M114 Riemannian Geometry Notes

Based on the 2013 spring lectures by Dr J Lotay

Riemannian Geometry

4 problem sheets due weeks 3, 5, 9, 11 in Monday Lectures

Course outline:

Part A: Manifolds, general theory of smooth geometric objects

Part B: Riemannian manifolds, curved objects with striking applications

Aim: What is a manifold?

$\mathbb{R}^n$ n-dimensional metric Euclidean space

Coordinates $(x_1, \dots, x_n)$

Standard orthonormal basis $e_1, \dots, e_n$

$B_r(x) = \{ y \in \mathbb{R}^n : d(x,y) < r \}$.

$d(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$

$U \subseteq \mathbb{R}^n$ open if $\forall x \in U \ \exists r > 0$ st $B_r(x) \subseteq U$.

$\Rightarrow$ every open set is the union of open balls

$\Rightarrow$ open balls form a basis for open sets.

An n-dimensional manifold "looks like" $\mathbb{R}^n$ near each point.

"Abstract" object which does not depend on the ambient space.

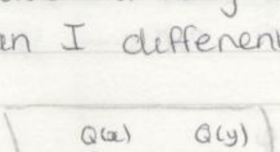
$\subseteq \mathbb{R}^3$ $\quad \{ (\cos\theta, \sin\theta, \cos\psi, \sin\psi) : \theta, \psi \in \mathbb{R} \}$

$\subseteq \mathbb{R}^4$

We want to consider smooth objects

How does Q vary from x to y?

ie. Can I differentiate Q?



fold

Definition:

An n-dimensional manifold M is a metric space such that $\exists A = \{(u_i, \varphi_i) : i \in I\}$ with

u_i is open in M $\forall i \in I$ and $\bigcup_{i \in I} u_i = M$

$\varphi_i : u_i \longrightarrow \mathbb{R}^n$ is a continuous bijection

$\varphi_i(u_i)$ open in $\mathbb{R}^n$ with continuous bijection

ie $\varphi_i : u_i \longrightarrow \varphi_i(u_i)$ is a homeomorphism

If $u_i \cap u_j \neq \emptyset$ then $\varphi_j \circ \varphi_i^{-1} : \underset{\mathbb{R}^n}{\underset{\cap}{\varphi_i(u_i \cap u_j)}} \longrightarrow \varphi_j(u_i \cap u_j)$

is smooth (infinitely differentiable) bijection with smooth inverse (diffeomorphism)

A is an atlas and (u_i, φ_j) are called charts (or coordinate charts)
$\varphi_j \circ \varphi_i^{-1}$ are called transition maps

$\mathbb{R}^n$ is an n-dimensional manifold - take $M = \mathbb{R}^n$, $\varphi = id$.

Any open set $U \subseteq \mathbb{R}^n$ $M \subseteq \mathbb{R}^n$ is an n dimensional n dimensional manifold as well

If M is an n-dimensional manifold any open $U \subseteq M$ is also an n-dimensional manifold as well, take atlas.
$\{(v_i = u_i \cap U, \varphi_i) : i \in I\}$.

Example 1.3: $S^n = \{x \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i^2 = 1\}$

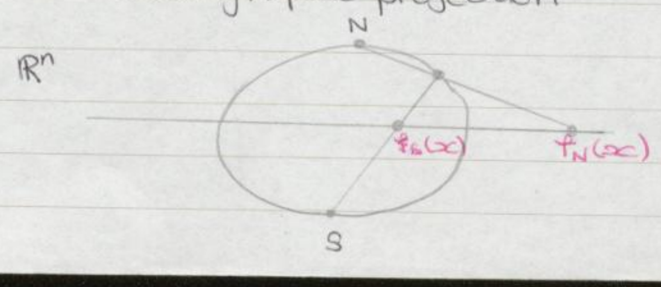
S^n is a seperable metric space because $\mathbb{R}^{n+1}$ is.

Let $N = (0, 0, 0 \ldots, 1)$, $S = (0, \ldots 0, -1)$

ie North and south poles

Let $U_N = S^n \setminus \{N\}$ $U_S = S^n \setminus \{S\}$ these are open and
$U_N \cup U_S = S^n$

Now use "stereographic projection"



$\varphi_N : U_N \longrightarrow \mathbb{R}^n$

$\varphi_N(x_1, \dots, x_{n+1}) = \dfrac{(x_1, \dots, x_n)}{1 - x_{n+1}}$ (continuous)

$\varphi_S(x_1, \dots, x_{n+1}) = \dfrac{(x_1, \dots, x_n)}{1 + x_{n+1}}$ (continuous)

$\varphi_N^{-1} : \mathbb{R}^n \longrightarrow U_N$

$\varphi_N^{-1}(y_1, \dots, y_n) = \dfrac{(2y_1, \dots, 2y_n, |y|^2 - 1)}{1 + |y|^2}$ (continuous)

$\varphi_S^{-1}(y_1, \dots, y_n) = \dfrac{(2y_1, \dots, 2y_n, 1 - |y|^2)}{1 + |y|^2}$ (continuous)

So we have the second property.

$U_N \cap U_S = S^n \setminus \{N, S\}$

$\varphi_N(U_N \cap U_S) = \mathbb{R}^n \setminus \{0\}$
$= \varphi_S(U_N \cap U_S)$

$\varphi_S \circ \varphi_N^{-1}(y) = \dfrac{y}{|y|^2}$ diffeomorphism because we have removed $y = 0$

$\Rightarrow S^n$ is an n-dim manifold.

Example 1.4

Let $\mathbb{RP}^n$ be the set of straight lines through 0 in $\mathbb{R}^{n+1}$

Equivalently $\mathbb{R}^{n+1} \setminus \{0\} / \sim$ where $x \sim y$ iff $y = \lambda x$ for some $\lambda \in \mathbb{R} \setminus \{0\}$.

We denote points in $\mathbb{RP}^n$ as $[x]$ - the equivalence class of x

We can represent $[x]$ by $x \in S^n$ unique upto $\pm x$

So we can define

$d([x], [y]) = \min\{d(x, y), d(-x, y)\}$.

$\Rightarrow \mathbb{RP}^n$ is a seperable metric space.

For $i = 1, \dots, n+1$ let $U_i = \{[x_1, \dots, x_{n+1}] \in \mathbb{RP}^n \mid x_i \neq 0\}$

open because $[x] \in U_i$ then $B_{\frac{|x_i|}{2}}([x]) \subseteq U_i$

and $\bigcup_{i=1}^{n+1} U_i = \mathbb{RP}^n$

because $\forall [x] \in \mathbb{RP}^n$ since $x_i \neq 0$

$\varphi_i : U_i \longrightarrow \mathbb{R}^n$ $\varphi_1([x_1, \dots, x_{n+1}]) = \dfrac{(x_2, \dots, x_{n+1})}{x_1}$

$\vdots$

$\varphi_{n+1}([x_1, \dots, x_{n+1}]) = \dfrac{(x_1, \dots, x_n)}{x_{n+1}}$

2 maps here are continuous and well defined.

diffeom

if $y \in [x] \Rightarrow y = \lambda x$

so $\varphi_1(y) = \dfrac{(y_2, \dots, y_{n+1})}{y_1} = \dfrac{\lambda(x_2, \dots, x_{n+1})}{\lambda x_1} = \varphi_1(x)$

Again equivalence classes

$$\left.\begin{aligned} \varphi_1^{-1}(y_1, \dots, y_n) &= [1, y_1, \dots, y_n] \\ &\vdots \\ \varphi_{n+1}^{-1}(y_1, \dots, y_n) &= [y_1, \dots, y_n, 1] \end{aligned}\right\} \text{all continuous.}$$

wlog $i > j$

$U_i \cap U_j = \{ [x] \in \mathbb{R}P^n : x_i \neq 0 \neq x_j \}$.

$\varphi_i(U_i \cap U_j) = \{ (y_1, \dots, y_n) \in \mathbb{R}^n : y_j \neq 0 \}$

$\varphi_j \circ \varphi_i^{-1} = \varphi_j([y_1, \dots, y_j, \dots, \underset{i^{\text{th}}\text{ pos}}{1}, \dots, y_n])$

$= \dfrac{(y_1, \dots, 1, \dots, y_n)}{y_j}$ as $y_j \neq 0$ this is smooth with a smooth inverse

So $\mathbb{R}P^n$ is an n-dim manifold.

Remark: An atlas is ~~an~~ manifold structure on the metric space. An ~~atla~~ atlas A and an ~~at~~ atlas A' are equivalent if $A \cup A'$ is an ~~atl~~ atlas. Equivalent atlases define the same manifold.

Theorem 1.5

Let $F : \mathbb{R}^{n+m} \to \mathbb{R}^m$ be a smooth map such that $\forall p \in F^{-1}(0)$ $dF_p : \mathbb{R}^{m+n} \to \mathbb{R}^m$ is surjective
(ie 0 is a regular value of F)
Then $F^{-1}(0)$ is an n-dimensional manifold

Proof: $F^{-1}(0)$ is a seperable metric space because $\mathbb{R}^{n+m}$ is.
Implicit function theorem $\Rightarrow \forall p \in F^{-1}(0)$ $\exists$ splitting of $\mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m$ such that if $p = (a, b)$ $a \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ then $\exists$ open $V_p \subseteq \mathbb{R}^n$ st $a \in V_p$, $W_p \subseteq \mathbb{R}^m$ st $b \in W_p$ and a smooth map $G_p : V_p \to W_p$ with $G_p(a) = b$
and $F^{-1}(0) \cap (V_p \times W_p) = \{ (q, G_p(q)) : q \in V_p \}$
Let $U_p = F^{-1}(0) \cap (V_p \times W_p)$
note U_p is open on $p \in U_p$
$\Rightarrow \bigcup_{p \in F^{-1}(0)} U_p = F^{-1}(0)$

$\varphi_p : U_p \longrightarrow \mathbb{R}^n \qquad \varphi_p(\underset{\mathbb{R}^n}{\underset{\cap}{q}}, \underset{\mathbb{R}^m}{\underset{\cap}{G_p(q)}}) = q$ (projection)

$\varphi_p(U_p) = V_p$ - open

$\varphi_p^{-1} : V_p \longrightarrow U_p$

$\varphi_p^{-1}(q) = (q, G_p(q))$ continuous as $G_p(q)$ is continuous.

Thus φ_p is a homeomorphism.

If $U_p \cap U_q \neq \emptyset$ then if the splitting map of $\mathbb{R}^{m+n} = \mathbb{R}^n \times \mathbb{R}^m$ are the same for p and q then

$\varphi_q \circ \varphi_p^{-1} : V_p \cap V_q \longrightarrow V_p \cap V_q$ is the identity and is a diffeomorphism

If the splittings are different $\varphi_p \circ \varphi_q^{-1}$ is $\pi \circ \varphi_p^{-1}$ where π is a projection map to an $\mathbb{R}^n$ factor in $\mathbb{R}^{n+m}$ (given by implicit function theorem) which amounts to a change in coordinate in $\mathbb{R}^n$ defined by G_p which is smooth

$\Rightarrow \varphi_q \circ \varphi_p^{-1}$ is a diffeomorphism

so $F^{-1}(0)$ is an n-dim manifold.

Remark: This works just as well for F defined just on an open set and for any other regular value of F.

Example 1.6

Let $F : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ be defined by

$F(x) = \sum_{i=1}^{n+1} x_i^2 - 1$ then $F^{-1}(0) = S^n$ and

$dF_x = (2x_1, \dots, 2x_n) \neq 0 \quad \forall x \in F^{-1}(0)$

$\Rightarrow dF_x : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ is surjective

and thus by theorem 1.5 $\Rightarrow S^n$ is an n-dim manifold.

Example 1.7

Let $F : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^n$ be defined by

$F(x_1, \dots, x_{2n}) = (x_1^2 + x_2^2 - 1, \dots, x_{2n-1}^2 + x_{2n}^2 - 1)$

$$dF_x = \begin{pmatrix} 2x_1 & 2x_2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2x_3 & 2x_4 & \cdots & & \\ \vdots & \vdots & 0 & 0 & & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & 2x_{2n-1} & 2x_{2n} \end{pmatrix}$$

diffeon

rank of matrix is n for $x \in F^{-1}(0)$ because $(x_{2i-1}, x_{2i}) \neq (0,0) \ \forall i \Rightarrow 0$ is a regular value.

theorem 1.5 $\Rightarrow f^{-1}(0)$ is a regular n-dim manifold which is in fact T^n the standar n-tgons in $\mathbb{R}^{2n}$

Example 1.8

$F : \mathbb{R}^2 \to \mathbb{R}$ $\quad F(x,y) = x^3 - y^3$

then $dF_{(x,y)} = (3x^2 - 3y^2) = (0,0)$ at any $(x,y) = (0,0) \in F^{-1}(0)$

$\Rightarrow 0$ is not a regular point of F.

but $F^{-1}(0) = \{(x,y) : x^3 = y^3\} = \{(x,x) \in \mathbb{R} : x \in \mathbb{R}\}$.

which is a smooth 1-dim manifold.

Example 1.9

$F : \mathbb{R}^3 \to \mathbb{R}$ by $F(x,y,z) = x^2 + y^2 - z^2$

then $dF_{(x,y,z)} = (2x, 2y, -2z) \neq (0,0,0)$

except when $(x,y,z) = (0,0,0)$

$\Rightarrow c \in \mathbb{R}$ is a regular point of F iff $c \neq 0$

$F^{-1}(c)$ is a 2-dim manifold (hyperboloid) for $c \neq 0$.

$F^{-1}(0)$ is a constant and we will show that this is not a manifold

Example 1.10

let $f : \mathbb{R}^2 \to \mathbb{R}$ be $f(x,y) = x^2 + y^2$

Define $F : \mathbb{R}^3 \to \mathbb{R}$ by $F(x,y,z) = f(x,y) - z$

$dF_{(x,y,z)} = (2x, 2y, -1) \neq 0$

$\Rightarrow 0$ is a regular ~~point~~ value $(-1 \neq 0)$ of F.

$\Rightarrow F^{-1}(0) = \{(x,y,z) : z = x^2 + y^2\}$ is a 2-dim manifold.

Example 1.11

In general given $\mathbb{R}^n \to \mathbb{R}^m$ we can define $F : \mathbb{R}^{n+m} \to \mathbb{R}^m$ by $F(x,y) = f(x) - y$ and $dF_{(x,y)} = (df_x - I)$ is surjective $\forall (x,y) \in \mathbb{R}^{n+m}$

$\Rightarrow F^{-1}(0) =$ Graph of $f = \{(x, f(x)) : x \in \mathbb{R}^n\}$.

is an n-dim manifold.

Example 1.12

Let $M_n(\mathbb{R})$ be $n \times n$ real matrices

$Sym_n(\mathbb{R})$ $n \times n$ symmetric matrices

$M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ $\quad Sym_n(\mathbb{R}) \cong \mathbb{R}^{\frac{1}{2}n(n+1)}$

$F: M_n(\mathbb{R}) \to Sym_n(\mathbb{R})$ by $F(A) = A^T A - I$

$dF_A(B) = A^T B + B^T A$

so if $A \in F^{-1}(0)$ ($\Rightarrow AA^T = I$)

and $C \in Sym_n(\mathbb{R})$, then $dF_A(\frac{1}{2}AC) = C$

$\Rightarrow dF_A$ is surjective $\forall A \in F^{-1}(0)$

$\Rightarrow F^{-1}(0) = O(n)$ is a $\frac{1}{2}n(n-1)$ dim-manifold

orthogonal matrices

$O(n)$ splits into 2 manifolds (one with $\det = +1$ and one with $\det = -1$)

$SO(n) = \{A \in O(n) : \det(A) = 1\}$

special orthogonal matrices is a $\frac{1}{2}n(n-1)$ dim manifold

(could look at $F: GL_n^+(\mathbb{R}) \to Sym(\mathbb{R})$ det >0 rather than look at all $n \times n$ matrices)

Proposition 1.13

A surface in $\mathbb{R}^3$ is a 2-dim manifold

Proof: $M \subseteq \mathbb{R}^3$ is a surface if $\forall p \in M$ $\exists$ open $W \ni p$ ($\subseteq \mathbb{R}^3$)

and open $V_p \subseteq \mathbb{R}^2$ and smooth map $\underline{x}_p : V_p \to W_p \cap M$ such that

i. $\underline{x}_p$ is a homeomorphism

ii. $d(\underline{x}_p)_q : \mathbb{R}^2 \to \mathbb{R}^3$ injective $\forall q \in V_p$

Let $U_p = W_p \cap M$ open and $\bigcup_{p \in M} U_p = M$

Let $\varphi_p : U_p \to \mathbb{R}^2$ be $\varphi_p = \underline{x}_p^{-1}$ so $\varphi_p(U_p) = V_p$ open

$\Rightarrow \varphi_p$ homeomorphism.

$U_p \cap U_{p'} \neq \emptyset \Rightarrow \varphi_{p'} \circ \varphi_p^{-1} = \underline{x}_{p'}^{-1} \circ \underline{x}_p$

Curves and surface courses $\Rightarrow \underline{x}_{p'}^{-1} \circ \underline{x}_p$ is a diffeomorphism (change of coordinates)

Proof uses ii + Inverse function Thm $\square$

diffeom

Example 1.14:

The torus in $\mathbb{R}^3$, $\{((2+\cos\theta)\cos\phi, (2+\cos\theta)\sin\phi, \sin\theta) : \theta, \phi \in \mathbb{R}\}$ is a 2-dim manifold.

Proposition 1.15

An n-(sub)manifold M in $\mathbb{R}^{n+m}$ is an n-dim manifold.

Proof: Multivariable analysis course $\Rightarrow \forall p \in M \; \exists$ open $W_p \subseteq \mathbb{R}^{n+m}$ ($p \in W_p$)
open ~~in~~ $V_p \subseteq \mathbb{R}^n$ and smooth map $x_p : V_p \rightarrow W_p \cap M$ such that
i x_p homeomorphism
ii $d(x_p)_q : \mathbb{R}^n \rightarrow \mathbb{R}^{n+m}$ injective

So same proof as proposition 1.13

Definition 1.16

A map $f: M \rightarrow N$ between manifolds is smooth at $p \in M$ if $\exists$ constant charts (U, φ) on M around p and (V, ψ) on N around $f(p)$ with $f(U) \subseteq V$ such that

$\psi \circ f \circ \varphi^{-1} : \varphi(U) \rightarrow \psi(V)$ is smooth.

Suppose (U', φ') is another chart around p and (V', ψ') another chart around $f(p)$ with $f(U') \subseteq V$ then

$$\psi' \circ f \circ (\varphi')^{-1} = \underset{\text{smooth by def of manifold}}{(\psi' \circ \psi^{-1})} \circ \psi \circ f \circ \varphi^{-1} \circ \underset{\text{smooth}}{(\varphi \circ (\varphi')^{-1})}$$

$\Rightarrow$ definition of smooth makes sense for f.
(independent of choice of chart).

f is smooth if it is smooth at all $p \in M$.

diffeomorphism

Example 1.17

(U, φ) chart on M.

$\varphi : U \rightarrow \mathbb{R}^n$ is smooth because in definition 1.16 we let $(U, \varphi) = (U, \varphi)$ and $(V, \psi) = (\mathbb{R}^n, id) \Rightarrow id \circ \varphi \circ \varphi^{-1} = id$ is smooth on $\varphi(U) \subseteq \mathbb{R}^n$

Similarly $\varphi^{-1} : \underset{\mathbb{R}^n}{\varphi(U)} \rightarrow M$ is smooth

Example 1.18 Any "constant" map $f: M \rightarrow N$ (ie q fixed $f(p) = q \; \forall p \in M$) is smooth because the corresponding map between Euclidean spaces is a constant map

Definition: 1.19

$f: M \rightarrow N$ is diffeomorphism if it is a smooth bijection with smooth inverse. (We then say M and N are diffeomorphic)

$f: M \to N$ is a local diffeomorphism at $p \in M$ if $\exists$ open $U \ni p$, open $V \ni f(p)$ such that $f: U \to V$ is a diffeomorphism.

Example 1.20

(U, φ) chart on M then $\varphi: U \to \mathbb{R}^n$ is a local diffeomorphism

Theorem 1.21

Let G be a discrete (ie countable) group and M an n-dimensional manifold st $\forall g \in G$, $\exists$ a diffeomorphism $\phi_g: M \to M$ with

i If $e \in G$ is the identity then $\phi_e = id$
ii $\phi_{gh} = \phi_g \circ \phi_h \quad \forall g, h \in G$ $\Big\}$ G acting on M
iii $\forall p \in M \; \exists$ open $V \ni p$ st $V \cap \phi_g(V) = \emptyset \quad \forall g \neq e$.
iv $\forall p, q \in M$ with $p \neq q$ $\exists$ open $V \ni p$, open $W \ni q$ such that $V \cap \phi_g(W) = \emptyset \quad \forall g \in G$.

G acts freely (ie with no fixed points) and properly discontinuously on M.
Then $M/G = M/\sim$ (where $p \sim q$ iff $\exists g \in G$ st $q = \phi_g(p)$).
is an n-dim manifold

Example 1.22

$G = \mathbb{Z}_2 = \{1, -1\}$ acting on $\mathbb{R}^n$ with $\phi_{-1} = -id$
This action is not free on $\mathbb{R}^n$ because 0 is fixed

a) G acts freely and properly discontinuously on S^n ($0 \notin S^n$)
(take V, W to be subsets of a hemisphere) $\Rightarrow S^n/\mathbb{Z}_2$ is a n-dim manifold which is $\mathbb{RP}^n$

b) G acts .... on cylinder ~~on~~ $C = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1 \; |z| = 1\}$.
$\Rightarrow C/\mathbb{Z}_2$ is a 2-dim manifold = mobius band

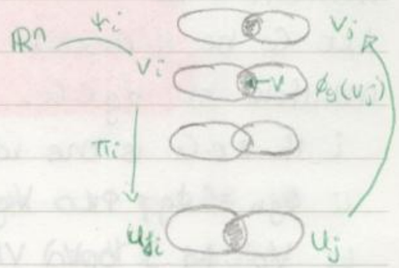
c) G acts .... on T^2 in $\mathbb{R}^3 \Rightarrow T^2/\mathbb{Z}_2$ is 2-dim manifold = Klien bottle.

proof: M/G is a second countable metric ~~subspace~~ space with "quotient" metric $d([p], [q]) = \min_{g \in G}(d(p, \phi_g(q)))$ (by G discrete and iv).

Let $\{(V_i, \psi_i)\; i \in I\}$ be an atlas for M st
$V_i \cap \phi_g(V_i) = \emptyset \quad \forall g \neq e \quad \forall i \in I$ (by iii).
Let $U_i = \pi(V_i)$ where $\pi: M \to M/G$ is projection
$\pi^{-1}(U_i) = \bigcup_{g \in G} \phi_g(V_i)$ disjoint union of open sets and is open

$\Rightarrow U_i$ is open in M/G by definition and $\bigcup_{i\in I} U_i = M/G$
because $\bigcup_{i\in I} V_i = M$

Let $\pi_i : \pi|_{V_i} : V_i \longrightarrow U_i$ which is a homeomorphism (it is injective
because $V_i \cap \phi_g(V_i) = \emptyset \ \forall g \neq e$)
Let $\varphi_i = \psi_i \circ \pi_i^{-1} : U_i \longrightarrow \mathbb{R}^n$ homeomorphism onto its image $\psi_i(V_i)$



Suppose $U_i \cap U_j \neq \emptyset$

$$\varphi_i(U_i \cap U_j) = \psi_i \circ \pi_i^{-1}(U_i \cap U_j) = \psi_i(V_i \cap \pi^{-1}(U_j)) = \psi_i(V_i \cap \underbrace{\bigcup_{g\in G}(V_j)}_{\text{disjoint union}})$$

$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \longrightarrow \varphi_j(U_i \cap U_j)$ is a homeomorphism
Want to show $\varphi_j \circ \varphi_i^{-1}$ (and its inverse) is smooth
Let $p \in \varphi_i(U_i \cap U_j) \Rightarrow \exists!\ g \in G$ st $p \in \psi_i(V_i \cap \phi_g(V_j)) = W$

$$\varphi_j \circ \varphi_i^{-1}|_W = \underbrace{\psi_j}_{\text{smooth}} \circ \pi_j^{-1} \circ \pi_i \circ \underbrace{\psi_i^{-1}}_{\text{smooth}}|_W$$

is smooth if $\pi_j^{-1} \circ \pi_i$ is smooth on $V = \psi_i^{-1}(W) = V_i \cap \phi_g(V_j)$

Let $q \in V$ then $q' = \pi_j^{-1} \circ \pi_i(q) \in V_j$ and $\pi_j(q') = \pi_i(q)$
So $\exists!\ g_q \in G$ st $\phi_{g_q}(q') = q$
$\Rightarrow q \in \phi_{g_q}(V_j) \cap \phi_g(V_j) \Rightarrow g_q = g$.
$\Rightarrow \pi_j^{-1} \circ \pi_i = \phi_{g^{-1}}$ (ie $\phi_g(q') = q$) over V
So it is smooth
$\Rightarrow \varphi_j \circ \varphi_i^{-1}$ smooth (smooth on inverse also) ☐

Proposition 1.23
If a discrete group ~~acts~~ G acts freely and properly discontinuously on M then $\pi : M \longrightarrow M/G$ is a local diffeomorphism

Proof: Let $p \in M$. Use notation from thm 1.21
$\pi|_{V_i} = \pi_i : V_i \longrightarrow U_i$ is a diffeomorphism because $\varphi_j \circ \pi_i \circ \psi_i^{-1} = \psi_j \circ \pi_j^{-1} \circ \pi_i \circ \psi_i^{-1}$
is smooth
$\Rightarrow \pi_i$ is smooth. Choose $i \in I$ st $p \in V_i$

Example : 1.24

Ex 1.22 and prop 1.23

$\Rightarrow$ we have local diffeomorphisms between S^n & $\mathbb{RP}^n$, cylinder & mobius band and the torus on $\mathbb{R}^3$ & the Klien bottle.

These are not diffeomorphisms (not injective)

But $S^1 \cong \mathbb{RP}^1$

...ent vectors and the tangent bundle.

Definition: 2.1

A curve in M (through p) is a smooth map $\alpha : (-\varepsilon, \varepsilon) \longrightarrow M$ (with $\alpha(0) = p$) such that $\forall t \in (-\varepsilon, \varepsilon)$ $\exists$ a chart (U, φ) and $\delta > 0$ such that $\alpha((t-\delta, t+\delta)) \subseteq U$ and

$\varphi \circ \alpha : (t-\delta, t+\delta) \longrightarrow \mathbb{R}^n$ is a curve (in the usual sense).

Definition 2.2

Let α be a curve on M through p, let $U \ni p$ be open and $f: U \longrightarrow \mathbb{R}$ be smooth at p.
Then $f \circ \alpha : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}$ is smooth at 0 (if we choose $\varepsilon > 0$ small enough that $\alpha(-\varepsilon, \varepsilon) \subseteq U$).
Call $\alpha'(0) : f \longmapsto (f \circ \alpha)'(0)$ the tangent vector to α at 0.

Suppose (U, φ) is a chart around $p \Rightarrow \varphi \circ \alpha(t) = (a_1(t), \ldots, a_n(t)) \subseteq \mathbb{R}^n$

$$(f \circ \alpha)'(0) = \frac{d}{dt}(f \circ \varphi^{-1} \circ \varphi \circ \alpha)(t)\Big|_{t=0}$$

$$= \frac{d}{dt}(f \circ \varphi^{-1})(a_1(t), \ldots, a_n(t))\Big|_{t=0}$$

$$= \sum_{i=1}^{n} a_i'(0) \frac{\partial (f \circ \varphi^{-1})}{\partial x_i}\Big|_{\varphi(p) = (a_1(0), \ldots, a_n(0))}$$

$$(f \circ \alpha)'(0) = \left(\sum_{i=1}^{n} a_i'(0) \frac{\partial}{\partial x_i}\Big|_{\varphi(p)} \right)(f \circ \varphi^{-1}).$$

$\Rightarrow \alpha'(0)$ is the differential operator $\sum_{i=1}^{n} a_i'(0) \partial_i |_{\varphi(p)}$ $\frac{\partial}{\partial x_i}$ acting on $f \circ \varphi^{-1}$ (which is how we identify f with a function on $\mathbb{R}^n$)
Moreover if we use $\partial_i |_{\varphi(p)}$ as a basis (tangent vector to curve $\varphi^{-1}(0, \ldots, 0, t, 0, \ldots 0)$ at 0 assuming $\varphi(p) = 0$) then $\alpha'(0)$ is identified with the vector $(a_1'(0), \ldots, a_n'(0)) \in \mathbb{R}^n$

Definition 2.3

X is a tangent vector to M at p if $\exists$ curve α on M through p such that $X = \alpha'(0)$

Definition 2.4

Let α, β be curves on M through p.
We say $\alpha \sim \beta$ if and only if $\exists$ chart (U, φ) around p such that $(\varphi \circ \alpha)'(0) = (\varphi \circ \beta)'(0)$

Suppose (V, ψ) is another chart around p. Then
$(\psi \circ \alpha)'(0) = (\psi \circ \varphi^{-1} \circ \varphi \circ \alpha)'(0) = d(\psi \circ \varphi^{-1})_{\varphi(p)}(\varphi \circ \alpha)'(0)$ * $'$ for $\mathbb{R} \to \mathbb{R}^n$, d for $\mathbb{R}^n \to \mathbb{R}^n$
$\Rightarrow$ definition is independent of the coordinate chart as $d(\psi \circ \varphi^{-1})_{\varphi(p)}$ is an isomorphism.

We say that $[\alpha]$ is a tangent vector to M at p.

Given $[\alpha]$ define $X = \alpha'(0)$
Suppose $\beta \in [\alpha]$
Then $(f \circ \beta)'(0) = (f \circ \varphi^{-1} \circ \varphi \circ \beta)'(0)$
$= d(f \circ \varphi^{-1})_{\varphi(p)}(\varphi \circ \beta)'(0)$
$= d(f \circ \varphi^{-1})_{\varphi(p)}(\varphi \circ \alpha)'(0) = (f \circ \alpha)'(0)$
So X is well defined.

Given tangent vector X at p $\exists$ ~~some~~ curve α in M through p such that $\alpha'(0) = X$. So we map X to $[\alpha]$ (this is well defined by essentially the same argument as above)

Definition: 2.5
Let (U, φ), (V, ψ) be charts around $p \in M$ and let $u, v \in \mathbb{R}^n$
We say $(U, \varphi, u) \sim (V, \psi, v)$ if and only if $d(\psi \circ \varphi^{-1})_{\varphi(p)}(u) = v$.

We call $[(U, \varphi, u)]$ a tangent vector to M at p.

Given $[\alpha]$ and chart (U, φ) take $u = (\varphi \circ \alpha)'(0) \in \mathbb{R}^n$
* $\Rightarrow$ we get a map $[\alpha] \mapsto [(U, \varphi, u)]$
Given $[(U, \varphi, u)]$ take $[\alpha]$ for any curve through p such that $u = (\varphi \circ \alpha)'(0)$

Definition 2.6
Let T_pM denote the set of tangent vectors to M at p.

Proposition 2.7
T_pM is an n-dim vector space.

Proof: If $X = [\alpha] = [(U, \varphi, u)] \in T_pM$
Define λX, $\lambda \in \mathbb{R}$ by $\lambda X = [(U, \varphi, \lambda u)]$

If $X' = [(U, \varphi, u')] \in T_pM$

then define $X + X' = [(U, \varphi, u+u')]$

So T_pM is a vector space

$[(U, \varphi, e_i)]$ are linearly independent on T_pM

$\Rightarrow T_pM$ is an n-dimensional vector space $\square$.

Proposition 2.8

Let $F: \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^m$ be a smooth map with 0 a regular value of F.

Then $T_pF^{-1}(0) = \ker dF|_p$.

proof: Let ~~When~~ $X = [\alpha] \in T_pF^{-1}(0) \Rightarrow F(\alpha(t)) = 0 \ \forall t$

$0 = \frac{d}{dt} F(\alpha(t))\Big|_{t=0} = dF_p(\alpha'(0))$

vector in $\mathbb{R}^{n+m}$

$\Rightarrow \alpha'(0) \in \ker dF_p$

$\dim \ker dF_p = n$ because 0 is a regular value.

The map $T_pF^{-1}(0) \longrightarrow \ker dF_p$ is injective because $\alpha'(0) = 0 \Leftrightarrow X = 0$

$\Rightarrow T_pF^{-1}(0) = \ker dF_p$. $\square$

Example 2.9

$S^n = F^{-1}(0)$ where $F(x_1, \dots, x_{n+1}) = \sum_{i=1}^{n+1} x_i^2 - 1$

$dF_{(x_1, \dots, x_{n+1})} = 2(x_1, \dots, x_{n+1})$

$\Rightarrow T_pS^n = \{q \in \mathbb{R}^{n+1} \ \langle p, q \rangle = 0\}$

$= \langle p \rangle^{\perp}$

Example 2.10

Let $f: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a smooth map, then $\mathrm{Graph}(f) = F^{-1}(0)$

where $F(x, y) = f(x) - y$

$dF_{(x,y)} = (df_x \ \ -I) \Rightarrow T_{(x, f(x))}\mathrm{Graph}(f) = \{(u, v) \in \mathbb{R}^{n+m} : v = df_x(u)\}$

$\mathbb{R}^n + \mathbb{R}^m$

$= \mathrm{Graph}(df_x)$

Example 2.11

$SL_n(\mathbb{R}) = F^{-1}(0)$ where $F(A) = \det A - 1$

$\forall A \in SL_n(\mathbb{R}) \quad dF_A(B) = \mathrm{tr}(A^{-1}B) \neq 0^{*}$ for $B = A$ for example

$\Rightarrow 0$ is a regular value $\Rightarrow SL_n(\mathbb{R})$ is an (n^2-1)-dim manifold

$T_A SL_n(\mathbb{R}) = \{B \in M_n(\mathbb{R}) : \mathrm{t}(A^{-1}B) = 0\}$

*

$* \det(A+B) - \det(A) = \det A(\det(I + A^{-1}B) - 1)$

$\det\begin{pmatrix} 1+c_{11} & & \\ & 1+c_{22} & \\ & & \ddots \\ & & & 1+c_{nn}\end{pmatrix} = 1 + (c_{11} + \dots + c_{nn}) + O(c^2)$

$= 1 + tr(A^{-1}B) + O(\|B\|^2)$

$C = A^{-1}B$. $\Rightarrow d\det_A(B) = tr(A^{-1}B)$

Example 2.12

Let $C = \{(x,y,z) \in \mathbb{R} : x^2+y^2 = z^2\}$.

$t \mapsto (t, 0, t)$ and $t \mapsto (0, t, t)$

are curves on C through O, with tangent vectors at O $(1,0,1)$ and $(0,1,1)$

$\Rightarrow (1,-1,0) \in \operatorname{span}\{(1,0,1),(0,1,1)\}$ but not tangent to a curve in C through O (because curve is $t \mapsto (t,-t,0)$)

$\Rightarrow$ tangent vectors to C at O do not form vector space

$\Rightarrow C$ is not a manifold.

Definition 2.13

Let f be a smooth map $f: M \to N$.

Let $X = [\alpha] = [(U, \varphi, u)] \in T_pM$

$f \circ \alpha : (-\varepsilon, \varepsilon) \to N$ is a curve on N through $f(p)$

Define the differential of f at p by

$df_p : T_pM \to T_{f(p)}N$, $\quad df_p([\alpha]) = [f \circ \alpha]$

$\alpha'(0) = X \quad = (f\circ\alpha)'(0)$

Let $[(V, \psi, v)] = [f \circ \alpha] \Rightarrow v = d(\psi \circ f \circ \varphi^{-1})_{\varphi(p)}(u)$

Proposition 2.14 (kinda inverse function thm for manifolds)

A smooth map $f: M \to N$ is a local diffeomorphism at p $\Leftrightarrow$ df_p is an isomorphism

Proof: $\Rightarrow$ $\exists$ open $U \ni p$, $V \ni f(p)$ st $f: U \to V$ is a diffeomorphism

Consider $id: M \to M$ then $d\,id_p = id$ on T_pM.

Consider $f_1: M_1 \to M_2$, $f_2: M_2 \to M_3$

$d(f_2 \circ f_1)_p = df_{2_{f_1(p)}} \circ d(f_1)_p$ by chain rule

$f: U \to V$ diffeomorphism $\Rightarrow f^{-1} \circ f = id : U \to U$

$f \circ f^{-1} = id : V \to V$

$$\left.\begin{aligned} \Rightarrow d(f^{-1}\circ f)_p &= df^{-1}_{f(p)} \circ df_p = id \\ d(f\circ f^{-1})_{f(p)} &= df_p \circ df^{-1}_{f(p)} = id \end{aligned}\right\} \Rightarrow df_p \text{ is an isomorphism}$$

$\Leftarrow$ df_p is an isomorphism $\Rightarrow \dim M = \dim N = n$
First part of this proof gives that, if (U,φ) is a chart around p and (V,ψ) is a chart around $f(p)$
$\Rightarrow d\varphi_p : T_pM \to \cancel{T_{\varphi p}}\,\mathbb{R}^n$ is an isomorphism
$d\psi_{f(p)} : T_{f(p)}N \to \mathbb{R}^n$ is an isomorphism
$\Rightarrow d(\psi\circ f\circ\varphi^{-1})_{\varphi(p)}$ is an isomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$
$= d\psi_{f(p)}\circ df_p\circ (d\varphi_p)^{-1}_{\varphi(p)}$
Inverse function thm $\Rightarrow \exists$ open $U'\ni p$, $V'\ni f(p)$ st
$\psi\circ f\circ\varphi^{-1} : \varphi(U') \to \psi(V')$ is a diffeomorphism
$\Rightarrow f : U' \to V'$ is a diffeomorphism. $\square$

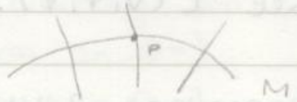
Example 2.15
Let $f : \mathbb{R}^2 \to T^2 \subseteq \mathbb{R}^3$
by $f(\theta,\phi) = ((2+\cos\theta)\cos\phi, (2+\cos\theta)\sin\phi, \sin\theta)$

$$df_{(\theta,\phi)} = \begin{pmatrix} -\sin\theta\cos\phi & -(2+\cos\theta)\sin\phi \\ -\sin\theta\cos\phi & (2+\cos\theta)\cos\phi \\ \cos\theta & 0 \end{pmatrix}$$

$\Rightarrow df_{(\theta,\phi)} : \mathbb{R}^2 \to T_{f(\theta,\phi)}T^2$ is an isomorphism
$\Rightarrow f$ is a local diffeomorphism by prop 2.14
(but not a diffeomorphism).

Definition: 2.16
The tangent bundle of M is

$$TM = \bigcup_{p\in M} T_pM = \{(p,X) : p\in M,\ X\in T_pM\}.$$



Theorem 2.17:
TM is a $2n$-dim manifold.

Proof: Let $\{(U_i,\varphi_i) : i\in I\}$ be an atlas for M
and $\pi : TM \to M$ be $\pi(p,X) = p$.
- TM is a second countable metric space (by topology)
- Let $V_i = \pi^{-1}(U_i)$ open in TM by definition.
and $\bigcup_{i\in I} V_i = TM$

- $\psi_i : V_i \to \mathbb{R}^n\times\mathbb{R}^n$ be $\psi_i(p, \underset{=X}{[(U_i,\varphi_i,u)]}) = (\varphi_i(p), \underset{=d(\varphi_i)_p(X)}{u})$

φ_i diffeomorphism, $d(\varphi_i)_p$ is an isomorphism

$\Rightarrow \psi_i : V_i \longrightarrow U_i \times \mathbb{R}^n$ is a homeomorphism.

- ~~Later~~ if $V_i \cap V_j \neq \emptyset$ then $\psi_j \circ \psi_i^{-1}(\varphi_i(p), d(\varphi_i)_p(X))$

$$= (\varphi_j(p), d(\varphi_j)_p(X))$$

$\Rightarrow \psi_j \circ \psi^{-1}(q, u) = (\underbrace{\varphi_j \circ \varphi^{-1}}_{\text{diff}}(q), d(\underbrace{\varphi_j \circ \varphi^{-1}_i}_{\text{iso}})_q(u))$

$\Rightarrow \psi_j \circ \psi_i^{-1}$ is a diffeomorphism

Examples: 2.18

$T_p\mathbb{R}^n = \mathbb{R}^n$ and $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$

Example 2.19

$TS^2 = \{(p, q) \in S^2 \times \mathbb{R}^3 : \langle p, q\rangle = 0\}$.

affine, oriented straight lines in $\mathbb{R}^3$ $\longleftrightarrow$ direction $p \in S^2$, closest point to origin $q \in \langle p \rangle^{\perp}$

$\Rightarrow \left\{ \begin{array}{l} \text{affine (oriented)} \\ \text{straight lines in } \mathbb{R}^3 \end{array} \right\} \hat{=} TS^2 \; (T\mathbb{RP}^2)$

Definition: 2.20

If $f : M \to N$ is a smooth map, then the pushforward $f_* : TM \to TN$ is $f_*(p, X) = (f(p), df_p(X))$.

Remark: If $f : M \to N$ is a diffeomorphism then $f_* : TM \to TN$ is a diffeomorphism st $f_* : T_pM \to T_{f(p)}N$ is an isomorphism.

ie it is a bundle isomorphism.

Remark: Chain rule $\Rightarrow (f \circ g)_* = f_* \circ g_*$

Definition 2.21

A manifold E is a vector bundle over M if there exists:

- a smooth surjective map $\pi : E \to M$
- $\pi^{-1}(p)$ is a vector space for all $p \in M$
- $\forall p \in M$ $\exists$ open $U \ni p$ and diffeomorphism $\psi : \pi^{-1}(U) \to U \times \mathbb{R}^m$ st $\psi : \pi^{-1}(p) \longmapsto \{p\} \times \mathbb{R}^m$ is an isomorphism

m is the same $\forall p \in M$ and is called the rank of ~~the~~ E

E is the total space and M is the base.

(n+m) dim manifold — n-dim manifold

Example 2.22

Given any M let $E = M \times \mathbb{R}^m$ is a vector bundle.

Simplest example is the cylinder $S^1 \times \mathbb{R}$

Example 2.23

TM is a vector bundle of rank n over M (an n-dim manifold)

Definition 2.24

A vector bundle E of rank m over M is trivial if $\exists$ diffeomorphism $\Psi : E \to M \times \mathbb{R}^m$ st $\Psi : \pi^{-1}(p) \to \{p\} \times \mathbb{R}^m$ is an isomorphism $\forall p \in M$

Ψ is a bundle isomorphism.

Example 2.25

$TS^1 = \{ (\cos\theta, \sin\theta, -\lambda \sin\theta, \lambda\cos\theta) : \lambda, \theta \in \mathbb{R} \}$.

$\cong S^1 \times \mathbb{R} \Rightarrow TS^1$ is trivial.

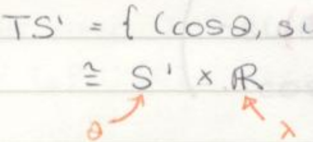
θ λ

Similarly TT^n is trivial (or $T(M \times N) = TM \times TN$)

Definition 2.26

Let $E \xrightarrow{\pi} M$ be a vector bundle over M.

A section of E is a smooth map $s : M \to E$ st $\pi \circ s(p) = p$ $\forall p \in M$



Let $\Gamma(E) = \{$sections of $E\}$ is a vector space.

Example 2.27.

Let $C = S^1 \times \mathbb{R}$

Then $s : S^1 \to C$ given by $s(\theta) = (\cos\theta, \sin\theta, z)$ for z fixed is a section.

Another section is $s(\theta) = (\cos\theta, \sin\theta, \cos\theta)$ (slanted circle)

Proposition 2.28

A ~~real~~ vector bundle of rank m is trivial if and only if it has m linearly independent sections.

Proof: Problem sheet 2.

Example 2.29

We have n linearly independent sections of $T\mathbb{R}^n$ given by

$\partial_i(p) = e_i \in T_p\mathbb{R}^n = \mathbb{R}^n \quad \forall p \in \mathbb{R}^n$

If $f: \mathbb{R}^n \to T \subseteq \mathbb{R}^{2n}$ is $f(\theta_1, \dots, \theta_n) = (\cos\theta_1, \sin\theta_1, \dots, \cos\theta_n, \sin\theta_n)$

~~si~~ Define $s_i(f(p)) = df_p(e_i) \in T_{f(p)}T^n$

is well defined because $df_p = df_q$ if $f(p) = f(q)$ and $s_i \in \Gamma(TT^n)$ everywhere $\Rightarrow TT^n$ is trivial.

Vector fields

Definition: 3.1

A vector field X on a manifold M is a section of TM
ie a smooth mapping $X : M \longrightarrow TM$ st $X(p) \in T_pM \ \forall p \in M$

Example 3.23

On $\mathbb{R}^n$ we have ~~sets~~ vector fields $\partial_i : p \mapsto e_i$, $\partial_i(p) = [\alpha_i]$
where $\alpha_i(t) = p + te_i$

$$\Rightarrow \alpha_i'(0)(f) = (f \circ \alpha_i)'(0)$$
$$= \frac{d}{dt} f(p + te_i)\Big|_{t=0}$$
$$= \frac{\partial f}{\partial x_i}(p)$$

$\Rightarrow \partial_i$ is the differential operator $\frac{\partial}{\partial x_i}$ on functions on $\mathbb{R}^n$ to $\mathbb{R}$.

If $X \in \Gamma(TM)$ and (U, φ) is a chart
then $d\varphi_p(X(p)) = \sum_{i=1}^{n} X_i(\varphi(p))\, \partial_i(\varphi(p))$ for some smooth
$X_i : \varphi(U) \longrightarrow \mathbb{R}$
So we can identify $X|_U$ with $\sum_{i=1}^{n} X_i \partial_i$ on $\varphi(U)$

Example 3.3

On S^n a vector field is a smooth map $X : S^n \longrightarrow \mathbb{R}^{n+1}$ such that
$X(p) \in \mathrm{Span}\{p\}^{\perp} \ \forall p \in S^n$

Hairy Ball Thm: Every vector field on S^{2n} has at least one zero.
$\Rightarrow$ no linearly independent vector fields on S^{2n}
$\Rightarrow TS^{2n}$ is not trivial.

Definition: 3.4

Let $f : M \longrightarrow N$ be a diffeomorphism. Then the pushfoward
$f_* : \Gamma(TM) \longrightarrow \Gamma(TN)$ is given by $f_*(X)(f(p)) = df_p(X(p)) \ \forall p \in M$
Well defined because f is a diffeomorphism.

Example 3.5

Let $f : \mathbb{R}^+ \times S^1 \longrightarrow \mathbb{R}^2 \setminus \{(0,0)\}$ be $f(r, \theta) = (r\cos\theta, r\sin\theta)$
f is a diffeomorphism.

$$df_{(r,\theta)} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \quad f_*\left(\frac{\partial}{\partial r}\right) = (co$$

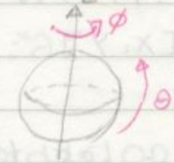
$\Rightarrow f_*\left(\frac{\partial}{\partial r}\right) = \cos\theta\, \partial x + \sin \partial y$

$f_*\left(\frac{\partial}{\partial \theta}\right) = -r\sin\theta\, \partial x + r\cos\theta\, \partial y.$

Example 3.6

$f : (0,\pi)\times(0,2\pi) \longrightarrow S^2$

$f(\theta,\phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta).$



f is a diffeomorphism.

$f_*\left(\partial_\theta\right) = \cos\theta\cos\phi\,\partial x + \cos\theta\sin\phi\,\partial y - \sin\theta\,\partial z$

$f_*(\partial_\phi) = -\sin\theta\sin\phi\,\partial x + \sin\theta\cos\phi\,\partial y$

Remark: $\varphi_* X = \sum_{i=1}^{n} X_i \partial_i$

$X, Y \in \Gamma(TM)$ $\varphi_* X = \sum_{i=1}^{n} X_i\partial_i$, $\varphi_* Y = \sum_{j=1}^{n} Y_i \partial_i$

$\varphi_* X \varphi_* Y = \sum_{i=1}^{n} X_i \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n} Y_j \frac{\partial}{\partial y_j}\right) = \sum_{i,j=1}^{n} X_i Y_j \frac{\partial^2}{\partial x_i \partial y_j} + \sum_{i,j=1}^{n} X_i \frac{\partial Y_j}{\partial x_i}\frac{\partial}{\partial x_i}$

ish

$\Rightarrow XY$ is not a vector field.

Definition:

Given $X, Y \in \Gamma(TM)$ the Lie Braket $[X,Y] = XY - YX$.

In a chart (U,φ) $\varphi_*[X,Y] = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} X_j \frac{\partial Y_i}{\partial x_i} - Y_j \frac{\partial X_i}{\partial x_j}\right)\frac{\partial}{\partial x_i}$.

so $[X,Y] \in \Gamma(TM)$

Example 3.8

$[\partial_i, \partial_j] = 0$

Example 3.9

$X = z\partial_y - y\partial_z$, $Y = x\partial_z - z\partial_x$, $Z = y\partial_x - x\partial_y$ be a vector field on $\mathbb{R}^n$

$[X,Y] = (z\partial_y - y\partial_z)(x\partial_z - z\partial_x) - (x\partial_z - z\partial_x)(z\partial_y - y\partial_z)$

$= -y\partial_z(-z)\partial_x - x\partial_z(z)\partial_y$

$= y\partial_x - x\partial_y = Z.$

Similarly $[Y,Z] = X$, $[Z,X] = Y$

$(a,b,c) \longmapsto aX + bY + cZ$ is an isomorphism between $\mathbb{R}^3$ and $\operatorname{Span}\{X, Y, Z\}$.
$\underset{\mathbb{R}^3}{\wedge}$

$[aX+bY+cZ, a'X+b'Y+c'Z] = (bc'-cb')X + (ca'-ac')Y + (ab'-ba')Z$
$= f((a,b,c) \times (a',b',c'))$

f identifies Lie braket with vector cross product on $\mathbb{R}^3$ here.

Proposition 3.10

If $f: M \to N$ is a diffeomorphism, then
$f_*[X,Y] = [f_*X, f_*Y] \quad \forall X, Y \in \Gamma(TM)$

Proof: Let (U, φ) be a chart on M. Then $(f(U), \varphi \circ f^{-1})$ is a chart on N

Chain rule $\Rightarrow (\varphi \circ f^{-1})_* \circ f_* [X,Y] = \varphi_* [X,Y]$

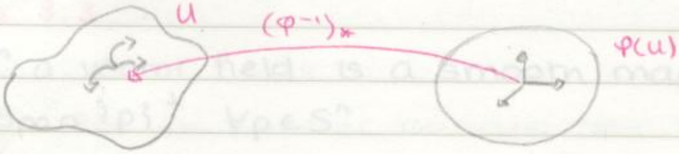
$\therefore$ it is enough to show that $\varphi_*[X,Y] = [\varphi_* X, \varphi_* Y]$.

But this is true by definition 3.7. $\square$.

Example: ~~Let~~ 3.11

Let (U, φ) be a chart on M.

Then we have vector field $X_i = (\varphi^{-1})_* \partial_i$ on U



$[X_i, X_j] = [(\varphi^{-1})_* \partial_i, (\varphi^{-1})_* \partial_j] = (\varphi^{-1})_* [\partial_i, \partial_j] = 0$

Proposition 3.12

The Lie braket satisfies the Jacobi identity:

$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \qquad \forall X, Y, Z \in \Gamma(TM)$

Proof: Local coordinate calc. but we'll see nicer proof in Sheet 2.

Given a curve $\alpha : (-\varepsilon, \varepsilon) \longrightarrow M$ we can define $\alpha'(t) \in T_{\alpha(t)}$
$\Rightarrow$ we have a vector field α' along α.
Let $X \in \Gamma(TM)$ and $p \in M$ $\exists$ unique curve $\alpha_p : (-\varepsilon, \varepsilon)$ through p $(\alpha(0) = p)$ such that $\alpha_p'(t) = X(\alpha_p(t))$ † $\forall t \in (-\varepsilon, \varepsilon)$ because if α_p is contained in a chart (U, φ) then

$\varphi_*(\alpha_p'(t)) = (\varphi \circ \alpha)'(t)$
$= \sum_{i=1}^{n} a_i(t)\partial_i$

If $(\varphi \circ \alpha)(t) = (a_1(t), \dots, a_n(t))$ and

$\varphi_*(X(\alpha_p(t)) = \sum_{i=1}^{n} X_i(a_i(t))\partial_i$ if $\varphi_* X = \sum_{i=1}^{n} x_i \partial_i$

hence † is $a_i(t) = X_i(a_i(t))$
and $(a_1(0), \dots, a_n(0)) = \varphi(p)$ is a system of 1st order ODEs with initial conditions and this has a unique solution.

Definition 3.13
Let $X \in \Gamma(TM)$ and $p \in M$, $\exists$ open $U \ni p$ such that $\forall q \in U$ we have a unique curve $\alpha_q : (-\varepsilon, \varepsilon) \longrightarrow M$ such that $\alpha_q(0) = q$ and $\alpha_q'(t) = X(\alpha_q(t))$
Those curves are called the intergral curves of X in U.

Example 3.14.
Let $X = z\partial_y - y\partial_z$ in $\mathbb{R}^3$ and let $(a, b, c) \in \mathbb{R}^3$
Intergral curves $\alpha(t) = (x(t), y(t), z(t))$ of X satisfy †
$\alpha'(t) = x'(t)\partial_x + y'(t)\partial_y + z'(t)\partial_z$
$\Leftrightarrow x'(t) = 0 \quad y'(t) = z(t), \; z'(t) = -y(t)$
If $\alpha(0) = (a, b, c)$ then $\alpha(t) = (a, b\cos t - c\sin t, c\cos t - b\sin t)$
which is a circle in $x = a$ plane.

Definition: 3.15
Let X, p, U be as in definition 3.13
We define the flow of X on U as

$\{\phi_t^X : U \longrightarrow M \mid t \in (-\varepsilon, \varepsilon)\}$

given by $\phi_t^X(q) = \alpha_q(t)$
The ϕ_t^X are smooth by theory of ODEs

Example 3.16
Let $Z = y\partial_x - x\partial_y$ in $\mathbb{R}^3$
This restricts to a vector field on $C = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$

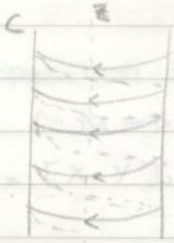
The intergral curve of Z through (a,b,c) is

$\alpha(t) = (a\cos t + b\sin t,\ b\cos t - a\sin t,\ c)$

so the flows of Z on C is given by

$\phi_t^Z(\cos\theta, \sin\theta, z) = (\cos(\theta - t), \sin(\theta - t), z)$

ie rotation clockwise around the circle $z = \text{constant}$



Take $W = y\partial_x - x\partial_y + \partial_z$ which is also a vector field on C.

Intergral curves of W satisfy

$(\alpha(t) = x(t), y(t), z(t))$

$x'(t) = y(t) \quad y'(t) = -x(t) \quad z'(t) = 1$

$\Rightarrow \alpha(t) = (a\cos t + b\sin t,\ b\cos t - a\sin t,\ c + t)$

so the flow of W on C is,

$\phi_t^W(\cos\theta, \sin\theta, z) = (\cos(\theta - t), \sin(\theta - t), z + t)$

$T_{(x,y,z)}C = \{(u,v,w) \in \mathbb{R}^3\} : \langle (u,v,w), (x,y,0)\rangle = 0\}$.

$= \text{span}\{(y,-x,0), (0,0,1)\}$

and W and Z are just linear combinations of this

Proposition 3.17

Let $\{\phi_t^X : U \to M : t \in (-\varepsilon, \varepsilon)\}$ be the flow of X on U.

The $\phi_s^X \circ \phi_t^X = \phi_{s+t}^X$ $\phi_0^X = \text{id}$ and ϕ_t^X is a local diffeomorphism.

Proof: $\phi_s^X \circ \phi_t^X(q) = \phi_s^X(\alpha_q(t)) = \alpha_{\alpha_q(t)}(s)$

$\phi_{s+t}^X(q) = \alpha_q(s+t)$ is the unique solution to $\dagger$ $\alpha'(a) = X(\alpha(a))$ with $\alpha(0) = q$

but then it also solves the equation with $\alpha(t) = \alpha_q(t)$

$\alpha_{\alpha_q(t)}(s)$ is the unique solution to $\dagger$ with $\alpha_{\alpha_q(t)}(0) = \alpha_q(t)$

so $\alpha_{\alpha_q(t)}(s) = \alpha_q(s+t)$

$\phi_0^X(q) = \alpha_q(0) = q \Rightarrow \phi_0^X = \text{id}$

$\phi_{-t}^X \circ \phi_t^X = \phi_0^X = \text{id}$

$\Rightarrow d(\phi_{-t}^X)_{\phi_t^X(q)} \cdot d(\phi_t^X)_q = \text{id}$

$\Rightarrow d(\phi_t^X)_q$ is invertible

$\Rightarrow$ by prop^n 2.14 ϕ_t^X is a local diffeomorphism. $\square$

$X, Y \in \Gamma(TM)$, $p \in M$ then $Y(\phi_t^X(p)) \in T_{\phi_t^X(p)} M$

$d(\phi_{-t}^X)_{\phi_t^X(p)} Y(\phi_t^X(p)) \in T_p M$

$\downarrow$

$Y(p)$ as $t \to 0$

So we can compare this to $Y(p)$

Definition: 3.18

For $X, Y \in \Gamma(TM)$, the Lie derivative of Y with respect to X is

$$\mathcal{L}_X Y(p) = \lim_{t \to 0} \frac{d(\phi_{-t}^X)_{\phi_t^X(p)} Y(\phi_t^X(p)) - Y(p)}{t}$$

$\mathcal{L}_X Y \in \Gamma(TM)$

Proposition 3.19

$\mathcal{L}_X Y = [X, Y]$.

Example 3.20

$Z = y\partial_x - x\partial_y$ on $\mathbb{R}^3$

$\mathcal{L}_{\partial_z} Z = [\partial_z, Z] = 0$

Example 3.21

Example 3.9 with vector fields X, Y, Z then $\mathcal{L}_X Y = [X, Y] = Z$

Example 3.22

~~Let~~ If (U, φ) is a chart on M and $X_i = (\varphi^{-1})_* \partial_i$ are the coordinate vector fields

(so ∂_i are the standard vector fields on $\mathbb{R}^n$) then

$\mathcal{L}_{X_i} X_j = [X_i, X_j] = 0$ by example 3.11

Differential Forms

~~Def~~

Definition 4.1:

For $p \in M$ let T_p^*M be the dual space of T_pM

$T_p^*M = \{$linear maps $\xi : T_pM \longrightarrow \mathbb{R}\}$

T_p^*M is an n-dim vector space

If $x_1, \dots, x_n$ is a basis for T_pM (ie a frame) then I define $\xi_1, \dots, \xi_n$ a basis for T_p^*M by

$$\xi_i(x_j) = \delta_{ij}$$

T_p^*M is the cotangent space of M at p.

Example 4.2

If $f : M \longrightarrow R$ is a smooth function and $x \in T_pM$ we defined $X(f) \in \mathbb{R}$.

$X : f \longmapsto (f \circ \alpha)'(0)$ where $X = \alpha'(0)$

But we can also define $\underset{T_pM}{\overset{\Uparrow}{X}} \longmapsto X(f) \in R$ which is a cotangent vector

If (U, φ) is a chart and $X_1, \dots, X_n$ coordinate frame for T_pM and we write $X = \sum_{i=1}^n a_i X_i$ then

$$X(f) = \sum_{i=1}^n a_i X_i(f) = \sum_{i=1}^n a_i \frac{\partial f}{\partial x_i} = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

$$= df_p(X)$$

So $df_p \in T_p^*M$

Recall: $f : M \longrightarrow N \rightsquigarrow df_p : T_pM \longrightarrow T_{f(p)}N$.

Definition:

Let $f : M \longrightarrow N$ be a smooth map

The pullback $df^*_p : T^*_{f(p)}N \longrightarrow T_p^*M$ is given by

$$df_p^*(\eta)(X) = \eta(df_p(X)) \qquad \eta \in T^*_{f(p)}N \quad X \in T_pM$$

which is a linear map.

Definition:

$T^*M = \bigcup_{p \in M} T_p^*M$ is a rank n vector bundle over M (so 2n-dim manifold) called the cotangent bundle

Proof that T^*M is a manifold is the same as thm 2.17 with $(d\varphi_p^*)^{-1}$ instead of $(d\varphi_p)$

Definition 4.5
A 1-form is a section of T^*M

Example 4.6
Suppose TM is trivial so $\exists$ n linearly independent vector fields $X_1, \dots, X_n$ on M. Define $\xi_1, \dots, \xi_n$ 1-forms on M by
$\xi_i(X_j) = \delta_{ij}$
The ξ_i are linearly independent so T^*M is trivial.

Example 4.7
On $\mathbb{R}^n$ we have a basis for the 1-forms denoted dx_i so that
$dx_i\left(\frac{\partial}{\partial x_j}\right) = \delta_{ij}$

In a chart (U, φ), given a 1-form ξ then
$$\varphi^* \xi = \sum_{i=1}^{n} \xi_i \, dx_i \quad \text{on } \varphi(U) \subseteq \mathbb{R}^n$$

Example 4.8
Given a smooth function $f: M \to \mathbb{R}$ we get a 1-form df on M by
$df_{(p)}(X) = df_p(X) \in \mathbb{R} \quad p \in M, X \in T_pM$

Definition 4.9
Let $f: M \to N$ be a smooth map
If η is a 1-form on N then the pullback $f^*\eta$ is the 1-form on M given by
$$(f^*\eta)_{(p)}(X) = \eta(f_{(p)})(df_p(X)) \quad p \in M, X \in T_pM$$

Remark: $(f \circ g)_* = f_* \circ g_*$
Claim: $(f \circ g)^* = g^* \circ f^*$
$$M \xrightarrow{g} N \xrightarrow{f} P$$
$$\xleftarrow{g^*} \quad \xleftarrow{f^*}$$

$$\begin{aligned}(f \circ g)^*\eta_{(p)}(X) &= \eta((f \circ g)(p))(d(f \circ g)_p(X)) \\ &= \eta(f(g(p)))(df_{g(p)}(dg_p(X))) \\ &= (f^*\eta)_{g(p)}(dg_p(X)) \\ &= g^*(f^*\eta)_{(p)}(X)\end{aligned}$$

Definition 4.10

The tensor products $\otimes^k T_p^* M$ is the set of multilinear maps
$T: \underbrace{T_pM \times \cdots \times T_pM}_{k} \longrightarrow \mathbb{R}$

Remark: If $S \in \otimes^k T_p^* M$, $T \in \otimes^l T_p^* M$, the tensor product
$S \otimes T(X_1, \ldots, X_{k+l}) = S(X_1, \ldots, X_k) T(X_{k+1}, \ldots, X_{k+l})$
$S \otimes T \in \otimes^{k+l} T_p^* M$

We define $S^k T_p^* M$ = symmetric tensors on $\otimes^k T_p^* M$ ~~by~~
ie $T(X_{\sigma(1)}, \ldots, X_{\sigma(k)}) = T(X_1, \ldots, X_k) \quad \forall \sigma \in S_k$
$\Lambda^k T_p^* M$ = alternating tensors in $\otimes^k T_p^* M$
ie $T(X_{\sigma(1)}, \ldots, X_{\sigma(k)}) = \operatorname{sign}(\sigma) T(X_1, \ldots, X_k) \quad \forall \sigma \in S_k$.

Example 4.11

If $g \in S^2 T_p^* M$ then $g(X, Y) = g(Y, X) \quad \forall X, Y \in T_pM$
If $\omega \in \Lambda^2 T_p^* M$ then $\omega(X, Y) = -\omega(Y, X) \quad \forall X, Y \in T_pM$
(In particular $\omega(X, X) = 0$)

We can form bundles $\otimes^k T^* M$, $S^k T^* M$, $\Lambda^k T^* M$ over M
rank $\binom{n}{k}$ vector bundle.

Define the tensor product on these bundles pointwise

Definition 4.12

A section of $\Lambda^k T^* M$ is a k-form

Notice that $\Lambda^1 T^* M = T^* M$

Example 4.13

A 0-form is a smooth map s on M such that ~~s(p)~~
$s(p) \in \Lambda^0 T_p^* M = \mathbb{R}$
ie smooth function $S: M \longrightarrow \mathbb{R}$

Example 4.14

$\Lambda^n T^* M$ is a rank 1 vector bundle over M but it is not necessarily trivial (ie $M \times \mathbb{R}$)
This will be important later.

Remark We can also define $\otimes^k TM$ and $\otimes^k TM \otimes \otimes^l T^*M$
- this last one is the bundle of (k, l)-tensors.

In $\mathbb{R}^n$ we defined the wedge product of forms, so if ω is a k-form, η is an l-form, then $\omega \wedge \eta$ is a $(k+l)$-form
($\omega \wedge \eta$ is the alternating part of $\omega \otimes \eta$)
We then have a basis for the k-forms on $\mathbb{R}^n$
$\{dx_{i_1} \wedge \ldots \wedge dx_{i_k} : i_1, \ldots, i_k \in \{1, \ldots, n\} \; i_1 < \ldots < i_k\}$.

If $\omega \in \Gamma(\Lambda^k T^*M)$ and (U, φ) is a chart, then

$$\varphi^* \omega = \sum_{i_1 \ldots i_k} \omega_{i_1 \ldots i_k} dx_{i_1} \wedge \ldots \wedge dx_{i_k}$$

eg On $\mathbb{R}^4$ $dx_1 \wedge dx_2 + dx_3 \wedge dx_4$ is a 2 form.

Definition 4.15
If $f : M \to N$ is a smooth map and η is a k-form on N then
$(f^*\eta)(p)(X_1, \ldots, X_k) = \eta(f(p))(df_p(X_1), \ldots, df_p(X_k))$ $\forall p \in M$ $X_1, \ldots, X_k \in T_pM$
defines a k-form $f^*\eta$ on M which is the pullback.
Rmk

Example 4.16
If $i : M \to N$ is the inclusion map and η is a k-form on N then $i^*\eta$ is called the restriction of η to M
If η is a 1-form then

$$(i^*\eta)(p)(X) = \eta(p)(di_p X) = \eta(p)(X) \quad (\text{since } di = id)$$

for $p \in M$, $X \in T_pM$
So $i^*\eta$ is η only evaluated on the tangent vectors to M.

In particular if $M = \mathbb{R}^n$, $N = \mathbb{R}^n \times \mathbb{R}$ $i : M \to N$ then
$i^* dx_{n+1} = 0$.

Example:
Let $\xi = \dfrac{x\,dy - y\,dx}{x^2 + y^2}$ be a 1-form on $\mathbb{R}^2 \setminus \{0\}$.
Let $i : S^1 \to \mathbb{R}^2$ be the map $i(\theta) = (\cos\theta, \sin\theta)$
Then $i_*(\partial_\theta) = -\sin\theta \partial_x + \cos\theta \partial_y = x\partial_y - y\partial_x$ on S^1

$$(i^*\xi)(\partial_\theta) = \xi(i_*\partial_\theta) = \left(\frac{x\,dy - y\,dx}{x^2 + y^2}\right)(x\partial_y - y\partial_x) = 1$$

$\Rightarrow i^* \xi = d\theta$ the 1-form dual to θ.

Recall on $\mathbb{R}^n$ we have an operation from k-forms to (k+1)-forms called exterior derivative

$$d(f\,dx_{i_1} \wedge \dots \wedge dx_{i_k}) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}\, dx_j \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

and extend linearly.

Example 4.18

Let ξ be as in Example 4.17.

Then $d\xi = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) dx \wedge dy + \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) \cancel{dy \wedge dy}$

$$+ \frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) \cancel{dx \wedge dx} + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) dy \wedge dx$$

$$d\xi = \frac{(y^2-x^2)}{(x^2+y^2)^2}(dx \wedge dy + dy \wedge dx) = 0.$$

Theorem 4.19

We can define the exterior derivative

$d : \Gamma(\Lambda^k T^*M) \longrightarrow \Gamma(\Lambda^{k+1} T^*M)$ by requiring that if ω is a k-form and (U,φ) is a chart on M then

$$d\omega|_U = \varphi^* d[(\varphi^{-1})^*(\omega|_U)]$$

Then

$d(d\omega) = 0$

ω k-form, η l-form then $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$

if $f : M \to N$ smooth map $\Rightarrow f^* d\eta = d(f^*\eta)$

Proof: The properties of d all follow because d has these properties on $\mathbb{R}^n$, as long as d is well defined, so that is we have to prove. Suppose we have two overlapping charts (U,φ) and (V,ψ)

$$(\varphi^{-1})^*\omega = (\psi^{-1} \circ \psi \circ \varphi^{-1})^*\omega$$
$$= (\psi \circ \varphi^{-1})^* \circ (\psi^{-1})^*\omega$$

$\psi \circ \varphi^{-1}$ is a smooth map on $\mathbb{R}^n$

$\Rightarrow d((\psi \circ \varphi^{-1})^*(\psi^{-1})^*\omega) = (\psi \circ \varphi^{-1})^* d((\psi^{-1})^*\omega)$

$\therefore \varphi^* d((\varphi^{-1})^*\omega) = \varphi^* \circ (\psi \circ \varphi^{-1})^* d((\psi^{-1})^*\omega)$

$$= (\psi \circ \varphi^{-1} \circ \varphi)^* d((\psi^{-1})^*\omega)$$
$$= \psi^* d((\psi^{-1})^*\omega) \quad \square$$

Example 4.20

Let $f: M \to \mathbb{R}$ be a smooth ~~map~~ function.

$p \in M$
$X \in T_pM$

The 1-form df satisfies $df(p)(X) = (\varphi^* d(\varphi^{-1})^* f)(X)$

$$= (d(\varphi^{-1})^* f)(\varphi_{(p)})(d\varphi_p(X))$$
$$= d(f \circ \varphi^{-1})_{\varphi(p)}(d\varphi_p(X))$$
$$= d(f \circ \varphi^{-1} \circ \varphi)_p(X)$$
$$= df_p(X)$$

So $d: f \mapsto df$ as defined before

Remark: We say ω is closed if $d\omega = 0$ and ω is exact if $\omega = d\eta$.

Example 4.21

Example 4.18 shows that the 1-form $d\theta$ on S^1 is closed.
but $d\theta$ is not exact because $\int_{S^1} d\theta = 2\pi \neq 0$ and by Stokes thm
$d\theta$ is not exact.

Example 4.22

Let $\pi: T^*M \to M$ be the projection

$\xi \in T^*M \Rightarrow d\pi_{(x,\eta)}^* \xi \in T^*_{(x,\eta)} T^*M \quad \forall (x,\eta) \in T^*M$

Define $\underbrace{\tau(x,\xi)}_{\in T^*M} = d\pi^*_{(x,\xi)} \xi \in T^*_{(x,\xi)} T^*M$

$\Rightarrow \tau$ is a 1-form on T^*M

in local coordinates $(x_1, \dots x_n, \xi_1, \dots, \xi_n)$ τ is $\sum_{i=1}^n \xi_i dx_i$

Let $\omega = -d\tau$ so $d\omega = 0$

ω in local coordinates is $-d\left(\sum_{i=1}^n \xi_i dx_i\right) = \sum_{i=1}^n dx_i \wedge d\xi_i$

$\Rightarrow \omega^n = \omega \wedge \dots \wedge \omega$ is locally (proportional to)
$dx_1 \wedge \dots \wedge dx_n \wedge d\xi_1 \wedge \dots \wedge d\xi_n$
which is nowhere vanashing, so ω^n is nowhere vanashing
(call it nondegenerate)
So ω is a non degenerate closed 2-form on T^*M
$\Rightarrow (T^*M, \omega)$ is a symplectic manifold.

Definition 4.23

X vector field, ω is a k-form then the Lie derivative of ω in the direction of X is

$$\mathcal{L}_X \omega(p) = \lim_{t \to 0} \frac{(\varphi^X_t)^* \omega(\varphi^X_t(p)) - \omega(p)}{t} \in \Lambda^k T^*_p M$$

where $\{\phi_t^X, t \in (-\varepsilon, \varepsilon)\}$ is the flow of X map.

$\mathcal{L}_X \omega$ is a k-form.

Example 4.24:

Let $f : M \to \mathbb{R}$ be a smooth function

$$\mathcal{L}_X f(p) = \lim_{t \to 0} \frac{(\phi_t^X)^* f(\phi_t^X(p)) - f(p)}{t}$$

$$(\phi_t^X)^* f(\phi_t^X(p)) = [(\phi_t^X)^* f](p) = (f \circ \phi_t^X)(p)$$

$$= \lim_{t \to 0} \frac{f(\phi_t^X(p)) - f(p)}{t}$$

$$= \frac{d}{dt} f(\phi_t^X(p)) \Big|_{t=0}$$

$$= \frac{d}{dt} f(\alpha_p(t)) \Big|_{t=0} \quad \text{where } \alpha_p'(t) = X(\alpha_p(t))$$

$$= df_p(\alpha_p'(0)) = df_p(X) = X(f)(p)$$

$$\Rightarrow \mathcal{L}_X f = df(X) = X(f).$$

Example 24.25

On $\mathbb{R}^n$ $$(\mathcal{L}_X dx_j)(\partial_k) = \lim_{t \to 0} \frac{dx_j((\phi_t^X)_* \partial_k) - dx_j(\partial_k)}{t}$$

$$= dx_j \left(\lim_{t \to 0} \frac{(\phi_t^X)_* \partial_k - \partial_k}{t} \right)$$

$$(\mathcal{L}_X dx_j)(\partial_k) = dx_j(-\mathcal{L}_X \partial_k)$$

$$= dx_j(-[X, \partial_k])$$

$$= dx_j\left(-\left[\sum_{i=1}^n X_i \partial_i, \partial_k\right]\right) \qquad -\sum x_i \partial_i \partial_k + \partial_k \sum x_i \partial_i$$

$$= dx_j\left(-\sum_{i=1}^n X_i \underbrace{[\partial_i, \partial_k]}_{=0} + \sum_{i=1}^n (\partial_k X_i) \partial_i\right)$$

$$= \frac{\partial X_j}{\partial x_k} \quad \text{because } dx_j(\partial_i) = \delta_{ij}$$

So $\mathcal{L}_X dx_j = d(dx_j(X))$ because $\mathcal{L}_X dx_j = \sum_{i=1}^n \frac{\partial x_j}{\partial x_i} dx_i$

$$-\sum X_i(\partial_i \partial_k - \partial_k \partial_i) + \sum (\partial_k X_i) \partial_i$$

$$-\sum X_i(\partial_i \partial_k) + \sum X_i(\partial_k \partial_i) + \sum (\partial_k X_i) \partial_i$$

Proposition: 4.26 (Cartan's formula)

Let X be a vector field and ω a k-form on M. Then we define $i_X \omega$ be the interior product of X and ω

$$i_X \omega (Y_1, \ldots, Y_{k-1}) = \omega(X, Y_1, \ldots, Y_{k-1})$$

for tangent vectors $Y_1, \ldots, Y_{k-1}$ so $i_X \omega$ is a $(k-1)$-form

Then $\mathcal{L}_X \omega = d(i_X \omega) + i_X d\omega$

Proof: Example 4.24 & 4.25, local coords + induction give result.

Example: 4.27

Let $\xi = \dfrac{x\,dy - y\,dx}{x^2+y^2}$ on $\mathbb{R}^2 \setminus \{0\}$

By Ex 4.18, $d\xi = 0$, $\mathcal{L}_X \xi = d(i_X \xi)$

If $X = x\partial_x + y\partial_y \Rightarrow i_X \xi = 0 = \left(\dfrac{xy}{x^2+y^2} - \dfrac{yx}{x^2+y^2}\right) \Rightarrow \mathcal{L}_X \xi = 0$

If $Y = x\partial_y - y\partial_x \Rightarrow i_Y \xi = 1$

$\Rightarrow \mathcal{L}_Y \xi = d(i_Y \xi) = d(1) = 0.$

ie ξ is invariant under X, Y

5. Orientation and Riemannian metrics.

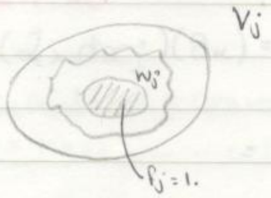
Theorem 5.1

Let M be a manifold with an atlas $\{(U_i, \varphi_i)\ i \in I\}$ $\exists$ an equivalent atlas $\{(V_j, \psi_j) : j \in J\}$ and smooth functions $\{f_k : M \to \mathbb{R},\ k \in K\}$ st

- $\forall j \in J\ \exists i \in I$ st $V_j \subseteq U_i$ (open refinement)
- $\forall p \in M\ \exists$ open $W \ni p$ st $W \cap V_j \neq \emptyset$ for only finitely many $j \in J$ (locally finite)
- $f_k \geq 0$ on M $\forall k \in K$
- $\forall k \in K\ \exists j \in J$ such that $\operatorname{supp} f_k = \{p \in M : f_k(p) \neq 0\} \subseteq V_j$
- $\sum_{k \in K} f_k(p) = 1\ \forall p \in M$ (always finite by local finiteness)

We call $\{f_k : k \in K\}$ a partition of unity subordinate to the atlas $\{(U_i, \varphi_i)\}$ (and $\{(V_j, \psi_j)\}$)

Moreove we can choose $J = K = \mathbb{N}$
$\psi_j(V_j) = B_3(0)$ and $W_j = \psi_j^{-1}(B_1(0))$ such that
$\bigcup_{j=1}^{\infty} W_j = M$ and $f_j|_{W_j} = 1$.



Proposition 5.2

Let $B_r(0)$, $\overline{B_r(0)} \subseteq \mathbb{R}^n$ be the open and closed balls of radius $r > 0$
$\exists$ smooth $g_r : \mathbb{R}^n \to \mathbb{R}$ such that

$g_r \geq 0$
$g_r = 1$ on $\overline{B_{r/2}(0)}$
$g_r = 0$ $\mathbb{R}^n \setminus B_r(0)$
($\Rightarrow \operatorname{supp} g_r \subseteq \overline{B_r(0)}$)

Proof: Consider $h : \mathbb{R} \to \mathbb{R}$ given by $h(t) = \begin{cases} e^{-1/t} & t > 0 \\ 0 & t \leq 0 \end{cases}$

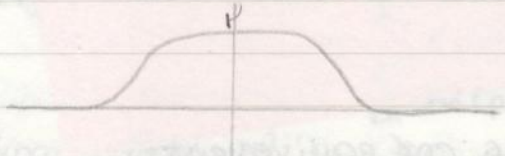
$h'(t) = \frac{1}{t^2} e^{-1/t} > 0$ for $t > 0$ so h is increasing and we know h is smooth

Consider $h_r(t) = \dfrac{h(r^2 - t^2)}{h(r^2 - t^2) + h(t^2 - \frac{1}{4} r^2)}$

This is well defined because of $h(r^2 - t^2) = 0$ then $|t| \geq r$

So $t^2 - \frac{1}{4}r^2 > 0 \Rightarrow h(t - \frac{1}{4}r^2) > 0$

and similarly if $h(t^2 - \frac{1}{4}r^2) = 0$ then $h(r^2 - t^2) > 0$
$\Rightarrow h_r$ is smooth
Now $0 \le h_r \le 1$ and $h_r(t) = 0 \Leftrightarrow |t| \ge r$
and $h_r(t) = 1 \Leftrightarrow |t| \le r/2$



Let $g_r(x) = h_r(|x|)$

Definition 5.3
A manifold M is orientable if $\exists$ an atlas $\{(U_i, \varphi_i)\ i \in I\}$ st whenever $U_i \cap U_j \neq \emptyset$ $\det(d(\varphi_j \circ \varphi_i^{-1})_q) > 0$ $\forall q \in \varphi_i(U_i \cap U_j)$
An orientation is a choice of such an atlas.

Example 5.4
Take the atlas for S^n given in example 1.3
Then the transition map is the map $F: y \longmapsto y/|y|^2$ for $y \in \mathbb{R}^n \setminus \{0\}$.

$$dF_j = \begin{pmatrix} |y|^2 - 2y_1^2 & -2y_1y_2 & \cdots & -2y_1y_n \\ -2y_1y_2 & |y|^2 - 2y_2^2 & \cdots & \cdot \\ \vdots & & & \\ -2y_1y_n & \cdot & \cdots & |y|^2 - 2y_n^2 \end{pmatrix}$$

To discover the sign of the determinant it is enough to check one point, so choose $y = (1, 0, \dots, 0)$
then $dF_y = \begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix} \Rightarrow \det dF_y < 0$

So we change φ_N to $\varphi_N(x_1, \dots, x_{n+1}) = \dfrac{(-x_1, x_2, \dots, x_n)}{1 - x_{n+1}}$

$\Rightarrow \det dF_y > 0$ everywhere (as we changed sign of first column.

Example 5.5
Let $C = \{(\cos\theta, \sin\theta, z) : \theta, z \in \mathbb{R}\}$.
Let $U = C \setminus \{(-1, 0, z) : z \in \mathbb{R}\}$.
$\varphi(\cos\theta, \sin\theta, z) = \left(\dfrac{\theta + \pi}{2\pi}, z\right)$ so $\varphi : U \longrightarrow (0,1) \times \mathbb{R}$ is diffeomorphism

$\theta \in (-\pi, \pi)$

$\psi(\cos(\theta+\pi), \sin(\theta+\pi), z) = \left(\frac{\theta+\pi}{2\pi}, z\right)$ so $\psi : V \to (0,1) \times \mathbb{R}$ is diffeomorphism

$\Rightarrow \psi \circ \varphi^{-1} = id$ so $\det(\varphi(\psi \circ \varphi^{-1})_q) > 0 \ \forall q \in (0,1) \times \mathbb{R}$

$\Rightarrow C$ is orientable.

Example 5.6

The mobius band and the klien bottle are not orientable.

Theorem 5.7

For an n-dim manifold M the following are equivalent

- M is orientable
- $\exists$ nowhere vanishing n-form on M (called a volume form)
- $\Lambda^n T^* M$ is trivial.

Proof: The last two are equivalent by Proposition 2.28

Suppose $\exists \Omega$ nowhere vanishing n-form on M.

Let $\{(U_i, \varphi_i) \ i \in I\}$ be an atlas st $\varphi_i(U_i)$ is connected.

Let $\Omega_0 = dx_1 \wedge \ldots dx_n$ on $\mathbb{R}^n$.

Then $(\varphi_i^{-1})^* \Omega = \lambda_i \Omega_0$ for some nowhere zero function

$\lambda_i : \varphi_i(U_i) \to \mathbb{R} \Rightarrow \lambda_i > 0$ everywhere or $\lambda_i < 0$ everywhere

(because $\varphi_i(U_i)$ is connected)

If $\lambda_i < 0$ we change $\varphi_i : p \mapsto (x_1(p), \ldots, x_n(p))$

to $\varphi_i : p \mapsto (-x_1(p), x_2(p), \ldots, x_n(p))$

which then changes λ_i to $-\lambda_i$ (as $x_1 \mapsto -x_1$ changes Ω_0 to $-\Omega_0$).

Now $(\varphi_j \circ \varphi_i^{-1})^* \Omega_0(q) = \det(d(\varphi_j \circ \varphi_i^{-1})_q) \Omega_0$ (on $\mathbb{R}^n$)

so $(\varphi_j \circ \varphi_i^{-1})^* \circ (\varphi_j^{-1})^* \Omega = (\varphi_j^{-1} \circ \varphi_j \circ \varphi_i^{-1})^* \Omega$

$= (\varphi_i^{-1})^* \Omega$

$\Rightarrow \det(d(\varphi_j \circ \varphi_i^{-1})_q) \lambda_j = \lambda_i \Rightarrow \det(d(\varphi_j \circ \varphi_i^{-1})_q) = \frac{\lambda_i}{\lambda_j} > 0 \quad \forall q \in \varphi_i(U_i \cap U_j)$

So M is orientable.

Suppose that M is orientable and let $\{(U_i, \varphi_i) : i \in I\}$ be an orientation

Let $\{f_k : k \in \mathbb{N}\}$ be a partition of unity subordinate to

$\{(U_i, \varphi_i) : i \in I\}$ given by Thm 5.1

$\forall k \in \mathbb{N} \ \exists i(k) \in I$ st $\operatorname{supp} f_k \subseteq U_{i(k)}$

Since $\sum_{k=1}^{\infty} f_k = 1$, $\forall p \in M \ \exists k \in \mathbb{N}$ st $f_k(p) \neq 0$

$\Rightarrow p \in U_{i(k)}$

So $\bigcup_{k=1}^{\infty} U_{i(k)} = M$ and $\{(U_{i(k)}, \varphi_{i(k)}) : k \in \mathbb{N}\}$ is an orientation

Define $\Omega_1 = \sum_{k=1}^{\infty} f_k \varphi_{i(k)}^* \Omega_0$ where $f_k \varphi_{i(k)}^* \Omega_0$ is zero outside $U_{i(k)}$ which is a n-form because the sum if finite near any given point.
Let $p \in M$ and open $W \ni p$ st $W \cap \operatorname{supp} f_k \neq \emptyset$ for only finitely many k.
By taking the intersection with a coordinate chart if necessary, we can assume $\exists j \in I$ st $W \subseteq U_j$

$$(\varphi_j^{-1})^* \Omega_1(\varphi_j(p)) = \sum_{k=1}^{\infty} f_k(p)(\varphi_j^{-1})^* \circ (\varphi_{i(k)})^* \Omega_0$$
$$= \sum_{k=1}^{\infty} f_k(p)(\varphi_{i(k)} \circ \varphi_j^{-1})^* \Omega_0$$
$$= \sum_{k=1}^{\infty} f_k(p) \det(d(\varphi_{i(k)} \circ \varphi_j^{-1})_{\varphi_j(p)}) \Omega_0$$

Since $\det(d(\varphi_{i(k)} \circ \varphi_j^{-1})_{\varphi_j(p)}) > 0$ and $f_k(p) \neq 0$ for some $k \in \mathbb{N}$.
$\Rightarrow (\varphi_j^{-1})^* \Omega_1(\varphi_j(p)) \neq 0 \Rightarrow \Omega_1(p) \neq 0$. □

Example 5.8

On $\mathbb{R}^n$, we have the standard orientation $\Omega_0 = dx_1 \wedge \dots dx_n$
We say an ordered basis $\{X_1, \dots, X_n\}$ for $\mathbb{R}^n$ is positively oriented if $\Omega_0(X_1, \dots, X_n) > 0 \Rightarrow$ if $x_i = \partial_i$, then this is positively oriented and given an ordered basis $\{Y_1, \dots Y_n\}$ we can write $Y_i = \sum_{j=1}^{n} a_{ij} \partial_i$
So $\Omega_0(Y_1, \dots, Y_n) = \det(a_{ij})$
and hence $\{Y_1, \dots, Y_n\}$ are positively oriented if $\det(a_{ij}) > 0$

The usual notion of orientation on $\mathbb{R}^n$ is an equivalence class of ordered bases $\{X_1, \dots, X_n\}$ with $\{X_1, \dots X_n\} \sim \{Y_1, \dots Y_n\}$ iff $Y_i = \sum_{j=1}^{n} a_{ij} X_j$

Hence the definition of orientation using Ω_0 corresponds with the usual one in $\mathbb{R}^n$
Similarly for an n-dim oriented manifold M with volume form Ω_1, we define an ordered basis $\{X_1, \dots, X_n\}$ for T_pM to be positively oriented if $\Omega_1(p)(X_1, \dots, X_n) > 0 \Rightarrow \Omega_1$ defines an orientation on T_pM varying smoothly with p.

Definition 5.9

Two orientations on M given by volume form Ω and Ω_1' are the same if $\Omega_1' = \lambda \Omega_1$ for some positive smooth function $\lambda : M \to \mathbb{R}$.
A diffeomorphism $f: M \to N$ between oriented manifolds is

orientation preserving if given volume forms Ω_M & Ω_N on M and N, we have $f^*\Omega_N = \lambda \Omega_M$ for some positive smooth function $\lambda : M \to \mathbb{R}$

Note: The key point here is that definition 5.9 is horrible to say in charts, this is why we use volume forms

Remark: $(f^*\Omega_N)(p)(X_1, \dots, X_n) = \Omega_N(df_p(X_1), \dots, df_p(X_n))$ so if $\{X_1, \dots, X_n\}$ is a basis for T_pM then $\{df_p(X_1), \dots, df_p(X_n)\}$ is a basis for $T_{f(p)}N$ as df_p is an isomorphism.
Hence $(f^*\Omega_N)(p) \neq 0 \ \forall p \in M \Rightarrow f^*\Omega_N$ is a volume form.

Example 5.10
The identity $id : M \to M$ is orientation preserving since $id^*\Omega = \Omega$ for a volume form Ω
However consider $-id : \mathbb{R}^n \to \mathbb{R}^n$, then
$(-id)^*\Omega_0 = \det(-id)\Omega_0 = (-1)^n \Omega_0$
So $-id$ is orientation preserving/reversing if n is even/odd

Definition 5.11
A Riemanian metric on a manifold M is a section g of S^2T^*M which is positive definate, ie $\forall p \in M$ g_p is a symmetric bilinear positive definate map from $T_pM \times T_pM \to \mathbb{R}$ (so an inner product on T_pM) varying smoothly with p.
In particular if X, Y vector fields on M then if $(X, Y) = g(Y, X) : M \to \mathbb{R}$ is a smooth function and $g(X, X)(p) \geq 0$ and $= 0$ iff $X(p) = 0$.

Note: So we are taking each tangent space (which is a vector space) and making it an inner product space

Remark: $d_p(X, Y) = \sqrt{g_p(X-Y, X-Y)}$ is a metric on T_pM

Example 5.12
On $\mathbb{R}^n$ we have the standard Riemanian metric g_0 defined by $g_0(\partial_i, \partial_j) = \delta_{ij}$ ie g_0 is dot product.

Example 5.13
Let $M \subseteq \mathbb{R}^{n+m}$, we can define the induced Riemanian metric on g on M by
$$g_p(X,Y) = g_0(X,Y) \quad \forall X,Y \in T_pM \subseteq \mathbb{R}^{n+m} \ \forall p \in M$$

In particular S^n has a Riemannian metric induced from $\mathbb{R}^{n+1}$

Recall Example 3.9 it easy to see that on $\mathbb{R}^3$ the vector fields X, Y, Z satisfying
$$g_0(X,Y) = g_0(Y,Z) = g_0(Z,X) = 0$$

Proposition 5.14
Let $f: M \to N$ be an immersion (ie derivative is injective at every point) and let h be a Riemannian metric on N. Then $g = f^* h$ is a Riemannian metric on M.

Proof: $h \in \Gamma(S^2 T^* N) \Rightarrow g = f^* h \in \Gamma(S^2 T^* M)$
so we only need to check that g is positive definite.
Let $p \in M$, $X \in T_pM$ then
$$g_p(X,X) = h_{f(p)}(df_p(X), df_p(X)) \geq 0$$
and equality iff $df_p(X) = 0$, but f is an immersion ie df_p injective, hence $X = 0$.

Theorem 5.15
Every manifold has a Riemannian metric.

Proof: By theorem 5.1 $\exists$ a countable locally finite atlas $\{(V_i, \psi_i) \ i \in I\}$ with $\psi(V_i) = B_3(0)$ and $W_i = \psi_i^{-1}(B_1(0))$ such that $\bigcup_{i=1}^{\infty} W_i = M$ and partition of unity $\{f_i \mid i \in \mathbb{N}\}$ subordinate to the atlas with $f_i = 1$ on W_i
On V_i, $g_i = \psi_i^* g_0$ is a Riemannian metric.
Define $g = \sum_{i=1}^{\infty} f_i g_i$ (well defined because atlas is locally finite so sum is finite near any point & $f_i = 0$?)
So $g \in \Gamma(S^2 T^* M)$
Let $p \in M$, $X \in T_pM$
$$g_p(X,X) = \sum_{i=1}^{\infty} \underbrace{f_i(p)}_{\geq 0} \underbrace{(g_i)_p(X,X)}_{\geq 0} \geq 0 \text{ and equality iff}$$

$f_i(p)(g_i)_p(X,X) = 0 \quad \forall i$

But $\exists X \in N$ st $f_j(p) = 1$ since $\bigcup_{i=1}^{\infty} W_i = M$

So $(g_j)_p(X,X) = 0 \Rightarrow X = 0$ QED.

6. Riemannian manifolds, definitions and examples.

Definition 6.1:
A Riemannian manifold (M, g) is a manifold M with a Riemannian metric g ie. a positive definate section of $S^2 T^*M$

Example 6.2
$M \subseteq \mathbb{R}^3$ is a surface then the first fundemental form is a Riemannian metric on M
ie $X, Y \in T_pM \subseteq \mathbb{R}^3 \Rightarrow g_p(X, Y) = g_0(X, Y)$ ← dot product.
If $i: M \longrightarrow \mathbb{R}^3$ is continuous the $g = i^* g_0$.

Example 6.3
Let (M, g_M), (N, g_N) be Riem. manifolds
$T_{(p,q)}(M \times N) = T_pM \times T_qN$
$\Rightarrow$ define g on $M \times N$ by
$$g_{(p,q)}((X, U), (Y, V)) = (g_M)_p(X, Y) + (g_N)_q(U, V)$$
$X, Y \in T_pM \quad U, V \in T_qN$.
Clearly g is bilinear, symmetric and smooth because g_M, g_N are.
$$g_{(p,q)}((X, U), (X, U)) = \underbrace{(g_M)_q(X, X)}_{\geqslant 0} + \underbrace{(g_N)_q(U, U)}_{\geqslant 0}$$

and it equals zero iff $(g_M)_p(X, X) = 0 = (g_N)_q(U, U)$
iff $X = 0 = U$.
So g is a Riem. metric.

Example 6.4
Suppose G is a discrete group acting freely and properly discontinuously on a manifold M
Suppose h is a Riem. metric on M/G. Then $g = \pi^* h$ is a Riem. metric on M, where $\pi: M \longrightarrow M/G$ is projection map.
(since $d\pi_p$ is an isomorphism $\forall p \in M$)

Let (U, φ) be a chart on n-dim (M, g) and ∂_j be the standard vector fields on $\mathbb{R}^n$ for $j = 1, \dots, n$. Then we have vector fields $X_j = (\varphi^{-1})_* \partial_j$ on U which form a basis for $\Gamma(TU)$, called the coordinate frame field, and X_j are coordinate vector fields.
So if $Y, Z \in \Gamma(TU)$ we can write $Y = \sum_{j=1}^{n} y_j X_j$, $Z = \sum_{j=1}^{n} z_j X_j$
and $g(Y, Z) = g\left(\sum_{i=1}^{n} y_i X_i, \sum_{j=1}^{n} z_j X_j\right)$
$= \sum_{i,j=1}^{n} y_i z_j g(X_i, X_j)$
We let $g_{ij} = g(X_i, X_j) : U \longrightarrow \mathbb{R}$
Then $(\varphi^{-1})^* g = \sum_{i,j=1}^{n} g_{ij} \, dx_i dx_j$ on $\varphi(U)$
So g is given locally by matrix of smooth functions (g_{ij})

Example 6.5
For $\mathbb{R}^2$, $g_0 = dx^2 + dy^2$
So the matrix is $((g_0)_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
ie $g_0(\partial_x, \partial_x) = g_0(\partial_y, \partial_y) = 1$ $g_0(\partial_x, \partial_y) = 0$.
Let (r, θ) be polar coordinates (Ex 3.5)
$g_0(\partial_r, \partial_r) = g_0(\cos\theta \partial_x + \sin\theta \partial_y, \cos\theta \partial_x + \sin\theta \partial_y) = 1$
$g_0(\partial_\theta, \partial_\theta) = g_0(-r\sin\theta \partial_x + r\cos\theta \partial_y, -r\sin\theta \partial_x + r\cos\theta \partial_y) = r^2$
$g_0(\partial_r, \partial_\theta) = 0$
$\Rightarrow$ in polar coordinates $g_0 = dr^2 + r^2 d\theta^2$ ie the matrix is
$$((g_0)_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$
So $g_0 = dr^2 + r^2 g_{S^1}$ on $\mathbb{R}^2 \setminus \{0\}$.
In general on $\mathbb{R}^{n+1} \setminus \{0\}$ $g_0 = dr^2 + r^2 g_{S^n}$

Example 6.6
Let (θ, ϕ) be coordinates on S^2
ie $p \in S^2$ is given by $p = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$
Example 3.6 $\Rightarrow$ $\partial_\theta = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)$
$\partial_\phi = (-\sin\theta\sin\phi, \sin\theta\cos\phi, 0)$
Let g be the induced Riem. metric on S^2
Then $g(\partial_\theta, \partial_\theta) = 1$ $g(\partial_\phi, \partial_\phi) = \sin^2\theta$, $g(\partial_\theta, \partial_\phi) = 0$
$\Rightarrow$ in these coordinates g is given by $d\theta^2 + \sin^2\theta \, d\phi^2$, or as a matrix
$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$

Definition 6.7:
A smooth map $f: (M, g) \to (N, h)$ is an isometry if f is a diffeomorphism and $f^* h = g$
f is a local isometry at $p \in M$ if $\exists$ open $U \ni p$, open $V \ni f(p)$ such that $f: U \to V$ is an isometry
f is a local isometry if it is a local isometry at all $p \in M$.

Example 6.8
Let $f: \mathbb{R}^n \to \mathbb{R}^n$ be linear on $f(x) = Ax + y$ for $A \in M_n(\mathbb{R})$, $y \in \mathbb{R}^n$
Then $df_x(u) = Au$

$$\Rightarrow (f^* g_0)(\partial_i, \partial_j) = g_0(A\partial_i, A\partial_j)$$
$$= g_0\Big(\sum_{k=1}^n a_{ki}\partial_k, \sum_{l=1}^n a_{lj}\partial_l\Big)$$
$$= \sum_{k,l=1}^n a_{ki} a_{lj}\, g_0(\partial_k, \partial_l) = \sum_{k=1}^n a_{ki} a_{kj} = (A^T A)_{ij}$$

So $f^* g_0 = g_0 \Leftrightarrow A^T A = I$ ie $A \in O(n)$
Moreover f is orientation preserving iff $\det(df_u) > 0$
iff $\det(A) > 0$
iff $A \in SO(n)$.

Suppose (U, φ) is a chart on (M, g) and (V, ψ) is a chart on (N, h) with $\varphi(U) = \psi(V) = W \subseteq \mathbb{R}^n$ (M, N are n-dim) and $(\varphi^{-1})^* g = (\psi^{-1})^* h$ on W.
Then

$$(\psi^{-1} \circ \varphi)^* h = \varphi^* [(\psi^{-1})^* h]$$
$$= \varphi^* [(\varphi^{-1})^* g]$$
$$= (\varphi^{-1} \circ \varphi)^* g$$
$$= g$$

$\Rightarrow f = \psi^{-1} \circ \varphi : U \to V$ is an isometry
Equivalently $g_{ij}(p) = h_{ij}(f(p))$ $\forall p \in U$
So we can detect local isometries using coordinates.

Example 6.9
Let G be a Lie group. Then TG is trivial and bundle isomorphic to $G \times \mathfrak{g}$ where $\mathfrak{g} = T_e G$ is the Lie algebra of G.
Suppose h_e is an inner product on $\mathfrak{g}$
Then define $h_g(X, Y) = h_e(d(L_{g^{-1}})_g X, d(L_{g^{-1}})_g Y)$ $X, Y \in T_g G$
h is a Riem. metric on G such that $L_g^* h = h$ $\forall g \in G$
so h is left invariant and L_g is an isometry $\forall g \in G$

Example 6.10

Let $SU(n) = \{A \in M_n(\mathbb{C}) : \bar{A}^T A = I, \det A = 1\}$. which is a Lie group, with Lie algebra $su(n) = \{A \in M_n(\mathbb{C}) : \bar{A}^T = -A \; tr(A)=0\}$.

Define a map $su(n) \times su(n) \longrightarrow \mathbb{R}$ by $(X, Y) \longmapsto tr(X, Y)$

Clearly this is symmetric and bilinear and

$\overline{tr(XY)} = tr(\bar{X}\bar{Y}) = tr(X^T Y^T) = tr((YX)^T) = tr(YX) = tr(XY) \in \mathbb{R}$.

Let $x_1, \dots, x_n$ be columns of X, then

$-tr(X^2) = tr(\bar{X}^T X) = \sum_{j=1}^n |x_j|^2 \geq 0$ and $=0$ iff $X = 0$

$\Rightarrow h_e(X, Y) = -tr(XY)$ is an inner product on $su(n)$

So define left-invariant Riem metric h on $SU(n)$ by example 6.9

In fact, h is also righ-invariant

So h is bi-invariant

$h([X, Y], Z) = -tr((XY - YX)Z) = -tr(X(YZ) - Z(YX))$

$= -tr(X(YZ) - X(ZY))$

$= h(X, [Y, Z])$

This is true for any bi-invariant h.

Example 6.11

The helicoid $\{(s\cos t, s\sin t, t) \; t \in \mathbb{R}\}$ and the catonoid $\{(\cosh z \cos\theta, \cosh z \sin\theta, z) : z, \theta \in \mathbb{R}\}$ are locally isometric (Can check by $s = \cosh z$, $t = \theta$ in def of helicoid).

Example 6.12

Let $M = \{(t - \tanh t, \frac{\cos\theta}{\cosh t}, \frac{\sin\theta}{\cosh t}) : t, \theta \in \mathbb{R}\}$.

be the pseudo sphere.

Let $f : \mathbb{R}^+ \times S^1 \longrightarrow M$ be the map

$f_*(\partial_t) = \left(\tanh^2 t, -\frac{\cos\theta \sinh t}{\cosh^2 t}, -\frac{\sin\theta \sinh t}{\cosh^2 t}\right)$

$f_*(\partial_\theta) = \left(0, -\frac{\sin\theta}{\cosh t}, \frac{\cos\theta}{\cosh t}\right)$

The induced Riem. metric g on M is defined:

$(f^* g)(\partial_t, \partial_t) = \tanh^4 t + \tanh^2 t \operatorname{sech}^2 t - \tanh^2 t$

$(f^* g)(\partial_\theta, \partial_\theta) = \operatorname{sech}^2 t$

$(f^* g)(\partial_t, \partial_\theta) = 0$.

So $f^*(g) = \tanh^2 t \, dt^2 + \operatorname{sech}^2 t \, d\theta^2$

Let $x = \theta$, $y = \cosh t$ then in these coordinates g is given locally by

$\frac{dx^2+dy^2}{y^2}$ $x \in \mathbb{R}$, $y \geq 1$

$(M \setminus \{(0, \cos\theta, \sin\theta)\}, g)$ is locally isometric to the upper half plane with the hyperbolic metric (Problem sheet 3)

Theorem 6.13

Let G be a discrete group acting freely and properly discontinuously on a Riem. manifold (M, g) such that the diffeomorphism ϕ_g for all $g \in G$ are isometries. Then $\exists$ Riem. metric h on M/G such that $\pi : M \to M/G$ is a local isometry ie $\pi^* h = g$.

Proof: Define $h_q(X, Y) = g_p((d\pi_p)^{-1}(X), (d\pi_p)^{-1}(Y))$

$q \in M/G$, $X, Y \in T_q M$, $\pi(p) = q$.

Suppose $\pi(p) = \pi(p') \Rightarrow p' = \phi_g(p)$ for some $g \in G$.

$\Rightarrow d(\pi_{p'})^{-1} = d(\phi_g)_p \circ (d\pi_p)^{-1}$ since $\pi(p) = (\pi \circ \phi_g)(p)$

$\Rightarrow g_{p'}((d\pi_{p'})^{-1}X, (d\pi_{p'})^{-1}Y) = g_{\phi_g(p)}(d(\phi_g)_p \circ (d\pi_p)^{-1}(X) d(\phi_g)_p \circ (d\pi_p)^{-1}(Y))$

$= (\phi_g^* g)((d\pi_p)^{-1}(X), (d\pi_p)^{-1}(Y))$

$= g_p((d\pi_p)^{-1}(X), (d\pi_p)^{-1}(Y))$

so h_q is well defined

h_q is bilinear and injective because g_p is and $(d\pi_p)^{-1}$ is linear

$h_q(X, X) = g_p((d\pi_p)^{-1}X, (d\pi_p)^{-1}X) \geq 0$

and equality if and only if $(d\pi_p)^{-1}X = 0$ if and only if $X = 0$ since $(d\pi_p)^{-1}$ is an isomorphism.

π local diffeomorphism $\Rightarrow \exists$ open $U \ni q$ open $V \ni p$ such that $\pi : V \to U$ is a diffeomorphism.

Let $f = \pi^{-1} : U \to V$

Then $\forall q' \in U$, $h_q(X, Y) = (f^* g)_{q'}(X, Y)$

since $(f^* g)_{q'}(X, Y) = g_{f(q')}(df_q X, df_q Y)$

$= g_{p'}((d\pi_{p'})^{-1}X, (d\pi_{p'})^{-1}Y)$ where $\pi(p') = q'$

$h|_U = f^*(g|_V) \Rightarrow h$ is smooth as g is smooth, f is a diffeomorphism

$\Rightarrow h$ is a Riem. metric and $\pi^* h = g$ by definition $\square$

Example 6.14

Since id, $-$id are isometries on $\mathbb{R}^{n+1}$, we have that $\mathbb{RP}^n$, Mobius band, Klien bottle obtain Riem. metrics from S^n, cylinder and $T^2 \subseteq \mathbb{R}^3$ respectively.

Example 6.15

Problem sheet 1 $\Rightarrow$ $\mathbb{R}^2/\mathbb{Z}^n$ inherits a Riem. metric ~~from~~ g from $\mathbb{R}^n$ since translations are isometries of $\mathbb{R}^n$.

if $f: \mathbb{R}^n/\mathbb{Z}^n \longrightarrow T^n$ is diffeomorphism from problem sheet 1 and has Riem. metric $T^n \subseteq \mathbb{R}^{2n}$ then $f^*h = 4\pi^2 g$.

$\Rightarrow f$ is conformal, but rescaling translations (ie $\mathbb{Z}^n \longrightarrow 2\pi\mathbb{Z}^n$) makes f an isometry.

7. The Levi-Civita connection

Definition 7.1
For $X \in T_pM$ let $X^b \in T_p^*M$ be $X^b(Y) = g_p(X, Y)$ for $Y \in T_pM$
For $\xi \in T_p^*M$ let $\xi^\# \in T_pM$ be st $\xi(Y) = g_p(\xi^\#, Y)$ for $Y \in T_pM$

Suppose $X^b = 0 \Rightarrow X^b(Y) = 0 \ \forall Y \Rightarrow g_p(X, Y) = 0 \Rightarrow g_p(X, X) = 0$
$\Rightarrow X = 0$. $X \longmapsto X^b$ is linear injective
$\dim T_pM = \dim T_p^*M \Rightarrow X \longmapsto X^b$ isomorphism and so is
$\xi \longmapsto \xi^\#$.

Example 7.2
On $\mathbb{R}^n$ $g_0(\partial_i, \partial_j) = \delta_{ij} = dx_i(\partial_j) \Rightarrow \partial_i^{\,b} = dx_i$ and $dx_i^{\#} = \partial_i$

Theorem 3.7 Fundemental Theorem of Riemannian Geometry

$\exists!\ \nabla : \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM)$ denoted
$(X, Y) \longmapsto \nabla_X Y$ such that if $X, Y, Z \in \Gamma(TM)$ and
a, b are smooth functions on M then:

i $\nabla_{aX+bY} Z = a\nabla_X Z + b\nabla_Y Z$
ii $\nabla_X (Y+Z) = \nabla_X Y + \nabla_X Z$
iii $\nabla_X (aY) = a\nabla_X Y + X(a)Y$ diff a in direction of X.
iv $X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$
v $\nabla_X Y - \nabla_Y X = [X, Y]$

We call ∇_X the covariant derivative of Y with respect to X
and ∇ the Levi-Civita connection.

Proof: Suppose ∇ exists
$\text{iv} \Rightarrow X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y))$
$\text{v} \Rightarrow \quad = 2g(\nabla_X Y, Z) + g(X, [Y, Z]) - g(Y, [Z, X]) - g(Z, [X, Y])$

So
$$g(\nabla_X Y, Z) = \frac{1}{2}\Big(X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) - g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]) \Big) \quad *$$

$* \Rightarrow \nabla$ is unique if it exists.
Goal: Define $\nabla_X Y$ by $*$ and show that i-v are satisfied
Let $W \in \Gamma(TM)$ as well

$$g(\nabla_{aX+bY} Z, W) = \frac{1}{2}\Big((aX+bY)(g(Z, W)) + Z(g(W, aX+bY)) - W(g(aX+bY, Z)) - g(aX+bY, [Z, W]) + g(Z, [W, aX+bY]) - g(W, [aX+bY, Z]) \Big)$$

$= g(a\nabla_X Y + b\nabla_X Z, W) + \frac{1}{2}(Z(a)g(W,X) + Z(b)g(W,Y)) - W(a)g(X,Z)$
$- W(b)g(Y,Z) + g(Z, W(a)X + W(b)Y) - g(W, Z(a)X + Z(b)Y))$
$= g(a\nabla_X Y + b\nabla_X Z, W) \Rightarrow i$

ii is obvious

iii similar to i

iv last four terms in * are skew symmetric in Y, Z
$\Rightarrow g(\nabla_X Y, Z) + g(Y, \nabla_X Z) = X(g(Y,Z)) \Rightarrow iv$

v. first five terms in * are symmetric in X, Y
$\Rightarrow g(\nabla_X Y, Z) - g(\nabla_Y X, Z) = g([X,Y], Z) \Rightarrow v.$

$\square$.

Example 7.4
On $\mathbb{R}^n$ $[\partial_i, \partial_j] = 0$ and $g_0(\partial_i, \partial_j) = \delta_{ij}$ which is a constant function.
$\Rightarrow g_0(\nabla_{\partial_i}\partial_j, \partial_k) = 0 \Rightarrow \nabla_{\partial_i}\partial_j = 0$

Example 7.5
Let G be a Lie group with a bi-invariant Reim. metric
Problem sheet 4 $\Rightarrow \nabla_X Y = \frac{1}{2}[X,Y] \quad \forall X, Y \in \mathfrak{g}$

We can define $\nabla : \Gamma(TM) \times \Gamma(\otimes^m T^*M) \to \Gamma(\otimes^m T^*M)$
by $(X, T) \mapsto \nabla_X T$ where
$\nabla_X T(Y_1, \dots, Y_m) = X(T(Y_1, \dots, Y_m)) - \sum_{j=1}^{m} T(Y_1, \dots, Y_{j-1}, \nabla_X Y_j, Y_{j+1}, \dots, Y_m)$
$\forall Y_1, \dots, Y_m \in \Gamma(TM)$.

Example 7.6
Let $X, Y, Z \in \Gamma(TM)$. Then
$(\nabla_X Y^\flat)(Z) = X(Y^\flat(Z)) - Y^\flat(\nabla_X Z)$
$= X(g(Y,Z)) - g(Y, \nabla_X Z)$
$= g(\nabla_X Y, Z)$ by iv
$= (\nabla_X Y)^\flat(Z)$
$(\nabla_X Y^\flat)^\# = \nabla_X Y$

Example 7.7
$X, Y, Z \in \Gamma(TM) \Rightarrow (\nabla_X g)(Y,Z) = X(g(Y,Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z)$
$= 0$ by property iv of ∇.

Definition 7.8
Suppose (U, φ) is a chart on n-dim (M, g) and let $X_i = (\varphi^{-1})_* \partial_i$ be coordinate vector field on U.
Then there exist functions Γ_{ij}^k on U such that
$$\nabla_{X_i} X_j = \sum_{k=1}^n \Gamma_{ij}^k X_k.$$
Γ_{ij}^k are the Christoffel symbols of g (U, φ)
Caveat: Γ_{ij}^k depends on φ!

Example 7.9
On $\mathbb{R}^n$ $\nabla_{\partial_i} \partial_j = 0 \Rightarrow \Gamma_{ij}^k = 0$.

Proposition 7.10
Let (U, φ) be a chart on (M, g), let $g = (g_{ij})$ on U.
Then $\Gamma_{ji}^k = \Gamma_{ij}^k$ and if $g^{-1} = (g^{ij})$ on U and we write $\partial_k g_{ij} = X_k(g_{ij})$ then
$$\Gamma_{ij}^k = \frac{1}{2} \sum_{l=1}^n g^{kl} (\partial_i g_{lj} + \partial_j g_{li} - \partial_l g_{ij})$$

Proof: $\nabla_{X_i} X_j - \nabla_{X_j} X_i = [X_i, X_j] = 0$ by ex 3.11
$\Leftrightarrow \sum_{k=1}^n (\Gamma_{ij}^k - \Gamma_{ji}^k) X_k = 0 \Rightarrow \Gamma_{ji}^k = \Gamma_{ij}^k$
$$g(\nabla_{X_i} X_j, X_l) = \sum_{k=1}^n g(\Gamma_{ij}^k X_k, X_l)$$
$$= \sum_{k=1}^n \Gamma_{ji}^k g_{kl}$$
$$= \frac{1}{2}\Big(X_i(g(X_j, X_l)) + X_j(g(X_l, X_i)) - X_l(g(X_i, X_j))\Big)$$

using the formula for ∇ and $[X_i, X_j] = 0$
Finally $\Gamma_{ij}^k = \sum_{m=1}^n \Gamma_{ij}^m \underbrace{g_{ml} g^{kl}}_{\delta^k_m}$

□

Example 7.11
For S^2 take (θ, ϕ) coords as in ex 6.6 so X_1, X_2 images of $\partial_\theta, \partial_\phi$ and
$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} \Rightarrow (g^{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \operatorname{cosec}^2\theta \end{pmatrix}$$
$\Gamma_{11}^1 = \Gamma_{11}^2 = 0$, $\Gamma_{22}^1 = -\sin\theta\cos\theta$ $\Gamma_{22}^2 = 0 = \Gamma_{12}^1$, $\Gamma_{12}^2 = \cot\theta$
$$\Gamma_{12}^2 = \frac{1}{2} \sum_{l=1}^2 g^{2l} (\partial_1 g_{l2} + \partial_2 g_{l1} - \partial_l g_{12})$$

$g_{12} = 0$ $g_{l2} g_{l1} = 0$ since g_{l1} is 1 or 0 constant

$g_{i2} \neq 0$ only if $i = 2$

$$\Gamma_{12}^{2} = \frac{1}{2} g^{22} \partial_1 g_{22}$$

$$= \frac{1}{2} \operatorname{cosec}^2\theta \, \partial_\theta (\sin^2\theta)$$

$$= \frac{1}{2} \frac{1}{\sin^2\theta} 2\sin\theta\cos\theta = \cot\theta.$$

$$\therefore \nabla_{X_1} X_1 = \sum_{k=1}^{2} \Gamma_{11}^{k} X_k = 0$$

$$\nabla_{X_2} X_2 = \sum_{k=1}^{2} \Gamma_{22}^{k} X_k = -\sin\theta\cos\theta X_1$$

$$\nabla_{X_1} X_2 = \sum_{k=1}^{2} \Gamma_{12}^{k} X_k = \cot\theta X_2.$$

Definition 7.12

Let α be a curve in (M, g) and let X be a vector field along α ie $X(\alpha(t)) \in T_{\alpha(t)}M$ $\forall t$ and $t \mapsto X(\alpha(t))$ is smooth. The covariant derivative of X along α is

$$\frac{DX}{Dt} = \nabla_{\dot\alpha} X$$

X is parallel if $\frac{DX}{Dt} = 0$.

Suppose α is contained in a chart (U, φ)

Write $\alpha(t) = \varphi^{-1}(a_1(t), \dots, a_n(t))$

$$X(t) = \sum_{i=1}^{n} x_i(t) X_i \qquad (X_i = (\varphi^{-1})_* \partial_i)$$

Then $\alpha' = \sum_{i=1}^{n} a_i' X_i$ and

$$\frac{DX}{Dt} = \nabla_{\sum_{i=1}^n a_i' X_i} \sum_{j=1}^{n} x_j X_j$$

$$= \sum_{i=1}^{n} a_i' \nabla_{X_i} \sum_{j=1}^{n} x_j X_j$$

$$= \sum_{i,j=1}^{n} a_i' X_i(x_j) X_j + \sum_{k,i,j=1}^{n} \Gamma_{ij}^{k} a_i' x_j X_k$$

$$\alpha'(x_j) = \sum_{i=1}^{n} a_i' X_i(x_j) = \alpha_*\left(\frac{d}{dt}\right)(x_j)$$

$$\frac{DX}{Dt} = \sum_{j=1}^{n} \frac{dx_j}{dt} X_j + \sum_{i,j,k=1}^{n} \Gamma_{ij}^{k} a_i' x_j X_k$$

$$= \sum_{k=1}^{n} \left(\frac{dx_k}{dt} + \sum_{i,j=1}^{n} \Gamma_{ij}^{k} a_i' x_j \right) X_k \quad *$$

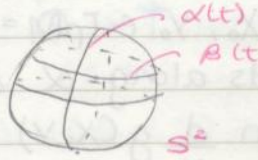
In particular, $\frac{D\alpha'}{Dt} = \sum_{k=1}^{n} \left(a_k'' + \sum_{i,j=1}^{n} \Gamma_{ij}^{k} a_i' a_j' \right) X_k$.

Example 7.13

On $\mathbb{R}^n$, $\Gamma_{ij}^{k} = 0$ so $\frac{DX}{Dt} = \sum_{k=1}^{n} \frac{dx_k}{dt} X_k$.

Example 7.14

On S^2 fix $\theta_0, \phi_0 \in \mathbb{R}$ and let



$\alpha(t) = (\sin t \cos\phi_0, \sin t \sin\phi_0, \cos t)$

$\beta(t) = (\sin\theta_0 \cos t, \sin\theta_0 \sin t, \cos\theta_0)$

$\alpha' = X_1$ as in ex 7.11 and $\beta = X_2$

$\overset{\text{on } \alpha}{\Longrightarrow}$ $\frac{DX_1}{Dt} = \nabla_{X_1} X_1 = 0$ $\quad \frac{DX_2}{Dt} = \nabla_{X_1} X_2 = \cot t \, X_2$

$\overset{\text{on } \beta}{\Longrightarrow}$ $\frac{DX_1}{Dt} = \nabla_{X_2} X_1 = \cot\theta_0 X_2$ $\quad \frac{DX_2}{Dt} = -\sin\theta_0 \cos\theta_0 X_1$

X_1 is parallel along α, X_1, X_2 are parallel along β iff $\theta_0 = \pi/2$

Theorem 7.15

Let $p, q \in (M, g)$ let $\alpha : [0, L] \longrightarrow M$ be a curve such that $\alpha(0) = p$, $\alpha(L) = q$ and let $X_0 \in T_pM$

$\exists!$ parallel vector field X along α such that $X(p) = X_0$

The map $\tau_\alpha : T_pM \longrightarrow T_qM$ given by

$$\tau_\alpha(X_0) = X(q)$$

is an isometry ie $g_p(X_0, Y_0) = g_q(\tau_\alpha(X_0), \tau_\alpha(Y_0))$

called the parallel transport along α.

proof: It is enough to consider curves contained in charts, because we can cover α with charts and since $[0, L]$ is compact we can take finitely many and the uniqueness of X implies agreement on overlapping charts.

X is parallel along α $\Longleftrightarrow$ RHS of * is zero.

But this equivalent to n first order ODEs in variables $(x_1, \ldots, x_n)$ with n initial conditions $(x_1(0), \ldots, x_n(0)) = X_0$.

$\Longrightarrow \exists!$ solution X as claimed.

So $\tau_\alpha : T_pM \longrightarrow T_qM$ is well defined

Let $\beta(t) = \alpha(L - t)$ and consider $\tau_\beta : T_qM \longrightarrow T_pM$

$\exists!$ parallel vector field ~~along~~ Y along β such that $Y(q) = X(q)$
But $\beta'(t) = \alpha'(L-t)$ so $\nabla_{\alpha'} X = \nabla_{\beta'} X = 0$
Y is unique $\Rightarrow Y(p) = X_0$ so $\tau_\beta \circ \tau_\alpha = \mathrm{id}$ so τ_α is an isomorphism
Let X, Y be vector fields along α.

$$\frac{d}{dt}(g(X,Y)) = \alpha'(g(X,Y))$$
$$= g(\nabla_{\alpha'} X, Y) + g(X, \nabla_{\alpha'} Y)$$
$$= g\left(\frac{DX}{Dt}, Y\right) + g\left(X, \frac{DY}{Dt}\right)$$

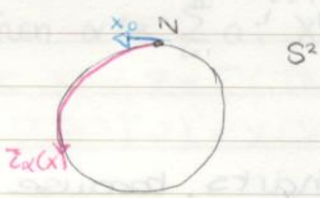
If $X_0, Y_0 \in T_pM$ then let X, Y be the unique parallel vector fields along α such that $X(p) = X_0$, $Y(p) = Y_0$
Then $\frac{d}{dt} g(X,Y) = 0 \Rightarrow g(X,Y)$ is constant along α

$$\Rightarrow g_p(X_0, Y_0) = g_p(X(p), Y(p)) = g_q(X(q), Y(q))$$
$$= g_q(\tau_\alpha(X_0), \tau_\alpha(Y_0))$$

$\square$

Example 7.16
On $\mathbb{R}^n$ if α is a curve from p to q and $X_0 \in T_p\mathbb{R}^n = \mathbb{R}^n$ then the parallel vector field X is $X(\alpha(t)) = X_0 \; \forall t$.
So $\tau_\alpha = \mathrm{id}$.

Example 7.17
If $\alpha: [0, \pi/2] \to S^2$ is given by $\alpha(t) = (\sin t, 0, \cos t)$
Let $X_0 = (1,0,0) \in T_N S^2$
$\nabla_{\alpha'} \alpha' = 0$, $\alpha'(0) = X_0$ so $\tau_\alpha(X_0) = \alpha'(\pi/2) = (0,0,-1)$



Geodesics

Definition 8.1:
A curve γ in (M,g) is a geodesic if $\nabla_{\gamma'}\gamma' = 0$

Since $\frac{d}{dt} g(\gamma',\gamma') = 2g(\nabla_{\gamma'}\gamma',\gamma') = 0$ so $g(\gamma',\gamma')$ is constant

γ is normalised if $g(\gamma',\gamma') = 1$ parametrised by arclength.

Recall in a chart (U,φ), if we write $\gamma(t) = \varphi^{-1}(a_1(t),\dots,a_n(t))$
then γ is a geodesic $\Leftrightarrow a_k'' + \sum_{i,j=1}^{n} \Gamma_{ij}^k a_i' a_j' = 0$ †
These are the geodesic equations

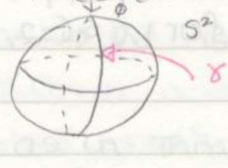
Example 8.2
On $\mathbb{R}^n$, $\Gamma_{ij}^k = 0$ so † $\Leftrightarrow a_k'' = 0 \Leftrightarrow \gamma$ geodesic is a straight line.

Example 8.3
On S^2, take $\gamma(t) = (\sin\theta(t)\cos\phi(t), \sin\theta(t)\sin\phi(t), \cos\theta(t))$
Ex 7.11 $\Rightarrow \gamma$ is a geodesic if and only if

$\theta'' - \sin\theta\cos\theta(\phi')^2 = 0$

$\phi'' - \cot\theta\,\phi'\theta' = 0$

if $\phi' = 0$ then $\theta(t) = at+b$, $\phi(t) = \phi_0$ satisfy the equations
$\Rightarrow$ ~~great~~ great circles with ϕ constant are geodesics
maximal radius



Example 8.4
Thm 6.13 $\Rightarrow$ projection map $\pi: S^2 \to \mathbb{RP}^2$ is a local isometry. The conditions to be geodesic is local so if α is a geodesic on S^2 then $\gamma = \pi\circ\alpha$ is a geodesic in $\mathbb{RP}^2$
Notice α 2π-periodic $\Rightarrow \gamma$ is π-periodic.

Example 8.5
We have a bi-invariant Riem. metric on $SU(2)$ by Ex 6.10
Problem sheet 3 $\Rightarrow$ if $X = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in$ ~~su(2)~~ $su(2)$

then $\nabla_X X = 0$, so intergral curves of X are geodesic
ie curves γ st $\gamma'(t) = X(\gamma(t)) \Rightarrow \gamma(t) = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}$
is a geodesic through I $\gamma(0) = I$ $\gamma'(0) = X$

Example 8.6

On the standard n-torus $T^n \subseteq \mathbb{R}^{2n}$ we have a global coordinate frame given by

$$X_i = -\sin\theta_i \partial_{2i-1} + \cos\theta_i \partial_{2i}$$

then on this frame $g = (g_{ij}) = (\delta_{ij}) \Rightarrow \Gamma^x_y = 0$

so geodesic equations are: $\theta_i'' = 0 \Rightarrow$ geodesics are curves with $\theta_i = a_i t + b$.

Theorem 8.7

Let $p \in (M, g)$. $\exists$ open $U \ni p$, $\varepsilon > 0$ and a smooth map $\Gamma : (-2, 2) \times V \longrightarrow M$ where $V = \{(q, X) : q \in U, X \in B_\varepsilon(0) \subseteq T_qM\}$ st $\gamma_{(q,X)}(t) = \Gamma(t, q, X)$ is the unique geodesic with $\gamma_{(q,X)}(0) = q$, $\gamma_{(q,X)}'(0) = X$

Proof: The geodesic equations are a system of 2nd order ODEs linear in second derivatives so $\exists$ open $U \ni p$ $\varepsilon' > 0$, $\delta > 0$ so that $\forall q \in U$, $Y \in B_{\varepsilon'}(0) \subseteq T_qM$.

$\exists!$ geodesic $\alpha_{(q,Y)} : (-\delta, \delta) \longrightarrow M$

Moreover, the map $(t, q, Y) \longmapsto \alpha_{(q,Y)}(t)$ is smooth

If $\delta \geq 2$ we are done.

Suppose $\delta < 2$.

Define $\gamma_{(q,X)}(t) = \alpha_{(q, \frac{2X}{\delta})}\left(\frac{\delta t}{2}\right)$ for $t \in (-2, 2)$, $X \in B_\varepsilon(0) \subseteq T_qM$

where $\varepsilon = \frac{\delta \varepsilon'}{2} < \varepsilon'$

$\Rightarrow \gamma_{(q,X)}(0) = q$, $\gamma'_{(q,X)}(0) = \frac{\delta}{2} \alpha'_{(q, \frac{2X}{\delta})}(0) = \frac{\delta}{2} \frac{2X}{\delta} = X$

and $\nabla_{\gamma'} \gamma' = \frac{\delta^2}{4} \nabla_{\alpha'} \alpha' = 0 \Rightarrow \gamma_{(q,X)}$ is our required geodesic and is unique because α is unique

Example 8.8

Let $p \in S^n$ and $X \in T_pS^n = \langle p \rangle^\perp$

Consider plane $\Pi = \operatorname{span}\{p, X\} \subseteq \mathbb{R}^{n+1}$

Then the intersection $\Pi \cap S^n$ is a great circle α through p and by appropriate parameterisation has tangent vector X at p.

Let γ be the unique geodesic st $\gamma(0) = p$, $\gamma'(0) = X$

Let ρ be a rotation in the plane $\Pi \Rightarrow \rho$ is an isometry $\Rightarrow \rho \circ \gamma$ is a geodesic

Thm 8.7 $p \circ \gamma = \gamma \Rightarrow \gamma = \alpha$

Example 8.9

$\exists!$ geodesic γ in $\mathbb{RP}^n$ such that $\gamma(0) = [p] \in \mathbb{RP}^n$ and $\gamma'(0) = X \in T_{[p]}\mathbb{RP}^n$

Take $p \in S^n$ and $\exists!\ Y \in T_pS^n$ st $d\pi_p(Y) = X$ — π local diffeo. $\Rightarrow d\pi_p$ is an isomorphism.

So $\exists!$ great circle α ~~st~~ st $\alpha(0) = p$, $\alpha'(0) = Y$

then $(\pi \circ \alpha)(0) = \pi(p) = [p]$ and $(\pi \circ \alpha)'(0) = d\pi_p(Y) = X$

π is local isometry $\Rightarrow \pi \circ \alpha$ geodesic

Thm 8.7 $\Rightarrow \pi \circ \alpha = \gamma$

So geodesics in $\mathbb{RP}^n$ are projection of great circles in S^n.

~~Definition 8.1~~

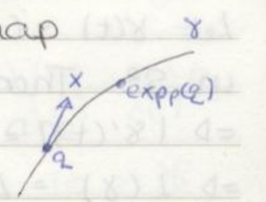
Definition 8.10

Use the notation of Thm 8.7. Define the exponential map

$\exp_p : V \longrightarrow M$ by $\exp_p(q, X) = \gamma_{(q,X)}(1)$

We often restrict to $\exp_p : \underbrace{B_\varepsilon(0)}_{\sqrt{g_p(X,X)} < \varepsilon} \subseteq T_pM \longrightarrow M$

ie $\exp_p(X) = \gamma_{(p,X)}(1)$



Theorem 8.11

Given $p \in M$ $\exists$ open $W \ni p$, $\delta > 0$ such that $\forall q \in W$

$\exp_q : B_\delta(0) \subseteq T_qM \longrightarrow W$ is a diffeomorphism onto is image.

Proof: Let U, V, ε be as in Thm 8.7

Define $F : V \subseteq TM \longrightarrow M \times M$ by $(q, X) \longmapsto (q, \exp_q(X))$

Let $X \in T_0(T_pM) = T_pM$ then

$$d(\exp_p)_0(X) = \frac{d}{dt}\exp_p(tX)\Big|_{t=0} = \frac{d}{dt}\gamma_{(p,X)}(t)\Big|_{t=0}$$

$$= \gamma'_{(p,X)}(0) = X$$

$d(\exp_p)_0 = id$! $\overset{T_pM}{\cong}$

Hence $dF_{(p,0)} : T_pM \times T_0(T_pM) \longrightarrow T_pM \times T_pM$

$dF_{(p,0)} = \begin{pmatrix} I & I \\ 0 & I \end{pmatrix}$ is an isomorphism

So F is a local diffeomorphism at $(p, 0)$.

Thus ~~the~~ $\exists\ \delta \in (0, \varepsilon)$, open $\hat{U} \subseteq U$, open $\hat{W} \subseteq M \times M$ such that if

$\hat{V} = \{(q, X) : q \in \hat{U},\ X \in B_\delta(0) \subseteq T_qM\} \subseteq V$

then $F : \hat{V} \longrightarrow \hat{W}$ is a diffeomorphism. Choose open $W \ni p$ st $W \times W \subseteq \hat{W}$

□

Definition 8.12

A piecewise smooth curve $\alpha : [0,L] \longrightarrow M$ is a continuous curve such that $\exists\ 0 = t_0 < t_1 < \dots < t_k < t_{k+1} = L$ with α smooth on $[t_i, t_{i+1}]$ for $i = 0, \dots k$.

Definition 8.13

Length of a piecewise smooth curve α is $L(\alpha) = \int_0^L \underbrace{|\alpha'(t)|}_{\sqrt{g_{\alpha(t)}(\alpha'(t), \alpha'(t))}}\, dt$

Example 8.14

Let $\alpha(t) = at + b$ be a straight line in $\mathbb{R}^n$ for $t \in [0,L]$ then

$L(\alpha) = \int_0^L |\alpha'(t)| dt$

$= \int_0^L |a|\, dt$

$= L|a|$

Example 8.15

Let $\gamma(t) = (\cos t \cos\phi_0, \cos t \sin\phi_0, \sin t)$ for $t \in [0,L]$ be a geodesic in S^2. Then $\gamma'(t) = (-\sin t\cos\phi_0, -\sin t \sin\phi_0, \cos t)$

$\Rightarrow |\gamma'(t)| = 1$

$\Rightarrow L(\gamma) = L \quad (= \int_0^L 1\, dt)$

Since $\gamma(t+\pi) = -\gamma(t)$ so a half-circle has length π.

Definition 8.16

Let $p \in M$. An open ~~set~~ $U \ni p$ is a normal nbd of p if $\exists$ open ~~$V \subseteq T$~~ $V \subseteq T_pM$ such that $\exp_p : V \longrightarrow U$ is a diffeomorphism

If $B_\varepsilon(0) \subseteq V$ then we call $B_\varepsilon(p) = \exp_p(B_\varepsilon(0))$ and $S_\varepsilon(p) = \partial B_\varepsilon(p) = \exp_p(\partial\, \overline{B_\varepsilon(0)})$ the geodesic ball and geodesic sphere of volume ε ~~[illegible]~~ around p, respectively.

Open $W \subseteq M$ is a totally normall nbd of p if it is a normal nbd of every $q \in W$

Remarks:

- Thm 8.11 $\Rightarrow$ totally normal nbds exist
- Geodesics in a normal nbd starting at p are called radial geodesic
- Given q ~~on~~ in normal nbd of p, the radial geodesics from p to q is the unique geodesic from p to q.

Example 8.17

For $p \in \mathbb{R}^n$, $X \in T_p\mathbb{R}^n = \mathbb{R}^n$ $\exp_p(tX) = p + tX$ and $\exp_p(T_p\mathbb{R}^n) = \mathbb{R}^n$ so $\mathbb{R}^n$ is totally normal nbd of all all $p \in \mathbb{R}^n$ and $B_\varepsilon(p)$ is the usual metric ball.

Example 8.18

If $N \in S^n$, $X \in T_N S^n$ such that $|X| = \pi$ then $\exp_N(X) = S$
$\Rightarrow \exp_N : B_\pi(0) \subseteq T_N S^n \longrightarrow S^n \setminus \{S\}$ is a diffeomorphism and $S^n \setminus \{S\}$ is a normal nbd of N

Lemma 8.19 (Gauss Lemma)

Let $p \in M$, $X \in T_pM$ such that $\exp_p(X)$ is well defined ~~$\forall Y \in T$~~
$\forall Y \in T_X T_pM = T_pM$.

$$g_{\exp_p(X)}\left(d(\exp_p)_X(X), d(\exp_p)_X(Y)\right) = g_p(X, Y)$$

Remark: Lemma 8.19 says that radial geodesics from p are orthogonal to geodesic spheres around p.

Proof: Write $Y = Y^T + Y^\perp$ where $Y^T \in \operatorname{span}\{X\}$ and $Y^\perp \in \operatorname{span}\{X\}^\perp$

Then $g_{\exp_p(X)}(d(\exp_p)_X(X), d(\exp_p)_X(Y^T)) = g_p(X, Y^T)$

because $d(\exp_p)_X(\lambda X) = \lambda X$, so it is enough to show the lemma for $Y = Y^\perp \neq 0$

$\exists \varepsilon > 0$ such that if $X(t) = X\cos t + Y^\perp \sin t$ then $\exp_p(sX(t))$ is well defined $\forall s \in [0,1]$ $\forall t \in (-\varepsilon, \varepsilon)$

Let $f(s,t) = \exp_p(sX(t))$ so $s \longmapsto f(s,t)$ are ~~rad~~ radial geodesics

$$\frac{\partial f}{\partial s} = d(\exp_p)_{sX(t)}(X(t)) \qquad \frac{\partial f}{dt} = d(\exp_p)_{sX(t)}(sX'(t))$$

$$\frac{\partial f}{\partial s}(1,0) = d(\exp_p)_X(X) \qquad \frac{\partial f}{\partial t}(1,0) = d(\exp_p)_X(Y^\perp)$$

Note:
$g(d(\exp_p)_X X, d(\exp_p)_X X)$
$= g(\gamma'(1), \gamma'(1))$ γ geodesic
$= g(\gamma'(0), \gamma'(0))$
$= g(X, X)$

First $\frac{D}{Ds}\frac{\partial f}{\partial s} = 0$ as $s \longmapsto f(s,t)$ is a geodesic

Choose a chart (U, φ) around $f(s_0, t_0)$ and write
$\varphi \circ f(s,t) = (x_1(s,t), \ldots, x_n(s,t))$

Then $$\frac{D}{Ds}\frac{\partial f}{\partial t} = \frac{D}{Ds}\sum_{j=1}^{n} \frac{\partial x_j}{\partial t}\partial_j$$

$$= \sum_{j=1}^{n} \frac{\partial^2 x_j}{\partial s \partial t} \partial_j + \sum_{j,k=1}^{n} \frac{\partial x_j}{\partial t} \nabla_{\frac{\partial x_k}{\partial s}\partial_k} \partial_j$$

$$= \sum_{j=1}^{n} \frac{\partial^2 x_j}{\partial s \partial t} \partial_j + \sum_{j,k,l=1}^{n} \frac{\partial x_j}{\partial t} \frac{\partial x_k}{\partial s} \Gamma_{kj}^{l} \partial_l$$

This is symmetric in s,t since $\Gamma_{jk}^{l} = \Gamma_{kj}^{l}$

$\Rightarrow \frac{D}{Ds}\frac{\partial f}{\partial t} = \frac{D}{Dt}\frac{\partial f}{\partial s}$

$$\Rightarrow \frac{\partial}{\partial s}\left(g\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)\right) = g\left(\frac{D}{Ds}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right) + g\left(\frac{\partial f}{\partial s}, \frac{D}{Ds}\frac{\partial f}{\partial t}\right)$$

$$= g\left(\frac{\partial f}{\partial s}, \frac{D}{Dt}\frac{\partial f}{\partial s}\right)$$

$$= \frac{1}{2}\frac{\partial}{\partial t} g\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial s}\right)$$

$$= \frac{1}{2}\frac{\partial}{\partial t} |X(t)|^2 = 0$$

$X(t) = X\cos t + Y^{\perp}\sin t$
so $|X(t)|$ does not depend on t.

$g\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)(1,0) = g\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)(s,0) \quad \forall s$

$\frac{\partial f}{\partial s}(s,0) = d(\exp_p)_{sX}(sY^{\perp}) \longrightarrow 0$ as $s \longrightarrow 0$

$\Rightarrow g\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)(1,0) = 0$ $\square$

Theorem 8.20

Geodesics $\gamma : [0,L] \longrightarrow M$ in $B_\varepsilon(p)$ with $\gamma(0) = p$ are minimising ie if $\alpha : [0,L] \longrightarrow M$ is a piecewise smooth curve with $\alpha(0) = p$, $\alpha(L) = \gamma(L)$ then $L(\alpha) \geq L(\gamma)$

Moreover, if $L(\alpha) = L(\gamma)$ then $\alpha([0,L]) = \gamma([0,L])$

Proof: Suppose wlog that $\gamma(0) \neq \gamma(L)$ and let α be a comparison curve. Suppose $\alpha([0,L]) \not\subseteq B_\varepsilon(p)$. $\exists\, T \in [0,L]$ least such that $\alpha(T) \in S_\varepsilon(p)$. Then $L(\alpha) \geq L(\alpha|_{[0,T]})$ and $\alpha|_{[0,T]}$ is contained in $B_\varepsilon(p)$

We can reparmetrsise $\alpha|_{[0,T]}$ so that it is defined on $[0,L]$ and not change its length. so it is enough to consider α such that $\alpha([0,L]) \subseteq B_\varepsilon(p)$. wlog we can assume $\alpha(t) \neq p$ $\forall t > 0$

Write $\alpha(t) = \exp_p(r(t)X(t))$ for $t \in (0,L]$ with $r : (0,L] \longrightarrow \mathbb{R}^+$ piecewise smooth and $X(t)$ a curve on T_pM with $|X(t)| = 1$

Proof of Gauss Lemma $\Rightarrow \alpha(t) = f(r(t), t)$

$\Rightarrow \alpha'(t) = \frac{\partial f}{\partial s} r' + \frac{\partial f}{\partial t}$

Gauss Lemma $\Rightarrow g\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right) = 0$ and $g\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial s}\right) = |X(t)|^2 = 1$

So $|\alpha'|^2 = |r'|^2 + \left|\frac{\partial f}{\partial t}\right|^2$

So $L(\alpha) = \int_0^L |\alpha'(t)|\, dt \geq \int_0^L |r'(t)|\, dt$

$\geq \int_0^L r'(t)\, dt = r(L)$

$\gamma(s) = \exp_p\left(\frac{s\, r(L) X(L)}{L}\right)$ because $\gamma(L) = \exp_p(r(L)X(L)) = \alpha(L)$

$\Rightarrow L(\gamma) = r(L)$ by Gauss Lemma

$\Rightarrow L(\alpha) \geq L(\gamma)$ and so γ is minimising

Moreover $L(\alpha) = L(\gamma) \Rightarrow \frac{\partial f}{\partial t} = 0$ and $|r'| = r'$

$\frac{\partial f}{\partial t} = 0 \Rightarrow X'(t) = 0 \Rightarrow X(t) = X$ constant

So $\alpha(t) = \exp_p(r(t)X)$

$\gamma(s) = \exp_p\left(\frac{s\, r(L) X}{L}\right)$

$r' \geq 0 \Rightarrow \alpha$ is a monotonic reparametrisation of γ so $\alpha([0,L]) = \gamma([0,L])$

Proposition 8.21

If $\gamma : [0,L] \to M$ is piecewise smooth curve with $|\gamma'|$ constant and locally minimising then it is a geodesic.

Problem sheet 3

1. $\forall p \in (M,g)\ \exists$ chart (U, φ) st $g_{ij}(p) = \delta_{ij}$ and $\Gamma_{ij}^k(p) = 0$ and geodesics through p and straight lines on $\varphi(U)$
2. $\mathbb{RP}^n$ is orientable iff n is odd.
3. $\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^n x_i^2 - x_{n+1}^2 = -1\}$ with restriction of $\sum_{i=1}^n dx_i^2 - dx_{n+1}^2$ is Riem. and isometric to $B_1(0) \subseteq \mathbb{R}^n$ with $\sum_{i=1}^n \frac{4\, dy_i^2}{(1 - \sum_{j=1}^n y_j^2)}$ and isometric to upper half plane H^n with $\sum_{i=1}^n \frac{dz_i^2}{z_n^2}$.
4. Exponential map is the exponential function on the Lie algebra.

proof: Let $t \in [0, L]$. Let W be a totally normal nbd of $\gamma(t)$.
$\exists \delta > 0$ such that $\gamma([t-\delta, t+\delta]) \subseteq W \Rightarrow$ ~~δ~~ γ joins $\gamma(t-\delta)$ to $\gamma(t+\delta)$ and lies in a geodesic ball around $\gamma(t-\delta)$. Thm 8.20 $\Rightarrow \cancel{\gamma(s)}$
$\Rightarrow \gamma(s) = \exp_p(r(s)X)$ $(p = \gamma(t-\delta))$ with $r' > 0$ for some X.
$|\gamma'|$ constant $\Rightarrow r'$ constant $\Rightarrow r(s) = ks$ for some $k \in \mathbb{R}$
$\Rightarrow \gamma$ is a geodesic on $[t-\delta, t+\delta] \Rightarrow \gamma$ is a geodesic.

9. Completeness

Assume (M, g) is a connected Riem. manifold.

Definition 9.1
(M, g) is (geodesically) complete if every geodesic exists $\forall t$
ie $\exp_p(X)$ is defined $\forall X \in T_pM$ $\forall p \in M$.

Example 9.2
$\mathbb{R}^n$ is complete but H^n is not complete with the standard Riem. metric
Consider $\gamma(t) = te_n = (0, \dots, 0, t)$ is only defined for $t > 0$.
$(H^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\})$.

Example 9.3
S^n is complete but $S^n \setminus \{N\}$ is not complete. Great circles are defined $\forall t$ but any great circle that went through N (ie any one through S) is now not defined $\forall t$ in $S^n \setminus \{N\}$

Example 9.4
Problem sheet 3 $\Rightarrow$ any Lie group with a bi-invariant Riem. metric is complete $\Rightarrow$ eg. $SO(n)$ is complete.

Example 9.5
Problem sheet 4 $\Rightarrow$ $\mathbb{RP}^n$, T^n and the Klien bottle are complete.

Proposition 9.6
For $p, q \in (M, g)$ define
$$d(p,q) = \inf\{L(\alpha), \alpha \text{ piecewise smooth curve from } p \text{ to } q\}.$$
The (M, d) is a metric space.

Proof: $d(p,p) = 0$ clearly
If $p \neq q$ $\exists$ geodesic ball $B_\varepsilon(p)$ such that $q \notin B_\varepsilon(p)$ $\Rightarrow d(p,q) \geq \varepsilon \neq 0$.
$d(p,q) = d(q,p)$ clearly.
Let $p, q, r \in M$ the $d(p,r) \leq L(\alpha) + L(\beta)$ such that α is from p to q
β is from q to r
True for every curve $\alpha, \beta \Rightarrow d(p,r) \leq d(p,q) + d(q,r)$
Metric balls are geodesic balls, which are open sets.
If U is any open set in M $\forall p \in U$ $\exists \varepsilon > 0$ such that $B_\varepsilon(p) \subseteq U$
$\Rightarrow$ metric d is equivalent to the original metric on M

Theorem 9.7 (Hopf-Rinow Theorem)
Let (M,g) be a connected Riem. manifold. The following are equivalent,

a (M,g) is complete

b $\exp_p$ is defined on all of T_pM for some $p \in M$

c closed bounded subsets of M are compact

d (M,d) is a complete metric space.

Moreover, if (M,g) is complete then $\forall p,q \in M$ $\exists$ geodesic γ from p to q such that $L(\gamma) = d(p,q)$

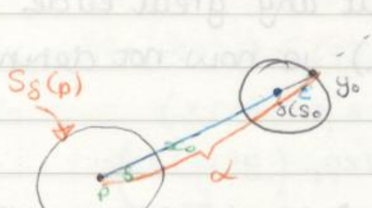
Proof: $a \Rightarrow b$ is trivial by definition

$b \Rightarrow c$ want to show: $\forall q \in M$ $\exists$ geodesics $\gamma: [0,L] \to M$ from p to q such that $L(\gamma) = d(p,q)$.

Let $q \in M$ and let $d(p,q) = R$

Let $\delta > 0$

The map $x \mapsto d(q,x)$ is continuous since d is a metric



$\Rightarrow \exists x_0 \in S_\delta(p)$ such that $d(q,x_0)$ is minimal.

$\Rightarrow x_0 = \exp_p(\delta X)$ for some $X \in T_pM$ with $|X| = 1$

Let $\gamma(s) = \exp_p(sX)$, defined $\forall s \in \mathbb{R}$.

Let $A = \{ s \in [0,R] : d(\gamma(s), q) = R - s \}$.

$0 \in A$ since $\gamma(0) = p$ and A is closed because d is continuous

Want to show: A is open $\Rightarrow A = [0,R] \Rightarrow \gamma(R) = q$

$\Rightarrow \gamma$ is a geodesic from p to q such that $L(\gamma) = d(p,q) = R$.

Suppose $s_0 \in A$ with $s_0 < R$

Let $\varepsilon > 0$ be such that $B_\varepsilon(\gamma(s_0))$ is well defined

Let $y_0 \in S_\varepsilon(\gamma(s_0))$ be such that $d(y_0,q)$ is minimum

Then $d(\gamma(s_0),q) = \varepsilon + \min_{y \in S_\delta(\gamma(s_0))} d(y,q) = \varepsilon + d(y_0,q) = R - s_0$ since $s_0 \in A$

$\Rightarrow d(y_0,q) = R - (s_0 + \varepsilon)$

If $y_0 = \gamma(s_0 + \varepsilon) \Rightarrow$ so $s_0 + \varepsilon \in A$ so A is open.

Now $d(p,y_0) \geq d(p,q) - d(q,y_0) = R - (R - (s_0 + \varepsilon)) = s_0 + \varepsilon$

Curve α from p to $\gamma(s_0)$ then $\gamma(s_0)$ to y_0 has length $s_0 + \varepsilon$ is minimising, $|\alpha'|$ constant and $d(p,y_0) = s_0 + \varepsilon \Rightarrow \alpha$ is a geodesic $\Rightarrow \alpha = \gamma$ by uniqueness

Let $S \subseteq M$ be closed and bounded $\Rightarrow S \subseteq B_r^d(p) = \{q \in M : d(p,q) < r\}$

but $B_r^d(p) \subseteq \exp_p(\overline{B_r(0)})$ for some R

ie there exists geodesic from p to any $q \in B_r^d(p)$

$\exp_p(\overline{B_r(0)})$ is compact because $\exp_p$ continuous and $\overline{B_R(0)}$ compact

$\Rightarrow S$ is compact $\therefore$ (b) $\Rightarrow$ (c)

c $\Rightarrow$ d Let (p_n) be a Cauchy sequence in $(M,d) \Rightarrow (p_n)$ bounded

$\Rightarrow S = \{p_n : n \in \mathbb{N}\}$ is closed bounded $\Rightarrow S$ is compact

$\Rightarrow \{p_n\}$ has convergent subsequence $\Rightarrow (M,d)$ complete

d $\Rightarrow$ a Suppose (M,g) is not complete

$\Rightarrow \exists$ normalized geodesic γ defined for $s < s_0$ but not for $s = s_0$

Let (s_n) be a strictly increasing sequence in $[0, s_0]$ converging to s_0

$\Rightarrow (s_n)$ is Cauchy. $\Rightarrow (\gamma(s_n))$ is Cauchy since ~~$d(\gamma(s_n), \gamma(s_n)$~~

$d(\gamma(s_n), \gamma(s_m)) = |s_n - s_m| \longrightarrow 0$ as $n,m \to \infty$

$\Rightarrow \exists$ convergent subsequence as (M,d) is complete

$\gamma(s_{n_k}) \longrightarrow p_0 \in M$

Choose totally normal nbd W of p_0 such that for some $\delta > 0$

$\exp_q(B_\delta(0)) \supseteq W \quad \forall q \in W.$

Choose N large such that $d(\gamma(s_{n_k}), \gamma(s_{n_l})) < \delta \quad \forall k,l \geq N$

Choose $k, l \geq N$. $\exists$ exists unique geodesic α in W from $\gamma(s_{n_k})$ to $\gamma(s_{n_l}) \Rightarrow \alpha$ and γ coincide where they are defined

Since $\exp_{\gamma(s_{n_k})} : B_\delta(0) \longrightarrow M$ contains W, α extends γ past s_0, giving us a contradiction.

Final conclusion is obvious from b $\Rightarrow$ c geodesic argument.

Remark: Minimising geodesics need not be unique.

eg S^2 and take S, N.

- For the upper half space $\exists$ minimising geodesic between any two points, but it is not complete.

Example 9.8

Any compact manifold is complete (by c of Thm 9.8)

$\Rightarrow T^n, S^n, \mathbb{R}P^n, \mathbb{C}P^n$ are complete.

10. Curvature

Let (M,g) be a Riem manifold with Levi-Civita connection ∇.

Proposition 10.1

For vector X, Y, Z on M we define

$$R(X,Y)Z = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})Z.$$

$R(\cdot,\cdot)$ is bilinear, $R(X,Y)$ is linear and $(R(X,Y),Z)(p)$ only depends on $X(p), Y(p), Z(p)$

$R(X,Y)$ is the Riemann curvature operator

$R(X,Y) = -R(Y,X)$

Example 10.2

On $\mathbb{R}^n$, $\nabla_{\partial_i} \partial_j = 0 = [\partial_i, \partial_j]$

$\Rightarrow R(\partial_i, \partial_j)\partial_k = 0 \Rightarrow R(X,Y) = 0$ $\forall$ vector fields X, Y on $\mathbb{R}^n$

Definition 10.3

Define $R \in \Gamma(\otimes^4 T^*M)$ by

$R(X,Y,Z,W) = g(R(X,Y)Z, W)$ for vector fields X, Y, Z, W

Well defined because $p \in M$ only dependents on g_p and values of X, Y, Z, W at p.

R is the Riemannian curvature tensor

Example 10.~~3~~4

On $\mathbb{R}^n$, $R=0$. (M,g) with $R=0$ is called flat

~~Let~~

Let (U,φ) be a chart on (M,g) and let $X_i = (\varphi^{-1})_* \partial_i$

Then $R(X_i, X_j)X_k = \sum_{l=1}^{n} R^l_{ijk} X_l$

Problem Sheet 4 $\Rightarrow R^l_{ijk} = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ki} + \sum_{m=1}^{n} (\Gamma^l_{im}\Gamma^m_{jk} - \Gamma^l_{jm}\Gamma^m_{ki})$

Define $R_{ijkl} = R(X_i, X_j, X_k, X_l) = \sum_{m=1}^{n} R^m_{ijk} g_{lm}$

$\Rightarrow R$ is preserved by local isometries.

Example 10.5

Projection $\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^n/\mathbb{Z}^n$ is a local isometry $\Rightarrow T^n \subseteq \mathbb{R}^{2n}$ which is isometric to $\mathbb{R}^n/\mathbb{Z}^n$ is flat

Example 10.6

The map $f: \mathbb{R}^2 \to S^1 \times \mathbb{R}$ given by $f(\theta, z) = (\cos\theta, \sin\theta, z)$ is a local isometry $\Rightarrow$ ~~cty~~ cylinder $S^1 \times \mathbb{R}$ is flat.

Example 10.7

Take coordinates (θ, ϕ) on S^2 as usual. Let X_1, X_2 be the image of $\partial_\theta, \partial_\phi$ so $g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$ and

$\Gamma_{11}^1 = \Gamma_{11}^2 = 0, \ \Gamma_{22}^1 = -\sin\theta\cos\theta \quad \Gamma_{22}^2 = \Gamma_{12}^1 = 0 \quad \Gamma_{12}^2 = \cot\theta.$

$R(X_1, X_2)X_1 = \sum_{l=1}^{2} R_{121}^{l} X_l$

Formula $\Rightarrow R_{121}^1 = 0$ and

$R_{121}^2 = \partial_1 \Gamma_{21}^2 - \partial_2 \cancel{\Gamma_{12}^2}^{0} + \sum_{m=1}^{2} (\Gamma_{1m}^2 \Gamma_{21}^m - \Gamma_{2m}^2 \cancel{\Gamma_{11}^m}^{0})$

$= \partial_1 \Gamma_{21}^2 + \Gamma_{12}^2 \Gamma_{21}^2$

$= \frac{\partial}{\partial\theta}(\cot\theta) + \cot^2\theta = -\operatorname{cosec}^2\theta + \cot^2\theta = -1$

$\Rightarrow R(X_1, X_2) = -X_2 \quad \Rightarrow R(X_1, X_2, X_1, X_2) = -g(X_2, X_2) = -g_{22} = -\sin^2\theta$

Similarly $R(X_1, X_2, X_2, X_1) = \sin^2\theta.$

Example 10.8

Let $H^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = -1, \ z > 0\}$ with Riem. metric given by restriction of $dx^2 + dy^2 - dz^2$ to H^2

Let $f(\theta, \phi) = (\sinh\theta\cos\phi, \sinh\theta\sin\phi, \cosh\theta)$

$X_1 = f_* \partial_\theta = (\cosh\theta\cos\phi, \cosh\theta\sin\phi, \sinh\theta)$

$X_2 = f_* \partial_\phi = (-\sinh\theta\sin\phi, \sinh\theta\cos\phi, 0)$

$g(X_1, X_1) = 1$ and $g(X_2, X_2) = \sinh^2\theta$ and $g(X_1, X_2) = 0$

so $g_{ij} \begin{pmatrix} 1 & 0 \\ 0 & \sinh^2\theta \end{pmatrix}$

$g(X_1, X_1) = \cosh^2\theta\cos^2\phi + \cosh^2\theta\sin^2\phi - \sinh\theta$

$= \cosh^2\theta - \sinh^2\theta = 1$

$\Gamma_{11}^1 = \Gamma_{11}^2 = 0 \quad \Gamma_{21}^1 \ \sinh\theta\cosh\theta \quad \Gamma_{22}^2 = \Gamma_{22}^1 = 0 \quad \Gamma_{12}^2 = \coth\theta$

$R(X_1, X_2)X_1 = \sum_{l=1}^{2} R_{121}^{l} X_l$

Formula $\Rightarrow R_{121}^1 = 0$ and

$R_{121}^2 = \partial_1 \Gamma_{21}^2 - \partial_2 \Gamma_{12}^2 + \sum_{m=1}^{2} (\Gamma_{1m}^2 \Gamma_{21}^m - \Gamma_{2m}^2 \Gamma_{11}^m)$

$= \partial_1 \Gamma_{21}^2 + \Gamma_{12}^2 \Gamma_{21}^2$

$= \frac{\partial}{\partial\theta}(\coth\theta) + \coth^2\theta = -\operatorname{cosech}^2\theta + \coth^2\theta = +1$

$\Rightarrow R(X_1, X_2)X_1 = X_2 \quad \Rightarrow R(X_1, X_2, X_1, X_2) = +g(X_2, X_2) = +g_{22} = \sinh^2\theta$

Similarly $R(X_1, X_2, X_2, X_1) = -\sinh^2\theta$

Example 10.9
For the Lie groups $SO(n)$ and $su(n)$ with Riem. metric $g(X,Y) = \mathrm{tr}(X,Y)$
$R(X,Y,Z,W) = -\frac{1}{4}\mathrm{tr}((XY-YX)(ZW-WZ))$ problem sheet 4

Proposition 10.10
Let X,Y,Z,W be vector fields on (M,g)
a $R(Y,X,Z,W) = -R(X,Y,Z,W)$
b $R(X,Y,W,Z) = -R(X,Y,Z,W)$
c $R(Z,W,X,Y) = R(X,Y,Z,W)$
d Bianchi identity $R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0$

Definition 10.11
Let $\sigma = \mathrm{Span}\{X,Y\} \subseteq T_pM$ be a 2-plane
The sectional curvature of σ is
$$K(\sigma) = K(X,Y) = \frac{R(X,Y,Y,X)}{|X|^2|Y|^2 - g(X,Y)^2}$$

This is independent of choice of X and Y.

Proposition 10.12
Let $\bar{R} \in \Gamma(\otimes^4 T^*M)$ with the same properties as R given in prop 10.10. Then if
$$\bar{K}(\sigma) = \frac{\bar{R}(X,Y,Y,X)}{|X|^2|Y|^2 - g(X,Y)} = K(\sigma)$$
$\forall \sigma = \mathrm{Span}\{X,Y\} \subseteq T_pM \ \forall p \in M$
Then $R = \bar{R}$

Sectional curvature determines riemann curvature.

Proof: $K = \bar{K} \Rightarrow \bar{R}(X+Z, Y, Y, X+Z) = R(X+Z, Y, Y, X+Z)$
$\bar{R}(X,Y,Y,X) + \bar{R}(Z,Y,Y,Z) = R(X,Y,Y,X) + R(Z,Y,Y,Z)$
$\quad + 2\bar{R}(X,Y,Y,Z) \qquad * \qquad + 2R(X,Y,Y,Z)$
$\Rightarrow \bar{R}(X,Y,Y,Z) = R(X,Y,Y,Z) \quad \forall X,Y,Z$
$\Rightarrow \bar{R}(X,Y+W,Y+W,Z) = R(X,Y+W,Y+W,Z)$
$\Rightarrow \bar{R}(X,Y,W,Z) + \bar{R}(X,W,Y,Z) = R(X,Y,W,Z) + R(X,W,Y,Z)$
$\quad -\bar{R}(X,Y,Z,W) + \bar{R}(Y,Z,X,W) = -R(X,Y,Z,W) + R(Y,Z,X,W)$
$\Rightarrow R(X,Y,Z,W) * - \bar{R}(X,Y,Z,W) = R(Y,Z,X,W) - \bar{R}(Y,Z,X,W)$
$\qquad = R(Z,X,Y,W) - \bar{R}(Z,X,Y,W)$

Bianchi $R(X,Y,Z,W)+R(Y,Z,X,W)+R(Z,X,Y,W)=0$
$\Rightarrow R(X,Y,Z,W)=\bar{R}(X,Y,Z,W)$

Example 10.13
For $\mathbb{R}^n$, $K=0$. Same for any any flat manifold.

Example 10.14
From ex 10.7, for S^2 we have $T_pS^2=\text{Span}\{X_1,X_2\}$ where
$g(X_1,X_1)=1$, $g(X_2,X_2)=\sin^2\theta$, $g(X_1,X_2)=0$
$$\Rightarrow K(T_pS^2)=\frac{R(X_1,X_2,X_2,X_1)}{g(X_1,X_1)g(X_2,X_2)-g(X_1,X_2)^2}=\frac{\sin^2\theta}{\sin^2\theta}=1$$
moreover, the same is true for $\mathbb{RP}^2$ since $\pi: S^2\to\mathbb{RP}^2$ is a local isometry

Example 10.15
From ex 10.8, for H^2 we have $T_pH^2=\text{Span}\{X_1,X_2\}$ where
$g(X_1,X_1)=1$, $g(X_2,X_2)=\sinh^2\theta$, $g(X_1,X_2)=0$
$$\Rightarrow K(T_pH^2)=\frac{R(X_1,X_2,X_2,X_1)}{g(X_1,X_1)g(X_2,X_2)-g(X_1,X_2)^2}=\frac{-\sinh^2\theta}{\sinh^2\theta}=-1$$

Example 10.16
Let (M,g_M), (N,g_N) be Riem. manifold
$\Rightarrow (M\times N,g)$ is a Riem. manifold where g is the product metric
If (X,Z), (Y,W) are vector fields on $M\times N$, then
$\nabla_{(X,Z)}(Y,W)=(\nabla^M_X Y,\nabla^N_Z W)$
$\Rightarrow R(\underset{(X,0)}{X},\underset{(0,Z)}{Z})\underset{(0,Z)}{Z}=0$ since $\nabla_{(X,0)}(0,Z)=0=\nabla_{(0,Z)}=[(X,0),(0,Z)]$
$\Rightarrow K(\text{Span}\{(X,0),(0,Z)\})=0$
so product $M\times N$ always has zero sectional curvature (eg $S^2\times S^2$ cf. Hopf conjecture).

Proposition 10.17
Let M be an oriented surface on $\mathbb{R}^3$. Then $K(T_pM)=K(p)$ the Gauss curvature of M at p.

Proof: Immediate from Christoffel symbol formula fo R.

Example 10.18

For $T^2 \subseteq \mathbb{R}^3$ if $p = ((2+\cos\theta)\cos\phi, (2\cos\theta)\sin\phi\}, \sin\theta)$

$\Rightarrow K(p) = K(T_pT^2) = \dfrac{\cos\theta}{2+\cos\theta}$

$\Rightarrow T^2 \subseteq \mathbb{R}^3$ is not isometric to $T^2 \subseteq \mathbb{R}^4$.

Let $p \in (M, g)$, let $\sigma \subseteq T_pM$ be a 2-plane, let U be a normal nbd of p ie $\exists$ open $V \subseteq T_pM$ st $\exp: V \to U$ is a diffeomorphism.

Let $S = \exp_p(\sigma \cap V)$ is a surface in M and $K(\sigma)$ is the Gaussian curvature of S at p.

Definition 10.19

Define $Ric \in \Gamma(S^2T^*M)$, the Ricci tensor by

$$Ric(X,Y)(p) = R(X(p), E_i, E_i, Y(p)) = R(E_i, X(p), Y(p), E_i)$$

where $E_1, \ldots E_n$ are an orthonormal basis for T_pM

In coordinates

$$R_{ij} = Ric(X_i, X_j) = \sum_{k,l=1}^{n} R_{ikjl}\, g^{kl}$$

Example 10.20

For a 2-dim Riem. manifold (M,g), if E_1, E_2 is an orthonormal basis for T_pM

$$K(T_pM) = R(E_1, E_2, E_2, E_1)$$
$$= Ric(E_1, E_1) = Ric(E_2, E_2)$$

In dimension 3, Ric encodes all of R, but in higher dimensions they are different.

Remark: manifold with $Ric = \lambda g$ are called Einstein.

Definition 10.21

The scalar curvature S of (M,g) is

$$S(p) = \sum_{i,j}^{n} R(E_i, E_j, E_j, E_i)$$

where $E_1, \ldots E_n$ is an orthonormal basis for T_pM

In coordinates

$$S = \sum_{i,j,k,l=1}^{n} R_{ikl j}\, g^{ij} g^{kl} = \sum_{i,j=1}^{n} R_{ij}\, g^{ij}$$

11. Constant curvature

Proposition 11.1
(M, g) has constant sectional curvature K iff and only if
$R(X, Y, Z, W) = K(g(X,W)g(Y,Z) - g(X,Z)g(Y,W))$

Proof: $\Leftarrow$ Let σ be a 2-plane on T_pM, $\sigma = \text{Span}\{X, Y\}$, then

$$K(\sigma) = \frac{R(X,Y,Y,X)}{g(X,X)g(X,Y) - g(X,Y)^2}$$

$$= \frac{K(g(X,X)g(Y,Y) - g(X,Y)^2)}{g(X,X)g(Y,Y) - g(X,Y)^2} = K$$

$\Rightarrow$ Define $\bar{R} = (X,Y,Z,W) = K(g(X,W)g(Y,Z) - g(X,Z)g(Y,W))$
$\Rightarrow \bar{K}(\text{Span}\{X,Y\}) = K(\text{Span}\{X,Y\})$ since M has constant sectional curvature K.
$\bar{R}$ has the same properties as R in prop 10.10, so prop 10.12
$\Rightarrow R = \bar{R}$

Proposition 11.2
If (M, g) has constant sectional curvature K, then
$Ric = (n-1)Kg$ and $S = n(n-1)K$

Proof: $Ric(X,Y)(p) = \sum_{i=1}^{n} R(X, E_i, E_i, Y)$ E_i orthonormal basis for T_pM.

$$= K\sum_{i=1}^{n}(g(X,Y)g(E_i,E_i) - g(X,E_i)g(Y,E_i))$$

$$= K(ng(X,Y) - g(X,Y))$$

$$S = \sum_{i,j=1}^{n} R(E_i, E_j, E_j, E_i)$$

$$= \sum_{i=1}^{n}(n-1)Kg(E_i,E_i) = n(n-1)K$$

$$g(X,Y) = \sum_{i=1}^{n} g(X,E_i)g(Y,E_i)$$

Example 11.3
$\mathbb{R}^n$ has constant sectional curvature 0 so $Ric = 0 = S$

Example 11.4
S^2 has constant sectional curvature 1 so $Ric = g$ and $S = 2$

Example 11.5
H^2 has constant sectional curvature -1 so ~~Ric g~~ $Ric = -g$ and $S = -2$

Remark:
If $g_t = tg$ where $t>0$, then $g_t^{-1} = t^{-1}g^{-1}$ the Christoffel symbol formula $\Rightarrow R_t = tR$ and $K_t = t^{-1}K$ so $Ric_t = Ric$ and $S_t = t^{-1}S$.
So if M has constant sectional curvature k, then we can always rescale so that $K \in \{-1, 0, 1\}$.

Remark:
$\mathbb{R}^n$ is complete with constant sectional curvature 0 and $O(n)$ together with translations give the isometries

Theorem 11.6
The unit ~~shape~~ sphere (S^n, g) with its induced Riem metric from $\mathbb{R}^{n+1}$, $S^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n} x_i^2 = 1\}$ is

- complete
- its geodesics are great circles given by $\Pi \cap S^n$ for 2-planes, Π in $\mathbb{R}^{n+1}$ through 0.
- it has constant sectional curvature $+1$
- its isometries are given by $O(n+1) = \{A \in M_{n+1}(\mathbb{R}) : A^T A = I\}$

Proof: Completness and geodesics where shown earlier
$O(n+1)$ gives isometries of $\mathbb{R}^{n+1}$ and these are the only ones which preserve S^n.
Let $p \in S^n$, ~~an~~ let σ be a 2-plane T_pS^n.
Since $O(n+1)$ gives isometries, we can rotate so that $p = (-1, 0, \dots 0)$ and we rotate such that $\sigma = \mathrm{Span}\{E_1, E_2\}$ where $E_1 = (0, \dots, 0, 1)$ and $E_2 = (0, -1, \dots, 0)$ since $\sigma \subseteq \mathrm{span}\{p\}^\perp$
Define $f(\theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, 0, \dots, 0, \cos\theta)$
for $\theta \in (0, \pi)$, $\phi \in (0, 2\pi)$ so that $f(\theta, \phi) = p \Leftrightarrow \theta = \pi/2, \phi = \pi$
Then $f_*\partial_\theta = E_1$ and $f_*\partial_\phi = E_2$
Hence by our calculations for S^2, $K(\sigma) = 1$

Theorem 11.7
The hyperbolic n-space (H^n, g) where
$H^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n} x_i^2 - x_{n+1}^2 = -1, x_{n+1} > 0\}$
and ~~the~~ g is the restriction of $\sum_{i=1}^{n} dx_i^2 - dx_{n+1}^2$, is

- complete
- the geodesics through $(0, \dots, 0, 1)$ are $\Pi \cap H^n$ for 2-planes

Π in $\mathbb{R}^{n+1}$, through O (and the rest are found using isometries)

- It has constant sectional curvature -1
- Its isometries are given by $O(n,1) = \{A \in M_{n+1}(\mathbb{R}) : A^T g A = g\}$ where $g = \begin{pmatrix} I & 0 \\ 0 & -1 \end{pmatrix}$

Proof:

Very similar to S^n

The isometries are as stated because $O(n,1)$ preserves the metric g by definition and H^n.

Given $p \in H^n$, $X \in T_p H^n$, let γ be the unique geodesic such that $\gamma(0) = p$, $\gamma'(0) = X$. Let $\rho \in O(n,1)$ be the "hyperbolic" rotation in $\pi = \text{span}\{p, X\}$ then ρ is an isometry so $\rho \circ \gamma$ is a geodesic with the same properties as γ so $\rho \circ \gamma = \gamma \Rightarrow \gamma = \pi \cap H^n$

eg. if $X = (0, \dots, 0, 1, 0)$ $p = (0, \dots, 0, 1)$ then

$$\rho = \begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix} \quad \text{where } A = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}$$

So in this case $\gamma(t) = (0, \dots, 0, \sinh t, \cosh t)$ defined $\forall t \in \mathbb{R}$

$\Rightarrow (H^n, g)$ is complete.

Same trick as for $S^n \Rightarrow$ sectional curvature of $\sigma \subseteq T_p H^n$ for any 2-plane σ is $K(\sigma) = -1$ since it is the same as the sectional curvature of H^2.

(H^n, g) where $H^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^n x_i^2 - x_{n+1}^2 = -1,\ x_{n+1} > 0\}$ and g is the restriction of $\sum_{i=1}^n dx_i^2 - dx_{n+1}^2$ is called the hyperboloid model of hyperbolic n-space

Example 11.8

Problem sheet 8 $\Rightarrow \exists$ isometry $f : (H^n, g) \to (B^n, h)$ where $B^n = \{(y_1, \dots, y_n) \in \mathbb{R}^n : \sum_{i=1}^n y_i^2 < 1\}$ and $h = \sum_{i=1}^n \frac{4\, dy_i^2}{(1 - \sum_{j=1}^n y_j^2)^2}$

given by $f(x_1, \dots, x_{n+1}) = \frac{(x_1, \dots, x_n)}{1 + x_{n+1}}$

(B^n, h) is called the Poincare disc model of hyperbolic n-space.

Example 11.9

Problem sheet 3 $\Rightarrow \exists$ an isometry ~~from~~ $f : (H^n, g) \to (H^n, h)$ where

$H^n = \{(z_1, \dots, z_n) \in \mathbb{R}^n : z_n > 0\}$

$$h = \sum_{i=1}^{n} \frac{dz_i^2}{z_n^2}$$

$$f(x_1, \dots, x_n) = \frac{(x_1, \dots, x_{n+1}, 1)}{x_n + x_{n+1}}$$

(H^n, h) is the upper half-space model of hyperbolic n-space

Example 11.10

Let $H^2 = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ and $g = \dfrac{dx^2 + dy^2}{y^2}$ ~~be the~~

be the hyperbolic Riem. metric

Then $(g_{ij}) = \begin{pmatrix} 1/y^2 & 0 \\ 0 & 1/y^2 \end{pmatrix} \Rightarrow (g^{ij}) = \begin{pmatrix} y^2 & 0 \\ 0 & y^2 \end{pmatrix}$

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{l=1}^{2} g^{kl} (\partial_i g_{lj} + \partial_j g_{li} - \partial_l g_{ij}) \quad \text{where } \partial_1 = \frac{\partial}{\partial x}, \ \partial_2 = \frac{\partial}{\partial y}$$

Clearly $\partial_1 g_{ij} = 0$ and $g^{ij} = g_{ij} = 0$ if $i \neq j$

$\Gamma_{11}^1 = 0 = \Gamma_{22}^1, \quad \Gamma_{11}^2 = \frac{-1}{2} g^{22} \partial_2 g_{11} = \frac{1}{y} = -\Gamma_{22}^2$

$\Gamma_{12}^1 = \frac{1}{2} g^{11} \partial_2 g_{11} = -\frac{1}{y} = \Gamma_{12}^2$

Recall $R_{ijkl} = \sum_{m=1}^{2} g_{lm} \left(\partial_i \Gamma_{jk}^m - \partial_j \Gamma_{ki}^m + \sum_{n=1}^{2} (\Gamma_{in}^m \Gamma_{jk}^n - \Gamma_{jn}^m \Gamma_{ki}^n) \right)$

The only one that matters is

$$R_{1221} = g_{11} \left(\partial_1 \Gamma_{22}^1 - \partial_2 \Gamma_{12}^1 + \sum_{n=1}^{2} (\Gamma_{1n}^1 \Gamma_{22}^n - \Gamma_{2n}^1 \Gamma_{12}^n) \right)$$

$$= \frac{1}{y^2} \left(0 - \frac{1}{y^2} + \left(-\frac{1}{y}\right)\left(-\frac{1}{y}\right) - \left(-\frac{1}{y}\right)^2 \right)$$

$$= \frac{-1}{y^4}$$

$$K(T_{(x,y)} H^2) = \frac{R_{1221}}{g_{11} g_{22} - g_{12}^2} = \frac{-1/y^4}{(\frac{1}{y^2})(1/y^2) - 0} = -1$$

so (H^2, g) has constant sectional curvature -1

Geodesics $\gamma(t) = (x(t), y(t))$ ~~satisfy~~ satisfy

$$x'' - \frac{2}{y} x' y' = 0$$

$$y'' + \frac{1}{y} ((x')^2 - 2x'y' - (y')^2) = 0$$

~~derivation~~ using $x_i'' + \sum_{k=1}^{2} \Gamma^k_{ij} x_i' x_j' = 0$

Clearly $x = \text{constant}$ and $y = e^t$ is a solution.

So vertical half lines (are (the images of) geodesics, which are defined for all $t \in \mathbb{R}$.

Let $z = x + iy$, so $g = \dfrac{dz\,d\bar{z}}{|\operatorname{Im} z|^2}$

If $f: H^2 \to H^2$ then $f^* dz = f'(z)\,dz$, so

$$f^* g = \frac{f'(z)\overline{f'(z)}\,dz\,d\bar{z}}{|\operatorname{Im} f(z)|^2} = \frac{|f'(z)|^2}{|\operatorname{Im} f(z)|^2}\,dz\,d\bar{z}$$

so $f^* g = g \iff \dfrac{1}{|\operatorname{Im} z|^2} = \dfrac{|f'(z)|^2}{|\operatorname{Im} f(z)|^2}$

Let $f(z) = \dfrac{az+b}{cz+d}$ where $ad - bc = 1$, $a, b, c, d \in \mathbb{R}$.

$$f(z) = f(x+iy) = \frac{ax + b + iay}{cx + d + icy}$$

$$= \frac{(acx^2 + acy^2 + bd) + i(ad-bc)y}{|cz+d|^2}$$

and $f'(z) = \dfrac{ad - bc}{(cz+d)^2} = \dfrac{1}{(cz+d)^2}$

So $\dfrac{|f'(z)|^2}{|\operatorname{Im}\{f(z)\}|^2} = \dfrac{1}{|\operatorname{Im} z|^2}$ and f is smooth with inverse $f^{-1}(z) = \dfrac{dz - b}{cz + a}$ which is also smooth, because f isometry.

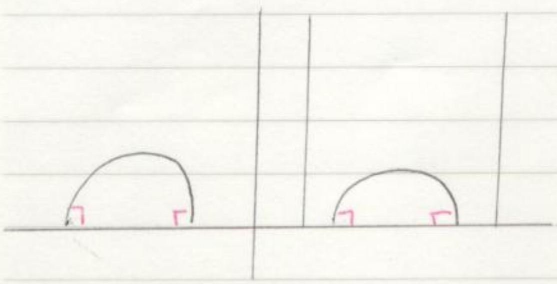
So what is the image of the half line $(0, e^t)$ under f?

$$f(iy) = \frac{acy^2 + bd + iy}{c^2y^2 + d^2} = u + iv$$

$\Rightarrow (2cdu - (ad+bc))^2 + (2cdv)^2 = 1$

so if $cd \neq 0$ we get a semi circle with centre on x-axis, otherwise get another half-line.

Hence the geodesics ~~are~~ of (H^2, g) are



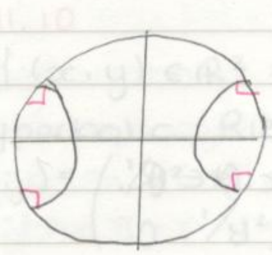
Example 11.11

Problem sheet 3 $\Rightarrow$ we have an isometry f from H^n to B^n with hyperbolic Riem. metric given by:

$$f(z_1, \dots, z_n) = \frac{(2z_1, \dots, 2z_{n-1}, 1 - \sum_{i=1}^{n} z_i^2)}{\sum_{i=1}^{n-1} z_i^2 + (z_n + 1)^2}$$

Notice f maps $z_n = 0$ to S^{n-1} so since f preserves angles and

$$f(0, e^t) = \frac{(0, 1 - e^{2t})}{(1+e^t)^2} \longrightarrow (0, \mp 1) \text{ as } t \longrightarrow \pm\infty$$



geodesics of Poincaré disc model.

Theorem 11.12

Let (M, g) be a complete Riem. manifold with constant sectional curvature $K \in \{-1, 0, 1\}$. Then $\exists$ discrete group G acting by isometries freely and properly discontinuously such that (M, g) is isometric to either S^n/G, $\mathbb{R}^n/G$ or $\mathcal{H}^n/G$.