M204 Representation Theory Notes

Based on the 2013 spring lectures by Mr J Nadim

Representation Theory

Goal: Represent finite groups as a group of invertible matrices over $\mathbb{F}$.
When you hear/read representation, think group homomorphism.

Definition:
Let $\mathbb{F}$ be a field, then define
$$GL_n(\mathbb{F}) = \{A \in M_n(\mathbb{F}) : \det(A) \neq 0\}.$$
ie the set of invertible $n \times n$ matrices with entries in $\mathbb{F}$.

Note: $GL_n(\mathbb{F})$ forms a group under matrix multiplication.

Definition:
Let $\mathbb{F}$ be a field, G a finite group and V a finite dimensional vector space over $\mathbb{F}$ such that $\dim V = n$
Then define an $\mathbb{F}$-representation of G as the homomorphism
$$\rho : G \longrightarrow GL(V) = \{\phi : V \longrightarrow V : \phi \text{ invertible linear map}\}.$$

If we fix a basis for V, say $\{e_1, \dots, e_n\}$ then $GL(V) \cong GL_n(\mathbb{F})$
So we define the $\mathbb{F}$-representation of G as the groups homomorphism
$$\rho : G \longrightarrow GL_n(\mathbb{F})$$
such that $\rho(gh) = \rho(g)\rho(h) \quad \forall g, h \in G$.

Definition:
If $\dim_{\mathbb{F}}(V) = n$, we call n the dimension/degree of the representation

Examples of $\mathbb{C}$-reps, $\mathbb{R}$-reps

1. Trivial representation
Let G be any finite group, fix $n \in \mathbb{N}$
Define $\rho : G \longrightarrow GL_n(\mathbb{F})$
$$g \longmapsto I_n \quad \forall g \in G$$

2. The trivial representation of cyclic groups
Let $G = C_m = \langle x : x^m = 1 \rangle$
Fix n, define $\rho : C_m \longrightarrow GL_n(\mathbb{C})$
$$\rho(x) = I_n$$

Only need to specify where generator goes because of group homomorphism

$\rho(x^s) = \rho(x)^s = \underbrace{I_n \cdots I_n}_{n \text{ times}} = I_n$

$G = C_3 = \langle x : x^3 = 1 \rangle$

Define $\rho : C_3 \to GL_2(\mathbb{C})$

$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$x \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

3. A non-trivial representation of C_m

let $G = C_m = \langle x : x^m = 1 \rangle$ and fix n

define $\rho : C_m \to GL_n(\mathbb{C})$

$x \mapsto A$

what conditions must A satisfy to be a group homorphism

The group law: $A^m = \rho(x)^m = I_n$

eg $\rho : C_m \to GL_n(\mathbb{C})$ of degree n

$x \mapsto \begin{pmatrix} \xi & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \xi \end{pmatrix}$ $\xi = e^{2\pi i/n}$

n^{th} roots of unity

eg. Classify all $\mathbb{C}$-reps of C_m of deg 1

$\rho_i : C_m \to GL_1(\mathbb{C}) = \mathbb{C}^* = \mathbb{C} \setminus \{0\}$

ρ_i are completely determined by roots of unity

$x \mapsto \xi_i \quad 0 \leq i \leq m$

4. Recall: Dihedreal group

$D_{2n} = \langle x, y \rangle : x^n = y^2 = 1, yx = x^{n-1}y \rangle$

Define $D_6 = \langle x, y : x^3 = y^2 = 1, yx = x^2 y \rangle$

$= \langle 1, x, x^2, y, xy, x^2y \rangle$

i) Trivial representation of deg 1

$\rho : D_6 \to GL_1(\mathbb{C}) = \mathbb{C}^*$

$1 \mapsto 1$

$x \mapsto 1$

$y \mapsto 1$

Check: $\rho(xy) = \rho(x^2y) \quad \rho(x^3) = 1 \quad \rho(y^2) = 1$

ii) A non-trivial rep of D_6 of deg 1

$\rho : D_6 \longrightarrow GL_1(\mathbb{C}) = \mathbb{C}^*$

$1 \longmapsto 1$

$x \longmapsto 1$ $\qquad\cong \rho : S_3 \longrightarrow \{\pm 1\} \subset \mathbb{C}^*$

$y \longmapsto -1$ $\qquad$ "sign of permutation"

Check this is well defined

$\rho(x^3) = \rho(x)^3 = 1 = 1$

$\rho(y^2) = \rho(y)^2 = (-1) = 1$

$\rho(xy) = \rho(x)\rho(y) = -1$

$\rho(x^2y) = \rho(x)^2\rho(y) = -1$

iii) $\mathbb{C}$-rep of deg 2 for D_6

Define $\rho : D_6 \longrightarrow GL_2(\mathbb{C})$

$1 \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ always

$x \longmapsto \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$

$y \longmapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\rho(x^2) = \rho(x)\rho(x)$

$= \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$

$\rho(x^3) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\rho(y^2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\rho(x^2y) = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$

$\rho(yx) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$

$\rho(xy) = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}$

$\therefore \langle \rho(1), \rho(x), \rho(x^2), \rho(y), \rho(xy), \rho(x^2y) \rangle$ is a realisation of D_6 as a group of matrices where the group structure remains

iv) Another 2-dim $\mathbb{C}$-rep of D_6

$\rho: D_6 \longrightarrow GL_2(\mathbb{C})$

$1 \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$x \longmapsto \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix} \quad \omega = e^{2\pi i/3}$

$y \longmapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Check $\rho(x^3) = \rho(y^2) = I_2$

$\rho(yx) = \rho(x^2y)$

v) 2-rep of D_6 over $\mathbb{R}$

Define $\rho: D_6 \longrightarrow GL_2(\mathbb{R})$

$1 \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

rotate by $2\pi/3$ $\quad x \longmapsto \begin{pmatrix} \cos 2\pi/3 & -\sin 2\pi/3 \\ \sin 2\pi/3 & \cos 2\pi/3 \end{pmatrix} = A \quad \det A = 1$

reflect $\quad y \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$x^k \longmapsto \begin{pmatrix} \cos 2\pi k/3 & -\sin 2\pi k/3 \\ \sin 2\pi k/3 & \cos 2\pi k/3 \end{pmatrix}$

Define symmetric group S_n, $|S_n| = n!$

Let $G = S_3 = \langle (1), (12), (23), (13), (123), (132) \rangle$

where $S_3 \cong D_6$

$1 \longmapsto 1$

$(1,2,3) \longmapsto x$

$(1,2) \longmapsto y$

Example: A rep of S_3 of deg 3

Using the fact that S_3 acts on the set $X = \langle 1, 2, 3 \rangle$ (by group action) we can construct a 3-dim representation of S_3 as follows:-

Let $V = \mathbb{F}^3$ be generated by $\langle e_1, e_2, e_3 \rangle$ the canonical basis over a field $\mathbb{F}$.

Then $\rho: S_3 \longrightarrow GL(V) = GL_3(\mathbb{F})$

$\rho(\sigma)(e_i) = e_{\sigma(i)} \quad \forall \sigma \in S_3$

Definition:

This representation is called the permutation representation and it works for any S_n

Check ρ defines a hom/rep:

$\rho(1)e_i = e_{1(i)} = e_i$

$\rho(1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Let $\tau, \sigma \in S_3$

$\rho(\sigma\tau)e_i = e_{\sigma\tau(i)}$

$= e_{\sigma(\tau(i))}$

$= \rho(\sigma)e_{\tau(i)}$

$= \rho(\sigma)\rho(\tau)e_i$

$\rho(\sigma\tau) = \rho(\sigma)\rho(\tau)$

Look at when caffeinated!

Let $\sigma = (1\,2\,3)$ $\tau = (1\,2)$

Write matrices for σ and τ

$\rho(\sigma) = \rho((1\,2\,3)) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

$\rho(\sigma)e_i = e_{\sigma(i)}$

$\rho(\sigma)e_1 = e_{\sigma(1)}$

$= e_{(1\,2\,3)(1)} = e_2$

$\rho(\tau) = \rho((1\,2)) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Check $\rho(\sigma)^3 = \rho(\tau)^2 = 1$

$\rho(\tau\sigma) = \rho(\sigma^2\tau)$

* There is a vector subspace $W \subseteq V$ st $\rho(g)W \subseteq W$ $\forall g \in S_3$
(ie W is invariant under the transformation $\rho(g)$)

$W = \text{span}\ \underbrace{(e_1 + e_2 + e_3)}_{w}$

$\rho(g)w = w$ $\forall g \in S_3$.

The permutation representation can be generalised further
Let G be a group acting on a finite set X by
$\circ : G \times X \longrightarrow X$
$1 \circ x = x$
$g \circ (h \circ x) = gh \circ x \quad \forall x \in X \quad \forall g, h \in G$
Choose vectors e_x for each $x \in X$ and form $V = \bigoplus_{x \in X} \mathbb{F} e_x$
$\mathbb{F}$ arbitrary span of basis vectors.
Then define $\rho(g) e_x = e_{g \circ x}$

Provided we know G well enough $G = \langle \dots \rangle$, then we will get a complete answer to the task of classifying all $\mathbb{F}$-reps of G.

However there are
provisos
1. G always finite
2. $\mathbb{F} = \mathbb{C}$ (later $\mathbb{R}$)
3. $|G| \neq 0$ in $\mathbb{F}$ ie $\mathrm{char}(\mathbb{F}) \nmid |G|$
   (ok for $\mathbb{F} = \mathbb{C}, \mathbb{R}$ since $\mathrm{char} = \infty$)
4. $\rho(g)$ is diagonalisable $\forall g \in G$ because $\exists n$ st $g^n = 1$
   $\therefore \rho(g^n) = \rho(g)^n = I \Rightarrow \rho(g)$ satisfies
   $x^n - 1 = 0 \Rightarrow m_\rho(x)$ divides $x^n - 1$ (factor)
   When $\mathbb{F} = \mathbb{C}$ we know FTA
   $x^n - 1 = \prod_{\xi_m = e^{2\pi i/m}} (x - \xi_m)$

is a product of distinct linear factors for any n
$\therefore m_\rho(x) = \dots$
$\Rightarrow \rho(x)$ is diagonalisable.
5. If $\mathbb{F} = \mathbb{R}$ $\rho(g)$ does not have to be diagonalisable
   / $x^n - 1$ does not have to split over $\mathbb{R}, \mathbb{Q}$ ie $x^3 - 1$
6. If G is infinite, $\rho(g)$ is not diagonalisable
eg if $G = \mathbb{Z}$
Define $\rho : \mathbb{Z} \longrightarrow GL_2(\mathbb{C})$
$$\rho(n) \longmapsto \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$
If $n \neq 0$ then $\rho(n)$ is not diagonalisable because
$m_\rho(x) = (x-1)^2$

Groups we will consider:

1. Finite abelian groups $C_n = \langle x : x^n = 1 \rangle$ and and $C_{n_1} \times C_{n_2} \times \dots \times C_{n_i}$
2. Dihedrial groups $D_{2n} = \langle x, y \mid x^n = y^2 = 1,\ yx = x^{n-1}y \rangle$
3. Quaternian groups $Q_{4n} = \langle x, y \mid x^n = y^2 \ \ y^{-1}xy = x^{-1} \rangle$
4. Alternating groups $A_n = \langle \sigma \in S_n \mid \mathrm{sgn}(\sigma) = 1 \rangle$
5. Symmetric groups S_n $n \leq 5, 6$

Distinguishing between representations.

Consider the following ~~rep representations~~ maps of D_6

1. $\sigma : D_6 \longrightarrow GL_3(\mathbb{F})$

$$x \longmapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \sim (1\ 2\ 3)$$

$$y \longmapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim (1\ 2)$$

2. $\tau : D_6 \longrightarrow GL_3(\mathbb{F})$

$$x \longmapsto \left(\begin{array}{cc|c} 0 & -1 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$\oplus$ of reps of D_6 from earlier

$$y \longmapsto \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

2 dim $\oplus$ 1 dim.

σ and τ are essentially the same!

Choose canonical basis for $V = \mathbb{F}^3 = \mathrm{Sp}_{\mathbb{F}} \langle e_1, e_2, e_3 \rangle$ and define new basis :-

$$\phi_1 = e_1 + \frac{e_3}{2} \qquad \phi_2 = e_2 + \frac{e_3}{2} \qquad \phi_3 = -e_1 - e_2 + \frac{e_3}{2}$$

Check $\langle \phi_1, \phi_2, \phi_3 \rangle$ is a basis

LI + span, $\mathbb{F} \neq \mathbb{F}_2$

$$\underset{\begin{pmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}{\underset{\shortparallel}{\tau(x)}}(e_1) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = e_2$$

$\tau(x)(e_2) = -e_1 - e_2$

$\tau(x)(e_3) = e_3$

Now apply $\tau(x)(\phi_1) = \tau(x)(e_1 + e_3/2)$

$= e_2 + \frac{e_3}{2} = \phi_2$

$\tau(x)(\phi_2) = \phi_3$

$\tau(x)(\phi_3) = \tau(x)(-e_1 - e_2 + e_3/2)$

$= -e_2 + e_1 + e_2 + \frac{e_3}{2} = e_1 + \frac{e_3}{2} = \phi_1$

Similarly $\tau(y)(\phi_1) = \phi_2$

$\tau(y)(\phi_2) = \phi_1$

$\tau(y)(\phi_3) = \phi_3$

τ does the same job on $\langle \phi_1, \phi_2, \phi_3 \rangle$ as σ does on $\langle e_1, e_2, e_3 \rangle$

Recall we chose $V = \mathbb{F}^n$ we implicitly choose the standard basis $\langle e_1, \dots, e_n \rangle$ and we ~~choose~~ change basis by conjugating with an invertible matrix.

$\rho \sim \phi : V \longrightarrow V$

Definition:
Two matrices A and B are equivalent if $\exists\, T \in GL_n(\mathbb{F})$ such that $B = T^{-1}AT$

Definition:
Given 2 representations of same group G
$\rho : G \longrightarrow GL_n(\mathbb{F})$, $\rho' : G \longrightarrow GL_n(\mathbb{F})$
We say ρ' is equivalent / conjugate / isometric to ρ if $\exists\, T \in GL_n(\mathbb{F})$ such that $\rho'(g) = T^{-1}\rho(g)T \;\; \forall g \in G$.

Example:

Let $G = D_8$, $\mathbb{F} = \mathbb{C}$.
Define $\rho : D_8 \longrightarrow GL_2(\mathbb{C})$
$x \longmapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\quad y \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

and $T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$ $T^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 0 & i \end{pmatrix}$

Find $\rho' : D_8 \longrightarrow GL_2(\mathbb{C})$

$\rho'(x) = T^{-1}\rho(x) T = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$

$\rho'(y) = T^{-1}\rho(x) T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Exercise: Find T for examples of D_6

$G = C_2 = \langle x : x^2 = 1 \rangle$

$\rho : C_2 \longrightarrow GL_2(\mathbb{C})$ by defining

$x : \longrightarrow A = \begin{pmatrix} -5 & 12 \\ -2 & 5 \end{pmatrix}$ $A^2 = I_2$

Take $T = \begin{pmatrix} 2 & -3 \\ 1 & -1 \end{pmatrix}$

$\rho' = T^{-1}\rho T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\rho'(x)^2 = I_2$

which is also a representation of C_2.

Let $\rho : G \longrightarrow GL_2(\mathbb{C})$ be a representation

Definition:

The kernal of ρ

$\ker(\rho) = \langle g \in G : \rho(g) = I_n \rangle$

* If $\ker(\rho) \lhd G \Rightarrow G/\ker(\rho) \cong Im(\rho) \subset GL_n(\mathbb{F})$

$\rho : G \longrightarrow GL_n(\mathbb{F})$

$1 \longmapsto I_n$

* If $\ker(\rho) = \langle 1 \rangle$

$\Rightarrow G$ is a subgroup of $GL_n(\mathbb{F})$

$G/\langle 1 \rangle \cong G \subseteq GL_n(\mathbb{F})$

Definition:
If $Ker(\rho) = \langle 1 \rangle$ (ρ injective) then we say ρ is faithfull representation of G.

Examples of faithfull representations:

1. The trivial representation is not faithfull unless $G = \langle 1 \rangle$
$\rho : G = Q_4 \longrightarrow GL_n(\mathbb{F}) \quad \Rightarrow Ker(\rho) = Q_4$
$1 \longmapsto I_n$ not faithfull.
$x \longmapsto I_n$
$y \longmapsto I_n$

2. 2-dim representation $\rho : D_{2n} \longrightarrow GL_2(\mathbb{R})$
given by $\rho(x) = \begin{pmatrix} \cos\ldots & -\sin\ldots \\ \sin\ldots & \cos\ldots \end{pmatrix}$
$\rho(y) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
ρ is faithfull by definition of representation because transformations $\rho(x)$ and $\rho(y)$ don't fix vertices.

3. Permutation representation of S_n
$\rho : S_n \longrightarrow GL_n(\mathbb{F})$
$\rho(\sigma)e_i = e_{\sigma(i)}$ is a faithful representation
proof: Show $ker(\rho) = \langle 1 \rangle$
Let $\sigma \in S_n$ be such that
$\rho(\sigma) \in I_n$
$\Leftrightarrow \rho(\sigma)e_i = e_{\sigma(i)} = e_i$
$\Leftrightarrow \sigma(i) = i \ \forall i$
$\Leftrightarrow \sigma = (1)$
$\Leftrightarrow ker(\rho) = \langle (1) \rangle$ faithful.

4. 1-dim rep D_{2n}
$\rho : D_{2n} \longrightarrow GL_1(\mathbb{C}) = \mathbb{C}^*$
$\rho(x) = 1$
$\rho(y) = -1$
is not faithfull
proof: $\langle x \rangle \subseteq Ker(\rho) \quad \rho(x) = 1$

Is $\ker(\rho) = \langle x \rangle$ or are there other elements in $\ker(\rho)$

No other elements

Let $g \in D_{2n}$

$g = x^i y^j \quad i \leq n \quad j \leq 2$

$\rho(x^i y^j) = \rho(x)^i \rho(y)^j$

$= 1^i (-1)^j$

$= (-1)^j$

Suppose $g \in \ker(\rho)$ then $(-1)^j = 1 \Rightarrow j = 0$

$\Rightarrow \underset{\substack{\wr\wr \\ C_n}}{\langle x \rangle} = \ker(\rho)$

Menu:

1. Given any finite group G, find all $\mathbb{C}$-reps of G upto conjugancy
2. Charter Theory
3. Construct representations using tensor products
4. Real representation theory.

The Road map.

$$\begin{array}{lcl} G & \longrightarrow & \text{Construct group ring } \mathbb{F}[G] \text{ and structure} \\ \downarrow & & \downarrow \\ \mathbb{F}\text{-rep of } G & & \text{Classifying matrices over } \mathbb{F}[G] \end{array}$$

Semisimple rings, modules and algebras

Definition:

A ring R is a set with two operations $+$ and $\times$ such that the following axioms holds $\forall a, b \in R$

1. $a+b = b+a$
2. $(a+b)+c = a+(b+c) = a+b+c$
3. $\exists 0 \in R$ st $a+0 = a = 0+a$
4. $\forall a \in R \ \exists -a \in R$ st $a+(-a) = 0 = (-a)+a$
5. $\exists 1 \in R$ st $1 \cdot a = a = 1 \cdot a \ \forall a$
6. $a(bc) = (ab)c = abc$
7. $a(b+c) = ab+ac$
8. $(a+b)c = ac+bc$

If, also $ab = ba$, then R is a commutative ring.

Examples:

i). Commutative rings

$R = \mathbb{Z}, \ \mathbb{F}, \ \mathbb{Z}_n, \ \mathbb{F}[x], \ \mathbb{F}[x]/I$

2) Non commutative rings

- Matrix ring $M_n(R)$ where R is any ring, $n \geq 2$
- Upper/lower triangular matrices
- Group rings: $\mathbb{F}[G]$ are generally non-commutative

3). Products of rings form a ring

Let R, S be rings then the direct product, $R \times S$

$(r_1, s_1) + (r_2, s_2) = (r_1 + r_2, s_1 + s_2)$

$(r_1, s_1)(r_2, s_2) = (r_1 r_2, s_1 s_2)$

$R \times S$ is also a ring.

Definition:

A subset $I \subseteq R$ is called a (left) ideal, $I \triangleleft R$, if

1. $(I, +)$ is a subgroup of R
2. $\forall x \in I, \forall r \in R \; rx \in I$

Examples

1. $I = n\mathbb{Z} \subset \mathbb{Z}$

   $I = 2\mathbb{Z} \subset \mathbb{Z}$

2. $(p(x)) \subset \mathbb{F}[x]$

Definition:

Let R and S be 2 rings

If the map $\phi : R \longrightarrow S$ satisfies

$$\left.\begin{array}{l} 1\ \phi(r_1 + r_2) = \phi(r_1) + \phi(r_2) \\ 2\ \phi(r_1 r_2) = \phi(r_1)\phi(r_2) \end{array}\right\} \forall r_1, r_2 \in R.$$

Then ϕ is called a ring homomorphism.

Q: $\phi(I_r) = I_s$ always?

No unless ϕ is surjective (epimorphism) or S is an ID (commutative)

Counter example:

$\phi : M_2(\mathbb{F}) \longrightarrow M_3(\mathbb{F})$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$I_2 \not\longmapsto I_3$$

Definition:

If ϕ is bijective then ϕ is called a ring isomorphism

ie $\exists \phi^{-1} : S \longrightarrow R$ such that

$\phi \circ \phi^{-1} = I_S$

$\phi^{-1} \circ \phi = I_R$.

Example:

1. $\phi: \mathbb{Z} \to \mathbb{Z}_n$
$x \mapsto x \bmod n$
$Ker(\phi) = n\mathbb{Z}$
$Im(\phi) = \mathbb{Z}_n$

2. Are there any rings homomorphism from $\mathbb{Z}_n \to \mathbb{Z}$?
No.

Modules over rings

Definition:
let R be a ring, a left R-module M is an abelian group combined with a map
$\varphi: R \times M \to M$
$\varphi(r,m) = rm$
satisfying
1. $1 \cdot m = m$
2. $r(m+n) = rm + rn$
3. $(r+s)m = rm + sm$
4. $(rs)m = r(sm)$

Definition:
External direct sum of modules
Let M and N be 2 modules over R then $M \oplus N$ is a module over R constructed as follows:
As a set $M \oplus N = M \times N$ (m,n)
$(m_1, m_2) + (m_2, n_2) = (m_1 + m_2, n_1 + n_2)$
$\lambda(m,n) = (\lambda m, \lambda n)$.

Definition:

N is a submodule of M when

1. $0 \in N$
2. $n_1 + n_2 \in N \quad \forall n_1, n_2 \in N$
3. $\lambda \cdot n \in N \quad \forall \lambda \in R \ \forall n \in N$

N closed under scalar multiplication.

Example:

1. Any vector subspace = submodule
2. M = abelian groups, submodules = subgroups.

Note: $I \triangleleft R$ is an (left) R-submodule of ${}_R R$.

Definition:

Let M and N be 2 R-modules. We say that they are homomorphic/ $\exists$ an R-module homomorphism if $\exists$ map

$\phi : M \to N$ st

1 $\phi(0) = 0$

2 $\phi(m_1 + m_2) = \phi(m_1) + \phi(m_2)$

3 $\phi(\lambda m) = \lambda \phi(m)$ $\forall \lambda \in R$ $\forall m_1, m_2 \in M$.

We say that $M \cong_R N$ are isomorphic if

4 ϕ is bijective

ie $\exists \phi^{-1} : N \to M$ st $\phi \circ \phi^{-1} = Id_N$ $\phi^{-1} \circ \phi = Id_M$.

Examples of left modules and submodules.

0. $\mathbb{Q}$ is a $\mathbb{Z}$-module $\mathbb{Z} \times \mathbb{Q} \to \mathbb{Q}$.

1. Any vector space V over $\mathbb{F}$ is an $\mathbb{F}$-module

2. Any finite abelian group over $\mathbb{Z}$ is a $\mathbb{Z}$-module

$A = \mathbb{Z}_{n_1} \times \ldots \times \mathbb{Z}_{n_r}$ (no basis)

eg $\mathbb{Z} \times \mathbb{Z}_3 \to \mathbb{Z}_3$.

$\bar{0} = \bar{3} = \bar{6} \ldots$

3. R any ring, $M = R^n = \oplus R$ n summands. $n \geq 1$ is an R-module

$R \times R^n \to R^n$

$R = \mathbb{Z}$ $\therefore$ $\mathbb{Z}$ is a $\mathbb{Z}$-module.

4 If $I \triangleleft R$, then I is a left submodule/ideal over R

$R \times I \to I$

5 Let R be any ring, $a \in R$ then define the principle ideal gen. by a. $(a) = \langle ra : r \in R \rangle = Ra$, is a left submodule over R

6 $M_n(R)$ is an R-module and an $M_n(R)$-module

$R \times M_n(R) \to M_n(R)$ "matrices as

$M_n(R) \times M_n(R) \to M_n(R)$ vectors"

7. Quaternions: the real vector space generated by $\langle i,j \rangle$ $\langle 1, i, j, k \rangle$

$\mathbb{H} = \langle a \cdot 1 + bi + cj + dk \mid a,b,c,d \in \mathbb{R} \rangle$ $i^2 = j^2 = 1$ $ij = k = -ji$

$\mathbb{H}$ forms a ring which is a vector space over $\mathbb{R}$ (dim 4) and a $\mathbb{C}$-vector space of dim 2 :-

Proof: Let $\alpha \in \mathbb{H}$

$\alpha = a \cdot 1 + bi + cj + dk$

$= (a\cdot 1 + bi) + (cj + dij)$
$= (a+bi) + (c+di)j$
$= z_1 + z_2 j$

By setting $\mathbb{C} = \langle x+iy : x, y \in \mathbb{R} \rangle$
$\mathbb{H} = \mathbb{C} + \mathbb{C}j$, basis $Sp_{\mathbb{R}}\langle i, j \rangle$

Left $\mathbb{C}$-modules differ from right $\mathbb{C}$-mods for $\mathbb{H}$:-
$jz = \bar{z}j$

Check: Let $z = a+bi$ $\bar{z} = a - bi$
$jz = ja + bji = aj - bij$
$\bar{z}j = (a - bi)j = aj - bij$

Definition:
An R-module M is finitely generated if $\exists$ finitely many elements $\langle m_1, \dots, m_k \rangle$ st any $m \in M$ can be written as
$$m = \sum_{i=1}^{k} \lambda_i m_i \quad \lambda_i \in R.$$

Examples of finitely generated modules.

1. Any vector space of finite dimension, over $\mathbb{F}$ is finitely generated by its basis
   $\equiv$ Any module over a division ring.

2. $M_n(\mathbb{F})$ is f.g over $\mathbb{F}$ by E_{ij}

3. Any fg abelian group as a $\mathbb{Z}$-module
   $A = \mathbb{Z}^r \times \mathbb{Z}_{n_1} \times \dots \times \mathbb{Z}_{n_k}$ primary decomposition thm. Just look at commutative

4. $\mathbb{F}[x]$ is not f.g over $\mathbb{F}$

6. $\mathbb{Q}$ is not fg over $\mathbb{Z}$

proof: Suppose $\mathbb{Q}$ is fg by $\langle q_1, \dots, q_s \rangle$.
Then let n be st it is coprime to all the denominators.
Then $1/n$ can't be written as a linear combination of $\langle q_1, \dots, q_s \rangle$.

Note: modules in this course will be fg! 

Definition:
Let ~~$\boxtimes$~~ $N \leq M$ be an R-submodule
Then define $M/N = \langle x + N : x \in M \rangle$

Rule of equality of cosets: $x + N = y + N$ iff $x - y \in N$.

Proposition / Definition:
M/N is an R-module called the quotient module

proof:
Obviously M/N is an abelian additive group
$(x+N) + (y+N) = (x+y) + N$ id $= N$.
R-action? Define as follows
$\lambda(x+N) = \lambda x + N$ $\lambda \in R$.
is this well defined.
Let $x + N = y + N$
$x + y \in N$
$\lambda(x - y) \in N$ $N \leq M$
$\lambda x - \lambda y \in N$
$\lambda x + N = \lambda y + N$
well define
$\therefore$ M/N is an R-module.

Example:
$I \triangleleft R$ then the additive quotient group R/I is an R-module by $r(a + I) = ra + I$ $\forall a \in R$.

Definition:
If $N_1, N_2 \leq M$ are R-submodules then define their sum by
$N_1 + N_2 = \langle x + y : x \in N_1, y \in N_2 \rangle$
If $N_1 \cap N_2 = \langle 0 \rangle$, then we call the sum an (internal) direct sum.
Denoted $N_1 \oplus N_2$ where $N_1 \oplus N_2$ is an R-module.

Definition:
Say $N \leq M$ is a direct summand of M if $\exists N' \leq M$ st
$M = N \oplus N'$

Definition:
If $I \triangleleft R$ is both a left and right ideal then I is called a 2-sided ideal.

Definition:
A ring R is called simple if its only 2-sided ideals are $<0>$ and R.

Example of a simple ring.
Any field $\mathbb{F}$ is a ~~ring~~ simple ~~R~~ ring because only 2-sided ideals are $<0>$ and ~~R~~ $\mathbb{F}$.

Proposition:
Let R be a ring, then the 2-sided ideals of $M_n(R)$ are of the form $M_n(I)$ where I is a 2-sided ideal of R.

Proof Ex 2.

Consequences of proposition.

I If $R = \mathbb{F}$ then the 2-sided ideals of $M_n(\mathbb{F})$ are what?
The 2-sided ideal of $M_n(\mathbb{F})$ are $<0>$ and $M_n(\mathbb{F})$.
$\therefore$ $M_n(\mathbb{F})$ is a simple ring.

II The same holds if $R = D$ is a division ring.
$M_n(D)$ is a simple ring.

III Does $M_n(\mathbb{F})$ have any non trivial left ideals?
If we take $c_j = \left\{ \begin{pmatrix} 0 & \begin{matrix} a_{1j} \\ \vdots \\ a_{nj} \end{matrix} & 0 \end{pmatrix} : a_{ij} \in \mathbb{F} \right\}$

then c_j is a left $M_n(\mathbb{F})$-module because $\therefore$ $C_j \triangleleft M_n$.
$M_n(\mathbb{F}) \cancel{\times C_j} \longrightarrow C'_j$

eg $M_2(\mathbb{F})$ $\quad C_2 = \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}$ $\quad A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & a+2b \\ 0 & 3a+4b \end{pmatrix} \in C_2$.

$\therefore$ C_j absorbs $R = M_n(\mathbb{F})$ action!
(In fact these are essentially the only left ideals)

Note: Note if we want to consider right ideals just look at rows.

Definition:

An R-module (left) M is called simple if its only submodules are $\langle 0 \rangle$ and M.

Definition:

An R-module M is called semisimple if it can be written as $M = \bigoplus_{i \in I} M_i$ where M_i's are simple modules; for some ~~[illegible]~~ I.

~~Definition:~~

~~Let M be a (left) R-module, then M is call~~

Examples

1. $\mathbb{F}$ considered as an $\mathbb{F}$-module (1-dim vs) is simple ($\nexists$ any vector subspaces)

2. $C_j = \begin{array}{c} \begin{pmatrix} 0 & \begin{bmatrix} \\ \\ \end{bmatrix} & 0 \end{pmatrix} \\ j^{th} \end{array} \leq M_n(\mathbb{F})$ considered as left $M_n(\mathbb{F})$-modules ($M_n(\mathbb{F}) \times C_j \to C_j$) is a simple module.

3. $M_n(\mathbb{F})$ considered as an $M_n(\mathbb{F})$-module is semisimple
 $\exists$ submodules $C_j \leq M_n(\mathbb{F})$ which are simple
 ($M_n(\mathbb{F})$ is simple as a ring)
 $M_n(\mathbb{F}) = \bigoplus C_j$

4. C_k is not simple as an $\mathbb{F}$-module.
 It is semisimple $C_k \cong \mathbb{F}^n$

5. $\mathbb{F}^n$ is semisimple as an $\mathbb{F}$-module
 Let $\langle e_1, \ldots, e_n \rangle$ be the standard basis
 Then $\mathbb{F}^n \cong \bigoplus_{i=1}^{n} \mathbb{F}e_i = \underbrace{\mathbb{F}e_1}_{\substack{\cong \\ \mathbb{F} \\ \text{simple}/\mathbb{F}}} \oplus \ldots \oplus \underbrace{\mathbb{F}e_n}_{\substack{\cong \\ \mathbb{F} \\ \text{simple}/\mathbb{F}}}$

6. $\mathbb{Z}$ considered as a module over itself is not simple because $\exists$ submodules $n\mathbb{Z} \leq \mathbb{Z}$.

7. $\mathbb{Q}$ considered as a $\mathbb{Z}$-module is not simple,
 $\exists$ a submodule $\mathbb{Z} \leq \mathbb{Q}$

Beware! Some modules are neither simple nor semisimple.

eg. For example $\mathbb{Z}/4\mathbb{Z}$ as a $\mathbb{Z}$-module

Its not simple : $\mathbb{Z}/2\mathbb{Z} \leq \mathbb{Z}/4\mathbb{Z}$

But $\mathbb{Z}/2\mathbb{Z} = \mathbb{Z} \oplus \underset{\text{not simple.}}{\dots ?}$

Real Question:

Is $\mathbb{Z}$ semisimple ? as a $\mathbb{Z}$-module?

$n\mathbb{Z} \leq \mathbb{Z}$ $\quad M = M_1 \oplus \dots \oplus M_n$

$\mathbb{Z}/4\mathbb{Z} \leq \mathbb{Z}$

$\mathbb{Z}/2\mathbb{Z} \oplus$??? so no.

Recall:

Let M and N be 2 R-mods

Then $\varphi: M \to N$ is a R-mod hom if

$\varphi(0) = 0$

$\varphi(m_1 + m_2) = \varphi(m_1) + \varphi(m_2)$

$\varphi(rm) = r\varphi(m)$.

Definition:

1 $\mathrm{Ker}(\varphi) = \{m \in M : \varphi(m) = 0\} \leq M$

$\mathrm{Ker}(\varphi) = \langle 0 \rangle \iff \varphi$ injective

$\mathrm{Ker}(\varphi)$ measures φ injectivity

2. $\mathrm{Im}(\varphi) = \{\varphi(m) \in N : m \in M\} \leq N$

$\mathrm{Im}(\varphi) = N \iff \varphi$ surjective

$\mathrm{Im}(\varphi)$ measures surjectivity.

Proposition:

1 $\mathrm{Ker}(\varphi) \leq M$

2 $\mathrm{Im}(\varphi) \leq N$

proof: 1. $\varphi(0) = 0 \Rightarrow 0 \in \mathrm{Ker}$ and $0 \in \mathrm{Im}(\varphi)$

...

Schur's Lemma (VII)

Let M and N be 2 non-zero, simple modules over R.
Let $\varphi: M \to N$ be an R-module homomorphism
Then either
1. φ is an isomorphism

or 2. $\varphi = \underline{0}$

Proof:
Suffices to show that if $\varphi \neq 0$, then φ is an isomorphism
So suppose $\varphi \neq 0$.
Injectivity: $Ker(\varphi) \leq M$ but since M is simple
$Ker(\varphi) = \langle 0 \rangle$ or M
since $\varphi \neq 0$ $Ker(\varphi) \neq M \Rightarrow Ker(\varphi) = \langle 0 \rangle$
$\Rightarrow \varphi$ injective.
Surjectivity: $Im(\varphi) \leq N$.
By simplicity of N, $Im(\varphi) = \langle 0 \rangle$ or N
But since $\varphi \neq 0 \Rightarrow Im(\varphi) \neq \langle 0 \rangle \Rightarrow Im(\varphi) = N$
$\Rightarrow \varphi$ surjective.
$\Rightarrow \varphi$ is an isomorphism.

Definition:
Let M be an R-module, then define the endomorphism of M by $End_R(M) := Hom_R(M, M)$
$= \{\varphi: M \to M : \varphi \text{ is } R\text{-mod homomorphism}\}$.

* $End_R(M)$ encodes useful info about M.

Proposition:
$End_R(M)$ is naturally a ring.

proof: Let $\alpha, \beta \in End_R(M)$
$(\alpha + \beta)(m) = \alpha(m) + \beta(m) \quad \forall m \in M$ (addition)
Multiplication is composition of maps
$End_R(M) \times End_R(M) \longrightarrow End_R(M)$
$(\alpha, \beta)(m) \longmapsto \alpha \circ (\beta(m))$
zero: $0(m) = 0 \quad \forall m \in M$ Unit: $Id(m) = m \quad \forall m \in M$

- Can we consider $End_R(M)$ as module over R?
Yes iff R is commutative Exercise

Definition:
D is called a division ring if $\forall x \in D$ $x \neq 0$ $\exists y$ st $xy = 1$
(skew field)
Except elements do not have to commute).

Examples:
1. Any field $\mathbb{F}$ is a division ring.
2. $\mathbb{Z}$ is not a division ring.
3. $\mathbb{F}[x]$ is not a division ring.
4. $M_n(\mathbb{F})$ is not a division ring
5. The quaternions $\mathbb{H}$ over $\mathbb{R}$ is a division ring
Let $\alpha \in \mathbb{H}$, $\alpha = a + bi + cj + dk$ $\alpha \neq 0$
define its conjugate $\bar{\alpha} = a - bi - cj - dk$
define $Norm(\alpha) = \alpha\bar{\alpha} = a^2 + b^2 + c^2 + d^2$
$\alpha^{-1} = \dfrac{\bar{\alpha}}{Norm(\alpha)}$

5. $\left(\dfrac{3,-1}{\mathbb{Q}}\right)$ is a $\mathbb{Q}$-vector space of dim 4 with basis $\langle 1, i, j, k \rangle$ with $i^2 = 3$ $j^2 = -1$, $ik = -k$.

Note:
- $M_n(D)$ is a semisimple module over $M_n(D) = \oplus C_j$
- Let $I \triangleleft M_n(D)$, then $I = \langle 0 \rangle$ or $M_n(D)$ $\therefore$ $M_n(D)$ is a simple ring
Let $\alpha \in I$ $\exists \alpha^{-1}$ st $\alpha\alpha^{-1} = 1$.

Schur's Lemma (V2)

Let M be a non-zero simple R-module
Then $End_R(M)$ is a division ring.

proof: Let $\alpha \in End_R(M)$ such that $\alpha \neq 0$
$\alpha : M \longrightarrow M$ non zero homomorphism
By Schur's lemma V1 α is an isomorphism
$\Rightarrow \exists \alpha^{-1}$ st $\alpha \circ \alpha^{-1} = \alpha^{-1} \circ \alpha = Id_M$

Note: There is an even more powerful version Schur's Lemma, which we'll prove later that applies to $\mathbb{F}(G)$-modules.

* $End_R(M)$ is a tool to measure simplicity of M.

Examples of applications of $End_R(M)$

1. Let $\mathbb{F}$ be a field, define the $\mathbb{F}$-linear map $\varphi_\lambda : \mathbb{F} \to \mathbb{F}$
(simple $\mathbb{F}$-module $\mathbb{F}$) $x \to \lambda x$

Then $End_{\mathbb{F}}(\mathbb{F}) \cong \mathbb{F}$ (division ring)
$\varphi_\lambda \mapsto \lambda$.

2. $M = \mathbb{F} \times \mathbb{F}$ as an $\mathbb{F}$-module, is not simple $\mathbb{F}^2 = \mathbb{F}e_1 \oplus \mathbb{F}e_2$
$End_{\mathbb{F}}(\mathbb{F}^2)$ is not a division ring
$End_{\mathbb{F}}(\mathbb{F}^2) = M_2(\mathbb{F})$

3. Remember $C_j = \{ \sum_{k=1}^{n} c_k E_{kj} : c_k \in \mathbb{F} \} = \begin{pmatrix} 0 & \begin{matrix} c_1 \\ \vdots \\ c_n \end{matrix} & 0 \end{pmatrix} \leq M_n(\mathbb{F})$
is a left $M_n(\mathbb{F})$ module.
and is simple.
$End_{M_n(\mathbb{F})}(C_j)$ = division ring.
pf: 1 Compute $End_{M_n(\mathbb{F})}(C_j) = \{ f : C_j \to C_j \mid f \text{ is } M_n(\mathbb{F})\text{-homomorphism} \}$
and hope that $End_{M_n(\mathbb{F})}(C_j) \cong \mathbb{F}$!
By confirming C_j is simple $\Rightarrow M_n(\mathbb{F}) = \bigoplus_{j=1}^{n} C_j$ semisimple.
2 Choose canonical basis $\langle e_1, \dots, e_n \rangle$
$\therefore \mathbb{F}^n = \bigoplus_{i=1}^{n} \mathbb{F}e_i$
3. Identify C_j with $\mathbb{F}^n$ as $\mathbb{F}$-modules
$f : C_j \to \mathbb{F}^n$
$\begin{pmatrix} 0 & \begin{matrix} a_{1j} \\ \vdots \\ a_{nj} \end{matrix} & 0 \end{pmatrix} \mapsto \begin{pmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{pmatrix} = a_{ij} e_i$

4. Since f is a linear map $\Rightarrow$ can be represented by matrix.
$\exists \Phi = (\varphi_{ij}) \in M_n(\mathbb{F})$ st
$f_\lambda \begin{pmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{pmatrix} = \begin{pmatrix} \varphi_{11} & \cdots & \\ \vdots & \ddots & \vdots \\ & \cdots & \varphi_{nn} \end{pmatrix} \begin{pmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{pmatrix} = \begin{pmatrix} \lambda a_{1j} \\ \vdots \\ \lambda a_{nj} \end{pmatrix}$ for some $\lambda \in \mathbb{F}$.

$f_\lambda : C_j \to C_j$

5. ~~Using~~ ~~prowant~~ $\Phi A = A \Phi \ \forall A \in M_n(\mathbb{F})$ and the only matrix that does this commuting is the scalar matrix
$\therefore \Phi = \begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix} \quad \lambda \in \mathbb{F}$

6. $\mathrm{End}_{M_n(\mathbb{F})}(C_j') = \{f : C_j \longrightarrow C_j' \mid f \text{ commutes with all } M_n(\mathbb{F})\}$.
$= \{B \in M_n(\mathbb{F}) \mid AB = BA \quad \forall A \in M_n(\mathbb{F})\}$.
$= \{\Phi = (\lambda \cdots \lambda) \quad \lambda \in \mathbb{F}\}$.
$\therefore \mathrm{End}_{M_n(\mathbb{F})}(C_j) \cong \mathbb{F}$.

Question!
$\mathrm{End}_{\mathbb{F}}(C_j) \cong \mathbb{F}^n$ or $M_n(\mathbb{F})$
1 C_j as an $\mathbb{F}$-module semi simple $C_j \cong \mathbb{F}$
2. $\mathrm{End}_{\mathbb{F}}(C_j) = M_n(\mathbb{F})$.

Exercise

In V2 of Schur's Lemma, the converse statement does not generally hold.
ie $\mathrm{End}_R(M) = D \Rightarrow M$ is simple as shown by following example.

4. $\mathbb{Q}$ as a $\mathbb{Z}$-module is not simple because $\exists$ submodule $\mathbb{Z} \leq \mathbb{Q}$
Compute $\mathrm{End}_{\mathbb{Z}}(\mathbb{Q}) = \{f : \mathbb{Q} \rightarrow \mathbb{Q} \mid f \ \mathbb{Z}\text{-hom}\}$.
Let $f \in \mathrm{End}_{\mathbb{Z}}(\mathbb{Q})$
Take $n \in \mathbb{Z}$
$f(n) = f(n \cdot 1) = n f(1)$
If $n \neq 0$ $f(\frac{1}{n}) = \frac{1}{n} f(1)$
$f(\frac{n}{n}) = f(1) = n f(\frac{1}{n})$
If $\frac{m}{n} \in \mathbb{Q}$ $f\left(\frac{m}{n}\right) = m f\left(\frac{1}{n}\right) = \frac{m}{n} f(1)$

So $\forall q \in Q$ $f(q) = q f(1)$
Hence $\mathrm{End}_{\mathbb{Z}}(\mathbb{Q}) \xrightarrow{\cong} \mathbb{Q}$ which is a division ring
$f \longmapsto f(1)$
But $\mathbb{Q}$ is not simple as a $\mathbb{Z}$-module.

Proposition:
If M is a simple module over $_RR$, then M is generated by any $m \in M$ where $m \neq 0$

proof:
Let $0 \neq m \in M$, $Rm \leq M$ is a ~~simple~~ submodule
$Rm \neq 0$ since $m \neq 0$
Since M is simple $\Rightarrow Rm = M$.

Example: Characterise all simple $\mathbb{Z}$ modules.

Let M be a simple $\mathbb{Z}$ module

Let $m \in M$ be st $m \neq 0$

Define $\varphi: \mathbb{Z} \to M$

$n \mapsto n \cdot m$

By last proposition, φ is surjective.

$\ker(\varphi) \triangleleft \mathbb{Z} \Rightarrow \ker(\varphi) = n\mathbb{Z}$

$\therefore \varphi$ induces an isomorphism

$\varphi_*: \mathbb{Z}/n\mathbb{Z} \xrightarrow{\cong} M$

case 1: $n = n_1 n_2$ $(n_1, n_2) = 1$ coprime

$\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/n_1\mathbb{Z} \times \mathbb{Z}/n_2\mathbb{Z}$

$\therefore \mathbb{Z}/n\mathbb{Z}$ is not simple as $\mathbb{Z}/n_1\mathbb{Z} \leq \mathbb{Z}/n\mathbb{Z}$.

Case 2 $n = p^k$ p = prime $k > 1$

$\mathbb{Z}/p^k\mathbb{Z}$ is not simple because $\mathbb{Z}/p\mathbb{Z} \leq \mathbb{Z}/p^k\mathbb{Z}$ is a proper submod.

case 3. $n = p$

$\mathbb{Z}/p\mathbb{Z} \cong \mathbb{F}_p$ char(p)

The simple $\mathbb{Z}$-mods are exactly the $\mathbb{Z}/p\mathbb{Z}$ as $\mathbb{Z}$-mods

So the semisimple $\mathbb{Z}$-mods look like

$M = \mathbb{Z}/p_1\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/p_k\mathbb{Z}$

Classification Theorem of semisimple modules

Let M be a finitely generated module. Then the following are equivalent

1. M is semisimple
2. $\forall N \leq M$ $\exists$ complimentary submodule $N' \leq M$ such that $M = N \oplus N'$

Example:

$M = \mathbb{F}^n$ as an $\mathbb{F}$-module

Then for any sub vector space $V \leq M = \mathbb{F}^n$ $\exists$ V' st $\mathbb{F}^n = V \oplus V'$

proof:

$1 \Rightarrow 2$ $M = \bigoplus_{i \in I} M_i$ for a finite indexing set I.

Let $N \lneq M$ be a proper submodule

Take $J \subsetneq I$: $J \neq \emptyset$ be a maximal subset

$M^* = N \oplus \bigoplus_{i\in J} M_i$ ie $N \cap \bigoplus_{i\in J} M_i = \langle 0 \rangle$

Show $M = M^* \Rightarrow$ can take $N' = \bigoplus_{i\in J} M_i$

Suppose $M \neq M^*$, take $i \in I / J$ and consider $N \cap M_i$

1. If $N \cap M_i = \langle 0 \rangle \Rightarrow$ can add M_i to $\bigoplus_{i\in J} M_i$

$\Rightarrow J$ is not maximal. contradiction

2. So $N \cap M_i = \langle M_i \rangle$

$\Rightarrow M_i \subset N \subset M^* \quad \forall i \in I/J$

$\Rightarrow$ contradiction $M \neq M^* \Rightarrow M = M^*$

$\textcircled{2} \Rightarrow \textcircled{1}$ $\forall N \leq M \; \exists N' \leq M$ st $M = N \oplus N' = \bigoplus M_i$

$\Rightarrow N$ and N' are semisimple *

Lemma: Any non-zero M satisfying * contains a non-zero simple submodule

Let $M_0 \leq M$ be a submodule = sum of all simple submodules

Show $M = M_0$

By lemma $M_0 \neq \langle 0 \rangle$. Suppose $M \neq M_0$

$\Rightarrow \exists W \neq \langle 0 \rangle$ st $M = M_0 \oplus W$

By lemma $W \neq \langle 0 \rangle \Rightarrow W$ contains a non-zero simple submodule contradiction.

$\therefore M = M_0 = \bigoplus M_i$, M_i simple submodules and sum is finite as M is fg.

Definition:

Let $N \leq M$ be a submodule, then N is called 'maximal' if $\forall K \leq M$ st $N \leq K \leq M \Rightarrow K = N$ or $K = M$.

Example:

Let R be an R-module, any maximal ideal is a maximal submod of R.

Fact 1: A submodule of $N \leq M$ is maximal $\Leftrightarrow M/N$ is simple

Fact 2: Any proper submod of a fg mod is contained in a max submod.

Fact 2 fails if not fg

proof of lemma:

1. Let M be st $\forall N \leq M \; \exists N'$ st $M = N \oplus N'$
2. Take $v \in M$ $v \neq 0$ and look at mod homomorphism
$\varphi : {}_R R \twoheadrightarrow Rv \leq M$
$\lambda \longmapsto \lambda v$
3. φ is surjective $\therefore$ $Im(\varphi) = Rv$ and $Ker(\varphi) \triangleleft R$ is an R-submodule (ideal)
4. By fact 2, we know $Ker(\varphi) \leq I$ where I is maximal idea/submod of R.
5. $\therefore$ By definition, Iv is maximal submodule of Rv
6. By construction: $M = Iv' \oplus M'$ $\cap Rv$
$\cap Rv$ $Rv = Iv \oplus (M' \cap Rv)$
$x = y + z$ unique direct sum.
7. The module $M' \cap Rv$ is simple submodule of M:
$M' \cap Rv \cong Rv/_{Iv} \cong R/_{I_R} \cong$ field, by fact 1 quotient is simple as I is maximal.
maximal.

Proposition

Every submodule and every quotient module of a semisimple module M is semisimple.

proof: 1. Let M be semisimple and $N \leq M$ and show N is semisimple.
Let $W \leq N \Rightarrow W \leq M$ also
$\therefore M = W \oplus W'$
$M \cap N = W \cap N \oplus W' \cap N$
$N = W \oplus (W' \cap N)$
$\overset{\psi}{n} = \overset{\psi}{w} + \overset{\psi}{w'}$ unique
$\therefore$ N is semisimple by characterisation Thm.
2. Assume $M = W \oplus W'$ is semisimple
and $N \leq W \leq M$
$\therefore W/_N \leq M/_N$

as before $M/_N = W/_N \oplus \dfrac{W'}{W' \cap N}$

By characterisation thm $M/_N$ is semisimple.

New concept: an algebra A is a ring which is automatically a VS over $\mathbb{F}$

Point: Look at modules over A which are also gonna have a VS / $\mathbb{F}$.

Goal: Classify all semisimple algebras over division rings $\cong M_{n_i}(D_i)$

Definition: An algebra A over $\mathbb{F}$ is a ring which has the structure of a VS over $\mathbb{F}$, such that.

1. $a + b = (a + b$

A-ring element addition $\longleftrightarrow$ A = VS over $\mathbb{F}$ vector addition (= mod element addition)

2. $\lambda(ab) = \cancel{\lambda(a\lambda)}(\lambda a)b = a(\lambda b)$

$\forall \lambda \in \mathbb{F} \quad \forall a, b \in A$.

Examples

1. $\mathbb{F}$ over $\mathbb{F}$ is an $\mathbb{F}$-algebra
2. $\mathbb{F}[x]$ is an $\mathbb{F}$-algebra
3. $\mathbb{F}[x]/_I$ is an $\mathbb{F}$-algebra
4. $M_n(\mathbb{F})$ is a $\mathbb{F}$-algebra with scalar ~~algebra~~ multiplication
5. $\mathbb{H}$ is a $\mathbb{R}$-algebra but not a $\mathbb{C}$-algebra since $jz = \bar{z}j$.
6. (General) Every ring is a $\mathbb{Z}$-algebra

$End_R(M)$ is a R-algebra if R is commutative ring

$\varphi : R \longrightarrow End_R(M)$

$a \longmapsto a \cdot Id_M$.

Definition: An algebra A is finite dimensional iff its dimension as an $\mathbb{F}$-vector space is finite.

Note: $\varphi : A_1 \longrightarrow A_2$ algebra homomorphism $\cong$ ring homomorphism $\cong$ linear map.

"Proposition"

Let A be an algebra, an A-module, M is a module over A $\therefore$ it is automatically an $\mathbb{F}$-vector space.

Definition:
An algebra is semisimple if any non-zero fg A-module is semisimple

Example:
1. $\mathbb{F}$ a field is a semisimple algebra since any fg $\mathbb{F}$-module = VS ie isomorphic to $\mathbb{F}^n \cong \oplus \mathbb{F}e_i$ which is semisimple.

"Proposition" characterisation Theorem of Semisimple algebras.
A is semisimple (as ring) iff A viewed as an A-module is semisimple.

proof: $\Rightarrow$ Trivial by definition.
$\Leftarrow$ 1. Suppose A is semisimple as a module
2. Let $M \neq \langle 0 \rangle$ be an A-module (another module)
3. Choose a set of generators $\langle m_1, \dots, m_r \rangle$
4. Let $\varphi : A^r \twoheadrightarrow M$ be a homomorphism of A-modules.
$(a_1, \dots, a_r) \mapsto \sum a_i m_i$
where $A^r = \underbrace{A \oplus \dots \oplus A}_{r \text{ times}}$
5. Since A is semisimple A-module
$\Rightarrow A = S_1 \oplus \dots \oplus S_t$ S_i simple $\forall i$
$A^r = (S_1 \oplus \dots \oplus S_t) \oplus \dots \oplus (S_1 \oplus \dots \oplus S_t)$
6 Since φ is surjective as the m_i generate M over A
7 By 1st iso Thm $Im(\varphi) \cong M \cong A^r / Ker(\varphi)$
$\therefore$ M is semisimple since its a quotient of a semisimple
So A is semisimple as a ring by definition.

Examples of consequences:
1. $\mathbb{D}$ a division algebra, then $M_n(\mathbb{D})$ is semisimple $\mathbb{D}$-algebra.
2. Let $A = \mathbb{Z}/p^2\mathbb{Z}$ be an algebra over $\mathbb{F}_p = (\mathbb{Z}/p\mathbb{Z})$, A is not semisimple as no complement for $\mathbb{Z}/p\mathbb{Z} \oplus ? \cong \mathbb{Z}/p^2\mathbb{Z}$.

Proposition.
Let A be a semisimple algebra over $\mathbb{F}$ such that
$A = A_1 \oplus \dots \oplus A_r$ where A_i simple $\forall i$
Then any simple A-module $S \cong A_i$ for some i

proof: 1. Let S be a simple A-module and show $S \cong A_i$

2. Let $s \in S$, $s \neq 0$, then $\varphi : A \twoheadrightarrow As = S$

$a \mapsto as$.

3. $\varphi \neq 0$ because $s \neq 0$

Let $\varphi_i = \varphi|_{A_i} : A_i \longrightarrow A_i s = S$

$a_i \mapsto a_i s$

4 $Im(\varphi) \leq S$ and $\varphi_i \neq 0$ $\forall i$ otherwise $\varphi = 0$

must have some non-trivial φ_i-map.

5. Let i be st $\varphi_i : A_i \longrightarrow S$

6. Since A_i and S are simple and $\varphi_i \neq 0$, by Scher's Lemma

$\Rightarrow \varphi_i : A_i \xrightarrow{\cong} S$ is isomorphism.

Proposition:

Let A be a semisimple algebra and $\{S_i\}$ be a collection of simple modules

Let M be an A-module $\Rightarrow M$ is semisimple

ie $M = S_1^{n_1} \oplus \dots \oplus S_r^{n_r}$ and decomposition is unique

ie if $M = T_1^{m_1} \oplus \dots \oplus T_s^{m_s}$

$\Rightarrow r = s$ and $T_i \cong S_i$ $\forall i$

no proof.

Definition:

An algebra D is called a division algebra if D is a division ring.

Examples:

1 $\mathbb{F}$ is a division algebra

2 $\mathbb{H}$ is "

3. $M_n(\mathbb{F})$ is not

4. $D_1 \times D_2$ is not.

Hurwitz / Frobenius Theorem.

Characterises fd division algebras over $\mathbb{R}$

If D is a finite dimensional division algebra over $\mathbb{R}$ then

1 $D \cong \mathbb{R}$ or 2 $D \cong \mathbb{C}$ or 3 $D = \mathbb{H}$

Fact: If $\mathbb{D}$ is a ~~field~~ fd $\mathbb{F}$-algebra, for any n, $M_n(D)$ is an $\mathbb{F}$-algebra of dimension $= n^2 \dim_{\mathbb{F}}(\mathbb{D})$

eg. $\dim(M_n(\mathbb{H})) = 4n^2$

Proposition: *

Let $\mathbb{D}$ be a division algebra, $n \geq 1$

$M_n(D)$ as usual. Then,

1 Any simple $M_n(D)$-module is $\cong \mathbb{D}^n$

2 $M_n(D)$ is a direct sum of $\mathbb{D}^n$'s

$\therefore$ $M_n(D)$ is semisimple.

proof:

$$\underbrace{\begin{pmatrix} 0 & \boxed{\,} & 0 \end{pmatrix}}_{C_j} \longleftrightarrow \underbrace{\begin{pmatrix} \, \end{pmatrix}}_{D^n}$$ $\therefore$ both simple $M_n(D)$ modules.

$\mathbb{D}^n$ is the only non-zero $M_n(D)$-submod of $\mathbb{D}^n$ (ie $\mathbb{D}^n$ simple)

Any simple $M_n(D)$ module $\cong C_j \cong D^n$

Since $M_n(D) \cong \bigoplus_{i=1}^{n} C_j$

$\cong D^n \oplus \ldots \oplus D^n$

~~Theorem~~ Definition:

A field is algebraically closed $\overline{\mathbb{F}}$, if every polynomial $f(x) \in \mathbb{F}[x]$ of degree ≥ 1 has a root in $\mathbb{F}$.

eg. $< \mathbb{C} = \overline{\mathbb{R}}, \overline{\mathbb{Q}}, \overline{\mathbb{F}_p} >$

Burnside's Theorem

Let S be an ~~algebra~~ simple mod over A where A is an algebra over $\overline{\mathbb{F}}$

Then $\mathrm{End}_A(S) \cong \overline{\mathbb{F}}$ (division ring)

Proof:

1. Let $\varphi \in \mathrm{End}_A(S)$ $\varphi \neq 0$, S is a $\overline{\mathbb{F}}$-vs.

$\therefore$ φ is $\overline{\mathbb{F}}$-linear map.

Let $ch_\varphi(x) \in \mathbb{F}[x]$ be charac. poly

$\mathbb{F}$ algebraically closed, $ch_\varphi(x)$ has a non-zero eigenvalue $\lambda \in \overline{\mathbb{F}}$

$\varphi v = \lambda v$ for $\lambda \neq 0$

$(\varphi - \lambda Id_S)v = 0$

$\varphi - \lambda Id_S : S \longrightarrow S$

Not invertible - has non zero kernel. ($v \in \ker$)

$\Rightarrow \varphi - \lambda Id = 0$. (Schur's lemma)

$\Rightarrow \varphi = \lambda Id$.

$\therefore End_A(S) \xrightarrow{\cong} \overline{\mathbb{F}}$

$\varphi \longmapsto \lambda$.

Definition:

Let A be an algebra define A^{op} by:

1. As sets $A^{op} = A$
2. $+$ is the same as in A $(A^{op}, +) = (A, +)$
3. $\times = *$ is different in A^{o} $a * b = ba$.

Proposition: $(A^{op}, +, *)$ is an algebra.

Properties of A^{op}

1. A division algebra $\Leftrightarrow A^{op}$ division algebra
2. $(A^{op})^{op} \cong A$
3. If A is commutative, then $A^{op} = A$
4. If $B = B_1 \oplus B_2$ then $B^{op} = B_1^{op} \oplus B_2^{op}$

Lemma:

Let A be an $\mathbb{F}$-algebra then $End_A(A) \cong A^{op}$

Proof: o $End_A(A)$ is an $\mathbb{F}$-algebra

Let $\lambda \in \mathbb{F}$ $[\lambda(fg)](x) = \lambda f(g(x))$

$= \lambda f(g(x \cdot 1))$

$= \lambda g(x) f(1)$

$= g(x) \lambda f(1)$

$= g(x) f(1) \lambda$

1 Let $G \in End_A(A)$ $\varphi : A \longrightarrow A$ over A

2 Let $b \in A$, by def of ring/algebra homomorphism

$\varphi(b) = \varphi(b \cdot 1) = b\varphi(1)$

3 Let $a = \varphi(1) \in A \Rightarrow \varphi(b \cdot 1) = b\varphi(1) = ba$

4. Let $\varphi = \rho_a$ be the endomorphism given by right $\times$ by a

5. $\therefore \Psi : \mathrm{End}_A(A) \longleftrightarrow A^{op} = \{\rho_a : a \in A\}$ is a byjection.

Obviously Ψ is surjective and Ψ injective if $\varphi \in \mathrm{End}_A(A)$ st $\varphi(1) = 0$ then $\varphi(b) = b\varphi(1) = b \cdot 0 = 0$ $\Rightarrow \varphi \equiv 0$.

6 $\rho_a(\rho_b(x)) = x(ba)$
$= \rho_{ba}(x)$
$= \rho_{a * b}(x)$

$\Psi : \mathrm{End}_A(A) \xrightarrow{\cong} A^{op}$
$\varphi \longmapsto \varphi(1)$.

Lemma:

If B is an algebra then. $M_n(B)^{op} \cong M_n(B^{op})$

Proof: 1 Let $\Psi : M_n(B)^{op} \longrightarrow M_n(B^{op})$
$X \longmapsto X^{+}$

2. Clear that Ψ is a byjection.

3. $\Psi(X * Y) = (YX)^{T} = X^{T} Y^{T}$
$= \Psi(X)\Psi(Y)$

$\therefore \Psi$ is an algebra morphism
$\Rightarrow \Psi$ is isomorphism.

Lemma:

Let S be a A-module, then $\forall n$ $\mathrm{End}_A(S^n) \cong M_n(\mathrm{End}_A(S))$

proof: Ex 3.

Lemma:

If M is an R-module and $U_1, U_2 \leq M$ are submodules with $U_1 \cap U_2 = \langle 0 \rangle$ then $\mathrm{End}(U_1 \oplus U_2) = \mathrm{End}(U_1) \oplus \mathrm{End}(U_2)$.

Proof: Boring.

Artin-Wedderburn Theorem

An algebra A is semisimple if and only if
$A \cong M_{n_1}(D_1) \oplus \ldots \oplus M_{n_r}(D_r)$
where D_i's are division algebras over $\mathbb{F}$.

Proof: $\Leftarrow$ By proposition * a direct sum of semisimple $M_{n_i}(D_i)$'s is semisimple.
$\Rightarrow$ Suppose A is semisimple ie $A = S_1^{n_1} \oplus \ldots \oplus S_r^{n_r}$ where each S_i is simple
By property 4 A^{op} is semisimple
$A^{op} \cong End_A(A)$
$\cong End_A(S_1^{n_1} \oplus \ldots \oplus S_r^{n_r})$
$\cong End_A(S_1^{n_1}) \oplus \ldots \oplus End_A(S_n^{n_r})$
$\cong M_{n_1}(End_A(S_1)) \oplus \ldots \oplus M_{n_r}(End_A(S_r))$
Take oppisites
$A = (A^{op})^{op} = (M_{n_1}(End_A(S_1)) \oplus \ldots \oplus M_{n_r}(End_A(S_r)))^{op}$
$= M_{n_1}(End_A(S_1))^{op} \oplus \ldots \oplus M_{n_r}(End_A(S_r))^{op}$
$= M_{n_1}(End_A(S_1)^{op}) \oplus \ldots \oplus M_{n_r}(End_A(S_r)^{op})$.
Since S_i's simple modules by Schur's Lemma (V2), $End_A(S_i)$ and $End_A(S_i)^{op}$ are division algebras
so set $D_i = End_A(S_i)^{op}$
$\therefore A \cong M_{n_1}(D_1) \oplus \ldots \oplus M_{n_r}(D_r)$.

Applications:
If $\mathbb{F}$ is algebraically closed (hint hint $\mathbb{C}$) then D_i's $\cong \mathbb{C}$ over $\mathbb{C}$,
then $A \cong \underset{\mathbb{C}}{\underset{\shortparallel}{M_{n_1}(\mathbb{F})}} \oplus \ldots \oplus \underset{\mathbb{C}}{\underset{\shortparallel}{M_{n_r}(\mathbb{F})}}$

$\therefore$ A is simple as an algebra iff $A \cong M_n(\mathbb{F})$.

* If $\mathbb{F} = \mathbb{R}$ D_i's $= \mathbb{R}, \mathbb{C}, \mathbb{H}$ *

Group algebras - show they are semisimple.

Definition: Let $\mathbb{F}$ be a field, G a finite group, define the group ring/group algebra as
$$\mathbb{F}[G] = \{ \underbrace{\sum_{g \in G} \lambda_g g}_{\text{linear comb of group elements } / \mathbb{F}.} : \lambda_g \in \mathbb{F} \} \quad \text{where } \lambda_g g = g \lambda_g$$

Proposition:
$\mathbb{F}[G]$ is a ring

Proof: Addition : $\sum_{g\in G} \lambda_g g + \sum_{g\in G} \mu_g g = \sum_{g\in G} (\lambda_g + \mu_g) g$

Multiplication : $\sum_{g\in G} \lambda_g g (\sum_{h\in G} \mu_h h) = \sum_{g,h\in G} (\lambda_g \mu_h)(gh)$

$= \sum_{g,h\in G} \lambda_g \mu_{h^{-1}g}\, g$

Zero : $0 \in \mathbb{F}[G]$ where $\sum \lambda_g g = 0 \Rightarrow \lambda g = 0 \quad \forall g\in G$ since $g_i \neq 0$
Unit : $1 \in \mathbb{F}[G]$ (LI $\langle 1, g, \dots, g^{n-1} \rangle$ form basis).
The set $(\mathbb{F}(G), +, \circ)$ is a ring. $\square$.

$\dim_{\mathbb{F}}(\mathbb{F}[G]) = |G|$ since group elements form a basis for $\mathbb{F}[G]$ as an $\mathbb{F}$-vector space.
Hence $\mathbb{F}[G]$ is an $\mathbb{F}$-algebra.

Fact: The algebra $\mathbb{F}[G]$ is non-commutative unless G is commutative.

Fact: clear that basis elements $g\in G$ are invertible in $\mathbb{F}[G]$

Example Let $G = C_2 = \langle x \mid x^2 = 1 \rangle$
$\mathbb{F}[C_2] = \langle a + bx : a, b \in \mathbb{F} \rangle$
add obvious $(2+3x) + (6-2x) = 8 + x$.
multiply using group laws :

$(2+x)\cdot(3-4x) = 2+x$

$$\begin{array}{r} 2+x \\ \underline{3-4x} \\ 6+3x \\ \underline{-8x-4x^2} \\ = 2-5x \in \mathbb{F}[G] \end{array}$$

$x^2 = 1$.

Q What is $(1+x)^{-1}$?
Doesn't exist.

$(a+bx)(c+dx) = (ac+bd)\cdot 1 + (ad+bc)\cdot x$
$(\sum \lambda_g g)(\sum \mu_h h) = \sum \lambda_{h^{-1}g}\, g$

Lemma: If $|G|>1$ then $\mathbb{F}[G]$ is not a division algebra.

proof: If $|G|=1$, $\mathbb{F}[G]=\mathbb{F}[1]=\mathbb{F}$ which is a division algebra.
So suppose $|G|>1$, then its easy to find zero divisors.
Let $g\in G$, since G is finite $\exists n$ st $g^n=1$
In $\mathbb{F}[G]$ $\underset{\neq 0}{(1-g)}\underset{\neq 0}{(1+g+\dots+g^{n-1})} = 1-g^n$
$= 1-1$
$= 0$

This is not possible in a division algebra.

Definition:
By an $\mathbb{F}[G]$-module, I will always mean a module V over the ring $\mathbb{F}[G]$ and these will always be finitely generated.

Example:
The left regular $\mathbb{F}[G]$-module $(V=)\,\mathbb{F}[G]$ acting on itself by left group multiplication.

Definition:
Let V, W be 2 $\mathbb{F}[G]$-modules.
A map $\varphi: V \longrightarrow W$ is called an $\mathbb{F}[G]$-homomorphism if it is $\mathbb{F}[G]$ linear, ie it satisfies

$$\left.\begin{array}{l} 1.\ \varphi(v+v') = \varphi(v)+\varphi(v') \\ 2.\ \varphi(\lambda v) = \lambda\varphi(v) \end{array}\right\}\ \lambda\in\mathbb{F}[G]\quad v, v'\in V$$

3. $\varphi(gv) = g\varphi(v)\ \forall g\in G$

Remember since $\mathbb{F}[G]$ is an $\mathbb{F}$-algebra
$\varphi: V\longrightarrow V$ can be considered as an $\mathbb{F}$-linear map of vs over $\mathbb{F}$

Proposition:
Let $\varphi: V\longrightarrow W$ be an $\mathbb{F}[G]$-homomorphism
Then $Ker(\varphi)$ and $Im(\varphi)$ are $\mathbb{F}[G]$-submodules of V and W respectively

Correspondence Theorem

Let G be a finite group, V fd vector space over $\mathbb{F}$

$\rho: G \longrightarrow GL(V) = GL_n(\mathbb{F})$ an $\mathbb{F}$-representation of G.

Then there exists 1-1 correspondence between representations of G over $\mathbb{F}$ and fg left $\mathbb{F}[G]$-modules

$\{M \text{ an } \mathbb{F}[G]\text{-module}\} \longleftrightarrow \{\rho: G \longrightarrow GL(V)\}$.

proof: $\Leftarrow$ Let V be a fg $\mathbb{F}[G]$-module

$\therefore$ V is a fd $\mathbb{F}$-vector space.

$\forall g \in G$ define an $\mathbb{F}$-representation of G by the $\mathbb{F}$-automorphism

$\psi: V \longrightarrow V$

$\psi(v) \longmapsto gv \quad \forall v \in V = \mathrm{Sp}_{\mathbb{F}}\{b_1, \dots, b_n\} \cong \mathbb{F}^n$

Write the map ψ wrt basis as a matrix $[\psi]_B = \rho(g)$

Show $\rho(g) \in GL(V)$ is a rep/linear map.

$$\begin{aligned}\rho(g)(\lambda v + w) &= g(\lambda v + w)\\ &= \lambda g v + g w\\ &= \lambda \rho(g)(v) + \rho(g) w\end{aligned}$$

Check $\rho(g)$ is a homomorphism.

Factorise the map

$$\begin{tikzcd} V \arrow[r, "\rho(h)"] \arrow[rr, bend right, "\rho(gh)"'] & V \arrow[r, "\rho(g)"] & V \end{tikzcd}$$

$$\begin{aligned}\rho(gh)(v) = (gh)v &= g(h(v))\\ &= \rho(g)(hv)\\ &= \rho(g)\rho(h)(v)\end{aligned}$$

$\therefore$ Composition of $\rho(g)$ and $\rho(h)$ as linear maps = mult. of matrices.

$\rho(1)$ is the matrix corresponding to identity map $\mathrm{id}: V \longrightarrow V$

$$\rho(1) = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = I_n$$

Show $\rho(g)$ is invertible!

Let $g \in G$ since $g \cdot g^{-1} = 1$ in G and in $\mathbb{F}[G]$ then the map

$$\begin{tikzcd} V \arrow[r, "\rho(g)"] \arrow[rr, bend right, "\rho(1)"'] & V \arrow[r, "\rho(g^{-1})"] & V \end{tikzcd}$$

$$\begin{aligned}\rho(g^{-1})\rho(g)(v) &= \rho(g^{-1})(gv)\\ &= (g^{-1}g)v\\ &= v\\ &= \rho(1)v\end{aligned}$$

$\therefore \rho(g^{-1})\rho(g) = \rho(1)$ $\therefore$ $\rho(g)$ is invertible

$\therefore$ $\mathbb{F}[G]$-module corresponds to representation of G

$\Rightarrow$ Let $\rho: G \longrightarrow GL_n(\mathbb{F})$ be an $\mathbb{F}$-map
Then associate to it an $\mathbb{F}[G]$-module which we construct from $\mathbb{F}^n = V$ by keeping the same addition structure and defining scalar multiplication on it by letting $\alpha = \sum \lambda_g g \in \mathbb{F}[G]$ $v \in V = \mathbb{F}^n$

$\alpha v = (\sum \lambda_g g)(v) = \sum \lambda_g (gv)$
$= \sum \lambda_g \rho(g)(v)$

making $\mathbb{F}^n$ into an $\mathbb{F}$-module. QED.

Examples

1. Let $G = D_8 = \langle x, y \mid x^4 = y^2 = 1, yx = x^3 y \rangle$

Define $\rho: D_8 \longrightarrow GL_2(\mathbb{R})$

$$x \longmapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad y \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Let $V = \mathbb{R}^2 = Sp_{\mathbb{F}}\langle v_1, v_2 \rangle$

$\rho(x)(v_1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -v_2$

$\rho(x)(v_2) = v_1 \qquad \rho(y)(v_1) = v_1 \qquad \rho(y)(v_2) = -v_2$

This defines the structure of $V = \mathbb{R}^2$ as an $\mathbb{R}[D^8]$-module
Conversly using the above, write matrices for $\rho(x)$ and $\rho(y)$ wrt $\langle b_1, b_2 \rangle$ to recover the representation from module.

2. Let $G = S^n$

Define $\rho: S^n \longrightarrow GL_n(\mathbb{C})$ the permutation rep on $\mathbb{C}^n = V$ by
$\rho(\sigma)(e_i) = e_{\sigma(i)}$ where $V = \mathbb{C} = Sp_{\mathbb{C}}\langle e_1, \dots, e_n \rangle$
This makes $\mathbb{C}^n$ into a module over $\mathbb{C}[S^n]$ called the permutation module.

eg. $n = 4$ $\quad B = \langle e_1, \dots, e_4 \rangle$ basis of $\mathbb{F}^4$

Let $\sigma = (1\,2) \in S_4$

$\rho(\sigma)(e_1) = e_{\sigma(1)} = e_2$

$\rho(\sigma)(e_2) = e_1 \quad \rho(\sigma)(e_3) = e_3 \quad \rho(\sigma)(e_4) = e_4$

$$[\rho(\sigma)]_B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Definition:
Let G be a finite group, V fd $\mathbb{F}$-vector space, An $\mathbb{F}$-representation $\rho: G \to GL(V)$ is called irreducible if $V \neq \langle 0 \rangle$ and the only invariant subspace of V under ρ are the trivial ones $\langle 0 \rangle$ & V.
The representation is called reducible if $\exists W \subsetneq V$ $W \neq 0$ st $\rho(g)W \subseteq W$ $\forall g \in G$
ie $\exists$ an invariant subspace.

Fact: By correspondence Thm
$\rho: G \to GL(V)$ irreducible $\Leftrightarrow$ V is simple $\mathbb{F}[G]$-module.

Equivalent definition:
An $\mathbb{F}$-representation $\rho: G \to GL_n(\mathbb{F})$ is called reducible if $\exists$ $T \in GL_n(\mathbb{F})$ st $\forall g \in G$ we have an equivalent matrix representation of the form
$\rho'(g) = T^{-1}\rho(g)T = \begin{pmatrix} X_g & Y_g \\ 0 & Z_g \end{pmatrix}$ where X_g is a $\dim W \times \dim W$ matrix.

Examples of Irreducible / reducible ~~reps~~ reps.

1. $\rho: D_8 \to GL_2(\mathbb{R})$ is irreducible 2-dim rep
$\rho(x) \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\rho(y) \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
Apply definition: Suppose ρ reducible
$\therefore \exists W \subsetneq$ st $\rho(G)W \subseteq W$ where $\dim(W) = 1$
ie W is $\rho(g)$ invariant
$\rho(x)W = W$ $\rho(y)W = W$
Let $W = \text{span}\langle \lambda v_1 + \mu v_2 \rangle$ $V = \mathbb{R}^2 = \langle v_1, v_2 \rangle$
Let $w = \lambda v_1 + \mu v_2$
$\rho(x)w =$
$\rho(y)w =$
will get some sort of contradiction.

2. If $\mathbb{F} = \mathbb{F}_2$ $\rho: D_8 \to GL_2(\mathbb{F}_2)$ then $W = \text{span}_{\mathbb{F}_2}(v_1 + v_2) \subseteq \mathbb{F}_2^2$
is $\rho(G)$-stable
$\rho(x)(W) = \rho(x)(v_1 + v_2) = -v_2 + v_1 = v_1 + v_2$
$\rho(y)(W) = \rho(y)(v_1 + v_2) = v_1 - v_2 = v_1 + v_2$
$\therefore \rho$ is reducible over $\mathbb{F}^2$

3. Let $G = C_3 = \langle x \mid x^3 = 1 \rangle$ and consider the $\mathbb{R}[C_3]$-module V, of $\dim(V) = 3$.

This has a permutation representation on $\mathbb{R}^3 = V$ given by:-

$\rho : C_3 \longrightarrow GL_3(\mathbb{R})$

$\rho(\sigma)(e_i) = e_{\sigma(i)}$

Fix $B = \langle e_1, e_2, e_3 \rangle = \mathbb{R}^3$

In standard basis $[\rho(x)]_B = \begin{array}{c} \begin{matrix} 1 & x & x^2 \end{matrix} \\ \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \end{array}$

Claim: this representation is reducible.

ie $V = \mathbb{R}^3$ is a semisimple $\mathbb{F}[C_3]$-module

Let $W = \text{span}(w) = \mathbb{R}w$ where $w = e_1 + e_2 + e_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

W is an $\mathbb{R}[C_3]$-submodule which is $\rho(G)$-stable

$\rho(x)(w) = \rho(x)(e_1 + e_2 + e_3)$

$= e_2 + e_3 + e_1 = w$

$\rho(x)W \subseteq W$.

Choose a different basis $B' = \{w, e_2, e_3\}$.

Apply $\rho(x)$

$\rho(x)w = w$

$\rho(x)e_2 = e_3$

$\rho(x)e_3 = e_1 = w - e_2 - e_3$

$\rho'(x) = \left(\begin{array}{c|cc} 1 & 0 & 1 \\ \hline 0 & 0 & -1 \\ 0 & 1 & -1 \end{array}\right)$

Algebra 3 projection

If $V = U \oplus V$, then we can construct a special endomorphism of V that depends on the expression $V = U \oplus W$

Proposition

Suppose $V = U \oplus W$, define

$\pi : V \longrightarrow U \subset V$ by

$(u + w) \longmapsto u \quad \forall u \in U, w \in W$

then π is an endomorphism of V

Furthermore: $\operatorname{Im}\pi = U$, $\operatorname{Ker}\pi = W$ and $\pi^2 = \pi$

Definition:

An endomorphism π of a vector space V such that $\pi^2 = \pi$ is called a projection of V

Proposition

Suppose π is a projection of V, then $V = \ker \pi \oplus \operatorname{Im} \pi$

Maschke's Theorem

Let G be a finite group, $\mathbb{F}$ a field such that $\operatorname{char}(\mathbb{F}) \nmid |G|$ ~~(examples)~~

Let V be an $\mathbb{F}[G]$, then for any $\mathbb{F}[G]$-submodule $U \le V$, then $\exists$ $\mathbb{F}[G]$-submodule W such that $V = U \oplus W$ (an $\mathbb{F}[G]$-mods)

In english, any $\mathbb{F}[G]$-module V is semisimple/reducible

Proof: Let V be an $\mathbb{F}[G]$-module and $U \le V$ be a $\mathbb{F}[G]$-submodule

Assume $U \ne \{0\}$ or V otherwise nothing to prove

Since U is an $\mathbb{F}$-subspace of V $\exists$ W_0 which can be any other $\mathbb{F}$-subspace

$V = U \oplus W_0$ ($\mathbb{F}$-vector space)

Choose any projection onto U

$\pi : V \longrightarrow U$ is an $\mathbb{F}$-linear map

$u + v \longmapsto u$

Secret: Turn π into an $\mathbb{F}[G]$-module homomorphism by defining an averaging proven as follows:

$\varphi : V \longrightarrow U$ as $\mathbb{F}[G]$-mods

$$\varphi(v) = \frac{1}{|G|} \sum_{g \in G} g\pi(g^{-1}v)$$

If we prove that φ is an $\mathbb{F}[G]$-homomorphism such that $\varphi^2 = \varphi$ and $\operatorname{Im}\varphi = U$, then $\ker\varphi$ will have to be an $\mathbb{F}[G]$-submod complementing U such that

$$V = U \oplus \underbrace{\ker\varphi}_{W}$$

Claim: φ is an $\mathbb{F}[G]$-hom

(check: $\varphi(gv) = g\varphi(v)$ $\forall g \in G$ $\forall v \in V$)

Let $x \in G$ and set $h = x^{-1}g \Leftrightarrow g = xh$
$\Leftrightarrow h^{-1} = g^{-1}x$

Let $v \in V$

$$\varphi(xv) = \frac{1}{|G|} \sum_{g \in G} g\pi(g^{-1}(xv))$$ as g ~~run~~ are all elements of G

$$= \frac{1}{|G|} \sum_{h \in G} (xh)\pi(h^{-1}v)$$

$$= \frac{x}{|G|} \sum_{h \in G} h\pi(h^{-1}v)$$

$$= x\varphi(v)$$

Claim: $\varphi^2 = \varphi$ (ie show $\mathrm{Im}\varphi = U$)

1. $\mathrm{Im}\varphi \subset U$

since π projects onto U, $\pi(v) = u$, $\pi(u) = u$

$$\varphi(u) = \frac{1}{|G|} \sum g \underbrace{\pi(g^{-1}u)}_{\in U} \in U$$

U

since $gu \in U$ $\forall u \in U$ as U is an $\mathbb{F}[G]$-submod

2. $U \subset \mathrm{Im}\varphi$

$$\varphi(u) = \frac{1}{|G|} \sum_{g \in G} g\pi(g^{-1}u)$$

$$= \frac{1}{|G|} \sum_{g \in G} g(g^{-1}u)$$

$$= \frac{1}{|G|} \sum_{g \in G} u = \frac{|G|}{|G|} u = u$$

$\therefore U = \mathrm{Im}\,\varphi$

3. $\varphi^2 = \varphi$

Take $v \in V$, $\varphi^2(v) = \varphi(\varphi(v))$
$= \varphi(v)$

since $\varphi(v) \in U$ and $\mathrm{Im}\varphi = U$

$\therefore \varphi^2 = \varphi$

$\therefore \varphi$ is an $\mathbb{F}[G]$-hom, $U = \mathrm{Im}\varphi$ is then an $\mathbb{F}[G]$-submod

Let $W = \ker\varphi$ which is also an $\mathbb{F}[G]$-submod

$\therefore V = \mathrm{Im}\,\varphi \oplus \ker\varphi = U \oplus W$ $\square$

Definition:

1. An $\mathbb{F}[G]$-module V is called completely reducible if $V = U_1 \oplus \dots \oplus U_r$ where each U_i is an irreducible $\mathbb{F}[G]$ submodule of V
2. An $\mathbb{F}$-representation $\rho : G \to GL(V)$ is completely reducible if $\forall U \subseteq V$ which is invariant under ρ (ie $\rho(G) \cdot U \subseteq U$) $\exists$ another $\rho(G)$ invariant subspace W such that $V = \underset{\{u_1,\dots,u_m\}}{U} \oplus \underset{\{w_1,\dots,w_n\}}{W}$.

Then $\{u_1,\dots,u_m, w_1,\dots,w_n\}$ is a basis for V and $\forall g$
$$\rho(g) = \begin{pmatrix} U_g & 0 \\ 0 & W_g \end{pmatrix}.$$

So an $\mathbb{F}$-rep is completely reducible if $\exists\, T \in GL_n(\mathbb{F})$, $\rho : G \to GL_n(\mathbb{F})$
$$T^{-1}\rho(g)T = \underbrace{\begin{pmatrix} X_g & Y_g \\ 0 & Z_g \end{pmatrix}}_{\text{reducible}} \sim \underbrace{\begin{pmatrix} U_g & 0 \\ 0 & W_g \end{pmatrix}}_{\text{completely reducible}}$$

Maschke's Corollary

Keep breaking U, W into irreducibles.
If G is a finite group, $\mathrm{char}(\mathbb{F}) \nmid |G|$ then for every non-zero $\mathbb{F}[G]$-mod is completely reducible

Proof: Let $V \neq \{0\}$ be an $\mathbb{F}[G]$-module. By induction on $\dim(V)$ and use Maschke's Theorem.
If $\dim V = 1 \Rightarrow V$ is irreducible
so suppose V is reducible ($\dim V > 1$)
$\Rightarrow \exists\, U \subseteq V$ such that $U \neq \{0\}$ or V and by Maschke's $\exists\, W \subseteq U$ st $V = U \oplus W$. Since $\dim U \leq \dim V$, ~~thenal~~ and $\dim W < \dim V$, then we have by induction hypothesis
$U = U_1 \oplus \dots \oplus U_s \qquad W = W_1 \oplus \dots \oplus W_t$
$\Rightarrow V = U_1 \oplus \dots \oplus U_s \oplus W_1 \oplus \dots \oplus W_t$
all irreducible submodules of any dimension

Definition:
Let A, B, C be R-mods. Say a sequence of homos
$A \xrightarrow{\varphi} B \xrightarrow{\psi} C$ is exact iff
$\operatorname{Ker}(\psi) = \operatorname{Im}\varphi$.
A sequence $\dots \to A_{n+1} \xrightarrow{\phi_{n+1}} A_n \xrightarrow{\phi_n} A_{n-1} \to \dots$
is exact iff $\forall n$ $\operatorname{Ker}\phi_n = \operatorname{Im}\phi_{n+1}$

Example:
Let A and C be R-mods, then
$0 \to A \xrightarrow{i} A \oplus C \xrightarrow{\pi} C \to 0$
is exact where $i(a) = (a, 0)$, $\pi(a, c) = c$

Proposition:
$0 \to A \xrightarrow{i} A \oplus C \xrightarrow{\pi} C \to 0$
is exact if i injective and π is surjective

An exact sequence $0 \to A \to B \to C \to 0$ is a short exact sequence SES

Definition
The trivial SES is exact for any ring R.
Let A, C be R-mods. Then
$0 \to A \xrightarrow{i} A \oplus C \xrightarrow{\pi} C \to 0$
is exact for $i(a) = (a, 0)$, $\pi(a, c) = c$

Definition:
Say that an SES $0 \to A \to B \to C \to 0$ splits when $\exists$ isomorphism $\psi : A \oplus C \to B$

$$\begin{array}{ccccccccc} 0 & \to & A & \to & A \oplus C & \to & B & \to & 0 \\ & & \downarrow \mathrm{id} & & \downarrow \psi & & \downarrow \mathrm{id} & & \\ 0 & \to & A & \to & B & \to & C & \to & 0 \end{array}$$

Usually SES don't split.

Splitting Theorem for Vector Spaces ($\cong$ Basis theorem)
If $\mathbb{F}$ is a field, then every SES of modules over $\mathbb{F}$ (vs) splits

Masckel's Theorem v2 (modern form)
Let G be a finite group, $\mathbb{F}$ a field st $\mathrm{char}(\mathbb{F}) \nmid |G|$
If $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$
is a SES of $\mathbb{F}[G]$-modules
Then it splits ie $B = A \oplus C$

proof: want to find an $\mathbb{F}[G]$-homomorphism st $g\,\psi(g(a,c)) = g\,\psi(a,c)$ $\forall g \in G$, ψ splitting isomorphism and apply splitting theorem for vs.

Mascke's Theorem V1
Let G be a ~~group~~ finite group, $\mathbb{F}$ a field st $\mathrm{char}(\mathbb{F}) \nmid |G|$
Let V be an $\mathbb{F}[G]$-module then for any $\mathbb{F}[G]$-submod $U \leq V, \exists W \leq V$ st $V = U \oplus W$
(ie V is semisimple - characteristic theorem).

Conditions that falsify Mascke's Theorem.

1. if G is infinite
Let $G = \mathbb{Z}$ ($= C_\infty$) and $\mathbb{F} = \mathbb{C}$
Define $\rho : \mathbb{Z} \longrightarrow GL_2(\mathbb{C})$ by
$$n \longmapsto \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$
ρ is a $\mathbb{C}$-rep
$$\rho(n+m) = \begin{pmatrix} 1 & n+m \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} = \rho(n)\rho(m)$$
Why does Mascke's fail?
Let $U \leq \mathbb{C}^2$ be a G-invariant subspace where
$U = \mathrm{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\} = \mathrm{span}\{u\}$.
$\therefore \dim_{\mathbb{C}}(U) = 1$
recall G invariant subspace $\forall g \in G$ $\rho(g)U \leq U$ ie $\rho(g)u = \lambda u$
U is an ~~eigenspace~~ eigensubspace $\forall\ \rho(g)$ and its the only one
(ie no complement)
If there was a G-invariant complement W, then W would also be

Abbra.

1-dim eigenspace $\forall g \in G \Rightarrow \rho(g)$ is diagonalisable
$\Rightarrow V = U \oplus W$
But $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ is not diagonalisable unless $n=0$
since $m_{\rho(g)}(x) = (x-1)^2$
$\therefore$ $U = \text{span}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \text{span}\{u\}$ has no complement

2. If $|G| = 0$ in $\mathbb{F}$ ie $\text{char}(\mathbb{F}) \mid |G|$

- Let $G = C_p = \langle x \mid x^p = 1 \rangle$ and $\mathbb{F} = \mathbb{F}_p$
$\overset{\shortparallel}{\mathbb{Z}/p\mathbb{Z}}$

Define $\rho: C_p \longrightarrow GL_2(\mathbb{F}_p) = \text{Aut}(C_p \times C_p)$

$$x^j \longmapsto \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} \quad j = 0, 1, \dots, p-1$$

$$x^p \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \rho(1)$$

If Maskke's Theorem holds, then V would decompose
$V = \mathbb{F}_p^2 = U \oplus W$ where $U \leq \mathbb{F}_p^2$ is a 1-dim G-invarient subspace
$U = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\} = \text{span}\{u\}$.

$$\forall x^j \in C_p \quad \rho(x^j)u = \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = u$$

But there is no $\mathbb{F}_p[C_p]$-submodule W st $\mathbb{F}_p^2 = U \oplus W$
If $\exists$ ~~W~~ such a W, ~~th~~ W is also an eigenspace
but

- Let $G = D_8$, $\mathbb{F} = \mathbb{F}_2$
Define $\rho: D_8 \longrightarrow GL_2(\mathbb{F}_2)$

$$x \longmapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad y \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Let $U = \text{span}(e_1 + e_2) = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\} = \text{span}\{u\}$.
Then $U \leq V$ is D_8-invarient
Does it have a complement?
Consider options for complement
$V = \mathbb{F}_2^2$, let $W = \text{span}(w)$
$= \text{span}(\lambda e_1 + \mu e_2) = \begin{pmatrix} \lambda \\ \mu \end{pmatrix}$
where λ & μ are both not 0.
If $\lambda = 0 \Rightarrow \mu = 1 \Rightarrow w = e_2$ ✗
$\mu = 0 \Rightarrow \lambda = 1 \Rightarrow w = e_1$ ✗

If $\lambda, \mu \neq 0 \Rightarrow W = e_1 + e_2 = U$
$\therefore$ only options for W are $\text{span}\{e_1\}$ and $\text{span}\{e_2\}$ but both are not invariant by G.

$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = e_2 \neq \lambda e_1$

$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = e_1 \neq \lambda e_2$

$\therefore$ $\text{span}(e_1)$ and $\text{span}(e_2)$ are not $\mathbb{F}^2[D_8]$-submods because they are not D_8 invariant
ie $\mathbb{F}_2^2 \neq U \oplus W$ for D_8

The proof of Mascke's theorem gives us a procedure to find complimentary subspace of U (W) if the conditions are satisfied and we know a submodule $U \leq V$ which is G-invariant already

Example:
Let $G = S_3 (\cong D_6)$ $\mathbb{F} = \mathbb{C}$
Define $\rho : S_3 \longrightarrow GL_3(\mathbb{C})$ by
$\rho(\sigma)e_i = e_{\sigma(i)}$
$\therefore V = \mathbb{C}^2 = \text{span}\langle e_1, e_2, e_3 \rangle$ is a $\mathbb{C}[S_3]$-module has structure given by $\sigma \cdot e_i = e_{\sigma(i)}$
Q: Is the rep ρ irreducible?
3-dim = 1-dim $\oplus$ 2-dim or
~~is S_3-invariant~~
If $u = e_1 + e_2 + e_3$ then $U = \text{span}\{u\} = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$ is S_3-invariant
$\therefore$ U is $\mathbb{C}[S_3]$-submodule
$\therefore$ $\mathbb{C}^3 = V \cong U \oplus W$ by Mascke's
Let $\pi : V \twoheadrightarrow U$ be the projection map given by

$\left.\begin{aligned} \pi(e_1) &= 0 \\ \pi(e_2) &= 0 \\ \pi(e_3) &= e_1 + e_2 + e_3 \end{aligned}\right\}$ $\text{Ker}(\pi) = W_0$ $\mathbb{F}$-subspace
$= \text{span}\{e_1, e_2\}$ is an $\mathbb{F}$-complement of U.

Find W a unique $\mathbb{F}[G]$-submodule
$S_3 = \langle (1), (1\,2), (1\,3), (2\,3), (1\,2\,3), (1\,3\,2) \rangle$
The $\mathbb{F}[G]$-homo given in the proof of Mascke's VI is
$\varphi(v) = \frac{1}{|G|} \sum_{g \in G} g\pi(g^{-1}v)$ computed on basis e_i

$\varphi(e_i) = \frac{1}{G} \sum_{\sigma \in S_3} \sigma \pi(\sigma^{-1} e_i)$

$= \frac{1}{6} \sum_{\sigma \in S_3} \sigma \pi(e_{\sigma^{-1}(i)})$

Compute $\varphi(e_i)$

$\varphi(e_1) = \frac{1}{6}\Big[(1)\cancel{\pi(e_1)} + (12)\cancel{\pi}(e_2) + (13)\pi(e_3) + (23)\cancel{\pi}(e_1) + (123)\cancel{\pi}(e_2) + (132)\pi(e_3) \Big]$

$= \frac{1}{6}\left[(13)\pi(e_3) + (132)\pi(e_3) \right]$

$= \frac{1}{6}\left[(13)(e_1+e_2+e_3) + (132)(e_1+e_2+e_3) \right]$

$= \frac{1}{6}\left[2(e_1+e_2+e_3) \right]$

$= \frac{1}{3}(e_1+e_2+e_3)$

Similarly $\varphi(e_2) = \varphi(e_3) = \frac{1}{3}(e_1+e_2+e_3)$

$\therefore W = \ker(\varphi)$
$= \{ \sum_{i=1}^{3} \lambda_i e_i : \sum_{i=1}^{3} \lambda_i = 0 \}$
$= \operatorname{span}\{ e_1 - e_2, e_3 - e_2 \}$. $\dim W = 2$

φ takes these to 0

$\therefore V = U \oplus W$

Lemma

Let V and W be R-mods such that $\operatorname{Hom}_R(V,W) = \operatorname{Hom}_R(W,V) = 0$

Then $\operatorname{End}_R(V \oplus W) \cong \operatorname{End}_R(V) \times \operatorname{End}_R(W)$

proof: $\operatorname{End}_R(V \oplus W) \cong$ ring of matrices of the form $\begin{pmatrix} \alpha_{VV} & \alpha_{WW} \\ \alpha_{WV} & \alpha_{WW} \end{pmatrix}$

where $\alpha_{VV} : V \to V$ $\alpha_{VW} : V \to W$

$\alpha_{WV} : W \to V$ $\alpha_{WW} : W \to W$

Since $\operatorname{Hom}(V,W) = \operatorname{Hom}(W,V) = 0$ then

$\operatorname{End}(V \oplus W) \xrightarrow{\cong} \operatorname{End}(V) \times \operatorname{End}(V)$

$\alpha \longmapsto (\alpha_{VV}, \alpha_{WW})$

Schur's Lemma revisited for $\mathbb{F}[G]$-modules.

V1 Let V and W be 2 simple non-zero $\mathbb{F}[G]$-modules
Let $\varphi : V \to W$ be an $\mathbb{F}[G]$-homomorphism.
Then either $\varphi = 0$ or φ is an isomorphism.

V2 If V is simple $\mathbb{F}[G]$-module then $End_{\mathbb{F}[G]}(V)$ is a division ring
ie if $\varphi \in End_{\mathbb{F}[G]}(V)$ ~~w~~ then if $\varphi \neq 0 \Rightarrow \varphi = \lambda Id$ where $\mathbb{F}$ is algebraically closed since we need $ch_\varphi(n)$ to have at ~~most~~ root λ in $\mathbb{F}$ $\therefore$ $\exists \varphi^{-1}$ st $\varphi \circ \varphi^{-1} = Id$

* In V1 we don't need $\mathbb{F}$ algebraically closed

Schur's Lemma V3

Let V be a semisimple fg $\mathbb{F}[G]$-module st $char(\mathbb{F}) \nmid |G|$ then
V is simple $\Leftrightarrow$ $\underset{\overset{\text{III}}{D}}{End_{\mathbb{F}[G]}(V)}$ is a division ring

proof: $\Rightarrow$ By schur's V2
$\Leftarrow$ Let $V = V_1^{n_1} \oplus \dots \oplus V_m^{n_m}$ (V semisimple) where $V_1, \dots, V_m$ are simple $\mathbb{F}[G]$-modules and $V_i \not\cong V_j$ if $i \neq j$

$$
\begin{aligned}
End_{\mathbb{F}[G]}(V) &= End_{\mathbb{F}[G]}(V_1^{n_1} \oplus \dots \oplus V_m^{n_m}) \\
&= \prod_{i=1}^{m} End_{\mathbb{F}[G]}(V_i^{n_i}) \\
&= \prod_{i=1}^{m} M_{n_i}(End_{\mathbb{F}[G]}(V_i)) \\
&= \prod_{i=1}^{m} M_{n_i}(D_i)
\end{aligned}
$$

♡ Wedder Lemma HW3

Schur's V2.

where D_i's are division rings
The only way for the RHS to be a division ring is if $\exists$ unique r st

$$n_i = \begin{cases} 1 & i = r \\ 0 & i \neq r \end{cases}$$

$\Rightarrow V \cong V_r$ is simple. $\square$

Schur's V3 is a pratical tool for ~~dacting~~ detecting irreducible reps $\longleftrightarrow$ simple $\mathbb{F}[G]$-modules (of any dim)

Elegant examples :

1. $\rho : D_8 \longrightarrow GL_2(\mathbb{C})$ is a 2-dim rep of D_8

$\rho(x) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\rho(y) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Lets show that this 2-dim rep of D_8 is irreducible

ie $\mathbb{C}^2$ is a simple $\mathbb{C}[D_8]$-module, ~~by~~

by computing its endomorphism ring $\text{End}_{\mathbb{C}[D_8]}(\rho)$

ie all complex 2×2 matrices that commute with $\rho(x)$ and $\rho(y)$ simultaneously

ie find 2×2 complex A st

$A\rho(g) = \rho(g)A \quad \forall g \in G$ ~~(only need to look at generators)~~

Only need to look at generators

$A\rho(x) = \rho(x)A \quad A\rho(y) = \rho(y)A$

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$A\rho(y) = \begin{pmatrix} a & -b \\ c & d \end{pmatrix}$ $\rho(y)A = \begin{pmatrix} a & b \\ -c & d \end{pmatrix}$

$\Rightarrow b = 0, \ c = 0, \ a = a, \ d = d$

$A = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$

$A\rho(x) = \begin{pmatrix} 0 & a \\ -d & 0 \end{pmatrix}$ $\rho(x)A = \begin{pmatrix} 0 & d \\ -a & 0 \end{pmatrix}$

$\Rightarrow a = d$

So $\text{End}_{\mathbb{C}[D_8]}(\rho) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in \mathbb{C} \right\}$

$\cong \mathbb{C} \ni a$ which is a division ring

$\therefore \rho$ is irreducible by ~~sha~~ Schur's V3

2. Let $\sigma : D_6 \longrightarrow GL_3(\mathbb{C})$

$\sigma(x) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \sim (1\ 3\ 2)$ $\sigma(y) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \sim (2\ 3)$

Q is σ irreducible?

Compute $\text{End}_{\mathbb{C}[D_6]}(\sigma)$

ie 3×3 matrices A st

$A\sigma(x) = \sigma(x)A \quad A\sigma(y) = \sigma(y)A$

$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}$

$A\sigma(x) = \begin{pmatrix} c & a & b \\ f & d & e \\ k & g & h \end{pmatrix}$ $\sigma(x)A = \begin{pmatrix} d & e & f \\ g & h & k \\ a & b & c \end{pmatrix}$

$\Rightarrow a = e = k, \ b = f = g, \ c = d = h$

$\Rightarrow A = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}$

$A\sigma(y) = \begin{pmatrix} a & c & b \\ c & b & a \\ b & a & c \end{pmatrix}$ $\sigma(y)A = \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}$

$\therefore a = a \Rightarrow b = c$

$\Rightarrow A = \begin{pmatrix} a & b & b \\ b & a & b \\ b & b & a \end{pmatrix}$

$\dim_{\mathbb{C}}(\mathrm{End}_{\mathbb{C}[D_6]}(\sigma)) = 2$ $\mathrm{End}_{\mathbb{C}[D_6]}(\sigma) \cong \mathbb{C}^2$ which is not a division ring

$\therefore \sigma$ is reducible

ie the $\mathbb{C}[D_6]$-module $\mathbb{C}^3$ is semisimple.

Definition

Let $\rho_1 : G \to GL(U)$

$\rho_2 : G \to GL(W)$

be two $\mathbb{F}$-reps of G.

Define the $\oplus$ of reps $\rho_1 \oplus \rho_2 : G \to GL(U \oplus V)$ of G over $\mathbb{F}$ with the rep space $U \oplus W$ by

$(\rho_1 \oplus \rho_2)(g) = \rho_1(g) \oplus \rho_2(g) \quad \forall g \in G.$

If we choose basis $\{u_1, \dots, u_n\}$ and $\{w_1, \dots, w_m\}$

then $\rho_1 : G \to GL_n(\mathbb{F})$ $\rho_2 : G \to GL_m(\mathbb{F})$

$g \mapsto A$ $g \mapsto B$

So the matrix rep wrt $\{(u_1, 0), \dots, (u_n, 0), (0, w_1), \dots, (0, w_m)\}$.

$\rho_1 \oplus \rho_2 : G \to GL_{n+m}(\mathbb{F})$

$g \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$

Question: How many distinct irreducible reps are for each G over $\mathbb{C}$?
ie find all simple $\mathbb{C}[G]$-mods for G.

Representations of finite abelian groups over $\mathbb{C}$.

$G = C_{n_1} \times \dots \times C_{n_r}$

$$\begin{array}{c} \mathbb{C}[x]/_{x^n-1} \cong \mathbb{C}[x]/_{(x-1)} \times \dots \times \mathbb{C}[x]/_{(x-\xi_n)} \\ \cong \\ \mathbb{C}[C_n] \end{array}$$

Let G be a finite abelian group and let V be a $\mathbb{C}[G]$-mod
Since G is abelian: $\forall x, g \in G$, $\forall v \in V$
$x(g \cdot v) = (xg) \cdot v = g(xv)$
Fix $x \in G$ define the $\mathbb{C}[G]$-ends
$\varphi_x : V \longrightarrow V$
$\varphi_x(v) = x \cdot v$
Suppose that V is simple $\mathbb{C}[G]$-mod by Schur's V2
$\varphi_x \in \mathrm{End}_{\mathbb{C}[G]}(V)$ is such that
$\varphi_x = \lambda_x \mathrm{Id}$ for some $\lambda_x \in \mathbb{C}$.
ie $\underbrace{\varphi_x(v) = \lambda_x v}_{\text{eigenspace}}$ $\forall v \in V$

$\therefore$ Any one dim eigenspace of V is a $\mathbb{C}[G]$-submod
but since V is simple $\Rightarrow \dim(V) = 1$
$\therefore$ We've proved that all irreducible reps of finite abelian groups have degree 1
ie $V \cong \mathbb{C}$ as $\mathbb{C}[G]$-module.

$$\mathbb{C}[C_n] \cong \mathbb{C}[x]/_{(x^n-1)} = \underset{\mathbb{C}}{\underset{\cong}{\mathbb{C}[x]/_{(x-1)}}} \times \dots \times \underset{\mathbb{C}}{\underset{\cong}{\mathbb{C}[x]/_{(x-\xi_n)}}}$$

Examples:
Recall any finite abelian group $G = C_{n_1} \times \dots \times C_{n_r}$
Good to show $\rho : G \longrightarrow GL_n(\mathbb{C})$
1. $G = C_n = \langle x \mid x^n = 1 \rangle$.
Let $\lambda_n = e^{2\pi i/n}$ then the irreducible reps of G are all of dim 1, which looks like

$\rho_{\lambda_n} : C_n \longrightarrow GL_1(\mathbb{C}) = \mathbb{C}^*$
$\rho_{\lambda_n}(x^k) \longmapsto \lambda_n^k \qquad 0 \leq k \leq n-1$
and $\rho : C_n \longrightarrow GL_n(\mathbb{C})$
is given by $x \longmapsto \begin{pmatrix} \lambda_n^0 & & 0 \\ & \lambda_n^1 \ddots & \\ 0 & & \lambda_n^{n-1} \end{pmatrix}$

2. $G = C_2 \times C_2$
How many irreducible reps are there?
They are all of dim 1.
$C_2 \times C_2 = \{(x_1, x_2) : x_1^2 = x_2^2 = 1, \ x_1 x_2 = x_2 x_1\}$
There are only $4 = |C_2 \times C_2|$ irreducible reps of $C_2 \times C_2$, which are all of dim 1

$\rho_1 : x_1 \longmapsto 1 \quad \rho_1 : x_2 \longmapsto 1 \quad \longleftrightarrow \quad x_1 u_1 = u_1, \ x_2 u_2 = u_2$
$\rho_2 : x_1 \longmapsto -1 \quad \rho_2 : x_2 \longmapsto 1 \quad \longleftrightarrow \quad x_1 u_1 = -u_1, \ x_2 u_2 = u_2$
$\rho_3 : x_1 \longmapsto 1 \quad \rho_3 : x_2 \longmapsto -1$
$\rho_4 : x_1 \longmapsto -1 \quad \rho_4 : x_2 \longmapsto -1$

$\mathbb{C}[C_2 \times C_2] = U_1 \oplus U_2 \oplus U_3 \oplus U_4$

Definition (example of regular representation)
$\rho_{reg} \equiv \mathbb{C}[G]$ as a $\mathbb{C}[G]$-module
Let G be a finite group of order n.
We know $V = \mathbb{C}[G]$ is a $\mathbb{C}$-algebra
$\Rightarrow$ $\mathbb{C}$-vector space of dimension $|G|$
Let $G = \{1, g_2, g_3, \dots, g_n\}$ be basis of $\mathbb{C}[G] = V$
Define $\rho_{reg} : G \longrightarrow GL(\mathbb{C}[G]) = GL_n(\mathbb{C})$
by $\rho_g(g_i) = g g_i$
ie assign to each chosen $g \in G$, a map/matrix ρ_g which acts on the $\mathbb{C}[G]$ basis by left multiplication
Key point: ρ_{reg} is always reducible because it contains the trivial rep.

Example of ρ_{reg}
Let $G = C_3 = \langle 1, x, x^2 \rangle$, with $1 = g_1$, $x = g_2$, $x^2 = g_3$

Find image of x under $\rho_{reg} : C_3 \longrightarrow GL_3(\mathbb{C})$
$\rho_x(g_1) = \rho_x(1) = x \cdot 1 = x = g_2$
$\rho_x(g_2) = \rho_x(x) = x^2 = g_3$

$\rho_{x^2}(g_0) = \rho_{x^2}(g_3) = x^3 = 1 = g_1$

$\rho_{reg}(x) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ Find $\rho_{reg}(x^2)$ $(\rho_{reg}(1))$

By correspondence theorem, $\mathbb{C}[G]$ viewed as a $\mathbb{C}[G]$-mod is called the regulcr module

The regulcr module is always semisimple, use mascke's.

Example:

Let $G = C_3 = \langle x \mid x^3 = 1 \rangle$

Let $\omega = e^{2\pi i/3}$

Define $u_1 = 1 + x + x^2$

$u_2 = 1 + \omega^2 x + \omega x^2$

$u_3 = 1 + \omega x + \omega^2 x^2$

Apply x to the u_i

$x \cdot u_1 = x + x^2 + 1 = 1 u_1$

$\therefore U_1 = \operatorname{span}\{u_1\} \leq \mathbb{C}[C_3]$ -submod

which is a 1-dim $\Leftrightarrow$ reducible rep of C_3

the trivial rep of C_3 $\rho_1 : C_3 \to GL_1(\mathbb{C})$

$x \mapsto 1$

$x \cdot u_2 = \ldots = \omega u_2$

$U_2 = \operatorname{Span}\{u_2\} \leq \mathbb{C}[C_3]$

is a simple C_3-invcrient $\mathbb{C}[C_3]$-submod

corresponding to $\rho_\omega : C_3 \to GL_1(\mathbb{C})$

$x \mapsto \omega$

$x \cdot u_3 = \ldots \omega^2 u_3$

$U_3 = \operatorname{Span}\langle u_3 \} - \ldots$

$\rho_{\omega^2} : C_3 \to GL_1(\mathbb{C})$

$x \mapsto \omega^2$

$\therefore \mathbb{C}[C_3] = U_1 \oplus U_2 \oplus U_3$

$\rho_1 \oplus \rho_\omega \oplus \rho_{\omega^2}$

$= \begin{pmatrix} 1 & & 0 \\ & \omega & \\ 0 & & \omega^2 \end{pmatrix} \sim (\rho_{reg})$

~~The converse for finite abelian groups.~~

The converse for finite abelian groups

If all irreducible reps of G over $\mathbb{C}$ are of degree 1 then G is abelian.

proof:

1 View $\mathbb{C}[G]$ as $\mathbb{C}[G]$-mod $\leadsto \rho_{reg}$

2 Decompose $\mathbb{C}[G] = U_1 \oplus \dots \oplus U_r$ where each U_i is a simple $\mathbb{C}[G]$-mod

3 $\dim(U_i) = 1$ $\forall i$ by assumption

4 Choose basis $\langle u_1, \dots, u_r \rangle$ st $U_i = \mathrm{Span}\{u_i\}$.

5 Let $g \in G$, the matrix of g on $\mathbb{C}[G]$ is diagonal on the basis u_i

$g u_i = \lambda_g^i u_i$

$$\rho(g) = \begin{pmatrix} \lambda_g^1 & & 0 \\ & \ddots & \\ 0 & & \lambda_g^r \end{pmatrix}$$

6 Diagonal matrice commute $\Rightarrow G$ is commutative because ρ_{reg} is faithful $\Rightarrow G/_{\ker(\rho_{reg})} \cong G/_{\langle 1 \rangle} \cong G \subset \underset{GL_n(\mathbb{C})}{\text{Diagonals}}$

$\therefore$ G is abelian $\square$

Consequence

Any non-abelian group must have an irreducible rep of degree ≥ 2

Artin-Wedderburn Theorem revisited.

Let G be a finite group, $\mathbb{C}[G]$ is a semisimple algebra

$\Leftrightarrow$ $\mathbb{C}[G]$ as a $\mathbb{C}[G]$-module is semisimple

(by characterisation for semisimple algebras)

and by Mascke's theorem

$$\mathbb{C}[G] = \underbrace{U_1 \oplus \dots \oplus U_r}_{\text{simple}} = \underbrace{S_1^{n_1} \oplus \dots \oplus S_r^{n_r}}_{\text{simple}} \quad \text{, degree}$$

where the S_i are simple non-pairwise isomorphic $\mathbb{C}[G]$-mods.

proof: Let $A = \mathbb{C}[G]$ in Wedderburn for algebras

$\mathbb{C}[G]^{op} = \mathrm{End}_{\mathbb{C}[G]}(\mathbb{C}[G])$

$= \mathrm{End}_{\mathbb{C}[G]}(S_1^{n_1} \oplus \dots \oplus S_r^{n_r})$

$= \mathrm{End}_{\mathbb{C}[G]}(S_1^{n_1}) \oplus \dots \oplus \mathrm{End}_{\mathbb{C}[G]}(S_r^{n_r})$

$= M_{n_1}(\underset{\mathbb{C}}{\underset{\wr\wr}{\mathrm{End}(S_1)}}) \oplus \dots \oplus M_{n_r}(\underset{\mathbb{C}}{\underset{\wr\wr}{\mathrm{End}(S_r)}})$

$= M_{n_1}(\mathbb{C}) \oplus \dots \oplus M_{n_r}(\mathbb{C})$

$$((\mathbb{C}[G])^{op})^{op} = \mathbb{C}[G] = (M_{n_1}(\mathbb{C}) \oplus \dots \oplus M_{n_r}(\mathbb{C}))^{op}$$
$$= M_{n_1}(\mathbb{C})^{op} \oplus \dots \oplus M_{n_r}(\mathbb{C})^{op}$$
$$= M_{n_1}(\mathbb{C}^{op}) \oplus \dots \oplus M_{n_r}(\mathbb{C}^{op})$$
$$= M_{n_1}(\mathbb{C}) \oplus \dots \oplus M_{n_r}(\mathbb{C})$$

$$\therefore \mathbb{C}[G] = S_1^{n_1} \oplus \dots \oplus S_r^{n_r}$$
$$= M_{n_1}(\mathbb{C}) \oplus \dots \oplus M_{n_r}(\mathbb{C})$$

QED.

Definition:

$\dim_{\mathbb{C}}(S_i) = n_i$ are the degrees of all the irreducible reps of G.

Corollary

$|G| = n_1^2 + \dots + n_r^2$

proof: $|G| = \dim_{\mathbb{C}} \mathbb{C}[G]$
$$= \dim_{\mathbb{C}} \left(\bigoplus_{i=1}^{r} M_{n_i}(\mathbb{C})\right)$$
$$= \sum_{i=1}^{n} \dim_{\mathbb{C}} (M_{n_i}(\mathbb{C}))$$
$$= \sum_{i=1}^{n} n_i^2$$

Fact: The trivial $\mathbb{C}[G]$-module, $V = \mathbb{C} \longleftrightarrow$ trivial rep $\rho: G \longrightarrow GL_1(\mathbb{C})$

$\mathbb{C}$ is 1 dim and hence simple

G always has a 1 dim rep

So we can always set $n_1 = 1 \quad \forall G$

in $\mathbb{C}[G] = M_{n_1}(\mathbb{C}) \times \dots \times M_{n_r}(\mathbb{C})$

$= \mathbb{C} \times M_{n_2}(\mathbb{C}) \times \dots \times M_{n_r}(\mathbb{C})$

Game time

Rules of the Game

1 use Wedderburn to decompose $\mathbb{C}[G] \cong M_{n_1}(\mathbb{C}) \times \dots \times M_{n_r}(\mathbb{C})$
2 r is the number of conjugacy classes in G
ie number of $\langle x^G \rangle = \{g^{-1}xg : g \in G\}$.
3 $|G| = n_1^2 + \dots + n_r^2$
4 We can always take $n_1 = 1$ ∵ trivial rep exists $\forall G$
5 Each $n_i \mid |G|$ exactly

Goal: find $n_1, \dots, n_r$ for a specific rep.

Examples:

1. $G = C_2 = \langle 1, x \mid x^2 = 1 \rangle$
conjugacy classes: $\langle 1 \rangle, \langle x \rangle$
Solve $|G| = 2 = n_1^2 + n_2^2$
$2 = 1^2 + n_2^2$
$\Rightarrow n_2 = 1$ only solution
$\therefore \mathbb{C}[G] \cong M_1(\mathbb{C}) \times M_1(\mathbb{C})$
$\cong \mathbb{C} \times \mathbb{C}$
$\Rightarrow C_2$ has 2 distinct 1-dim irreducible reps
$\rho_1 : C_2 \to GL_1(\mathbb{C})$ $\quad$ $\rho_2 : C_2 \to GL_1(\mathbb{C})$
$x \mapsto 1$ $\quad$ $x \mapsto -1$

~~Example~~ Examples:

2. $G = C_3 = \langle 1, x, x^2 : x^3 = 1 \rangle$
Cong classes $\langle 1 \rangle, \langle x \rangle, \langle x^2 \rangle$
Solve $|G| = 3 = n_1^2 + n_2^2 + n_3^2$
$3 = 1^2 + n_2^2 + n_3^2$ $\quad n_i \geq 1$
$3 = 1^2 + 1^2 + 1^2$
$\therefore \mathbb{C}[C_3] \cong \mathbb{C} \times \mathbb{C} \times \mathbb{C}$
$\Rightarrow$ C_3 has 3 distinct 1-dim irreducible reps only
$\rho_1 : C_3 \to GL_1(\mathbb{C})$
$x \mapsto 1$ $\quad$ $\rho_2 : x \mapsto \omega$ $\quad$ $\rho_3 : x \mapsto \omega^2$

3. $G = C_2 \times C_2 = \langle 1, x, y, xy : x^2 = y^2 = 1 \; xy = yx \rangle$
Conj classes $\langle 1 \rangle, \langle x \rangle \; \langle y \rangle \; \langle xy \rangle$
Solve $(|G| =) 4 = n_1^2 + n_2^2 + n_3^2 + n_4^2$
$= 1^2 + 1^2 + 1^2 + 1^2$

$\therefore \mathbb{C}[C_2 \times C_2] \cong \mathbb{C} \times \mathbb{C} \times \mathbb{C} \times \mathbb{C}$.

$\Rightarrow$ 4 irreducible 1-dim reps

$\rho_1 : x \mapsto 1 \ y \mapsto 1$ $\quad \rho_2 : x \mapsto 1 \ y \mapsto -1$ $\quad \rho_3 : x \mapsto -1 \ y \mapsto 1$

$\rho_4 : x \mapsto -1, \ y \mapsto -1$.

4. $G = D_6 = \langle x, y \mid x^3 = x^2 = 1, \ yx = x^2 y \rangle$

Conj classes : $\langle 1 \rangle, \langle x, x^2 \rangle, \langle y, xy, x^2 y \rangle$

$\Rightarrow r = 3$

Solve $|G| = 6 = n_1^2 + n_2^2 + n_3^2$

$6 = 1^2 + 1^2 + 2^2$

$\Rightarrow \mathbb{C}[D_6] = \mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C})$

$\Rightarrow D_6$ has 2 distinct 1-dim reps $\rho_1 : x \mapsto 1 \ y \mapsto 1$ $\quad \rho_2 : x \mapsto 1 \ y \mapsto -1$

and 1 2-dim rep

eg $\rho_3 : x \mapsto \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}$ $\quad y \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

5. $D_8 = \langle x, y \mid x^4 = y^2 = 1, \ yx = x^3 y \rangle$

Conj classes : $\langle 1 \rangle, \langle x^2 \rangle, \langle x, x^3 \rangle, \langle y, x^2 y \rangle, \langle xy, x^3 y \rangle$

$\Rightarrow r = 5$

Solve $8 = n_1^2 + n_2^2 + n_3^2 + n_4^2 + n_5^2$

$8 = 1^2 + n_2^2 + n_3^2 + n_4^2 + n_5^2$

$7 = n_2^2 + n_3^2 + n_4^2 + n_5^2$

$7 = 1 + 2^2 + 1 + 1$

$\therefore \mathbb{C}[D_8] \cong \mathbb{C} \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C})$

$\Rightarrow D_8$ has 4 distinct irreducible 1-dim reps

and 1 2-dim irreducible rep.

6. $Q_8 = \langle x, y \mid x^4 = 1, \ x^2 = y^2, \ y^{-1} x y = x^{-1}$

conj classes are same ~~for~~ D_8

Solve same equations $\Rightarrow$ same answers

$\mathbb{C}[Q_8] \cong \mathbb{C} \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C})$.

7. A_4 = even permutations of 4 letters

$= \langle \sigma \in S_4 : \mathrm{sgn}(\sigma) = +1 \rangle$

$s = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$ $\quad t = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$ $\Rightarrow st = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$

$x = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$

$xsx^{-1} = st \quad xtx^{-1} = s \quad xstx^{-1} = t$

Conj classes: $\langle 1 \rangle, \langle s, t, st \rangle, \langle x, xs, xt, xst \rangle$
$\langle x^2, x^2 s, x^2 t, x^2 st \rangle$

$\Rightarrow r = 4$

Solve $12 = n_1^2 + n_2^2 + n_3^2 + n_4^2$

$12 = 1^2 + 1^2 + 1^2 + 3^2$

$\therefore \mathbb{C}[A_4] = \mathbb{C} \times \mathbb{C} \times \mathbb{C} \times M_3(\mathbb{C})$.

Note: 1 Complex reps theory is "easy" since the only associative division algebras that occur over $\mathbb{C}$ is $\mathbb{C}$ itself

No $\mathbb{H}$ over $\mathbb{C}$ $\Rightarrow jz = \bar{z}j$

$\therefore \mathbb{H}$ is not a $\mathbb{C}$-algebra

2 Real rep theory is harder

$\mathbb{R}[G] \cong M_{n_1}(D_1) \times \dots \times M_{n_r}(D_r)$
$\underset{\mathbb{R}, \mathbb{C}, \mathbb{H}}{\text{\rotatebox{90}{$\in$}}}$

Recall: If $x \in G$, then its conjugacy class is $x^G = \langle g^{-1}xg : g \in G$ and conjuganay classes are disjoint.

Definition:

$Z(\mathbb{C}[G]) = \{ z \in \mathbb{C}[G] : xz = zx \ \forall x \in \mathbb{C}[G] \}$.

which is a subalgebra of $\mathbb{C}[G]$ and a $\mathbb{C}$-vector subspace of $\mathbb{C}[G]$ by definition of algebra.

Lemma:

$\dim_{\mathbb{C}}(Z(\mathbb{C}[G])) = r$

Proof: We know we have an isomorphism of algebras

$\mathbb{C}[G] \cong M_{n_1}(\mathbb{C}) \oplus \dots \oplus M_{n_r}(\mathbb{C})$

which is also an isomorphism of $\mathbb{C}$-vector spaces

$\therefore Z(\mathbb{C}[G]) \cong Z(M_{n_1}(\mathbb{C})) \oplus \dots \oplus Z(M_{n_r}(\mathbb{C}))$

$\cong \mathbb{C} \oplus \dots \oplus \mathbb{C}$.

since $Z(M_{n_i}(\mathbb{C})) = \left\{ \begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix} \ \lambda \in \mathbb{C} \right\} \cong \mathbb{C}$

$\therefore \dim_{\mathbb{C}}(Z(\mathbb{C}[G])) = r$.

Theorem:
If G is finite then $\mathbb{C}[G] \cong M_{n_1}(\mathbb{C}) \times \ldots \times M_{n_r}(\mathbb{C})$
$\cong S_1^{n_1} \oplus \ldots \oplus S_r^{n_r}$
where r is the number of conjugancy classes.

proof: Let $x = \sum \lambda_g g \in Z(\mathbb{C}[G])$
and conjugate $\forall h \in G$
$h^{-1}(\underbrace{\sum \lambda_g g}_{\in Z(\mathbb{C}[G])})h = \sum \lambda_g h^{-1}gh$
$\sum \lambda_g g = \sum \lambda_{h^{-1}gh}\, g$
$\Rightarrow \lambda_g = \lambda_{h^{-1}gg} \quad \forall g \in G \quad \forall h \in G$.
$\therefore$ coefficents of elements of $Z(\mathbb{C}[G])$ are constant on conjugacy classes.
So a basis for $Z(\mathbb{C}[G])$ is the set of class sums of the form x which is a linear combanction of $\sum_{g \in k_i} g$
where k_i are conjugacy classes
$\therefore \dim_{\mathbb{C}} Z(\mathbb{C}[G]) =$ no. of conjugacy classes
$= r$

Example of basis for $Z(\mathbb{C}[D_6])$
$D_6 = \langle x, y \mid x^3 = y^2 = 1, yx = x^2 y \rangle$
$1^{D_6} = \langle 1 \rangle \quad x^{D_6} = \langle x, x^2 \rangle \quad y^{D_6} = \langle y, xy, x^2 y \rangle$
$\mathbb{C}[D_6] \cong \mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C})$
A basis for $Z(\mathbb{C}[D_6])$ is $Sp_{\mathbb{C}}\langle 1, x + x^2, y + xy + x^2 y \rangle$
$\therefore \dim_{\mathbb{C}}(Z(\mathbb{C}[D_6]) = 3$.

Consequence
If G is finite abelian, each conj class k_i has one element so
G has exactly $|G|$ conj classes
$\therefore$ G has exactly $|G|$ irreducible reps
ie all $n_i = 1$
eg $|G| = \sum_{i=1}^{r=|G|} n_i^2 \Rightarrow n_i = 1$
$G = C_n \quad \mathbb{C}[C_n] \cong \underbrace{\mathbb{C} \times \mathbb{C} \times \ldots \times \mathbb{C}}_{n \text{ times}}$ (n conj classes)

Nice formulas for conjugacy classes for $C_{n_1} \times \dots C_{n_s}$, D_{2n}, S_n.

1. If G is finite abelian
Let $a \in G$ then $g^{-1}ag = a$ $\forall g \in G$ $\therefore$ $a^G = \langle a \rangle$
Examples. • $G = C_n = \langle x \mid x^n = 1 \rangle$
Let $x^i \in C_n$ the $x^{-1}x^i x = x^i$ $\forall i$
$(x^i)^{C_n} = \langle x^i \rangle$

2. i) G dihedral D_{2n} n is odd.
If n is odd, D_{2n} has $\frac{n+3}{2}$ conjugacy classes
$\langle 1 \rangle, \langle x, x^{-1} \rangle, \langle x^2, x^{-2} \rangle, \dots, \langle x^{\frac{n-1}{2}}, x^{-\frac{n-1}{2}} \rangle, \langle y, xy, \dots, x^{n-1}y \rangle$
eg. D_6 has $\frac{3+3}{2} = 3$ conjugacy classes
$\langle 1 \rangle, \langle x, x^{-1} \rangle, \langle y, xy, x^2y \rangle$
ii) G dihedral D_{2n}, n is even (trick $n = 2m$)
If n is even, then D_{2n} has $m+3$ conjugacy classes
$\langle 1 \rangle, \langle x^m \rangle, \langle x^i, x^{-i} : 1 \leq i \leq m-1 \rangle$
$\langle x^{2j}y : 0 \leq j \leq m-1 \rangle, \langle x^{2j+1}y : 0 \leq j \leq m-1 \rangle.$
eg. D_8 $n = 4 = 2 \times 2$
$\Rightarrow$ D_8 has $m+3 = 2+3 = 5$ conjugacy classes
$\langle 1 \rangle, \langle x^2 \rangle, \langle x^1, x^{-1} \rangle, \langle y, x^2y \rangle, \langle xy, x^3y \rangle$

3. Conjugacy classes in S_n
Group elements that decompose into cycles of the same shape are in the same conj class and there are $p(n)$ of them, where $p(n)$ = partitions of n.
eg. 1 S_3 has 3 conjugacy classes $\langle 1 \rangle$, $\langle (12), (13), (23) \rangle$
$\langle (123), (132) \rangle$
2 S_4 $|S_4| = 4! = 24$ S_4 has 5 conjugacy classes
4, 3+1, 2+2, 2+1+1, 1+1+1+1
$\langle (1) \rangle$, $\langle (12), (13), (14), (23), (24), (34) \rangle$
$\langle (123), (124), (134), (234) \rangle$
$\langle (1234) \rangle$ $\langle$products$\rangle$.

Tensor products:

Beware ⚠

These are not direct products

$V \times W (\cong V \oplus W)$ since fd vs $/\mathbb{F}$.

~~then~~

$\bigoplus_{i=1}^{\infty} V \subset \prod_{i=1}^{\infty} V_i$ is a proper subset for infinite indices

but $\bigoplus_{i=1}^{\infty} V \neq \prod_{i=1}^{\infty} V_i$

eg. Take $V_i = \mathbb{R}$ $\forall i$

$(1,0,\dots,0) \in \bigoplus_{i=1}^{\infty} \mathbb{R}$ but $(1,1,\dots,1) \notin \bigoplus_{i=1}^{\infty} \mathbb{R}$

however both elements are in $\prod_{i=1}^{\infty} \mathbb{R}$

* for finite indices $\bigoplus_{i=1}^{n} V_i = \prod_{i=1}^{n} V_i$ hence no distinction with $\mathbb{C}[G]$.

Tensor product construction for vector spaces.

The idea is to construct an $\mathbb{F}$-vector space $V \otimes_{\mathbb{F}} W$ whose elements look like

$\sum_{i=1}^{k} v_i \otimes w_i = v_1 \otimes w_1 + v_2 \otimes w_2 + \dots + v_k \otimes w_k$.

where k is arbitrary, not unique

where $v_i \otimes w_i$ are the generators called simple tensors

and we want $- \otimes_{\mathbb{F}} -$ to obey equivalence relations

1. $(v+v') \otimes w = v \otimes w + v' \otimes w$
2. $v \otimes (w+w') = v \otimes w + v \otimes w'$
3. key rule $\lambda(v \otimes w) = \lambda v \otimes_{\mathbb{F}} w = v \otimes \lambda w$ where $\lambda \in \mathbb{F}$, $v, v' \in V$, $w, w' \in W$

Generally if $v = \sum \lambda_i v_i$ $w = \sum \mu_j w_j$ then $v \otimes w = \sum \lambda_i \mu_j (v_i \otimes v_j)$

so $(2v_1 - v_2) \otimes (w_1 + w_2) = 2v_1 \otimes w_1 + 2v_1 \otimes w_2 - v_2 \otimes w_1 - v_2 \otimes w_2$

$+ v_2 \otimes -w_1$

and if given a basis $\langle e_1, \dots, e_m \rangle$ and $\langle f_1, \dots, f_n \rangle$ for V and W respectively, then we want $\langle e_i \otimes f_j \rangle_{\substack{1 \le i \le m \\ 1 \le i \le n}}$ to be basis for $V \otimes_{\mathbb{F}} W$ over new space

$\therefore \dim_{\mathbb{F}}(V \otimes W) = \dim_{\mathbb{F}}(V) \dim_{\mathbb{F}}(W)$

Example: $\dim(\mathbb{R}^m \otimes_{\mathbb{F}} \mathbb{R}^n) = \dim_{\mathbb{R}}(\mathbb{R}^m) \dim_{\mathbb{R}}(\mathbb{R}^n) = mn$

can we make such a space exist?

Yes & main use is to external scalars

- Tensor products is a machine to turn bilinear forms into linear

Definition:

Given a vector space V and W, by a tensor product $(V \otimes_{\mathbb{F}} W)$ over $\mathbb{F}$ we mean

1. a vector space $V \otimes W$
2. a bilinear map $- \otimes_{\mathbb{F}} - : V \times W \longrightarrow V \otimes W$
$$(v, w) \longmapsto v \otimes w$$
3. given any bilinear map $f : V \times W \longrightarrow U$,
$\exists$ a unique linear map $\tilde{f} : V \otimes W \longrightarrow U$ such that

$$\begin{tikzcd} V \times W \arrow[r, "f"] \arrow[dr, dashed, "-\otimes-"'] & U \\ & V \otimes W \arrow[u, "\tilde{f}"'] \end{tikzcd}$$

commutes $\}$ ie $f = \tilde{f} \circ - \otimes -$

ie every bilinear map can be factored through $- \otimes -$

This is the universal property U.P

Note: 1. To prove two tensor product spaces are isomorphic just define maps that satisfy U.P

2 Calculate using bilinearity (key rule) and don't forget, not all tensors are simple $(v_1 \otimes w_1) + (v_2 \otimes w_2) \neq v \otimes w$.

Examples of $\mathbb{Z}$-mods tensored over $\mathbb{Z}$.

1. $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$

Use key rule : $ra \otimes_{\mathbb{Z}} b = a \otimes_{\mathbb{Z}} rb$

Let $\mathbb{Z}/2\mathbb{Z} = \{e, a\}$ $\quad a + a = 2a = e$

$\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots, \}$.

$\Rightarrow$ hence two generators $e \otimes \mathbb{Z}$, and $a \otimes \mathbb{Z}$

Consider $a \otimes 4 = a \otimes 2 \cdot 2 = 2a \otimes 2$
$= e \otimes 2$

$\Rightarrow a \otimes 2m = e \otimes m$

$a \otimes 9 = a \otimes 9 \cdot 1 = 9 \cdot a \otimes 1 = a \otimes 1$

Can add $(a \otimes 2) + (a \otimes 2) = (a + a) \otimes 2 = 2a \otimes 2 = e \times 2$

only have 2 elements $(a \otimes n) =$ $(e \otimes m)$ if n even and $(a \otimes k)$ when k is odd

$\Rightarrow \mathbb{Z}/2\mathbb{Z}$.

2. $\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/3\mathbb{Z} = \langle 0 \rangle$

Let $\mathbb{Z}/2\mathbb{Z} = \langle e, a \rangle$ $\quad 2a = e$

$\mathbb{Z}/3\mathbb{Z} = \langle e, b, 2b \rangle$ $\quad 3b = e$

6 elements to consider, $e\otimes e$, $e\otimes b$, $e\otimes 2b$, $a\otimes e$, $a\otimes b$, $a\otimes 2b$.

consider $e\otimes b = 2e\otimes_{\mathbb{Z}} b = e\otimes 2b$

$e\otimes b = 3e\otimes b = e\otimes 3b = e\otimes e$

$\therefore e\otimes e = e\otimes b = e\otimes 2b$

$a\otimes_{\mathbb{Z}} b = 3a\otimes_{\mathbb{Z}} b = a\otimes 3b = a\otimes e$

$a\otimes e = a\otimes 2e = 2a\otimes e = e\otimes e$

$a\otimes 2b = 2a\otimes b = e\otimes b = e\otimes e$.

$\therefore e\otimes e = a\otimes b = a\otimes 2b$

$e\otimes e = \langle 0\rangle$

$\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/3\mathbb{Z} \longrightarrow \{0\}$.

$e\otimes e \longmapsto 0$

3. $\mathbb{Z}^2\otimes\mathbb{Z}^2 \cong \mathbb{Z}^4$ since $\mathbb{Z}^2$ is a 2-dim module over $\mathbb{Z}$

Formal properties:

Let U, W, V be R-mods

1 Let R be an R-mod, then $R\otimes_R V \cong V$

$\lambda\otimes v \longmapsto \lambda v$

2. $V\otimes W \cong W\otimes V$

$v\otimes w \longmapsto w\otimes v$

3. $U\otimes(V\otimes W) \cong (U\otimes V)\otimes W \cong U\otimes V\otimes W$

$u\otimes(v\otimes w) \longmapsto (u\otimes v)\otimes w$

4 $U\otimes(V\oplus W) \cong (U\otimes V)\oplus(U\otimes W)$

$u\otimes(v,w) \longmapsto (u\otimes v, u\otimes w)$

proof 1. $R\otimes_R V \xrightarrow{\cong} V$ $\lambda\otimes v \longmapsto \lambda v$

Danger ⚠ $\sum \lambda_i\otimes v_i \longmapsto \sum\lambda_i v_i$ is not well defined since $\lambda_i\otimes v_i$ as generators do not form a basis, this is why we need to use U.P.

The ~~map~~ map $f: R\times V \longrightarrow V$ is bilinear

$(\lambda, v) \longmapsto \lambda v$

consider $-\otimes- : R\times V \longrightarrow R\otimes_R V$

$(\lambda, v) \longmapsto 1\otimes v$ is a homomorphism of modules

So $\exists$ a linear map $\hat{f}: R\otimes_R V \longrightarrow V$ st $\hat{f}(\lambda\otimes v) = \lambda v$

$\hat{f}(\sum\lambda_i\otimes v_i) = \sum \lambda_i v_i$ ~~an~~ where $-\otimes-$ and $\hat{f}$ are mutual inverses

$-\otimes- \circ \hat{f}(\lambda\otimes v) = -\otimes-(\lambda v)$

$= \lambda\otimes v$

$\tilde{f} \circ - \otimes - (v) = \bar{f}(1 \otimes v) = 1 \cdot v = v$

Tensor products of matrices over a field $\mathbb{F}$.

Let $\mathbb{F}$ be a field, then $M_n(\mathbb{F}) \otimes M_n(\mathbb{F}) \cong M_{nm}(\mathbb{F})$

proof: Define the "secret" bilinear map $f : M_m(\mathbb{F}) \times M_n(\mathbb{F}) \to M_{mn}(\mathbb{F})$
by thinking of $M_{mn}(\mathbb{F})$ as $M_m(M_n(\mathbb{F}))$
ie let $A = (a_{ij}) \in M_m(\mathbb{F})$ and $B \in M_n(\mathbb{F})$

$$f(A,B) = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ \vdots & \ddots & & \vdots \\ a_{m1}B & - & - & a_{mn}B \end{pmatrix}$$

~~Swn~~ Since ~~F~~ f is bilinear, $\exists$ a linear map
$\bar{f} : M_m(\mathbb{F}) \otimes_{\mathbb{F}} M_n(\mathbb{F}) \to M_{mn}(\mathbb{F})$
which is an isomorphism as it maps basis to basis
Let $\langle e_{ij} \rangle$ and ~~...~~ $\langle e'_{hk} \rangle$ be the basis of elementry matrices of $M_m(\mathbb{F})$ and $M_n(\mathbb{F})$ respectively
From def of $\bar{f}$, we can see $\bar{f}(e_{ij} \otimes e'_{hk})$ = an elementry matrix in $M_{mn}(\mathbb{F})$.
Further more $\bar{f}$ is 1-1 mapping of the set $\langle e_{ij} \otimes e'_{hk} \rangle$ onto the set of all elementry matrices in $M_{mn}(\mathbb{F})$

Examples.

1. $I_n \otimes I_m \cong I_{mn}$
2. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes I_2 = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix}$ 4x4

Tensor products of algebras $\mathbb{C}[G]$ over $\mathbb{C}$.

Let A and B be two algebras over a field $\mathbb{F}$
Then $A \otimes_{\mathbb{F}} B$ is a vector space over $\mathbb{F}$ which becomes an algebra over $\mathbb{F}$ by defining the following multiplication
$(a_1 \otimes b_1)(a_2 \otimes b_2) = (a_1 a_2 \otimes b_1 b_2)$ ²
$(\sum a_i \otimes b_i) \cdot (\sum a_j \otimes b_j) = \sum_{i,j} a_i a_j \otimes b_i b_j$
with identity $1 \otimes 1$

Examples :

1. $\mathbb{C} \otimes \mathbb{C}_{\mathbb{C}} \cong \mathbb{C}$
2. Recall $\mathbb{C}[G]$ is a $\mathbb{C}$-algebra
   Let V & W be $\mathbb{C}[G]$ -modules, then one can define a structure of $\mathbb{C}[G]$ mods on $V \otimes W$ by defining
   $g \cdot (v \otimes w) = gv \otimes gw$

Further playtime with Wedderburn - decompose direct products of groups using Wedderburn.

We know how many irreducible reps G has since
$\mathbb{C}[G] \cong M_{n_1}(\mathbb{C}) \times \dots \times M_{n_r}(\mathbb{C})$
What about $\mathbb{C}[G \times H] \xrightarrow{\cong} \mathbb{C}[G] \otimes_{\mathbb{C}} \mathbb{C}[H]$
$(g, h) \longmapsto g \otimes h$ check dim a VS$/\mathbb{C}$

Example :

1. We known $\mathbb{C}[D_6] \cong \mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C})$
   Think of $\mathbb{C}$ as $M_1(\mathbb{C})$ and use $M_m(\mathbb{F}) \otimes M_n(\mathbb{F}) \cong M_{mn}(\mathbb{F})$

$$
\begin{aligned}
\mathbb{C}[D_6 \times D_6] &\cong \mathbb{C}[D_6] \otimes_{\mathbb{C}} \mathbb{C}[D_6] \\
&\cong [\mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C})] \otimes_{\mathbb{C}} [\mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C})] \\
&\cong ((\mathbb{C} \otimes \mathbb{C}) \times (\mathbb{C} \otimes \mathbb{C}) \times (\mathbb{C} \times M_2(\mathbb{C})) \\
&\quad (\mathbb{C} \otimes \mathbb{C} \times \mathbb{C} \otimes \mathbb{C} \times \mathbb{C} \otimes M_2(\mathbb{C})) \\
&\quad (M_2(\mathbb{C}) \otimes \mathbb{C}) \times (M_2(\mathbb{C}) \otimes \mathbb{C}) \times (M_2(\mathbb{C}) \otimes M_2(\mathbb{C})) \\
&\cong \mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C}) \times \mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C}) \times M_2(\mathbb{C}) \times M_2(\mathbb{C}) \\
&\quad \times M_4(\mathbb{C}) \\
&\cong \mathbb{C}^{(4)} \times M_2(\mathbb{C})^{(4)} \times M_4(\mathbb{C})
\end{aligned}
$$

$36 = 4 + 16 + 16$

$\therefore$ $D_6 \times D_6$ has 4 distinct simple 1-dim reps
4 2-dim reps
1 4-dim reps

2. Binary Dihedral group $D_6^* \cong C_3 \rtimes C_4$ has order 12
   $D_6^* = \langle x, y \mid x^3 = y^4 = 1,\ yx = x^2 y \rangle$
   Conj classes : $\langle 1 \rangle$, $\langle x, x^2 \rangle$, $\langle y, xy, x y^2 \rangle$, $\langle y^2 \rangle$ $\langle xy^2, x^2y^3 \rangle$
   $\langle y^3, xy^3, x^3 \rangle \Rightarrow r = 6$
   $12 = 1^2 + n_2^2 + n_3^2 + n_4^2 + n_5^2 + n_6^2$
   $= 1^2 + 1^2 + 1^2 + 1^2 + 2^2 + 2^2$

$\therefore \mathbb{C}[D_6^*] \cong \mathbb{C}^{(4)} \times M_2(\mathbb{C})^{(2)}$

$\mathbb{C}[D_6^* \times D_6^*] \cong \mathbb{C}[D_6^*] \otimes_{\mathbb{C}} \mathbb{C}[D_6^*]$

$\cong [\mathbb{C}^{(4)} \times M_2(\mathbb{C})^{(2)}] \otimes [\mathbb{C}^{(4)} \times M_2(\mathbb{C})^{(4)}]$

$\cong (\mathbb{C}^{(4)} \times \mathbb{C}^{(4)}) \times (\mathbb{C}^{(4)} \otimes M_2(\mathbb{C})^{(2)}] \times (M_2(\mathbb{C})^{(2)} \times \mathbb{C})$

$\times (M_2(\mathbb{C})^{(2)} \otimes M_2(\mathbb{C})^{(2)})$

$\cong \mathbb{C}^{(16)} \times M_2(\mathbb{C})^{(8)} \times M_2(\mathbb{C})^{8} \times M_4(\mathbb{C})^{(4)}$

$144 = 16 + 64 + 64.$

Try $\mathbb{C}[D_6^* \times D_6]$.

Induced representations.

Goal: To construct a representation of G by inducing a known rep of a subgroup of G and using G's structure to make the large rep. a rep of G. Going to pick easy reps of cyclic subgroups to induce from.

construction:

Let G be a finite ~~subgroup~~ group $H \subset G$ a subgroup and V a left $\mathbb{C}[H]$-module. Then construct a $\mathbb{C}[G]$-module

$\mathrm{Ind}_H^G(V) = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V.$

Think of V as $\mathbb{C}[G]$-mod by letting G act trivially on V ie $g \cdot V \cong V \cong \mathbb{C}$.

by doing the following :-

R $g(g_i \otimes v) = gg_i \otimes v = g_i h \otimes v = g_i \otimes hv.$

Make a $\mathbb{C}$-vector space $\mathbb{C}[G] \otimes_{\mathbb{C}} V$ by considering

$Y = \langle gh \otimes v - g \otimes hv \mid g \in G, h \in H, v \in V \}.$

and let $\mathrm{Ind}_H^G(V) = \mathbb{C}[G] \otimes_{\mathbb{C}} V / Y$

1. Define the left cosets $G/H = \langle gH : g \in G \rangle$ where $g_1 H = g_2 H$ iff $g_2^{-1} g \in H$.
2. Suppose $|G/H| = n$, then take $\langle g_1, \ldots, g_n \rangle$ to be coset representatives, for G/H ie $G = \coprod_{i=1}^{n} g_i H$.
3. Let $\mathbb{C}[G/H]$ be the $\mathbb{C}$-vector space with basis $\langle g_1, \ldots, g_n \rangle$
4. Define the induced $\mathbb{C}[G]$-module

$\mathrm{Ind}_H^G(V) = \mathbb{C}[G/H] \otimes_{\mathbb{C}} V$

regarding each as a $\mathbb{C}$-vector space.

where $g(g_i \otimes v) = gg_i \otimes v$ where $g \in G$

(G-action on tensors)

5. Using group relations to express simple tensors as basis in terms of $\mathbb{C}$-tensors

Example:

$G = D_6 = \langle 1, x, x^2, y, xy, x^2y \rangle \quad x^3 = y^2 = 1 \quad yx = x^2y$

$H = C_3 = \langle 1, x, x^2 \rangle \quad x^3 = 1$

$H \trianglelefteq G$ (but not neccessary)

Let V be the 1-dim $\mathbb{C}[C_3]$ module where x acts by $\omega = e^{2\pi i/3}$ (trivially)

$\rho: C_3 \longrightarrow GL_1(\mathbb{C})$

$x \longmapsto \omega$

$|G/H| = |D_6/C_3| = 2$ cosets $\Rightarrow H \trianglelefteq G$

I can construct the induced rep by taking

$Q = \langle 1, y \rangle$ as coset reps

So $\mathrm{Ind}_{C_3}^{D_6}(V) = \mathbb{C}[D_6] \otimes_{\mathbb{C}[C_3]} V$

$= \mathbb{C}[D_6/C_3] \otimes_{\mathbb{C}} V$

$= \mathbb{C}[Q] \otimes_{\mathbb{C}} V$

with basis $\langle 1\otimes 1, y \otimes 1 \rangle$

How do group elements act?

$x(1\otimes 1) = x \cdot 1 \otimes 1 = 1 \otimes \underbrace{x \cdot 1}_{\in V} = 1 \otimes \omega \cdot 1 = (1\otimes 1)\omega$

$x(y\otimes 1) = xy \otimes 1 = yx^2 \otimes 1 = y \otimes x^2 1 = y \otimes \omega^2 \cdot 1 = (y\otimes 1)\omega^2$

$x \sim \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix} \qquad \rho_{D_6}: D_6 \longrightarrow GL_2(\mathbb{C})$

$x \longmapsto \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$

$y(1\otimes 1) = y1 \otimes 1 = 1 \otimes y1 = 1 \otimes y \ (= e_2)$

$y(y\otimes 1) = y^2 \otimes 1 = 1 \otimes 1 \quad (= e_1)$

$\rho_{D_6}(y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

So we've constructed a ~~dim~~ 2-dim rep of D_6. But is it the irreducible one occuring in $\mathbb{C}[D_6] \cong \mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C})$.

Calculate $\mathrm{End}_{\mathbb{C}[D_6]}(\rho_{D_6}) = \left\{ A \in GL_2(\mathbb{C}) : \begin{array}{l} A\rho(x) = \rho(x)A \\ A\rho(y) = \rho(y)A \end{array} \right\}$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \cong \mathbb{C} = \langle a \in \mathbb{C} \rangle$

Exercise $G = D_6$, $H = \langle 1, y \rangle = C_2$

Take 1-dim $\mathbb{C}[C_2]$-module corresponding to the 1-dim trivial rep $\rho_V: C_2 \longrightarrow GL_1(\mathbb{C})$

$y \longmapsto 1$

$1 \longmapsto 1$

$V \cong \mathbb{C}$ with basis 1 $\quad D_6/C_2 \cong \langle 1, x, x^2 \rangle = Q$

Answer to exercise overleaf:

Notice that $C_2 \ntriangleleft D_6$

Take 1-dim $\mathbb{C}[C_2]$-irreducible rep V corresponding to the trivial rep of C_2

$\rho: C_2 \to \mathbb{C}$

$1 \mapsto 1$ $\qquad y \cdot 1 = 1$ where $V \cong \mathbb{C}$ with basis $\{1\}$.

$y \mapsto 1$

In this case $|D_6/C_2| = 3$ so we can identify it with $Q = \langle 1, x, x^2 \rangle$ ie the set of coset reps for D_6/C_2

Construct the induced rep:-

$$\mathrm{Ind}_{C_2}^{D_6}(\rho) = \mathbb{C}[D_6] \otimes_{\mathbb{C}[C_2]} V$$
$$= \mathbb{C}[D_6/C_2] \otimes_{\mathbb{C}} V$$
$$= \mathbb{C}[Q] \otimes_{\mathbb{C}} V$$

with basis $\langle 1\otimes 1, x \otimes 1, x^2 \otimes 1 \rangle$

$x(1\otimes 1) = x \cdot 1 \otimes 1 = x \otimes 1$

$x(x \otimes 1) = x^2 \otimes 1$ $\qquad x \sim \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

$x(x^2 \otimes 1) = x^3 \otimes 1 = 1 \otimes 1$

$y(1\otimes 1) = y \cdot 1 \otimes 1 = 1 \cdot y \otimes 1$
$= 1 \otimes y \cdot 1$
$= 1 \otimes 1$

$y(x \otimes 1) = x^2 y \otimes 1 = x^2 \otimes y \cdot 1$ $\qquad y \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$
$= x^2 \otimes 1$

$y(x^2 \otimes 1) = yx^2 \otimes 1 = xy \otimes 1$
$= x \otimes y 1$
$= x \otimes 1$

$$\mathrm{End}_{\mathbb{C}[D_6]}(\mathrm{Ind}\uparrow_{C_2}^{D_6}(V)) = \left\{ \begin{pmatrix} a & b & b \\ b & a & b \\ b & b & a \end{pmatrix} a, b \in \mathbb{C} \right\}$$

$= \mathbb{C} \oplus \mathbb{C}$ which is not a division ring.

$\therefore \mathrm{Ind}_{C_2}^{D_6}(V)$ is reducible $\sim$ 1 dim irred $\oplus$ 2-dim irred,

$\mathbb{C}[D_6] = \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C})$

Example Q_8 non abelian, $\forall H \triangleleft Q_8$

$G = Q_8 = \langle x, y \mid x^2 = y^2, x^4 = 1, yx = x^3 y \rangle$ $\quad xy = yx^3$

Let $H = C_4 = \langle 1, x, x^2, x^3 \rangle$ $\quad x^4 = 1$

Let V be the 1-dim $\mathbb{C}[C_4]$-module where x acts as $i = \sqrt{-1}$

$\rho: C_4 \to \mathbb{C}$ $\qquad x \cdot 1 = i$

$x \mapsto i$

$|Q_8/C_4| = 2$ Let $Q = \langle 1, y \rangle$ $C_4 \lhd Q_8$

Construct $\mathrm{Ind}_{C_4}^{Q_8}(V) = \mathbb{C}[Q] \otimes_{\mathbb{C}} V$

with basis $\{1\otimes 1, y\otimes 1\}$.

$$x(1\otimes 1) = x\cdot 1\otimes 1 = 1\cdot x\otimes 1$$
$$= 1\otimes x\cdot 1$$
$$= 1\otimes i$$
$$= (1\otimes 1)i \qquad x \sim \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$x(y\otimes 1) = yx^3\otimes 1$$
$$= y\otimes x^3 1$$
$$= y\otimes -i = (y\otimes 1)-i$$

$$y(1\otimes 1) = y\otimes 1$$
$$y(y\otimes 1) = y^2\otimes 1 \qquad y \sim \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
$$= x^2\otimes 1$$
$$= 1\otimes x^2\cdot 1 = 1\otimes -i^2$$
$$= (1\otimes 1)-1$$

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\mathbb{C}[Q_8] \cong \mathbb{C}^{(4)} \times M_2(\mathbb{C})$

$$A\rho(x) = \rho(x)A$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} ai & -bi \\ ci & -di \end{pmatrix} = \begin{pmatrix} ai & bi \\ -ci & di \end{pmatrix}$$
$$A = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$$
$$A\rho(y) = \begin{pmatrix} 0 & -a \\ d & 0 \end{pmatrix} = \begin{pmatrix} 0 & -d \\ a & 0 \end{pmatrix} = \rho(y)A \quad \Rightarrow a = d$$

$A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ $\therefore$ $\mathrm{End}_{\mathbb{C}[Q_8]}(\mathrm{Ind}_{C_4}^{Q_8}(V)) = \left\{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in \mathbb{C}\right\} \cong \mathbb{C}$
which is a division ring

$\therefore$ The rep is irreducable.

Note: $\{1\} \subseteq G$, ~~whatever~~ $\mathrm{Ind}_{\{1\}}^{G}(V)$ = regular rep of G.

Real Representation Theory

In general $|G| \neq 0$ in $\mathbb{F}$ and G finite. Maschke's Theorem still holds so $\mathbb{F}[G]$ is still semisimple.
$\mathbb{F}[G] \cong M_{n_1}(D_1) \times \ldots \times M_{n_r}(D_r)$ where D_i are division rings over $\mathbb{F}$.
We are not going to put an interpration on r.
But over $\mathbb{R}$ you can say exactly what the division rings are!

Frobenius Theorem:
The only finite dimensional assosiative division algebras that occur over $\mathbb{R}$ are either i) $\mathbb{R}$ or ii) $\mathbb{C}$ or iii) $\mathbb{H}$.

In $\mathbb{R}[G]$ all three types occur.

Examples:

1. $\mathbb{R}[C_2] \cong \mathbb{R} \times \mathbb{R}$
2. $\mathbb{R}[C_3] = \mathbb{R} \times \mathbb{C}$
3. $\mathbb{R}[Q_8] \cong \mathbb{R}^{(4)} \times \mathbb{H}$.

$\mathbb{R}[C_3] \cong \mathbb{R} \times \mathbb{C}$

$$\mathbb{R}[x]/(x^3-1) \cong \mathbb{R}[x]/x-1 \times \mathbb{R}[x]/(x^2+x+1)$$
$$\cong \mathbb{R} \times \mathbb{C}$$

We know $\mathbb{R}[x]/(x^2+1) \cong \mathbb{C}$

$$\mathbb{R}[x]/x^2+a^2 \cong \mathbb{R}[y]/(y^2+1) \text{ if you put } y=\frac{x}{a}$$
$$\cong \mathbb{C}$$

$\therefore x^2+x+1 = (x+\frac{1}{2})^2 + (\sqrt{3}/2)^2 = t^2+1$

Let $t = \frac{x+1/2}{\sqrt{3}/2}$ $\therefore \mathbb{R}[x]/x^2+x+1 \cong \mathbb{R}[t]/t^2+1 \cong \mathbb{C}$.

$\mathbb{R}[C_3] \cong \mathbb{R} \times \mathbb{C}$.

From Wedderburn, we know $\mathbb{C}[G] \cong M_{n_1}(\mathbb{C}) \times \ldots \times M_{n_r}(\mathbb{C})$
We are going to construct an explicit isomorphism
$G = D_6$. We know $\mathbb{C}[D_6] \cong \mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C})$

$\rho_1 : D_6 \longrightarrow \mathbb{C}$	$\rho_2 : D_6 \longrightarrow \mathbb{C}$	$\rho_3 : D_6 \longrightarrow M_2(\mathbb{C})$
$x \longmapsto 1$	$x \longmapsto 1$	$x \longmapsto \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$ $\quad \omega = e^{2\pi i/3}$
$y \longmapsto 1$	$y \longmapsto -1$	$y \longmapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Let $\Phi : \mathbb{C}[D_6] \to \mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C})$

Let $\alpha = a \cdot 1 + b \cdot x + c \cdot x^2 + dy + exy + fx^2 y$

$\rho_1(\alpha) = a+b+c+d+e+f$ augmentation map.

$\rho_2(\alpha) = a+b+c-d-e-f$

$\rho_3(\alpha) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + b\begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix} + c\begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix} + \dots$

$\Phi(\alpha) = (\rho_1(\alpha), \rho_2(\alpha), \rho_3(\alpha))$

Proposition: $\mathbb{R}[Q_8] \cong \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{H}$

Proof: Write down irreducible reps of Q_8

$\rho_1 : Q_8 \to \mathbb{R}$, $x \mapsto 1$, $y \mapsto 1$

$\rho_2 : Q_8 \to \mathbb{R}$, $x \mapsto 1$, $y \mapsto -1$

$\rho_3 : Q_8 \to \mathbb{R}$, $x \mapsto -1$, $y \mapsto 1$

$\rho_4 : Q_8 \mapsto \mathbb{R}$, $x \mapsto -1$, $y \mapsto -1$

$\rho_5 : Q \mapsto \mathbb{H}$, $x \mapsto i$, $y \mapsto j$

$\Phi : \mathbb{R}(Q_8) \xrightarrow{\cong} \mathbb{R}^{(4)} \times \mathbb{H}$

$\alpha \mapsto (\rho_1(\alpha), \rho_2(\alpha), \rho_3(\alpha), \rho_4(\alpha), \rho_5(\alpha))$.

Applications:

1. $\mathbb{R}[Q_8 \times C_3] \cong \mathbb{R}[Q_8] \otimes_{\mathbb{R}} \mathbb{R}[C_3]$
$\cong (\mathbb{R}^{(4)} \times \mathbb{H}) \otimes_{\mathbb{R}} (\mathbb{R} \times \mathbb{C})$

What is $\mathbb{H} \otimes \mathbb{C}$?

We know $\mathbb{R} \otimes_{\mathbb{R}} \mathbb{H} = \mathbb{H}$

$\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}$.

2. $\mathbb{R}[Q_8 \times Q_8] \cong \mathbb{R}(Q_8) \otimes \mathbb{R}(Q_8)$
$\cong (\mathbb{R}^{(4)} \times \mathbb{H}) \otimes_{\mathbb{R}} (\mathbb{R}^{(4)} \times \mathbb{H})$

What is $\mathbb{H} \otimes \mathbb{H}$?

~~W~~

We have tensored so far over base fields $\mathbb{C}$ and $\mathbb{R}$.

The distinction is that $\lambda x \otimes_{\mathbb{R}} y = x \otimes \lambda y$ $\lambda \in \mathbb{R}$

But I don't know what $i x \otimes y$ will be.

i can't ship i over in $- \otimes_{\mathbb{R}} -$ but I can in $- \otimes_{\mathbb{C}} -$

$-\otimes_{\mathbb{R}}-$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$
$\mathbb{R}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$
$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}\times\mathbb{C}$	$M_2(\mathbb{C})$
$\mathbb{H}$	$\mathbb{H}$	$M_2(\mathbb{C})$	$M_4(\mathbb{C})$

Proposition: $\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C} \cong \mathbb{C}\times\mathbb{C}$ is an isomorphism of $\mathbb{R}$ algebras.

proof: Let $\{1\otimes 1,\ i\otimes 1,\ 1\otimes i,\ i\otimes i\}$ be a basis for $\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C}$ over $\mathbb{R}$

$1\otimes 1$ is identity on LHS. For an iso I require $1\otimes 1 \longmapsto (1,1)$

Try $\varepsilon_1 = \dfrac{(1\otimes 1)+(i\otimes i)}{2}$ $\quad \varepsilon_2 = \dfrac{(1\otimes 1)-(i\otimes i)}{2}$

$$\varepsilon_1\varepsilon_1 = \frac{\{(1\otimes 1)+(i\otimes i)\}\cdot((1\otimes 1)\mp(i\otimes i))}{4}$$

$$= \frac{2(1\otimes 1)+2(i\otimes i)}{4} = \varepsilon_1$$

$\therefore\ \varepsilon_1^2 = \varepsilon_1$ ie ε_1 is independent

Similarly check $\varepsilon_2^2 = \varepsilon_2$ and clearly $\varepsilon_1+\varepsilon_2 = 1\otimes 1$

$\therefore\ \varphi : \mathbb{C}\times\mathbb{C} \longrightarrow \mathbb{C}\otimes_{\mathbb{R}}\mathbb{C}$

$(1,1) \longmapsto (1\otimes 1)$

$(1,0) \longmapsto \varepsilon_1$

$(0,1) \longmapsto \varepsilon_2$

$i\varepsilon_1 = \dfrac{i((1\otimes 1)+(i\otimes i))}{2} = \dfrac{i\otimes 1 - 1\otimes i}{2}$

$i\cdot\varepsilon_2 = \dfrac{i((1\otimes 1)-(i\otimes i))}{2} = \dfrac{i\otimes 1 + 1\otimes i}{2}$

$$\left.\begin{array}{l} \varphi : \mathbb{C}\times\mathbb{C} \xrightarrow{\cong} \mathbb{C}\otimes_{\mathbb{R}}\mathbb{C} \\ (1,0) \longmapsto \varepsilon_1 \\ i(1,0) = (i,0) \longmapsto \dfrac{i\otimes 1 - 1\otimes i}{2} \\ (0,1) \longmapsto \varepsilon_2 \\ (0,i) \longmapsto \dfrac{i\otimes 1 + 1\otimes i}{2} \end{array}\right\} \text{form basis for } \mathbb{C}\otimes_{\mathbb{R}}\mathbb{C}$$

Proposition: $\mathbb{H}\otimes_{\mathbb{R}}\mathbb{C} \cong M_2(\mathbb{C})$

proof: $\{1, i, j, k\}$ $\quad \{1, i\}$

$1\otimes 1 \qquad 1\otimes i$

$i\otimes 1 \qquad i\otimes i \qquad 1\otimes 1 \sim \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad 1\otimes i \sim \begin{pmatrix}i & 0\\ 0 & i\end{pmatrix}$

$j\otimes 1 \qquad j\otimes i \qquad i\otimes 1 \sim \begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix} \qquad i\otimes i \sim \begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix}$

$k\otimes 1 \qquad k\otimes i \qquad j\otimes 1 \sim \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} \qquad j\otimes i \sim \begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}$

$k\otimes 1 \sim \begin{pmatrix}0 & i\\ i & 0\end{pmatrix} \qquad k\otimes i \sim \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$

These matrices form basis $M_2(\mathbb{C})$ & preserve multiplication.

We saw that $\mathbb{R}[C_3] \cong \mathbb{R} \times \mathbb{C}$

$\mathbb{C}[C_3] \cong \mathbb{R}[C_3] \otimes_{\mathbb{R}} \mathbb{C}$

$\cong (\mathbb{R} \times \mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C}$

$\cong (\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C}) \times (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})$

$\cong \mathbb{C} \times \mathbb{C} \times \mathbb{C}.$

$\mathbb{R}[Q_8] \cong \mathbb{R}^{(4)} \times \mathbb{H}$

$\mathbb{R}[D_8] \cong \mathbb{R}^{(4)} \times M_2(\mathbb{R})$

So real rep theory allows us to distinguish between Q_8 and D_8.

whereas $\mathbb{C}[Q_8] \cong \mathbb{C}[D_8]$.

$\mathbb{C}[D_8] \cong \mathbb{R}[D_8] \otimes_{\mathbb{C}} \mathbb{C}$

$\cong \mathbb{C}^{(4)} \times M_2(\mathbb{C})$

$\mathbb{C}[Q_8] \cong \mathbb{R}[Q_8] \otimes_{\mathbb{R}} \mathbb{C}$

$\cong (\mathbb{R}^{(4)} \times \mathbb{H}) \otimes_{\mathbb{R}} \mathbb{C}$

$\cong \mathbb{C} \times \mathbb{C} \times \mathbb{C} \times \mathbb{C} \times (\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C})$

$\cong \mathbb{C}^{(4)} \times M_2(\mathbb{C})$

Proposition: Let A and B be algebras over $\mathbb{F}$, then

1. $M_n(A) \otimes_{\mathbb{F}} B \cong M_n(A \otimes_{\mathbb{F}} B)$ $\quad a_{ij}E(i,j) \otimes b_{ij} \longmapsto (a_{ij} \otimes b_{ij})E(i,j)$
2. $M_n(A) \otimes M_n(B) \cong M_n(A \times B)$ $\quad [a_{ij}E(i,j), b_{ij}E(i,j)] \longmapsto (a_{ij}, b_{ij})E(i,j)$
3. $M_n(\mathbb{F}) \otimes M_m(\mathbb{F}) \cong M_{mn}(\mathbb{F})$

~~Examples. A and B algebras over $\mathbb{F}$ ~ D_i over $\mathbb{R}$~~ $\mathbb{R}\ \mathbb{C}\ \mathbb{H}$

Example: Find Wedderburn decomp of $\mathbb{R}[C_5 \times D_6 \times Q_8] \cong \mathbb{R}[C_5] \otimes \mathbb{R}[D_6] \otimes \mathbb{R}[Q_8]$

Do it in steps: - $\mathbb{R}[C_5] \cong \mathbb{R}[x]/(x^5-1) \cong \mathbb{R}[x]/(x-1) \times \mathbb{R}[x]/(x^2+\frac{1+\sqrt{5}}{2}x+1) \times \mathbb{R}[x]/(x^2+\frac{1-\sqrt{5}}{2}x+1)$

$\cong \mathbb{R} \times \mathbb{C} \times \mathbb{C}$

$\cong \mathbb{R} \times \mathbb{C}^{(2)}$

$\mathbb{R}[D_6] \cong \mathbb{R}^{(2)} \times M_2(\mathbb{R})$

$\mathbb{R}[Q_8] \cong \mathbb{R}^{(4)} \times \mathbb{H}$

$*\ \mathbb{C}^{(2)} \times \mathbb{R}^{(2)}$
$= (\mathbb{C} \times \mathbb{C}) \otimes (\mathbb{R} \times \mathbb{R})$
$\cong \mathbb{C}^{(4)}$

First step: $\mathbb{R}[C_5 \times D_6] \cong \mathbb{R}[C_5] \otimes_{\mathbb{R}} \mathbb{R}[D_6]$

$\cong (\mathbb{R} \times \mathbb{C}^{(2)}) \otimes_{\mathbb{R}} (\mathbb{R}^{(2)} \times M_2(\mathbb{R}))$

$\cong (\mathbb{R} \otimes \mathbb{R}^{(2)}) \times (\mathbb{R} \otimes M_2(\mathbb{R})) \times (\mathbb{C}^{(2)} \overset{*}{\otimes} \mathbb{R}^{(2)}) \times (\mathbb{C}^{(2)} \otimes M_2(\mathbb{R}))$

$\cong \mathbb{R}^{(2)} \times M_2(\mathbb{R}) \times \mathbb{C}^{(4)} \times M_2(\mathbb{C})^{(2)}$

Second step: $\mathbb{R}[C_5\times D_6]\otimes\mathbb{R}[Q_8] = [\mathbb{R}^{(2)}\times M_2(\mathbb{R})\times\mathbb{C}^{(4)}\times M_2(\mathbb{C})^{(2)}]\times[\mathbb{R}^{(4)}\times\mathbb{H}]$

$= \mathbb{R}^{(8)}\times(M_2(\mathbb{R})\otimes\mathbb{R}^{(4)})\times(\mathbb{C}^{(4)}\otimes\mathbb{R}^{(4)})\times(M_2(\mathbb{C})^{(2)}\otimes\mathbb{R}^{(4)})$

$\times(\mathbb{R}^{(2)}\otimes\mathbb{H})\times(M_2(\mathbb{R})\otimes\mathbb{H})\times(\mathbb{C}^{(4)}\otimes\mathbb{H})\times(M_2(\mathbb{C})^{(2)}\otimes\mathbb{H})$

$= \mathbb{R}^{(8)}\times M_2(\mathbb{R})^{(4)}\times\mathbb{C}^{(16)}\times M_2(\mathbb{C})^{(8)}\times\mathbb{H}^{(2)}\times M_2(\mathbb{H})\times M_2(\mathbb{C})^{(4)}\times M_4(\mathbb{C})^{(2)}$

$M_2(\mathbb{C})\otimes\mathbb{H} = M_2(\mathbb{C}\otimes\mathbb{H}) = M_2(M_2(\mathbb{C})) = M_4(\mathbb{C})$.

1

Character Theory ~~over~~ $\mathbb{C}$.

So the theory of $\mathbb{F}[G]$ is sound so far
The Wedderburn Decomp of $\mathbb{C}[C_4] \cong \mathbb{C}[C_2 \times C_2]$ but $C_4 \not\cong C_2 \times C_2$, $\mathbb{C}[Q_8] \cong \mathbb{C}[D_8]$ but $Q_8 \not\cong D_8$
So we need some sort of invariant to distinguish between groups and their group rings. Character tables!

Theorem:
If two groups G_1 and G_2 have the same character table $\Rightarrow \mathbb{C}[G_1] \cong \mathbb{C}[G_2]$

Theorem:
$\mathbb{Z}[G] \cong \mathbb{Z}[H] \Rightarrow G \cong H$ if finite.

Character Theory of finite groups over $\mathbb{C}$.

Basics:
Proposition: $Tr(AB) = Tr(BA)$ if $A, B \in M_n(\mathbb{F})$

Corollary: If A and B are equivalent $Tr(A) = Tr(B)$
proof: $Tr(B) = Tr(T^{-1}AT) = Tr(AT^{-1}T) = Tr(A)$.

Definition:
Let G be a finite group, let V be a fd vector space over $\mathbb{C}$ of dim n, let $\rho: G \rightarrow GL_n(\mathbb{C})$ be a rep of G
Then define mapping $\chi_\rho : G \rightarrow \mathbb{C}$ by
$$\chi_\rho(g) = Tr(\rho(g)) \quad \forall g \in G.$$

Jargon:
1. If ρ is an irreducible rep, the χ_ρ is called an irreducible character.
2. The degree of the rep $\rho: G \rightarrow GL_n(\mathbb{C})$ is also called the degree of the character.
   $\deg(\chi_\rho) = [V : \mathbb{C}] = n$.
   To find degree ~~comp~~ compute $\chi_\rho(1)$ for $1 \in G$.
   $\chi_\rho(1) = Tr(\rho(1)) = Tr(I_n) = n = \deg(\chi_\rho)$
3. A character ~~θ~~ of degree 1 is called linear and its irreducible.
4. Characters are **not** in general group homomorphisms!

$\chi_\rho : G \to \mathbb{C}$ is not a group homo unless ρ is linear/1-dim rep
ie $\rho : G \to GL_1(\mathbb{C})$
$\chi_\rho(gh) = Tr(\rho(gh)) = Tr(\rho(g)\rho(h))$
$= Tr(\rho(g))Tr(\rho(h))$
$= \chi_\rho(g)\chi_\rho(h).$

Example:
$G = C_3 = \langle x \mid x^3 = 1 \rangle$ has 3 conjugacy classes $\langle 1 \rangle, \langle x \rangle \langle x^2 \rangle$
$\Rightarrow r = 3 \Rightarrow 3 = 1^2 + 1^2 + 1^2$
$\mathbb{C}[C_3] \cong \mathbb{C} \times \mathbb{C} \times \mathbb{C}.$
Define $\rho_i : C_3 \to GL_1(\mathbb{C})$
$x \mapsto \omega^{i-1}$ $1 \le i \le 3.$

$\rho_1 : x \mapsto 1$ $\rho_2 : x \mapsto \omega$
$\rho_3 : x \mapsto \omega^2$

	Conjugacy classes		
Characters	$\langle 1 \rangle$	$\langle x \rangle$	$\langle x^2 \rangle$
χ_1	1	1	1
χ_2	1	ω	ω^2
χ_3	1	ω^2	ω

Note: $(\chi_1 + \chi_2 + \chi_3) : G \to 0$ is a character $\sim \chi_{reg} = \begin{cases} 3 & g = 1 \\ 0 & g \neq 1 \end{cases}$

Recall: Two reps $\sigma : G \to GL_n(\mathbb{C})$, $\rho : G \to GL_n(\mathbb{C})$ are conjugate/equivalent if $\exists T \in GL_n(\mathbb{C})$ st $\sigma(g) = T^{-1}\rho(g)T$ $\forall g \in G$
ie we can go from ρ to σ by changing basis in $V = \mathbb{C}^n$

Proposition:
$\chi_\sigma = \chi_\rho$ if σ and ρ are equivalent

proof: Since σ and ρ are equivalent $\sigma(g) = T^{-1}\rho(g)T$
$\therefore$ $\chi_\sigma(g) = Tr(\sigma(g)) = Tr(T^{-1}\rho(g)T)$
$= Tr(\rho(g)T^{-1}T)$
$= Tr(\rho(g)) = \chi_\rho(g)$ QED

If $\chi_\sigma \neq \chi_\rho$ then σ and ρ are inequivalent reps.

Proposition characters are constant on conjugacy classes
χ_ρ is constant on conj. classes of G.

Proof: Suppose $g = x^{-1}hx$
$\rho(g) = \rho(x^{-1}hx) = \rho(x^{-1})\rho(h)\rho(x)$
$\chi_\rho(g) = Tr(\rho(g)) = Tr(\rho(x^{-1})\rho(h)\rho(x))$
$= Tr(\rho(h))$
$\therefore$ if $h \in (g)^G \Rightarrow \chi_\rho(g) = \chi_\rho(h)$ QED

Need an example with $|(g)^G| \geq 2$

Example:
$D_6 = \langle x, y \mid x^3 = y^2 = 1, yx = x^2 y \rangle$
$= \langle 1 \rangle \amalg \langle x, x^2 \rangle \amalg \langle y, xy, x^2 y \rangle$
Consider $\rho : D_6 \to GL_2(\mathbb{C})$
$x \mapsto \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$

$y \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\Rightarrow \rho(x^2) = \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix}$

$\chi_\rho(x) = Tr(\rho(x)) = Tr\begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix} = \omega + \omega^2 = -1$

$\chi_\rho(x^2) = Tr(\rho(x^2)) = Tr(\rho(x^2)) = \omega^2 + \omega = -1$
Note $x^2 \in (x)^{D_6}$
Do the same for $\langle y, xy, x^2 y \rangle$
$\chi_\rho(y) = 0$
$\chi_\rho(xy) = 0$

Conj classes

Character	$\langle 1 \rangle$	$\langle x, x^2 \rangle$	$\langle y, xy, xy \rangle$
χ_1	1	1	1
χ_2	1	1	-1
χ_3	2	-1	0

Theorem:

Let G be any group

Let $\rho : G \longrightarrow GL_m(\mathbb{C})$ be a complex rep of G

$V = \mathbb{C}$ as $\mathbb{C}[G]$-mod.

Let $\chi_\rho : G \to \mathbb{C}$ be the charactered afforded by ρ

The $\forall g \in G$ we have

1. $\rho(g)$ diagonaliscble
2. $\chi_\rho(g)$ is sum of roots of unity
3. $\chi_\rho(g^{-1}) = \overline{\chi_\rho(g)}$
4. $|\chi_\rho(g)| \leq m = \chi_\rho(1) = \dim_{\mathbb{C}} V$

proof:

1. Let G be finite st $|G| = m$. Then $\exists g$ st $g^m = 1$ so $\rho(g)^m = I_n$
$\Rightarrow \rho(g)$ is a root of the poly $x^m - 1$ but by FTA
$x^m - 1 = (x-1)(x-\omega_1)\cdots(x-\omega_m)$ where $\omega_i = e^{2\pi i/m}$ are
the ~~eigenvalues~~ m^{th} roots of unity
Since the minamal poly of $\rho(g)$ divides $x^m - 1$ it is also
a product of distinct linear factors which contain the
eigenvalues.
$\therefore \exists$ basis in $\mathbb{C}[G]$ with respect to the matix
$\rho(g) = \begin{pmatrix} \omega_1 & & 0 \\ & \ddots & \\ 0 & & \omega_m \end{pmatrix}$ is diagonal.

2. $\chi_\rho(g) = Tr(\rho(g)) = Tr \begin{pmatrix} \omega_1 & & 0 \\ & \ddots & \\ 0 & & \omega_m \end{pmatrix}$

$= \omega_1 + \ldots + \omega_m$

3. Since eigenvalues associated to g^{-1} is $\omega_i^{-1} \Rightarrow \omega_i^{-1} = \overline{\omega_i}$
since the ω_i are roots of unity $|\omega_i| = 1$
$\therefore \rho(g) = \begin{pmatrix} \omega_1 & & 0 \\ & \ddots & \\ 0 & & \omega_m \end{pmatrix}$ $\chi_\rho(g) = \omega_1 + \ldots + \omega_m$

$\Rightarrow \rho(g^{-1}) = \begin{pmatrix} \omega_1^{-1} & & \\ & \ddots & \\ & & \omega_m^{-1} \end{pmatrix} = \begin{pmatrix} \overline{\omega_1} & & \\ & \ddots & \\ & & \overline{\omega_m} \end{pmatrix}$

$\therefore \chi_\rho(g^{-1}) = \sum_{i=1}^{m} \omega_i^{-1} = \sum_{i=1}^{m} \overline{\omega_i} = \overline{\omega_1} + \ldots + \overline{\omega_m}$
$= \overline{\omega_1 + \ldots + \omega_m}$

4) By triangle inequality
$|\chi_\rho(g)| = |\omega_1 + \dots + \omega_m| \leq |\omega_1| + \dots + |\omega_m|$
$= 1 + \dots + 1$
$= m = \chi_\rho(1)$ QED

Consequence.
If g and g^{-1} are in the same conjugacy class then $\chi_\rho(g) \in \mathbb{R}$
since $g = x^{-1} g^{-1} x \Rightarrow \underset{z}{\chi_\rho(g)} = \chi_\rho(g^{-1})$
$= \underset{\bar{z}}{\chi_\rho(\bar{g})}$

$\therefore \chi_\rho(g) \in \mathbb{R}$
$\chi_\rho(g^{-1}$

Example:
$\rho : D_6 \to GL_2(\mathbb{C})$
$\rho(x) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$
$\chi_\rho(x) = \chi_\rho(x^2) = \omega + \omega^2$
$= \omega + \omega^{-1}$
$= \omega + \bar{\omega}$
$= 2\cos\left(\frac{2\pi}{3}\right) \in \mathbb{R}$

Example
If $G = S_n$ conj classes - permutations of same size
then each g is such that $g^{-1} \in (g)^G$
$\Rightarrow \chi_\rho(g) \in \mathbb{R}\ \forall g$.

Proposition
Let $|G| = m$
Let $\rho : G \to GL_n(\mathbb{C})$
$\chi_\rho : G \to \mathbb{C}$
Then $\forall g \in G$ $|\chi_\rho(g)| = \chi_\rho(1) = n$
$\Leftrightarrow \rho(g) = \omega I_n$ $\omega \in \{\omega_1, \dots, \omega_m\}$

proof: $\Leftarrow$ Let $g \in G$ $g^m = 1$
If $\rho(g) = \omega I_n$ where $\omega^m = 1$, $|\omega| = 1$
$\chi_\rho(g) = n\omega$

$|\chi_\rho(g)| = |n\omega| = |n||\omega| = |n| = n$

$\Rightarrow$ Suppose $|\chi_\rho(g)| = \chi_\rho(1) = n$ wrt some basis

$\rho(g) = \begin{pmatrix} \omega_1 & & 0 \\ & \ddots & \\ 0 & & \omega_n \end{pmatrix}$ ω_i roots of 1

$|\chi_\rho(g)| = |\omega_1 + \dots + \omega_n| \leq |\omega_1| + \dots + |\omega_n|$
$= n$

with equality if ~~and only if~~ all ω_i lie on straight line in $\mathbb{C}$

Since roots of 1

$\Rightarrow \omega_1 = \omega_2 = \dots = \omega_n = \omega$

$\rho(g) = \omega I_n$ QED.

Definition:

Let $\rho : G \to GL_n(\mathbb{C})$

The kernal of a character $\chi_\rho : G \to \mathbb{C}$ is defined as the set

$\ker(\chi_\rho) = \langle g \in G : \chi_\rho(g) = \chi_\rho(1) = n \rangle$

Proposition:

In fact $\ker(\chi_\rho) = \ker(\rho) = \langle g \in G : \rho(g) = I_n \rangle$

Proof: $\ker(\rho) \subseteq \ker(\chi_\rho)$

Suppose g is st $\chi_\rho(g) = \chi_\rho(1) = n$

If $|\chi_\rho(g)| = |\chi_\rho(1)| = n \Rightarrow \rho(g) = \omega I_n$

$\Rightarrow \chi_\rho(g) = \omega \cdot n = n \Rightarrow \omega = 1$

$\therefore \rho(g) = \omega I_n = I_n \Rightarrow g \in \ker(\rho)$

$\therefore \ker(\chi_\rho) \subseteq \ker(\rho)$ QED.

Definition:

A character χ_ρ st $\ker(\chi_\rho) = \langle 1 \rangle$ is called faithfull character

Example:

$\ker D_6 = \langle 1, x, x^2, y, xy, x^2y \rangle$

	1	x	x^2	y	xy	x^2y
χ_1	1	1	1	1	1	1
χ_2	1	1	1	-1	-1	-1
χ_3	2	-1	-1	0	0	0
χ_{reg}	6	0	0	0	0	0

$\chi_1 + \chi_2 + 3\chi_3$

$\ker(\chi_1) = \ker(\rho_1) = D_6$

$\ker(\chi_2) = \ker(\rho_2) = \underset{\langle 1, x, x^2\rangle}{\langle x \rangle} \cong C_3$

Same as $Tr(I_n)$

n dim of rep.

$\ker(\chi_3) = \ker(\rho_3) = \langle 1 \rangle \therefore \chi_3$ is faithfull.

$\ker(\chi_{reg}) = \ker(\rho_{reg}) = \langle 1 \rangle$ χ_{reg} is faithful.

Now define another rep of D_6 of dim $= 2$

$\sigma : D_6 \longrightarrow GL_2(\mathbb{C})$

$x \longmapsto \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ $\quad y \longmapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad 1 \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\chi_\sigma(1) = 2$

$\chi_\sigma(x) = \chi_\sigma(x^2) = -1$

$\chi_\sigma(y) = \chi_\sigma(xy) = \chi_\sigma(x^2 y) = 0$

χ_{ρ_3} and χ_σ are equivalent.

The Regular Representation

Recall $\rho_{reg} : G \longrightarrow \underset{GL_{|G|}(\mathbb{C})}{GL(\mathbb{C}[G])}$

If $|G| = n$ $\rho_{reg} : G \longrightarrow GL_n(\mathbb{C})$ is given by

$\rho_g(g_i) = g \cdot g_i$

This corresponded to $\mathbb{C}[G]$ as a $\mathbb{C}[G]$-module

$\mathbb{C}[G] \times \mathbb{C}[G] \longrightarrow \mathbb{C}[G]$

$R \times M \longrightarrow M.$

Let $B = \langle \underset{g_1}{1}, g_2, \dots, g_n \rangle$ be a $\mathbb{C}[G]$-basis for $\mathbb{C}[G] = V$

Then if $g_i \neq g_1 = 1$

$$\rho_{reg}(g_i) = \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ 0 & & & & \vdots \\ \vdots & & \vdots & & \vdots \\ 0 & & 0 & & \\ 1 & & 0 & & \\ 0 & & 1 & & \\ \vdots & & 0 & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}$$

has zeroes along diagonal unless $g_i = g_1$

$g_i g_1 = g_i$

$g_i g_j = g_k$

If you have a non zero diagonal entry in the i^{th} place $\Rightarrow$

$g \cdot g_i = g_i \Rightarrow g = 1$

Only matrix that has non-zero entries is $\rho_{reg}(1) = J_n = I_{|G|}$

$\therefore \chi_{reg}$ and therefore ρ_{reg} are always faithfull and decomposable.

Corollary

$$\chi_{reg}(g) = \begin{cases} |G| & g = 1 \\ 0 & g \neq 1 \end{cases}$$

Example: $\rho_{reg} : D_6 \longrightarrow GL(\mathbb{C}(D_6)) = GL_6(\mathbb{C})$

$$\rho_{reg}(1) = \begin{pmatrix} 1 & & & & & 0 \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ 0 & & & & & 1 \end{pmatrix}$$

$\rho_x(g_i) = x g_i$

$\langle \underset{g_1}{1}, \underset{g_2}{x}, \underset{g_3}{x^2}, \underset{g_4}{y}, \underset{g_5}{xy}, \underset{g_6}{x^2y} \rangle$.

$$\rho_{reg}(x) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$\chi_{reg}(1) = 6$

$\chi_{reg}(x) = 0$

Theorem:

"irreducible χ's determine irreducible reps"

Two irreducible reps of G are equivalent iff their characteristics are equal

Proof $\Rightarrow$ Let ρ & σ be equivalent and irreducible

$\sigma(g) = T^{-1}\rho(g)T$ and apply trace

$\chi_\sigma(g) = Tr(\sigma(g)) = Tr(T^{-1}\rho(g)T)$

$= Tr(\rho(g))$

$\Leftarrow$ Let $U = S_1^{a_1} \oplus \ldots \oplus S_r^{a_r}$

and $V = S_1^{b_1} \oplus \ldots \oplus S_r^{b_r}$

be two $\mathbb{C}[G]$-mods which are semisimple

Show $a_i = b_i$ via χ's

Let $\chi_1, \ldots, \chi_r$ be the irreducible characters of S_i's

$\chi_u = a_1\chi_1 + \ldots + a_r\chi_r$

$\chi_v = b_1\chi_1 + \ldots + b_r\chi_r$

$\therefore \chi_u = \chi_v \Rightarrow a_i = b_i \ \forall i \Rightarrow U \cong V$ as $\mathbb{C}[G]$-mods

$\therefore$ corresponding reps are equivalent.

Tool: This is used as an invariant to determine if 2 reps are inequivalent which is faster than trying to find $\tau \in GL_n(\mathbb{C})$.

Example: $Q_8 = \langle x, y \mid x^4 = 1, x^2 = y^2, \ yx^2 = xy \rangle$

Conj classes $\langle 1 \rangle$, $\langle x^2 \rangle$ $\langle x, x^3 \rangle$ $\langle y, x^2 y \rangle$ $\langle xy, x^3 y \rangle$

$\mathbb{C}[Q_8] \cong \mathbb{C}^{(4)} \times M_2(\mathbb{C})$.

Consider the following 2-dim reps of Q_8

$\rho_1 : \ x \longmapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad y \longmapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \chi_1$

$\rho_2 : \ x \longmapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad y \longmapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \chi_2$

$\rho_3 : \ x \longmapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad y \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \chi_3$

	$\langle 1 \rangle$	$\langle x^2 \rangle$	$\langle x, x^3 \rangle$	$\langle y, x^2 y \rangle$	$\langle xy, x^3 y \rangle$
χ_1	2	-2	0	0	0
χ_2	2	-2	0	0	0
χ_3	2	2	0	0	-2

$\therefore \rho_1 \cong \rho_2$ but $\rho_1 \not\cong \rho_3$

$\therefore \rho_3$ decomposes into 1-dim reps

Let $B = \langle e_1, e_2 \rangle$ be basis for $\mathbb{C}^2$

$\rho_3(x) e_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -1 e_1$

$\rho_3(y) e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 e_1$

$\therefore \operatorname{span}(e_1) = U$ is stable by Q_8

$\therefore U \subseteq V = \mathbb{C}^2$ is a submodule ~ sub rep of ρ_3

$\rho_3(x) e_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = e_2$

$\rho_3(y) e_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -e_2$

$\therefore$ span$(e_2) = W$ is stable by Q_8

Define $\rho_3' : Q_8 \longrightarrow GL_1(\mathbb{C})$

$x \longmapsto +1$

$y \longmapsto 1$

$\rho_3'' : Q_8 \longrightarrow GL_1(\mathbb{C})$

$x \longmapsto 1$

$y \longmapsto -1$ $\quad V = \text{Span}(e_1) \oplus \text{Span}(e_2)$

$\therefore \rho_3 \sim \rho_3' \oplus \rho_3''$

If G and H have the same character tables $\Rightarrow \mathbb{C}[G] \cong \mathbb{C}[H]$

$\not\Rightarrow G \cong H$

Nilpotency and Idenpotency

Definition:

Let R be a ring, then say that an element $a \in R$ is nilpotent if $\exists n \in \mathbb{N}$ st $a^n = 0$

Proposition:

If R is an intergral domain the only nilpotent ~~divisor~~ element is 0.

Proof: Let $a \in R$ st $a^n = 0 \Rightarrow a(a^{n-1}) = 0$ but if $a \neq 0 \Rightarrow$ a is a zero divisor $\divideontimes$ contradiction, ID don't have zero divisors

Examples of nilpotent elements

1. $\mathbb{Z}_9 = \{\bar{0}, \bar{1}, \dots, \bar{8}\}$

$\bar{0}^n = 0$, $3^2 = 9 = 0$, $6^2 = 36 \equiv 0$. $\Rightarrow \mathbb{Z}_9$ not ID.

2. $R = M_2(\mathbb{F})$ $\quad a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad a^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $\quad a$ is nilpotent

$\Rightarrow M_2(\mathbb{F})$ is not an ID.

We are interested in central idempotents in group rings.

Definition:

An element $e \in R$ is called idenpotent if $e^2 = e$.

Examples:

1. $R = \mathbb{Z}_6$ 3 is idenpolent $3^2 = 9 \equiv 3 \pmod 6$
2. $R = M_2(\mathbb{F})$ $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $e^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ e is idenpolent

so is $e' = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

Proposition

If R is an intergral domain then the only idenpolent elements are the trivial ones 0 and 1.

Proof: Let $e \in R$ st $e^2 = e$ $\Rightarrow e^2 - e = 0$

$\Rightarrow e(e-1) = 0$

But since R is ID $e = 0$ or $e = 1$.

Definition:

The centre of R, $Z(R) = \{z \in R \text{ st } \cancel{\forall} rz = zr \ \forall r \in R\}$.

Definition:

An element $e \in R$ is called a central idenpolent if $e \in Z(R)$ and $e^2 = e$.

Definition:

If $e_i, e_j \in R$ st $i \neq j$ and e_i and e_j are idenpotents with $e_i \cdot e_j = 0$ we call e_i and e_j orthogonal idenpolents.

Example:

Suppose $R = R_1 \times R_2 \times R_3$ is a product of 3 rings

Then $1 \in R$ decomposes into a sum of orthogonal central idenpolents.

$1 = (1,1,1) = (1,0,0) + (0,1,0) + (0,0,1)$

$= e_1 + e_2 + e_3$

$e_i e_j = \begin{cases} 0 & i \neq j \\ e_i & i = j \end{cases}$

Theorem:
A ring R can be written as a product of r subrings $R_1, \dots, R_r$ iff $1 \in R$ can be written as a sum of centeral orthogonal idenpolents $\langle e_1, \dots, e_r \rangle$ and in this case $R_i = Re_i$
$1 = e_1 + \dots + e_r$.

Theorem:
A ring R is semisimple iff every left ideal $I \triangleleft R$ is of the form $I = \bigoplus_{i=1}^{r} Re_i$ where each e_i is an idenpolent.
$\therefore R = \bigoplus_{i=1}^{r} Re_i$

Proof: Long... show $R = Re \oplus R(1-e)$
$1 = e_1 + \dots + e_r$ $e_i = 1 - \sum_{i \neq j} e_j$ go on inductively.

Definition:
A central indenpolent is called primative if it cannot be written as a sum of 2 central orthogonal idenpolents

Our goal is to write $1 \in \mathbb{C}[G]$ as a sum of orthogonal central idenpotents.

By above theorems & Wedderburn + Mascke's
$\mathbb{C}[G] \cong \bigoplus_{i=1}^{r} M_{n_i}(\mathbb{C}) \cdot (0, \dots, 0, \underset{\epsilon_i}{I_{n_i}}, 0, \dots, 0)$

$\cong \bigoplus_{i=1}^{r} \mathbb{C}[G]\epsilon_i$

where $\{\epsilon_1, \dots, \epsilon_r\}$ is a complete set of orthogonal idenpolents.

Note: If $\rho_1, \dots, \rho_r : G \longrightarrow GL_{n_i}(\mathbb{C})$ are the corresponding reps of the subalgebra $\mathbb{C}[G]$
ie defining $\rho_i : \mathbb{C}[G] \xrightarrow{\cong} GL_{n_i}(\mathbb{C})$
$\rho_i : (\sum \alpha_g g) \longmapsto \sum \alpha_g \rho_i(g)$
Then $\rho_i(\epsilon_i) = I_{n_i} \longrightarrow \chi_{\rho_i}(\epsilon_i)$
$\rho_i(\epsilon_j) = 0 \quad i \neq j \longrightarrow \chi_{\rho_i}(\epsilon_j)$

Theorem: Idenpotent Formula.

Let $\rho_1 \dots \rho_r$ be the distinct simple reps of a finite group G, where $\rho_i : G \longrightarrow GL_{n_i}(\mathbb{C})$ and $\chi_1, \dots, \chi_r$ be the irreducible character where $\chi_i(g) = Tr(\rho_i(g))$

$$\mathbb{C}[G] \cong M_{n_1}(\mathbb{C}) \times \dots \times M_{n_r}(\mathbb{C})$$

$$1 \longmapsto \underset{\rho_1}{\epsilon_1} + \dots + \underset{\rho_r}{\epsilon_r}$$

Let ϵ_i be the central orthogonal idenpotent of $\mathbb{C}[G]$ associated with ρ_i given in terms of χ_i by the formula

$$\epsilon_i = \frac{\overset{d_i}{n_i}}{|G|} \sum_{h \in G} \chi_i(h^{-1}) h$$

Proof: Since $\epsilon_i \in \mathbb{C}[G]$ $\therefore$ can write $\epsilon_i = \sum_{g \in G} \alpha_g g$

$$\epsilon_i h^{-1} = \sum_{g, h \in G} a_g (gh^{-1})$$

Evaluate χ_{reg} on $\epsilon_i h^{-1}$

Recall $\rho_{reg} : G \longrightarrow GL_{\mathbb{C}}(\mathbb{C}[G])$

$$\rho(g) \longmapsto gg_i$$

Then $\chi_{reg} = \begin{cases} |G| & g = 1 \\ 0 & g \neq 1 \end{cases}$ 1

We know $\mathbb{C}[G] = S_1^{n_1} \oplus \dots \oplus S_r^{n_r}$ where S_i are simple modules of degree n_i corresponding to $\rho_i : G \longrightarrow GL_{n_i}(\mathbb{C})$

$$\chi_{reg} = n_1 \chi_1 + \dots + n_r \chi_r = \sum_{j=1}^{r} n_j \chi_j \qquad 2.$$

Apply both χ_{reg} formulas 1 and 2 to $\epsilon_i h^{-1}$

$$\chi_{reg}(\epsilon_i h^{-1}) = \chi_{reg}\left(\sum_g a_g g h^{-1}\right)$$
$$= \sum_g a_g \chi_{reg}(gh^{-1}) \qquad g = h \quad \chi_{reg}(1) = |G|.$$
$$= a_h |G| \qquad *$$

$$\chi_{reg}(\epsilon_i h^{-1}) = \sum n_j \chi_j(\epsilon_i h^{-1})$$
$$= n_i \chi_i(h^{-1}) \qquad **$$

Since $\chi_i(\epsilon_i h^{-1}) = Tr(S_i \xrightarrow{h^{-1}} S_i)$
$$= \chi_i(h^{-1})$$

$$\chi_j(\epsilon_i h^{-1}) = Tr(S_j \xrightarrow{\epsilon_i h^{-1}} S_j) = 0$$

$\therefore * = **$ $\quad a_h |G| = n_i \chi_i(h^{-1})$

$$\epsilon_i = \sum a_g g$$
$$= \sum a_h h$$
$$= \frac{n_i}{|G|} \sum_{h \in G} \chi_i(h^{-1}) h.$$

Example: Calculate central idempotents of G.

1. $G = D_6 = \langle x, y \mid x^3 = y^2 = 1,\ yx = x^2 y \rangle$

$\mathbb{C}[D_6] = \underset{\rho_1}{\mathbb{C}} \times \underset{\rho_2}{\mathbb{C}} \times \underset{\rho_3}{M_2(\mathbb{C})}$ $\quad$ ρ_1, ρ_2, ρ_3 usual reps.

$\cong \bigoplus_{i=1}^{3} \mathbb{C}[D_6]\epsilon_i$

	$\langle 1 \rangle$	$\langle x, x^2 \rangle$	$\langle y, xy, x^2y \rangle$
χ_1	1	1	1
χ_2	1	1	-1
χ_3	2	$\omega + \omega^{-1} = -1$	0

$$\epsilon_i = \frac{n_i}{|G|} \sum_{h \in G} \chi_i(h^{-1}) h$$

$$\epsilon_1 = \frac{n_1}{6} \sum_{h \in D_6} \chi_1(h^{-1}) h$$

$$= \frac{1}{6}\left(\chi_1(1^{-1}) \cdot 1 + \chi_1(x^{-1})x + \chi_1(x^{-2})x^2 + \chi_1(y^{-1})y + \chi_1((xy)^{-1})xy + \chi_1((x^2y)^{-1})xy \right)$$

$$= \frac{1}{6}\left(\chi_1(1) \cdot 1 + \chi_1(x^2)x + \chi_1(x)x^2 + \chi_1(y)y + \chi(xy)xy + \chi_1(x^2y)x^2y \right)$$

$$= \frac{1}{6}(1 + x + x^2 + y + xy + x^2y)$$

$\epsilon_1^2 = \epsilon_1$

$\epsilon_2 = \frac{1}{6}(1 + x + x^2 - y - xy - x^2y)$

$\rho_3 : D_6 \to GL_2(\mathbb{C})$

$x \mapsto \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}$

$y \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\epsilon_3 = \frac{2}{6}\left(\chi_3(1) \cdot 1 + \chi_3(x^2)x + \chi_3(x)x^2 + \chi_3(y)y + \chi_3(xy)xy + \chi_3(x^2y)x^2y \right)$$

$$= \frac{1}{3}(2 - x - x^2)$$

$$\epsilon_3^2 = \frac{2 - x - x^2}{3} \times \frac{2 - x - x^2}{3}$$

$$= \frac{6 - 3x - 3x^2}{9} = \frac{\cancel{3}(2 - x - x^2)}{\cancel{9}\,3} = \epsilon_3.$$

Check $\epsilon_1 \cdot \epsilon_2 = 0 \quad \epsilon_1 \cdot \epsilon_3 = 0 \quad \epsilon_2 \cdot \epsilon_3 = 0 \quad \epsilon_1 + \epsilon_2 + \epsilon_3 = 1$

2. $G = A_4 = \langle 1, s, t, st, x, xs, xt, xst, x^2, x^2s, x^2t, x^2st \rangle$

$x \sim (1\ 2\ 3) \quad s \sim (1\ 2) \quad t = (3\ 4)$

$s^2 = t^2 = (st)^2 = 1 \quad x^3 = 1$

$xsx^{-1} = st \quad xtx^{-1} = s$

4 conj. classes

$\langle 1 \rangle, \langle s, t, st \rangle, \langle x, xs, xt, xst \rangle, \langle x^2, x^2s, x^2t, x^2st \rangle$

$|A_4| = 12 = 1^2 + 1^2 + 1^2 + 3^2$

$\rho_1 : A_4 \longrightarrow \mathbb{C} \quad \rho_1(g) = 1 \quad \forall g \in A_4$

$\rho_2 : A_4 \longrightarrow \mathbb{C} \quad \rho_2(s) = \rho_2(t) = 1 \quad \rho_2(x) = \omega = e^{2\pi i/3}$

$\rho_3 : A_4 \longrightarrow \mathbb{C} \quad \rho_3(s) = \rho_3(t) = 1 \quad \rho_3(x) = \omega^2$

$\rho_4 : A_4 \longrightarrow GL_3(\mathbb{C})$

$\rho_4(s) = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix} \quad \rho_4(t) = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix} \quad \rho_4(x) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

Let $\chi_1, \chi_2, \chi_3, \chi_4$ be the corresponding irreducible characters associated to $\rho_1, \dots, \rho_4$. Use them to compute the central idempotents.

$$\epsilon_i = \frac{n_i}{|A_4|} \sum_{h \in A_4} \chi_i(h^{-1}) h$$

st $1 = \epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4$.

	$\langle 1 \rangle$	$\langle s, t, st \rangle$	$\langle x, xs, xt, xst \rangle$	$\langle x^2, x^2s, x^2t, x^2st \rangle$
χ_1	1	1	1	1
χ_2	1	1	ω	ω^2
χ_3	1	1	ω^2	ω
χ_4	3	-1	0	0

$\epsilon_1 = \frac{1}{12}(\chi_1(1^{-1})1 + \chi_1(s^{-1})s + \chi_1(t^{-1})t + \chi_1((st)^{-1})st + \chi_1(x^{-1})x + \dots + \chi_1((x^2st)^{-1})x^2st)$

$= \frac{1}{12}(1 + s + t + st + x + xs + \dots + x^2st)$

$\epsilon_1^2 = \epsilon_1$

$\epsilon_2 = \frac{1}{12}(1 + s + t + st + \omega(x + xs + xt + xst) + \omega^2(x^2 + x^2s + x^2t + x^2st))$

$\epsilon_3 = \frac{1}{12}(1 + s + t + st + \omega^2(x + xs + xt + xst) + \omega(x^2 + x^2s + x^2t + x^2st))$

$\epsilon_4 = 3/12\,(3 - s - t - st)$

We know that $\chi_j : G \longrightarrow \mathbb{C}$ are constant on conj. classes.

$\chi_\rho(x^{-1}hx) = \chi_\rho(g)$

Definition: A mapping $\varphi: G \to \mathbb{C}$ is called a class function of G if $g = x^{-1}hx$
$\Rightarrow \varphi(g) = \varphi(h)$

Example: Character of group reps
Let $\xi = \{\varphi: G \to \mathbb{C} \mid \varphi$ is class function$\}$. The ξ is a $\mathbb{C}$-vector space of $\dim = r$ (number of conj. classes)
Let $\xi = \text{span}_{\mathbb{C}}\{\chi_1, \ldots, \chi_r\}$ where χ_i are irreducible characters of G.

Theorem:
Every class function $\varphi_i: G \to \mathbb{C}$ can be written uniquely in the form
$\varphi = \sum_{j=1}^{r} \lambda_j \chi_j \quad \lambda_j \in \mathbb{C}$
Thus $\langle \chi_1, \ldots, \chi_r \rangle$ forms a basis for ξ over $\mathbb{C}$.

Proof: We know $\mathbb{C}[G] = M_{n_1}(\mathbb{C}) \times \ldots \times M_{n_r}(\mathbb{C})$
r = no. conj. classes = no. of simple reps = no. of irred characters.
Let $\{k_1, \ldots, k_r\}$ be the the conj classes of G.
Denote by $\chi_i: G \to \mathbb{C}$ the class fuction st $\chi_i(g) = 1$ if $g \in k_i$
and $\chi_i(g) = 0$ if $g \notin k$
$\therefore \{\chi_1, \ldots, \chi_r\}$ is basis of ξ if we show LI.
Suppose $\sum_{j=1}^{r} \lambda_j \chi_j = 0$
Let $\{\epsilon_1, \ldots, \epsilon_r\}$ be the central idempotents of $\mathbb{C}[G]$
$\therefore 0 = (\sum_{j=1}^{r} \lambda_j \chi_j)(\epsilon_i)$
$= \sum_{j=1}^{r} \lambda_j \chi_j(\epsilon_i)$
$= \lambda_i \chi_i(\epsilon_i) = \underset{=0}{\lambda_i} \underset{\neq 0}{\deg(\rho_i)} \quad \forall i$ $\square$

Positive Definite Hermitian Forms

Definition
Let V be a $\mathbb{C}$-vector space, the inner product space $(V, \langle , \rangle)$ is the map
$\langle , \rangle : V \times V \to \mathbb{C}$ statisfying
1 $\langle v, w \rangle = \overline{\langle w, v \rangle}$ conjugate linearity
2 $\langle v_1 + v_2, w \rangle = \langle v_1, w \rangle + \langle v_2, w \rangle$
$\langle \lambda v_1, w \rangle = \lambda \langle v_1, w \rangle$
3 $\langle v, v \rangle \geq 0$ with equality iff $v = 0$

Note: $\langle v, \lambda w_1 + w_2 \rangle = \bar{\lambda} \langle v, w_1 \rangle + \langle v, w_2 \rangle$

Example: $V = \mathbb{C}^n \quad \langle\, ,\, \rangle : \mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathbb{C}$

$$\langle x, y \rangle = \bar{y}^T A x$$

where A is an Hermitian positive ~~degenerate~~ definite matrix $a_{ij} = \overline{a_{ji}}$

$$A = \begin{pmatrix} 1 & 2-3i \\ 2+3i & 4 \end{pmatrix} \qquad \bar{A}^T = \begin{pmatrix} 1 & 2 \mp 3i \\ 2+3i & 4 \end{pmatrix}$$

Definition

Let φ, ψ be class functions of G. Then their inner product is the complex number

$$\langle \varphi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \varphi(g) \overline{\psi(g)}$$

since $\varphi, \psi : G \longrightarrow \mathbb{C}$ give complex number.

$\langle \chi_1, \dots, \chi_r \rangle$ is an orthonormal basis of the space $\mathcal{C}$ with respect to the inner product 1 & 2 are same in def

3. $\langle \varphi, \varphi \rangle = \frac{1}{|G|} \sum \varphi(g) \overline{\varphi(g)}$

$= \frac{1}{|G|} \sum |\varphi(g)|^2 \geq 0$ if $= 0 \rightarrow \varphi(g) = 0 \; \forall g$.

Example:

$G = C_3 = \langle x \mid x^3 = 1 \rangle$

	$\langle 1 \rangle$	$\langle x \rangle$	$\langle x^2 \rangle$
φ	1	1	1
ψ	2	i	-1

$\langle \varphi, \psi \rangle = \frac{1}{|C_3|} \sum_{g \in C_3} \varphi(g) \overline{\psi(g)}$

$= \frac{1}{3} \left(\varphi(1) \overline{\psi(1)} + \varphi(x) \overline{\psi(x)} + \varphi(x^2) \overline{\psi(x^2)} \right)$

$= \frac{1}{3} \left(1 \times \bar{2} + 1 \cdot \bar{i} + 1 \cdot (-1) \right)$

$= \frac{1}{3} (1 - i) = \overline{\langle \psi, \varphi \rangle}$

$\langle \psi, \varphi \rangle = \frac{1}{3}(1 + i)$

$\langle \varphi, \varphi \rangle = \frac{1}{3}(1 + 1 + 1) = 1$

$\langle \psi, \psi \rangle = \frac{1}{3} \left(2 \cdot \bar{2} + i \cdot \bar{i} + (-1)\overline{(-1)} \right)$

$= \frac{1}{3}(6) = 2.$

Example

~~A4~~ $G = A_4 = \{\sigma \in S_4 \mid sgn(\sigma) = +1\}$.

Conj class reps $\underset{1}{g_1 = (1)}$ $\underset{s\ t}{g_2 = (12)(34)}$ $\underset{x}{g_3 = (123)}$ $\underset{x^2}{g_4 = (132)}$

	g_1	g_2	g_3	g_4
χ	1	1	ω	ω^2
ψ	4	0	ω^2	ω

$\mathbb{C}[A_4] = \mathbb{C}^{(3)} \times M_3(\mathbb{C})$

$$\langle \chi, \psi \rangle = \frac{1}{|A_4|} \sum_{g \in A_4} \chi(g) \overline{\psi(g)}$$
$$= \frac{4}{12}(1 \cdot \bar{4} + 1 \cdot \bar{0} + \omega(\overline{\omega^2}) + \overline{\omega^2}\omega)$$
$$= 0 \quad \text{orthogonal}$$
$$\langle \chi, \chi \rangle = \frac{1}{12}(1\cdot 1 + 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 + 4(\omega\bar{\omega}) + 4(\omega^2 \cdot \overline{\omega^2}))$$
$$= 1.$$

Proposition

Let G be finite with r conj classes ~~representing~~ represented by $\langle g_1, \dots, g_r \rangle$. Let χ and ψ be two characters of G. Then

1 $\langle \chi, \psi \rangle = \langle \psi, \chi \rangle$
$$= \frac{1}{|G|} \sum_{g \in G} \chi(g) \psi(g^{-1})$$

2 $\langle \chi, \psi \rangle = \sum_{i=1}^{r} \frac{\chi(g_i)\overline{\psi(g_i)}}{|C_G(g_i)|}$ $\qquad C_G(g) = \{x \in G : gx = xg\}$.

3 $\langle \chi, \psi \rangle \in \mathbb{R}$

Proof: Since $\psi(g^{-1}) = \overline{\psi(g)}$

$$\therefore \langle \chi, \psi \rangle = \frac{1}{|G|} \sum \chi(g) \overline{\psi(g)}$$
$$= \frac{1}{|G|} \sum \chi(g) \psi(g^{-1})$$
$$= \frac{1}{|G|} \sum \chi(g^{-1}) \psi(g)$$
$$= \frac{1}{|G|} \sum \overline{\chi(g)} \psi(g) = \langle \psi, \chi \rangle$$

2. $|g_i^G| = \frac{|G|}{|C_G(g_i)|}$

Consider $\sum_{g \in G} \chi(g)\overline{\psi(g)} = \sum_{i=1}^{r} |g_i^G| \chi(g_i) \overline{\psi(g_i)}$

divide by $|G|$ we get

$$\langle \chi, \psi \rangle = \sum_{i=1}^{r} \frac{\chi(g_i)\overline{\psi(g_i)}}{|C_G(g_i)|}$$

3. Since $\overline{\langle \chi, \psi \rangle} = \langle \psi, \chi \rangle$
$= \langle \chi, \psi \rangle$

$\Rightarrow \langle \chi, \psi \rangle \in \mathbb{R}$.

Motivation:

We know characters are constant on conj. classes $K_1, \dots, K_r$ of G.
So if we choose class reps of K_i $\{g_1, \dots, g_r\}$ $g_i \in K_i$, then the characters $\{\chi_1, \dots, \chi_r\}$ are completely determined by $\chi_i(g_j)$
$\Rightarrow$ arrange values in $r \times r$ matrix

Definition:

The character table of G is the $r \times r$ which is invertible as irred characters are a basis / LI.

	g_1	g_2	$\dots$	g_r
χ_1	$\chi_1(g_1)$	$\chi_1(g_2)$	$\dots$	$\chi_1(g_r)$
χ_2	$\chi_2(g_1)$	$\chi_2(g_2)$	$\dots$	$\chi_r(g_r)$
$\vdots$	$\vdots$			$\vdots$
χ_r	$\chi_r(g_1)$	$\chi_r(g_2)$	$\dots$	$\chi_r(g_r)$

$= (\chi_i(g_j))_{ij}$

Example: Look at D_6 c. table.

Theorem on irreducibility of characters.

Let $\rho_1, \dots, \rho_r$ be distinct simple reps of G. Let $\chi_1, \dots, \chi_r$.
Let $\chi_1, \dots, \chi_r$ be their irreducible characters
Then wrt inner product, the characters form an orthonormal basis for $\mathcal{E}$.
This gives orthogonality relation between the rows of other characters

$$\langle \chi_i, \chi_j \rangle = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Orthogonality Relations

Use for constructing unknown characters from known ones, find $|G|$, $|C_G(g_i)|$

1. The Row orthogonality relation

"runs through conjugacy classe"

By the irreducibility Theorem $\langle \chi_i, \chi_j \rangle = \delta_{ij}$, we get

$$\sum_{k=1}^{r} \frac{\chi_i(g_k)\overline{\chi_j(g_k)}}{|C_G(g_k)|} = \delta_{ij}$$

2. The column orthogonality relation

"runs through characters"

~~Hence~~ $\displaystyle\sum_{k=1}^{r} \frac{\chi_k(g_i)\overline{\chi_k(g_j)}}{|C_G(g_j)|} = \delta_{ij}$ we get

$$\sum_{k=1}^{r} \chi_k(g_i)\overline{\chi_k(g_j)} = \delta_{ij}|C_G(g_j)| = \begin{cases} |C_G(g_i)| & \text{same column } g_i \sim g_j \text{ conjugate} \\ 0 & \text{different columns} \end{cases}$$

proof of column orthog. relation:

Define the class function $\psi_j(g_i) = \delta_{ij}$ for $1 \le i \le r$, $\{g_1, \dots, g_r\}$ conj. class reps. Since χ_k's form a basis for the space of class functions $\Rightarrow \psi_j$ is a linear combination of $\{\chi_1, \dots, \chi_r\}$, say

$\psi_j = \sum_{k=1}^{r} \lambda_k \chi_k \quad \lambda_k \in \mathbb{C}$.

Using $\langle \chi_i, \chi_j \rangle = \delta_{ik}$ $\therefore \lambda_k = \langle \psi_j, \chi_k \rangle = \frac{1}{|G|} \sum_{g \in G} \psi_j(g)\overline{\chi_k(g)}$

Now $\psi_j(g) = 1$ if g is conjugate to g_j and $\psi_j(g) = 0$ otherwise

And there are $|g_j^G| = \frac{|G|}{|C_G(g_j)|}$ elements conjugate to g_j

$$\therefore \lambda_k = \frac{1}{|G|} \sum_{g \in g_j^G} \psi_j(g)\overline{\chi_k(g)}$$

$$= \frac{\overline{\chi_k(g_j)}}{|C_G(g_j)|}$$

$$\delta_{ij} = \psi_j(g_i) = \sum_{k=1}^{r} \lambda_k \chi_k(g_i)$$

$$= \sum_{k=1}^{r} \frac{\chi_k(g_i)\overline{\chi_k(g_j)}}{|C_G(g_j)|}$$

Examples:

1. $G = D_6 = \{x, y \mid x^3 = y^2 = 1 \ yx = x^2 y\}$

$= \langle 1 \rangle \amalg \langle x, x^2 \rangle \amalg \langle y, xy, x^2 y \rangle$

$$\sum_{r=1}^{r=3} \chi_r(g_i)\overline{\chi_r(g_j)} = |C_G(g_i)|\,\delta_{ij}$$

For the partial character table

	$\langle 1 \rangle$	$\langle x \rangle$	$\langle y \rangle$
χ_1	1	1	1
χ_2	1	-1	-1
χ_3	?2	?-1	?0

$|g^G| = \dfrac{|G|}{|C_G(g)|}$ $\qquad C_G(g) = \{x \in G : xg = gx\}$.

$|C_{D_6}(1)| = 6/1 = 6 \quad |C_{D_6}(x)| = 6/2 = 3 \quad |C_{D_6}(y)| = 6/3 = 2$

$i = j = 1$ First column with itself

$\sum_{r=1}^{3} \chi_r(1)\overline{\chi_r(1)} = |C_{D_6}(1)| \cdot \delta_{11}$

$\chi_1(1)\overline{\chi_1(1)} + \chi_2(1)\overline{\chi_2(1)} + \chi_3(1)\overline{\chi_3(1)} = 6$

$1 \cdot \overline{1} + 1 \cdot \overline{1} + \chi_3(1)^2 = 6$

$\chi_3(1)^2 = 4$

$\chi_3(1) = 2$

can also use $\chi_{reg} = \sum_{i=1}^{r} n_i \chi_i(1) = \sum_{i=1}^{r} \chi_i(1)\chi_i(1) = |G|$

$i = 1 \ j = 2$ column 2 with complete 1st column.

$\sum_{r=1}^{3} \chi_r(1)\overline{\chi_r(x)} = |C_{D_6}(x)|\,\delta_{12} = 0$

$\chi_1(1)\overline{\chi_1(x)} + \chi_2(1)\overline{\chi_2(x)} + \chi_3(1)\overline{\chi_3(x)} = 0$

$1 \cdot \overline{1} \quad 1 \cdot \overline{1} + 2\overline{\chi_3(x)} = 0$

$\overline{\chi_3(x)} = -2/2 = -1$

$\chi_3(x) = -1$

$i = 1, j = 3$

$\sum \chi_i(1)\overline{\chi_r(y)} = \cdots \delta_{13} = 0$

$\chi_1(1)\overline{\chi_1(y)} + \chi_2(1)\overline{\chi_2(y)} + \chi_3(1)\overline{\chi_3(y)} = 0$

$1 \cdot \overline{1} + 1 \cdot (-1) + 2 \cdot \overline{\chi_3(y)} = 0$

$\overline{\chi_3(y)} = 0$

$\chi_3(y) = 0.$

Example: Column orthogonality

A group G of order 12 has the following 4 conj. classes $\{g_1, g_2, g_3, g_4\}$ with characters $\chi_1, \chi_2, \chi_3, \chi_4$ and partial character table :-

	g_1	g_2	g_3	g_4
χ_1	1	1	1	1
χ_2	1	1	ω	ω^2
χ_3	1	1	ω^2	ω
χ_4	? 3	? -1	?	?

$|C_G(g_1)| = 12$ $|C_G(g_3)| = 3$
$|C_G(g_2)| = 4$ $|C_G(g_4)| = 3.$

using $\sum_{k=1}^{r=4} \chi_k(g_i)\overline{\chi_k(g_j)} = \delta_{ij} |C_G(g_i)|$

$i=j=1$ 1st column with itself

$$\chi_1(g_1)\overline{\chi_1(g_1)} + \chi_2(g_1)\overline{\chi_2(g_1)} + \chi_3(g_1)\overline{\chi_3(g_1)} + \chi_4(g_1)\overline{\chi_4(g_1)} = 12$$
$$\chi_1(1)^2 + \chi_2(g_1)^2 + \chi_3(g_1)^2 + \chi_4(1)^2 = 12$$
$$1^2 + 1^2 + 1^2 + \chi_4(1)^2 = 12$$
$$\chi_4(1)^2 = 9$$
$$\chi_4(1) = 3$$

$i=1, j=2$ column 1, column 2 for $\chi_4(g_2)$

$$\chi_1(1)\overline{\chi_1(g_2)} + \chi_2(1)\overline{\chi_2(g_2)} + \chi_3(1)\overline{\chi_3(g_2)} + \chi_4(1)\overline{\chi_4(g_2)} = 0$$
$$1\cdot 1 + 1\cdot 1 + 1\cdot 1 + 3\,\overline{\chi_4(g_2)} = 0$$
$$\overline{\chi_4(g_2)} = -3/3$$
$$\chi_4(g_2) = -1$$

$i=1, j=3$

$$\chi_1(1)\overline{\chi_1(g_3)} + \chi_2(1)\overline{\chi_2(g_3)} + \chi_3(1)\overline{\chi_3(g_3)} + \chi_4(1)\overline{\chi_4(g_3)} = 0$$
$$\underbrace{1\cdot 1 + 1\cdot\overline{\omega} + 1 + \overline{\omega}^2}_{=0} + 3\,\chi_4(g_3) = 0$$
$$\chi_4(g_3) = 0$$

Irreducibility theorem for characters.

Let U and V be two simple $\mathbb{C}[G]$-modules with characters χ and ψ. Then

1. $\langle \chi, \chi \rangle = 1$ $(= \langle \psi, \psi \rangle$
2. $\langle \chi, \psi \rangle = 0$ orthogonal $(U \not\cong V)$

proof: Let $\mathbb{C}[G] = S_1^{n_1} \oplus \dots \oplus S_r^{n_r}$ as usual with S_i irreducible $\mathbb{C}[G]$-modules.

Let $m = \dim U$ be the number of $\mathbb{C}[G]$-mods, S_i which are isomorphic to U

Let $W \cong U^{(m)} = \underbrace{U \oplus \ldots \oplus U}_{m}$

Let X = sum of remaining $\mathbb{C}[G]$-submods

$\therefore \mathbb{C}[G] = W \oplus X$

$= e_1 + e_2$

$\chi_W = \underbrace{\chi + \ldots + \chi}_{m} = m\chi$

$\langle \chi_W, \chi_W \rangle = \langle m\chi, m\chi \rangle$

$= m^2 \langle \chi, \chi \rangle$

However by considering the idenpoient e_1 of W, where $e_1 = \frac{1}{|G|} \sum_{g \in G} \chi(g^{-1}) g$

we see that $\langle \chi_W, \chi_W \rangle = m\chi(1) = m^2$

$\therefore m^2 \langle \chi, \chi \rangle = m^2 \qquad (e^2 = e_1)$

$\langle \chi, \chi \rangle = 1$

Let Y = sum of $\mathbb{C}[G]$-submods S_i isomorphic to either U or V

Let Z = sum of remaining $\mathbb{C}[G]$-submods S_i.

Let $\dim(V) = n$, $\dim(U) = m$

Then $\mathbb{C}[G] \cong \underset{\overset{\shortparallel}{U^{(m)} \oplus V^{(n)}}}{Y} \oplus Z$

$\chi_Y = m\chi_U + n\chi_V$

$= m\chi + n\psi$

$\langle \chi_Y, \chi_Y \rangle = \chi_Y(1) = m(\chi(1)) + n\psi(1)$

$= m^2 + n^2$

$\langle \chi_Y, \chi_Y \rangle = \langle m\chi + n\psi, m\chi, n\psi \rangle$

$= m^2 \langle \chi, \chi \rangle + n^2 \langle \psi, \psi \rangle + mn\{\langle \chi, \psi \rangle + \langle \psi, \chi \rangle\}$

$= m^2 + n^2 + 2mn\langle \chi, \psi \rangle$

$\Rightarrow \langle \chi, \psi \rangle = 0.$

Summary:

If $\mathbb{C}[G] \cong S_1^{n_1} \oplus \ldots \oplus S_r^{n_r}$ where S_i are simple $\mathbb{C}[G]$-submods of $\dim(S_i) = n_i$ st $\chi_1, \ldots, \chi_r$ are the irreducible characters of G then

1. $\langle \chi_i, \chi_j \rangle = \delta_{ij}$
2. If ψ is any chrracter of G then $\psi = d_1\chi_1 + \ldots + d_r\chi_r$ for some ~~numbers~~ non-negative untergers $d_1 \ldots d_r$ st $\langle \psi, \chi_i \rangle = d_i$
3. $\langle \psi, \psi \rangle = \sum_{i=1}^{r} d_i^2$

$d_i = n_i$

Example

$S_3 = \{(1)\} \amalg \{(12),(13),(23)\} \amalg \{(123),(132)\}$

$\mathbb{C}[S_3] = \mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C})$

$(12) \longmapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$(123) \longmapsto \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}$

	(1)	(12) [(23)(13)]	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	$-1\ (=\omega+\omega^{-1})$

Recall $|g^G| = \dfrac{|G|}{|C_G(g)|}$

$|C_{S_3}((1))| = 6/1 = 6 \quad |C_{S_3}((12))| = 6/3 = 2 \quad |C_{S_3}((123))| = 6/2 = 3$

Let ψ_ρ be the character of the 3-dim permutation rep^n of S_3

$\rho : S_3 \longrightarrow GL_3(\mathbb{C}) \longleftrightarrow V \cong \mathbb{C}^3 = Sp(e_1, e_2, e_3)$

$\rho(1) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \quad \rho(12) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \rho(123) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

$\psi_\rho(1) = 3 \quad \psi_\rho(12) = 1 \quad \psi_\rho(123) = 0$

$\psi_\rho = d_1\chi_1 + d_2\chi_2 + d_3\chi_3$

$d_1 = \langle \psi_\rho, \chi_1 \rangle$

$= \sum_{i=1}^{r=3} \dfrac{\psi_\rho(g_i)\overline{\chi(g_i)}}{|C_G(g_i)|} = \dfrac{3 \cdot 1}{6} + \dfrac{1 \cdot 1}{2} + 0 = 1$

$d_2 = \langle \psi_\rho, \chi_2 \rangle = 0$

$d_3 = \langle \psi_\rho, \chi_3 \rangle = 1.$

$\therefore \underset{\text{1 dim} \oplus \text{2 dim}}{\psi_\rho} = \underset{\text{1 dim}}{\chi_1} + \underset{\text{2 dim}}{\chi_3}.$