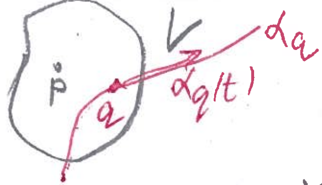
$$(a_1(t), a_2(t), a_3(t)) = (a_1(0), a_2(0)\cos t + a_3(0)\sin t, a_3(0)\cos t - a_2(0)\sin t)$$



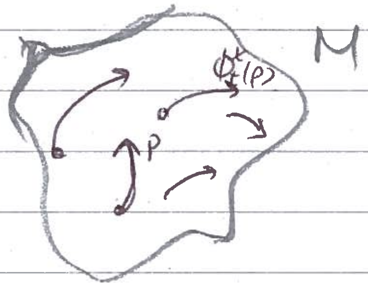
$\Rightarrow$ we should think of X as a rotation about x_1-axis

Definition 3.7: Let $X \in \Gamma(TM)$ & $p \in M$. Let $V \ni p$ open be s.t. integral curves $\alpha_q : (-\varepsilon, \varepsilon) \to M$ exist $\forall q \in V$. We define the flow of X on V (or we say near p) by

$$\boxed{\{\phi_t^X : V \to M : t \in (-\varepsilon, \varepsilon)\} \text{ where } \phi_t^X(q) = \alpha_q(t)}$$



Notice $\phi_0^X = id$ on V



Proposition 3.8

Let $\phi_t^X : V \to M$ s.t. $t \in (-\varepsilon, \varepsilon)$ be the flow of X on V then ϕ_t^X is a local diffeomorphism $\forall t \in (-\varepsilon, \varepsilon)$ and
$\phi_t^X \circ \phi_{t'}^X = \phi_{t+t'}^X$, when both sides are defined. (if $t, t' \in (\varepsilon, \varepsilon)$ and $t+t' \in (-\varepsilon, \varepsilon)$)
[This that the flow is a (one-parameter) group of local diffeos]

Proof:

$$\phi_t^X \circ \phi_{t'}^X (p) = \phi_t^X(\alpha_p(t')) = \alpha_{\alpha_p(t')}(t)$$

and $\phi_{t+t'}^X (p) = \alpha_p(t+t')$

α_p is the unique solution to $\alpha' = X(\alpha)$ with $\alpha(0) = p$. α_p is also the unique solution to $\alpha' = X(\alpha)$ with $\alpha(t') = \alpha_p(t')$
But so is $\alpha_{\alpha_p(t')}$ $\Rightarrow \alpha_p(t+t') = \alpha_{\alpha_p(t')}(t)$

with initial condition $\alpha(0) = \alpha_p(t')$

Now $\phi_t^X \circ \phi_{-t}^X = \phi_{-t}^X \circ \phi_t^X = \phi_0^X = id$

$$(\phi_t^X)_* \circ (\phi_{-t}^X)_* = (\phi_{-t}^X)_* \circ (\phi_t^X)_* = id$$

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$\Rightarrow (\phi_t^X)_*$ is an isomorphism $\Rightarrow \phi_t^X$ is a local diffeomorphism ∎

Examples:

- ∂_i on $\mathbb{R}^n \Rightarrow \phi_t^{\partial_i}(p) = p + te_i \Rightarrow$ flow of ∂_i is translation along the e_i direct

- X on $\mathbb{R}^3 \Rightarrow \phi_t^X(a_1, a_2, a_3) = (a_1, a_2\cos t + a_3 \sin t, a_3 \cos t - a_2 \sin t)$

  $\Rightarrow$ the flow is a rotation about the x_1-axis.

- Let $W = \partial_1 + x_3 \partial_2 - x_2 \partial_3$

  The integral curves are
  $$a_1' = 1$$
  $$a_2' = a_3$$
  $$a_3' = -a_2$$

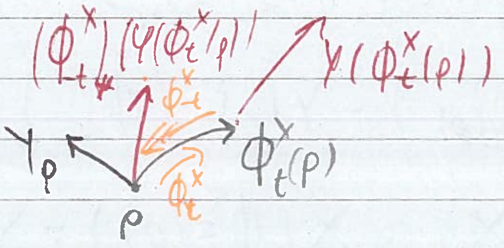
  $\Rightarrow \alpha(t) = (a_1(0) + t, a_2(0)\cos t + a_3(0) \sin t, a_3(0)\cos t - a_2(0) \sin t)$

  This is a spiral

  $\Rightarrow$ the flow is a spiral/screw motion around the x_1-axis

Let X, Y be vector fields i.e. $X, Y \in \Gamma(TM)$

The goal is to measure how Y changes w.r.t. X.



$(\phi^X_{-t})_* Y(\phi^X_t(p)) \in T_pM$ and $Y(p) \in T_p(M)$

So we should consider

$$\lim_{t\to 0} \frac{(\phi^X_{-t})_* Y(\phi^X_t(p)) - Y(p)}{t}$$

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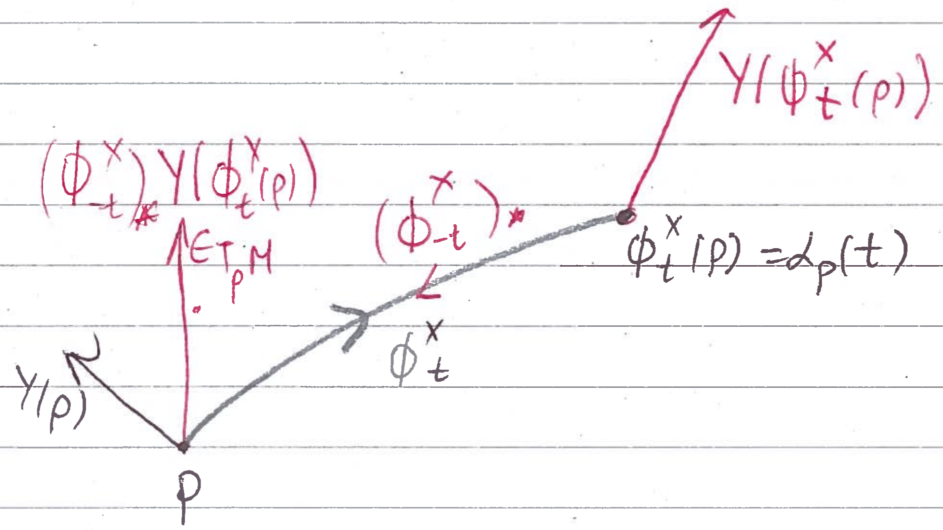

<u>Definition 3.9</u> The <u>Lie derivative of</u> Y w.r.t. X is

$$\boxed{\mathcal{L}_X Y(p) = \lim_{t\to 0} \frac{(\phi^X_{-t})_* Y(\phi^X_t(p)) - Y(p)}{t} \in T_pM}$$

where $\{\phi^X_t : t \in (-\varepsilon, \varepsilon)\}$ is the flow of X near p.

$\mathcal{L}_X Y$ is a vector field on M

<u>Example</u> : What is $\mathcal{L}_X X$?

$$(\phi^X_{-t})_* Y(\phi^X_t(p)) - Y(p) = (\phi^X_{-t})_* [Y(\phi^X_t(p)) - (\phi^X_t)_*(Y(p))]$$

Set $Y = X$: $X(\phi^X_t(p)) = X(\alpha_p(t)) = \alpha'_p(t)$

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$$(\phi_t^X)_*(X(p)) = (\phi_t^X)_*(\alpha_p'(0)) = (\phi_t^X \circ \alpha_p)'(0)$$

Look at $(\phi_t^X \circ \alpha_p)(s) = \phi_t^X(\alpha_p(s)) = \phi_t^X(\phi_s^X(p)) =$
$= \phi_{t+s}^X(p) =$
$= \alpha_p(s+t)$

$$(\phi_t^X \circ \alpha_p)'(0) = \alpha_p'(t)$$

And $(\phi_t^X)_*(X(p)) = X(\phi_t^X(p))$

$$\Rightarrow \mathcal{L}_X X(p) = \lim_{t\to 0} 0 = 0 \quad \forall p \in M$$

Proposition 3.10 $\mathcal{L}_X Y = [X, Y]$

Example • $Z = x_2 \partial_1 - x_1 \partial_2$

$$\mathcal{L}_{\partial_3} Z = [\partial_3, Z] = [\partial_3, \underline{x_2 \partial_1 - x_1 \partial_2}] =$$

since they do not depend on ∂_3

$$= 0$$

• $X = x_3 \partial_2 - x_2 \partial_3$, $Y = x_1 \partial_3 - x_3 \partial_1$

$$\Rightarrow \mathcal{L}_X Y = Z$$

§4. Differential Forms

Definition 4.1 Let $T_p^* M = \{\text{linear maps } \xi : T_p M \to \mathbb{R}\}$
i.e $T_p^* M$ is the dual space to $T_p M$

We call $T_p^* M$ the cotangent space and elements of $T_p^* M$ are cotangent vectors.

$T_p^* M$ is n-dim. v.s. if M is n-dim.

Example: Let $f : M \to \mathbb{R}$ be a smooth map
$\Rightarrow df_p : T_p M \to \mathbb{R}$ is linear $\Rightarrow$
$df_p \in T_p^* M$

Notice that $df_p(X) = df_p(\alpha'(0)) = (f \circ \alpha)'(0) =$
$= \alpha'(0)(f) =$
$= X(f)$

Remark:
In linear algebra if $T : V \to W$ is linear
then $T^* : W^* \to V^*$ $T^*(w^*)(v) = w^*(T(v))$

Definition 4.2: Let $f : M \to N$ be smooth then the pull back $df_p^* : T_{f(p)}^* N \to T_p^* M$ is

$$\boxed{df_p^* \underset{\in T_{f(p)}^* N}{(\eta)} \underset{\in T_p M}{(X)} = \eta(df_p(X))}$$

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Definition 4.3: The cotangent bundle T^*M is $T^*M = \bigsqcup_{p \in M} T_p^*M$ is a rank n vector bundle over M

Remark: Proof is the same as for TM except we use $(d(\varphi_i)_p^*)^{-1}$

Definition 4.4 A section of T^*M is called a *1-form*

Let $\xi \in \Gamma(T^*M)$ i.e. 1-form and X a vector field, $X \in \Gamma(M)$. $\xi(p) \in T_p^*M$ and $X(p) \in T_p(M) \Rightarrow \xi_{(p)}(X(p)) \in \mathbb{R}$ $\forall p \in M$

$\Rightarrow \xi(X) : M \to \mathbb{R}$ is a smooth function

Example: Suppose TM is trivial $\Leftrightarrow$ i.e. $TM \cong M \times \mathbb{R}^n$
$\exists$ n l.i. vector fields $X_1, \dots, X_n$

Consider $\xi_1, \dots \xi_n \in \Gamma(T^*M)$ given by

$$\xi_i(X_j) = \delta_{ij} : M \to \mathbb{R}$$

Then $\xi_1, \dots, \xi_n$ are l.i $\Rightarrow T^*M$ is trivial

Example: On $\mathbb{R}^n$ we have a basis for the 1-forms: $dx_1, \dots, dx_n$, which satisfies $dx_i(\partial_j) = \delta_{ij}$

Example: Let $f: M \to \mathbb{R}$ be smooth, we define a 1-form df on M by
$df(p) = df_p \in T_p^* M$

On $\mathbb{R}^n$: $df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$

Definition 4.5: Let $f: M \to N$ be any smooth map. Then the pullback $f^*: \Gamma(T^*N) \to \Gamma(T^*M)$ is given by $\underset{\substack{\cap \\ \Gamma(T^*N) \\ \text{1 form}}}{f^*(\eta)}(p) = df_p^*(\eta(p)) \in T_p^*M$ $\forall p \in M$

Remark $(f \circ g)^* = g^* \circ f^*$ (Chain Rule)

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Recall last time:

- Lie derivative of vector field $\mathcal{L}_X Y = [X,Y]$
  Exercise $X = x_3\partial_2 - x_2\partial_3$, $Y = x_1\partial_3 - x_3\partial_1$
  show directly $\mathcal{L}_X Y = Z = x_2\partial_1 - x_1\partial_2$

- 1-forms and pullback

Brief Review of some tensor Algebra

- V is n-dim vector space $\rightsquigarrow$ dual space $V^* =$
  $V^* = \{$ linear maps $T : V \to \mathbb{R}\}$

- $\bigotimes^k V^* = \{$multilinear maps $T : \underbrace{V \times .. \times V}_{k \text{ times}} \to \mathbb{R}\}$

- Tensor product $S \in \bigotimes^k V^*$ and $T \in \bigotimes^\ell V^* \Rightarrow$
  $S \otimes T \in \bigotimes^{k+\ell} V^*$

by $\boxed{S \otimes T(v_1, \dots, v_{k+\ell}) = S(v_1, \dots, v_k)\, T(v_{k+1}, \dots, v_{k+\ell})}$

- $T \in \bigotimes^k V^* \Rightarrow \boxed{\operatorname{Sym} T(v_1, \dots, v_k) = \frac{1}{k!} \sum_{\sigma \in S} T(v_{\sigma(1)}, \dots, v_{\sigma(k)})}$

$T \in \bigotimes^2 V^* \Rightarrow \operatorname{Sym} T(v_1, v_2) = \frac{1}{2} T(v_1, v_2) + \frac{1}{2} T(v_2, v_1)$

Let $S^k V^* = \{ T \in \bigotimes^k V^* : \operatorname{Sym} T = T\}$

- $T \in \bigotimes^k V^* \Rightarrow \boxed{\operatorname{Alt} T(v_1, \dots, v_k) = \frac{1}{k!} \sum_{\sigma \in S} \operatorname{sgn}(\sigma) . T(v_{\sigma(1)}, \dots, v_{\sigma(k)})}$

so $T \in \bigotimes^2 V^* \Rightarrow \operatorname{Alt} T(v_1, v_2) = \frac{1}{2}(T(v_1, v_2) - T(v_2, v_1))$

Let $\Lambda^k V^* = \{ T \in \bigotimes^k V^*$ s.t. $\operatorname{Alt}(T) = T\}$

Definition 4.6 We can define $\otimes^k T_p^*M$, $S^k T_p^* M$ and $\Lambda^k T_p^* M$ and hence bundles $\otimes^k T^*M$, $S^k T^*M$ and $\Lambda^k T^*M$ called the bundle of $(0,k)$-tensors, the bundle of symmetric $(0,k)$-tensors and the bundle of k-forms

Definition 4.7: A k-form is a section of $\Lambda^k T^* M$

i.e. if $\omega \in \Gamma(\Lambda^k T^* M)$ (ω is a k-form), then $\omega(p) \in \Lambda^k T_p^* M \quad \forall p \in M$ and $\omega(p)(X_1, \dots, X_k) \in \mathbb{R}$, $X_1, \dots, X_k \in T_p M$

Remarks: • $\Lambda^1 T^* M = T^* M$

• we define $\Lambda^0 V^* = \mathbb{R} \Rightarrow \Lambda^0 T^* M = M \times \mathbb{R}$ so a 0-form is a section of $M \times \mathbb{R}$ i.e. a smooth map $f: M \to M \times \mathbb{R}$ s.t. $f(p) \in \mathbb{R}$ $\forall p \in M$ so $f: M \to \mathbb{R}$ is a smooth function

• $\Lambda^n V^*$ is 1-dim., $\Lambda^n V^* \cong \mathbb{R}$ so $\Lambda^n T^* M$ is a rank 1 vector bundle over M, but it is not necessarily trivial i.e. it is not always bundle isomorphic to $M \times \mathbb{R}$

Examples:

• On $\mathbb{R}^n$ we can define $g_0 \in \Gamma(S^2 T^* \mathbb{R}^n)$ by $g_0(p)(X, Y) = \langle X, Y \rangle$ $\forall p \in \mathbb{R}^n$, $\forall X, Y \in T_p \mathbb{R}^n = \mathbb{R}^n$

dot product

• On $\mathbb{R}^2$ we can define $\omega_0 \in \Gamma(\Lambda^2 T^* \mathbb{R}^2)$ by $\omega_0(p) = (\underset{u_1\partial_1 + u_2\partial_2}{\underset{\parallel}{(u_1, u_2)}}, \underset{v_1\partial_1 + v_2\partial_2}{\underset{\parallel}{(v_1, v_2)}}) = u_1 v_2 - u_2 v_1$.

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The wedge product:

$\omega \in \Lambda^k V^*$ and $\eta \in \Lambda^l V^* \Rightarrow$

$$\boxed{\omega \wedge \eta = \frac{(k+l)!}{k!\, l!} \mathrm{Alt}(\omega \otimes \eta)}$$

Example $(dx_1 \wedge dx_2)(u_1\partial_1 + u_2\partial_2, v_1\partial_1 + v_2\partial_2) =$

$$= \frac{(1+1)!}{1!\;1!} \times \frac{1}{2}\Big[(dx_1 \otimes dx_2)(u_1\partial_1 + u_2\partial_2, v_1\partial_1 + v_2\partial_2)$$

$$- (dx_2 \otimes dx_1)(v_1\partial_1 + v_2\partial_2, u_1\partial_1 + u_2\partial_2)\Big]$$

$$= dx_1(u_1\partial_1 + u_2\partial_2) \cdot dx_2(v_1\partial_1 + v_2\partial_2) -$$

$$- dx_1(v_1\partial_1 + v_2\partial_2) \cdot dx_2(u_1\partial_1 + u_2\partial_2)$$

$$= u_1 v_2 - v_1 u_2 = \omega_0(u_1\partial_1 + u_2\partial_2, v_1\partial_1 + v_2\partial_2)$$

$dx_1\, \partial_1 = 1$
$dx_2\, \partial_1 = 0$

$$\Rightarrow (dx_1 \wedge dx_2) = \omega_0$$

We know $\Lambda^k V^* = \mathrm{Span}\{ \xi_1 \wedge \dots \wedge \xi_k,\ 1 < \dots < k,\ \xi_1, \dots, \xi_k \in V^* \}$

$$\Lambda^k V^* = \{0\} \quad \text{if } k > n$$

$\Rightarrow \Lambda^k T^* \mathbb{R}^n = \mathrm{Span}\{ dx_{i_1} \wedge \dots \wedge dx_{i_k},\ i_1 < \dots < i_k \}$

So given $\omega \in \Gamma(\Lambda^k T^* M)$ and (U, φ) a chart then $\varphi : \underset{M}{\overset{\cap}{U}} \to \underset{\mathbb{R}^n}{\overset{\cap}{\varphi(U)}}$. $\Rightarrow \varphi^{-1} : \underset{\mathbb{R}^n}{\overset{\cap}{\varphi(U)}} \to \underset{M}{\overset{\cap}{U}}$

$$\Rightarrow (\varphi^{-1})^* : \Lambda^k T_p^* M \longrightarrow \Lambda^k T^*_{\varphi(p)} \mathbb{R}^n$$

Definition 4.8 Given $f : M \to N$ smooth and $\eta \in \Gamma(\Lambda^k T^* M)$ then the pullback $f^*\eta$ of η by f is a k-form on M given by $\boxed{(f^*\eta)(\underset{\in M}{p})(X_1, \dots, X_k) = \eta(\underset{\in N}{f(p)})(df_p(X_1) \dots df_p(X_k))}$ $X_1, \dots, X_k \in T_pM$ and $df_p(X_i) \in T_{f(p)} N$

In particular $(\varphi^{-1})^* w = \sum_{i_1 < \dots < i_k} w_{i_1 \dots i_k} dx_{i_1} \wedge \dots \wedge dx_{i_k}$

e.g. if $w \in \Gamma(\Lambda^2 T^* M)$ and $\dim(M) = 2$
$(\varphi^{-1})^* w = f\, dx_1 \wedge dx_2 = f w_0$ for some $f : \varphi(U) \to \mathbb{R}$

Example: Let $i : M \to N$ be an embedding i.e. $i : M \to i(M)$ is a diffeomorphism. Then if $w \in \Gamma(\Lambda^k T^* N)$ then $i^* w \in \Gamma(\Lambda^k T^* M)$ is called the restriction of w to M

Let $N = \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$
$M = \mathbb{R}^n$
$i : M \to N$ be the inclusion
dx_{n+1} is a nowhere vanishing 1-form on N since $dx_{n+1}(\partial_{n+1}) = 1$

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But $i^* dx_{n+1}(u_1\partial_1 + \dots + u_n\partial_n) =$

since $i_*\partial_k = \partial_k$, $\forall k = 1,\dots,n$

$= dx_{n+1}(u_1\partial_1 + \dots + u_n\partial_n)$

$= 0$

<u>Recall</u>: if $\omega = \sum_{i_1<\dots<i_k} \omega_{i_1,\dots,i_k}\, dx_{i_1}\wedge\dots\wedge dx_{i_k}$

Then $d\omega = \sum_j \sum_{i_1<\dots<i_k} \frac{\partial \omega_{i_1,\dots,i_k}}{\partial x_j}\, dx_j \wedge dx_{i_1}\wedge\dots\wedge dx_{i_k}$

In particular if $f : \mathbb{R}^n \to \mathbb{R}$,

$$df = \sum_j \frac{\partial f}{\partial x_j}\, dx_j$$

<u>Remark</u>: $df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \frac{\partial f}{\partial x_3}dx_3$

if $x_2 = g_2(x_1)$
$x_3 = g_3(x_1)$

$\Rightarrow dx_2 = \frac{\partial g_2}{\partial x_1}dx_1$, $dx_3 = \frac{\partial g_3}{\partial x_1}dx_1$

$\Rightarrow df = \left(\frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2}\cdot\frac{\partial g_2}{\partial x_1} + \frac{\partial f}{\partial x_3}\cdot\frac{\partial g_3}{\partial x_1}\right)dx_1$

<u>Example</u> Let $\xi = \frac{x_1 dx_2 - x_2 dx_1}{x_1^2 + x_2^2}$ in $\mathbb{R}^2\setminus\{0\}$

Let $i : S^1 \to \mathbb{R}^2$ be $i(\theta) = (\cos\theta, \sin\theta)$

$\Rightarrow i_*\partial_\theta = (-\sin\theta\,\partial_1 + \cos\theta\,\partial_2 = (-x_2\partial_1 + x_1\partial_2)(i(\theta))$

$i^*\xi(\partial_\theta) = \xi(i_*\partial_\theta) = \xi(-x_2\partial_1 + x_1\partial_2)(i(\theta))$

$$= \frac{(x_1 dx_2 - x_2 dx_1)}{x_1^2 + x_2^2} (-x_2 \partial_1 + x_1 \partial_2) \Big|_{i(\theta)} =$$

since $dx_2 \partial_1 = 0$, $dx_2 \partial_2 = 1$

$$= \frac{x_1 \cdot x_1 - x_2(-x_2)}{x_1^2 + x_2^2} \Big|_{i(\theta)} =$$

$$= 1$$

$\Rightarrow i^* \xi = d\theta$ the 1-form dual of ∂_θ

$$d\xi = \frac{\partial}{\partial x_1}\left(\frac{x_1}{x_1^2+x_2^2}\right) dx_1 \wedge dx_2 + \frac{\partial}{\partial x_2}\left(\frac{-x_2}{x_1^2+x_2^2}\right) dx_2 \wedge dx_1$$

$$= \frac{x_2^2 - x_1^2}{(x_1^2+x_2^2)^2} dx_1 \wedge dx_2 + \frac{x_2^2 - x_1^2}{(x_1^2+x_2^2)^2} \underbrace{dx_2 \wedge dx_1}_{= -dx_1 \wedge dx_2}$$

$$= 0$$

In general if w is a k-form and η is a l-form then $\boxed{\eta \wedge w = (-1)^{kl} w \wedge \eta}$

<u>Fact</u>: $\underset{\underset{0}{\shortparallel}}{i^* d\xi} = d(i^*\xi) \Rightarrow d(d\theta) = 0$ i.e. $d\theta$ is a <u>closed</u> form but not <u>exact</u>

<u>Recall</u> $d(dw) = 0$ and $dw = 0$ iff w is <u>closed</u> and $w = d\eta$ if w is <u>exact</u>

$\int_{S^1} d\theta = 2\pi \neq 0$ so Stoke's Thm $\Rightarrow$ $d\theta$ is not exact

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Theorem 4.9 We can define the exterior derivative $d : \Gamma(\Lambda^k T^* M) \to \Gamma(\Lambda^{k+1} T^* M)$ by for any chart (U, φ):

$$\varphi^* d((\varphi^{-1})^* \omega|_U) = d\omega|_U$$

It enjoys the following properties

- $d(d\omega) = 0$
- $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$

  ω - k form, η - ℓ form
- if $f : M \to N$ smooth then

  $f^* d\eta = d(f^* \eta)$ for any form η on N

Proof: Since the exterior derivative on forms on $\mathbb{R}^n$ enjoys the properties stated, it is enough to show that the exterior derivative is well-defined i.e. does not depend on the choice of chart.

Suppose (U, φ), (V, ψ) are overlapping charts on M.

$$d\omega|_{U \cap V} = \psi^* d((\psi^{-1})^* \omega) =$$

$$= (\psi \circ \varphi^{-1} \circ \varphi)^* d((\psi^{-1})^* \omega) =$$

$$= \varphi^* \circ \underbrace{(\psi \circ \varphi^{-1})^*}_{\substack{\text{map from} \\ \varphi(U \cap V) \to \psi(U \cap V) \\ \in \mathbb{R}^n \qquad \in \mathbb{R}^n}} d(\underbrace{(\psi^{-1})^* \omega}_{\text{form on } \mathbb{R}^n}) =$$

$$= \varphi^* d((\psi \circ \varphi^{-1})^* (\psi^{-1})^* \omega)$$

We can do that since $(\psi \circ \varphi^{-1})^*$ & $d((\psi^{-1})^* \omega)$ is on $\mathbb{R}^n$

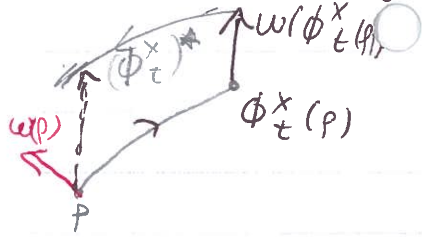
$$= \varphi^* d((\varphi^{-1})^* \omega)$$

as required ∎

Remark: $d\omega = 0 \Rightarrow \omega$ is closed
$\omega = d\eta \Rightarrow \omega$ is exact
If $f: M \to N$ is smooth then
$d(f^*\omega) = 0$ if $d\omega = 0$
and if $\omega = d\eta$ then
$$f^*\omega = f^* d\eta = d(f^*\eta)$$
i.e. if ω is exact $\Rightarrow f^*\omega$ is exact
$\rightsquigarrow$ de Rham cohomology

Back to Lie derivative: X vector field and ω a form on M.

$$(\phi_t^X)^* \omega(\phi_t^X(p)) \in \Lambda^k T_p^* M, \ \forall p \in M$$



Definition 4.10: Let $X \in \Gamma(TM)$, $\omega \in \Gamma(\Lambda^k T^* M)$. Then the Lie derivative of ω w.r.t. X is given by: $$\mathcal{L}_X \omega(p) = \lim_{t \to 0} \frac{(\phi_t^X)^* \omega(\phi_t^X(p)) - \omega(p)}{t}$$
where $\{\phi_t^X : t \in (-\varepsilon, \varepsilon)\}$ is the flow of X near $p \in M$
$\mathcal{L}_X \omega \in \Gamma(\Lambda^k T^* M)$

Example: $f: M \to \mathbb{R}$ smooth function
$$\Rightarrow (\phi_t^X)^* f = f \circ \phi_t^X$$

in general if $f: N \to \mathbb{R}$ and $g: M \to N$
$$g^* f = f \circ g$$

$$\Rightarrow \mathcal{L}_X f(p) = \lim_{t \to 0} \frac{(\phi_t^X)^* f - f}{t}(p) =$$
$$= \lim_{t \to 0} \frac{f \circ \phi_t^X(p) - f(p)}{t} =$$
$$= \lim_{t \to 0} \frac{f(\alpha_p(t)) - f(\alpha_p(0))}{t} =$$

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$= \frac{d}{dt}(f \circ \alpha_p)(0) = df_p(\alpha_p'(0)) = df_p(X(p)) =$

$= [df(X)](p) \quad \Rightarrow \mathcal{L}_X f = df(X) = X(f)$

Recall df the differential of f is the 1 form $d(f)$.

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<u>Last time:</u> $\mathcal{L}_X \omega$

<u>Lie Derivative Calculations on $\mathbb{R}^n$</u>

$X = \sum_i a_i \partial_i$

a_i, b_i are functions

$\xi = \sum_i b_i dx_i$

Compute : $\mathcal{L}_{\partial_i} X$, $\mathcal{L}_X \partial_i$,

$\mathcal{L}_{\partial_i} \xi$ $\quad \mathcal{L}_X dx_i$

<u>$\mathcal{L}_{\partial_i} X(p)$</u>

Step 1. Find flow of ∂_i

$\phi_t^{\partial_i}(p) = p + te_i$ where $e_i = (0, \dots, \underset{i^{th}}{1}, \dots 0)$

$\mathcal{L}_{\partial_i} X = \lim_{t \to 0} (\phi_{-t}^{\partial_i})_* \frac{X(\phi_t^{\partial_i}(p)) - X(p)}{t}$

$X(\phi_t^{\partial_i}(p)) = \sum_j a_j(\phi_t^{\partial_i}(p)) \partial_j = \sum_j a_j(p + te_i) \partial_j$

$$(\phi^{\partial_i}_{-t})_* \, \partial_j = \partial_j \qquad \text{since } (\phi^{\partial_i}_{-t})_* : \underset{\mathbb{R}^n}{\overset{\shortparallel}{T_{\phi_t(p)}\mathbb{R}^n}} \longrightarrow \underset{\mathbb{R}^n}{\overset{\shortparallel}{T_p\mathbb{R}^n}}$$

$\downarrow$ identity map $\qquad \mathbb{R}^n \xrightarrow{id} \mathbb{R}^n$

$$\Rightarrow \mathcal{L}_{\partial_i} X(p) = \lim_{t\to 0} \frac{\sum_j [a_j(p+te_i) - a_j(p)]}{t} \partial_j =$$

$$\Rightarrow \mathcal{L}_{\partial_i} X = \sum_j (\partial_i a_j)\, \partial_j = \sum_j (\mathcal{L}_{\partial_i} a_j) \partial_j = \mathcal{L}_{\partial_i} \underbrace{(\textstyle\sum_j a_j \partial_j)}_{X}$$

$$\underline{\mathcal{L}_X \partial_i} = [X, \partial_i] = [\sum_j a_j \partial_j , \partial_i] =$$

$$= -\sum_j \partial_i(a_j) \partial_j = -\mathcal{L}_{\partial_i} X$$

$$\underline{\mathcal{L}_{\partial_i} \xi}(p) = \lim_{t\to 0} \frac{(\phi^{\partial_i}_t)^* \xi(\phi^{\partial_i}_t(p)) - \xi(p)}{t}$$

$$(\phi^{\partial_i}_t)^* dx_j = dx_j (\phi^{\partial_i}_t)_* = dx_j$$

$$\lim_{t\to 0} \frac{\sum (b_j(p+te_i) - b_j(p))\, dx_j}{t} = \sum_j (\partial_i b_j)\, dx_j (p)$$

$$\mathcal{L}_X dx_i(p)(Y) = \lim_{t\to 0} \left[\frac{(\phi^X_t)^* dx_i(\phi^X_t(p)) - dx_i(p)}{t}\right](Y)$$

dx_i doesn't care which pts it acts on

Y is tangent vector at p

$$= \lim_{t\to 0} \frac{dx_i[(\phi^X_t)_*(Y) - Y]}{t}$$

$$= \lim_{-t\to 0} dx_i\left[\frac{(\phi^X_{-t})_*(Y) - Y}{-t}\right]$$

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$= dx_i(-\mathcal{L}_X Y)$ true $\forall Y$

Choose $Y = \partial_j$

$\Rightarrow (\mathcal{L}_X dx_i)(\partial_j) = -dx_i(\mathcal{L}_X \partial_j) =$

$= dx_i\left(\sum_k (\partial_j a_k)\partial_k\right)$

$\Rightarrow (\mathcal{L}_X dx_i)(\partial_j) = \partial_j a_i$

$\Rightarrow \mathcal{L}_X dx_i = \sum_j (\partial_j a_i)\, dx_j = da_i = d(i_X dx_i)$

Proposition 4.11 (Cartan's formula)

Let ω be a k-form on M and X a vectorfield on M. We define a $(k-1)$-form $i_X\omega$ by:

$$i_X\omega_p(Y_1, \dots, Y_{k-1}) = \omega(p)(X(p), Y_1, \dots, Y_{k-1})$$
$Y_i \in T_pM$

Then $\boxed{\mathcal{L}_X \omega = d(i_X \omega) + i_X d\omega}$

Example: Let $\xi = \dfrac{x_1 dx_2 - x_2 dx_1}{x_1^2 + x_2^2}$ on $\mathbb{R}^2 \setminus \{0\}$

Let $X = x_1\partial_1 + x_2\partial_2$ (flow dilation) $\quad Y = x_2\partial_1 - x_1\partial_2$ (flow rotation)

Recall $d\xi = 0$ by example

$\mathcal{L}_X \xi = d(i_X \xi)$

What is $i_X\xi = \xi(X) = \left(\dfrac{x_1 dx_2 - x_2 dx_1}{x_1^2 + x_2^2}\right)(x_1\partial_1 + x_2\partial_2)$

$$= \frac{x_1 x_2 - x_2 x_1}{x_1^2 + x_2^2} = 0$$

$$\Rightarrow \mathcal{L}_X \xi = d(0) = 0$$

$$\mathcal{L}_Y \xi = d(i_Y \xi)$$

$$i_Y \xi = \xi(Y) = \left(\frac{x_1 dx_2 - x_2 dx_1}{x_1^2 + x_2^2} \right)(x_2 \partial_1 - x_1 \partial_2) =$$

$$= -1$$

$$\mathcal{L}_Y \xi = d(-1) = 0$$

§5. Orientation and Riemannian Metrics

Theorem 5.1 Let M be a manifold and let $A = \{(U_i, u_i) : i \in I\}$ be an atlas on M. $\exists \{f_j : M \to \mathbb{R} : j \in \mathbb{N}\}$ smooth s.t.:

- $\forall j \in \mathbb{N}$ $\exists i \in I$ s.t. support $f_j = \overline{\{p \in M : f_j(p) \neq 0\}}^{\text{closure}} \subseteq U_i$
- $\forall p \in M$ $\exists$ open $W \ni p$ s.t. $W \cap \text{support } f_j = \emptyset$ for all but finitely many $j \in \mathbb{N}$
- $f_j(p) \geq 0$ $\forall p \in M$, $\forall j \in \mathbb{N}$
- $\sum_{j=1}^{\infty} f_j(p) = 1$ $\forall p \in M$ notice this sum is always finite since there are only finitely many j for which $f_j \neq 0$

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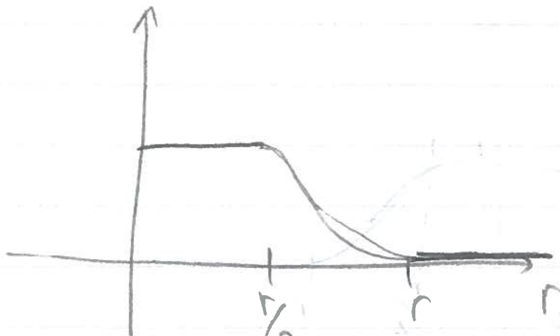
We call $\{f_j : j \in \mathbb{N}\}$ a partition of unity (subordinate to the atlas A)

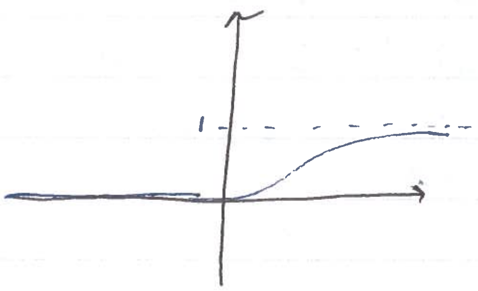
Proposition 5.2 : Let $B_r(0)$ be the open ball in $\mathbb{R}^n$ of radius $r>0$. $\exists$ smooth function $g_r : \mathbb{R}^n \to [0,1]$ s.t:

- $g_r \geq 0$
- $g_r = 1$ on $\overline{B_{r/2}(0)}$
- $g_r = 0$ on $\mathbb{R}^n \setminus B_r(0) \Rightarrow$ support $g_r \subseteq B_r(0)$



No way to have this function in terms of power series since if not on the ~~left~~ right of r is the 0 function but on the left this is false

Proof : Consider $h(t) = \begin{cases} e^{-\frac{1}{t}} & t>0 \\ 0 & t \leq 0 \end{cases}$



Smooth for $t \neq 0$
Notice, h is continuous and $0 \leq h < 1$. For $t>0$
$h'(t) = \frac{1}{t^2} e^{-\frac{1}{t}} > 0 \Rightarrow$

h is increasing for $t>0$.

Notice $0 < t^{-k} \cdot e^{-\frac{1}{t}} = t \cdot t^{-(k+1)} e^{-\frac{1}{t}} = t(k+1)! \frac{(t^{-1})^{k+1}}{(k+1)!} \cdot e^{-\frac{1}{t}} \leq$
$t>0$

$$\leq t(k+1)! \sum_{m=0}^{\infty} \frac{(t^{-1})^m}{m!} \cdot e^{-\frac{1}{t}} = t \cdot (k+1)! \to 0 \text{ as } t \to 0$$

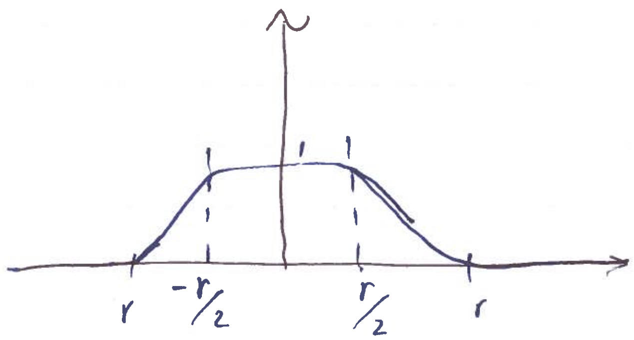
$\Rightarrow h'$ is continuous at $t=0$

$h^{(k)}(t) = p_k(t^{-1}) \cdot e^{-\frac{1}{t}}$ for a poly p_k which

means $h^{(k)}(t) \to 0$ as $t \to 0$ $\forall k$

So h is smooth at $t=0$ and all of its derivatives $h^{(k)}(0)=0$

Goal is to construct



Define $h_r(t) = \dfrac{h(r^2-t^2)}{h(r^2-t^2)+h(t^2-\frac{1}{4}r^2)}$ $\Rightarrow 0 < h_r < 1$ when it is defined

if $h(r^2-t^2)=0 \iff r^2-t^2 \le 0 \iff t^2 \ge r^2 \ge \frac{1}{4}r^2$

$\Rightarrow h(t^2-\frac{1}{4}r^2) > 0$ whenever $h(r^2-t^2)=0$

Similarly $h(t^2-\frac{1}{4}r^2)=0$ means $h(r^2-t^2)>0$

So $h_r(t)$ is defined $\forall t \in \mathbb{R}$ and

$h_r(t)=0$ iff $t^2 > r^2$ and if $t^2 < \frac{1}{4}r^2$

$\Rightarrow t^2-\frac{1}{4}r^2 \le 0 \Rightarrow h(t^2-\frac{1}{4}r^2)=0 \Rightarrow h_r(t)=1$.

Then $g_r : \mathbb{R}^n \to \mathbb{R}$ is $g_r(x) = h_r(|x|)$

since x is a vector and h takes numbers we need to apply h on $|x|$

Riemannian Geometry

Orientation:

Definition 5.3: A manifold is orientable if $\exists\ \mathcal{A} = \{(U_i, \varphi_i)_{i\in I}\}$ s.t. whenever $U_i \cap U_j \neq \emptyset$ we have: $\varphi_j \circ \varphi_i^{-1} : \underset{\subseteq \mathbb{R}^n}{\varphi_i(U_i \cap U_j)} \to \underset{\subseteq \mathbb{R}^n}{\varphi_j(U_i \cap U_j)}$

satisfies $\det(\varphi_j \circ \varphi_i^{-1})_* > 0$ on $\varphi_i(U_i \cap U_j)$

, linear map think of it as matrix

↑ det of the derivative

Example: $\mathbb{R}^n$ is orientable as $\{(\mathbb{R}^n, id)\}$ is an atlas

Example: S^n is orientable: take the atlas $\{(U_n, \varphi_n), (U_s, \varphi_s)\}$, $n \geq 2$

$\varphi_s \circ \varphi_N^{-1} : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ is $y \to \frac{y}{|y|^2}$

What is $(\varphi_s \circ \varphi_N^{-1})_*(y) = \frac{1}{|y|^4} \begin{pmatrix} |y|^2 - 2y_1^2 & -2y_1y_2 & \cdots & -2y_1y_n \\ -2y_1y_2 & |y|^2 - 2y_2^2 & & \vdots \\ \vdots & & \ddots & \vdots \\ -2y_1y_n & \cdots & \cdots & |y|^2 - 2y_n^2 \end{pmatrix}$

Take $y = (1, 0, \dots, 0) \Rightarrow (\varphi_s \circ \varphi_N^{-1})_*(y) = \frac{1}{1}\begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$

$\det(\varphi_s \circ \varphi_N^{-1})_* = -1 < 0$

$\det(\varphi_s \circ \varphi_N^{-1})_* \neq 0$ since $\varphi_s \circ \varphi_N^{-1}$ is a diffeo thus the matrix is an isomorphism.
Hence $\det(\varphi_s \circ \varphi_N^{-1})_*$ has the same sign everywhere otherwise by IVT it must be 0 at some pt

To get an atlas with $\det(\varphi_j \circ \varphi_i^{-1})_* > 0$
Change φ_N to $\tilde{\varphi}_N(x_1, \dots, x_{n+1}) = \varphi_N(-x_1, \dots, x_{n+1})$
and leave φ_S as it is.

Example: If there are only at most two charts in the atlas then it is always orientable

Example: Möbius band and Klein bottle are not orientable. (PS 3)

Example: $\mathbb{R}P^n$ is orientable iff n is odd (Sheet 3)

Volume Forms

Definition: A volume form on M (n-dim) is a nowhere vanishing n-form Ω.
i.e. $\Omega(p)(X_1, \dots, X_n) \neq 0$ for any (some) basis $\{X_1, \dots, X_n\}$ of T_pM

Example On $\mathbb{R}^n$ we have $\Omega_0 = dx_1 \wedge \dots \wedge dx_n$ is a volume form
Let $f: \mathbb{R}^n \to \mathbb{R}^n$ be smooth $\Rightarrow$

$$f^*\Omega_0 = \Omega_0(f_* X_1, \dots, f_* X_n) = \\ = \det f_* \, \Omega_0(X_1, \dots, X_n)$$

Here f_* is the matrix of the differential of f w.r.t. $\{X_1, \dots, X_n\}$

Theorem 5.4 The following are equivalent
- M is orientable
- $\exists$ a volume form on M
- $\Lambda^n T^* M$ is trivial

Proof: The equivalence of the existance of volume forms and the triviality of $\Lambda^n T^* M$ is Prop. 2.15 since $\Lambda^n T^* M$ is of rank 1

Now Suppose M has a volume form Ω and let $\{(U_i, \varphi_i) : i \in I\}$ be an atlas for M s.t. $\varphi_i(U_i)$ is connected $\forall i$ wlog.
Let $\Omega_0 = dx_1 \wedge \cdots \wedge dx_n$, $\varphi_i^{-1} : \underset{\cap \mathbb{R}^n}{\varphi_i(U_i)} \to \underset{\cap M}{U_i}$
$\Rightarrow (\varphi_i^{-1})^* \Omega = \lambda_i \Omega_0$ for some function $\lambda_i : \varphi_i(U_i) \to \mathbb{R}$ since $\Lambda^n T^* \mathbb{R}^n$ has rank 1 and Ω_0 is nowhere vanishing

Ω is a volume form means λ_i is nowhere vanishing $\Rightarrow \lambda_i > 0$ or $\lambda_i < 0$ on $\varphi_i(U_i)$ since it is connected

If $\lambda_i > 0$ we leave (U_i, φ_i) unchanged and if $\lambda_i < 0$ we change (U_i, φ_i) to $(U_i, \tilde{\varphi}_i)$, where $\tilde{\varphi}_i^{-1}(x_1, \dots, x_n) = \varphi_i^{-1}(-x_1, \dots x_n)$ (i.e. change the sign of the 1st coordinate)
Relabeling $\tilde{\varphi}_i$ to φ_i
Then we have $(\varphi_i^{-1})^* \Omega = \lambda_i \Omega_0$ where $\lambda_i : \varphi_i(U_i) \to \mathbb{R}^+ \ \forall i$

Now WTS: $\det(\varphi_j \circ \varphi_i^{-1})_* > 0$ on $\varphi_i(U_i \cap U_j)$
$\lambda_i \Omega_0 = (\varphi_i^{-1})^* \Omega$

$= (\varphi_j^{-1} \circ \varphi_j \circ \varphi_i^{-1})^* \Omega$

$= (\varphi_j \circ \varphi_i^{-1})^* \circ (\varphi_j^{-1})^* \Omega =$

$= \lambda_j (\varphi_j \circ \varphi_i^{-1})^* \Omega_0 =$) change of variables

$= \lambda_j \det (\varphi_j \circ \varphi_i^{-1})_* \Omega_0$

$\Rightarrow \det (\varphi_j \circ \varphi_i^{-1})_* = \dfrac{\lambda_i}{\lambda_j} > 0 \Rightarrow M$ is orientable

Now suppose M is orientable and let $\{(U_i, \varphi_i) : i \in I\}$ be an atlas s.t. $\det (\varphi_j \circ \varphi_i^{-1})_* > 0$. Theorem 5.1 $\Rightarrow \exists$ a partition of unity $\{f_j : M \to \mathbb{R} : j \in \mathbb{N}\}$ subordinate to $\mathcal{A}$
$\Rightarrow \forall j \in \mathbb{N}\ \exists\ i \in I$ s.t. support $f_j \subset U_{i(j)}$

Relabel $(U_j, \varphi_j) = (U_{i(j)}, \varphi_{i(j)})$. Then

$\mathcal{B} = \{(U_j, \varphi_j)\ j \in \mathbb{N}\}$ is an atlas with $\det (\varphi_j \circ \varphi_i^{-1})_* > 0$ because $\forall p \in M\ \exists j \in \mathbb{N}$ s.t. $f_j(p) \neq 0$ since $\sum_{k \in \mathbb{N}} f_k(p) = 1$ so $\bigcup_{j=1}^{\infty} U_j = M$

Define $\Omega = \sum_{j=1}^{\infty} f_j \varphi_j^* \Omega_0$ is well defined

because $\varphi_j^* \Omega_0$ is an n-form on U_j, f_j is 0 outside U_j so $f_j \varphi_j^* \Omega_0$ is 0 outside U_j and only finitely many f_j are non zero at any given $p \in M$.

Ω is a section of $\Lambda T^* M$ so we only need to show Ω is nowhere vanishing.

Riemannian Geometry

Let $p \in M$, then $\exists\, k$ s.t. $p \in U_k$ and $f_k(p) \neq 0$

$$(\varphi_k^{-1})^* \Omega = \sum_{j=1}^{\infty} f_j (\varphi_k^{-1})^* \circ \varphi_j^{\,*} \Omega_0 =$$

$$= \sum_{j=1}^{\infty} f_j \, (\varphi_j \circ \varphi_k^{-1})^* \Omega_0 =$$

$$= \sum_{j=1}^{\infty} \underbrace{f_j \det(\varphi_j \circ \varphi_k^{-1})_*}_{\geq 0 \quad \text{since } f_j \geq 0 \,\&\, \det > 0} \Omega_0$$

and in fact >0 at p since $f_k(p) \neq 0$

$\Rightarrow \Omega$ is a volume form ∎

Def. 5.5:

We say that an orientation on M is a choice of atlas $\mathcal{A} = \{(U_i, \varphi_i)\}$ s.t. $\det(\varphi_j \circ \varphi_i^{-1})_* > 0$ or equivalently a choice of a volume form Ω

We say two orientations given by volume forms Ω and Ω' are the same if $\Omega' = \lambda \Omega$ for a positive smooth function λ

Let $f: M \to N$ be a diffeomorphism and Υ be a volume form on N.

$$(f^*\Upsilon)(p)(X_1, \dots, X_n) = \Upsilon(f(p))(df_p(X_1), \dots, df_p(X_n)) \neq 0$$

$p \in M$ $\{X_1, \dots, X_n\}$ basis of T_pM

Because $\{df_p(X_1), \dots, df_p(X_n)\}$ is a basis since $df_p : T_pM \to T_{f(p)}N$ is an isomorphism

Notice that if f is a local diffeo, the same argument $\Rightarrow$ $(f^*\Upsilon)$ is a volume form

Suppose $f: M \to N$ is a diffeomorphism and Ω is a volume form on M, Υ is a volume form on N then we say f is orientation preserving if $(f^* \Upsilon) = \lambda \Omega$ for positive function $\lambda : M \to \mathbb{R}$

Examples : Suppose $f: \mathbb{R}^n \to \mathbb{R}^n$ is a diffeo Then f is orientation preserving iff $f^* \Omega_0 = \det f_* \Omega_0 = \lambda \Omega_0$, for $\lambda : \mathbb{R}^n \to \mathbb{R}^+$ iff $\det f_* > 0$

Example : $id : \mathbb{R}^n \to \mathbb{R}^n$ has $\det id_* = 1$ so it is orientation preserving.

$-id : \mathbb{R}^n \to \mathbb{R}^n$ has $\det -id_* = (-1)^n$

so $-id$ is orientation preserving iff n even

Definition 5.6 A Riemannian metric g on M is a section of S^2T^*M (Sym M) which is a positive definite i.e. $\forall p \in M$ we have $g_p \in S^2T_p^*M$, so $g_p(x,y) = g_p(Y,X)$ $\forall X, Y \in T_pM$ and $g_p(X,X) \geq 0$ $\forall X \in T_pM$ and $g_p(X,X) = 0$ iff $X = 0$

Thus g_p is a positive definite inner product on T_pM

Example : • The Euclidean metric g_0 on $\mathbb{R}^n$ is given by $g_0(X,Y) = \langle X, Y \rangle = X \cdot Y$ dot prod.
$\forall X, Y \in \mathbb{R}^n \cong T_p\mathbb{R}^n$

• If $M \subseteq \mathbb{R}^n$ then the induced Riemannian metric by $g_p(X,Y) = g_0(X,Y)$ $\forall p \in M$ $\forall X, Y \in T_pM \subseteq \mathbb{R}^n$

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Last time: defined Riemannian metric.
$g \in \Gamma(S^2 T^* M)$ s.t. $g_p(X,X) \geq 0$ $\forall p \in M$
$\forall X \in T_pM$ and $g_p(X,X) = 0$ iff $X = 0$

Example: $S^n \subseteq \mathbb{R}^{n+1} \Rightarrow S^n$ has a Riemannian metric g, given by: for $p \in S^n$, $X, Y \in T_p S^n$
$T_pS^n \cong \{ q \in \mathbb{R}^{n+1} : \langle q, p \rangle = 0 \}$ Then
$g_p(X,Y) = \langle X, Y \rangle$

Let $X = x_3 \partial_2 - x_2 \partial_3$, $Y = x_1 \partial_3 - x_3 \partial_1$
$Z = x_2 \partial_1 - x_1 \partial_2$

Notice that $X(p)$, $p \in S^2$, then
let $p = (x_1, x_2, x_3)$, then $\langle (0, x_3, -x_2), (x_1, x_2, x_3) \rangle = 0$
$\Rightarrow X(p) \in T_p S^2$. $\Rightarrow X_{\text{to } S^2}$ is a vectorfield on S^n

Remark: $W = \partial_1$ restricted to S^2 is not a vectorfield in S^2 because $W(1,0,0) \notin T_{(1,0,0)} S^2$

Similarly Y and Z restrict to the vector field on S^2

$$g(X,X) = \langle (0, x_3, -x_2), (0, x_3, -x_2) \rangle =$$
$$= x_2^2 + x_3^2 \geq 0$$

And $g(X,X)(p) = 0$ iff $x_2 = x_3 = 0$
iff $p = (\pm 1, 0, 0)$
$X(p) = 0$

Similarly $g(Y,Y)(p) = 0$ iff $p = (0, \pm 1, 0)$
$Y(p) = 0$

$g(Z,Z)(p) = 0$ iff $p = (0, 0 \pm 1)$
$Z(p) = 0$

$$g(X,Y) = \langle (0, x_3, -x_2), (-x_3, 0, +x_1) \rangle$$
$$= -x_1 x_2$$

So $g(X,Y)(p) = 0$ iff $p = (0, x_2, x_3)$ or
$p = (x_1, 0, x_3)$

$\Rightarrow$ fits in with the fact that the flows of X, Y are rotations about the axis defined by the points where X, Y are 0.

In fact almost any vectorfield on S^2 has exactly 2 zeros

Notice: Euler characteristic of S^2 is 2

In general, if M is a surface of genus k then $\chi(M) = 2 - 2k$ = # zeros of almost any vectorfield on M

$$\chi(M) = \frac{1}{2\pi} \int_M K \quad \text{Gauss Bonnet}$$
K curvature of M

Proposition 5.7: Let $f: M \to N$ be an immersion (i.e. df_p is inj $\forall p \in M$) and let h be a Riemannian metric on N. Then $g = f^* h$ is a Riemannian metric on M

Proof: $h \in \Gamma(S^2 T^* N) \Rightarrow g = f^* h \in \Gamma(S^2 T^* M)$
We only need to show that g_p is positive definite $\forall p \in M$

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Let $p \in M$, $X \in T_pM \Rightarrow g_p(X,X) = (f^*h)_p(X,X) =$

$= h_{f(p)}(df_p(X), df_p(X)) \geq 0$ since h is a positive definite

and $g_p(X,X) = 0 \Leftrightarrow df_p(X) = 0$ but df_p inj $\Leftrightarrow X = 0$

■

<u>Example</u>: if $M \subseteq \mathbb{R}^n$ then let $i: M \to \mathbb{R}^n$ be the inclusion map. Prop. 5.7 $\Rightarrow$ $g = i^* g_0$ is a Riemannian metric on M called the Restriction of the Euclidean metric g_0.

· if M is a surface in $\mathbb{R}^3$ then g is the first fundamental form

<u>Theorem 5.8</u>: Every manifold has a Riemannian metric

<u>Proof</u>: Let M be a manifold and let $\{(U_i, \varphi_i) : i \in I\}$ be an atlas. Thm 5.1. $\Rightarrow \exists$ partition of unity $\{f_j : M \to \mathbb{R} : j \in \mathbb{N}\}$ subordinate to the atlas

$\forall j \in \mathbb{N}$ $\exists i(j) \in I$ s.t. $\operatorname{supp} f_j \subseteq U_{i(j)}$. Let $(U_j, \varphi_j) = (U_{i(j)}, \varphi_{i(j)})$ which is still an atlas in particular $\bigcup_{j=1}^{\infty} U_j = M$ since $\forall p \in M$ $\exists j \in \mathbb{N}$ s.t. $f_j(p) \neq 0$

Recall $\varphi_j : \underset{M}{\underset{\cap\mid}{U_j}} \to \underset{\mathbb{R}^n}{\underset{\cap\mid}{\varphi_j(U_j)}}$ is a diffeo $\Rightarrow$ inversion

$\Rightarrow \varphi_j^* g_0$ is a Riem. metric on U_j

Let $g = \sum_{j=1}^{\infty} f_j \varphi_j^* g_0 \in \Gamma(S^2 T^* M)$ because

$\varphi_j^* g_0 \in \Gamma(S^2 T^* U_j)$ $\forall j$ and $f_j = 0$ when

we are outside U_j, and only finitely many f_j are non-zero at any $p \in M$

Let $p \in M$, $X \in T_p M$, $g_p(X,X) = \sum_{j=1}^{\infty} \underset{\underset{0}{\vee\!|}}{f_j(p)} \underbrace{(\varphi_j^*(g_0))_p(X,X)}_{\underset{0}{\vee\!|} \text{ when defined}}$

since $\varphi_j^* g_0$ is a Riem. metric on U_j

$\Rightarrow g_p(X,X) \geq 0$

And $g_p(X,X)$ iff $f_j(p) \cdot (\varphi_j^*(g_0))_p(X,X) = 0$

$\forall j \in \mathbb{N}$, $\exists k \in \mathbb{N}$ s.t. $f_k(p) \neq 0 \Rightarrow g_p(X,X) = 0$

iff $f_k(p)(\varphi_k^*(g_0))(p)(X,X) = 0$

$\Rightarrow (\varphi_k^* g_0)(p)(X,X) = 0$

But $\varphi_k^* g_0$ is a Riem metric $\Rightarrow X = 0$

∎

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Definition 6.1 : A Riemannian manifold (M, g) where M is manifold and g is a Riemannian metric on M, i.e. $g \in \Gamma(S^2 T^* M)$ which is positive definite

Example: Suppose (M, g_M), (N, g_N) Riemannian manifolds $\Rightarrow$ $M \times N$ is manifold and $T_{(p,q)}(M \times N) \cong T_p M \times T_q N$

This means we can define $g_{M\times N}$ - Riemannian metric on $M \times N$ by $(g_{M\times N})_{(p,q)}((X, U), (Y, N))$

for $X, Y \in T_p M$ and $U, V \in T_q N$

$\Rightarrow (g_{M\times N})_{(p,q)}((X,U),(Y,V)) = (g_M)_p(X,Y) + (g_N)_q(U,V)$

Positive definite:

$$(g_{M\times N})_p((X,U),(X,U)) = \underbrace{(g_M)_p(X,X)}_{\geq 0} + \underbrace{(g_N)_q(U,U)}_{\geq 0}$$

$$\geq 0$$

Since $(g_M)_p(X,X) \geq 0$
$(g_N)_q(U,U) \geq 0$
because they are Riemannian metrics $\Rightarrow$ positive definite

And $(g_{M\times N})_p((X,U),(X,U)) = 0$

$\Leftrightarrow (g_M)_p(X,X) = 0$ and $(g_N)_q(U,U) = 0$ $\Leftrightarrow X = 0$ and $U = 0$

Because they are Riem. metrics

Thus $g(M_1 \times N)$ is a Riem. metric

Example: Suppose, G discrete group acts freely and properely discontinuously by diffeos. on M $\Rightarrow$ M/G is a manifold and $\pi : M \to M/G$ is a local diffeo. $\Rightarrow$ if h is a Riem. metric on M/G then $\pi^* h = g$ is a Riem. metric on M.

Remarks: Notice we now have different Riem. metrics on $T^2 = S^1 \times S^1$

- product metric
- induced metric from inclusion $f_1 : T^2 \to \mathbb{R}^3$
- induced metric from inclusion $f_2 : T^2 \to \mathbb{R}^4$

Are they related?

Example: Let $i : S^n \to \mathbb{R}^{n+1}$ be the standard inclusion of the unit n-sphere and let $g = i^* g_0$

Let $\delta_r : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be $\delta_r(x) = rx$ for $r > 0$

what is $(\delta_r \circ i)^* g_0$?

$(\delta_r \circ i)^* = i^* \circ \delta_r^*$ $\rightsquigarrow$ look at

$$\delta_r^* g_0(X, Y) = g_0((\delta_r)_*(X), (\delta_r)_*(Y)) =$$

$(\delta_r)_* = \delta_r$

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$= g_0(\delta_r(X), \delta_r(Y)) = g_0(rX, rY) =$
since bilinear

$= r^2 g_0(X, Y)$

$\Rightarrow (\delta_r \circ i)^* g_0 = i^*(\delta_r^* g_0) = i^*(r^2 g_0) = r^2 i^* g_0 =$
$= r^2 g$

How do we write down Riem. metrics?

Euclidean metric $g_0 \in \Gamma(S^2 T^* \mathbb{R}^n)$

Recall $dx_1, \ldots, dx_n$ form a basis of sections for $T^*\mathbb{R}^n$

$\Rightarrow \{dx_i dx_j : i, j \in \{1, \ldots, n\}\}$ is a basis of sections for $S^2 T^* \mathbb{R}^n$

e.g. $dx_1 dx_2 = Sym(dx_1 \otimes dx_2)$

$\Rightarrow g_0 = \sum a_{ij} dx_i dx_j \Rightarrow g_0(\partial_k, \partial_l) = \delta_{kl} = a_{kl}$
where $A = (a_{ij})$ is symmetric matrix $\Rightarrow A = I$

$\Rightarrow g_0 = dx_1^2 + \ldots + dx_n^2$ or equivalently the matrix I

Example $\mathbb{R}^2$, $g_0 = dx_1^2 + dx_2^2$, matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\partial_r = \cos\theta \partial_1 + \sin\theta \partial_2$
$\partial_\theta = -r\sin\theta \partial_1 + r\cos\theta \partial_2$

Let $x_1 = r\cos\theta$, $x_2 = r\sin\theta$

compute
$g_0(\partial_r, \partial_r)$
$g_0(\partial_r, \partial_\theta)$
$g_0(\partial_\theta, \partial_\theta)$

$g_0(\partial_r, \partial_r) = \cos^2\theta + \sin^2\theta = 1$

$g_0(\partial_r, \partial_\theta) = 0$

$g_0(\partial_\theta, \partial_\theta) = r^2$

$\Rightarrow \quad g_0 = 1dr^2 + 0\,drd\theta + r^2 d\theta^2 = dr^2 + r^2 d\theta^2$

In general on $\mathbb{R}^{n+1}$, $g_0 = dr^2 + r^2 g_{S^2}$: the matrix is $\begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$ w.r.t. $\partial_r, \partial_\theta$

where g_{S^n} is the standard induced metric on S^n.

For any (M, g), let (U, φ) be a chart

$\Rightarrow$ $\boxed{X_i = (\varphi^{-1})_* \partial_i}$ are called the coordinate vector fields

$\Rightarrow$ g is determined on U by $g_{ij} = g(X_i, X_j)$
functions on U

$g_{ij} : U \longrightarrow \mathbb{R}$

$$\begin{array}{ccc} U & \underset{\varphi^{-1}}{\overset{\varphi}{\rightleftarrows}} & \varphi(U) \\ \cap & & \cap \\ M & & \mathbb{R}^n \end{array}$$

$(\varphi^{-1})^* : U \to \varphi(U)$

$\Rightarrow (\varphi^{-1})^* g$ is a Riemannian metric on $\varphi(U)$

And $(\varphi^{-1})^* g = \sum_{i,j} g_{ij}\, dx_i\, dx_j$

Exercise : Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be
$f(\theta, \varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$

So $f(\mathbb{R}^2) = S^2$ $\quad f : (0, \pi) \times (0, 2\pi) \to S^2$ is an immersion

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$$\Rightarrow f^* g_{S^2} = d\theta^2 + \sin^2\theta \, d\varphi^2$$

$$f_* \partial_\theta = \cos\theta\cos\varphi\,\partial_1 + \cos\theta\sin\varphi\,\partial_2 - \sin\theta\,\partial_3$$

$$f_* \partial_\varphi = -\sin\theta\sin\varphi\,\partial_1 + \sin\theta\cos\varphi\,\partial_2$$

$$g_0(f_*\partial_\theta, f_*\partial_\theta) = 1$$

$$g_0(f_*\partial_\theta, f_*\partial_\varphi) = 0$$

$$g_0(f_*\partial_\varphi, f_*\partial_\varphi) = \sin^2\theta$$

and as a matrix $\begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} = g_{ij}$

This is a positive definite: look at $\det(g_{ij}) = \sin^2\theta$ which works unless $\theta = 0, \pi$, but $0, \pi \notin (0, \pi)$. So it stops being a Riem. metric when the stops being an immersion.

Definition 6.2 A map $f : (M, g) \to (N, h)$ is an isometry if f is a diffeo and $\boxed{f^* h = g}$

A map $f : (M, g) \to (N, h)$ is a local isometry at $p \in M$ if $\exists\ U \ni p$ open, $V \ni f(p)$ s.t. $f : U \to V$ is an isometry.

We say f is a local isometry if it is a local isometry at all $p \in M$.

Example: $id : (M, g) \to (M, g)$ is an isometry

Example: Let $A \in M_n(\mathbb{R})$ and define $f : \mathbb{R}^n \to \mathbb{R}^n$ by $f(x) = Ax$

$f_*(X) = AX$, $X \in \Gamma(T\mathbb{R}^n)$
Because its linear

$$(f^* g_0)(\partial_i, \partial_j) = g_0(f_* \partial_i, f_* \partial_j) = g_0(A\partial_i, A\partial_j) =$$

$$= g_0\left(\sum_k a_{ik} \partial_k , \sum_e a_{je} \partial_e\right) =$$

$$= \sum_{k,e} a_{ik} a_{je} \underbrace{g_0(\partial_k, \partial_e)}_{= \delta_{ke}} =$$

$$= \sum_k a_{ik} a_{jk}$$

$$= (AA^T)_{ij}$$

This is an isometry iff $AA^T = Id$

$\Rightarrow$ $f^* g_0 = g_0 \iff AA^T = Id \Rightarrow A \in O(n)$ -group

f_* is orientation preserving iff $\det f_* > 0$ iff $\det A > 0$ $\Rightarrow$ f is an orientation preserving isometry iff A $SO(n)$

Let (M, g) and (N, h) be Riem. Manifolds and (U, φ) chart on M and (V, ψ) chart on N s.t. $\varphi(U) = \psi(V)$

$\Rightarrow \psi^{-1} \circ \varphi : U \to V$ is a diffeo.

Suppose $(\varphi^{-1})^* g = (\psi^{-1})^* h$ $\iff$

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$g = \varphi^* \circ (\psi^{-1})^* h = (\psi^{-1} \circ \varphi)^* h \iff \psi^{-1} \circ \varphi$ is an isometry

This means if (g_{ij}) and (h_{ij}) are equal then $\psi^{-1} \circ \varphi$ defines a local isometry between (U, g) and (V, h)

Example: Let $M = \{(s\cos t, s\sin t, t) = s, t \in \mathbb{R}\}$ helicoid, let $N = \{(\cosh z \cos\theta, \cosh z \sin\theta, z)$ catenoid $z, \theta \in \mathbb{R}$

Claim: M, N with their induced Riem metrics ~~from~~ $\mathbb{R}^3$ are locally isometric.

Let $f_1 : \mathbb{R}^2 \to M$ be
$f_1(x_1, x_2) = (\sinh x_2 \cos x_1, \sinh x_2 \sin x_1, x_1)$
$f_2 : \mathbb{R}^2 \to N$ be
$f_2(x_1, x_2) = (\cosh x_2 \cos x_1, \cosh x_2 \sin x_1, x_2)$

$(f_1)_* \partial_1 = -\sinh x_2 \sin x_1 \partial_1 + \sinh x_2 \cos x_1 \partial_2 + \partial_3$

$(f_1)_* \partial_2 = \cosh x_2 \cos x_1 \partial_1 + \cosh x_2 \sin x_1 \partial_2$

$f_1^* g_0(\partial_1, \partial_1) = \sinh^2 x_2 + 1 = \cosh^2 x_2$

$f_1^* g_0(\partial_1, \partial_2) = 0$

$f_1^* g_0(\partial_2, \partial_2) = \cosh^2 x_2$

$\Rightarrow f_1^* g_0 = \cosh^2 x_2 (dx_1^2 + dx_2^2)$

$(f_2)_* \partial_1 = -\cosh x_2 \sin x_1 \,\partial_1 \;+ \cosh x_2 \cos x_1 \,\partial_2$

$(f_2)_* \partial_2 = \sinh x_2 \cos x_1 \,\partial_1 \;+ \sinh x_2 \sin x_1 \,\partial_2 + \partial_3$

$\Rightarrow (f_2^* g_0)(\partial_1, \partial_1) = \cosh^2 x_2$

$(f_2^* g_0)(\partial_1, \partial_2) = 0$

$(f_2^* g_0)(\partial_2, \partial_2) = 1 + \sin^2 x_2 = \cos^2 x_2$

$\Rightarrow \; f_2^* g_0 = \cosh^2 x_2 \,(dx_1^2 + dx_2^2) = f_1^* g_0$ so they are locally isometric.

Theorem 6.3. Let G be a discrete group acting freely and properly discontinuously on (M, g) by isometries (i.e. $\phi_* : M \to M$ are isometries) Then $\exists!$ Riemannian metric h on M/G s.t. $\pi^* h = g$, where $\pi : M \to M/G$ is the proj.

Motivation $(\pi^* h)_p (X, Y) = h_{\pi(p)} (\pi_* X, \pi_* Y) =$

$= h_{\pi(p)} (d\pi_p X, d\pi_p Y) \quad \forall p \in M, X, Y \in T_p M$

This suggests defining $h_{\pi(p)} (u, v) = g_p(d\pi_p^{-1}(u), d\pi_p^{-1}(v))$
Since $d\pi_p : T_p M \to T_{\pi(p)} M/G)$ is iso

then $\pi^* h = g$.

But we need to check it does not depend on the choice of $p \in \pi^{-1}(\pi(p))$

Proof: Define $h_{\pi(p)}(u,v) = g_p(d\pi_p^{-1}(u), d\pi_p^{-1}(v))$ for $p \in M$, $u, v \in T_{\pi(p)} M/G$

We first show this is well defined.

Let $\pi(p) = \pi(q)$ then $\exists\ x \in G$ s.t. $q = \phi_x(p)$

$$\Rightarrow \pi(p) = (\pi \circ \phi_x)(p) \Rightarrow d\pi_p = d\pi_{\phi_x(p)} \circ d(\phi_x)(p)$$
$$= d\pi_q \circ d(\phi_x)(p)$$

$$\Rightarrow d\pi_q^{-1} = d(\phi_x)_p \circ d\pi_p^{-1}$$

$$g_q(d\pi_q^{-1}(u), d\pi_q^{-1}(v)) = g_{\phi_x(p)}(d(\phi_x)_p \circ d\pi_p^{-1}(u), (d(\phi_x)_p \circ d\pi_p^{-1}(v))$$
$$= (\phi_x^* g)_p(d\pi_p^{-1}(u), d\pi_p^{-1}(v)) = g_p(d\pi_p^{-1}(u), d\pi_p^{-1}(v))$$

Since $\phi_x^* g = g$ $\forall\ u, v \in T_{\pi(p)} M/G$

$\Rightarrow$ h is well defined and h is a section of $(S^2 T^* M/G)$ since $g \in \Gamma(S^2 T^* M)$ and π is local diffeo

By construction $\pi^* h = g$.

Remains to show it is a positive definite
$h_{\pi(p)}(u,u) = g_p(d\pi_p^{-1}(u), d\pi_p^{-1}(u)) \geq 0$

since g is Riem. metric and $= 0$ iff $d\pi_p^{-1}(u) = 0 \iff u = 0$ since $d\pi_p$ is an iso. $\blacksquare$

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Example: id, −id are isometries on $\mathbb{R}^n$ $\Rightarrow$ $\mathbb{R}P^n$, Möb, K^2 inherit Riemannian metrics form S^n, cylinder and $T^2 \subseteq \mathbb{R}^3$

Example: $\mathbb{Z}^n$ acting by translations on $\mathbb{R}^n$ is an action by isometries $\Rightarrow$ $\mathbb{R}^n/\mathbb{Z}^n$ inherits a Riem. metric h

$$f : \mathbb{R}^n/\mathbb{Z}^n \longrightarrow T^n \subseteq \mathbb{R}^{2n}$$ diffeo given by

$$f([x_1, \dots, x_n]) = (\cos(2\pi x_1), \sin(2\pi x_1), \dots, \cos(2\pi x_n), \sin(2\pi x_n))$$

If g is the Riem. metric on $T^n \subseteq \mathbb{R}^{2n}$ then $f^* g = 4\pi^2 h$

The metrics are the same up to rescaling!

§7 The Levi-Civita Connection

Motivation V inner product space $\Rightarrow$ $\exists$ a canonical isomorphism $V \cong V^*$ given by $v \in V \longmapsto \langle v, \cdot \rangle \in V^*$.

Definition 7.1 $X \in T_pM \Longrightarrow \underbrace{g_p(X, \cdot)}_{\overset{\shortparallel}{X^\flat}} \in T_p^*M$ called X flat

And $\xi \in T_p^*M \Rightarrow \xi(Y) = \underbrace{g_p(\xi^\#, Y)}_{\text{sharp}}$ defines $\xi^\# \in T_pM$

Notice $(X^\flat)^\# = X$ and $(\xi^\#)^\flat = \xi \Rightarrow \flat, \#$ are "musical" isomorphisms

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The same works for vectorfields and 1-forms

<u>Example</u> $\partial_i^\flat = dx_i$ and $dx_i^\# = \partial_i$

$$X = \sum a_i \partial_i \Rightarrow X^\flat = \sum a_i$$

<u>Theorem 7.2</u> (The Fundamental Thm of Riem. Geom)

There exists a unique map $\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ denoted by $(X, Y) \longmapsto \nabla_X Y$ s.t. if $X, Y, Z \in \Gamma(TM)$, a, b are smooth functions on M then:

i) $\nabla_{aX+bY} Z = a \nabla_X Z + b \nabla_Y Z$

ii) $\nabla_X (Y+Z) = \nabla_X Y + \nabla_X Z$

iii) $\nabla_X (aY) = a \nabla_X Y + X(a) Y$

iv) $X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$

v) $\nabla_X Y - \nabla_Y X = [X, Y]$

∇ is called the Levi-Civita connection

<u>Proof</u>: Suppose ∇ exists and write down iv) three times

$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

$$Y(g(Z,X)) = g(\nabla_Y Z, X) + g(Z, \nabla_Y X)$$

$Z(g(X,Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$

$X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) =$

$= 2g(\nabla_X Y, Z) + g(Y, [X,Z]) + g(X,[Y,Z]) -$

$- g(Z,[X,Y])$ by v)

$\Rightarrow \underbrace{g(\nabla_X Y, Z)}_{=(\nabla_X Y)^{\#}(Z)} = \frac{1}{2}\Big(X(g(Y,Z)) + Y(g(\cancel{Z}\cancel{X})) - Z(g(X,Y)) -$

$- g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y])$ (*)

$\Rightarrow$ * defines $\nabla_X Y$ uniquely so if ∇ exists it is unique.

Now define ∇ by (*) we then just need to check i) – v) are satisfied

ii) is obvious because g and the Lie bracket are bilinear over $\mathbb{R}$

iv) first term on RHS of * is symmetric in Y & Z whereas the last five terms are skew symmetric in Y and Z since $[Y,Z] = -[Z,Y] \Rightarrow g(\nabla_X Y, Z) + g(\nabla_X Z, Y) = 2 \times \frac{1}{2} X(g(Y,Z)) = X(g(Y,Z))$

v): The first five terms on RHS of * are symmetric in X & Y and the last term is skew symmetric in X, Y since $[X,Y] = -[Y,\cancel{X}]$

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$$g(\nabla_X Y, Z) - g(\nabla_Y X, Z) = g([X,Y], Z)$$

$$\| $$

$$g(\nabla_X Y - \nabla_Y X, Z)$$

$$\Rightarrow \nabla_X Y - \nabla_Y X = [X,Y]$$

So vi holds.

iii) $g(\nabla_X (aY), Z) = \frac{1}{2}\big(X(g(aY,Z)) +$
$+ aY(g(Z,X)) - Z(g(X,aY)) - g(X,[aY,Z]) +$
$+ g(aY,[Z,X]) + g(Z,[X,aY]) =$

$= a\nabla_X Y + \frac{1}{2}\big(X(a)g(Y,Z) - Z(a)g(X,Y) +$

Remark : $[aY, Z] = (aY)\circ Z - Z\circ(aY) =$
$= a[Y,Z] - Z(a)Y$

$+ Z(a)g(X,Y) - X(a)g(Y,Z)\big) =$

$= a\, g(\nabla_X Y, Z) + X(a)g(Y,Z) = g(a\nabla_X Y + X(a), Z)$

i) Exercise

Example : On $\mathbb{R}^n$: $\nabla_{\partial_i}\partial_j = 0$ since

$g(\partial_i, \partial_j) \overset{=\delta_{ij}}{=}$ const. so $\partial_k(g(\partial_i,\partial_j)) = 0 \quad \forall k$

Here you choose $X = \partial_i$, $Y = \partial_j$, $Z = \partial_k$

$\Rightarrow g(\nabla_{\partial_i}\partial_j, \partial_k) = 0 \Rightarrow \nabla_{\partial_i}\partial_j = 0$

if $X = \sum a_i \partial_i$ and $Y = \sum b_j \partial_j$

Then $\nabla_X Y = \nabla_{\sum_i a_i \partial_i} \sum_j b_j \partial_j = \sum_i a_i \nabla_{\partial_i} \sum_j b_j \partial_j =$

$= \sum_{i,j} (a_i b_j \underbrace{\nabla_{\partial_i} \partial_j}_{0} + a_i \partial_i (b_j) \partial_j) =$

$= \sum_{i,j} a_i \partial_i (b_j) \partial_j = \sum_j X(b_j) \partial_j$

$\Rightarrow \nabla_X Y \neq \mathcal{L}_X Y$ in general

Example : Let $f : (0,\pi) \times (0, 2\pi) \longrightarrow S^2 \subseteq \mathbb{R}^3$ be $f(\theta, \varphi) = (\sin\theta \cos\varphi, \sin\theta \sin\varphi, \cos\theta)$

Let $X_1 = f_* \partial_\theta$ and $X_2 = f_* \partial_\varphi$

$\Rightarrow [X_1, X_2] = 0$, $g(X_1, X_1) = 1$, $g(X_1, X_2) = 0$

and $g(X_2, X_2) = \sin^2\theta$

$\nabla_{X_1} X_1 = 0$

$\nabla_{X_2} X_2 = -\sin\theta \cos\theta X_1$. from the formula

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7. Levi-Civita Connection

(M, g) Riemannian Manifold

Theorem 7.2 There exist a unique map
$\nabla : \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM)$
$(X, Y) \longrightarrow \nabla_X Y$ - covariant derivative

In the proof we have

$$(*)\quad g(\nabla_X Y, Z) = \frac{1}{2}\Big(X(g(Y,Z)) + Y(g(Z,X)) -$$

$$- Z(g(X,Y)) - g(X,[Y,Z]) + g(Y,[Z,X]) +$$

$$+ g(Z,[X,Y])\Big)$$

Examples:

- $\mathbb{R}^n$, $\{\partial_i\}$ coordinate vector fields
  $g(\partial_i, \partial_j) = \delta_{ij}$, $[\partial_i, \partial_j] = 0 \Rightarrow \nabla_{\partial_i}\partial_j = 0$

- S^2 : $f(\theta, \varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$

  $X_1 = f_*(\partial_\theta) = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta)$

  $X_2 = f_*(\partial_\varphi) = (-\sin\theta\sin\varphi, \sin\theta\cos\varphi, 0)$

  Compute $\nabla_{X_1} X_1$, $\nabla_{X_1} X_2$ and $\nabla_{X_2} X_2$, $\nabla_{X_2} X_1$

Because f_* is diffeo we have

$$[X_1, X_2] = [f_* \partial_\theta, f_* \partial_\varphi] = f_* \underbrace{[\partial_\theta, \partial_\varphi]}_{0} = 0$$

$g(X_1, X_1) = 1$ and $g(X_1, X_2) = 0$ and

$g(X_2, X_2) = \sin^2\theta$

Compute $\nabla_{X_1} X_1$, From $*$ we have

$$g(\nabla_{X_1} X_1, X_1) = \frac{1}{2} X_1(g(X_1, X_1)) = 0$$

$$g(\nabla_{X_1} X_1, X_2) = \frac{1}{2}\left(2X_1(g(X_1, X_2)) - X_2(g(X_1, X_1))\right)$$

$$= 0$$

$\{X_1, X_2\}$ is a basis for $\Gamma(TS^2)$ $\Rightarrow$

$\nabla_{X_1} X_1 = 0$

Compute $\nabla_{X_2} X_2$

$$g(\nabla_{X_2} X_2, X_1) = -\sin\theta\cos\theta$$

$$g(\nabla_{X_2} X_2, X_2) = 0$$

$$\Rightarrow \quad \nabla_{X_2} X_2 = -\sin\theta\cos\theta \, X_1$$

Suppose (U, φ) is a local chart for (M, g)
Then we have

$X_i = \varphi_*^{-1}(\partial_i)$, $i = 1, \dots, n$

are coordinate vector fields in U

Riemannian Geometry

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$\nabla : \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM)$

$(X, Y) \longrightarrow \nabla_X Y \in \Gamma(TM)$

$\{X_1, X_2, \dots, X_n\}$ is a basis for $\Gamma(TU)$

$$\nabla_{X_i} X_j = \sum_{k=1}^{n} \Gamma_{ij}^{k} X_k$$

because this is a vector field $\in \Gamma(TU)$

Γ_{ij}^{k} is called the Christoffel symbol of ∇ in a coordinate chart (U, φ)

Example • $\mathbb{R}^n$ $\nabla_{\partial_i} \partial_j = 0$ $\{\partial_i, i=1, \dots, n\}$ is a basis for $\Gamma(T\mathbb{R}^n)$

$\Rightarrow \Gamma_{ij}^{k} = 0$

• S^2, $\nabla_{X_1} X_1 = 0$ and $\nabla_{X_2} X_2 = -\sin\theta\cos\theta X_1$

we know $\{X_1, X_2\}$ is a basis

$\Rightarrow \Gamma_{11}^{1} = \Gamma_{11}^{2} = 0$ since $\nabla_{X_1} X_1 = \Gamma_{11}^{1} X_1 + \Gamma_{11}^{2} X_2$

Similarly $\Gamma_{22}^{1} = -\sin\theta\cos\theta$ and $\Gamma_{22}^{2} = 0$

Proposition: Suppose (U, φ) is a local coordinate chart of (M, g) and $X_i = \varphi^{-1}_{*}(\partial_i)$ is a coordinate vector field.

$g = (g_{ij})$, $g_{ij} = g(X_i, X_j)$

$g^{-1} = (g^{ij})$. Then:

1) $\Gamma_{ij}^{k} = \Gamma_{ji}^{k}$

ii) $\Gamma_{ij}^{k} = \frac{1}{2} \sum_{\ell=1}^{n} g^{k\ell} (\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij})$

where $\partial_k g_{ij} = X_k (g_{ij})$

$$\nabla_{X_i} X_j = \sum_{k=1}^{n} \Gamma_{ij}^{k} X_k$$

$$\forall \; X = \sum_{i=1}^{n} a_i X_i$$

$$Y = \sum_{j=1}^{n} b_j X_j$$

$$\Rightarrow \nabla_X Y = \nabla_{\sum a_i X_i} (\sum b_j X_j) \overset{\text{by Property}}{=} \sum_{i=1}^{n} a_i \sum_{j=1}^{n} (X_i(b_j) + b_j \nabla_{X_i} X_j)$$

Proof :

i) $\nabla_{X_i} X_j = \sum_{k=1}^{n} \Gamma_{ij}^{k} X_k$

$\nabla_{X_j} X_i = \sum_{k=1}^{n} \Gamma_{ji}^{k} X_k$

$$\Rightarrow \nabla_{X_i} X_j - \nabla_{X_j} X_i = \sum_{k=1}^{n} (\Gamma_{ij}^{k} - \Gamma_{ji}^{k}) X_k$$

By the Property of Levi-Civita connection

$LHS = [X_i, X_j]$ because $X_i = \varphi_*^{-1}(\partial_i)$

$$\Rightarrow [X_i, X_j] = [\varphi_*^{-1}(\partial_i), \varphi_*^{-1}(\partial_j)] =$$

$$= \varphi_*^{-1} \underbrace{[\partial_i, \partial_j]}_{\overset{\shortparallel}{0}} = 0$$

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Since $\{X_k\}_{k=1,\dots,n}$ form a basis for $\Gamma(TU)$ then

$\Gamma_{ij}^{\ \ k} - \Gamma_{ji}^{\ \ k} = 0 \quad \forall\ k=1,\dots,n$

$\Rightarrow \Gamma_{ij}^{\ \ k} = \Gamma_{ji}^{\ \ k}$ ∎

ii) $\nabla_{X_i} X_j = \sum_{k=1}^{n} \Gamma_{ij}^{\ \ k} X_k$

$$g(\nabla_{X_i} X_j, X_\ell) = \sum_{k=1}^{n} \Gamma_{ij}^{\ \ k} g(X_k, X_\ell) =$$
$$= \sum_{k=1}^{n} \Gamma_{ij}^{\ \ k} g_{k\ell}$$

We can compute the LHS using (∗)
Note $[X_i, X_j] = 0 \quad \forall\ i,j$

$$g(\nabla_{X_i} X_j, X_\ell) = \frac{1}{2}\Big(X_i(g(X_j, X_\ell)) + X_j(g(X_i, X_\ell)) - X_\ell(g(X_i, X_j))\Big) =$$
$$= \frac{1}{2}\left(X_i(g_{j\ell}) + X_j(g_{i\ell}) - X_\ell(g_{ij})\right)$$
$$= \frac{1}{2}\left(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}\right)$$

So $\sum_{k=1}^{n} \Gamma_{ij}^{\ \ k} g_{k\ell} = \frac{1}{2}\left(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}\right)$

multiply by $g^{\ell m}$ on both sides and sum $\ell = 1,\dots,n$

Note $\sum_{l=1}^{n} g_{kl} \cdot g^{lm} = \delta_{km}$

$\Rightarrow$ ~~[illegible]~~ $\Gamma_{ij}^{m} = \frac{1}{2} \sum_{l=1}^{n} g^{lm} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$

Example :

· on S^2 $f(\theta, \varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$

$X_1 = f_*(\partial_\theta) = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta)$

$X_2 = f_*(\partial_\varphi) = (-\sin\theta\sin\varphi, \sin\theta\cos\varphi, 0)$

$g = (g_{ij})$

$$g_{11} = g(X_1, X_1) = 1$$
$$g_{12} = g_{21} = g(X_1, X_2) = 0$$
$$g_{22} = g(X_2, X_2) = \sin^2\theta$$

$\Rightarrow g = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$ $g^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sin^2\theta} \end{pmatrix}$

$\Gamma_{12}^{1} = \frac{1}{2} \sum_{l=1}^{2} g^{1l} (\partial_1 g_{2l} + \partial_2 g_{1l} - \partial_l g_{12}) =$

$= \frac{1}{2} g^{11} (\partial_1 g_{21} + \partial_2 g_{11} - \partial_1 g_{12})$ since

$g^{12} = 0$

$= 0$

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$$\Gamma_{12}^{2} = \frac{1}{2}\sum_{l=1}^{2} g^{2l}(\partial_1 g_{2l} + \partial_2 g_{1l} - \partial_l g_{12}) =$$

$$= \frac{1}{2} g^{22}(\partial_1 g_{22} + \partial_2 g_{12} - \partial_2 g_{12}) =$$

$$= \frac{1}{2}\frac{1}{\sin^2\theta}\partial_\theta(\sin^2\theta) = \cot\theta$$

$$\nabla_{X_1} X_2 = \Gamma_{12}^{1} X_1 + \Gamma_{12}^{2} X_2 = \cot\theta . X_2$$

For tensors $\Gamma(\otimes^m T^*M)$

$$\nabla : \Gamma(TM) \times \Gamma(\otimes^m T^*M) \to \Gamma(\otimes^m T^*M)$$
$$(X, T) \longmapsto \nabla_X T$$

is defined by $\nabla_X T(Y_1, \dots, Y_m)$

$$\nabla_X T(Y_1, \dots, Y_m) = X(T(Y_1, \dots, Y_m)) - \sum_{k=1}^{m} T(Y_1, \dots, Y_{k-1}, \nabla_X Y_k, Y_{k+1}, \dots, Y_m)$$

Example:

- 1-forms, suppose $Y \in \Gamma(TM)$, then $Y^\flat \in \Gamma(T^*M)$

  For $X, Z \in \Gamma(TM)$ $\nabla_X Y^\flat(Z) = X(Y^\flat(Z)) - Y^\flat(\nabla_X Z)$

by def of $\flat$ $Y^\flat(Z) = g(Y, Z)$ and $Y^\flat(\nabla_X Z) = g(Y, \nabla_X Z)$

$$\Rightarrow \nabla_X Y^\flat(Z) = X(g(Y,Z)) - g(Y, \nabla_X Z) =$$

$$\overset{\text{IV prop of L-C}}{=} g(\nabla_X Y, Z) + \cancel{g(Y, \nabla_X Z)} - \cancel{g(Y, \nabla_X Z)}$$

$$= g(\nabla_X Y, Z) = (\nabla_X Y)^\flat(Z)$$

$$\Rightarrow \forall Z \in \Gamma(TM),\ \nabla_X(Y^\flat) = (\nabla_X Y)^\flat$$

- metric $g \in \Gamma(S^2(T^*M))$

$$\nabla_X g(Y,Z) = X(g(Y,Z)) - g(\nabla_X Y, Z) -$$
$$- g(Y, \nabla_X Z) = 0 \quad \text{by IV prop of L-C}$$

$$\Rightarrow \underline{\nabla_X g = 0 \quad \forall X \in \Gamma(TM)}$$

<u>g is parallel</u>

<u>Example Computing Γ for Torus</u>

Torus $T^n \subset \mathbb{R}^{2n}$ $\qquad T^n = S^1 \times S^1 \times \ldots \times S^1$

$$f(x_1, x_2, \ldots, x_n) = (\cos x_1, \sin x_1, \cos x_2, \sin x_2, \ldots, \cos x_n, \sin x_n)$$

$$X_i = f_*(\partial_i) = (0, 0, \ldots, -\sin x_i, \cos x_i, 0, \ldots 0)$$

$$g_{ij} = g(X_i, X_j) = \delta_{ij} \quad \Rightarrow \Gamma_{ij}^k = 0 \quad \forall i,j,k$$

$\Rightarrow$ Torus is flat Riemannian manifold

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Definition: If $\alpha(t)$ is a smooth curve on (M, g) and Y is a vectorfield along the curve α. (i.e. $Y(\alpha(t)) \in T_{\alpha(t)}M$), $\alpha(t) = \varphi^{-1}\underbrace{(a_1(t), \dots, a_n(t))}_{\in \mathbb{R}^n}$

(i) The covariant derivative of Y along $\alpha(t)$ is defined by $\boxed{\frac{DY}{Dt} = \nabla_{\alpha'(t)} Y}$

ii) Y is parallel along α if $\boxed{\frac{DY}{Dt} = 0}$

In a coordinate chart (U, φ) of (M, g)

$X_i = \varphi^{-1}_*(\partial_i)$, $i = 1, \dots, n$

$$\Rightarrow Y = \sum_{i=1}^{n} y_i X_i \quad , \text{ so } \alpha'(t) = \sum_{i=1}^{n} \varphi^{-1}_*(\partial_i)\, a_i'(t) = \sum_{i=1}^{n} a_i'(t) X_i(\alpha(t))$$

Note $\alpha : [0, L] \to (U, \varphi)$

$\alpha(t) = \varphi^{-1}\underbrace{(a_1(t), \dots, a_n(t))}_{\text{curve in } \mathbb{R}^n}$

$$\frac{DY}{Dt} = \nabla_{\alpha'(t)}\left(\sum_{j=1}^{n} y_j X_j\right) = \sum_{j=1}^{n} y_j'(t) X_j(\alpha(t)) +$$

$$+ \sum_{j=1}^{n} y_j(t) \nabla_{\sum_{i=1}^{n} a_i'(t) X_i} X_j =$$

$$= \sum_{j=1}^{n} y_j'(t) X_j + \sum_{j=1}^{n} y_j(t) \sum_{i=1}^{n} a_i' \nabla_{X_i} X_j =$$

$$= \sum_{k=1}^{n} \left(y_k' + \sum_{i,j=1}^{n} y_j a_i' \Gamma_{ij}^{k} \right) X_k$$

If Y is parallel $\Rightarrow \frac{DY}{Dt} = 0 \Rightarrow$

$$y_k' + \sum_{i,j=1}^{n} y_j\, a_i' \Gamma_{ij}^{k} = 0$$ first order ODE on y_k

Example

- on S^2 $\quad f(\theta, \varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$

X_1, X_2

$\Gamma_{11}^1 = \Gamma_{11}^2 = 0, \quad \Gamma_{22}^1 = -\sin\theta\cos\theta$

$\Gamma_{22}^2 = 0, \quad \Gamma_{12}^1 = 0, \quad \Gamma_{12}^2 = \cot\theta$

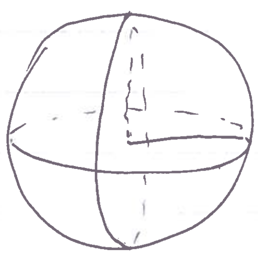
Suppose $\alpha(t) = f(a_1(t), a_2(t))$ is a curve on the sphere S^2

$Y = y_1 X_1 + y_2 X_2$, $\quad Y$ vector field

Y parallel on $\alpha(t)$ $\Leftrightarrow 0 = \dfrac{DY}{Dt} \Rightarrow$

$$0 = \frac{DY}{Dt} = (y_1' - \sin a_1 \cos a_1\, y_2\, a_2')\, X_1 + \\ + (y_2' - \cot a_1 (y_1 a_2' + y_2 a_1'))\, X_2$$

This is a bit too complicated so we will just consider the longtitude



$\Rightarrow \varphi = \text{const.}$ $\quad$ So $a_2 = \text{const}$ $\Rightarrow a_2' = 0$

$\Rightarrow \dfrac{DY}{Dt} = 0 \Rightarrow \quad y_1' = 0$ and $\quad y_2' - (\cot a_1) y_2 a_1' = 0$

$\Rightarrow y_1 = \text{const.}$

$$y_2' + \frac{\cos a_1 \cdot a_1'}{\sin a_1} \cdot y_2 = 0$$

$$\Rightarrow (\sin a_1 \, y_2)' = 0$$

$$\Rightarrow y_2 \cdot \sin a_1 = \text{const}$$

But at N $\sin a_1 = 0$

$$\Rightarrow y_2 = 0$$

$\Rightarrow$ on longtitude the parallel vector field $Y = \lambda X_1$

- latitude $\Rightarrow \cos\theta = \text{const} \Rightarrow \theta$ is const
  $\Rightarrow a_1(t) = \text{const.}$
  $\Rightarrow a_1' = 0$

Similarly $\frac{DY}{Dt} = 0 \iff \begin{cases} y_1' - \sin a_1 \cos a_1 \, y_2 a_2' = 0 & (1) \\ y_2' + (\cot a_1) y_1 a_2' = 0 & (2) \end{cases}$

Let's check whether X_1, X_2 are parallel along α (latitude)

$$Y = X_1 \iff y_1 = 1 \; , \; y_2 = 0$$

$\frac{DY}{Dt} = 0 \iff \begin{cases} (1) \text{ is satisfied for } y_1 = 1, y_2 = 0 \\ (2) \text{ becomes } 0 + (\cot a_1) \cdot a_2' = 0 \end{cases}$

iff $a_1 = \pi/2$

So only on the Equator X_1 parallel along α

Similarly if $Y = X_2$ is parallel $\Longleftrightarrow a_1 = \frac{1}{2}$

<u>Parallel transport</u> 27/11

<u>Theorem</u> : Suppose $p, q \in (M, g)$ and $\alpha : [0, L] \longrightarrow M$ is a curve connecting p and q i.e. $\alpha(0) = p$ and $\alpha(L) = q$. Let $Y^0 \in T_pM$, then (1) $\exists$ a parallel vector field Y on M s.t. $Y(\;) = Y_0$ (2) Define a map $\tau_\alpha : T_pM \longrightarrow T_qM$

$$\tau_\alpha(Y_0) = Y(q)$$

τ_α is an isometry

So it is an isomorphism s.t.

$$g_p(Y_0, Y_0) = g_q(\tau_\alpha(Y_0), \tau_\alpha(Y_0)).$$

τ_α is called parallel transport along $\alpha(t)$

<u>Recall</u> : Y is parallel $\Longleftrightarrow \frac{DY}{Dt} = 0 = \nabla_{\alpha'(t)} Y$

Proof : Assume $\alpha(t)$ is contained in a coordinate chart (U, φ). [if not we can cover the curve α by a finite selection of charts]

$$\alpha(t) = \varphi^{-1}(a_1(t), \dots, a_n(t))$$

$$\frac{DY}{Dt} = 0 \Longleftrightarrow y_K'(t) + \sum_{i,j=1}^{n} \Gamma_{ij}^K y_i a_i' = 0 \quad (*1) \quad , K = 1, \dots, n$$

Here $Y(\alpha(t)) = \sum y_K(t) X_K(t)$, $X_K(t) = \varphi_*^{-1} \partial_i$

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(*1) is a 1^{st} order ODE with initial condition $(y_1(0), \dots y_n(0)) = Y_0$

ODE theory $\Rightarrow$ solution $y(t)$ on $[0, L]$ which is unique.

For (2) To show τ_α is invertible

Define $\beta(t) = \alpha(L-t)$ $\quad \beta(t) : q \to p$

$Y_0 \in T_p M$

$$\tau_\alpha(Y_0) = Y(q),$$

By (1) $\Rightarrow$ $\exists$ parallel vector field Z along $\beta(t)$ with initial condition $Z(\beta(0)) = Y(q)$

$$\Rightarrow \tau_\beta : T_q M \longrightarrow T_p M$$

$$\tau_\beta(Y(q)) = Z(p)$$

Because $Y(\alpha(t))$ is parallel along α then Y is also parallel along β

By the uniqueness part in (1) gives us

$$Z(p) = Y(p)$$

$$\Rightarrow \quad \tau_\beta \circ \tau_\alpha(Y(p)) = Y(p)$$

so $\tau_\beta \circ \tau_\alpha = id$ $\Rightarrow$ τ_α is an isomorphism

Finally we need to show $g_p(Y_0, Y_0) = g_q(\tau_\alpha(Y_0), \tau_\alpha(Y_0))$

Define $g_{\alpha(t)}(Y(\alpha(t)), Y(\alpha(t)))$

$$\frac{d}{dt} g_{\alpha(t)}(Y(\alpha(t)), Y(\alpha(t))) = 2 g_{\alpha(t)}(\nabla_{\alpha'(t)} Y, Y) = 0$$

Example :

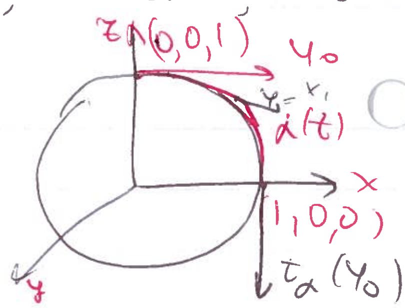
• on $\mathbb{R}^n$, $\Gamma_{ij}^k = 0 \Rightarrow (*1): y_k'(t) = 0$
$\forall K$

$$\Rightarrow y_k = \text{const} = C_K$$

So parallel transport is just translation

• on S^2 $\quad f(\theta, \varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$

Consider longtitutude



$\alpha(t) = f(\theta(t), \varphi(t))$
Take $\theta(t) = t$ and $\varphi(t) = 0$

$$\Rightarrow \alpha(t) = (\sin t, 0, \cos t)$$
$$Y_0 = (1, 0, 0)$$
$\tau_\alpha(Y_0)$?

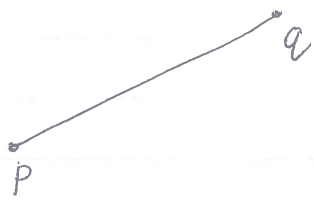
Along the longtitude we have Y is parallel iff $Y = \lambda X_1$, λ - const

$$X_1(\alpha(t)) = f_*(d_\theta)(\alpha(t)) = (\cos t, 0, -\sin t)$$
$$X_1(\alpha(0)) = (1, 0, 0) = Y_0$$

$$\underset{\text{by uniqueness}}{\tau_\alpha(Y_0) = X_1(\alpha(\pi/2))} = (0, 0, -1)$$

§8 Geodesics

in $\mathbb{R}^n$



shortest path is a straight line

<u>Definition 8.1</u> A curve $\gamma(t)$ on M is called a <u>geodesic</u> if $\boxed{\nabla_{\gamma'}\gamma' = 0}$ (*2)

<u>Remark</u> if γ is a geodesic then $|\gamma'(t)| = \sqrt{g(\gamma'(t), \gamma'(t))} = \text{const}$

<u>Proof</u>: $\frac{d}{dt} g(\gamma', \gamma') = 2g(\underbrace{\nabla_{\gamma'}\gamma'}_{=0}, \gamma') = 0$ ∎

If $|\gamma'| = 1$, we call $\gamma(t)$ is normalised or $\gamma(t)$ is parametrised by arc-length.

In a coordinate chart (U, φ)

if $\gamma(t) = \varphi^{-1}(c_1(t), \dots, c_n(t))$
then $*(2) \iff c_k''(t) + \sum_{i,j=1}^{n} \Gamma_{ij}^{k} c_i' c_j' = 0$ (*3)

$\forall k = 1, \dots, n$

<u>Example:</u>

- On $\mathbb{R}^n$ $\Gamma_{ij}^k = 0$ so $\overset{*3}{\Rightarrow} c_k''(t) = 0 \ \forall k = 1, \dots, n$
  $c_k(t) = a_k t + b_k$

- On $T^n \subset \mathbb{R}^{2n}$, $f(x_1, \dots, x_n) = (\cos x_1, \sin x_1, \dots, \cos x_n, \sin x_n)$

  $\Gamma_{ij}^k = 0 \quad \forall i, j, k$

  $\gamma(t) = f(x_1(t), \dots, x_n(t))$

γ is a geodesic iff $x_i(t) = a_i t + b_i$

- S^2 $f(\theta, \varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$

$\gamma(t) = f(\theta(t), \varphi(t))$ is a geodesic iff

$$\begin{cases} \theta''(t) - \sin\theta(t)\cos\theta(t)(\varphi'(t))^2 = 0 \\ \varphi''(t) + 2\cot\theta(t)\,\varphi'(t)\,\theta'(t) = 0 \end{cases}$$

Special solutions:

1) Longtitude : $\varphi = \varphi_0$ const.
$\theta' = c$
$\therefore \theta(t) = ct + \theta_0$

2) Equator : $\theta = \pi/2$ const
$\varphi''(t) = 0$
$\varphi(t) = bt + \varphi_0$

Theorem 8.2

Let $p \in M$. There exists a neighbourhood $U \ni p$ and $\varepsilon > 0$ and a smooth map $\Gamma : (-2, 2) \times V \longrightarrow M$, here $V = \{(q, X) : q \in U, X \in B_\varepsilon(0) \subseteq T_q M\}$

$V \subseteq TM$ s.t $\gamma_{(q,X)}(t) = \Gamma(t, q, X)$ is the unique geodesic with $\gamma_{(q,X)}(0) = q$, $\gamma'_{(q,X)}(0) = X$

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Proof: Because the geodesic equation is a 2^{nd} order ODE. By ODE theory we can always find a solution of the geodesic in a small neighbourhood.

Namely $\exists\ \mathcal{U} \ni p$ and $\varepsilon' > 0$, $\delta > 0$ small s.t. $\forall q \in \mathcal{U}$, $Y \in B_{\varepsilon'}(0) \subseteq T_q M$

$\exists!$ geodesic $\alpha_{(q,Y)} : (-\delta, \delta) \longrightarrow M$

$$\alpha_{(q,Y)}(0) = q \quad \text{and} \quad \alpha'_{(q,Y)}(0) = Y$$

If $\delta > 2$, we are done

If $\delta < 2$ we can rescale:

define a curve $\gamma_{(q,X)}(t) = \alpha_{(q, \frac{2}{\delta}X)}\left(\frac{\delta t}{2}\right)$

for $X \in B_{\frac{\delta\varepsilon'}{2}} \subseteq T_q M$, $t \in (-2, 2)$

$\gamma_{(q,X)}(0) = q$ and $\gamma'_{(q,X)}(0) = X$ ■

<u>Application of uniqueness</u>

1) All geodesies on S^2 are great circles i.e. intersection of S^2 with II-dim plane through origin

At north pole $(0,0,1)$, $X = (1,0,0)$ the geodesic $\gamma(t) =$ longtitude. So at any $p \in S^2$ and any $Y \in T_p S^2$ the geodesic is a great circle $\gamma(t)$ by Thm

Proof: Find an isometry map $T \in SO(3)$
s.t. $T(0,0,1) = P$
$T(X) = Y$

$\Rightarrow$ because T is an isometry, $T(\alpha(t))$ is also a geodesic

By uniqueness $\gamma(t) = T(\alpha(t))$

2) All geodesics in S^n are also great circles

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Example: Let $p \in S^n$, $X \in T_p S^n$ with $|X| = 1$

$\exists\ T \in SO(n+1)$ s.t. $T(p) = (0, 0, \dots, 0, 1) = e_{n+1}$
$T(X) = (0, 0, \dots, 1, 0) = e_n$

By uniqueness, the geodesic $T(\gamma_{(p,X)}) = (\gamma_{e_{n+1}, e_n})$

This means we only need to find $\gamma_{(e_{n+1}, e_n)}$

Consider $p : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}$ given by
$p(x_1, .., x_{n+1}) = (-x_1, .., -x_{n-1}, x_n, x_{n+1})$

This is the reflection on span $\{e_n, e_{n+1}\}$. This means p is an isometry. Thus $p(e_n) = e_n$ and $p(e_{n+1}) = e_{n+1}$

Thus $p \circ \gamma = \gamma_{(e_{n+1}, e_n)}$

This means $\gamma_{(e_{n+1}, e_n)} \subseteq \text{span}\{e_n, e_{n+1}\} \cap S^n$ which is a great circle.

Example : Let $[p] \in \mathbb{R}P^n$, $X \in T_{[p]}\mathbb{R}P^n$ with $|X| = 1$. We can choose $p \in S^n$ and
Then $d\pi_p : T_pS^n \longrightarrow T_{[p]}\mathbb{R}P^n$ where $\pi : S^n \to \mathbb{R}P^n$
$d\pi_p$ is an isomorphism, in fact an isometry

So $Y = d\pi_p^{-1}(X) \in T_pS^n$ and $|Y| = 1$

$\Rightarrow \exists!$ great circle $\alpha \in S^n$ s.t. $\alpha(0) = p$
$\alpha'(\ ) = Y$

Then $\gamma = \pi \circ \alpha$ is a geodesic in $\mathbb{R}P^n$ since π is a local isometry, $\gamma(0) = \pi(\alpha(0)) =$
$= \pi(p) = [p]$
and $\gamma'(0) = d\pi_p(\alpha'(0)) = d\pi_p(Y) = X$

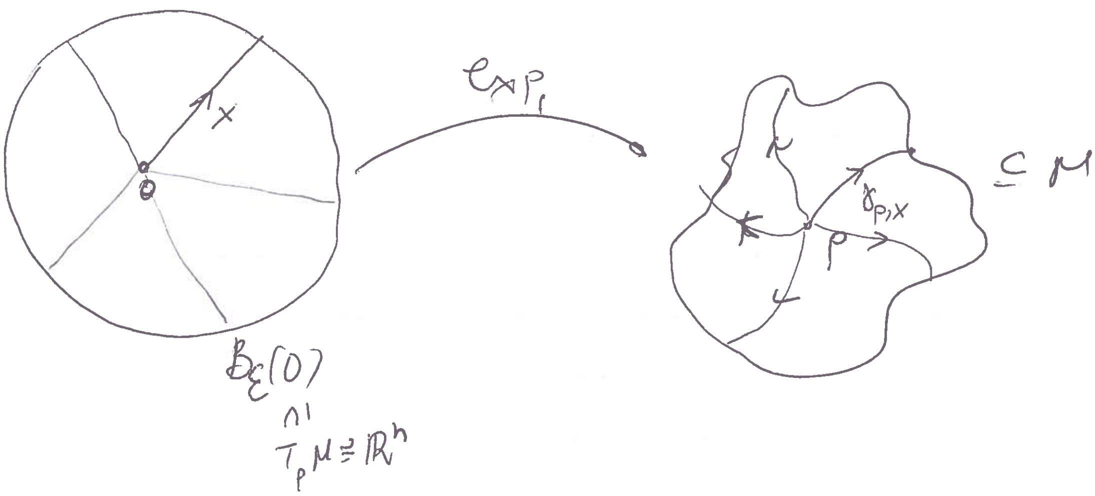
So γ is the unique geodesic $\gamma = \gamma_{([p],X)}$
The geodesic in $\mathbb{R}P^n$ are the projections of the great circles.

Definition 8.3 : Use the notation of Thm 8.2
Define $\exp_p : \underset{\text{open}}{V} \subseteq TM \longrightarrow M$ by

$\exp_p(q, X) = \gamma_{(q,X)}(1)$ for $(q, X) \in V$

We often restrict to $\exp_p : B_\varepsilon(0) \subseteq T_pM \to M$
given by $\exp_p(X) = \gamma_{(p,X)}(1)$ for $X \in B_\varepsilon(0) \subseteq T_pM$

This is the exponential map



Remark: Notice $\exp_p(tX) = \gamma_{(p,tX)}(1) = $

$$\overset{\text{Thm 8.2}}{=} \gamma_{(p,X)}(t)$$

$SO(n) = \{ A \in M_n(\mathbb{R}) : A^TA = I, \ \det A = 1 \}$

$T_I SO(n) = \{ B \in M_n(\mathbb{R}) : B^T = -B \}$

Given $B \in T_I SO(n)$ i.e. $B^T = -B$

Consider $A = \exp(B)$

$$A^TA = \exp(B^T) . \exp(B) = \exp(B^T + B) = $$
$$= \exp(0) = I$$

This implies $\exp_I(B) = \exp(B) = e^B$

Example: $\mathbb{R}^n$. If $p \in \mathbb{R}^n$, $X \in T_p\mathbb{R}^n \cong \mathbb{R}^n$

Then $\gamma_{p,X}(t) = p + tX$. So $\exp_p(X) = \gamma_{(p,X)}(1) =$
$= p + X$

Riemannian Geometry

<u>Example</u> : Let $p \in (0,0,1) \in S^2$
$X = (0,1,0) \in T_p S^2$

$$\Rightarrow \gamma_{(p,X)}(t) = (0, \sin t, \cos t)$$

$$\Rightarrow \exp_p(X) = \gamma_{(p,X)}(1) = (0, \sin 1, \cos 1)$$

Notice $\gamma_{(p,X)}(\pi) = (0,0,-1) = -p$

$\gamma_{(p,X)}(2\pi) = p$

The normalised geodesic has length 2π.

<u>Theorem 8.4</u> : Let $p \in M$, $\exists$ open $W \ni p$ and $\delta > 0$ s.t. $\forall q \in W$, $\exp_q : B_\delta \subset T_q M \to \exp_q(B_\delta(0)) \subseteq M$ is a diffeomorphism.

Q: Why is this true?
A: $d(\exp_p)_0$ is an isomorphism, so $\exp_p$ is a local diffeo.

$$\frac{d}{dt}\left(\exp_p(tX)\right)\Big|_{t=0} = d(\exp_p)_0(X)$$
$$\parallel$$
$$\frac{d}{dt}\left(\gamma_{p,X}(t)\right)\Big|_{t=0} = \gamma'_{p,X}(0) = X$$

$$\Rightarrow \boxed{d(\exp)_0 = id}$$ so it's an iso.

Definition 8.5 : The length of a piecewise smooth curve $\alpha : [0,L] \longrightarrow M$ is

$$\boxed{L(\gamma) = \int_0^L |\alpha'(t)|\, dt} = \int_0^L \sqrt{g_{\alpha(t)}(\alpha'(t), \alpha'(t))}\, dt$$

For a normalised geodesic $\gamma : [0,L] \longrightarrow M$,
$L(\gamma) = L$ since $|\gamma'(t)| \equiv 1$

Example $\mathbb{R}^2$, $\alpha(t) = (x_1(t), x_2(t))$

$$\alpha : [0,L] \longrightarrow \mathbb{R}^2$$

$$\alpha'(t) = x_1'(t)\,\partial_1 + x_2'(t)\,\partial_2$$

$$g_0(\alpha'(t), \alpha'(t)) = g_0(x_1'(t)\partial_1 + x_2'(t)\partial_2,\ x_1'(t)\partial_1 + x_2'(t)\partial_2)$$

$$= (x_1'(t))^2\, g_0(\partial_1, \partial_1) + 2(x_1'(t))(x_2'(t))\, g_0(\partial_1, \partial_2)$$

$$+ (x_2'(t))^2\, g_0(\partial_2, \partial_2) =$$

$$= (x_1'(t))^2 + (x_1'(t))^2$$

Since $g_0(\partial_1, \partial_1) = g_0(\partial_2, \partial_2) = 1$ and $g_0(\partial_1, \partial_2) = 0$

$$\Rightarrow \quad |\alpha'(t)| = \sqrt{(x_1'(t))^2 + (x_2'(t))^2}$$

$$\Rightarrow \quad L(\alpha) = \int_0^L \sqrt{(x_1'(t))^2 + (x_2'(t))^2}\, dt$$

Example : Normalized geodesic $\gamma : [0, 2\pi] \longrightarrow S^n$ has length 2π and $\gamma(0) = \gamma(2\pi)$

Normalised geodesic $\gamma : [0, \pi] \longrightarrow \mathbb{R}P^n$ has length π and $\gamma(0) = \gamma(\pi)$

Riemannian Geometry

Definition 8.6 An open set $U \subseteq M$, $p \in U$ is a normal neighbourhood of p if $\exists$ open $V \subseteq T_pM$ s.t. $\exp_p : \underset{\underset{0}{\Downarrow}}{V} \longrightarrow U$ is a diffeo.

Example: • $\mathbb{R}^n$ is a normal neighbourhood since $\exp_p : T_p\mathbb{R}^n = \mathbb{R}^n \to \mathbb{R}^n$ is a diffeo as it is a translation by p.

• $S^n \setminus \{S\}$ is a normal neighbourhood of $N \in S^n$ but S^n is not a normal neighbourhood because $\exp_N(X) = S$ $\forall X \in T_N S^n$ with $|X| = \pi$ so $\exp_N$ is not injective on T_pS^n, but it is a diffeo $\exp_p : B_\pi(0) \subseteq T_N S^n \to S^n \setminus \{S\}$

If $\exp_p$ is a diffeo from $B_\varepsilon(0) \subseteq T_pM$ onto its image, we let $B_\varepsilon(p) = \exp_p(B_\varepsilon(0))$

which is the geodesic ball of radius ε centered at p

Moreover we say that the geodesics $\gamma_{(p,X)}(t)$ in $B_\varepsilon(p)$ are radial geodesics in $B_\varepsilon(p)$

The boundary $S_\varepsilon(p)$ of $B_\varepsilon(p)$ is called a geodesic sphere (if it is $(n-1)$-dim)

An open set $W \subseteq M$ is a totally normal neighbourhood if it is a normal neighbourhood of all $q \in W$.

Example: • $\mathbb{R}^n$ is a totally normal neighbourhood

• in S^n the upper hemisphere say

$\{x \in \mathbb{R}^{n+1} : x_{n+1} > 0\} \cap S^n$ is a totally normal neighbourhood since $\forall p \in W$ $-p \in W$

Theorem 8.4 $\Rightarrow$ totally normal neighbourhoods always exist.

Definition: A piecewise curve $\alpha : [0, L] \to M$ is length-minimising if $\forall$ piecewise curves $\beta : [0, L] \to M$ with $\beta(0) = \alpha(0)$ and $\beta(L) = \alpha(L)$ we have $L(\alpha) \leq L(\beta)$

Lemma 8.7 (Gauss Lemma)

Let $p \in M$, $X \in T_pM$ s.t. $\exp_p(X)$ is defined. Then $g_{\underbrace{\exp_p(X)}_{\text{point}}}\left(d(\exp_p)_X(X), d(\exp_p)_X(Y)\right) =$

$$= g_p(X, Y) \qquad \forall\, Y \in T_pM$$

Remark: The Gauss lemma implies that if $Z \in T_pM$ with $|Z| = 1$, then the radial geodesic $\gamma_{(p,Z)}(r)$ is orthogonal to

any geodesic sphere $S_\varepsilon(p)$. This implies we have geodesic polar coordinates given by r and $Z \in T_pM$ with $|Z| = 1$

Proof: Suppose $X \neq 0$, otherwise trivial.

$Y = Y^T + Y^\perp$, where $Y^T \in \mathrm{Span}\{X\}$

$g_p(Y^\perp, X) = 0$

$Y^T = \lambda X$ so $g_{\exp_p}((d(\exp_p)_X(X), d(\exp_p)_X(Y^T)) =$

$= \lambda g_{\exp_p}(d(\exp_p)_X(X), d(\exp_p)_X(X))$

and $g_p(X, Y^T) = \lambda g_p(X, X)$

Recall $\gamma_{(p,X)}(t)$ satisfies $\gamma_{(p,X)}(0) = p$

$\gamma'_{(p,X)}(0) = X$ and $\gamma_{(p,X)}(1) = \exp_p(X)$

And $|\gamma'_{(p,X)}(t)|^2 = |\gamma'_{(p,X)}(0)|^2 = |X|^2 = g_p(X, X)$

$$\gamma'_{(p,X)}(1) = \frac{d}{dt}(\exp_p(tX))\Big|_{t=1} = \frac{d}{dt}(\exp_p)_X(X)$$

$$\Rightarrow |\gamma'_{(p,X)}(1)|^2 = g_{\underset{\gamma_{(p,X)}(1)}{\underset{\parallel}{\exp_p(X)}}}(d(\exp_p)_X(X), d(\exp_p)_X(X))$$

$\Rightarrow$ Gauss lemma holds for $Y = Y^T$

Remains to show for $Y = Y^\perp$ by linearity

Questions Session 4/12

A tangent vector $X \in T_pM$ is $X = \alpha'(0)$ for some curve $\alpha : (-\varepsilon, \varepsilon) \to M$ with $\alpha(0) = p$

$$\alpha'(0) : \underset{\text{function}}{f} \longrightarrow (f \circ \alpha)'(0) = \frac{d}{dt} f(\alpha(t))\Big|_{t=0} \quad \text{number}$$

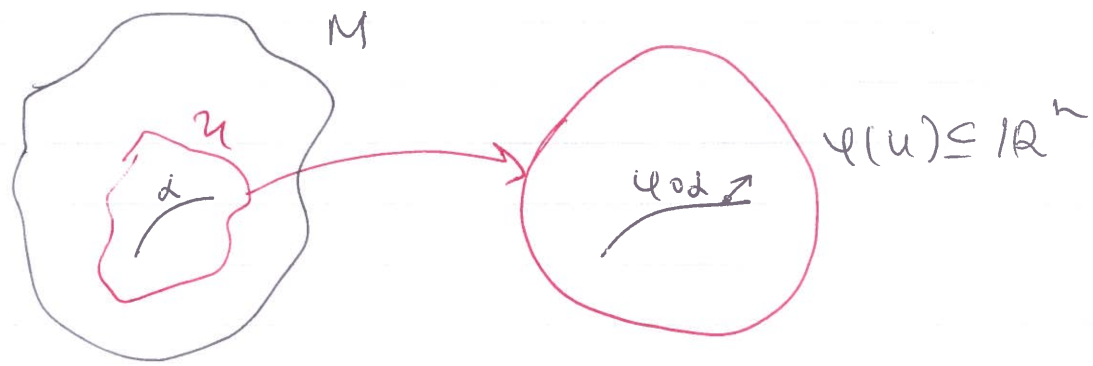
$f : M \to \mathbb{R}$

So a tangent vector is a differential operator.

To think of it as a vector, we need a chart (U, φ)

$\varphi \circ \alpha : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}^n$ curve in $\mathbb{R}^n$

We want to think of X as $(\varphi \circ \alpha)'(0) = \frac{d}{dt} \varphi(\alpha(t)) \Big|_{t=0} \in \mathbb{R}^n$



Write α as $\varphi^{-1} \circ \varphi \circ \alpha = \alpha$

$\Rightarrow X = \alpha'(0) = d\varphi^{-1}_{\varphi(p)}\left((\varphi \circ \alpha)'(0)\right) \in T_pM$, where $\varphi(p) = \varphi \circ \alpha(0)$

Once (U, φ) is chosen we can ~~write~~ identify X with the column/row vector $(\varphi \circ \alpha)'(0) \in \mathbb{R}^n$

Riemannian Geometry 4/12

On $\mathbb{R}^n$ we have a global chart $(\mathbb{R}^n, id)$ so we can identify $X = a_1\partial_1 + \dots + a_n\partial_n$ with $(a_1, \dots, a_n)$

$f: M \longrightarrow N$, $df_p : T_pM \longrightarrow T_{f(p)}N$

$$df_p(\alpha'(0)) = (f\circ\alpha)'(0)$$

if $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$ then

$$df_p(\alpha'(0)) = (f\circ\alpha)'(0) = \frac{d}{dt} f(\alpha(t))\Big|_{t=0}$$

$$\frac{|f(\alpha(t)) - f(\alpha(0)) - df_p(\alpha'(0))|}{|t|} \longrightarrow 0 \text{ as } t \to 0$$

Note $df_p(\alpha'(0)) = df_p(\varphi^{-1}\circ\varphi\circ\alpha)'(0) =$

$$= (df_p \circ d\varphi^{-1})(\varphi\circ\alpha)'(0) =$$

$$= \underbrace{d(f\circ\varphi^{-1})_{\varphi(p)}}_{\text{map } \mathbb{R}^n \to \mathbb{R}^m} \underbrace{((\varphi\circ\alpha)'(0))}_{\text{vector in } \mathbb{R}^n}$$

Helps to compute

$$\underbrace{d\psi_{f(p)}(df_p(\alpha'(0)))}_{\mathbb{R}^n} = \underbrace{d(\psi\circ f\circ\varphi^{-1})_{\varphi(p)}}_{\text{This is what we compute}}((\varphi\circ\alpha)'(0))$$

E.g. to prove df is a submersion show $d(\psi\circ f\circ\varphi^{-1})$ is a submersion

$$d(\psi\circ f\circ\varphi^{-1})_{\varphi(p)} : \mathbb{R}^m \longrightarrow \mathbb{R}^n$$

Pull-back:

$$f : M \longrightarrow N$$
$$df^*_p : T^*_{f(p)}(N) \longrightarrow T^*_p(M)$$

$\eta \in T^*_{f(p)}(N)$ let $X \in T_p(M)$

$$\mathbb{R} \ni \underbrace{df^*_p(\eta)}_{\in T^*_{f(p)}(N)}(\underbrace{X}_{\in T_p(M)}) = \eta(df_p(X)) \in \mathbb{R}$$

Flow and integral curves

Integral curve: X is vector field on M, $p \in M$

Find curve $\alpha_p : (-\varepsilon, \varepsilon) \longrightarrow M$

$$\alpha'_p(t) \in T_{\alpha_p(t)} M \qquad X : M \longrightarrow TM$$
$$\alpha_p(t) \longrightarrow X(\alpha_p(t)) \in T_{\alpha_p(t)} M$$

$$\alpha_p'(t) = X(\alpha_p(t))$$
$$\alpha_p(0) = p$$

α_p is the integral curve of X through p.

Take X on $\mathbb{R}^2$ to be $X = x_1 \partial_1$

Choose $p = (c_1, c_2)$ in $\mathbb{R}^2$

$\alpha_p(t) = (x_1(t), x_2(t))$ some curve in $\mathbb{R}^2$

$$\alpha_p'(t) = x_1'(t)\partial_1 + x_2'(t)\partial_2 = X(\alpha_p(t)) = x_1(t)\partial_1$$

$$\begin{cases} x_1'(t) = x_1(t) \\ x_2'(t) = 0 \end{cases} \Rightarrow \begin{aligned} x_1(t) &= c_1 e^t \\ x_2(t) &= c_2 \end{aligned}$$

with i.c. $p = (c_1, c_2)$, $\alpha_p(0) = p$

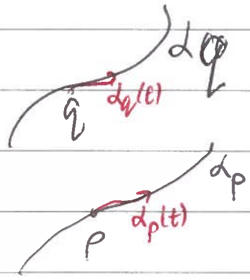
half lines or pts

$$\phi^X_t(c_1, c_2) = (c_1 e^t, c_2)$$

Riemannian Geometry 4/12

Flow of X near p

A family of smooth maps $\phi_t^X : \underset{p}{\overset{\text{open}}{V}} \longrightarrow M$ parametrised by t in $(-\varepsilon, \varepsilon)$



$$\phi_t^X(q) = \alpha_q(t) \quad \forall q \in V$$

t is fixed

$$\phi_0^X(q) = \alpha_q(0) = q$$

$$\phi_0^X = id$$

$$\phi^X_{t_1+t_2} = \phi^X_{t_1} \circ \phi^X_{t_2} \quad \text{so} \quad \phi^X_{-t} = (\phi^X_t)^{-1} \text{ when defined}$$

Lie Derivative : X, Y vector fields on M

$p \in M \rightsquigarrow$ flow of X near p $\phi_t^X : V \to M$ $t \in (-\varepsilon, \varepsilon)$

$$\mathcal{L}_X Y(p) = \lim_{t \to 0} \frac{(\phi^X_{-t})_*(Y(\phi^X_t(p))) - Y(p)}{t}$$

$(\phi^X_{-t})_* : T_q M \longrightarrow T_{\phi^X_{-t}(q)} M$

let $q = \phi_t^X(p) \rightarrow \phi^X_{-t}(q) = p$

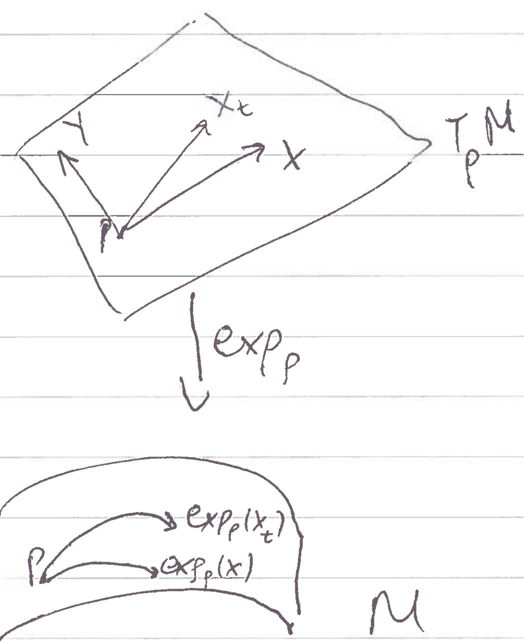
derivative map

Gauss lemma

Suppose we have a Riemannian manifold (M, g), $p \in M$. Let $\exp_p : T_pM \to M$, $X \in T_pM$, $d(\exp_p)_X : T_X(T_pM) \cong T_pM \to T_pM$.
Then $\forall\, Y \in T_X(T_pM) \cong T_pM$ we have

$$\boxed{g_{\exp_p(X)}\left(d(\exp_p)_X(X), d(\exp_p)_X(Y)\right) = g_p(X, Y)}$$

Proof: Last week we showed 1):
1) if $Y = \lambda X$, for some $\lambda = \text{const.}$
Now
2) $Y \perp X$, i.e. $g_p(X, Y) = 0$



$X_t = X \cos t + Y \sin t$
for t small, $t \in (-\varepsilon, \varepsilon)$

So we can define $f(s,t) = \exp_p(s X(t))$ for $t \in (-\varepsilon, \varepsilon)$ and $s \in [0, 1]$

$$\frac{\partial f}{\partial s} = d(\exp_p)_{sX(t)}(X(t))$$

$$\frac{\partial f}{\partial t} = d(\exp_p)_{sX(t)}(s X'(t))$$

Riemannian Geometry

For $s=1$, $t=0$

$$\frac{\partial f}{\partial s} = d(\exp_p)_X(X) \quad \text{and} \quad \frac{df}{dt} = d(\exp_p)_X(Y)$$

Since $X'(t) = -X \sin t + Y \cos t$

at $t=0$ $= Y$

So we need to prove

$$g\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)(1,0) = 0$$

<u>Claim</u> $\frac{\partial}{\partial s} g\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)(s,t) = 0$ for each fixed t

<u>Proof</u>: $LHS = g\left(\frac{D}{Ds}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right) + g\left(\frac{\partial f}{\partial s}, \frac{D}{Ds}\frac{\partial f}{\partial t}\right) =$

For each fixed t $f(s,t)$ is a geodesic because it is defined as $\exp_p(sX(t))$

$\Rightarrow \frac{D}{Ds}\frac{\partial f}{\partial s} = 0$, Furthermore $\frac{D}{Ds}\frac{\partial f}{\partial t} = \frac{D}{Dt}\frac{\partial f}{\partial s}$

$$\Rightarrow LHS = 0 + \frac{1}{2}\frac{\partial}{\partial t} g\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial s}\right) = \frac{1}{2}\frac{\partial}{\partial t} g\big(d(\exp_p)_{sX(t)}(X(t)),$$

by 1st part

$$d(\exp_p)_{s\cancel{X}(t)}(X(t))\big) \underset{\text{of Gauss lemma}}{=} \frac{1}{2}\frac{\partial}{\partial t} g_p(X(t), X(t)) \underset{\text{by def of } X(t)}{=}$$

$$= \frac{1}{2}\frac{\partial}{\partial t}\left(|X|^2 + |Y|^2\right) = 0$$

$$\Rightarrow \frac{\partial}{\partial s} g\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)(s,t) = 0$$

$$\Rightarrow g\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)(1,0) = g\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)(0,0) = 0$$

since if $s=0$ $\frac{\partial f}{\partial t} = 0$ $\Rightarrow g\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)(1,0) = 0$ ∎

Remark : (U, φ) is a chart on M

$\varphi : U \subset M \longrightarrow \mathbb{R}^n$

$f(s,t) = \varphi^{-1}(a^1(s,t), \dots, a^n(s,t))$
curve in $\mathbb{R}^n$

$X_j = \varphi^{-1}_*(\partial_i)$

$$\frac{D}{Ds}\frac{\partial f}{\partial t} = \frac{D}{Ds}\left(\sum_{j=1}^{n} \frac{\partial a^j}{\partial t} X_j\right) =$$

$$= \sum_{j=1}^{n} \frac{\partial^2 a^j}{\partial s \partial t} X_j + \sum_{j,k=1}^{n} \frac{\partial a^j}{\partial t}\frac{\partial a^k}{\partial s} \nabla_{X_k} X_j =$$

$$= \frac{D}{Dt}\frac{\partial f}{\partial s}$$

Using the symmetry of $\nabla_{X_k} X_j = \sum_{i=1}^{n} \Gamma_{kj}^{i} X_i$

and the symmetry of $\frac{\partial^2 a}{\partial s \partial t}$.

Thus $\frac{D}{Ds}\frac{\partial f}{\partial t} = \frac{D}{Dt}\frac{\partial f}{\partial s}$

Riemannian Geometry 10/12

$\gamma(s) = \exp_p(sX)$ is a geodesic, $p \in M$, $X \in T_pM$, $s \in [0; L]$

$$\underline{L(\gamma)} = \int_0^L |\gamma'(s)|\, ds = \int_0^L |d(\exp_p)_{sX}(X)|\, ds =$$

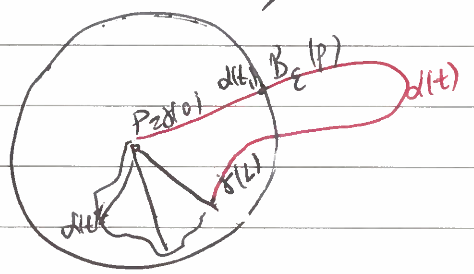
$$= \int_0^L \sqrt{g_{\exp_p(sX)}\left(d(\exp_p)_{sX}(X),\ d(\exp_p)_{sX}(X)\right)}\, ds$$

By Gauss lemma

$$= \int_0^L \sqrt{g_p(X, X)}\, ds = \underline{|X| \cdot L}$$

Theorem

Geodesics $\gamma : [0, L] \longrightarrow B_\varepsilon(p) \subset M$ with $\gamma(0) = p$ are locally minimising, i.e. for other curve $\alpha : [0, L] \longrightarrow M$ connecting $\gamma(0)$, $\gamma(L)$ ($\alpha(0) = \gamma(0)$ and $\alpha(L) = \gamma(L)$) then $L(\alpha) \geq L(\gamma)$. Moreover, if $L(\alpha) = L(\gamma)$, then $\alpha([0, L]) = \gamma([0, L])$



$\gamma(t) = \exp_p(tX)$

Proof:

Case 1 : $\alpha \subseteq B_\varepsilon(p)$

Case 2 : $\alpha \nsubseteq B_\varepsilon(p)$

We will first prove <u>case 2</u> assuming <u>case 1</u>

$\exists$ a point $\alpha(t_1) \in \partial B_\varepsilon(p) \Rightarrow L(\alpha) \geq L(\alpha|_{[0,t_1]})$

$\overset{\text{By } 1^{st}\text{ part}}{\geq} \varepsilon \geq L(\gamma)$

Next we prove <u>case 1</u> :

Because $\exp_p$ is a local diffeomorphism we can write $\alpha(t) = \exp_p(s(t)X(t))$
Also $\alpha(t) = f(s(t), t)$ as in the proof of Gauss lemma

$$L(\gamma) = \int_0^L |\alpha'(t)|\, dt = \int_0^L \left| s'(t)\frac{\partial f}{\partial s} + \frac{\partial f}{\partial t} \right| dt$$

$$|\alpha'(t)|^2 = |s'(t)|^2 g\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial s}\right) + g\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\right) + g\left(s'(t)\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)$$

$X(t) \in T_pM$ and $|X(t)| = 1$

$$\frac{\partial f}{\partial s} = d(\exp_p)_{sX(t)}(X(t)) \quad , \quad \frac{\partial f}{\partial t} = d(\exp_p)_{sX(t)}(sX'(t))$$

$$g\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial s}\right) = g_p(X(t), X(t)) \quad \text{By Gauss lemma}$$

$$= 1 \quad \text{since } |X(t)| = 1$$

$$g\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right) = g_p(X(t), sX'(t)) \quad \text{by Gauss lemma}$$

$$= s(t)\frac{1}{2}\frac{d}{dt}|X(t)|^2 = 0$$

$$\Rightarrow |\alpha'(t)|^2 = |s'(t)|^2 + \underbrace{g\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\right)}_{\geq 0}$$

$$\Rightarrow |\alpha'(t)|^2 \geq |s'(t)|^2$$

$$\Rightarrow L(\alpha) \geq \int_0^L |s'(t)|\, dt$$

$$\Rightarrow L(\alpha) \geq \int_0^L |s'(t)|\, dt \geq \int_0^L s'(t)\, dt =$$

$$= s(L) - s(0) = s(L) \quad \text{since } s(0)=0$$

Our assumption is $\gamma(L) = \alpha(L) = \exp_p(s(L) X(L))$

$\gamma(t) = \exp_p\left(\frac{t}{L} s(L) X(L)\right)$ since γ is a geodesic for any $t < L$

By gauss' lemma $L(\gamma) = L \cdot \left|\frac{s(L) . X(L)}{L}\right| =$

$= s(L)$ since $|X(L)| = 1$

$$\Rightarrow L(\alpha) \geq s(L) = L(\gamma)$$

If $L(\alpha) = L(\gamma) \Rightarrow s'(t) \geq 0$

and $|\alpha'(t)|^2 = |s'(t)|^2 \Rightarrow$

$\Rightarrow g\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\right) = 0$

$$g\left(\frac{\partial f}{\partial z}, \frac{\partial f}{\partial t}\right) = 0 \Rightarrow |X'(t)| = 0$$

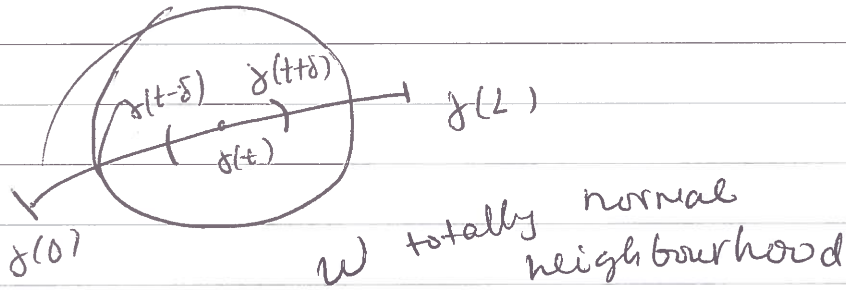
so $X(t) = $ const. vector, $X \in T_pM$

$$\alpha(t) = \exp_p(s(t)X(t)) = \exp_p(s(t)X(L))$$

since $s'(t) \geq 0 \Rightarrow s(t)$ is monotone increasing and $X(L)$ points in the same direction of $\gamma \Rightarrow \alpha(t)$ is a monotonic reparametrisation of $\gamma \Rightarrow \alpha([0,L]) = \gamma([0,L])$ ∎

Proposition Let $\gamma : [0, L] \to M$ be a curve and is piecewise smooth with $|\gamma'| = $ const, and it is locally minimising. Then γ is a geodesic

Proof:



$\gamma([t-\delta, t+\delta]) \subset W$
$p = \gamma(t-\delta)$, $q = \gamma(t+\delta)$

$\exists\ \beta(s) = \exp_p(sX)$ connecting p & q

i.e. $\beta(0) = p$ and $\beta(L') = q$

By the previous theorem $\alpha(s) = \gamma|_{[t-\delta, t+\delta]}$ is locally minimising and $\beta(s)$ is locally minimising

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$\Rightarrow \alpha(s)$ is a monotonic reparametrisation of $\beta(s)$

$\alpha(s) = \exp_p(u(s) X)$

$u'(s) \geq 0$ so $|\alpha'(s)| = |\gamma'(s)| = g(d(\exp_p)_{uX}(u'(s) X),$

$d(\exp_p)_{u(s)X}(u'(s)X)) = |u'(s)|^2$ by Gauss Lemma

since $|X| = 1$

By assumption $|u'(s)| = |\gamma'(s)| = \text{const}$
$u'(s) = \lambda$ const

$\therefore \alpha(s) = \exp_p(\lambda s X) \Rightarrow \alpha$ is a geodesic ■

Completeness

Suppose (M, g) is a connected Riemannian manifold. We can view (M, g) as a metric space.

Definition: $\forall\ p, q \in M$ $d(p, q) = \inf\{L(\alpha) : \alpha$ is piecewise smooth curve connecting p & $q\}$

Proposition (M, d) is a metric space

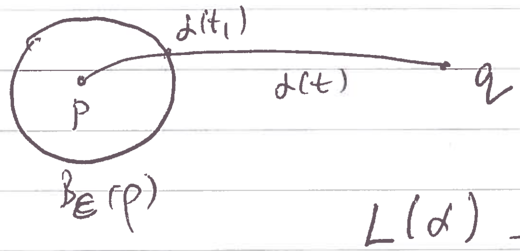
That is: 1) $d(p, q) \geq 0$, and $d(p, q) = 0 \iff p = q$

2) $d(p, q) = d(q, p)$

3) $d(p, q) \leq d(p, t) + d(t, q)$

$p, q, t \in M$

Proof : (1) if $p \neq q \in M$



$$L(\alpha) \geq L(\alpha|_{[0,t_1]}) \geq \varepsilon > 0$$

$\Rightarrow d(p,q) > 0$ if $p \neq q$

Definition (M, g) is geodesically complete if $\exp_p : T_pM \to M$ is well-defined for all $p \in M$ and all $X \in T_pM$.

Equivalently.

$\gamma(t) = \exp_p(tX)$ is defined $\forall t \in \mathbb{R}$

Example :

1. $\mathbb{R}^n$ geodesics are straight lines so $\mathbb{R}^n$ is complete

2. $H_+^n = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n, x_n > 0\}$ is non-complete

   $\gamma(t) = t\vec{e_n}$ only defined for $t \in (0, +\infty)$

3. S^n is complete since the geodesics are great circles and great circles are defined $\forall t$.

4. $S^n \setminus \{N\}$ non-complete because the geodesic through S are only defined in $t \in (-\pi, \pi)$

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Proposition: The topology of M w.r.t. the metric d coincides with the original topology of M

Proof: $\forall p \in M$ if r is sufficiently small then we have a geodesic ball $B_r(p)$

$B_r(p) = \exp_p(B_r(0))$, $\exp_p : B_r(0) \subseteq T_pM \cong \mathbb{R}^n \to B_r(p) \subseteq M$

- metric Ball $B_r^d(p) = \{x \in M, d(x,p) < r\}$

Claim: $B_r(p) = B_r^d(p)$ for r small

I show $B_r(p) \subseteq B_r^d(p)$

If $q \in B_r(p)$, then $q = \exp_p(r_1 X)$ for some $X \in T_pM$ and $r_1 < r$

$\beta(s) = \exp_p(sX)$ is a geodesic connecting p and q.

Locally the geodesic is a minimising

$\Rightarrow d(q,p) \leq L(\beta) = r_1 < r$

$\Rightarrow p \in B_r^d(p) \Rightarrow B_r(p) \subseteq B_r^d(p)$

II show $B_r^d(p) \subseteq B_r(p)$ ...

Completeness

- (M,d) metric completeness
- (M,g) (geodesically) completeness

Theorem (Hopf - Rinow)

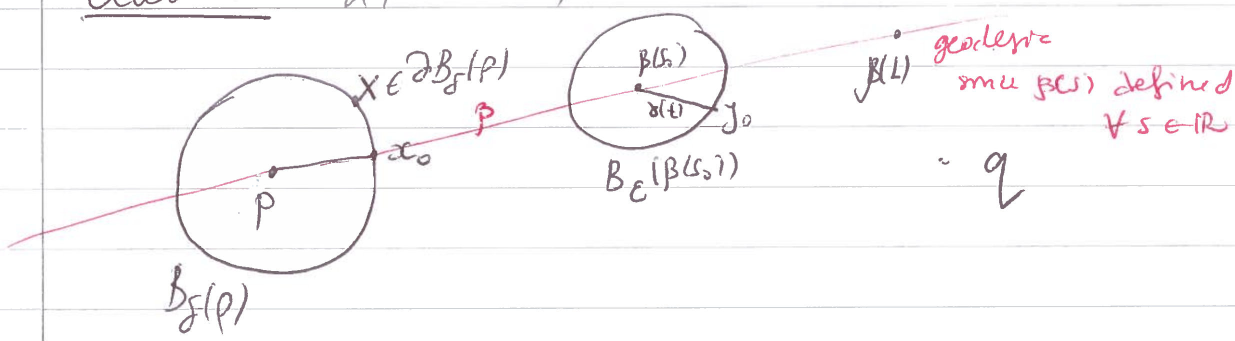
If (M,g) is a Riemannian manifold the following properties are equivalent:

1) (M,g) is (geodesically) complete
2) $\exp_p$ is well defined $\forall X \in T_pM$ for some $p \in M$
3) Any closed and bdd subset of M is compact
4) (M,d) is complete

✱ Moreover: If (M,g) is geodesically complete then $\forall p,q \in M$ we can find a minimizing geodesic $\gamma: [0,L] \to M$ s.t. $\gamma(0)=p$ and $\gamma(L)=q$, $L(\gamma)=d(p,q)=L$

Proof: 1) $\Rightarrow$ 2) by definition

2) $\Rightarrow$ 3)
Claim 2) $\Rightarrow$ ✱)



$d(x,q)$ is continuous function

$\Rightarrow \exists\, x_0 \in \partial B_\delta(p)$ s.t. $d(x_0,q) = \min_{x \in \partial B_\delta(p)} d(x,q)$

Because $x_0 \in \partial B_\delta(p)$ we can find a geodesic

$\beta(s) = \exp_p(sX)$, $s \in [0,\delta]$ and $|X|=1$, $X \in T_pM$

By assumption 2 $\beta(s)$ is well defined $\forall s \in \mathbb{R}$

We claim that $\beta(L) = q$

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Define $A = \{ s \in [0, L] \mid d(\beta(s), q) = L - s \}$

If $L \in A$ then $d(\beta(L), q) = L - L = 0$

If we show $L \in A$ we are done.

- A is non empty since $0 \in A$
  $d(\beta(0), q) = d(p, q) = L$

- A is closed since the distance function is continuous

if A is also open then the connectedness of $[0, L] \Rightarrow A = [0, L]$

<u>Proof A is open</u>:

$\forall\ s_0 \in A \quad \exists\ y_0 \in \partial B_\varepsilon(\beta(s_0))$ s.t. $d(q, y_0) =$
$= \min\limits_{x \in \partial B_\varepsilon(\beta(s_0))} d(x, q)$

Because $y_0 \in \partial B_\varepsilon(\beta(s_0))$ we can find a geodesic $\gamma(t) = \exp_{\beta(s_0)}(tY)$, $t \in [0, \varepsilon]$, $Y \in T_{\beta(s_0)}M$

$\gamma(0) = \beta(s_0)$ and $\gamma(\varepsilon) = y_0$

Because $s_0 \in A$ $\quad d(\beta(s_0), q) = L - s_0$

$$\begin{aligned} &\shortparallel \\ &\min_{x \in \partial B_\varepsilon(\beta(s_0))} \{ d(\beta(s_0), x) + d(x, q) \} \\ &\shortparallel \\ &\varepsilon + \min_{x \in \partial B_\varepsilon(\beta(s_0))} d(x, q) = \varepsilon + d(y_0, q) \end{aligned}$$

$\Rightarrow \quad d(y_0, q) = L - (s_0 + \varepsilon)$

$$d(p, y_0) \leq d(p, \beta(s_0)) + d(\beta(s_0), y_0)$$
$$= s_0 + \varepsilon$$

$$d(p, y_0) \geq d(p, q) - d(q, y_0)$$
$$L - L + (s + \varepsilon_0)$$
$$= s_0 + \varepsilon$$

$$\Rightarrow d(p, y_0) = s_0 + \varepsilon$$

The piecewise geodesic $(s) = \beta|_{[0, s_0]} \cup \gamma|_{[0, \varepsilon]}$

$$L(\alpha) = L(\beta|_{[0,s_0]}) + L(\gamma|_{[0,\varepsilon]}) =$$
$$= s_0 + \varepsilon$$

So $L(\alpha) = d(p, y_0)$

By previous thm α minimises the distance between two pts so α is a smooth geodesic.

By the uniqueness result $\Rightarrow \alpha = \beta|_{[0, s_0+\varepsilon]}$

So $d(\beta(s_0+\varepsilon), q) = L - (s_0+\varepsilon)$

$\Rightarrow s_0 + \varepsilon \in A \Rightarrow A$ is open

Now we have $2) \Rightarrow *)$

Now we prove $2) \Rightarrow 3)$. Suppose C is a closed bounded subset of M. Because C is bdd, $\exists$ metric ball $B_R^d(p)$ s.t. $C \subset B_R^d(p)$

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(2) $\Rightarrow$ ~~*~~ $\Rightarrow B^d_R(p) \subseteq \exp_p(B_{R'}(0))$

where $B_{R'}(0) \subseteq T_pM \cong \mathbb{R}^n$

Because $\forall q \in B^d_R(p)$ $\exp_p : T_pM \to M$

Because $\overline{B_{R'}(0)}$ is closed bdd subset in $\mathbb{R}^n$ it is compact. But $\exp_p$ is a diffeomorphism $\Rightarrow \exp_p(\overline{B_{R'}(0)})$ is compact

C is closed bdd & subset of compact set
C is compact ∎

(3) $\Rightarrow$ 4) : (M, d) is complete

For any cauchy seq $\{p_n\}$ let $C = \overline{\{p_n\}}$
Since $\{p_n\}$ is cauchy it is bdd. So C is closed and bdd by 3) $\Rightarrow$ C is compact
Thus we can find a converging subseq.
Hence the metric space is complete. ∎

(4) $\Rightarrow$ (1) Proof by contradiction:
Suppose (M, g) is not complete. Then we can find a normalised geodesic $\gamma(s)$, only defined in $s < s_0$ but not for $s = s_0$

$\exists$ seq. $s_n \to s_0$ as $n \to s_0$

$$d(\gamma(s_n), \gamma(s_m)) = |s_n - s_m| \to 0$$
if $n, m \to \infty$

by 4) we can find $p_0 \in M$ s.t. $d(\gamma(s_n), p) \to 0$ as $n \to \infty$

Choose W is a totally normal neighbourhood of p_0.

$\exp_p : B_\delta(0) \longrightarrow \exp_p(B_\delta(0))$ is well defined
$$\exp_p(B_\delta(0)) \subseteq W$$

$\exists\ \beta(s) = \exp_p(sX)$, $\beta(0) = \gamma(s_n)$

$\exists\ N > 0$, if $n, m > N$ s.t. $d(\gamma(s_n), \gamma(s_m)) < \delta$

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§10. Curvature

g - Riemannian metric
∇ Levi-Civita Connection ⟶ 1^{st} derivative of g

Suggests we need 2^{nd} derivatives of g for curvature

Proposition 10.1 For vector fields X, Y, Z on (M, g) we define:

$$R(X,Y)Z = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})Z,$$

which is a vector field.

- $R(-,-)$ is bilinear
- $R(Y,X) = -R(X,Y)$

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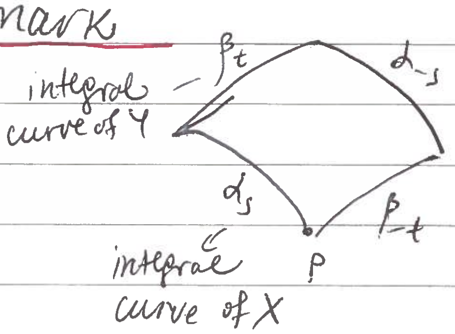
- $R(X,Y)Z \longrightarrow R(X,Y)Z$ linear
- $R(X,Y)Z(p)$ only depends on $X(p)$, $Y(p)$, $Z(p)$ for $p \in M$
- $R(X,Y)$ is called the Riemann curvature operator

Remark



$\lim_{s,t \to 0}$ parallel transport $Z(p)$ around the parallelogram $=$
$= R(X,Y)Z$

Example • $\mathbb{R}^n$ $\nabla_{\partial_i} \partial_j = 0$ and $[\partial_i, \partial_j] = 0$

$\Rightarrow R(\partial_i, \partial_j)\partial_k = 0$

• S^2 $f(\theta, \varphi) = (\sin\theta \cos\varphi, \sin\theta \sin\varphi, \cos\theta)$

$X_1 = f_* \partial_\theta = \cos\theta \cos\varphi\, \partial_1 + \cos\theta \sin\varphi\, \partial_2 - \sin\theta\, \partial_3$
$X_2 = f_* \partial_\varphi = -\sin\theta \sin\varphi\, \partial_1 + \sin\theta \cos\varphi\, \partial_2$

$[X_1, X_2] = f_* [\partial_\theta, \partial_\varphi] = 0$

$\nabla_{X_1} X_1 = 0$

$\nabla_{X_1} X_2 = \cot\theta\, X_2 = \nabla_{X_2} X_1$ since Lie bracket is 0

$\nabla_{X_2} X_2 = -\sin\theta \cos\theta\, X_1$

If $[X,Y] = [Y,Z] = [Z,X] = 0$ then

Then $g(\nabla_X Y, Z) = \frac{1}{2}\big(X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y))\big)$

$$g(\nabla_{X_1} X_2, X_1) = \frac{1}{2}\{X_1(g(X_2, X_1)) + X_2(g(X_1, X_1)) -$$

$$- X_1(g(X_1, X_2))\} = 0 \quad \text{since}$$

$$g(X_2, X_1) = 0 \quad \text{and} \quad g(X_1, X_1) = 1$$

$$g(\nabla_{X_1} X_2, X_2) = \frac{1}{2}(X_1(g(X_2, X_2)) + X_2(g(X_1, X_2)) -$$

$$- X_2(g(X_1, X_2))\} =$$

$$= \frac{1}{2} X_1(g(X_2, X_2)) = \frac{1}{2}\frac{\partial}{\partial\theta} \sin^2\theta = \sin\theta\cos\theta$$

$$\Rightarrow \nabla_{X_1} X_2 = \frac{\sin\theta\cos\theta . X_2}{g(X_2, X_2)} = \cot\theta . X_2$$

$$R(X_1, X_2)X_1 = (\nabla_{X_1}\nabla_{X_2} - \nabla_{X_2}\nabla_{X_1})X_1 \quad \text{Lie bracket is 0}$$

$$= \nabla_{X_1}(\nabla_{X_2} X_1) - \nabla_{X_2}(\nabla_{X_1} X_1) =$$

$$= \nabla_{X_1} \cot\theta X_2 - 0 =$$

$$= \cot\theta \, \nabla_{X_1} X_2 + \underbrace{X_1(\cot\theta)}_{\frac{\partial}{\partial\theta}(\cot\theta)} . X_2 =$$

$$= \cot^2\theta X_2 - \operatorname{cosec}^2\theta X_2 =$$

$$= \cot^2\theta X_2 - (1 + \cot^2\theta) X_2 =$$

$$= - X_2$$

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$R(X_1, X_2)X_2 = (\nabla_{X_1}\nabla_{X_2} - \nabla_{X_2}\nabla_{X_1})(X_2) =$

$= \nabla_{X_1}(-\sin\theta\cos\theta X_1) - \nabla_{X_2}\cot\theta X_2 =$

$= X_1(-\sin\theta\cos\theta)X_1 + \cot\theta\sin\theta\cos\theta X_1 - \underbrace{X_2(\cot\theta)}_{\frac{\partial}{\partial\varphi}(\cot\theta)=0}X_2$

$= (\sin^2\theta - \cos^2\theta)X_1 + \cos^2\theta X_1 =$

$= \sin^2\theta X_1$

Definition 10.2 Define the Riemann curvature tensor R by:

$$\boxed{R(X, Y, Z, W) = g(R(X,Y)Z, W)}$$

where X, Y, Z, W are vector fields

$R \in \Gamma(\otimes^4 T^*M)$

Example: • $\mathbb{R}^n$: $R=0$ we call (M,g) flat if $R=0$

• S^2: $R(X_1, X_2, X_1, X_1) = g(-X_2, X_1) = 0$

$R(X_1, X_2, X_1, X_2) = g(-X_2, X_2) = -\sin^2\theta$

$R(X_1, X_2, X_2, X_1) = g(\sin^2\theta X_1, X_1) = \sin^2\theta$

$R(X_1, X_2, X_2, X_2) = g(\sin^2\theta X_1, X_2) = 0$

Remarks: 1) Let (U, φ) be the chart s.t. $g_{ij} = \delta_{ij}$ at pt U and $\Gamma_{ij}^{k} = 0$ $\forall p \in U$, $\varphi(p) = 0$

$$X_i = (\varphi^{-1})_* \partial_i$$

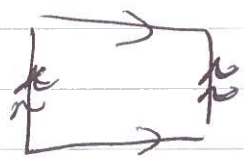
$$R_{ijkl} = R(X_i, X_j, X_k, X_l)$$

Then $g_{ij} = \delta_{ij} - \frac{1}{3}\sum_{kl} R_{ijkl}\, x_k x_l + O(|x|^3)$

2) R is determined by g and ∇, hence it is preserved by local isometries

$\Rightarrow$ since $\pi : \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ is a (surjective)

local isometry $\Rightarrow$ $\underset{\mathbb{R}^{2n}}{\overset{\cap}{T^n}} \cong \mathbb{R}^n/\mathbb{Z}^n$ is flat

$T^2 \subseteq \mathbb{R}^4$ is $S^1 \times S^1$  flat

Proposition 10.3

a) $R(Y, X, Z, W) = -R(X, Y, Z, W)$

b) $R(X, Y, W, Z) = -R(X, Y, Z, W)$

c) $R(Z, W, X, Y) = R(X, Y, Z, W)$

d) $R(X, Y, Z, W) + R(Y, Z, X, W) + R(Z, X, Y, W) = 0$

The First Bianchi identity

Definition 10.

Let $\sigma = \operatorname{Span}\{X, Y\} \subseteq T_pM$ be a 2-plane then the sectional curvature of σ is

$$K(\sigma) = K(X,Y) = \frac{R(X,Y,Y,X)}{\underbrace{g(X,X)\,g(Y,Y) - g(X,Y)^2}_{= \det \text{ of } g_{ij}}}$$

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Notice: This does not depend on the choice of basis X, Y for σ

Proposition 10.5 Let $\bar{R} \in \Gamma(\otimes^4 T^*M)$ with the same symmetries as R in Proposition 10.3. Suppose $\forall p \in M$, $\forall \sigma$ 2-planes in T_pM we have $\bar{K}(\sigma) = \dfrac{\bar{R}(X,Y,Y,X)}{g(X,X)g(Y,Y)-g(X,Y)^2} = K(\sigma)$

Then $R = \bar{R}$ i.e. "K determines R"

Examples • S^2, $K(T_{p \notin \{N,S\}} S^2) = K(X_1, X_2) =$

$$= \frac{R(X_1, X_2, X_2, X_1)}{g(X_1,X_1)g(X_2,X_2)-g(X_1,X_2)^2} = \frac{\sin^2\theta}{1.\sin^2\theta - 0} = 1 \quad \forall p$$

Example: Let $\mathcal{H}^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 - x_3^2 = -1, \ x_3 > 0\}$

with Riemannian metric g given by the restriction of $dx_1^2 + dx_2^2 - dx_3^2$.

Let $f(\theta, \varphi) = (\sinh\theta\cos\varphi, \sinh\theta\sin\varphi, \cosh\theta)$
$f : \mathbb{R}^2 \to \mathcal{H}^2$

$X_1 = f_* \partial_\theta = \cosh\theta\cos\varphi\, \partial_1 + \cosh\theta\sin\varphi\, \partial_2 + \sinh\theta\, \partial_3$
$X_2 = f_* \partial_\varphi = -\sinh\theta\sin\varphi\, \partial_1 + \sinh\theta\cos\varphi\, \partial_2$

$g(X_1, X_1) = 1$ $\qquad g(X_2, X_2) = \sinh^2\theta$

$g(X_1, X_2) = 0$

$[X_1, X_2] = 0$

$$g(\nabla_{X_1} X_1, X_1) = \frac{1}{2} X_1(g(X_1, X_1)) = 0$$

$$g(\nabla_{X_1} X_1, X_2) = \frac{1}{2}\Big(X_1(g(X_1,X_2)) + X_1(g(X_1,X_2)) -$$

$$- X_2(g(X_1,X_1))\Big) = 0$$

as $g(X_1, X_2) = 0$ and $g(X_1, X_1) = 1 = \text{const}$

$\nabla_{X_1} X_1 = 0$

$$g(\nabla_{X_1} X_2, X_1) = \frac{1}{2}(X_2\, g(X_1, X_1)) = 0$$

or $g(\nabla_{X_1} X_2, X_1) = X_1(g(X_2, X_1)) - g(X_2, \nabla_{X_1} X_1)$

$= 0$

$$g(\nabla_{X_1} X_2, X_2) = \frac{1}{2} X_1(g(X_2, X_2)) =$$

$$= \frac{1}{2} \underset{\substack{\| \\ \frac{\partial}{\partial\theta}}}{X_1}(\sinh^2\theta) =$$

$$= \sinh\theta \cosh\theta$$

so $\nabla_{X_1} X_2 = \dfrac{\sinh\theta\cosh\theta\, X_2}{g(X_2, X_2)} = \coth\theta\, X_2 =$

$= \nabla_{X_2} X_1$ since $[X_1, X_2] = 0$

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$$g(\nabla_{X_2} X_2, X_1) = X_2(g(X_2, X_1)) - g(X_2, \nabla_{X_2} X_1)$$

$$= -g(X_2, \coth\theta X_2) =$$

$$= -\sinh\theta\cosh\theta \quad \text{since } g(X_2, X_2) = \sinh^2\theta$$

$$g(\nabla_{X_2} X_2, X_2) = 0 = \frac{1}{2} X_2(g(X_2, X_2)) =$$

$$= \frac{1}{2}\frac{\partial}{\partial\varphi}(\sinh^2\theta)) = 0$$

$$\Rightarrow \nabla_{X_2} X_2 = -\sinh\theta\cosh\theta X_1$$

We need $R(X_1, X_2, X_2, X_1) = g(R(X_1, X_2)X_2, X_1)$

$$R(X_1, X_2)X_2 = (\nabla_{X_1}\nabla_{X_2} - \nabla_{X_2}\nabla_{X_1})(X_2) =$$

$$= \nabla_{X_1}(-\sinh\theta\cosh\theta)X_1 - \nabla_{X_2}(\coth\theta X_2)$$

$$= -\sinh\theta\cosh\theta \underbrace{\nabla_{X_1} X_1}_{=0} - X_1(\sinh\theta\cosh\theta)X_1$$

$$- \coth\theta \nabla_{X_2} X_2 - \underbrace{\frac{\partial}{\partial\varphi}(\coth\theta)}_{=0} X_2 =$$

$$= -(\sinh^2\theta + \cosh^2\theta)X_1 + \cosh^2\theta X_1 =$$

$$= -\sinh^2\theta X_1$$

$$K(X_1, X_2) = \underset{p \neq (0,0,1)}{K(T_p \mathcal{H})} = \frac{R(X_1, X_2, X_2, X_1)}{g(X_1, X_1)g(X_2, X_2) - g(X_1, X_2)^2}$$

$$= \frac{(-\sinh^2\theta X_1, X_1)}{\sinh^2\theta - 0} = -1 \quad \forall p$$

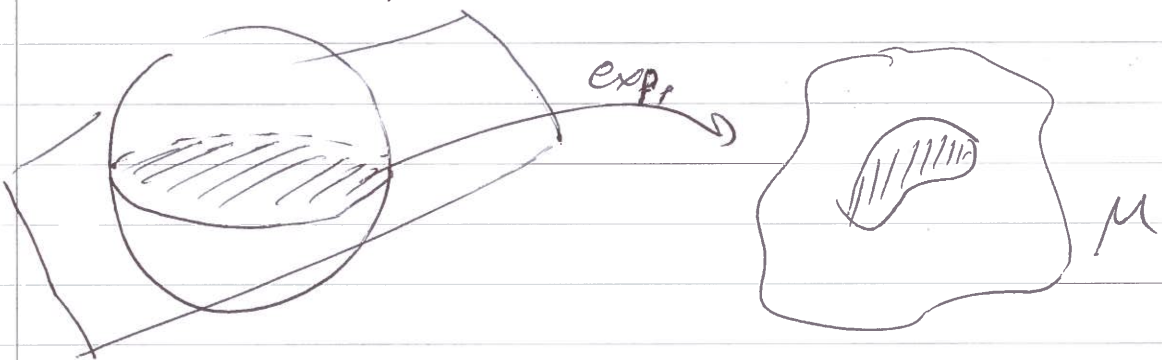
Proposition 10.6: Let M be an oriented surface in $\mathbb{R}^3$. Then $K(T_pM)$ is $K(p)$ the Gauss Curvature at $p \in M$

Example $T^2 \subseteq \mathbb{R}^3$

$$T^2 = \{((2+\cos\theta)\cos\varphi, (2+\cos\theta)\sin\varphi, \sin\theta), \theta, \varphi \in \mathbb{R}\}$$

Then $K(p) = \dfrac{\cos\theta}{2+\cos\theta} = K(T_p T^2)$

If σ is a 2-plane in T_pM then $K(\sigma)$ is the Gauss curvature of the surface $\exp_p(\sigma \cap B_\varepsilon(0))$ at p.



not in exam **Definition 10.7** The Ricci curvature $Ric \in \Gamma(S^2T^*M)$ is given by $Ric(X,Y) = \sum_{i=1}^{n} R(X, E_i, E_i, Y)$

where $X, Y \in T_pM$ and $E_1, .., E_n$ is an orthonormal basis of T_pM.

$$Ric(Y,X) = \sum_{i=1}^{n} R(Y, E_i, E_i, X) = \sum_{i=1}^{n} R(E_i, X, Y, E_i) =$$

$$= \sum_i R(X, E_i, E_i, Y) = Ric(X,Y)$$

Definition 10.8: Scalar curvature S is $S = \sum_{j,i=1}^{n} R(E_i, E_j, E_j, E_i)$ $\{E_1, \dots, E_n\}$ orthonormal basis.