# M204 Representation Theory Notes

Based on the 2018 spring lectures by Dr J Lamplugh

MATHM204 Dr J. Lamplugh

10-01-18 # Representation Theory

Exam 90%
Coursework 10%
Friday → Friday 11am
Problem class: Wed 12pm
Office hour: Thurs 2pm
Room 603

Books:

Fulton & Harris: Representation Theory, A First Course.

James & Liebeck: Representations and characters of Groups

Informally

Usually, given some object $X$ we ask "what are the symmetries?" Representation Theory asks "Given a group $G$, what objects does $G$ act on? Can we classify them up to isomorphism?"

Aim

To understand how finite groups can act on finite dimensional vector spaces.

Def

Suppose $F$ is a field and $V$ a vector space over $F$. Then define $GL(V) := \{\theta : V \to V \mid \theta \text{ is } F\text{-linear and invertible}\}$ with composition of linear maps being the group law.

Remark

If $V$ is a finite dimensional vector space over $F$ (fdvs / $F$) of dimension $n$, then after choosing an ordered basis of $V$, we have isomorphisms $V \cong F^n$ and $GL(V) \cong GL_n(F) = \{A \in M_n(F) : \det A \neq 0\}$

$M_n(F)$: $n \times n$ matrices with coeffs in $F$

Notation

When $n=1$, write $GL_1(F) = F^\times = F \setminus \{0\}$.

Examples

1). The trivial representation.

$G$ any finite group, $F$ any field, $V$ a fdvs / $F$.

The trivial representation on $V$ is

$\rho: G \to GL(V)$, $g \mapsto Id_V$ $\forall g \in G$

When $V = F$, then call $\rho$ the trivial representation.

We often write $\rho = \mathbb{1}$ for the trivial representation.

$\mathbb{1}: G \to F^\times$, $g \mapsto 1$ $\forall g \in G$

2). Cyclic group.

Let $G = C_m = \langle x \mid x^m = 1 \rangle$, $F = \mathbb{C}$, $V = \mathbb{C}^n$

What are the representations $\rho: C_m \to GL_n(\mathbb{C})$?

$\rho$ is determined by $\rho(x) =: A$, since $\rho(x^i) = A^i$.

We also must have $A^m = I_n$.

If $A \in GL_n(\mathbb{C})$ is any matrix such that $A^m = I_n$,

then there is a representation $\rho: C_m \to GL_n(\mathbb{C})$, $x^i \mapsto A^i$.

3). Dihedral groups

Recall: $D_{2n} = \langle x, y \mid x^n = 1 = y^2, \ yx = x^{-1}y \rangle$

Non-trivial representation of dimension 1:

$$\varepsilon: D_{2n} \to F^\times = GL_1(F)$$

$x \mapsto 1$

$y \mapsto -1$ $\qquad x^a y^b \mapsto \varepsilon(x)^a \varepsilon(y)^b$

To check $\varepsilon$ defines a representation, need to check

$\varepsilon(x)^n = \varepsilon(y)^2 = 1$ and $\underset{-1}{\underset{\shortparallel}{\varepsilon(y)}}\varepsilon(x) = \underset{-1}{\underset{\shortparallel}{\varepsilon(x)^{-1}}}\varepsilon(y)$

$\varepsilon$ is non trivial if char $F \neq 2^{(-1=1)}$ (e.g. $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$)

Recall: Let $F$ be a field and let $\phi: \mathbb{Z} \to F$, $1 \mapsto 1$, be the unique ring homomorphism. Then

$$\text{char}\, F = \begin{cases} 0, & \text{if } \ker\phi = 0 \ (\phi \text{ injective}) \quad \text{e.g. } \mathbb{Q}, \mathbb{R}, \mathbb{C}, \ldots \\ p, & \text{if } \ker\phi = p\mathbb{Z} \quad \text{e.g. } \mathbb{F}_p, \mathbb{F}_{p^2}, \mathbb{F}_p(x), \ldots \end{cases}$$

A 2-dimensional representation over $\mathbb{R}$:

$$\rho: D_{2n} \to GL_2(\mathbb{R}), \quad x \mapsto \begin{pmatrix} \cos 2\pi/n & -\sin 2\pi/n \\ \sin 2\pi/n & \cos 2\pi/n \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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Check $\rho(x)^n = \rho(y)^2 = I_2$, $\rho(y)\rho(x) = \rho(x)^{-1}\rho(y)$

When $n=3$, $\rho(x) = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$

Also for $n=3$

$\rho' : x \longmapsto \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$, $y \longmapsto \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$

check $\rho'(x)^3 = \rho'(y)^2 = I_2$ and $\rho'(x)\rho'(y) = \rho'(x)^{-1}\rho'(y)$

so $\rho'$ defines a representation.

Check that $\varepsilon = \det \circ \rho : D_{2n} \xrightarrow{\rho} GL_2(\mathbb{R}) \xrightarrow{\det} \mathbb{R}^\times$

Permutation representations

Suppose $G$ is a group acting on (the left of) some finite set $X$, with $|X| = n$. Let $V = F[X] := \bigoplus_{x \in X} F \cdot e_x$ be the free vector space on $X$ (so $V$ has dimension $n = |X|$ over $F$, and has a given basis, one basis element for each element of $X$).

Define $\rho_X : G \to GL(V) = GL(F[X])$
$g \longmapsto \rho_X(g) = (e_x \longmapsto e_{g.x})$

group action of $G$ on $X$.

identity in $G$

Clearly $1 \longmapsto \mathrm{Id}_V$ because $\rho_X(1) e_x \longmapsto e_{1.x} = e_x \quad \forall x \in X$

since $G$ acts on $X$

Also $\rho_X(gh) = \rho_X(g)\rho_X(h)$

since $\rho_X(gh) e_x = e_{(gh).x}$

and $\rho_X(g)\rho_X(h) e_x = \rho_X(g) e_{h.x} = e_{g.(h.x)} = e_{gh.x}$

since $G$ acts on $X$.

Example

$F$ any field, $G = S_3 = \{e, (1\,2), (1\,3), (2\,3), (1\,2\,3), (1\,3\,2)\}$

$X = \{1, 2, 3\}$, $V = F[X]$

$G$ acts on $X$ in the natural way:

$\rho_X(e) = I_3$, $\rho_X((1\,2)) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\rho_X((1\,2\,3)) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, etc.

Example (ctd)

Let $W \subseteq F[X]$ be $W = F \cdot (e_1 + e_2 + e_3)$

and let $W' \subseteq F[X]$ be $W' = \{ \sum a_i e_i \mid a_1 + a_2 + a_3 = 0 \}$

Check (if $\operatorname{char} F \neq 3$) $F[X] = W \oplus W'$ as vector spaces.

$\rho_X$ stabilises $W$ i.e. $g \in G$, $w \in W$ then $\rho_X(g) w \in W$. restrict to $W$

Write $\rho_W$ for the representation $\rho_W : G \to GL(W)$, $g \mapsto \rho_X(g)|_W$.

Also $\rho_X$ stabilises $W'$.

Pick a basis of $W'$: $w_1 = e_1 - e_2$, $w_2 = e_2 - e_3$

With respect to this basis, check $\rho_{W'}((1\,2\,3)) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$, $\rho_{W'}((1\,2)) = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$.

Def

A matrix representation is a homomorphism $\rho : G \to GL_n(F)$.

Def

Given a representation $\rho : G \to GL(V)$, a subrepresentation of $\rho$ (or $V$) is a vector subspace $W \subseteq V$ such that $W$ is stabilised by $G$ (i.e. $\rho(g) w \in W$ $\forall g \in G, w \in W$).

A subrepresentation is a representation $\rho_W : G \to GL(W)$, $g \mapsto \rho(g)|_W$ $\forall g \in G$.

If the only subrepresentations of $V$ are $0$ and $V$, we say that $V$ is irreducible. Otherwise $V$ is reducible.

Def

Given two representations $\rho : G \to GL(V)$ and $\rho' : G \to GL(V')$, and $\phi : V \to V'$ a linear map, we say $\phi$ is a $G$-homomorphism or that $\phi$ intertwines $\rho$ and $\rho'$ if

$\phi \circ \rho(g) = \rho'(g) \circ \phi$ in $\operatorname{Hom}_F(V, V')$ $\forall g \in G$

i.e.
$$\begin{array}{ccc} V & \xrightarrow{\rho(g)} & V \\ \phi \downarrow & & \downarrow \phi \\ V' & \xrightarrow{\rho'(g)} & V' \end{array}$$
commutes $\forall g \in G$.

We say that $\phi$ is a $G$-isomorphism if $\phi$ is bijective.

Exercise: Check that if $\phi$ is a $G$-isomorphism, so is $\phi^{-1}$.

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In this case we say $\rho$ and $\rho'$ are isomorphic (or equivalent or conjugate).

Definition

Given two representations of $G$ over $F$, $\rho_1 : G \to GL(V_1)$, $\rho_2 : G \to GL(V_2)$, we define the direct sum $\rho_1 \oplus \rho_2 : G \to GL(V_1 \oplus V_2)$, $g \mapsto (\rho_1(g), \rho_2(g))$, $(v_1, v_2) \mapsto (\rho_1(g)v_1, \rho_2(g)v_2)$.

[In matrices: $\rho_1 \oplus \rho_2 (g) \mapsto \begin{pmatrix} (\rho_1(g)) & 0 \\ 0 & (\rho_2(g)) \end{pmatrix}$]

Def

Two matrices $A, B \in M_n(F)$ are equivalent if $\exists T \in GL_n(F)$ such that $B = T^{-1}AT$

Def

Two matrix representations $\rho_1 : G \to GL_n(F)$, $\rho_2 : G \to GL_n(F)$ are isomorphic/equivalent/conjugate if $\exists T \in GL_n(F)$ such that $\rho'(g) = T^{-1}\rho(g)T \quad \forall g \in G$.

Remark

For $\rho$ and $\rho'$ to be conjugate it is necessary that $\det(X - \rho(g)) = \det(X - \rho'(g)) \quad \forall g \in G$ (same characteristic polynomials)

Example

$G = C_2 = \langle x \mid x^2 = 1 \rangle$, $\rho : C_2 \to GL_2(\mathbb{Q})$

$x \mapsto A = \begin{pmatrix} 7 & -12 \\ 4 & -7 \end{pmatrix} \quad (A^2 = I_2)$

Let $T = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$ check $T^{-1}AT = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\Rightarrow \rho \cong \mathbb{1} \oplus \varepsilon$ where $\varepsilon : C_2 \to \mathbb{Q}^\times$, $x \mapsto -1$.

Def

Let $\rho : G \to GL(V)$ be a representation.

normal subgroup

The kernel of $\rho$ is $Ker\rho = \{g \in G : \rho(g) = Id_V\} \triangleleft G$

Remark

We have an isomorphism $G/Ker(\rho) \cong Im(\rho) \leqslant GL(V) \cong GL_n(F)$

Def

If $Ker(\rho) = 1$ then we say $\rho$ is faithful

Examples

1). The trivial representation is not faithful (unless $G = \{1\}$)

2). $D_{2n} \longrightarrow GL_2(\mathbb{R})$, $x \longmapsto \begin{pmatrix} \cos 2\pi/n & -\sin 2\pi/n \\ \sin 2\pi/n & \cos 2\pi/n \end{pmatrix}$, $y \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

is faithful

3). $G = S_n$, $X = \{1, 2, \ldots, n\}$, $G \overset{\text{acts on}}{\curvearrowright} X$ naturally.

Then $\rho_X$ is faithful.

4). $\varepsilon : D_{2n} \longrightarrow F^\times$, $x \longmapsto 1$, $y \longmapsto -1$, char $F \neq 2$

$D_{2n} = \{x^i y^j : 0 \leq i \leq n-1, 0 \leq j \leq 1\}$

$\varepsilon(x^i y^j) = (-1)^j$

so $Ker(\rho) = \{x^i : 0 \leq i \leq n-1\} = \underset{\cong C_n}{\langle x \rangle} \triangleleft D_{2n}$

$D_{2n}/\langle x \rangle \cong C_2 \xrightarrow{\varepsilon} F^\times$

Next time:

- Define a ring $F[G]$ (group ring of $G$ over $F$)
- Show there is a 1:1-correspondence

$$\left\{ \begin{matrix} \text{representations} \\ \rho : G \to GL(V) \\ \text{over } F \end{matrix} \right\} \xleftrightarrow{1:1} \left\{ \begin{matrix} \text{finitely generated modules} \\ \text{over the ring } F[G] \end{matrix} \right\}$$

- Use the structure theory for $F[G]$-modules to deduce a structure theory for representations.

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Last time:

Defined representations $\rho: G \to GL(V)$

Today:

Define rings and modules

Later we'll describe representations as modules over the group ring $F[G]$.

## Rings, algebras and modules

Def

A ring $R$ is a set with two operations $+$ and $\cdot$ satisfying $\forall a, b, c \in R$ such that

- $\exists 0 \in R$ st. $(R, +, 0)$ is an abelian group
- $\exists 1 \in R$ st. $1 \cdot a = a = a \cdot 1$
- $(ab)c = a(bc)$
- $a(b+c) = ab + ac$
- $(a+b)c = ac + bc$

In general $ab \neq ba$ $\forall a, b \in R$.

If $ab = ba$ $\forall a, b \in R$, then we say $R$ is a commutative ring.

Examples

1). Commutative rings: $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}/n\mathbb{Z}$, $\mathbb{C}$, ...

2). Non-commutative rings:

(a) Matrix rings. Let $R$ be a ring.

$M_n(R) = \{ (a_{ij})_{1 \le i,j \le n} \mid a_{ij} \in R \}$

addition: $(a+b)_{ij} = (a_{ij}) + (b_{ij})$

multiplication: $(ab)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

$(1)_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases} = \delta_{ij}$

$(0)_{ij} = 0 \ \forall i,j$

(b) $R$ ring. Define $B_n(R) = \{$upper triangular matrices in $M_n(R)\}$

Def

We say a ring $R$ is an integral domain if
$a \cdot b = 0 \Rightarrow$ either $a = 0$ or $b = 0$.
We say $R$ is a division ring if for all $a \in R$,
$a \neq 0$, $\exists b \in R$ s.t. $a \cdot b = b \cdot a = 1$.
If $R$ is a commutative division ring, we say $R$ is a field.

Def (Products of rings)

Let $R, S$ be rings. Define their product $R \times S$ as the ring with underlying set the cartesian product $R \times S$ with componentwise addition and multiplication;

$(r_1, s_1) + (r_2, s_2) = (r_1 + r_2, s_1 + s_2)$ $\qquad 1 = (1, 1)$

$(r_1, s_1) \cdot (r_2, s_2) = (r_1 r_2, s_1 s_2)$ $\qquad 0 = (0, 0)$.

More generally, if $R_i$ are rings $\forall i \in I$, then $\prod_{i \in I} R_i$ is the ring with componentwise addition and multiplication.

Def

A subset $I \subseteq R$ is called a (left)-ideal if $(I, +)$ is an abelian group and for all $x \in I$ and $r \in R$, $rx \in I$ (closed under multiplication on the left).

Remark

$I$ is a right-ideal if it has the property $x \cdot r \in I$ $\forall x \in I, r \in R$

Examples

$n\mathbb{Z} \subseteq \mathbb{Z}$, $\quad (x^2+1) \subseteq \mathbb{R}[x]$

$\leftarrow$ ideal generated by $(x^2+1)$

i.e. abelian group generated by $f(x) \cdot (x^2+1)$ $\forall f(x) \in \mathbb{R}[x]$

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If an ideal is a left ideal and a right ideal then say $I$ is a two-sided ideal.

<u>Definition</u> (Quotient Rings)
Let $I \in R$ be a two-sided ideal.
The quotient ring $R/I$ is defined as follows:
$I$ gives an equivalence relation on $R$:
$a \sim b$ if $a - b \in I$.
Elements of $R/I$ are equivalence classes under $\sim$ written $[a] = a + I$.
Addition: $(a+I) + (b+I) = (a+b) + I$
Multiplication: $(a+I)(b+I) = ab + I$
Identities: $1 = 1 + I$, $0 = 0 + I$

Note that $(a+I) = (a'+I)$ iff $a - a' \in I$, so need to check addition and multiplication is well-defined.
(Need $I$ to be a two-sided ideal for multiplication to be well-defined)

<u>Morphisms of rings</u>
Suppose $R, S$ are rings. A <u>ring homomorphism</u> $\phi: R \to S$ is a map such that
$\phi(r_1 + r_2) = \phi(r_1) + \phi(r_2)$
$\phi(r_1 r_2) = \phi(r_1)\phi(r_2)$
$\phi(1_R) = 1_S$.
$\phi$ is a <u>ring isomorphism</u> if $\exists\ \psi: S \to R$ a ring homomorphism such that $\phi \circ \psi = Id_S$ and $\psi \circ \phi = Id_R$.
In fact, $\phi$ is a ring isomorphism $\Leftrightarrow$ $\phi$ is a bijective ring homo.
$\Leftrightarrow$ $\operatorname{Ker} \phi = 0$ (ie. $\{r \in R : \phi(r) = 0\} = 0$) and $\operatorname{Im} \phi = S$ (ie. $\{s \in S : s = \phi(r), r \in R\} = S$)

Examples

1). $\phi : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$

$R$ ring, $I$ two sided ideal, $\phi : R \to R/I$, $r \mapsto r + I$.

2). $R_1, R_2$ rings $R_1 \to R_1 \times R_2$, $r \mapsto (r, 0)$ is not a ring homomorphism since $1 \mapsto (1, 0) \neq (1, 1)$.

Modules over Rings

Let $R$ be a ring. A (left) $R$-module $M$ is an abelian group with a map $\varphi : R \times M \to M$, $(r, m) \mapsto r.m$, such that $1 \cdot m = m$ $\forall m \in M$, $r(m+n) = rm + rn$, $(r+s) \cdot m = rm + sm$, and $(r \cdot s)m = r \cdot (s \cdot m)$, $\forall r, s \in R$, $m, n \in M$.

Examples

1). If $R$ is a field, modules are vector spaces.

2). If $I \subseteq R$ is a (left) ideal then $I$ is a (left) module

Def (External) Direct sums of modules

Let $M, N$ be two $R$-module. Then $M \oplus N$ is the following $R$ module:

The underlying abelian group is $M \times N$,

multiplication is $r \cdot (m, n) = (rm, rn)$.

More generally if $M_i$ are $R$-modules for $i \in I$, define $\prod_{i \in I} M_i$ as the module with abelian group $\prod_{i \in I} M_i$, with componentwise multiplication.

Also, $\bigoplus_{i \in I} M_i$ is the $R$-module whose underlying abelian group is $\{(m_i)_{i \in I} \in \prod_{i \in I} M_i : m_i \neq 0$ only finitely often$\}$.

Def (Submodules)

Suppose $M$ is an $R$-module and $N \subseteq M$. Then we say $N$ is a submodule if $0 \in N$, $(N, +, 0)$ is an abelian group, and $\forall n \in N, r \in R$

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$r \cdot n \in N$.

Write $N \leq M$ when $N$ is a submodule of $M$.

If $N \leq M$, we can form the quotient module $M/N$.
Elements are equivalence classes, $m_1 \sim m_2$ if $m_1 - m_2 \in N$
and $r \cdot (m+N) = rm + N$.

<u>Def</u> (Internal Direct Sum)
If $N_1, N_2 \leq M$ are submodules,

$N_1 + N_2 = \{ m \in M \mid m = n_1 + n_2,\ n_1 \in N_1,\ n_2 \in N_2 \}$

$N_1 \cap N_2 = \{ m \in M \mid m \in N_1 \text{ and } m \in N_2 \}$.

If $N_1 \cap N_2 = \{0\}$ then $N_1 + N_2 \cong N_1 \oplus N_2$,
$n_1 + n_2 = m \longmapsto (n_1, n_2)$
write $N_1 \oplus N_2$ for $N_1 + N_2$.

<u>Morphisms of Modules</u>
Suppose $R$ is a ring, $M, N$ are $R$-modules. An
<u>$R$-module homomorphism</u> $\phi : M \to N$ is a map satisfying
(i) $\phi(m_1 + m_2) = \phi(m_1) + \phi(m_2)$ $\quad m_1, m_2 \in M$
(ii) $\phi(rm) = r\phi(m)$ $\quad r \in R, m \in M$
$\mathrm{Ker}\,\phi = \{ m \in M \text{ st. } \phi(m) = 0 \} \leq M$
$\mathrm{Im}\,\phi = \{ n \in N \text{ st. } n = \phi(m) \} \leq N$.

We say $\phi$ is an <u>isomorphism</u> if $\exists$ an $R$-module homomorphism
$\psi : N \to M$ st. $\psi \circ \phi = \mathrm{Id}_M$ and $\phi \circ \psi = \mathrm{Id}_N$.

$\phi$ injective $\Leftrightarrow \mathrm{Ker}\,\phi = 0$.
$\phi$ is an isomorphism $\Leftrightarrow$ $\phi$ is bijective $\Leftrightarrow \mathrm{Ker}\,\phi = 0$ and $\mathrm{Im}\,\phi = N$.

<u>Examples</u>
1). $R = \mathbb{Z}$, $\mathbb{Z}$-modules are the same as abelian groups.
2). $R^n := \underbrace{R \oplus \ldots \oplus R}_{n \text{ times}}$ is an $R$-module

[modules over fields are vector spaces]

3). $R$ a ring, then $R^n$ is a $M_n(R)$-module (think of $R^n$ as column vectors).

Def

An $R$-module $M$ is finitely generated (f.g.) if $\exists$ a finite subset $\{m_1, \dots, m_n\} \subseteq M$ such that any $m \in M$ is of the form $m = r_1 m_1 + \dots + r_n m_n$, $r_i \in R$.

We say $M$ is cyclic if we can take $n = 1$.

Examples

1). Any submodule of $\mathbb{Z}$ is cyclic, i.e. all ideals of $\mathbb{Z}$ are cyclic. (ideals equivalent to submodules – ring module over itself)

2). $R = \mathbb{Z}$, $\mathbb{Q}$ is not f.g. as a $\mathbb{Z}$-module.

3). $R^n$ is a f.g. $R$-module.

4). $R[x] = \{\sum_{n=0}^{d} a_n x^n : d \geq 0, a_n \in R\}$ is not f.g. as an $R$-module.

Def (simple ring)

A ring $R$ is called simple if its only two-sided ideals are $0$ and $R$.

Example

Any field (or division ring) is a simple ring.

Proposition

Let $R$ be a ring. Then the two sided ideals of $M_n(R)$ are of the form $M_n(I) = \{(a_{ij}) \in M_n(R) : a_{ij} \in I \ \forall i,j\}$ for $I \leq R$ a two-sided ideal.

Proof

Suppose $J \leq M_n(R)$ is a two sided ideal.

Let $I = \{\alpha \in R \text{ s.t. } \exists A \in J \text{ s.t. } \underset{\text{some } i,j}{A_{ij} = \alpha}\}$

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<u>Claim:</u> $I$ is a two-sided ideal.
Let $E_{ij}$ be the matrix in $M_n(R)$ such that
$(E_{ij})_{k,l} = \begin{cases} 1 & \text{if } (k,l) = (i,j) \\ 0 & \text{otherwise.} \end{cases}$
Then if $A \in M_n(R)$, we have $(A = (a_{ij}))$
$E_{ij} A E_{kl} = a_{jk} E_{il}$. $(*)$
Since $J$ is a two-sided ideal, if $A = (a_{ij}) \in J$ then so is $a_{jk} E_{il}$ $\forall i,j,k,l$.
To see that $I$ is closed under addition:
Let $\alpha, \beta \in I$. Then $\exists A, B \in J$, $A = (a_{ij})$, $B = (b_{ij})$ s.t. $\alpha = a_{ij}$ some $i,j$, $\beta = b_{kl}$
$J$ is two sided so $E_{1i} A E_{j1} = \alpha E_{11} \in J$
$E_{1k} B E_{l1} = \beta E_{11} \in J$
$J$? $J$ is an ideal $\Rightarrow (\alpha+\beta) E_{11} \in J \Rightarrow \alpha + \beta \in J$.
$I$ is closed under left multiplication:
Suppose $r \in R$, then $r \cdot I_n = \begin{pmatrix} r & & 0 \\ & \ddots & \\ 0 & & r \end{pmatrix}$
If $A = (a_{ij}) \in J$ then $(r \cdot I_n) \cdot (a_{ij}) = (r a_{ij})$.
Also $(a_{ij})(r \cdot I_n) = (a_{ij} r)$ so $I$ is closed under right multiplication.
$\Rightarrow I$ is a two-sided ideal.
Clearly $J \subseteq M_n(I)$. To see that $M_n(I) \subseteq J$:
If $A = (a_{ij}) \in M_n(I)$ then each $a_{ij}$ occurs as a component in some matrix $B_{ij} \in J$.
By $(*)$ $a_{ij} E_{ij} \in J$
$\Rightarrow \sum_{i,j} a_{ij} E_{ij} = A \in J \Rightarrow M_n(I) \subseteq J$ so $M_n(I) = J$.

Conversely any such $M_n(I)$ is a two-sided ideal. $\square$

<u>Example</u>
If $R$ is a field, then $M_n(F)$ is a simple ring.

Note

$M_n(F)$ has non-trivial (left) ideals,

e.g. $C_i = \left\{ \begin{pmatrix} & c_{1i} & \\ 0 & c_{2i} & 0 \\ & \vdots & \\ & c_{ni} & \end{pmatrix} : c_{ii} \in F \right\}$ (i-th column space)

is a left ideal.

Def (simple module)

Let $M$ be a non-zero $R$-module. Then we say $M$ is simple if its only submodules are $0$ and $M$.

Def

A module $M$ is called semisimple if $M \cong \bigoplus_{i \in I} M_i$ where each $M_i$ is a simple module.

Examples

1). $R = F$ is a field

All modules are vector spaces.

Simple modules are 1-dimensional vector-spaces

All vector spaces have a basis, i.e. all modules over fields are semisimple.

(e.g. $\mathbb{R}$ is a $\mathbb{Q}$-vector space)

$\mathbb{R} = \bigoplus_{i \in I} \mathbb{Q}_i$

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Recall:

- A non-zero $R$-module $M$ is simple if its only submodules are $0$ and $M$
- A module $M$ is semisimple if $M = \bigoplus_{i \in I} M_i$, where each $M_i$ is a simple submodule.

Examples

1). Every module (vector space) over a field is semisimple.

2). Consider $R = M_n(F)$ as an $R$-module.

As an $R$-module, $M_n(F) = c_1 \oplus \dots \oplus c_n$, each $c_i$ is the $i^{th}$ column space. In fact $c_i$ is a simple $R$-module
$\Rightarrow M_n(F)$ is a semisimple $M_n(F)$-module.

[Notation warning:
$M_n(F)$ is a simple ring but $M_n(F)$ is not a simple $M_n(F)$-module.]

3) ⓐ $R = \mathbb{Z}$, $\mathbb{Z}$ is not semisimple because its submodules are of the form $n\mathbb{Z}$ for $n \in \mathbb{Z}$.

As $\mathbb{Z}$-modules, $n\mathbb{Z} \cong \begin{cases} 0, & n=0 \\ \mathbb{Z}, & n \neq 0 \end{cases}$

ⓑ $\mathbb{Z}/4\mathbb{Z}$ is not semisimple because its submodules are $0$, $2\mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$

$2\mathbb{Z}/4\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \leftarrow$ simple (the only one).

Schur's Lemma

Let $M, N$ be two non-zero simple $R$-modules and let $\phi: M \to N$ be an $R$-module homomorphism.

Then either

(i) $\phi$ is an isomorphism

(ii) $\phi$ is zero, ie. $\phi(m) = 0 \ \forall m \in M$.

Proof

Recall $\phi$ is an isomorphism $\Leftrightarrow$ $\ker\phi = 0$ and $\mathrm{Im}\,\phi = N$.

Suppose that $\phi \neq 0$.

Consider: $\mathrm{Ker}\,\phi \leq M$ is a submodule of $M$,
so $\mathrm{Ker}\,\phi = 0$ or $M$ because $M$ is simple.

Since $\phi \neq 0$, $\mathrm{Ker}\,\phi \neq M \Rightarrow \mathrm{Ker}\,\phi = 0$.

Consider: $\mathrm{Im}\,\phi \leq N$ is a submodule of $N$,
so $\mathrm{Im}\,\phi = 0$ or $N$ because $N$ is simple.

Since $\phi \neq 0$, $\mathrm{Im}\,\phi \neq 0 \Rightarrow \mathrm{Im}\,\phi = N$

$\mathrm{Ker}\,\phi = 0$ and $\mathrm{Im}\,\phi = N \Rightarrow \phi$ is an isomorphism $\square$

Prop

semisimple has unique decomp.

Suppose $R$ is a ring and $M = S_1 \oplus \dots \oplus S_m$, $N = S_1' \oplus \dots \oplus S_n'$ are $R$-modules with each $S_i, S_i'$ simple $R$-modules.
Then if $\phi: M \to N$ is an isomorphism, we have $m = n$ and after reordering $S_i \cong S_i'$ for $i = 1, \dots, m$.

Proof

Let $\psi : N \to M$ be the inverse. Define $\iota_i : S_i \to M$,
$s \mapsto (0, \dots, 0, \underset{\uparrow\, i\text{th place}}{s}, 0, \dots, 0)$, $\pi_j : M \to S_j$, $(s_1, \dots, s_m) \mapsto s_j$.

Note that $\pi_j \circ \iota_i : S_i \to S_j$, $\pi_j \circ \iota_i = \begin{cases} \mathrm{Id}_{S_i}, & i = j \\ 0, & \text{otherwise} \end{cases}$

and $\sum_{i=1}^{m} \iota_i \pi_i = \mathrm{Id}_M$ ; $(s_1, \dots, s_m) = (s_1, 0, \dots, 0) + \dots + (0, \dots, 0, s_m)$.

Similarly define $\iota_i' : S_i' \to N$, $\pi_j' : N \to S_j'$.

Let $\phi_{ij} = \pi_j' \circ \phi \circ \iota_i : S_i \to S_j'$, $S_i \xrightarrow{\iota_i} M \xrightarrow{\phi} N \xrightarrow{\pi_j'} S_j'$
and $\psi_{ij} = \pi_i \circ \psi \circ \iota_j' : S_j' \to S_i$

Fix $i$, e.g. $i = 1$.

Consider $\sum_{j=1}^{n} \psi_{ij} \circ \phi_{ji} = \sum_{j=1}^{n} (\pi_i \psi \iota_j')(\pi_j' \phi \iota_i)$

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$$= \pi_i \psi \Big( \underbrace{\sum_{j=1}^{n} \iota'_j \pi'_j}_{Id_N} \Big) \phi \iota_i$$

$$= \pi_i \underbrace{\psi \phi}_{Id_M} \iota_i = \pi_i \iota_i = Id_{S_i}$$

For each $1 \le j \le n$, $\psi_{ij} \phi_{ji} : S_i \to S_i$
and $\sum_{j=1}^{n} \psi_{ij} \phi_{ji} = Id_{S_i}$

$\Rightarrow$ for some $1 \le j \le n$, $\psi_{ij} \phi_{ji} \neq 0$.
Since $S_i$ is simple, $\psi_{ij} \phi_{ji} : S_i \to S_i$ is an isomorphism.
$\psi_{ij} \phi_{ji}$ is the composition $S_i \xrightarrow{\phi_{ji}} S_j' \xrightarrow{\psi_{ij}} S_i$
and both $S_i$ and $S_j'$ are simple and neither
$\phi_{ji}$ nor $\psi_{ij}$ can be zero
$\Rightarrow$ $\phi_{ji}$ and $\psi_{ij}$ are isomorphisms.
W.l.o.g. $\psi_{11} \phi_{11} : S_1 \to S_1$ is an isomorphism
Let $B = S_2 \oplus \ldots \oplus S_m$ and $B' = S_2' \oplus \ldots \oplus S_n'$.
Let $f$ be the composition
$B \xhookrightarrow{\iota_B} M \xrightarrow{\phi} N \xrightarrow{\pi_B'} B'$,

? $(s_2, \ldots, s_m) \longmapsto (0, s_2, \ldots, s_m) \longmapsto (0, s_2, \ldots, s_n) \longmapsto (s_2, \ldots, s_n)$

<u>Claim:</u> $f$ is an isomorphism.
$\mathrm{Ker} f = 0$ :
Suppose $b \in B$ st. $f(b) = 0$. Then $\phi(b) \in S_1'$
and $\psi_{11} \phi(b) = (\pi_1 \circ \psi \circ \iota_1') \pi_1' \phi(b)$

$$= \pi_1 \psi \underbrace{(\iota_1' \pi_1')}_{Id \text{ on } S_1 \le S_1 \oplus \ldots + S_n = N} \phi(b)$$

$$= \pi_1 \underbrace{\psi \phi}_{Id_N}(b) = \pi_1(b) = 0 \qquad (B = \mathrm{Ker}\, \pi_1)$$

Since $\psi_{11}$ is an isomorphism, $\phi(b) = 0$.
Since $\phi$ is an isomorphism, $b = 0$
$\Rightarrow \mathrm{Ker} f = 0$.
$\mathrm{Im} f = B'$ :
Given $b' \in B'$.

Let $\psi(b') = b + s_1$ for $b \in B$, $s_1 \in S_1$.
Choose $s_1' \in S_1'$ s.t. $\psi_{11}(s_1') = s_1$ (because $\psi_{11} : S_1' \to S_1$ is an isomorphism).
Then $\psi(b' - s_1') = \psi(b') - \psi(s_1') = b^* \in B$
since $\pi_1(\psi(b') - \psi(s_1')) = s_1 - s_1 = 0$.
$\phi(b^*) = \phi\psi(b' - s_1') = b' - s_1'$
$\Rightarrow f(b^*) = \pi_{B'}(b' - s_1') = b'$
$\Rightarrow \operatorname{Im} f = B'$
$\Rightarrow f : B \to B'$ is an isomorphism.
$\Rightarrow$ done by induction on $m$.
$\square$

Definition

Suppose $M$ is an $R$-module. We define
$\mathrm{End}_R(M)$ to be the ring $\{\phi : M \to M,\ R\text{-module homomorphisms}\}$
with $(\phi + \psi)(m) = \phi(m) + \psi(m)$, $\phi\psi(m) = \phi(\psi(m))$,
$\mathbb{1} = \mathrm{Id}_M$ ($\mathrm{Id}_M(m) = m \ \forall m$), $0 = 0$ ($0(m) = 0 \ \forall m$).

Example

$R$ a ring, $\mathrm{End}_R(R) = \{\phi : R \to R,\ R\text{-module homomorphisms}\}$
Note that $\phi \in \mathrm{End}_R(R)$ is determined by $\phi(1)$, since
then $\phi(s) = s \cdot \phi(1) \ \forall s \in R$.
Moreover, for each $r \in R$ we can define $\phi_r \in \mathrm{End}_R(R)$
$\phi_r(s) = sr$, note that $\phi_r(1) = r$.

$\Rightarrow R \longrightarrow \mathrm{End}_R(R)$, $r \mapsto \phi_r$ is a bijection.
Clearly $\phi_r + \phi_{r'} = \phi_{r+r'}$ since $(\phi_r + \phi_{r'})(1) = r + r' = \phi_{r+r'}(1)$.
However, $\phi_r\phi_{r'}(1) = \phi_r(\phi_{r'}(1)) = \phi_r(r') = r'\phi_r(1) = r'r$
so $\phi_r\phi_{r'} = \phi_{r'r}$.

Def

Given a ring $R$, we define $R^{op}$ (the opposite ring) to be
the ring with the same elements and addition as $R$ but $r \cdot s = sr$
mult. in $R^{op}$ ↑ ↑ mult. in $R$

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Then $R^{op} \longrightarrow End_R(R)$, $r \longmapsto \phi_r$, $\phi(1) \longleftarrow \phi$
is an isomorphism of rings.

Examples

1). $M_n(R)^{op} \cong M_n(R^{op})$
$A \longmapsto A^T$ ~~when R is commutative.~~
$a_{ij} \longmapsto a_{ji}$

2). If $D$ is a division ring, so is $D^{op}$.

3). $End_R(R^n) = M_n(R)^{op}$ (exercise)

4). $(R^{op})^{op} = R$

Schur's Lemma (2)

If $M$ is a simple $R$-module, then $End_R(M)$ is a division ring.

Proof

Suppose $\alpha : M \to M$ is non-zero. Then by Schur's Lemma (version 1) $\alpha$ is an isomorphism; so $\exists\, \alpha^{-1} : M \to M$ s.t. $\alpha^{-1}\alpha = \alpha\alpha^{-1} = Id_M$.
$\Rightarrow End_R(M)$ is a division ring. $\square$

Examples

1). If $D$ is a division ring then $End_D(D) = D^{op}$ is a division ring.

2). $F$ is a field, $End_F(F^2) = M_2(F)$ is not a division ring.
($\Rightarrow F^2$ is not simple.)

3). $R = \mathbb{Z}$, $M = \mathbb{Q}$, $End_{\mathbb{Z}}(\mathbb{Q}) \cong \mathbb{Q}$, $\phi \mapsto \phi(1)$ [$\mathbb{Q}^{op} = \mathbb{Q}$]
but $\mathbb{Q}$ is not a simple $\mathbb{Z}$-module, so the converse to Schur's Lemma does not hold.

4). $End_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}) \cong (\mathbb{Z}/n\mathbb{Z})^{op} = \mathbb{Z}/n\mathbb{Z}$, $n \geq 1$.
which is a field $\Leftrightarrow n$ is prime.

Prop (Classification of simple modules)
If $M$ is a simple module over $R$ then $M$ is a cyclic module (i.e. $M = R\cdot m$ for some $m \in M$) and $M \cong R/I$ where $I$ is the left ideal $Ann_R(m) = \{r \in R : rm = 0\}$ for all $m$ s.t. $M = Rm$.

Proof
Let $0 \neq m \in M$. Then $Rm \leq M$ is a submodule of $M$
$\Rightarrow Rm = M$ since $M$ is simple and $m \neq 0$.
$\Rightarrow M = R\cdot m \;\; \forall\, 0 \neq m \in M$.
Pick $0 \neq m \in M$. Define $I = Ann_R(m) = \{r \in R : r\cdot m = 0\}$
Claim: $\phi : R/I \to M$, $(r+I) \mapsto rm$, is an isomorphism of $R$-modules.
Note that $\phi$ is well defined, since $(r+I) = (r'+I) \Leftrightarrow r'-r \in I$
so $rm - r'm = (r-r')m = 0 \Rightarrow rm = r'm$ when $(r+I) = (r'+I)$.
It is surjective because $M = Rm$.
Also if $\phi((r+I)) = 0 \Rightarrow rm = 0 \Rightarrow r \in I \Rightarrow (r+I) = (0+I)$.
$\square$

The proof shows that if $M$ is cyclic, then $M = R/I$ for some ideal $I$.

Lemma
Let $R$ be a ring and $I$ be a left ideal, then $R/I$ is a simple $R$-module $\Leftrightarrow$ $I$ is a maximal left ideal (i.e. $I \neq R$ and if $J$ is an ideal $I \leq J \subsetneq R \Rightarrow J = I$).

Proof
We have a one-to-one correspondence

$$\left\{\begin{matrix}\text{submodules} \\ \text{of } R/I\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{ideals} \\ I \leq J \leq R\end{matrix}\right\}$$

$$N \leq R/I \longmapsto J = \{r \in R : (r+I) \in N\}$$
$$J/I \longleftarrow J$$

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This shows that $R/I$ is simple $\Leftrightarrow$ $I$ is maximal.

$\square$

Example

$R = \mathbb{Z}$. The maximal ideals in $\mathbb{Z}$ are of the form $p\mathbb{Z}$ for $p$ prime.

$\Rightarrow$ the simple $\mathbb{Z}$-modules are $\mathbb{Z}/p\mathbb{Z}$ for $p$ prime.

Example

$R = \mathbb{C}[X] = \{ \sum_{i=0}^{d} a_i X^i : d \geq 0,\ a_i \in \mathbb{C} \}$.

By the FToA, the maximal ideals of $\mathbb{C}[X]$ are $(X-z) = (X-z)\mathbb{C}[X]$, $z \in \mathbb{C}$

$\Rightarrow$ simple $\mathbb{C}[X]$-modules are of the form $\mathbb{C} \cong \mathbb{C}[X]/(X-z)$, $z \longleftarrow\!\shortmid X$

with $f(X) \in \mathbb{C}[X]$ acts on $\mathbb{C}$ by multiplication by $f(z)$.

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Characterisation of semisimple modules

Recall: an $R$-module $M$ is semisimple if $M = \bigoplus_{i \in I} M_i$ where each $M_i$ is a simple submodule.

Theorem

Let $M$ be an $R$-module. Then the following are equivalent:

(i) $M$ is a direct sum of simple submodules ($M$ semisimple)

(ii) $M$ is a sum of simple submodules

(iii) Every submodule $N \leq M$ is a direct summand of $M$ i.e. there exists a complement $P \leq M$ s.t. $M = N \oplus P$.

Proof

Omitted. (see later)

Example

$F$ is a field, $V$ a vector space over $F$, then
$V = \sum_{0 \neq v \in V} F \cdot v$.

Since (ii) $\Rightarrow$ (i), $V$ has a basis.

Remark:

By (iii) every quotient of a semisimple module is isomorphic to a submodule and vice versa. (If $M = N \oplus P$ then $M/N \cong P$).

Proposition

Every submodule (and therefore every quotient) of a semisimple module is semisimple.

Proof

Suppose $M$ is semisimple and $N \leq M$.
If $P \leq N$ is a submodule, take $Q \leq M$ such that $P \oplus Q = M$ ($M$ is semisimple) and let $W = Q \cap N$.
Claim: $P \oplus W = N$.
$P, W \leq N$ so $P + W \subseteq N$ and $P \cap W = 0$ since $P \cap Q = 0$ and $W \subseteq Q$.
So we must show $P + W = N$.
Suppose $n \in N$, then $n = p + q$ where $p \in P$, $q \in Q$.
Then $q = n - p \in N$, so $q \in N \cap Q = W$.
Hence (iii) holds for $N$, so $N$ is semisimple. $\square$

Def

A ring $R$ is (left) semisimple if every non-zero (left) $R$-module is semisimple.

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Prop

$R$ is a semisimple ring $\Leftrightarrow$ $R$ is a semisimple $R$-module.

Proof

[$\Rightarrow$] by definition.

[$\Leftarrow$] Let $M \neq 0$ be an $R$-module.

Choose a surjective $R$-module homomorphism

$\varphi : \bigoplus_{i \in I} R_i \longrightarrow M$ (e.g. $I = M$)

Since $R = \bigoplus_{j \in J} S_j$ with each $S_j$ simple,

we have a surjection

$\varphi : \bigoplus_{i \in I} \left( \bigoplus_{j \in J} S_j \right) \longrightarrow M$

$\Rightarrow$ $M$ is a quotient of a semisimple module $\left( M = \left( \bigoplus_{i \in I} \bigoplus_{j \in J} S_j \right) \Big/ \ker \varphi \right)$

$\Rightarrow$ $M$ is semisimple. $\square$

Prop

Let $R$ be a semisimple ring and let $R = \bigoplus_{i \in I} R_i$ where each $R_i$ is a simple $R$-module.

Then any simple $R$-module is isomorphic to $R_i$ for some $i \in I$.

Proof

Suppose $S$ is a simple $R$-module. Pick $0 \neq s \in S$, then we have a surjection $\varphi : R \longrightarrow S$, $r \longmapsto rs$

and so we have a surjection $\varphi : \bigoplus_{i \in I} R_i \longrightarrow S$.

Consider $\varphi_i = \varphi|_{R_i} : R_i \longrightarrow S$.

Each is an $R$-module homomorphism between simple $R$-modules, so either $\varphi_i = 0$ or $\varphi_i$ is an isomorphism.

Since $\varphi$ is surjective, not all $\varphi_i = 0$, so $\varphi_i$ is an isomorphism for some $i \in I$. $\square$

Definition

Let $R$ be a commutative ring. An (associative) algebra over $R$ (an $R$-algebra) is a ring $A$ with the structure of an $R$-module s.t.

(i) $\underset{\text{ring addition in } A}{a+b} = \underset{\text{R-module addition}}{a+b} \quad \forall a,b \in A.$

(ii) $(\lambda \cdot a)b = \lambda \cdot (ab) = a(\lambda \cdot b) \quad \forall a,b \in A, \lambda \in R.$

Equivalently, $A$ is a ring together with a ring homomorphism $\varphi: R \to A$ s.t. $\varphi(R)$ is in the centre of $A$ (i.e. $a\varphi(r) = \varphi(r)a \quad \forall r \in R, a \in A$), $r \mapsto r \cdot \mathbb{1}_A$

Examples

1). Any ring $R$ is a $\mathbb{Z}$-algebra, $\varphi: \mathbb{Z} \to R$, $1 \mapsto 1_R$.

2). $R$ is a commutative ring, then $R[x]$, the polynomial ring over $R$, is an $R$-algebra. Also $M_n(R)$ are $R$-algebras, $\varphi: R \to M_n(R)$, $r \mapsto r \cdot I_n$

Note: If $M$ is an $A$-module, where $A$ is an $R$-algebra, then $M$ also has the structure of an $R$-module. (Define $r \cdot m := \varphi(r)m$).

3). $\mathbb{H}$ - quaternion algebra over $\mathbb{R}$:

$\mathbb{H} = \mathbb{R} \oplus \mathbb{R}\underline{i} \oplus \mathbb{R}\underline{j} \oplus \mathbb{R}\underline{k}$, with multiplication $\mathbb{R}$-bilinear and

$$\underline{i} \cdot \underline{j} = \underline{k}, \quad \underline{j} \cdot \underline{k} = \underline{i}, \quad \underline{k} \cdot \underline{i} = \underline{j}$$
$$\underline{j} \cdot \underline{i} = -\underline{k}, \quad \underline{k} \cdot \underline{j} = -\underline{i}, \quad \underline{i} \cdot \underline{k} = -\underline{j}$$

In fact $\mathbb{H}$ is a division ring (division algebra over $\mathbb{R}$) since if $\alpha = a + b\underline{i} + c\underline{j} + d\underline{k}$

let $\beta = a - b\underline{i} - c\underline{j} - d\underline{k}$

and then $\alpha\beta = \beta\alpha = a^2 + b^2 + c^2 + d^2$

$\Rightarrow$ if $\alpha \neq 0$, $\dfrac{\beta}{a^2+b^2+c^2+d^2}$ is a left and right inverse.

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Def

If $A$ is an $F$-algebra where $F$ is a field, then we say that $A$ is finite dimensional if $\dim_F A < \infty$, and say $\dim_F A$ is the dimension of $A$.

Def

We say a field $F$ is algebraically closed if every polynomial $p(x) \in F[x]$ has a root (i.e. $\exists \alpha \in F$ s.t. $p(\alpha) = 0$).

Theorem

Let $A$ be a finite dimensional algebra over an algebraically closed field $F$. Then if $S$ is a simple $A$-module we have

$$End_A(S) = F$$

Proof

Note that $A$-modules are $F$-vector spaces.
Let $0 \neq \varphi : S \to S$, $\varphi \in End_A(S)$, and let $ch_\varphi(X) \in F[X]$ be its characteristic polynomial as an $F$-linear map.
Since $F$ is algebraically closed, $ch_\varphi(X)$ has a root $\alpha \in F$.
Then $\varphi - \alpha Id_S$ is not invertible
$\Rightarrow \varphi - \alpha Id_S = 0$ by Schur's lemma
$\Rightarrow \varphi = \alpha Id_S$.
$\square$

Remark

In general (when $F$ is not algebraically closed) then $End_A(S)$ is a division algebra over $F$.

## Lemma

Let $M$ be an $R$-module, $n \geqslant 1$, $M^n = \bigoplus_{i=1}^{n} M$.

Then we have $End_R(M^n) \cong M_n(End_R M)$,

$\phi \longmapsto (\phi_{ij})$

where $\phi_{ij} = \pi_i \phi \iota_j$ where $\pi_i : M^n \to M$, $(m_1, \dots, m_n) \mapsto m_i$

$\iota_j : M \to M^n$, $m \mapsto (0, \dots, 0, \underset{\substack{\uparrow \\ j\text{th summand.}}}{m}, 0, \dots, 0)$

## Proof

Omitted.

## Lemma

If $S_1, \dots, S_k$ are pairwise non-isomorphic simple $R$-modules and $n_1, \dots, n_k \geqslant 1$ then

$$End_R(S_1^{n_1} \oplus \dots \oplus S_k^{n_k}) = End_R(S_1^{n_1}) \times \dots \times End_R(S_k^{n_k})$$
$$= M_{n_1}(End_R S_1) \times \dots \times M_{n_k}(End_R S_k)$$

as rings.

## Proof

Previous lemma and Schur's Lemma. $\square$

## Theorem (Artin-Wedderburn decomposition theorem)

Suppose $A$ is a finite dimensional $F$-algebra for some field $F$. Then $A$ is a semisimple ring $\iff A \cong M_{n_1}(D_1) \times \dots \times M_{n_k}(D_k)$ (as rings) where each $D_1, \dots, D_k$ are finite dimensional division algebras over $F$.

## Remark

In the case when $F$ is algebraically closed then each $D_i = F$ and all semisimple finite dimensional $F$-algebras are of the form $\prod_{i=1}^{k} M_{n_i}(F)$.

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Proof

[$\Leftarrow$] on problem sheet 2 (show $M_{n_i}(D_i)$ is semisimple $\forall i$).

[$\Rightarrow$]

Suppose $A$ is a semisimple ring.
Then $A = S_1^{n_1} \oplus \dots \oplus S_k^{n_k}$ where $n_1, \dots, n_k \geq 1$
with $S_1, \dots, S_k$ pairwise non-isomorphic.
(The sum is finite because $\dim_F A < \infty$ and
$\dim_F A = \sum_{i=1}^{k} n_i \dim_F(S_i)$.)

Then
$$\begin{aligned} \operatorname{End}_A(A) &\cong \operatorname{End}_A(S_1^{n_1} \oplus \dots \oplus S_k^{n_k}) \\ &\cong \operatorname{End}_A(S_1^{n_1}) \times \dots \times \operatorname{End}_A(S_k^{n_k}) \\ &\cong M_{n_1}(\operatorname{End}_A(S_1)) \times \dots \times M_{n_k}(\operatorname{End}_A(S_k)) \end{aligned}$$

Let $D_i = \operatorname{End}_A(S_i)$.
By Schur's Lemma each are division algebras over $F$.
(Note that when $F$ is algebraically closed each
$\operatorname{End}_A(S_i) = F$).

$\operatorname{End}_A A = A^{op}$
$$\begin{aligned} \Rightarrow A = (A^{op})^{op} &\cong (M_{n_1}(D_1) \times \dots \times M_{n_k}(D_k))^{op} \\ &\cong M_{n_1}(D_1)^{op} \times \dots \times M_{n_k}(D_k)^{op} \\ &\cong M_{n_1}(D_1^{op}) \times \dots \times M_{n_k}(D_k^{op}) \end{aligned}$$
Since $D_i$ is a division ring, so is $D_i^{op}$.
$\square$

Representations and $F[G]$-modules

Def

Suppose $G$ is a group, then the group ring of $G$ over a field $F$ is the following $F$-algebra:
The $F$-vector space is $F[G] = \bigoplus_{g \in G} F \cdot g$
The multiplication is defined to be $F$-bilinear
and $\underset{\text{mult. in } F[G]}{g \cdot h} = \underset{\text{group mult in } G}{hg}$

Elements of $F[G]$ are of the form $\sum_{g \in G} a_g g$ ($a_g \neq 0$ only finitely often).

$$\left(\sum_{g\in G} a_g g\right)\cdot\left(\sum_{h\in G} b_h h\right) = \sum_{(g,h)\in G\times G} a_g b_h gh$$

$$= \sum_{t\in G}\left(\sum_{g\in G} a_g b_{g^{-1}t}\right)t$$

$$\left(\sum_{g\in G} a_g g\right) + \left(\sum_{g\in G} b_g g\right) = \sum_{g\in G}(a_g+b_g)g$$

$$0 = \sum_{g\in G} 0\cdot g$$

$\mathbb{1} = e$ where $e \in G$ is the identity in $G$.

The corresponding ring homomorphism which gives $F[G]$ the structure of an $F$-algebra is $F \to F[G]$, $\lambda \mapsto \lambda\cdot e$. (usually write $\lambda e$ as $\lambda \in F[G]$).

Prop

There is a natural one-to-one correspondence between representations of $G$ over $F$ and finitely generated $F[G]$ modules.

$$\left\{\begin{matrix}\text{f.dim. rep}^{\text{n}}\text{s} \\ \rho: G \to GL(V)\end{matrix}\right\} \xleftrightarrow{1:1} \left\{\begin{matrix}\text{f.g. } F[G] \\ \text{modules}\end{matrix}\right\}$$

$(\rho, V) \longmapsto V$ with multiplication

$$F[G]\times V \longrightarrow V$$

$$\left(\sum_{g\in G} a_g g, v\right) \longmapsto \sum_{g\in G} a_g \rho(g)(v)$$

$(\rho, V) \longleftarrow V$

$\rho(g): V \to V$ is $v \mapsto gv$

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## Representations and $F[G]$ modules

Prop

There is a natural 1-1 correspondence between representations of $G$ over a field $F$ and finitely generated $F[G]$-modules.

$$\left\{\begin{array}{l}\text{rep}^n\text{s}\\ \rho: G \to GL(V)\end{array}\right\} \xleftrightarrow{1:1} \left\{\text{f.g. } F[G]\text{-modules } V\right\}$$

$$(\rho, V) \longmapsto V \text{ with multiplication}$$

$$F[G] \times V \longrightarrow V$$

$$\left(\sum a_g g, v\right) \longmapsto \sum a_g \rho(g) v$$

$$(\rho, V) \longleftarrow\!\shortmid V$$

where $\rho(g): V \to V$
$\quad v \longmapsto g(v)$

Proof

We need to check that if $\rho$ is a representation, then $F[G] \times V \to V$, $\left(\sum a_g g, v\right) \mapsto \sum a_g \rho(g) v$, gives $V$ the structure of an $F[G]$-module.

We'll check that if $\alpha, \beta \in F[G]$ and $v \in V$, then $\alpha(\beta v) = (\alpha\beta)v$.

$\alpha = \sum_{g \in G} a_g g$, $\beta = \sum_{h \in G} b_h h$, then

$$\sum_{g\in G} a_g g \cdot \left(\sum_{h \in G} b_h h \cdot v\right) = \sum_{g \in G} a_g g\left(\sum_{h\in G} b_h \rho(h) v\right)$$

$$= \sum_{g\in G} a_g \left(\rho(g)\left(\sum_{h\in G} b_h \rho(h) v\right)\right) \qquad \begin{array}{r}\rho(g) \in GL(V)\\ F\text{-linear}\end{array}$$

$$= \sum_{g \in G} a_g \sum_{h\in G} b_h \rho(g)(\rho(h) v)$$

$$= \sum_{g\in G} a_g \sum_{h \in G} b_h \rho(gh) v$$

$$= \sum_{(g,h) \in G\times G} a_g b_h \rho(gh) v$$

$$= \sum_{(g,h)\in G \times G} \rho(a_g b_h gh) \cdot v = \left(\left(\sum_{g\in G} a_g g\right)\left(\sum_{h \in G} b_h h\right)\right) \cdot v$$

So $\alpha(\beta v) = (\alpha\beta)v$.
Conversely, if we are given an $F[G]$ module $V$, then we need to show $\rho(g): \underset{v \mapsto gv}{V \to V}$ gives a representation $\rho: G \to GL(V)$. (= $F\cdot e$ where $e$ is the group id).
Firstly note that $F$ lies in the centre of $F[G]$, i.e. if $\lambda \in F$, $\alpha \in F[G]$ then $\lambda\alpha = \alpha\lambda$

$$\Rightarrow g(\lambda v + \mu u) = g(\lambda v) + g(\mu u) \qquad g \in G,\ \lambda,\mu \in F,\ u,v \in V$$
$$= (g\lambda)v + (g\mu)u$$
$$= (\lambda g)v + (\mu g)u \qquad (F \text{ is the centre of } F[G])$$
$$= \lambda g(v) + \mu g(u)$$

So $\rho(g): V \to V$, $v \mapsto gv$ is an $F$-linear map.
Since $V$ is an $F[G]$-module, $(gh)\cdot v = g(h\cdot v)$
$\Rightarrow \rho(g)\circ\rho(h) = \rho(gh)$.
Also since $V$ is an $F[G]$-module
$\Rightarrow e\cdot v = v \quad \forall v \in V$ where $e \in G$ is the group identity
$\Rightarrow \rho(e) = Id_V$
Note that $\rho(g)$ is invertible because $\rho(g)\rho(g^{-1}) = \rho(e) = Id_V$.
So we have shown that $\rho: G \to GL(V)$ is a representation.
The maps given in the proposition are mutual inverses of each other, so this is a one-to-one correspondence. $\square$

<u>Maschke's Theorem</u> (coming soon!)

The aim of this section is to prove that if $G$ is a finite group and $F$ is a field with $char(F) \nmid |G|$, then $F[G]$ is a semisimple ring.
We'll show that if $V$ is an $F[G]$-module, and $W \le V$ a submodule, then there exists $U \le V$ another submodule with $U \oplus W = V$ (ie. $U + W = V$, $U \cap W = 0$)

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## Definition

Let $R$ be a ring, $M$ an $R$-module. We say $\pi \in End_R(M)$ is a *projection* (or idempotent) if $\pi^2 = \pi$.

## Note

If $\pi$ is a projection, so is $1-\pi$:

$(1-\pi)^2 = 1 - 2\pi + \pi^2 = 1 - 2\pi + \pi = 1 - \pi$ since $\pi^2 = \pi$

## Proposition

Given an $R$-module $M$, we have a one-to-one correspondence:

ordered $\Rightarrow P \oplus Q \neq Q \oplus P$

$$\left\{\begin{array}{c}\text{ordered decompositions} \\ M = P \oplus Q\end{array}\right\} \longleftrightarrow \left\{\text{projections } \pi \in End_R(M)\right\}$$

$$P \oplus Q \longmapsto \pi : M \to M,\ m \mapsto p$$

where $m = p + q$, $p \in P$, $q \in Q$.

$$Im\,\pi \oplus Ker\,\pi \longleftarrow \pi$$

## Proof

[$\rightarrow$]

Given $M = P \oplus Q$, define $\pi : M \to M$ s.t. $\pi(m) = p$ where $m = p + q$, $p \in P$, $q \in Q$. Note that this is well defined since $M = P \oplus Q$ so $p, q$ exist and are unique.

Suppose $m = p + q$, then

$\pi^2(m) = \pi(\pi(m)) = \pi(p) = \pi(p + 0) = p$

$\Rightarrow \pi^2 = \pi$

$\Rightarrow \pi$ is a projection in $End_R(M)$.

[$\leftarrow$]

Given $\pi \in End_R(M)$ a projection, we must show (i) $Im(\pi) + Ker(\pi) = M$ and (ii) $Im(\pi) \cap Ker(\pi) = 0$.

(i) Note that $m = \pi m + (1-\pi)m$, $\forall m \in M$ and

$\pi(1-\pi)m = (\pi - \pi^2)m = 0$ so $(1-\pi)m \in Ker\,\pi$

$\Rightarrow Im\,\pi + Ker\,\pi = M$.

(2) Suppose $m \in \mathrm{Im}\,\pi \cap \mathrm{Ker}\,\pi$.
Since $m \in \mathrm{Im}\,\pi$, $m = \pi(n)$ for some $n \in M$.
Also $m \in \mathrm{Ker}\,\pi \Rightarrow 0 = \pi(m) = \pi^2(n) = \pi(n) = m \Rightarrow m = 0$
$\Rightarrow \mathrm{Im}\,\pi \cap \mathrm{Ker}\,\pi = 0$, as claimed.
Each of the maps are inverses of each other, so this is a one-to-one correspondence. □

**Theorem** (Maschke)
Let $G$ be a finite group, $F$ a field with $\mathrm{char}(F) \nmid |G|$. Let $V$ be an $F[G]$-module.
Then for any $F[G]$-submodule $U \leq V$ $\exists$ an $F[G]$-submodule $W \leq V$ s.t. $U \oplus W = V$.
In particular $F[G]$ is a semisimple ring.

**Proof**
Assume $U \neq 0$. Consider $U, V$ as $F$-vector spaces.
$F$ is a field $\Rightarrow V$ is a semisimple $F$-module
($V$ has a basis).
$\Rightarrow \exists\, W_0 \leq V$ s.t. $U \oplus W_0 = V$ as $F$-module (as vector spaces).
Let $\pi_0 \in \mathrm{End}_F(V)$ be the corresponding projection,
$\pi_0^2 = \pi_0$, $\mathrm{Im}\,\pi_0 = U$, $\mathrm{Ker}\,\pi_0 = W_0$.
We turn $\pi_0$ into an $F[G]$-module homomorphism by an averaging process.
Define $\pi : V \to V$, $v \mapsto \frac{1}{|G|} \sum_{g \in G} g \pi_0 (g^{-1} v)$
[note that we use $|G| < \infty$ and $|G|$ is invertible in $F$ ($\mathrm{char}\, F \nmid |G|$)].
Claim: $\pi^2 = \pi \in \mathrm{End}_{F[G]}(V)$ and $\mathrm{Im}(\pi) = U$
(Then we are done since $W := \mathrm{Ker}\,\pi$).

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<u>Theorem</u> (Maschke)
Assume char $F \nmid |G|$. If $V$ is an $F[G]$ module, then for every submodule $U \leq V$, $\exists W \leq V$ such that $U \oplus W = V$.

<u>Proof</u>
As $F$-vector spaces, $\exists$ an $F$-vector space $W_0$ s.t. $U \oplus W_0 = V$.
Let $\pi_0 \in End_F(V)$ be the projection such that
$Im(\pi_0) = U$, $Ker(\pi_0) = W_0$.
Define $\pi : V \longrightarrow V$, $v \longmapsto \frac{1}{|G|} \sum_{g \in G} g\pi_0(g^{-1}v)$.
<u>Claim</u>: $\pi^2 = \pi \in End_{F[G]}(V)$ and $Im(\pi) = U$.
Clearly $\pi \in End_F(V)$. If $x \in G$ and $v \in V$, then

$$\pi(x \cdot v) = \frac{1}{|G|} \sum_{g \in G} g\pi_0(g^{-1}xv)$$

$$= \frac{x}{|G|} \sum_{g \in G} x^{-1}g\pi_0(g^{-1}xv)$$

as $g$ runs over $G$ then $x^{-1}g$ runs over $G$ and $(x^{-1}g)^{-1} = g^{-1}x$

$$= \frac{x}{|G|} \sum_{h \in G} h\pi_0(h^{-1}v)$$

$$= x \cdot \pi(v)$$

$\Rightarrow \pi(xv) = x\pi(v) \quad \forall x \in G, v \in V$
$\Rightarrow$ if $\alpha \in F[G]$ then $\pi(\alpha \cdot v) = \alpha\pi(v) \quad \forall v \in V$
since $\pi \in End_F(V)$
$\Rightarrow \pi \in End_{F[G]}(V)$
$Im(\pi) \subseteq U$ since $U$ is an $F[G]$-submodule and $Im(\pi_0) \subseteq U$.
Also note that if $u \in U$, then since $\pi_0|_U = Id_U$, we have

$$\pi(u) = \frac{1}{|G|} \sum_{g \in G} g\pi_0(g^{-1}u)$$

$g^{-1}u \in U$, $\pi_0|_U = Id_U$

$$= \frac{1}{|G|} \sum_{g \in G} gg^{-1}u = \frac{1}{|G|} \sum_{g \in G} u = \frac{|G|}{|G|}u = u$$

$\Rightarrow Im(\pi) = U$ and $\pi|_U = Id_U$
$\Rightarrow \pi$ is a projection, so $\pi^2 = \pi \in End_{F[G]}(V)$ with $Im\,\pi = U$. Define $W = Ker\,\pi$. Then $W \leq V$ is an $F[G]$-submodule with $U \oplus W = V$. $\square$

We'll now prove the classification theorem for finitely generated semisimple modules over a finite dimensional $F$-algebra where $F$ is a field.

Proposition

Let $R$ be a finite dimensional $F$-algebra, for a field $F$. Let $M$ be a finitely generated $R$-module ($\Rightarrow \dim_F M < \infty$).

[If $R = \bigoplus_{i=1}^{n} Fe_i$ and $M$ is generated over $R$ by $r_1, \dots, r_m$, then $\{e_i r_j\}$, $i = 1, \dots, n$, $j = 1, \dots, m$ span $M$ as an $F$-vector space.]

Then the following are equivalent:

(i) $M = \sum_{i \in I} M_i$ for simple submodules

(ii) $M = \bigoplus_{i \in I} M_i$ for simple submodules

(iii) $\forall P \leq M$, $\exists N \leq M$ s.t. $P \oplus N = M$.

Proof

(ii) $\Rightarrow$ (i) is clear.

(i) $\Rightarrow$ (ii) :

Suppose that $M = \sum_{i \in I} M_i$ with $M_i$ simple submodules.

By considering $\dim_F M$, we can assume $I$ is finite.

Choose a maximal subset $K \subset I$ (maximal w.r.t. inclusion) such that $\sum_{k \in K} M_k$ is a direct sum.

Claim: $\bigoplus_{k \in K} M_k = M$.

If not then $\exists i \in I$ such that $M_i \not\subseteq \bigoplus_{k \in K} M_k$.

Consider $M_i \cap (\bigoplus_{k \in K} M_k)$. This is some submodule of $M_i$, and $M_i$ simple, $M_i \not\subseteq \bigoplus_{k \in K} M_k$ so $M_i \cap \bigoplus_{k \in K} M_k = 0$.

$\Rightarrow (\bigoplus_{k \in K} M_k) \oplus M_i$ is a direct sum, which contradicts the maximality of $K$ ($K \subset K \cup \{i\}$)

(ii) $\Rightarrow$ (iii)

Suppose $M = \bigoplus_{i \in I} M_i$ with $M_i$ simple submodules.

Note that $I$ is finite.

Suppose $P \leq M$. Let $J \subseteq I$ be maximal such that $P \cap (\bigoplus_{j \in J} M_j) = 0$.

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<u>claim</u>: $P \oplus (\bigoplus_{j\in J} M_j) = M$

If not, $\exists\, i \in I$ such that $M_i \not\subseteq P \oplus (\bigoplus_{j\in J} M_j)$

and so $M_i \cap P \oplus (\bigoplus_{j\in J} M_j) = 0$.

Consider $p \in P \cap ((\bigoplus_{j\in J} M_j) \oplus M_i)$

$\Rightarrow p = m_i + \sum_{j\in J} m_j$ with $m_i \in M_i$, $m_j \in M_j$ $\forall j \in J$.

$\Rightarrow m_i = p - \sum_{j\in J} m_j \in M_i \cap (P \oplus (\bigoplus_{j\in J} M_j)) = 0$

$\Rightarrow m_i = 0$ and $p = \sum_{j\in J} m_j \in P \cap (\bigoplus_{j\in J} M_j) = 0$.

$\Rightarrow P \cap ((\bigoplus_{j\in J} M_j) \oplus M_i) = 0$

which contradicts the maximality of $J$ ($J \subset J \cup \{i\}$).

(iii) $\Rightarrow$ (ii)

Recall that if a module $M$ satisfies (iii) then so does any submodule and any quotient.

If $M$ is simple then we are done.

Else choose $P \leq M$ such that $P \neq 0$ or $M$. Take $N \leq M$ such that $P \oplus N = M$

$\Rightarrow \dim_F M = \dim_F P + \dim_F N$, $\dim_F P < \dim_F M$, $\dim_F N < \dim_F M$.

By induction on $\dim_F M$, we have $P = \bigoplus_{i\in I} P_i$, $N = \bigoplus_{j\in J} N_j$

where $P_i$ and $N_j$ are simple submodules $\forall i \in I, j \in J$.

$\Rightarrow M = (\bigoplus_{i\in I} P_i) \oplus (\bigoplus_{j\in J} N_j)$ so $M$ satisfies (ii). $\square$

Let $G$ be a finite group and $F$ a field with char $F \nmid |G|$.

By Maschke's theorem $F[G]$ is a semisimple ring.

By the Artin-Wedderburn theorem,

$F[G] \cong M_{n_1}(D_1) \times \dots \times M_{n_r}(D_r)$ as rings, where each

$D_i$ is a finite dimensional division algebra over $F$.

<u>Note</u>: When $F$ is algebraically closed, each $D_i = F$.

We'll prove later that if $F = \mathbb{R}$, then each $D_i$ is either $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.

As an $M_n(D)$-module, $M_n(D) \cong S^n$ where $S = D^n$ and $M_n(D)$ acts on $S$ by matrix multiplication on column vectors.

$\Rightarrow F[G] \cong S_1^{n_1} \oplus \dots \oplus S_r^{n_r}$ as an $F[G]$-module, where each $S_i$ is simple, $S_i \cong D_i^{n_i}$.
$\Rightarrow$ each simple module over $F[G]$ is of the form $S_i$ for some $i$.

We have $\dim_F(S_i) = \dim_F(D_i^{n_i}) = n_i \dim_F(D_i)$.

Now assume $F = \mathbb{C}$ (char $F = 0$ and $F$ algebraically closed).
$\Rightarrow \mathbb{C}[G] \cong M_{n_1}(\mathbb{C}) \times \dots \times M_{n_r}(\mathbb{C})$ as rings (*)
$\cong S_1^{n_1} \times \dots \times S_r^{n_r}$ as $\mathbb{C}[G]$ modules
and $\dim_{\mathbb{C}}(S_i) = n_i$

Take $\dim_{\mathbb{C}}$ of both sides of (*)
$\Rightarrow |G| = n_1^2 + \dots + n_r^2$

We'll now show that $r = \#$ conjugacy classes of $G$.

Definition
The centre of a ring $R$ is $Z(R) = \{r \in R : xr = rx \ \forall x \in R\}$
It is a ring itself.

For $R = \mathbb{C}[G]$ it is clear that $Z(\mathbb{C}[G])$ is a $\mathbb{C}$-algebra, and so is a $\mathbb{C}$-vector space.

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**Lemma**

Suppose $\mathbb{C}[G] \cong M_{n_1}(\mathbb{C}) \times \ldots \times M_{n_r}(\mathbb{C})$, then
$\dim_{\mathbb{C}}(Z(\mathbb{C}[G])) = r$

**Proof**

$Z(\mathbb{C}[G]) = Z(M_{n_1}(\mathbb{C})) \times \ldots \times Z(M_{n_r}(\mathbb{C}))$

But $Z(M_n(\mathbb{C})) = \mathbb{C} \cdot I_n \cong \mathbb{C}$ is the subring of scalar matrices.

$\Rightarrow Z(\mathbb{C}[G]) \cong \mathbb{C} \times \ldots \times \mathbb{C}$ (r times)

$\Rightarrow \dim_{\mathbb{C}} Z(\mathbb{C}[G]) = r$. $\square$

**Theorem**

$\dim_{\mathbb{C}}(Z(\mathbb{C}[G])) = \#$ conjugacy classes.

Remark: This is true for all fields, not just $F = \mathbb{C}$.

**Proof**

Suppose $x = \sum_{g \in G} \lambda_g g \in Z(\mathbb{C}[G])$

Then for all $h \in G$ we have $h^{-1} x h = x$

$\Leftrightarrow \sum_{g \in G} \lambda_g h^{-1} g h = \sum_{g \in G} \lambda_g g \;\; \forall h \in G$

$\Leftrightarrow \lambda_g = \lambda_{hgh^{-1}} \;\; \forall g, h \in G$

Then as a $\mathbb{C}$-vector space $Z(\mathbb{C}[G])$ has a basis given by $\{\sum_{g \in K_i} g\}_{i \in I}$ where $\{K_i\}_{i \in I}$ are the conjugacy classes of $G$.

$\Rightarrow \dim_{\mathbb{C}} Z(\mathbb{C}[G]) = \#$ conjugacy classes of $G$. $\square$

**Finite abelian groups**

Suppose $G$ is a finite abelian group. Then $G \cong C_{d_1} \times \ldots \times C_{d_t}$

$\Rightarrow G \cong \langle \sigma_1 \rangle \times \ldots \times \langle \sigma_t \rangle$.

$\mathbb{C}[G] = M_{n_1}(\mathbb{C}) \times \ldots \times M_{n_r}(\mathbb{C})$

$|G| = n_1^2 + \ldots + n_r^2$ and $r = \#$ conjugacy classes $= |G|$.

$\Rightarrow$ each $n_i = 1$

$\Rightarrow \mathbb{C}[G] \cong \mathbb{C} \times \dots \times \mathbb{C}$ ($r$ copies) as rings

$\cong \mathbb{C} \oplus \dots \oplus \mathbb{C}$ ($r$ copies) as modules.

All simple modules of $\mathbb{C}[G]$ are 1-dimensional; and there are $r$ non-isomorphic simple modules.

$\Rightarrow$ There are exactly $r = |G|$ irreducible representations (up to isomorphism) of $G$ over $\mathbb{C}$, and each are one dimensional.

Explicitly if $G = C_{d_1} \times \dots \times C_{d_t} = \langle \sigma_1 \rangle \times \dots \times \langle \sigma_t \rangle$ then for each $t$-tuple $\underline{m} = (m_1, \dots, m_t) \in \mathbb{Z}^t$ with $0 \le m_i \le d_i - 1$, we have a representation $\rho_{\underline{m}} : G \to \mathbb{C}^\times = GL_1(\mathbb{C})$,

$\sigma_j \mapsto e^{2\pi i m_j / d_j} \quad \forall j = 1, \dots, t.$

In the 1-dimensional case, two representations $\rho_1, \rho_2 : G \to GL_1(\mathbb{C}) = \mathbb{C}^\times$ are isomorphic $\Leftrightarrow$ they are equal.

If $\exists\, T \in GL_1(\mathbb{C}) = \mathbb{C}^\times$ s.t. $T \rho_1(g) T^{-1} = \rho_2(g)$ $\forall g \in G$

$\Rightarrow \rho_1(g) = \rho_2(g)$ $\forall g \in G$ (since $GL_1(\mathbb{C})$ commutative).

There are $d_1 \times \dots \times d_t = |G|$ choices for $\underline{m}$ and each gives a different representation.

<u>Example</u>

$G = D_6 = \langle x, y \mid x^3 = y^2 = 1,\ yx = x^2 y \rangle$

<u>Notation</u>: If $g \in G$, write $g^G = \{h^{-1} g h : h \in G\}$ for the conjugacy class of $g$.

$1^{D_6} = \{1\}$, $x^G = \{x, x^2\}$, $y^G = \{y, xy, x^2 y\}$

$\Rightarrow \mathbb{C}[D_6] \cong M_{n_1}(\mathbb{C}) \times M_2(\mathbb{C}) \times M_3(\mathbb{C})$ and $n_1^2 + n_2^2 + n_3^2 = 6$

$\Rightarrow n_1 = n_2 = 1$ and $n_3 = 2$

$\Rightarrow D_6$ has exactly 3 irreducible representations over $\mathbb{C}$, up to isomorphism, two are 1-dimensional and there is a unique 2-dimensional representation.

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## Recap

$G$ finite group, $\mathbb{C}[G] \cong M_{n_1}(\mathbb{C}) \times .. \times M_{n_r}(\mathbb{C})$ as rings (Maschke + Artin-Wedderburn).
$\mathbb{C}[G] \cong S_1^{\oplus n_1} \oplus .. \oplus S_r^{\oplus n_r}$ as $\mathbb{C}[G]$-modules,
each $S_i$ is a simple $\mathbb{C}[G]$-module and $\dim_{\mathbb{C}} S_i = n_i$.
Under the correspondence

$$\left\{\begin{array}{c}\text{reps } \rho : G \to GL(V) \\ \text{over } \mathbb{C}\end{array}\right\} \longleftrightarrow \{\text{f.g. } \mathbb{C}[G] \text{ mods}\}$$

$$\text{irreducible reps} \longleftrightarrow \text{simple } \mathbb{C}[G]\text{-modules}$$

Recall that if $R$ is a semisimple ring and if $R = R_1 \oplus ... \oplus R_m$ as an $R$-module with each $R_i$ simple, then every simple module over $R$ is isomorphic to $R_i$ for some $i$.
$\Rightarrow \mathbb{C}[G]$ is semisimple and so any simple module is isomorphic to $S_i$ for some $i$.
Alternatively, any irreducible representation of $G$ is isomorphic to $\rho_i : G \to GL(S_i)$.
Any representation $\rho : G \to GL(V)$ is isomorphic to $\rho_1^{\oplus m_1} \oplus ... \oplus \rho_r^{\oplus m_r}$. $\dim_{\mathbb{C}} \rho_i = m_i$

We proved $\sum_{i=1}^{r} n_i^2 = |G|$ and $r = \#$ conjugacy classes of $G$.

## Remarks

1). Since the trivial representation $\rho : G \to GL_1(\mathbb{C}) = \mathbb{C}^\times$, $g \mapsto 1$ is irreducible we may assume $n_1 = 1$.
2). $\mathbb{C}[G]$ is abelian (commutative) $\Leftrightarrow$ $G$ is an abelian group
$M_n(\mathbb{C})$ commutative $\Leftrightarrow n = 1$
So $G$ is abelian $\Leftrightarrow \mathbb{C}[G]$ is commutative
$\Leftrightarrow \mathbb{C}[G] \cong \mathbb{C} \times ... \times \mathbb{C}$ (all $n_i = 1$)
$\Leftrightarrow$ all irreducible representations of $G$ are 1-dimensional
In particular if $G$ is not abelian, there exists an

irreducible representation $\rho : G \to GL(V)$ with $\dim \rho > 1$.

3). If $G, H$ are finite abelian groups, $\mathbb{C}[G] \cong \mathbb{C}[H]$
$\iff |G| = |H|$ (e.g. $\mathbb{C}[C_4] \cong \mathbb{C}[C_2 \times C_2]$)
but this isomorphism is non-canonical.

## New representations from old

### Lifting representations

Suppose $N \trianglelefteq G$ is a normal subgroup and
$\tilde{\rho} : G/N \to GL(V)$ is a representation.
Then we can <u>lift</u> $\tilde{\rho}$ to a representation
$\rho : G \to GL(V)$, $g \mapsto \tilde{\rho}(gN)$
To see that $\rho$ is a group homomorphism, $\rho = \tilde{\rho} \circ \pi$ where
$\pi : G \to G/N$, $g \mapsto gN$, is the quotient map, and the composition of two group homomorphisms is a group hom.
Conversely if $\rho : G \to GL(V)$ is a representation with
$N \leq \operatorname{Ker}\rho$ then $\rho$ is lifted from some $\tilde{\rho} : G/N \to GL(V)$,
$gN \mapsto \rho(g)$ ($\tilde{\rho}$ is well-defined since $N \leq \operatorname{Ker}\rho$).
$[\, g = g'n,\ n \in N \Rightarrow \rho(g) = \rho(g')\rho(n) \Rightarrow \rho(g) = \rho(g')$ since $N \leq \operatorname{Ker}(\rho)\,]$

In particular we have a one-to-one correspondence

$$\left\{ \begin{array}{l} \text{representations } \rho : G \to GL(V) \\ \text{with } \operatorname{Ker}\rho \geq N \end{array} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{l} \text{representations} \\ \tilde{\rho} : G/N \to GL(V) \end{array} \right\}$$

$$\left\{ \begin{array}{l} \text{irreducible reps } \rho : G \to GL(V) \\ \text{with } \operatorname{Ker}\rho \geq N \end{array} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{l} \text{irreducible reps.} \\ \tilde{\rho} : G/N \to GL(V) \end{array} \right\}$$

If $\rho : G \to GL(V)$ is a representation of $G$ with $\operatorname{Ker}\rho \overset{?}{\leq} N$
we say that $\rho$ <u>factors through</u> $G/N$.

$$\begin{tikzcd} G \arrow[rr, "\rho"] \arrow[dr, "\pi"'] & \arrow[d, phantom, "\circlearrowright"] & GL(V) \\ & G/N \arrow[ur, "\tilde{\rho}"'] & \end{tikzcd}$$

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## 1-dimensional representations of G

**Def**

G finite group. We define the <u>derived subgroup</u> of G as $[G,G] = \langle g^{-1}h^{-1}gh : g,h \in G \rangle$.

$[G,G]$ is also called the commutator subgroup.

$[G,G]$ is a normal subgroup.

Note that $N \trianglelefteq G$ is a normal subgroup of G, then

$$G/N \text{ is abelian} \iff (g^{-1}N)(h^{-1}N)(gN)(hN) = N \text{ in } G/N \quad \forall g,h \in G$$
$$\iff g^{-1}h^{-1}gh\, N = N \quad \forall g,h \in G$$
$$\iff g^{-1}h^{-1}gh \in N \quad \forall g,h \in G$$
$$\iff [G,G] \leq N$$

So in particular, $[G,G]$ is the smallest (w.r.t. inclusion) normal subgroup of G s.t. $G/N$ is abelian.

Also $G/[G,G]$ is an abelian group.

We call $G^{ab} := G/[G,G]$ the <u>abelianisation</u> of G.

Suppose $\rho : G \to GL_1(\mathbb{C}) = \mathbb{C}^\times$ is a one-dimensional representation. Then $G/\ker\rho \cong \operatorname{Im}\rho$ and since $\operatorname{Im}\rho \leq \mathbb{C}^\times$ is an abelian group, we have $G/\ker\rho$ is abelian $\Rightarrow [G,G] \leq \ker\rho$.

This shows we have a one-to-one correspondence

$$\{\text{1-dim reps of } G\} \xleftrightarrow{1:1} \{\text{irred. reps of } G^{ab}\}$$

In particular the number of 1-dimensional representations of G is $|G^{ab}|$.

## Restriction

$G$ is a finite group. $H \leq G$ a subgroup, then if $\rho: G \to GL(V)$ is a representation, we define

$\mathrm{Res}_H^G \rho : H \to GL(V)$, $h \mapsto \rho(h)$.

We write the corresponding $\mathbb{C}[H]$-module as $\mathrm{Res}_H^G V$.

More generally if $f: H \to G$ is a group homomorphism, then if $\rho: G \to GL(V)$ is a representation, then $\rho \circ f : H \to GL(V)$ is a representation.

Other examples of $f$ could be $f: G \to G \in \mathrm{Aut}(G)$.

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**Correction:**

$N \triangleleft G$

$$\left\{ \begin{array}{c} \text{representations} \\ \rho: G \to GL(V) \\ \text{with } N \leq \ker\rho \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{representations} \\ \tilde{\rho}: G/N \to GL(V) \end{array} \right\}$$

**Last time:**

Lifting and restricting representations.

$$\left\{ \begin{array}{c} \text{1-dim reps} \\ \rho: G \to \mathbb{C}^\times \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{irreducible reps} \\ \rho: G^{ab} \to \mathbb{C}^\times \end{array} \right\}$$

where $G^{ab} = G/[G,G]$, $[G,G] = \langle [g,h] : g,h \in G \rangle$, $[g,h] = g^{-1}h^{-1}gh$

**Remarks** (to help calculate $[G,G]$)

1). $[G,G]$ is the smallest normal subgroup s.t. $G/[G,G]$ is abelian, ie if $N \triangleleft G$ is normal with $G/N$ abelian, then $[G,G] \leq N$

2). If $g_1, \dots, g_t$ generate $G$ and if $N \triangleleft G$ is normal in $G$, then $g_1N, \dots, g_tN$ generate $G/N$. So $G/N$ is abelian $\Leftrightarrow [g_i, g_j] \in N \; \forall i,j$

Hence $[G,G]$ is the smallest normal subgroup containing $[g_i, g_j] \; \forall i,j$.

3). A subgroup $H \leq G$ is normal $\Leftrightarrow$ $H$ is a union of conjugacy classes

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<u>Example</u>

Let $n \geq 1$ be even.

$D_{2n} := \langle x, y : x^n = y^2 = 1,\ yx = x^{-1}y \rangle = \{x^i y^j : 0 \leq i \leq n-1,\ 0 \leq j \leq 1\}$

<u>Conjugacy classes</u>

$$x^{-1}yx = x^{-2}y = x^{n-2}y$$

$$x^{-1}(x^j y)x = x^{-2}x^j y = x^{j-2}y$$

$$y^{-1}xy = x^{-1}y^2 = x^{-1} = x^{n-1}$$

$$y^{-1}x^j y = (y^{-1}xy)^j = x^{-j} = x^{n-j}$$

$$y^{-1}(x^j y)y = x^{-j}y = x^{n-j}y$$

The conjugacy classes are

$\{1\}, \{x, x^{n-1}\}, \{x^2, x^{n-2}\}, \ldots, \{x^{n/2}\}$ (n even)

$\{y, x^{n-2}y, x^{n-4}y, \ldots, x^2y\}, \{xy, x^{n-1}y, \ldots, x^3y\}$

So $D_{2n}$ has $n/2 + 3 = \frac{n+6}{2}$ conjugacy classes.

$x, y$ generate $D_{2n}$.

$[x,y] = x^{-1}y^{-1}xy = yx^2y = x^{-2} = x^{n-2}$ $(y = y^{-1})$

Let $N = \langle x^{-2} \rangle = \{1, x^2, \ldots, x^{n-2}\}$.

Then $N$ is normal (it is a union of conjugacy classes) and since $N = \langle [x,y] \rangle$ this is the smallest normal subgroup containing $[x,y] \Rightarrow N = [G,G]$.

$|N| = n/2$ so $|G^{ab}| = |G|/|N| = 4$

$\Rightarrow$ If $n$ is even, $D_{2n}$ has exactly 4 1-dimensional representations.

$$G^{ab} \cong C_2 \times C_2 = \langle xN \rangle \times \langle yN \rangle$$

Consider $\mathbb{C}[D_{2n}] \cong M_{m_1}(\mathbb{C}) \times \ldots \times M_{m_r}(\mathbb{C})$, $m_1 \leq \ldots \leq m_r$

We have $r = \#$ conjugacy classes $= (n+6)/2$

$m_1 = \ldots = m_4 = 1$, $m_5, \ldots, m_r \geq 2$

We also have $|D_{2n}| = 2n = m_1^2 + \ldots + m_r^2 = 4 + \sum_{i=5}^{r} m_i^2$

$\geq 4 + 2^2(r-4)$ with equality $\Leftrightarrow m_5 = \ldots = m_r = 2$

$= 4 + 4\left(\frac{n+6}{2} + 4\right) = 4 + 4\left(\frac{n-2}{2}\right) = 4 + 2n - 4 = 2n$

$\Rightarrow m_5 = \ldots = m_r = 2$.

In conclusion, $D_{2n}$ (even) has 4 1-dim. reps., $\frac{n-2}{2}$ irred. 2-dim. reps. up to isomorphism, and no irred. reps. of dim > 2.

## The dual representation

**Definition**

Suppose that $V$ is a vector space over a field $F$, then its dual space is defined by $V^* := \mathrm{Hom}_F(V, F) = \{\phi : V \to F,\ F\text{-linear maps}\}$
$(\lambda\phi)(v) = \lambda\phi(v)$, $\forall \lambda \in F$, $\phi \in V^*$, $v \in V$, $(\phi + \psi)(v) = \phi(v) + \psi(v)$
$\forall \phi, \psi \in V^*$, $v \in V$.

If $V$ is a fdvs/$F$ with basis $e_1, \dots, e_n$ then $V^* \cong V$ (but not canonically), $e_i^* \longleftrightarrow e_i$, where $e_i^* : V \longrightarrow F$, $e_j \longmapsto \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & \text{else.} \end{cases}$
We call $e_1^*, \dots, e_n^*$ the dual basis of $V^*$ w.r.t. $e_1, \dots, e_n$

If $\rho : G \to GL(V)$ is a representation then the dual representation is $\rho^* : G \longrightarrow GL(V^*)$ defined by $\rho^*(g) : V^* \to V^*$, $\phi \mapsto \rho^*(g)\phi$ where $(\rho^*(g)\phi)(v) = \phi(\rho(g^{-1})v)$ $\forall g \in G$, $\phi \in V^*$, $v \in V$.

**Exercise**

Check $\rho^*$ is a representation, i.e. $\rho^*(1) = \mathrm{Id}_{V^*}$, $\rho^*(gh) = \rho^*(g)\rho^*(h)$.

## Tensor products

$F$ field, $V, W$ fdvs/$F$.
$\dim V = m$, $\dim W = n$. Choose a basis $\{v_1, \dots, v_m\}$ of $V$ and a basis $\{w_1, \dots, w_n\}$ of $W$.
The tensor product $V \otimes_F W = V \otimes W$ is an $mn$-dimensional vector space over $F$ with basis $\{v_i \otimes w_j : 1 \le i \le m, 1 \le j \le n\}$.
Addition: $\sum \lambda_{ij} v_i \otimes w_j + \sum \mu_{ij} v_i \otimes w_j = \sum (\lambda_{ij} + \mu_{ij}) v_i \otimes w_j$
multiplication: $\lambda(\sum \mu_{ij} v_i \otimes w_j) = \sum \lambda\mu_{ij} v_i \otimes w_j$
(free vector space on $v_i \otimes w_j$).

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If $v \in V$, $v = \sum_{i=1}^{m} a_i v_i$ and $w \in W$, $w = \sum_{j=1}^{n} b_j w_j$
then we define
$$v \otimes w := \sum_{i,j} a_i b_j (v_i \otimes w_j) \in V \otimes W.$$

This agrees with the notation $v_i \otimes w_j$.
We call elements of the form $v \otimes w \in V \otimes W$ <u>elementary tensors</u>.
Not all elements of $V \otimes W$ are elementary tensors if $m, n \geq 2$
e.g. $v_1 \otimes w_2 + v_2 \otimes w_2$ is not an elementary tensor.

<u>Universal property of $V \otimes W$</u>
We define the bilinear map
$\varphi : V \times W \longrightarrow V \otimes W$, $(v, w) \longmapsto v \otimes w$.
Suppose that $b : V \times W \to Z$ is a bilinear map, then there exists a unique linear map $\beta : V \otimes W \to Z$ such that $\beta \circ \varphi = b$, i.e. s.t. the following diagram commutes

$$\begin{tikzcd} V \times W \arrow[r, "\varphi"] \arrow[dr, "b"'] & V \otimes W \arrow[d, dashed, "\exists ! \beta"] \arrow[dl, phantom, "\circlearrowleft"] \\ & Z \end{tikzcd} \qquad \begin{matrix} v \otimes w \\ \downarrow \\ b(v,w) \end{matrix}$$

$\beta$ exists and is unique because the elementary tensors span $V \otimes W$.
The existence follows from defining $\beta$ on the basis $v_i \otimes v_j$ so $\beta\left(\sum_{i,j} \lambda_{ij} v_i \otimes w_j\right) = \sum_{i,j} \lambda_{ij} b(v_i, w_j)$.
If $v = \sum_i a_i v_i$, $w = \sum_j b_j w_j$ then
$\beta(v \otimes w) = \beta\left(\sum_{i,j} a_i b_j v_i \otimes w_j\right) = \sum_{i,j} a_i b_j b(v_i, w_j)$
But $b(v, w) = b\left(\sum a_i v_i, \sum b_j w_j\right) = \sum_{i,j} a_i b_j b(v_i, w_j)$ from the bilinearity of $b$.
So $\beta(v \otimes w) = b(v, w)$.

Suppose $X$ is another vector space with a bilinear map $\psi : V \times W \to X$ satisfying the universal property, then consider the diagram

$$\begin{tikzcd} V \times W \arrow[r, "\varphi"] \arrow[dr, "\psi"'] & V \otimes W \arrow[d, dashed, shift left, "\exists ! \beta"] \\ & X \arrow[u, dashed, shift left, "\exists ! \alpha"] \end{tikzcd}$$

Then $\alpha\circ\beta$ fits in the diagram

$$\begin{tikzcd} V\times W \arrow[r, "\varphi"] \arrow[dr, "\varphi"'] & V\otimes W \arrow[d] \\ & V\otimes W \end{tikzcd}$$

By the uniqueness property, $\alpha\circ\beta = \mathrm{Id}_{V\otimes W}$.
Similarly $\beta\circ\alpha = \mathrm{Id}_X$.
So $\alpha, \beta$ are isomorphism

In particular, the construction of $V\otimes W$ does not depend on a choice of basis.

$$\left\{\begin{array}{c}\text{linear maps} \\ \beta: V\otimes W \to Z\end{array}\right\} \xleftrightarrow{1:1} \left\{\begin{array}{c}\text{bilinear maps} \\ b: V\times W \to Z\end{array}\right\}$$

$$\beta \longmapsto b \qquad b(v,w) = \beta(v\otimes w)$$

$$\beta(v\otimes w) = b(v,w). \quad \beta \longleftarrow b$$

<u>Definition</u>
If $\rho: G\to GL(V)$ and $\rho': G\to GL(W)$ are representations, we define $\rho\otimes\rho': G\to GL(V\otimes W)$ by $[\rho\otimes\rho'(g) = \rho(g)\otimes\rho'(g)]$
$\rho\otimes\rho'(g): V\otimes W \to V\otimes W, \quad v\otimes w \mapsto \rho(g)v\otimes\rho'(g)w$
$\forall g\in G, v\in V, w\in W$.

Note that $V\times W \to V\otimes W, \quad (v,w)\mapsto \rho(g)v\otimes\rho'(g)w$
is bilinear, so $\rho(g)\otimes\rho'(g)$ is a linear map $V\otimes W\to V\otimes W$ by the universal property.
We have $\rho(1)\otimes\rho'(1)(v\otimes w) = \rho(1)v\otimes\rho'(1)w = v\otimes w \quad \forall v\in V, w\in W$.
So $\rho(1)\otimes\rho'(1) = \mathrm{Id}_{V\otimes W}$.
Also $\rho\otimes\rho'(g)(\rho\otimes\rho'(h)(v\otimes w)) = \rho\otimes\rho'(g)(\rho(h)v\otimes\rho'(h)w)$
$= \rho(g)\rho(h)v\otimes\rho'(g)\rho'(h)w$
$= \rho(gh)v\otimes\rho'(gh)w = (\rho\otimes\rho')(gh)(v\otimes w)$
Taking $h = g^{-1}$ we see that $\rho\otimes\rho'(g)\in GL(V\otimes W)$ and $\rho\otimes\rho'$ is a representation.

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## In terms of matrix representations

Pick a basis $\{v_1, \dots, v_m\}$ of $V$ and a basis $\{w_1, \dots, w_n\}$ of $W$.

Then $\{v_1 \otimes w_1, v_1 \otimes w_2, \dots, v_1 \otimes w_n, v_2 \otimes w_1, \dots, v_m \otimes w_n\}$ is a basis of $V \otimes W$.

Suppose $\rho(g)$ has matrix $A = (a_{ij}) \in GL_m(\mathbb{C})$, and $\rho'(g)$ has matrix $B = (b_{ij}) \in GL_n(\mathbb{C})$.

Then
$$\begin{aligned}
\rho \otimes \rho'(g)(v_j \otimes w_l) &= \rho(g) v_j \otimes \rho'(g) w_l \\
&= \left( \sum_{i=1}^{m} a_{ij} v_i \right) \otimes \left( \sum_{k=1}^{n} b_{kl} w_k \right) \\
&= \sum_{i,k} a_{ij} b_{kl} \, v_i \otimes w_k
\end{aligned}$$

Label the rows and columns of $\rho \otimes \rho'(g) \in GL_{mn}(\mathbb{C})$ by pairs $(i,k)$, $1 \le i \le m$, $1 \le k \le n$, corresponding to $v_i \otimes w_k$.

Let $C = \rho \otimes \rho'(g) \in GL_{mn}(\mathbb{C})$.

$C = (c_{(i,k)(j,l)})$ with $c_{(i,k)(j,l)} = a_{ij} b_{kl}$

$$\Rightarrow C = \begin{pmatrix} a_{11} B & a_{12} B & \cdots & a_{1m} B \\ a_{21} B & & & \vdots \\ \vdots & & & \vdots \\ a_{m1} B & \cdots & & a_{mm} B \end{pmatrix}$$

## Example

$\rho : D_8 \to GL_2(\mathbb{C})$, $x \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $y \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$,

$\varepsilon : D_8 \to \mathbb{C}^\times$, $x \mapsto 1$, $y \mapsto -1$, then

$$\rho \otimes \varepsilon(x) = \begin{pmatrix} 0 \cdot 1 & -1 \cdot 1 \\ 1 \cdot 1 & 0 \cdot 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\rho \otimes \varepsilon(y) = \begin{pmatrix} -1 \cdot (-1) & 0 \cdot (-1) \\ 0 \cdot (-1) & 1 \cdot (-1) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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Last time:

- How to calculate $[G,G]$ and therefore how to calculate $G^{ab}$ and the 1-dimensional representations of $G$
- Tensor product of representations

Today

- Induction of representations
- If time permits, $Hom(V,W)$.

Induced representations

$G$ a finite group, $H \leq G$ a subgroup.
Suppose that $\rho: H \to GL(V)$ is a representation.
We construct a representation $Ind_H^G \rho$ of $G$ in the following way:
Choose a set of left coset representatives of $H$, i.e. elements $t_1, \ldots, t_n \in G$ s.t. $t_1 H \sqcup \ldots \sqcup t_n H = G$ (where $n = [G:H]$).
The underlying vector space for $Ind_H^G \rho$ is

$$Ind_H^G V := \bigoplus_{i=1}^{n} t_i \otimes V.$$

Here $t_i \otimes V$ is the vector space $V$, but write $v \in V$ as $t_i \otimes v$.
Element of $Ind_H^G (V)$ are of the form $\sum_{i=1}^{n} t_i \otimes v_i$ for $v_i \in V$.

We define $Ind_H^G \rho : G \to GL(Ind_H^G(V))$ so that
$Ind_H^G \rho(g)(t_i \otimes v) = t_j \otimes \rho(t_j^{-1} g t_i) v$ where $g t_i H = t_j H$ for some $j \in \{1, \ldots, n\}$.

Remark

Recall that $G$ acts on the left cosets $G/H$, so if $g, h \in G$ and $h t_i H = t_j H$ and $g t_j H = t_k H$, then $(gh) t_i H = t_k H$.

Note that since $g t_i H = t_j H$, we have that $t_j^{-1} g t_i \in H$ and so $\rho(t_j^{-1} g t_i)$ makes sense.

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It is clear that $\mathrm{Ind}_H^G \rho(e) = \mathrm{Id}_{\mathrm{Ind}_H^G(V)}$ where $e \in G$ is the group identity.

Let's check $\mathrm{Ind}_H^G \rho(gh) = \mathrm{Ind}_H^G \rho(g) \circ \mathrm{Ind}_H^G \rho(h)$
We check that $\mathrm{Ind}_H^G V$ is an $F[G]$-module:

$$
\begin{aligned}
g(h(t_i \otimes v)) &= g(t_j \otimes t_j^{-1} h t_i v) \quad \text{where } t_j H = h t_i H \\
&= t_k \otimes (t_k^{-1} g t_j t_j^{-1} h t_i v) \quad \text{where } t_k H = g t_j H \\
&= t_k \otimes (t_k^{-1} g h t_i v) \\
&= (gh) t_i \otimes v \quad \text{since } g h t_i H = t_k H
\end{aligned}
$$

This is true for all $i \in \{1, \dots, n\}$ and $v \in V$, so
$\mathrm{Ind}_H^G \rho(gh) = \mathrm{Ind}_H^G \rho(g) \cdot \mathrm{Ind}_H^G \rho(h)$
since elements of the form $t_i \otimes v$ span $\mathrm{Ind}_H^G V$

Taking $h = g^{-1}$ we see that $\mathrm{Ind}_H^G \rho(g) \in GL(\mathrm{Ind}_H^G V)$ and so $\mathrm{Ind}_H^G \rho$ is a representation.

<u>Example</u>

$G = D_6 = \langle x, y \mid x^3 = y^2 = 1,\ yx = x^2 y \rangle$
$H = \langle x \rangle \cong C_3$
Let $\rho: H \to \mathbb{C}^\times$, $x \mapsto \omega = \exp(2\pi i/3)$
Calculate $\mathrm{Ind}_H^G \rho : D_6 \to GL_2(\mathbb{C})$.

$G = e.H \sqcup yH$ so take $t_1 = e$, $t_2 = y$.
$\mathrm{Ind}_H^G \rho$ acts on $(e \otimes \mathbb{C}) \oplus (y \otimes \mathbb{C})$ which has basis $e \otimes 1$ and $y \otimes 1$.
What is $\mathrm{Ind}_H^G \rho(x)$?
We have $x \cdot eH = eH$ and $xyH = yH$

$$
\begin{aligned}
x(e \otimes 1) &= e \otimes (e^{-1} x e) \cdot 1 \\
&= e \otimes \rho(x) \cdot 1 = e \otimes \omega = \omega(e \otimes 1)
\end{aligned}
$$

$$
\begin{aligned}
x(y \otimes 1) &= y \otimes (y^{-1} x y) \cdot 1 = y \otimes \rho(y^{-1} x y) \cdot 1 \qquad y^{-1} x y = x^2 \\
&= y \otimes \rho(x^2) \cdot 1 = y \otimes \omega^2 = \omega^2 (y \otimes 1).
\end{aligned}
$$

$\Rightarrow$ matrix for $\mathrm{Ind}_H^G \rho(x)$ is $\begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$

What is $\mathrm{Ind}_H^G \rho(y)$?

$y \cdot eH = yH \quad , \quad y \cdot yH = e \cdot H$

$y(e \otimes 1) = y \otimes (y^{-1} y e) \cdot 1$
$\quad = y \otimes \rho(e) \cdot 1 = y \otimes 1$

$y(y \otimes 1) = e \otimes (e^{-1} y y) \cdot 1$
$\quad = e \otimes \rho(e) \cdot 1 = e \otimes 1$

$\Rightarrow$ matrix for $\mathrm{Ind}_H^G \rho(y)$ is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

In terms of matrices, suppose that $\rho : H \to GL_d(\mathbb{C}) = GL(\mathbb{C}^d)$ is a matrix representation.
Let $t_1, \ldots, t_n$ be left coset representatives of $H$ in $G$.
If $e_1, \ldots, e_d$ denotes the standard basis for $\mathbb{C}^d$ then $\mathrm{Ind}_H^G \mathbb{C}^d$ has a $\mathbb{C}$-basis $t_1 \otimes e_1, t_1 \otimes e_2, \ldots, t_1 \otimes e_d, t_2 \otimes e_1, \ldots, t_n \otimes e_d$.

Recall that $\mathrm{Ind}_H^G \rho(g)(t_j \otimes e_k) = t_i \otimes \rho(t_i^{-1} g t_j) e_k$ where $g t_j H = t_i H$.
$\Rightarrow$ w.r.t. the basis $t_1 \otimes e_1, \ldots, t_n \otimes e_d$ the matrix of $\mathrm{Ind}_H^G \rho(g)$ is of the form $\begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix}$

where $A_{ij} \in M_d(\mathbb{C})$ and $A_{ij} = \begin{cases} \rho(t_i^{-1} g t_j) & , \text{ if } g t_j H = t_i H \\ 0 & , \text{ otherwise.} \end{cases}$

<u>$\mathrm{Hom}_F(V, W)$</u>

Given two fdvs/$F$, $V$ and $W$, we define the vector space $\mathrm{Hom}_F(V, W) = \{\phi : V \to W, \ F \text{ linear maps}\}$
addition: $(\phi + \psi)(v) = \phi(v) + \psi(v)$, scalar mult: $(\lambda \phi)(v) = \lambda \phi(v)$, $\forall \lambda \in F$, $\phi, \psi \in \mathrm{Hom}_F(V, W)$, $v \in V$.
If $\rho : G \to GL(V)$ and $\rho' : G \to GL(W)$ are representations, we define $\mathrm{Hom}(\rho, \rho') := G \to GL(\mathrm{Hom}_F(V, W))$ by $g \mapsto (\phi \mapsto \rho'(g) \circ \phi \circ \rho(g)^{-1})$
$\forall \phi \in \mathrm{Hom}(V, W)$ $\qquad \rho'(g) \circ \phi \circ \rho(g)^{-1} = \rho'(g) \circ \phi \circ \rho(g^{-1})$

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07-02-18 Next time we'll see that $\mathrm{Hom}(\rho,\rho')$ is a representation.

Note that when $W=F$ and $\rho': G \to GL(F) = F^{\times}$ is the trivial representation, then $\mathrm{Hom}_F(V,F) \cong V^*$.

—

09-02-18 $V, W$ finitely generated $F[G]$-modules.
$\mathrm{Hom}_F(V,W) = \{\phi : V \to W,\ F \text{ linear maps}\}$

If $\rho: G \to GL(V)$ and $\rho': G \to GL(W)$ are the corresponding representations we defined $\rho'' := \mathrm{Hom}(\rho,\rho') : G \to GL(\mathrm{Hom}_F(V,W))$, by $\rho''(g)\phi = \rho'(g)\circ\phi\circ\rho(g)^{-1} = \rho'(g)\circ\phi\circ\rho(g^{-1})$.
$\rho'(g)\circ\phi\circ\rho(g^{-1})$ is a composition of linear maps.
Also $\rho'(g)\circ(\phi+\psi)\circ\rho(g^{-1}) = \rho'(g)\phi\rho(g^{-1}) + \rho'(g)\psi\rho(g^{-1})$
and $\rho'(g)(\lambda\phi)\rho(g^{-1}) = \lambda(\rho'(g)\phi\rho(g^{-1}))$ for $\lambda \in F$.
Hence $\rho''(g)$ is a linear map on $\mathrm{Hom}_F(V,W)$.

Note that $\rho''(e)\phi = \rho'(e)\circ\phi\circ\rho(e) = \mathrm{Id}_W\circ\phi\circ\mathrm{Id}_V = \phi$
$\Rightarrow \rho''(e) = \mathrm{Id}_{\mathrm{Hom}_F(V,W)}$.
Also
$$\begin{aligned}\rho''(g)(\rho''(h)\phi) &= \rho'(g)(\rho''(h)\phi)\rho(g^{-1}) \\ &= \rho'(g)(\rho'(h)\phi\rho(h^{-1}))\rho(g^{-1}) = \rho'(g)\rho'(h)\phi\rho(h^{-1})\rho(g^{-1}) \\ &= \rho'(gh)\phi\rho(h^{-1}g^{-1}) \\ &= \rho'(gh)\phi\rho((gh)^{-1}) = \rho''(gh)\phi\end{aligned}$$
Taking $h = g^{-1}$ shows that $\rho''(g)\rho''(g^{-1}) = \rho''(e) = \mathrm{Id}$
so $\rho''(g)$ is invertible.
This proves that $\mathrm{Hom}(\rho,\rho')$ is a representation

<u>Remark</u>
If $W=F$ with the trivial action, then $\rho''$ is the dual representation $\rho^*: G \to GL(V^*)$.

Lemma

We have an isomorphism of $F[G]$-modules

$$f: V^* \otimes W \xrightarrow{\simeq} \mathrm{Hom}_F(V,W)$$
$$\phi \otimes \omega \longmapsto \phi\omega \qquad \text{where } \phi\omega : V \to W \text{ is the map } v \mapsto \underset{\in F}{\phi(v)} \cdot \omega$$

Proof

Note that $V^* \times W \to \mathrm{Hom}_F(V,W)$, $(\phi,\omega) \mapsto \phi\omega$ is a bilinear map, so $f$ is well defined.

Pick a basis $v_1, \ldots, v_m$ of $V$ over $F$ and a basis $w_1, \ldots, w_n$ of $W$.

Then $v_1^*, \ldots, v_m^*$ is a basis of $V^*$ $\left(v_i^*(v_j) = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & \text{else} \end{cases}\right)$

Then $\{v_j^* \otimes w_i\}$ is a basis for $V^* \otimes W$.

If $\alpha : V \to W$ ($\alpha \in \mathrm{Hom}_F(V,W)$) then there exist $a_{ij} \in F$, $1 \le i \le n$, $1 \le j \le m$ s.t. $\alpha(v_j) = \sum_{i=1}^{n} a_{ij} w_i$.

$\left(\mathrm{Hom}_F(F^m, F^n) = \mathrm{Mat}_{n \times m}(F)\right)$

Let $\beta = \sum_{i,j} a_{ij} v_j^* \otimes w_i$

Then $f(\beta) = \alpha$, since $f(\beta)(v_k) = \sum_{i,j} a_{ij} v_j^* w_i (v_k) = \sum_{i=1}^{n} a_{ik} w_i = \alpha(v_k)$ (since $v_j^*(v_k) = \delta_{jk}$).

$f(\beta)$ and $\alpha$ agree on a basis of $V$ so $f(\beta) = \alpha$.

Suppose $f\left(\sum a_{ij} v_j^* \otimes w_i\right) = 0$

Then $f\left(\sum a_{ij} v_j^* \otimes w_i\right)(v_k) = 0 \ \forall k$

$\Rightarrow \sum_{i=1}^{n} a_{ik} w_i = 0 \ \forall k \ \Rightarrow a_{ik} = 0 \ \forall i,k$.

Hence $f$ is injective.

So we have shown $f : V^* \otimes W \to \mathrm{Hom}_F(V,W)$ is an isomorphism of $F$ vector spaces.

To see this is an isomorphism of $F[G]$-modules:

Let $g \in G$, then

$$f(g \cdot (\phi \otimes \omega)) = f(\phi \circ \rho(g^{-1}) \otimes \rho'(g)\omega) \qquad v \mapsto \underset{\in F}{\phi(g^{-1}v)} \overbrace{\rho'(g)\omega}^{\in W} = \rho'(g)\phi(g^{-1}v)\omega$$

since $\rho'(g)$ is $F$-linear & $\phi(g^{-1}v) \in F$

$$= (\phi \circ \rho(g^{-1}))\rho'(g)\omega$$
$$= \rho'(g) \circ \phi\omega \circ \rho(g^{-1})$$
$$= g \circ (\phi\omega)$$

Hence $f$ is an $F[G]$-module homomorphism. $\square$

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Let $R$ be a ring, $M, N$ $R$-modules.
Then $\mathrm{Hom}_R(M,N) = \{\phi : M \to N,\ R\text{-module homomorphisms}\}$
It's an abelian group, $(\phi + \psi)(m) = \phi(m) + \psi(m)$.
$\mathrm{Hom}_R(M,M) = \mathrm{End}_R(M)$ is a ring.

If $F$ is a field, $G$ a finite group, $M, N$ $F[G]$-modules,
then $\mathrm{Hom}_{F[G]}(M,N)$ is an $F$-vector space, $(\lambda\phi)(m) = \lambda(\phi(m))$.

Suppose $M, N$ are finitely generated, $\dim_F M = m$, $\dim_F N = n$
Then $\mathrm{Hom}_F(M,N) \cong \mathrm{Hom}(F^m, F^n) = \mathrm{Mat}_{n\times m}(F)$
(after choosing a basis for $M, N$).

$\mathrm{Hom}_{F[G]}(M,N) \leq \mathrm{Hom}_F(M,N) \cong \mathrm{Mat}_{n\times m}(F)$
Let $\rho : G \to GL(M) \cong GL_m(F)$ and $\rho' : G \to GL(N) \cong GL_n(F)$
be the matrix representations associated to $M, N$.
Suppose $A \in \mathrm{Mat}_{n\times m}(F) \cong \mathrm{Hom}_F(M,N)$.
Then $A \in \mathrm{Hom}_{F[G]}(M,N) \iff \forall\, \alpha = \sum_{g\in G} a_g g \in F[G]$

$$A\cdot\Big(\sum_{g\in G} a_g \rho(g)\Big) = \Big(\sum_{g\in G} a_g \rho'(g)\Big)\cdot A$$

$$\iff A\rho(g) = \rho'(g)A \quad \forall g \in G$$

$$\iff A\rho(g) = \rho'(g)A \quad \text{for generators } g \in G$$

In particular when $N = M$, $A \in \mathrm{Hom}_F(M,M) = \mathrm{End}_F(M)$,
$A \in \mathrm{Hom}_{F[G]}(M,M) = \mathrm{End}_{F[G]}(M)$
$\iff \rho(g)A = A\rho(g)$ for generators $g \in G$.

If $M \cong F^m$, then $\mathrm{End}_{F[G]}(M) = \{A \in M_m(F) : A\rho(g) = \rho(g)A\ \forall g \in G\}$

We have isomorphisms

$$\mathrm{Hom}_R(N \oplus M, P) \cong \mathrm{Hom}_R(N,P) \oplus \mathrm{Hom}_R(M,P)$$
$$\phi \longmapsto (\phi|_N, \phi|_M)$$
$$\mathrm{Hom}_R(N, M \oplus P) \cong \mathrm{Hom}_R(N,M) \oplus \mathrm{Hom}_R(N,P)$$
$$\phi \longmapsto (\pi_M \circ \phi, \pi_P \circ \phi)$$

Prop

Suppose $U, V$ are $\mathbb{C}[G]$ modules for some finite group $G$. Suppose $S_1, \dots, S_r$ are the simple $\mathbb{C}[G]$-modules up to isomorphism ($r = \#$ conjugacy classes) and suppose $U \cong S_1^{a_1} \oplus \dots \oplus S_r^{a_r}$ $a_i \geq 0$, $V \cong S_1^{b_1} \oplus \dots \oplus S_r^{b_r}$ $b_i \geq 0$.

Then $\mathrm{Hom}_{\mathbb{C}[G]}(U, V) \cong \mathbb{C}^{\sum_{i=1}^{r} a_i b_i}$.

Proof

$$\mathrm{Hom}_{\mathbb{C}[G]}(U,V) = \mathrm{Hom}_{\mathbb{C}[G]}\left(\bigoplus_{i=1}^{r} S_i^{a_i}, \bigoplus_{i=1}^{r} S_i^{a_i}\right)$$
$$= \bigoplus_{i=1}^{r}\bigoplus_{j=1}^{r} \mathrm{Hom}_{\mathbb{C}[G]}(S_i, S_j)^{a_i b_j}$$

$$= \bigoplus_{i=1}^{r} \mathrm{Hom}_{\mathbb{C}[G]}(S_i, S_i)^{a_i b_i} \quad \text{(by Schur's Lemma)}$$
$$= \bigoplus_{i=1}^{r} \mathrm{End}_{\mathbb{C}[G]}(S_i)^{a_i b_i}$$

$$= \bigoplus_{i=1}^{r} \mathbb{C}^{a_i b_i} \quad (\mathbb{C} \text{ algebraically closed})$$
$$= \mathbb{C}^{\sum_{i=1}^{r} a_i b_i}$$

□

Corollary (Schur's Lemma 3)

Let $V$ be a finitely generated $\mathbb{C}[G]$-module.

Then $\mathrm{End}_{\mathbb{C}[G]}(V) = \mathbb{C} \iff V$ is simple.

Proof

$$\mathrm{End}_{\mathbb{C}[G]}(V) = \mathrm{Hom}_{\mathbb{C}[G]}(V,V) \cong \mathbb{C}^{\sum_{i=1}^{r} a_i^2} \quad \text{(in the notation above)}$$
$$= \mathbb{C}$$

$$\iff \sum_{i=1}^{r} a_i^2 = 1 \iff \exists\, 1 \leq j \leq r \text{ s.t. } a_i = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & \text{else} \end{cases}$$
$$\iff \exists\, 1 \leq j \leq r \text{ s.t. } V \cong S_j$$
$$\iff V \text{ is simple.}$$

□

$\mathrm{Hom}_{\mathbb{C}[G]}(U, V) \cong \mathbb{C}^{\sum_{i=1}^{r} a_i b_i}$

Character theory will give us a quick way of computing $\dim_{\mathbb{C}}(\mathrm{Hom}_{\mathbb{C}[G]}(U,V)) = \sum_{i=1}^{r} a_i b_i$.

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## Character Theory

$G$ a finite group, $F$ a field.

**Definition**

Let $A = (a_{ij})_{ij} \in M_n(F)$. Define the trace of $A$ to be
$Tr(A) = \sum_{i=1}^{n} a_{ii} \in F$.

**Proposition**

If $A, B \in M_n(F)$ then $Tr(AB) = Tr(BA)$.

**Proof**

$A = (a_{ij})$, $B = (b_{ij})$

$(AB)_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$

$$Tr(AB) = \sum_{i,j=1}^{n} a_{ij} b_{ji} = \sum_{i,j=1}^{n} b_{ji} a_{ij} = Tr(BA) \quad \square$$

**Corollary**

Two conjugate matrices have the same trace,
i.e. if $A \in M_n(F)$ and $T \in GL_n(F)$, then $Tr(T^{-1}AT) = Tr(A)$.

**Proof**

$Tr(\underbrace{T^{-1}}\underbrace{AT}) = Tr(\underbrace{AT}\underbrace{T^{-1}}) = Tr(A)$. $\square$

**Definition**

Suppose $V$ is a fdvs $/F$ with $\dim_F V = n$.
Then if $\alpha : V \to V$ is an $F$-linear map, i.e. $\alpha \in End_F(V)$, then we define $Tr_V(\alpha)$ to be $Tr(A)$ where $A \in M_n(F)$ is the matrix of $\alpha$ w.r.t. some basis of $V$ (it's well-defined by the previous corollary).

## Definition

Suppose that $\rho: G \to GL(V)$ is a representation.
Then the character of $\rho$ is the function
$\chi_\rho : G \to F$, $g \mapsto Tr_V(\rho(g))$.
So if $\rho: G \to GL_n(F)$, then $\chi_\rho(g) = Tr(\rho(g))$.
We say that $\chi_\rho$ is the character afforded by $\rho$ (associated to $\rho$).
We say that $\chi_\rho$ is irreducible if $\rho$ is.
We say that the function $\chi : G \to F$ is a character if $\exists$ a rep$^n$ $\rho: G \to GL_n(F)$ s.t. $\chi_\rho = \chi$.

If $\chi = \chi_\rho$ for $\rho: G \to F^\times$ a 1-dimensional rep$^n$ then we say $\chi$ is a linear character (in this case $\chi_\rho = \rho : G \to F^\times$).
Warning: Some authors use the word character to mean 1-dim$^L$ representations (often in number theory).

## Basic properties

1). If $\rho: G \to GL(V)$ is a representation, then $\chi_\rho(e) = Tr_V(Id_V) \overset{\in F}{=} n = \dim_F V$
So if char $F = 0$, we can recover dim$\rho$ from $\chi_\rho$.
2). If $\rho \cong \rho'$ are isomorphic then $\chi_\rho = \chi_{\rho'}$
Reason: If $\rho: G \to GL_n(F)$ and $\rho' : G \to GL_n(F)$ are equivalent, then $\exists T \in GL_n(F)$ s.t. $\rho'(g) = T^{-1}\rho(g)T$ and so $Tr(\rho'(g)) = Tr(T^{-1}\rho(g)T)$ $\forall g \in G$ $\Rightarrow \chi_{\rho'} = \chi_\rho$.
3). If $\chi = \chi_\rho$ is a character and $g, h \in G$, then
$\chi(h^{-1}gh) = \chi(g)$ since $Tr(\rho(h)^{-1}\rho(g)\rho(h)) = Tr(\rho(g))$

## Definition

If $\chi : G \to F$ is a function with the property $\chi(h^{-1}gh) = \chi(g)$ $\forall h, g \in G$, then we say that $\chi$ is a class function.
Note: all characters are class functions

4). $\chi_{\rho_1 \oplus \rho_2}(g) = \chi_{\rho_1}(g) + \chi_{\rho_2}(g)$ $\forall g \in G$.
ie. $\chi_{\rho_1 \oplus \rho_2} = \chi_{\rho_1} + \chi_{\rho_2}$

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From now on (unless stated otherwise) $F = \mathbb{C}$

Prop

$G$ a finite group.

(i) Suppose that $\rho: G \to GL_n(\mathbb{C})$ is a representation and $g \in G$ has order $d$. Then $\chi_\rho(g)$ is a sum of exactly $n$ $d$-th roots of unity.

(ii) $\chi_\rho(g^{-1}) = \overline{\chi_\rho(g)}$ (where $z \mapsto \bar{z}$ is complex conjugation).

(iii) $\chi_\rho(g) = \dim \rho \ (= \chi(1)) \Leftrightarrow \rho(g) = I_n$.

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Recall

If $\rho: G \to SGL_n(\mathbb{C})$ is a representation, the character $\chi_\rho$ afforded by $\rho$ is $\chi_\rho: G \to \mathbb{C}$, $g \mapsto Tr(\rho(g))$.

If $\chi: G \to \mathbb{C}$ is a function such that $\exists\ \rho: G \to GL_n(\mathbb{C})$ with $\chi = \chi_\rho$ then we say $\chi$ is a character.

Later we'll show that if $\rho_1: G \to GL_{n_1}(\mathbb{C})$, $\rho_2: G \to GL_{n_2}(\mathbb{C})$ are representations with $\chi_{\rho_1} = \chi_{\rho_2}$ then $\rho_1 \cong \rho_2$.

Definition

If $\chi = \chi_\rho$ is a character then define

$Ker(\chi) = Ker(\rho) = \{g \in G \mid \rho(g) = id\}$.

Proposition

Let $G$ be a finite group

(i) Suppose $\rho: G \to GL_n(\mathbb{C})$ is a representation and $g \in G$ has order $d$ then $\chi_\rho(g)$ is a sum of exactly $n$ $d$th roots of unity. $(\zeta \in \mathbb{C} : \zeta^d = 1)$

(ii) $\chi_\rho(g^{-1}) = \overline{\chi_\rho(g)}$ (complex conjugation)

(iii) $\chi_\rho(g) = \dim \rho \ (= \chi(1)) \Leftrightarrow \rho(g) = I_n \Leftrightarrow g \in Ker \rho \Leftrightarrow g \in Ker\, \chi_\rho$.

Proof

(i) Since $\rho$ is a homomorphism, we have $\rho(g)^d = I_n$, i.e. $\rho(g)$ satisfies the polynomial $X^d - 1$. $X^d - 1 = \prod_{\zeta^d=1}(X-\zeta)$ is a product of distinct linear factors.

By a theorem in linear algebra, $\rho(g)$ is diagonalisable, say $T^{-1}\rho(g)T = \mathrm{diag}(\lambda_1, \dots, \lambda_n) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

Since $\rho(g)^d = 1 \Rightarrow \lambda_i^d = 1$ for all $1 \le i \le n$.

$\Rightarrow \chi_\rho(g) = \sum_{i=1}^n \lambda_i$ is a sum of exactly $n$ $d$-th roots of unity.

(ii) Note that since $|\lambda_i| = 1$, we have $\lambda_i^{-1} = \overline{\lambda_i}$, so

$$T^{-1}\rho(g^{-1})T = \mathrm{diag}(\lambda_1^{-1}, \dots, \lambda_n^{-1})$$
$$= \mathrm{diag}(\overline{\lambda_1}, \dots, \overline{\lambda_n})$$

$\Rightarrow \chi_\rho(g^{-1}) = \overline{\lambda_1} + \dots + \overline{\lambda_n} = \overline{\chi_\rho(g)}$

(iii) $|\chi_\rho(g)| = |\sum_{i=1}^n \lambda_i| \le \sum_{i=1}^n |\lambda_i| = n$ with equality $\Leftrightarrow \lambda_1 = \dots = \lambda_n$ by the $\Delta$-inequality.

So if $\chi_\rho(g) = n$ then $\lambda_1 = \dots = \lambda_n = \lambda$ say, and so $\chi_\rho(g) = n\lambda \Rightarrow \lambda = 1$.

Conversely, if $g \in \mathrm{Ker}\rho$ then $\chi_\rho(g) = \mathrm{Tr}(I_n) = n$. □

Character tables

$G$ finite group. $\rho: G \to GL_n(\mathbb{C})$ a representation, $\chi = \chi_\rho$.

Recall
- $\chi_\rho$ only depends on $\rho$ up to isomorphism
- $\chi_\rho$ is a class function $(\chi_\rho(g) = \chi_\rho(h^{-1}gh)\ \forall g, h \in G)$.

Let $r$ be the number of conjugacy classes of $G$.

Let $\rho_1, \dots, \rho_r$ be the irreducible representations of $G$ up to isomorphism and let $\chi_i = \chi_{\rho_i}$.

The character table of $G$ is the following $r \times r$ table:

[$G = \bigsqcup_{i=1}^r x_i^G$, $x_1, \dots, x_r$ are conjugacy class representatives of $G$]

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| | $x_1^G$ | ... | $x_j^G$ | ... | $x_r^G$ conjugacy classes |
|---|---|---|---|---|---|
| $\chi_1$ | | | $\vdots$ | | |
| $\vdots$ | | | | | |
| $\chi_i$ | .... | | $\chi_i(x_j)$ | .... | |
| $\vdots$ | | | $\vdots$ | | |
| $\chi_r$ | | | | | |

irreducible characters

As a convention, $x_1 = e$ the identity on $G$, and $\chi_1 = \mathbb{1}$ is the trivial character $\chi_1(g) = 1 \ \forall g \in G$.

First remarks

(i) The first row is $(1, \ldots, 1)$

(ii) The first column is $(\chi_1(e), \ldots, \chi_r(e)) = (\dim \rho_1, \ldots, \dim \rho_r)$

So in particular, $\sum_{i=1}^{r} \chi_i(e)^2 = |G|$.

(iii) Can read off $Ker(\rho_i) = Ker(\chi_i)$ from the character table:

$\chi_i(g) = \chi_i(e) = \dim \rho_i \iff g \in Ker(\chi_i) = Ker(\rho_i)$

Examples

1). $G = C_3 = \langle x : x^3 = 1 \rangle$. Let $\omega = \exp(2\pi i/3)$

| | $1$ | $x$ | $x^2$ |
|---|---|---|---|
| $\chi_1$ | $1$ | $1$ | $1$ |
| $\chi_2$ | $1$ | $\omega$ | $\omega^2$ |
| $\chi_3$ | $1$ | $\omega^2$ | $\omega$ |

note $\bar{\omega} = \omega^2$, $x^{-1} = x^2$

2). $G = D_6 = \langle x, y \mid x^3 = y^2 = 1, \ yx = x^2 y \rangle$

Conjugacy classes:

$1^G = \{1\}$, $x^G = \{x, x^2\}$, $y^G = \{y, xy, x^2 y\}$ so $r = 3$.

$[G, G] = \langle x \rangle \Rightarrow G^{ab} = G/\langle x \rangle = C_2 = \langle y\langle x \rangle \rangle$

$G^{ab} \cong C_2$ has 2 irred. representations.

$\tilde{\rho}_1 : G^{ab} \to \mathbb{C}^\times$, $y\langle x \rangle \mapsto 1$

$\tilde{\rho}_2 : G^{ab} \to \mathbb{C}^\times$, $y\langle x \rangle \mapsto -1$

$\Rightarrow$ the one-dim reps of $G$ are

$\rho_1 : G \to \mathbb{C}^\times$, $x \mapsto 1$, $y \mapsto 1$

$\rho_2 : G \to \mathbb{C}^\times$, $x \mapsto 1$, $y \mapsto -1$

Recall: $\rho_3 : G \to GL_2(\mathbb{C})$, $x \mapsto \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$, $y \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\omega = e^{2\pi i/3}$

$(\omega + \omega^2 = -1)$ this is an irreducible representation

$Tr\begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix} = -1$, $Tr\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 0$

| | $1$ | $x$ | $y$ |
|---|---|---|---|
| $\chi_1$ | $1$ | $1$ | $1$ |
| $\chi_2$ | $1$ | $1$ | $-1$ |
| $\chi_3$ | $2$ | $-1$ | $0$ |

Definition

For class functions $\chi, \psi : G \to \mathbb{C}$, we define the inner product $\langle \chi, \psi \rangle_G = \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\psi(g)} = \frac{1}{|G|} \sum_{i=1}^{r} \chi(x_i) \overline{\psi(x_i)} \, |x_i^G|$

where $x_1, \dots, x_r$ are conjugacy class representatives.

Theorem (Row orthogonality)

Let $\chi_1, \dots, \chi_r$ be the irreducible characters of a finite group $G$. Then $\langle \chi_i, \chi_j \rangle_G = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

Proof

coming later.

Corollary

If $\rho_1, \dots, \rho_r$ are the irreducible representations of a finite group $G$, then if $\rho \cong \rho_1^{\oplus a_1} \oplus \dots \oplus \rho_r^{\oplus a_r}$ with $a_i \geq 0$, then $a_i = \langle \chi, \chi_i \rangle$ where $\chi = \chi_\rho$ is the character of $\rho$ and $\chi_i = \chi_{\rho_i}$.

So $\chi = \sum_{i=1}^{r} a_i \chi_i = \sum_{i=1}^{r} \langle \chi, \chi_i \rangle \chi_i$. In particular, $\rho$ is irreducible $\Leftrightarrow$

$\exists j$ s.t. $\langle \chi, \chi_i \rangle = \begin{cases} 1, & i = j \\ 0, & \text{otherwise} \end{cases}$ $\Leftrightarrow \langle \chi, \chi \rangle_G = 1$.

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Example

$D_6$

$$\langle \chi_3, \chi_3 \rangle = \frac{1}{6}\left(2^2 + 2(-1)^2\right) = 1$$

Proof (of corollary)

$$\chi = \sum_{i=1}^{r} a_i \chi_i$$

$$\Rightarrow \langle \chi, \chi_i \rangle = \left\langle \sum_{j=1}^{r} a_j \chi_j, \chi_i \right\rangle_G = \sum_{j=1}^{r} a_j \langle \chi_j, \chi_i \rangle = a_i$$

since $\langle \chi_j, \chi_i \rangle = \delta_{ji}$

$$\Rightarrow \chi = \sum_{i=1}^{r} \langle \chi, \chi_i \rangle \chi_i.$$

Also $\langle \chi, \chi \rangle = \left\langle \sum_{i=1}^{r} a_i \chi_i, \sum_{i=1}^{r} a_j \chi_j \right\rangle_G$

$$= \sum_{i=1}^{r} \sum_{j=1}^{r} a_i a_j \langle \chi_i, \chi_j \rangle$$

$$= \sum_{i=1}^{r} a_i^2 \qquad \text{since } \langle \chi_i, \chi_j \rangle = \delta_{ij}$$

Note that $\sum a_i^2 = 1 \iff \exists_j$ s.t. $a_i = \begin{cases} 1, & i = j \\ 0, & \text{otherwise} \end{cases}$

$\iff \exists_j$ s.t. $\rho \cong \rho_j \iff \rho$ is irreducible. $\square$

Definition

If $x \in G$ we define the centraliser of $x$ in $G$ to be $C_G(x) = \{ g \in G : [x, g] = e$ i.e. $xg = gx \}$.

Remark

left action

$G$ acts on $G$ by conjugation, $g \cdot x = gxg^{-1}$.

Then $C_G(x)$ is the stabiliser of $x$ under this action

$x^G$ is the orbit of $x$ under this action.

orbit-stabiliser thm $\Rightarrow |x^G| \cdot |C_G(x)| = |G|$.

__Theorem__ (column orthogonality)
Let $\chi_1, \dots, \chi_r$ be the irreducible characters of a finite group $G$. Then
$$\sum_{i=1}^{r} \chi_i(g)\overline{\chi_i(h)} = \begin{cases} |C_G(g)|, & \text{if } g^G = h^G \\ 0, & \text{otherwise.} \end{cases}$$

__Proof__
coming later.

__Example__
$G = A_4$
Conjugacy classes:

| | $e^G$, | $((1,2)(3,4))^G$, | $(1,2,3)^G$, | $(1,3,2)^G$ |
|---|---|---|---|---|
| sizes: | 1 | 3 | 4 | 4 |
| centraliser: | 12 | 4 | 3 | 3 |

By calculation $[G,G] = V_4 := \{e, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\} \cong C_2 \times C_2$

$G^{ab} = G/[G,G] \cong C_3$ (only group of order 3)
generated by $(1,2,3)V_4$
$\Rightarrow A_4$ 3 1-dim reps.

| | $e^G$ | $((1,2)(3,4))^G$ | $(1,2,3)^G$ | $(1,3,2)^G$ |
|---|---|---|---|---|
| $\chi_1$ | 1 | 1 | 1 | 1 |
| $\chi_2$ | 1 | 1 | $\omega$ | $\omega^2$ |
| $\chi_3$ | 1 | 1 | $\omega^2$ | $\omega$ |
| $\chi_4$ | $a=3$ | $b=-1$ | $c=0$ | $d=0$ |

$\omega = e^{2\pi i/3}$
$1+\omega+\omega^2 = 0$

Using column orthogonality,
$1^2+1^2+1^2+a^2 = 12 = |A_4| \Rightarrow a = 3$
1st + 2nd columns $\Rightarrow 1+1+1+3b = 0 \Rightarrow b = -1$
1st + 3rd columns $\Rightarrow 1+\omega+\omega^2+3c = 0 \Rightarrow 3c = 0 \Rightarrow c = 0$
1st + 4th column $\Rightarrow 1+\omega^2+\omega+3d = 0 \Rightarrow d = 0$

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Formulas for characters

1). $X$ is a finite set with a $G$-action, then let
$\chi_X = \chi_{\rho_X}$ (permutation representation), then
$\chi_X(g) = |\{x \in X : gx = x\}|$

2). If $\rho_1, \rho_2$ are rep$^n$s with characters $\chi_1, \chi_2$, then
$\rho_1 \otimes \rho_2$ has character $\chi_1 \chi_2$.
$\chi_1 \chi_2(g) = \chi_1(g) \chi_2(g)$

28-02-18 Formulas for characters

Permutation representations

If $G$ is a finite group acting on a finite set $X = \{x_1, \dots, x_n\}$ then $V = \mathbb{C}[X]$ has basis $e_{x_1}, \dots, e_{x_n}$ and

$\rho_X(g) e_x = e_{gx} \quad \forall g \in G, x \in X.$

$\Rightarrow$ matrix of $\rho_X(g)$ is $(a_{ij})$ where $a_{ij} = \begin{cases} 1, & \text{if } g(x_j) = x_i \\ 0, & \text{otherwise} \end{cases}$

$\Rightarrow$ trace of $\rho_X(g)$ is $\chi_X(g) = \operatorname{tr} \rho_X(g) = |\operatorname{fix}_X(g)| = |\{x \in X \mid g(x) = x\}|$

Lifts

Suppose $N \triangleleft G$ is a normal subgroup.
If $\tilde{\rho} : G/N \to GL_n(\mathbb{C})$ is a representation, its lift is the representation $\rho : G \to GL_n(\mathbb{C})$, $g \mapsto \tilde{\rho}(gN)$
$\rho$ has character $\chi = \chi_\rho$ given by
$\chi(g) = \operatorname{Tr}(\rho(g)) = \operatorname{Tr}(\tilde{\rho}(gN)) = \chi_{\tilde{\rho}}(g)$

Restriction

If $H \leq G$ is a subgroup and $\rho : G \to GL_n(\mathbb{C})$ a representation, then $\operatorname{Res}^G_H \rho : H \to GL_n(\mathbb{C})$, $h \mapsto \rho(h)$, is the restriction of $\rho$ to $H$. $H$ has character
$\operatorname{Res}^G_H \chi_\rho := \chi_{\operatorname{Res}^G_H \rho}$ given by $\operatorname{Res}^G_H \chi_\rho(h) = \chi_\rho(h)$.

## Tensor products

Suppose that $\rho_1 : G \to GL_n(\mathbb{C})$ and $\rho_2 : G \to GL_n(\mathbb{C})$ are two representations of $G$ with characters $\chi_1$ and $\chi_2$.

If $\rho_1(g) = (a_{ij})$ and $\rho_2(g) = (b_{ij})$, then $\rho_1 \otimes \rho_2(g)$ is the matrix

$$\begin{pmatrix} a_{11}\rho_2(g) & & * \\ & \ddots & \\ * & & a_{nn}\rho_2(g) \end{pmatrix}$$

Then
$$\begin{aligned}\chi_{\rho_1\otimes\rho_2}(g) &= Tr(\rho_1 \otimes \rho_2(g)) \\ &= a_{11} Tr\rho_2(g) + \dots + a_{nn} Tr\rho_2(g) \\ &= (a_{11} + \dots + a_{nn}) Tr\rho_2(g) \\ &= Tr\rho_1(g) Tr\rho_2(g) = \chi_{\rho_1}(g)\chi_{\rho_2}(g)\end{aligned}$$

We often write $\chi_1\chi_2$ for the character $\chi_{\rho_1\otimes\rho_2}$

## Dual representation

Suppose $\rho : G \to GL_n(\mathbb{C})$ is a representation and let $v_1, \dots, v_n$ be the standard basis of $\mathbb{C}^n$, let $v_1^*, \dots, v_n^*$ be the dual basis (Recall $v_i^*(v_j) = \delta_{ij}$).

### Claim

The matrix for $\rho^*(g)$ with respect to $v_1^*, \dots, v_n^*$ is $\rho(g^{-1})^t \leftarrow$ transpose.

### Proof

Let $\rho(g^{-1}) = (a_{ij})$.

$\rho^*(g) v_j^* = v_j^* \circ \rho(g^{-1})$

We have
$$\begin{aligned}(\rho^*(g) v_j^*)(v_k) &= v_j^*(\rho(g^{-1}) v_k) \\ &= v_j^*\left(\sum_{i=1}^n a_{ik} v_i\right) = a_{jk}\end{aligned}$$

Then we have

$$\rho^*(g) v_j^* = \sum_{k=1}^n a_{jk} v_k^*$$

since they both agree on $v_1, \dots, v_n$.

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$\Rightarrow$ matrix of $\rho^*(g)$ is $(a_{ji}) = (a_{ij})^t$

It follows that $\chi_{\rho^*}(g) = Tr(\rho(g^{-1})^t)$

$$= Tr(\rho(g^{-1}))$$
$$= \chi_\rho(g^{-1})$$
$$= \overline{\chi_\rho(g)} \quad \text{(complex conjugation)}$$

$\square$

Hom$_{\mathbb{C}}(V,W)$

Suppose $\rho: G \to GL(V)$ and $\rho': G \to GL(W)$ are representations with characters $\chi$ and $\chi'$.

We proved $Hom_{\mathbb{C}}(V,W) \cong V^* \otimes W$ as $\mathbb{C}[G]$-modules

$\Rightarrow Hom(\rho, \rho') \cong \rho^* \otimes \rho'$

$\Rightarrow \chi_{Hom(\rho,\rho')}(g) = \chi_\rho(g^{-1})\chi_{\rho'}(g)$

$$= \overline{\chi_\rho(g)}\,\chi_{\rho'}(g)$$

Induction

Suppose $H \leq G$ is a subgroup and $\rho: H \to GL_m(\mathbb{C})$ is a representation. Let $t_1, \dots, t_n$ be left coset representatives for $H$ in $G$, and let $v_1, \dots, v_m$ be the standard basis of $\mathbb{C}^m$.

With respect to the basis $t_1 \otimes v_1, \dots, t_1 \otimes v_m, t_2 \otimes v_1, \dots, t_n \otimes v_m$, the matrix of $Ind_H^G \rho(g)$ is

$$\begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix} \quad \text{where } A_{ij} \in M_m(\mathbb{C}) \text{ and } A_{ij} = \begin{cases} \rho(t_i^{-1} g t_j), & \text{if } \overbrace{g t_j H = t_i H}^{\text{equiv. } t_i^{-1} g t_j \in H} \\ 0, & \text{otherwise} \end{cases}$$

$\Rightarrow Ind_H^G \chi_\rho(g) = \chi_{Ind_H^G \rho}(g) = Tr(A_{11}) + \dots + Tr(A_{nn})$

$$= \sum_{i=1}^n \mathring{\chi}_\rho(t_i^{-1} g t_i)$$

where $\mathring{\chi}_\rho : G \to \mathbb{C}$ is the function defined by

$$\mathring{\chi}_\rho(g) = \begin{cases} \chi_\rho(g), & g \in H \\ 0, & \text{otherwise.} \end{cases}$$

Note that $\mathring{\chi}_\rho(t_i^{-1} g t_i)$ does not depend on the choice of coset representative for $t_i H$:

If $t_i' = t_i h$ for some $h \in H$, then $\mathring{\chi}_\rho(t_i'^{-1} g t_i') = \mathring{\chi}_\rho(h^{-1}(t_i^{-1} g t_i)h)$

and $t_i^{-1} g t_i \in H \Leftrightarrow h^{-1}(t_i^{-1} g t_i) h \in H$

and if $t_i^{-1} g t_i \in H$ then $\mathring{\chi}_\rho(h^{-1}(t_i^{-1} g t_i)h) = \chi_\rho(h^{-1}(t_i^{-1} g t_i)h)$

$= \chi_\rho(t_i^{-1} g t_i)$ since $\chi_\rho$ is a character $\Rightarrow$ class function

$= \mathring{\chi}_\rho(t_i^{-1} g t_i)$

$\Rightarrow \mathring{\chi}_\rho(t_i^{-1} \rho t_i) = \mathring{\chi}_\rho(t_i'^{-1} g t_i')$

In particular $\chi_{\mathrm{Ind}_H^G \rho}$ is independent of choice of left coset representatives.

By a consequence of row orthogonality, the representation $\mathrm{Ind}_H^G \rho$ is independent of choice of coset representatives (up to isomorphism).

We can rewrite $\mathrm{Ind}_H^G \rho(g)$ as

$\chi_{\mathrm{Ind}_H^G \rho}(g) = \frac{1}{H} \sum_{x \in G} \mathring{\chi}_\rho(x^{-1} g x)$.

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Proof of row orthogonality for characters

Let $G$ be a finite group, and let $V$ be a finitely generated $\mathbb{C}[G]$ module.

Notation: If $\alpha \in \mathbb{C}[G]$ we let $Tr_V(\alpha)$ denote the trace of the $\mathbb{C}$-linear map $V \to V$, $v \mapsto \alpha v$.

Note that if $\alpha, \beta \in \mathbb{C}[G]$ then $Tr_V(\alpha+\beta) = Tr_V(\alpha) + Tr_V(\beta)$, and if $\lambda \in \mathbb{C}$, then $Tr_V(\lambda\alpha) = \lambda Tr_V(\alpha)$.

Definition

Let $e_G \in \mathbb{C}[G]$ denote the element $e_G = \frac{1}{|G|}\sum_{g \in G} g$

Definition

If $V$ is a $\mathbb{C}[G]$-module, let $V^G := \{v \in V : gv = v \ \forall g \in G\}$

Lemma

We have $e_G \cdot V = V^G$

Proof

Note that $h \cdot e_G = e_G \ \forall h \in G$.

LHS $\subset$ RHS:

If $v \in V$ then $h(e_G \cdot v) = (he_G) \cdot v = e_G v$, so $e_G v \in V^G$

$\Rightarrow e_G V \subseteq V^G$

RHS $\subset$ LHS:

If $v \in V^G$ then $gv = v \ \forall g \in G$. Then $e_G v = \frac{1}{|G|}\sum gv = \frac{|G|}{|G|} v = v$

so $v \in e_G V \Rightarrow V^G \subseteq e_G V$. □

Corollary

If $V$ is a finitely generated $\mathbb{C}[G]$-module, then

$Tr_V(e_G) = \dim_{\mathbb{C}} V^G$

Proof

Note that $e_G^2 = \frac{1}{|G|}\sum_{h \in G} h e_G = \frac{|G|}{|G|} e_G = e_G$.

$\Rightarrow e_G \in End_{\mathbb{C}}(V)$ is a projection

$\Rightarrow V = Im(e_G) \oplus Ker(e_G)$

$= e_G V \oplus (1-e_G)V = V^G \oplus (1-e_G)V$

Let $f: V \to V$ denote $f(v) = e_G v$.

Then $f|_{e_G V} = Id_{e_G V}$ and $f|_{(1-e_G)V} = 0$

If we pick a basis for $V^G$ and $Ker(e_G)$, then w.r.t. this basis of $V$ the matrix for $f$ is $\begin{pmatrix} I_{\dim V^G} & 0 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow Tr_V(e_G) = Tr_V(f) = \dim_{\mathbb{C}} V^G$

$\square$

<u>Definition</u> (recall)

If $V, W$ are $\mathbb{C}[G]$-modules, we let

$Hom_{\mathbb{C}[G]}(V,W) = \{\phi : V \to W, \ \mathbb{C}[G]\text{-module homomorphisms}\}$

$= \{\phi : V \to W \text{ s.t. } \phi(v_1+v_2) = \phi(v_1)+\phi(v_2), \ \phi(\alpha v) = \alpha\phi(v)$

$\forall v_1, v_2, v \in V, \ \alpha \in \mathbb{C}[G]\}$

$= \{\phi : V \to W \ \mathbb{C}\text{-linear s.t. } \phi(gv) = g\phi(v) \ \forall v \in V, g \in G\}$

<u>Lemma</u>

Suppose $V, W$ are $\mathbb{C}[G]$-modules. Then

$Hom_{\mathbb{C}}(V,W)^G = Hom_{\mathbb{C}[G]}(V,W)$.

<u>Proof</u>

Recall that $G$ acts on $Hom_{\mathbb{C}}(V,W)$ by $(g\cdot\phi)(v) = g(\phi(g^{-1}(v)))$

$\forall g \in G, \ \phi \in Hom_{\mathbb{C}}(V,W), \ v \in V$.

Let $\phi \in Hom_{\mathbb{C}}(V,W)$. Then

$\phi \in Hom_{\mathbb{C}}(V,W)^G \iff g\cdot\phi = \phi \ \forall g \in G$

$\iff g(\phi(g^{-1}v)) = \phi(v) \ \forall g \in G, \forall v \in V$

$\iff g\phi(v) = \phi(gv) \ \forall g \in G \ \forall v \in V$ replace $v$ by $gv$

$\iff \phi \in Hom_{\mathbb{C}[G]}(V,W)$.

$\square$

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**Theorem** (Row orthogonality)
Suppose that $G$ is a finite group and that $V_1, \dots, V_r$ are the simple $\mathbb{C}[G]$-modules up to isomorphism, where $r = \#$ conjugacy classes.
Suppose that $V_1, \dots, V_r$ have characters $\chi_1, \dots, \chi_r$.
Then $\langle \chi_i, \chi_j \rangle_G = \delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$

**Proof**
By Schur's Lemma we have
$$\mathrm{Hom}_{\mathbb{C}[G]}(V_i, V_j) = \begin{cases} \mathbb{C} & i=j \\ 0 & \text{otherwise} \end{cases}$$
using that $\mathbb{C}$ is algebraically closed.
$$\Rightarrow \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}[G]}(V_i, V_j) = \begin{cases} 1, & i=j \\ 0, & \text{otherwise} \end{cases}$$
But $\mathrm{Hom}_{\mathbb{C}[G]}(V_i, V_j) = \mathrm{Hom}_{\mathbb{C}}(V_i, V_j)^G$
$$\begin{aligned} \Rightarrow \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}[G]}(V_i, V_j) &= \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}}(V_i, V_j)^G \\ &= \mathrm{Tr}_{\mathrm{Hom}_{\mathbb{C}}(V_i, V_j)}(e_G) \qquad \left[ e_G = \frac{1}{|G|} \sum_{g \in G} g \right] \\ &= \frac{1}{|G|} \sum_{g \in G} \mathrm{Tr}_{\mathrm{Hom}_{\mathbb{C}}(V_i, V_j)}(g) \\ &= \frac{1}{|G|} \sum_{g \in G} \chi_{\mathrm{Hom}_{\mathbb{C}}(V_i, V_j)}(g) \\ &= \frac{1}{|G|} \sum_{g \in G} \overline{\chi_i(g)} \chi_j(g) \\ &= \langle \overline{\chi_i(g)}, \overline{\chi_j(g)} \rangle_G \quad (\text{by def}^n \text{ of } \langle \chi, \psi \rangle_G) \end{aligned}$$
Since $\langle \overline{\chi_i}, \overline{\chi_j} \rangle_G = 0$ or $1$,
$\overline{\langle \overline{\chi_i}, \overline{\chi_j} \rangle_G} = \langle \overline{\chi_i}, \overline{\chi_j} \rangle_G$
but $\overline{\langle \overline{\chi_i}, \overline{\chi_j} \rangle_G} = \langle \overline{\overline{\chi_i}}, \overline{\overline{\chi_j}} \rangle_G = \langle \chi_i, \chi_j \rangle_G$ $\square$

(If $\psi$ is a class function $\psi : G \to \mathbb{C}$, then $\overline{\psi}$ is the class function $\overline{\psi}(g) = \overline{\psi(g)}$.)

Corollary

If $G$ is a finite group with irreducible complex characters $\chi_1, \dots, \chi_r$ then $\chi_1, \dots, \chi_r$ form an orthonormal basis for the $\mathbb{C}$-vector space of class functions $\{\psi : G \to \mathbb{C} \mid \psi(h^{-1}gh) = \psi(g) \ \forall g, h \in G\}$ w.r.t. $\langle \cdot, \cdot \rangle_G$.

Proof

Clearly the space of class functions has a basis $\psi_1, \dots, \psi_r$ where $K_1, \dots, K_r$ are the conjugacy classes of $G$ and $\psi_i(g) = \begin{cases} 1 & \text{if } g \in K_i \\ 0 & \text{otherwise} \end{cases}$

Hence the space of class functions has dimension $r$.

Therefore it is enough to show that $\chi_1, \dots, \chi_r$ are linearly independent.

If $\sum_{i=1}^{r} a_i \chi_i = 0$ for some $a_i \in \mathbb{C}$, then

$$0 = \langle \sum_{i=1}^{r} a_i \chi_i, \chi_j \rangle_G = \sum_{i=1}^{r} a_i \langle \chi_i, \chi_j \rangle_G = a_j \quad \text{by row orthogonality}$$

$\Rightarrow \chi_1, \dots, \chi_r$ are linearly independent, so $\chi_1, \dots, \chi_r$ form a basis.

It is orthonormal from the previous theorem. $\square$

Remark

This corollary tells us that if $\psi : G \to \mathbb{C}$ is a class function, then $\exists !\ a_1, \dots, a_r \in \mathbb{C}$ st. $\psi = \sum_{i=1}^{r} a_i \chi_i$.

Moreover $a_j = \langle \psi, \chi_j \rangle_G$ by row orthogonality.

$\psi$ is a character $\Leftrightarrow a_i \in \mathbb{Z}_{\geq 0}$ for $i = 1, \dots, n$.

Some authors say $\psi$ is a virtual character if $a_i \in \mathbb{Z}$ for $i = 1, \dots, n$.

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<u>Theorem</u> (Column orthogonality)

Let $G$ be a finite group and let $\chi_1, \dots, \chi_r$ be the irreducible complex characters of $G$.

Let $x_1, \dots, x_r$ be conjugacy class representatives of $G$.

Then for $1 \le j, k \le r$ we have

$$\sum_{i=1}^{r} \chi_i(x_j)\overline{\chi_i(x_k)} = \begin{cases} |C_G(x_j)| & \text{when } j=k. \\ 0 & \text{otherwise} \end{cases}$$

<u>Proof</u>

For $1 \le k \le r$ let $\psi_k : G \to \mathbb{C}$, $g \mapsto \begin{cases} 1, & g \in x_k^G \\ 0, & \text{otherwise} \end{cases}$

Then by the previous corollary

$$\psi_k = \sum_{i=1}^{r} \lambda_i \chi_i \quad \text{for some } \lambda_i \in \mathbb{C}$$

Since $\lambda_i = \langle \psi_k, \chi_i \rangle_G = \frac{1}{|G|} \sum_{g \in G} \psi_k(g)\overline{\chi_i(g)}$

$$= \frac{|x_k^G|}{|G|} \overline{\chi_i(x_k)}$$

$$= \frac{\overline{\chi_i(x_k)}}{|C_G(x_k)|} \quad \text{by the orbit-stabiliser thm.}$$

$$\Rightarrow \psi_k = \sum_{i=1}^{r} \left( \overline{\chi_i(x_k)} / |C_G(x_k)| \right) \chi_i$$

We have $\psi_k(x_j) = \delta_{jk}$

$$\Rightarrow \psi_k(x_j) = \frac{1}{|C_G(x_k)|} \sum_{i=1}^{r} \overline{\chi_i(x_k)} \chi_i(x_j) = \delta_{jk}$$

multiplying both sides by $|C_G(x_k)|$ gives the column orthogonality relation. □

## Permutation representations

$G$ is a finite group, $X$ a finite set upon which $G$ acts.

$\rho_X : G \to GL(\mathbb{C}[X])$ , $\rho_X(g)(e_x) = e_{g.x}$

Recall $\rho_X$ has character $\chi_X$ given by

$\chi_X(g) = |fix_X(g)| = |\{x \in X : gx = x\}|$.

Note that $\rho_X$ always contains a copy of the trivial representation, because

$\mathbb{C}[X]$ contains $\mathrm{span}_{\mathbb{C}}\{\sum_{x\in X} e_x\}$

and $g\cdot\left(\sum_{x\in X} e_x\right) = \sum_{x \in X} e_{g.x} = \sum_{x\in X} e_x$

complement is

$$\left\{\sum_{x\in X} a_x e_x : \sum_{x\in X} a_x = 0\right\}$$

### Lemma (Burnside's Lemma)

Let $G$ be a finite group and $X$ a finite set with a $G$-action

Then $\langle \chi_X, \mathbb{1}\rangle_G = \#$ orbits of $G$ on $X$.

where $\mathbb{1}$ is the character of the trivial representation.

### Proof

If $X = X_1 \sqcup \ldots \sqcup X_l$ are the $G$-orbits in $X$, then

$\chi_X = \chi_{X_1} + \ldots + \chi_{X_l}$ since
$$\begin{aligned}\chi_X(g) &= |\{x\in X : gx = x\}| \\ &= \left|\bigsqcup_{i=1}^{l}\{x\in X_i : gx = x\}\right| \\ &= \sum_{i=1}^{l} |\{x \in X_i : gx = x\}| \\ &= \sum_{i=1}^{l} \chi_{X_i}(g)\end{aligned}$$

$\Rightarrow$ wlog we may assume $l = 1$, i.e. $G$ has 1 orbit on $X$

i.e. $G$ is transitive on $X$.

Then
$$\begin{aligned}\langle \chi_X, \mathbb{1}\rangle_G &= \frac{1}{|G|}\sum_{g\in G}\chi_G(g) \\ &= \frac{1}{|G|}\sum_{g\in G}|\{x\in X : gx = x\}|\end{aligned}$$

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$$\Rightarrow \langle \chi_X, \mathbb{1} \rangle_G = \frac{1}{|G|} \left| \{ (g,x) \in G \times X : gx = x \} \right|$$

$$= \frac{1}{|G|} \sum_{x \in X} \left| \{ g \in G : gx = x \} \right|$$

$$= \frac{1}{|G|} \sum_{x \in X} |Stab_G(x)|$$

$$= \frac{1}{|G|} |X| \cdot |Stab_G(x)| \text{ for some } x \in X$$

$$= \frac{1}{|G|} \cdot |G| \text{ by the orbit stabiliser theorem}$$

$$= 1 \qquad \square$$

<u>Lemma</u>

Let $G$ act on two sets $X_1, X_2$. Then $G$ acts on $X_1 \times X_2$ via $g(x_1, x_2) = (gx_1, gx_2)$. The character $\chi_{X_1 \times X_2}$ is $\chi_{X_1} \chi_{X_2}$ and $\langle \chi_{X_1}, \chi_{X_2} \rangle_G = \#$ orbits of $G$ on $X_1 \times X_2$.

<u>Proof</u>

$$\chi_{X_1 \times X_2}(g) = \left| \{ (x_1, x_2) \in X_1 \times X_2 : gx_1 = x_1, \ gx_2 = x_2 \} \right|$$

$$= \left| \{ x_1 \in X_1 : gx_1 = x_1 \} \right| \cdot \left| \{ x_2 \in X_2 : gx_2 = x_2 \} \right|$$

$$= \chi_{X_1}(g) \chi_{X_2}(g).$$

Note that $\chi_{X_2}$ takes values in $\mathbb{Z}_{\geq 0}$, so $\overline{\chi_{X_2}} = \chi_{X_2}$.

$$\langle \chi_{X_1}, \chi_{X_2} \rangle_G = \frac{1}{|G|} \sum_{g \in G} \chi_{X_1}(g) \overline{\chi_{X_2}(g)}$$

$$= \frac{1}{|G|} \sum_{g \in G} \chi_{X_1}(g) \chi_{X_2}(g)$$

$$= \langle \chi_{X_1} \chi_{X_2}, \mathbb{1} \rangle_G$$

$$= \langle \chi_{X_1 \times X_2}, \mathbb{1} \rangle_G$$

$$= \# \text{ orbits by the previous lemma} \qquad \square$$

**Definition**

Let $G$ act on $X$ with $|X| \geq 2$. Then we say $G$ is **2-transitive** on $X$ if $G$ has exactly 2 orbits on $X \times X$, namely $\{(x,x) : x \in X\}$ and $\{(x,y) : x,y \in X, x \neq y\}$.

**Lemma**

Let $G$ act on $X$ with $|X| \geq 2$. Then $\chi_X = \mathbb{1} + \pi_X$ with $\pi_X$ a non-trivial irreducible character $\Leftrightarrow$ $G$ is 2-transitive on $X$.

**Proof**

Let $\psi_1, \dots, \psi_r$ be the irreducible characters of $G$.

Then $\chi_X = a_1\psi_1 + \dots + a_r\psi_r$ with $a_1 \geq 1$ (assuming $\psi_1 = \mathbb{1}$).

$\Rightarrow \pi_X = (a_1 - 1)\psi_1 + \sum_{i=2}^{r} a_i \psi_i$

$\pi_X$ is irreducible $\Leftrightarrow \langle \pi_X, \pi_X \rangle_G = 1$

We have $\langle \chi_X, \chi_X \rangle_G = \#$ orbits of $G$ on $X \times X$.

$\pi_X$ is a non-trivial irreducible character

$\Leftrightarrow \exists\, 2 \leq j \leq r$ s.t. $a_j = 1$ and $a_i = 0$ for $i \neq j$, $2 \leq i \leq r$ and $a_1 = 1$.

$$\langle \pi_X, \pi_X \rangle_G = \langle (a_1 - 1)\psi_1 + \sum_{i=2}^{r} a_i\psi_i , (a_1 - 1)\psi_1 + \sum_{i=2}^{r} a_i \psi_i \rangle_G$$

$$= (a_1 - 1)^2 + \sum_{i=2}^{r} a_i^2 .$$

The only way this sum can be 1, and $a_1 = 1$, is if $\exists\, 2 \leq j \leq r$ s.t. $a_j = 1$ and $a_i = 0$ if $i \neq j$, $2 \leq i \leq r$.

So $\pi_X$ irreducible and non-trivial $\Leftrightarrow \chi_X = \mathbb{1} + \chi$, $\chi$ non-trivial irred.

$$\Leftrightarrow \langle \chi_X, \chi_X \rangle_G = \langle \mathbb{1} + \chi, \mathbb{1} + \chi \rangle_G$$

$$= \langle \mathbb{1}, \mathbb{1} \rangle_G + \langle \mathbb{1}, \chi \rangle_G + \langle \chi, \mathbb{1} \rangle_G + \langle \chi, \chi \rangle_G$$

$$= 1 + 0 + 0 + 1 = 2 \qquad \chi_X = \sum a_i \psi_i$$

Note that $\langle \chi_X, \chi_X \rangle = \sum_{i=1}^{r} a_i^2 = 2 \Leftrightarrow a_1 = 1$ and $\exists\, 2 \leq j \leq r$ s.t. $a_j = 1$, $a_i = 0$ if $i \neq j$, $2 \leq i \leq r$. $\square$

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Recall

$G$ finite group, $\chi_1, \dots, \chi_r$ the irreducible characters of $G$.

If $\chi : G \to \mathbb{C}$ is a class function, then $\exists$ unique $a_1, \dots, a_r \in \mathbb{C}$ s.t. $\chi = \sum a_i \chi_i$

If $\psi = \sum b_j \chi_j$, then

$$\langle \chi, \psi \rangle_G = \sum_{i=1}^{r} \sum_{j=1}^{r} \langle a_i \chi_i, b_j \chi_j \rangle_G$$

$$= \sum_{i=1}^{r} \sum_{j=1}^{r} a_i \overline{b_j} \langle \chi_i, \chi_j \rangle_G \qquad \langle \chi_i, \chi_j \rangle_G = \delta_{ij}$$

$$= \sum_{i=1}^{r} a_i \overline{b_i}$$

$\chi$ is a character $\Leftrightarrow a_i \in \mathbb{Z}_{\geq 0} \; \forall i$

If $\chi$ is a character then $\langle \chi, \chi \rangle_G = \sum_{i=1}^{r} a_i^2 = 1 \Leftrightarrow \chi$ is irreducible

Permutation representation

$G$ finite group, $\mathbb{1} = \chi_1, \dots, \chi_r$ irreducible characters.

$X$ finite set with action of $G$.

Lemma

Suppose $|X| \geq 2$, then $G$ is 2-transitive on $X$

$\Leftrightarrow \chi_X = \chi_1 + \chi_i$ for some $2 \leq i \leq r$.

Proof

Let $\chi_X = \sum_{i=1}^{r} a_i \chi_i$ for some $a_i \in \mathbb{Z}_{\geq 0}$. Note that

$a_1 = \langle \chi_X, \chi_1 \rangle_G = \#$ orbits of $G$ on $X \geq 1$

$G$ is 2-transitive on $X \Leftrightarrow \#$ orbits of $G$ on $X \times X = 2$

$\Leftrightarrow \langle \chi_X, \chi_X \rangle_G = 2 \Leftrightarrow \sum_{i=1}^{r} a_i^2 = 2$

$\Leftrightarrow a_1 = 1$ & $\exists j \; 2 \leq j \leq r$, $a_j = 1$, $a_i = 0$ for $i \neq j$, since $a_1 \geq 1$

$\Leftrightarrow \chi_X = \chi_1 + \chi_j$ for some $2 \leq j \leq r$ $\square$

**Example**

$S_n$ acts 2-transitively on $X = \{1, \dots, n\}$,
so $\pi_X : \chi_X - \mathbb{1}$ is always an irreducible charater.

**Example**

$G = S_5$, character table:

| | $\overset{1}{e}$ | $\overset{15}{(12)(34)}$ | $\overset{20}{(123)}$ | $\overset{24}{(12345)}$ | $\overset{10}{(12)}$ | $\overset{20}{(123)(45)}$ | $\overset{30}{(1234)}$ |
|---|---|---|---|---|---|---|---|
| $\chi_1$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\chi_2$ | 1 | 1 | 1 | 1 | $-1$ | $-1$ | $-1$ |
| $\chi_3$ | $5-1=4$ | $1-1=0$ | $2-1=1$ | $-1$ | $3-1=2$ | $-1$ | $1-1=0$ |
| $\chi_4$ | 4 | 0 | 1 | $-1$ | $-2$ | 1 | 0 |
| $\chi_Y - \chi_1 - \chi_3 = \chi_5$ | 5 | 1 | $-1$ | 0 | $-1$ | $-1$ | 1 |
| $\chi_2\chi_5 = \chi_6$ | 5 | 1 | $-1$ | 0 | 1 | 1 | $-1$ |
| column orthog. $\chi_7$ | 6 | $-2$ | 0 | 1 | 0 | 0 | 0 |
| $\chi_Y$ | 10 | 2 | 1 | 0 | 4 | 1 | 0 |

$\chi_1$ is the trivial character.

Let $\chi_2$ be the sign character, $\chi_2 : G \to \underset{\overset{\|}{C_2}}{S_5/A_5} \to \mathbb{C}$
lifted from the non trivial character of $S_5/A_5$

Let $X = \{1, \dots, 5\}$. Then $\chi_3 = \chi_X - \chi_1$ is irreducible.

$\chi_2$ is 1-dimensional, so $\chi_2\chi_3$ is irreducible.

By inspection $\chi_2\chi_3 \neq \chi_1, \chi_2, \chi_3$. Let $\chi_4 = \chi_2\chi_3$

Consider $Y = \{ S \subset \{1,2,3,4,5\} : |S| = 2 \} = \{\text{unordered pairs of elements of } X\}$

So $|Y| = \binom{5}{2} = 10$

$\langle \chi_Y, \chi_1 \rangle_{S_5} = \#$ orbits of $S_5$ on $Y = 1$

$\langle \chi_Y, \chi_Y \rangle_{S_5} = \#$ orbits of $S_5$ on $Y \times Y = 3$

orbits on $Y \times Y$ are of the form

$\{ (\{i,j\}, \{i,j\}) : 1 \le i \neq j \le 5 \}$, $\{ (\{i,j\}, \{i,k\}), 1 \le i,j,k \text{ distinct} \le 5 \}$,

$\{ (\{i,j\}, \{k,l\}) : 1 \le i,j,k,l \text{ distinct} \le 5 \}$

$\langle \chi_Y, \chi_X \rangle_{S_5} = \#$ orbits of $S_5$ on $Y \times X = 2$

orbits are $\{ (\{i,j\}, i), 1 \le i,j \text{ distinct} \le 5 \}$,

$\{ (\{i,j\}, k), 1 \le i,j,k \text{ distinct} \}$

$\Rightarrow \chi_Y = \chi_1 + \chi_3 + \pi$ for some irreducible character $\pi$.

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Calculating $\pi$ we see $\pi \neq \chi_1, \dots, \chi_4$. Let $\chi_5 := \pi$.

$$\chi_\gamma \text{ on } \begin{cases} (12)(34) & \text{fixes } \{1,2\} \text{ and } \{3,4\} \\ (123) & \text{"} \quad \{4,5\} \\ (12) & \text{"} \quad \{1,2\}, \{3,4\}, \{4,5\}, \{5,3\} \\ (123)(45) & \text{"} \quad \{4,5\} \end{cases}$$

?

## Frobenius reciprocity

Suppose $G$ is a finite group with a subgroup $H$, and $\rho: H \to GL(V)$ is a representation over $\mathbb{C}$.

Let $\chi = \chi_\rho : H \to \mathbb{C}$ and let $\mathrm{Ind}_H^G \chi := \chi_{\mathrm{Ind}_H^G \rho} : G \to \mathbb{C}$.

Recall that $\mathrm{Ind}_H^G \chi(g) = \frac{1}{|H|} \sum_{x \in G} \mathring{\chi}(x^{-1} g x)$

where $\mathring{\chi} : G \to \mathbb{C}$ is given by $g \mapsto \begin{cases} \chi(g), & g \in H \\ 0 & \text{otherwise} \end{cases}$

**Theorem** (Frobenius Reciprocity)

Let $G$ be a finite group and $H \leq G$ a subgroup.

Let $\chi : H \to G$ be a character of $H$ and $\psi : G \to \mathbb{C}$ be a character of $G$. Then we have

$$\langle \mathrm{Ind}_H^G \chi, \psi \rangle_G = \langle \chi, \mathrm{Res}_H^G \psi \rangle_H .$$

**Example**

Recall the character table of $A_4$:

| | $e$ | $(12)(34)$ | $(123)$ | $(132)$ |
|---|---|---|---|---|
| $\chi_1$ | 1 | 1 | 1 | 1 |
| $\chi_2$ | 1 | 1 | $\omega$ | $\omega^2$ |
| $\chi_3$ | 1 | 1 | $\omega^2$ | $\omega$ |
| $\chi_4$ | 3 | -1 | 0 | 0 |

where $\omega = e^{2\pi i/3}$

$1 + \omega + \omega^2 = 0$.

Let $\rho_1, \dots, \rho_4$ be irreducible representations of $A_4$ with characters $\chi_1, \dots, \chi_4$ respectively.

Let $H = \{e, (12)(34)\} \cong C_2$ and let $\rho : H \to \mathbb{C}^\times$ be the non-trivial representation $(12)(34) \mapsto -1$.

Put $\chi = \chi_\rho : H \to \mathbb{C}$

Find $a_1, \dots, a_4$ s.t. $\mathrm{Ind}_H^{A_4} \rho = \rho_1^{\oplus a_1} \oplus \dots \oplus \rho_4^{\oplus a_4}$

We have $\mathrm{Ind}_H^{A_4} \chi = a_1 \chi_1 + \dots + a_4 \chi_4$

and $a_i = \langle \mathrm{Ind}_H^{A_4} \chi, \chi_i \rangle_{A_4}$

By Frobenius Reciprocity, $a_i = \langle \chi, \mathrm{Res}_H^{A_4} \chi_i \rangle_H$

$$\Rightarrow a_i = \frac{1}{2}\left(\chi(e)\overline{\chi_i(e)} + \chi((12)(34))\overline{\chi_i((12)(34))}\right)$$
$$= \frac{1}{2}\left(\overline{\chi_i(e)} - \overline{\chi_i((12)(34))}\right)$$

$$\Rightarrow \begin{cases} a_1 = \frac{1}{2}(1-1) = 0 \\ a_2 = \frac{1}{2}(1-1) = 0 \\ a_3 = \frac{1}{2}(1-1) = 0 \\ a_4 = \frac{1}{2}(3-(-1)) = 2 \end{cases}$$

$\Rightarrow \mathrm{Ind}_H^{A_4} \rho \cong \rho_4 \oplus \rho_4$

09-03-18 Theorem (Frobenius reciprocity)

Let $G$ be a finite group and let $H \leq G$ be a subgroup.
Let $\chi : H \to \mathbb{C}$ and $\psi : G \to \mathbb{C}$ be characters.
Then we have $\langle \mathrm{Ind}_H^G \chi, \psi \rangle_G = \langle \chi, \mathrm{Res}_H^G \psi \rangle_H$.

Proof

Recall that $\mathrm{Ind}_H^G \chi(g) = \frac{1}{|H|} \sum_{x \in G} \mathring{\chi}(x^{-1}gx)$

where $\mathring{\chi}(g) = \begin{cases} \chi(g), & g \in H \\ 0, & g \notin H \end{cases}$

$$\langle \mathrm{Ind}_H^G \chi, \psi \rangle_G = \frac{1}{|G|} \sum_{g \in G} \mathrm{Ind}_H^G \chi(g) \overline{\psi(g)}$$
$$= \frac{1}{|G||H|} \sum_{g \in G} \sum_{x \in G} \mathring{\chi}(x^{-1}gx) \overline{\psi(g)}$$
$$= \frac{1}{|G||H|} \sum_{x \in G} \sum_{g \in G} \mathring{\chi}(x^{-1}gx) \overline{\psi(g)}$$

[for fixed $x$, $y = x^{-1}gx$ runs over $G$ as $g$ runs over $G$
$G \to G$, $g \mapsto x^{-1}gx$ is a bijection]

$$= \frac{1}{|G||H|} \sum_{x \in G} \sum_{y \in G} \mathring{\chi}(y) \overline{\psi(xyx^{-1})}$$
$$= \frac{1}{|G||H|} \sum_{x \in G} \sum_{y \in G} \mathring{\chi}(y) \overline{\psi(y)} \quad \text{(since } \psi \text{ is a class function)}$$

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$$\Rightarrow \langle \mathrm{Ind}_H^G \chi, \psi \rangle_G = \frac{1}{|H|} \sum_{y \in G} \mathring{\chi}(y) \overline{\psi(y)}$$

$$= \frac{1}{|H|} \sum_{y \in H} \chi(y) \overline{\psi(y)} \quad \text{by def}^n \text{ of } \mathring{\chi}$$

$$= \langle \chi, \mathrm{Res}_H^G \psi \rangle_H$$

$\square$

<u>Symmetric square and antisymmetric square</u>

Suppose $V$ is a vector space over $\mathbb{C}$ with dimension $d$.

Note that $S_2 = \{e, (12)\} = \{e, \sigma\}$ acts on $V \otimes V$

$\sigma : V \otimes V \longrightarrow V \otimes V$, $u \otimes v \longmapsto v \otimes u$

($\sigma$ is a linear map by the universal property $V \times V \to V \otimes V$, $(u, v) \mapsto v \otimes u$ is bilinear).

We define the subspaces of $V \otimes V$:

$S^2 V = \{x \in V \otimes V \text{ s.t. } \sigma(x) = x\}$ symmetric square of $V$

$\Lambda^2 V = \{x \in V \otimes V \text{ s.t. } \sigma(x) = -x\}$ antisymmetric square of $V$

e.g. if $u, v \in V$ then $u \otimes u + v \otimes v \in S^2 V$, $u \otimes v - v \otimes u \in \Lambda^2 V$

<u>Proposition</u>

We have

(i) $V \otimes V = S^2 V \oplus \Lambda^2 V$

(ii) If $\{v_1, \dots, v_d\}$ is a basis for $V$, then a basis for $S^2 V$ is given by $\{v_i \otimes v_j + v_j \otimes v_i : 1 \le i, j \le d\}$

(iii) A basis for $\Lambda^2 V$ is given by $\{v_i \otimes v_j - v_j \otimes v_i : 1 \le i, j \le d\}$

<u>Proof</u>

(i) If $x \in V \otimes V$, then $\underbrace{\frac{x + \sigma(x)}{2}}_{\in S^2 V} + \underbrace{\frac{x - \sigma(x)}{2}}_{\in \Lambda^2 V} = x$

and so $V \otimes V = S^2 V + \Lambda^2 V$.

Suppose $x \in S^2 V \cap \Lambda^2 V$. Then $\sigma(x) = x$ and $\sigma(x) = -x$

$\Rightarrow x = -x \Rightarrow 2x = 0 \Rightarrow x = 0$

(ii) Let $U = \text{span}_{\mathbb{C}}\{v_i \otimes v_j + v_j \otimes v_i : 1 \le i, j \le d\}$

Then since $\{v_i \otimes v_j : 1 \le i, j \le d\}$ are a basis for $V \otimes V$, there exist linear maps $\pi_{i,j} : V \otimes V \to \mathbb{C}$

$$v_k \otimes v_l \longmapsto \begin{cases} 1 & \text{if } k = i,\ l = j \\ 0 & \text{otherwise.} \end{cases}$$

$\{v_i \otimes v_j + v_j \otimes v_i : 1 \le i \le j \le d\}$ is L.I. :

If $\sum_{1 \le i \le j \le d} a_{ij}(v_i \otimes v_j + v_j \otimes v_i) = 0$ then for $1 \le k \le l \le d$

$$\pi_{k,l}\left(\sum_{1 \le i \le j \le d} a_{ij}(v_i \otimes v_j + v_j \otimes v_i)\right) = \begin{cases} 2a_{kk}, & k = l \\ a_{kl}, & k \ne l \end{cases} = 0$$

$\Rightarrow a_{ij} = 0 \ \forall i, j$

Then $\dim_{\mathbb{C}} U = |\{(i,j) : 1 \le i \le j \le d\}| = \binom{d+1}{2}$

and $U \le S^2V$

Similarly if we let $W = \text{span}_{\mathbb{C}}\{v_i \otimes v_j - v_j \otimes v_i : 1 \le i < j \le d\}$ then $\dim_{\mathbb{C}} W = |\{(i,j) : 1 \le i < j \le d\}| = \binom{d}{2}$ and $W \le \Lambda^2 V$

Since $V \otimes V = S^2V \oplus \Lambda^2 V$ and $\dim(V \otimes V) = d^2$

and $\dim_{\mathbb{C}}(S^2V) \geq \binom{d+1}{2}$ with equality $\Leftrightarrow U = S^2V$

and $\dim_{\mathbb{C}}(\Lambda^2 V) \geq \binom{d}{2}$ with equality $\Leftrightarrow W = \Lambda^2 V$.

We have $U = S^2V$ and $W = \Lambda^2 V$ because

$$\binom{d+1}{2} + \binom{d}{2} = \frac{(d+1)d}{2} + \frac{d(d-1)}{2} = \frac{d^2 + d + d^2 - d}{2} = d^2$$

$\square$

Now suppose that $V$ is a finitely generated $\mathbb{C}[G]$-module for some group $G$.

Let $g \in G$ denote the $\mathbb{C}$-linear map $g : V \otimes V \to V \otimes V$, $v \otimes w \mapsto gv \otimes gw$. Then the following diagram commutes:

$$\begin{tikzcd} V \otimes V \arrow[r, "g"] \arrow[d, "\sigma"'] & V \otimes V \arrow[d, "\sigma"] \\ V \otimes V \arrow[r, "g"] & V \otimes V \end{tikzcd}$$

clockwise: $v \otimes w \mapsto gv \otimes gw \mapsto gw \otimes gv$

anticlockwise: $v \otimes w \mapsto w \otimes v \mapsto gw \otimes gv$

If $x \in S^2V = \{x \in V \otimes V : \sigma x = x\}$ then $\sigma(gx) = g(\sigma x) = gx \Rightarrow gx \in S^2V$.

If $x \in \Lambda^2 V = \{x \in V \otimes V : \sigma x = -x\}$ then $\sigma(gx) = g(\sigma x) = g(-x) = -gx$

$\Rightarrow gx \in \Lambda^2 V$.

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In particular, $S^2V$ and $\Lambda^2 V$ are $\mathbb{C}[G]$-submodules of $V\otimes V$ and $V\otimes V = S^2V \oplus \Lambda^2 V$ as $\mathbb{C}[G]$-modules.

**Proposition**

If $V$ is a finitely generated $\mathbb{C}[G]$-module, let $\chi_{S^2}$ and $\chi_{\Lambda^2}$ denote the characters of $S^2V$ and $\Lambda^2V$ respectively. Then we have

(i) $\chi_{S^2}(g) = \frac{1}{2}\left(\chi(g)^2 + \chi(g^2)\right)$

(ii) $\chi_{\Lambda^2}(g) = \frac{1}{2}\left(\chi(g)^2 - \chi(g^2)\right)$

where $\chi$ is the character of $V$.

**Proof**

Since $V\otimes V = S^2V \oplus \Lambda^2V$ and $V\otimes V$ has character $\chi^2 = (g \mapsto \chi(g)^2)$, it suffices to prove (i).

$(\chi_{V\otimes V} = \chi^2 = \chi_{S^2} + \chi_{\Lambda^2})$.

For a fixed $g\in G$, $\rho(g): V \to V$, $v \mapsto gv$, is diagonalisable (since $\rho(g)^{|G|} = Id$).

Pick a basis $v_1, \dots, v_d$ for $V$ such that $\rho(g)v_i = \lambda_i v_i$ for some $\lambda_i \in \mathbb{C}$.

Then $S^2V$ has basis $\{v_i\otimes v_j + v_j \otimes v_i : 1 \le i \le j \le d\}$, and we have

$$g(v_i\otimes v_j + v_j\otimes v_i) = (gv_i \otimes gv_j + gv_j\otimes gv_i) = \lambda_i\lambda_j(v_i\otimes v_j + v_j\otimes v_i)$$

$$\Rightarrow \chi_{S^2}(g) = \sum_{1\le i\le j\le d} \lambda_i\lambda_j$$

Note that $\chi(g) = \sum_{1\le i\le d} \lambda_i$ and $\chi(g^2) = \sum_{1\le i \le d}\lambda_i^2$

$$\Rightarrow \chi(g)^2 + \chi(g^2) = \left(\sum_{1\le i\le d}\lambda_i\right)^2 + \sum_{1\le i\le d}\lambda_i^2$$

$$= \left(\sum_{1\le i\le d}\lambda_i\right)\left(\sum_{1\le j\le d}\lambda_j\right) + \sum_{1\le i\le d}\lambda_i^2$$

$$= \sum_{1\le i,j\le d}\lambda_i\lambda_j + \sum_{1\le i\le d}\lambda_i^2$$

$$\Rightarrow \chi(g)^2 + \chi(g^2) = \sum_{1\le i\ne j\le d} \lambda_i\lambda_j + 2\sum_{1\le i\le d}\lambda_i^2$$

$$= 2\sum_{1\le i<j\le d} \lambda_i\lambda_j + 2\sum_{1\le i\le d}\lambda_i^2$$

$$= 2\sum_{1\le i\le j\le d} \lambda_i\lambda_j$$

$$= 2\chi_{S^2}(g)$$

$$\Rightarrow \chi_{S^2}(g) = \tfrac{1}{2}\left(\chi(g)^2 + \chi(g^2)\right) \qquad \square$$

## Real representation theory

- $\mathbb{C}$ has char $= 0$ $\Rightarrow$ Maschke ✓
- $\mathbb{C}$ algebraically closed

Recall that for an algebraically closed field $F$ and a finite dimensional algebra $A$ over $F$:
If $S$ is a simple $A$-module, then
$\mathrm{End}_A(S) = F$.

Applying this to $F = \mathbb{C}$, $A = \mathbb{C}[G]$ for a finite group $G$, we have $\mathrm{End}_{\mathbb{C}[G]}(V) = \mathbb{C}$ $\forall$ simple $\mathbb{C}[G]$-modules $V$.

For $F$ a general field, we know that for a simple $F[G]$-module $V$, we know $\mathrm{End}_{F[G]}(V)$ is a finite dimensional division algebra over $F$.
(It is finite dimensional because $\dim_F V < \infty$, so $\mathrm{End}_{F[G]}(V) \le \mathrm{End}_F(V)$, and $\mathrm{End}_F(V) \cong M_n(F)$ where $n = \dim_F V$).

**Theorem** (Frobenius)
The only finite dimensional division algebras over $\mathbb{R}$ are $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$. Proof omitted.

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Recall $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ $(x \in \mathbb{H} \Rightarrow x = a + bi + cj + dk)$

with multiplication: $i^2 = j^2 = k^2 = -1$

$ij = k$, $jk = i$, $ki = j$

$ji = -k$, $kj = -i$, $ik = -j$

Let $G$ be a finite group. Then by Mashke's Theorem

$\mathbb{R}[G] \cong S_1^{\oplus n_1} \oplus \dots \oplus S_d^{\oplus n_d}$ as $\mathbb{R}[G]$-modules

for some $d \geq 1$, simple $\mathbb{R}[G]$-modules $S_i$ where $S_i \cong S_j \Leftrightarrow i = j$

$$\Rightarrow \mathbb{R}[G]^{op} = End_{\mathbb{R}[G]}(\mathbb{R}[G])$$

$$\cong \prod_{i=1}^{r} End_{\mathbb{R}[G]}(S_i^{\oplus n_i})$$

$$\cong \prod_{i=1}^{r} M_{n_i}(\underbrace{End_{\mathbb{R}[G]}(S_i)}_{= \mathbb{R}, \mathbb{C}, \mathbb{H}})$$

Applying $(\cdot)^{op}$ we have

$\mathbb{R}[G] \cong M_{n_1}(\mathbb{R}) \times \dots \times M_{n_r}(\mathbb{R}) \times M_{p_1}(\mathbb{C}) \times \dots \times M_{p_s}(\mathbb{C}) \times M_{q_1}(\mathbb{H}) \times \dots \times M_{q_t}(\mathbb{H})$

for some $r, s, t \geq 0$, $n_i, p_i, q_i \geq 1$.

The number of non-isomorphic irreducible representations of $G$ over $\mathbb{R}$ is $r + s + t$.

It is not necessarily equal to the number of conjugacy classes

Taking $\dim_{\mathbb{R}}$ of both sides:

$$|G| = \sum_{i=1}^{r} n_i^2 + 2\sum_{i=1}^{s} p_i^2 + 4\sum_{i=1}^{t} q_i^2$$

$$\begin{bmatrix} \dim_{\mathbb{R}} M_n(\mathbb{R}) = n^2 \\ \dim_{\mathbb{R}} M_n(\mathbb{C}) = 2n^2 \\ \dim_{\mathbb{R}} M_n(\mathbb{H}) = 4n^2 \end{bmatrix}$$

Recall that for a division ring $D$,

$M_n(D) \cong (D^n)^{\oplus n}$ as $M_n(D)$-modules, where $D^n$ is the simple module of column vectors

$\Rightarrow$ the simple submodules of $\mathbb{R}[G]$ are of the form $\mathbb{R}^{n_i}$, $\mathbb{C}^{p_i}$, $\mathbb{H}^{q_i}$, so the dimensions of the irreducible representations of $G$ over $\mathbb{R}$ are $n_i$, $2p_i$, $4q_i$

$$\left[\begin{aligned}\mathbb{R}[G] \cong M_{n_1}(\mathbb{R}) \times \dots \times M_{n_r}(\mathbb{R}) \\ \times M_{p_1}(\mathbb{C}) \times \dots \times M_{p_s}(\mathbb{C}) \\ \times M_{q_1}(\mathbb{H}) \times \dots \times M_{q_t}(\mathbb{H})\end{aligned}\right]$$

Calculating the centre of both sides:
$Z(\mathbb{R}[G])$ has dimension = # conjugacy classes of $G$
(same proof as over $\mathbb{C}$).

$Z(M_n(\mathbb{R})) = \mathbb{R} \cdot I_n = \mathbb{R}$ $\quad \dim_{\mathbb{R}} = 1$
$Z(M_n(\mathbb{C})) = \mathbb{C} \cdot I_n = \mathbb{C}$ $\quad \dim_{\mathbb{R}} = 2$
$Z(M_n(\mathbb{H})) = Z(\mathbb{H}) \cdot I_n = \mathbb{R} \cdot I_n = \mathbb{R}$ $\quad \dim_{\mathbb{R}} = 1$
$\Rightarrow$ # conjugacy classes $= r + 2s + t$.

Definition
An element $g$ of a group $G$ is called real if
$g$ is conjugate to $g^{-1}$.
We say that $g^G$ is real.
Note if $x \in g^G$ then $x = h^{-1}gh$ for some $h \in G$
If $g$ is real then $g^{-1} = k^{-1}gk$ for some $k \in G$.
$\Rightarrow \quad x^{-1} = h^{-1}g^{-1}h = h^{-1}k^{-1}gkh$
$\qquad = (kh)^{-1}gkh$
$\Rightarrow x^{-1} \in g^G$ and $x \in g^G$ $\quad$ so $x^{-1} \in x^G$
Therefore if $g^G$ is a real conjugacy class then every element of the conjugacy class is real.

Definition
A character $\chi : G \to \mathbb{C}$ is real if $\chi$ takes values in $\mathbb{R}$.

Theorem
The number of real irreducible characters of $G$ is equal to the number of real conjugacy classes of $G$.

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Recall:

$G$ finite group.

$g \in G$, say $g$ is <u>real</u> if $g^{-1} \in g^G$ and if $g$ is real we say the conjugacy class is real. ($\Rightarrow hxh^{-1} = g$)

If $x \in g^G$ where $g$ is real then $x = h^{-1}gh$ for some $h \in G$ and $\exists\, k \in G$ s.t. $g^{-1} = k^{-1}gk$ since $g$ is real

$$\begin{aligned} \Rightarrow x^{-1} &= h^{-1}g^{-1}h \\ &= h^{-1}k^{-1}gkh = h^{-1}k^{-1}hxh^{-1}kh \\ &= (h^{-1}kh)^{-1}x(h^{-1}kh) \quad \text{so } x \text{ is real.} \end{aligned}$$

<u>Def</u>

A complex character is real (or real valued) if $\chi : G \to \mathbb{C}$ takes values in $\mathbb{R}$.

<u>Theorem</u>

The number of real irreducible ~~complex~~ characters of a finite group $G$ is equal to the number of real conjugacy classes of $G$.

Before proving the theorem, we make some remarks about permutation matrices.

<u>Definition</u>

Let $\sigma \in S_n$. We define the matrix $P = P_\sigma \in M_n(\mathbb{Z})$ by $P = (p_{ij})$ with $p_{ij} = \begin{cases} 1 & \text{if } \sigma(j) = i \\ 0 & \text{otherwise} \end{cases}$

We call such matrices permutation matrices.

Let $R$ be any ring. Note that if $A \in M_n(R)$ then $A = (a_{ij})$, $(PA)_{ik} = \sum_{j=1}^{n} p_{ij}a_{jk}$ $\quad p_{ij} \neq 0 \Leftrightarrow \sigma(j) = i \Leftrightarrow j = \sigma^{-1}(i)$

$= a_{\sigma^{-1}(i)k}$

i.e. left multiplication by $P_\sigma$ permutes the rows of $A$ by $\sigma$

i.e. $(PA)_{\sigma(i)k} = a_{ik}$

Also $(AP)_{ik} = \sum_{j=1}^{n} a_{ij} p_{jk} = a_{i\,\sigma^{-1}(k)}$

? $p_{jk} \neq 0 \iff \sigma(k) = j \iff j = \sigma^{-1}(k)$

i.e. right multiplication by $P$ permutes the columns of $A$ by $\sigma$.

Remark

$\mathrm{Tr}(P_\sigma) = \sum_{i=1}^{n} p_{ii} = \#\{1 \le i \le n : \sigma(i) = i\}$

Proof of Thm

Let $X = (\chi_i(g_j))_{i,j} \in M_r(\mathbb{C})$ be the matrix of the character table of $G$, i.e. $\chi_1, \dots, \chi_r$ are the irreducible characters of $G$ and $g_1, \dots, g_r$ are conjugacy class representatives.

Let $\overline{X} = (\overline{\chi_i(g_j)})_{i,j}$ be its complex conjugate.

Recall that if $\chi$ is an irreducible character, then $\overline{\chi}$ is a character because it is the character of the dual representation corresponding to $\chi$.

$\overline{\chi}$ is irreducible because $\langle \overline{\chi}, \overline{\chi} \rangle_G = \overline{\langle \chi, \chi \rangle_G} = \overline{1} = 1$

In particular $\exists$ a permutation matrix $P$ such that $PX = \overline{X}$.

We also have $\overline{\chi_i(g_j)} = \chi_i(g_j^{-1}) = \chi_i(g_k)$ where $g_j^{-1} \in g_k^G$

so there exists a permutation matrix $Q$ s.t.

$XQ = \overline{X}$.

Note that $X$ is invertible, because its rows are linearly independent by row orthogonality

$\sum a_i \chi_i = 0$ then $a_j = \langle \sum_{i=1}^{r} a_i \chi_i, \chi_j \rangle = 0$

$PX = XQ \Rightarrow X^{-1} P X = Q$

$\Rightarrow \mathrm{Tr}(P) = \mathrm{Tr}(Q)$

But $\mathrm{Tr}(P) = \#$ real irreducible characters of $G$

and $\mathrm{Tr}(Q) = \#$ real conjugacy classes of $G$

$\square$

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Definition

Suppose that $\chi : G \to \mathbb{C}$ is a character.

We say that $\chi$ is realisable over $\mathbb{R}$ if there exists some representation $\rho : G \to GL_n(\mathbb{R})$ with $\underbrace{\chi_\rho(g)}_{= Tr(\rho(g))} = \chi(g) \quad \forall g \in G$

If $\chi$ is realisable over $\mathbb{R}$ then $\chi$ is real, but we'll show that the converse fails.

$\mathbb{R}[G]$-modules and $\mathbb{C}[G]$-modules

Given a real representation $\rho : G \to GL_n(\mathbb{R})$ then we obtain a complex representation $\rho_\mathbb{C} : G \to GL_n(\mathbb{C})$, $g \mapsto \rho(g)$ (using $GL_n(\mathbb{R}) \subset GL_n(\mathbb{C})$).

In terms of modules:

If $V$ is a finitely generated $\mathbb{R}[G]$-module of dimension $d$ over $\mathbb{R}$, then if we consider $\mathbb{C}$ as a 2-dimensional $\mathbb{R}$-vector space, we can form the $\mathbb{R}$-vector space of dimension $2d$;

$$V_\mathbb{C} := V \otimes_\mathbb{R} \mathbb{C}$$

But $V_\mathbb{C}$ has the structure of a complex vector space.

If $\lambda \in \mathbb{C}$ and $\sum_{i=1}^{d} a_i(v_i \otimes 1) + \sum_{i=1}^{d} b_i(v_i \otimes \sqrt{-1}) = \sum_{i=1}^{d} v_i \otimes (a_i + b_i\sqrt{-1}) \in V_\mathbb{C}$

We define $\lambda \cdot \sum_{i=1}^{d} v_i \otimes (a_i + b_i\sqrt{-1}) = \sum_{i=1}^{d} v_i \otimes \lambda(a_i + b_i\sqrt{-1})$

So $V_\mathbb{C}$ is a $d$-dimensional $\mathbb{C}$-vector space with basis $\{v_1 \otimes 1, \dots, v_d \otimes 1\}$ where $\{v_1, \dots, v_d\}$ is a basis of $V$ over $\mathbb{R}$.

$V_\mathbb{C}$ has the structure of a $\mathbb{C}[G]$-module

If $g(v_j) = \sum_{i=1}^{d} a_{ij} v_i$ then $g(v_j \otimes 1) = \sum_{i=1}^{d} a_{ij}(v_i \otimes 1)$

So $Tr_V(g) = \sum_{i=1}^{d} a_{ii} = Tr_{V_\mathbb{C}}(g)$

Prop

If $V$ is an $\mathbb{R}[G]$-module with character $\chi$, then $V_{\mathbb{C}}$ is a $\mathbb{C}[G]$-module with character $\chi$.

We can also construct an $\mathbb{R}[G]$-module from a $\mathbb{C}[G]$-module.
Suppose $V$ is a finitely generated $\mathbb{C}[G]$-module with $\mathbb{C}$-basis $v_1, \dots, v_d$.
Then for each $g \in G$ $\exists Z = (z_{jk}) \in GL_d(\mathbb{C})$ st.

$$g(v_k) = \sum_{j=1}^{d} z_{jk} v_j$$

Write $V_{\mathbb{R}}$ for $V$ considered as an $\mathbb{R}$-vector space.
$V_{\mathbb{R}}$ is $2d$-dimensional with basis $v_1, \sqrt{-1}v_1, v_2, \sqrt{-1}v_2, \dots, v_d, \sqrt{-1}v_d$.
Write each $z_{jk} = x_{jk} + \sqrt{-1}\, y_{jk}$, $x_{jk}, y_{jk} \in \mathbb{R}$,
then

$$g(v_k) = \sum_{j=1}^{d}(x_{jk} + \sqrt{-1}\, y_{jk}) v_j$$

$$= \sum_{j=1}^{d}(x_{jk} v_j) + \sum_{j=1}^{d}(y_{jk}\sqrt{-1}\, v_j)$$

and

$$g(\sqrt{-1}\, v_k) = \sum_{j=1}^{d}\sqrt{-1}\,(x_{jk} + \sqrt{-1}\, y_{jk}) v_j$$

$$= \sum_{j=1}^{d}(-y_{jk} + \sqrt{-1}\, x_{jk}) v_j$$

$$= \sum_{j=1}^{k}(-y_{jk} v_j) + \sum_{j=1}^{d}(x_{jk}\sqrt{-1}\, v_j)$$

In terms of matrices:
The matrix of $g: V_{\mathbb{R}} \to V_{\mathbb{R}}$, $v \mapsto gv$, is given by

$$\begin{pmatrix} \begin{pmatrix} x_{11} & -y_{11} \\ y_{11} & x_{11} \end{pmatrix} & \cdots & \\ \begin{pmatrix} x_{21} & -y_{21} \\ y_{21} & x_{21} \end{pmatrix} & \ddots & \vdots \\ \vdots & & \begin{pmatrix} x_{dd} & -y_{dd} \\ y_{dd} & x_{dd} \end{pmatrix} \end{pmatrix}$$

"replace each $z$ with $\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$ where $z = x + \sqrt{-1}\, y$."

$$\Rightarrow Tr_{V_{\mathbb{R}}}(g) = 2x_{11} + \dots + 2x_{dd} = 2\,Re(z_{11}) + \dots + 2\,Re(z_{dd})$$

$$= 2\,Re(z_{11} + \dots + z_{dd}) = 2\,Re(Tr_V(g)) = Tr_V(g) + \overline{Tr_V(g)}$$

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i.e. if $V$ has character $\chi$, then $V_{\mathbb{R}}$ has character $\chi + \overline{\chi}$.

16-03-18 Given $V$ a finitely generated $\mathbb{C}[G]$-module we constructed an $\mathbb{R}[G]$-module $V_{\mathbb{R}}$

In terms of modules, $V_{\mathbb{R}} = V$ given the structure of an $\mathbb{R}[G]$-module using $\mathbb{R}[G] \subseteq \mathbb{C}[G]$.

In terms of matrices, if $V$ corresponds to $\rho : G \to GL_n(\mathbb{C})$, then $V_{\mathbb{R}}$ corresponds to $\rho_{\mathbb{R}} : G \to GL_{2n}(\mathbb{R})$, which is the composition $\rho : G \to GL_n(\mathbb{C})$ with $GL_n(\mathbb{C}) \to GL_{2n}(\mathbb{R})$, $(a_{ij}) \mapsto \begin{pmatrix} Re(a_{ij}) & -Im(a_{ij}) \\ Im(a_{ij}) & Re(a_{ij}) \end{pmatrix}$

Given $V$ a finitely generated $\mathbb{R}[G]$-module, we constructed $V_{\mathbb{C}}$ a $\mathbb{C}[G]$-module.

In terms of modules, $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$

In terms of matrices, if $V$ corresponds to $\rho : G \to GL_n(\mathbb{R})$, then $V_{\mathbb{C}}$ corresponds to $\rho_{\mathbb{C}} : G \to GL_n(\mathbb{C})$, using $GL_n(\mathbb{R}) \subseteq GL_n(\mathbb{C})$.

<u>~~Prop~~</u>

If $V$ is a finitely generated $\mathbb{C}[G]$-module with character $\chi$, then

(i) $V_{\mathbb{R}}$ is a f.g. $\mathbb{R}[G]$-module with character $\chi + \overline{\chi}$ (in particular $\dim_{\mathbb{R}} V_{\mathbb{R}} = 2 \dim_{\mathbb{C}} V$)

[only happens in real valued case] (ii) If $V$ is a simple $\mathbb{C}[G]$-module and $V_{\mathbb{R}}$ is not simple as an $\mathbb{R}[G]$-module, then $\chi$ can be realised over $\mathbb{R}$.

<u>Proof</u>

(i) We have already proved this

(ii) Suppose $V$ is a simple $\mathbb{C}[G]$-module with character $\chi$ and suppose that $V_{\mathbb{R}} = U \oplus W$ as $\mathbb{R}[G]$-modules with $U, W \neq 0$.

Consider $(V_{\mathbb{R}})_{\mathbb{C}} = U_{\mathbb{C}} \oplus W_{\mathbb{C}}$

$(V_{\mathbb{R}})_{\mathbb{C}}$ has character $\chi + \bar{\chi}$. Since $\chi$ and $\bar{\chi}$ are both irreducible characters, so $(V_{\mathbb{R}})_{\mathbb{C}} \cong S_1 \oplus S_2$ for simple $\mathbb{C}[G]$-modules $S_1, S_2$.

But $(V_{\mathbb{R}})_{\mathbb{C}} \cong U_{\mathbb{C}} \oplus W_{\mathbb{C}}$, and since decomposition into simple $\mathbb{C}[G]$-modules is unique, we must have $U_{\mathbb{C}}, W_{\mathbb{C}}$ are simple, $U_{\mathbb{C}} \cong S_1$, $W_{\mathbb{C}} \cong S_2$ after reordering.

Therefore $U_{\mathbb{C}}$ has character $\chi$.

But $U_{\mathbb{C}}$ has the same character as $U$, so $U$ has character $\chi$.

$\Rightarrow \chi$ is realisable over $\mathbb{R}$.

$\square$

<u>Examples</u>

1). $G = C_3 = \langle x : x^3 = 1 \rangle$

Let $V$ be the 1-dimensional $\mathbb{C}[G]$-module with basis $v$ and $x \cdot v = \omega v$ where $\omega = \exp(2\pi i/3) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

Then $V_{\mathbb{R}}$ has basis $v, iv$ and w.r.t. this basis, $x$ acts by the matrix
$$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

Since the character of $V$ is not real valued, $V_{\mathbb{R}}$ is simple.

$\Rightarrow \rho : C_3 \to GL_2(\mathbb{R})$ is an irreducible real representation of $C_3$ $\left[ x \longmapsto \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \right]$.

2). Let $G = D_8 = \langle x, y : x^4 = y^2 = 1, \ xy = yx^3 \rangle$

Let $V$ be the $\mathbb{C}[G]$-module corresponding to $\rho : G \to GL_2(\mathbb{C})$, $x \longmapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $y \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $tr(x) = 0$, $tr(y) = 0$.

$V$ is simple e.g. by character theory.

By calculation, $\rho$ has a real character.

$V_{\mathbb{R}}$ is the $\mathbb{R}[G]$-module corresponding to $\rho_{\mathbb{R}} : D_8 \to GL_4(\mathbb{R})$
$$x \longmapsto \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad y \longmapsto \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

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Claim:

$U = \left\{ \begin{pmatrix} a \\ b \\ b \\ a \end{pmatrix} : a, b \in \mathbb{R} \right\}$ is a subrepresentation

$\rho_{\mathbb{R}}(x) \begin{pmatrix} a \\ b \\ b \\ a \end{pmatrix} = \begin{pmatrix} -b \\ a \\ a \\ -b \end{pmatrix} \in U$ and $\rho_{\mathbb{R}}(y) \begin{pmatrix} a \\ b \\ b \\ a \end{pmatrix} = \begin{pmatrix} b \\ a \\ a \\ b \end{pmatrix} \in U$

so $U$ is a subrepresentation

$\Rightarrow V_{\mathbb{R}}$ is not simple

$\Rightarrow$ The character of $V$ is realisable over $\mathbb{R}$

Bilinear forms

$V$ a vector space over $F$ $(= \mathbb{R}$ or $\mathbb{C})$

Recall:

A bilinear form $\beta$ on $V$ is a map

$\beta : V \times V \to F$ s.t. $\beta(-, v_2) : V \to F$, $v_1 \to \beta(v_1, v_2)$

is $F$-linear $\forall v_2 \in V$ and $\beta(v_1, -) : V \to F$, $v_2 \to \beta(v_1, v_2)$

is $F$-linear $\forall v_1 \in V$.

We say $\beta$ is symmetric if $\beta(v_1, v_2) = \beta(v_2, v_1)$ $\forall v_1, v_2 \in V$

We say $\beta$ is skew-symmetric if $\beta(v_2, v_1) = -\beta(v_1, v_2)$ $\forall v_1, v_2 \in V$.

If $V$ is an $F[G]$-module for a finite group $G$,

then we say $\beta$ is $G$-invariant if

$\beta(gv_1, gv_2) = \beta(v_1, v_2)$ $\forall v_1, v_2 \in V$, $\forall g \in G$

Theorem

If $V$ is a finitely generated $\mathbb{R}[G]$-module,

$G$ a finite group, then there exists a real

symmetric G-invariant bilinear form $\beta$ on $V$ s.t. $\beta(v, v) > 0$ $\forall 0 \neq v \in V$.

In particular there exists a non-zero real symmetric

$G$-invariant bilinear form on $V$.

Proof

If $v_1, \ldots, v_n$ is a basis for $V$ over $\mathbb{R}$, then for $u = \sum_{j=1}^{n} \lambda_j v_j$, $v = \sum_{j=1}^{n} \mu_j v_j \in V$ then we define $\beta_0(u,v) = \sum_{j=1}^{n} \lambda_j \mu_j$.

Then $\beta_0$ is symmetric and $\beta_0(u,u) = \sum_{j=1}^{n} \lambda_j^2 > 0 \Leftrightarrow \lambda_i$ not all zero $\Leftrightarrow u \neq 0$.

But $\beta_0$ may not be $G$-invarient.

Define $\beta(u,v) = \sum_{g \in G} \beta_0(gu, gv) \quad \forall u, v \in V$

Then $\beta$ is symmetric, $\beta(u,u) = \sum_{g \in G} \beta_0(gu, gu) > 0$
$[\Leftrightarrow gu$ not zero $\forall g \in G \Leftrightarrow u \neq 0]$
and $\beta$ is $G$ invarient. $\square$

Prop

Let $V$ be a f.g. $\mathbb{R}[G]$-module for a finite group $G$ and let $\beta$ be a $G$-invarient bilinear form on $V$.

If $U$ is an $\mathbb{R}[G]$-submodule of $V$ then so is $W = \{ w \in V : \beta(u,w) = 0 \ \forall u \in U \}$.

Proof

Clearly $W$ is an $\mathbb{R}$-subspace since $\beta$ is bilinear.

If $w \in W$ and $g \in G$ and $u \in U$ then

$\beta(u, gw) = \beta(g^{-1}u, w) = 0$

since $\beta$ is $G$-invarient and $g^{-1}u \in U$.

$\Rightarrow gw \in W \quad \forall w \in W, g \in G$

$\Rightarrow W$ is an $\mathbb{R}[G]$-submodule. $\square$

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Lemma

If $A \in M_n(\mathbb{R})$ is a real symmetric matrix then $\exists$ an orthogonal matrix $P \in GL_n(\mathbb{R})$ (i.e. $PP^t = I_n$) s.t. $P^tAP = D$ where $D$ is a diagonal matrix

Proof

Omitted.

Lemma

If $\beta$ is a symmetric bilinear form on $\mathbb{R}^n$ such that $\beta(v,v) > 0 \ \forall\ 0 \neq v \in \mathbb{R}^n$, then there exists a basis $g_1, \dots, g_n$ of $\mathbb{R}^n$ such that $\beta(g_i, g_j) = \delta_{ij}$.

Proof

Let $B \in M_n(\mathbb{R})$ be defined by $B = (b_{ij})$ where $b_{ij} = \beta(e_i, e_j)$ where $e_1, \dots, e_n$ is the standard basis of $\mathbb{R}^n$.

Then $\beta(u,v) = u^t B v \quad \forall\ u, v \in \mathbb{R}^n$

$B$ is symmetric because $\beta$ is symmetric.

By the previous lemma $\exists P$ s.t. $P^tP = I_n$ and $P^tBP = D$ for some diagonal matrix $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

Then $\beta(Pu, Pv) = (Pu)^t B (Pv) = u^t P^t B P v = u^t D v$

i.e. if we put $f_i = Pe_i$ for $i = 1, \dots, n$

then $\beta(f_i, f_j) = \beta(Pe_i, Pe_j) = e_i{}^t D e_j = \begin{cases} \lambda_i & , i = j \\ 0 & , i \neq j \end{cases}$

Note that $\beta(f_i, f_i) > 0$ by assumption on $\beta$, so $\lambda_i > 0 \ \forall i$

Now define $g_i = \dfrac{f_i}{\sqrt{\lambda_i}}$ for $i = 1, \dots, n$

Then $\beta(g_i, g_j) = \delta_{ij}$. $\square$

**Prop**
Suppose $\gamma$ is a $G$-invariant symmetric bilinear form on an $\mathbb{R}[G]$-module $V$ and suppose $\exists\, u, v \in V$ st. $\gamma(u,u) > 0$ but $\gamma(v,v) < 0$.
Then $V$ is reducible (not simple) as an $\mathbb{R}[G]$-module.

**Proof**
Chose $\beta$ a $G$-invariant symmetric bilinear form on $V$ st. $\beta(w,w) > 0$, $\forall\, 0 \neq w \in V$.
Choose a basis $e_1, \dots, e_n$ for $V$ s.t. $\beta(e_i, e_j) = \delta_{ij}$
Let $C = (c_{ij}) \in M_n(\mathbb{R})$ be given by $c_{ij} = \gamma(e_i, e_j)$
Choose $P$ orthogonal st. $P^t C P = D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$
Then w.r.t. the basis $Pe_i = f_i$, $i = 1, \dots, n$, the matrix of $\beta$ is $P^t I_n P = I_n$ since $P$ is orthogonal
and the matrix for $\gamma$ is $P^t C P = D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$
i.e. $\beta(f_i, f_j) = \delta_{ij}$

$$\gamma(f_i, f_j) = \begin{cases} \lambda_i, & i = j \\ 0, & i \neq j \end{cases}$$

Since $\exists\, u$ st. $\gamma(u,u) > 0$ $\exists\, i$ s.t. $\lambda_i > 0$, say $\lambda_1 > 0$
Since $\exists\, v$ s.t. $\gamma(v,v) < 0$ $\exists\, i$ s.t. $\lambda_i < 0$, say $\lambda_2 < 0$.
So $\gamma(f_1, f_1) = \lambda_1 > 0$, $\gamma(f_2, f_2) = \lambda_2 < 0$.
Now consider the bilinear form

$$\delta(w, w') = \beta(w, w') - \frac{1}{\lambda_1}\gamma(w, w').$$

Then $\delta$ is $G$-invariant and symmetric since both $\beta$ and $\gamma$ are.
If $w = \sum_{i=1}^{n} a_i f_i \in V$ then

$$\delta(w, f_1) = \beta(w, f_1) - \frac{1}{\lambda_1}\gamma(w, f_1) = a_1 - \frac{\lambda_1 a_1}{\lambda_1} = 0.$$

So if we define

$$W := \{ w \in V \text{ st. } \delta(w, w') = 0 \ \forall\, w' \in V \}$$

then $W$ is an $\mathbb{R}[G]$-module, $W \neq 0$ since $f_1 \in W$.

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However $W \neq V$ since $\delta(f_2, f_2) = \beta(f_2, f_2) - \frac{1}{\lambda_1}\gamma(f_2, f_2)$
$= 1 - \frac{\lambda_2}{\lambda_1} > 1 \neq 0$

so $f_2 \notin W$.
Then $W \leq V$ is a non-trivial $\mathbb{R}[G]$-submodule of $V$ so $V$ is not simple. $\square$

**Theorem**
Let $\chi$ be an irreducible complex character of a finite group $G$. Then the following are equivalent
(i) $\chi$ can be realised over $\mathbb{R}$
(ii) There exists a $\mathbb{C}[G]$-module $V$ with character $\chi$ and a non-zero $G$-invariant symmetric bilinear form, $\beta: V \times V \to \mathbb{C}$

**Proof**
(ii) $\Rightarrow$ (i)
Let $V$ be a $\mathbb{C}[G]$-module with character $\chi$ and suppose $\beta: V \times V \to \mathbb{C}$ is a non-zero $G$-invarient symmetric bilinear form.
Take $u, v \in V$ s.t. $\beta(u,v) = \beta(v,u) \neq 0$
Since $\beta(u+v, u+v) = \beta(u,u) + 2\beta(u,v) + \beta(v,v)$
there exists $w \in V$ s.t. $\beta(w,w) \neq 0$
(can take $w$ to be one of $u, v$ or $u+v$).
If $\beta(w,w) = z$, then $\beta(z^{-1/2}w, z^{-1/2}w) = 1$
Put $v_1 := z^{-1/2}w$.
Extend $v_1$ to a basis $v_1, \dots, v_n$ of $V$.
Then $V_{\mathbb{R}}$ has basis $v_1, iv_1, \dots, v_n, iv_n$
Define $\tilde{\beta}: V_{\mathbb{R}} \times V_{\mathbb{R}} \to \mathbb{R}$, $(v,w) \mapsto Re(\beta(v,w))$
This defines a $G$-invarient symmetric bilinear form on $V_{\mathbb{R}}$.
Also $\tilde{\beta}(v_1, v_1) = Re(\beta(v_1, v_1)) = Re(1) = 1$
and $\tilde{\beta}(iv_1, iv_1) = Re(\beta(iv_1, iv_1)) = Re(i^2\beta(v_1, v_1)) = Re(-1) = -1$

So $V_{\mathbb{R}}$ has a $G$-invariant symmetric bilinear form $\tilde{\beta}$ and $u, v \in V_{\mathbb{R}}$ with $\tilde{\beta}(u,u) > 0$, $\tilde{\beta}(v,v) < 0$

$\Rightarrow V_{\mathbb{R}}$ is not simple

$\Rightarrow \chi$ is realisable over $\mathbb{R}$

(i) $\Rightarrow$ (ii)

Suppose $\chi$ is realisable over $\mathbb{R}$.

Let $U$ be an $\mathbb{R}[G]$-module with character $\chi$

Let $\beta : U \times U \to \mathbb{R}$ be a nonzero $G$-invariant symmetric bilinear form

Let $V = U_{\mathbb{C}}$ then $V$ has character $\chi$.

If $U$ has basis $u_1, \dots, u_n$ over $\mathbb{R}$ then $V = U_{\mathbb{C}}$ has basis $u_1, \dots, u_n$ over $\mathbb{C}$.

Then define $\tilde{\beta}_1 : V \times V \to \mathbb{C}$, $\left(\sum_{j=1}^{n} \lambda_j u_j, \sum_{k=1}^{n} \mu_k u_k\right) \mapsto \sum_{j,k} \lambda_j \mu_k \beta(u_j, u_k)$ for $\lambda_j, \mu_k \in \mathbb{C}$.

Then $\tilde{\beta}_1$ is a $G$-invariant symmetric bilinear form on $V$

It is non-zero since $\tilde{\beta}_1(u_1, u_1) = \beta(u_1, u_1) > 0$ (since $u_1 \neq 0$ and by choice of $\beta$). $\square$

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**Theorem**

Let $\chi$ be an irreducible complex character of a finite group $G$. The following are equivalent:

1). $\chi$ is realisable over $\mathbb{R}$

2). There exists a $\mathbb{C}[G]$ module $V$ with character $\chi$ and a non-zero $G$-invarient symmetric bilinear form $\beta: V \times V \to \mathbb{C}$

**Notation**

For a $\mathbb{C}$-vector space, let

$Bil(V, \mathbb{C}) = \{\beta: V \times V \to \mathbb{C} \text{ bilinear forms}\}$

and if $V$ is a $\mathbb{C}[G]$-module for a finite group $G$

$Bil_G(V, \mathbb{C}) = \{\beta \in Bil(V, \mathbb{C}) : \beta \text{ } G\text{-invarient}\}$ $\quad (\beta(gu, gv) = \beta(u,v))$

both are $\mathbb{C}$-vector spaces with

$(\beta + \beta')(u,v) = \beta(u,v) + \beta'(u,v)$, $\quad (\lambda\beta)(u,v) = \lambda\beta(u,v)$ $\quad \forall u,v \in V,$

$\beta, \beta' \in Bil(V, \mathbb{C})$

Recall that we have a bijection

$Hom_{\mathbb{C}}(V \otimes V, \mathbb{C}) \xrightarrow{\simeq} Bil(V, \mathbb{C})$

$b \mapsto \beta \qquad \beta(u,v) = b(u \otimes v)$

$b \leftarrow \beta$

$b(u \otimes v) = \beta(u,v)$ and extend linearly to $V \otimes V$.

Clearly this is an isomorphism of $\mathbb{C}$-vector spaces.

Suppose $V$ is a finitely generated $\mathbb{C}[G]$-module, $G$ a finite group.

**Lemma**

Under the above isomorphism, $Hom_{\mathbb{C}[G]}(V \otimes V, \mathbb{C})$ are identified with $Bil_G(V, \mathbb{C})$

**Proof**

If $b \in Hom_{\mathbb{C}[G]}(V \otimes V, \mathbb{C})$ then $\underbrace{b(g(u \otimes v))}_{= b(gu \otimes gv)} = g\, b(u \otimes v) = b(u \otimes v)$

since $G$ is acting trivially on $\mathbb{C}$

If $\beta$ is the corresponding bilinear form, we have
$\beta(gu, gv) = b(gu \otimes gv) = b(u \otimes v) = \beta(u, v)$.
Conversely if $\beta \in Bil_G(V, \mathbb{C})$, let $b$ be the corresponding element of $Hom_{\mathbb{C}}(V \otimes V, \mathbb{C})$.
Then if $u, v \in V$, $g \in G$, $\underset{\overset{\parallel}{b(g(u \otimes v))}}{b(gu \otimes gv)} = \beta(gu, gv) = \beta(u, v) = b(u \otimes v)$

Every element $x \in V \otimes V$ is of the form $x = \sum_{i=1}^{n} u_i \otimes v_i$
Then $b(gx) = b\left(\sum_{i=1}^{n} gu_i \otimes gv_i\right)$

$$= \sum_{i=1}^{n} b(gu_i \otimes gv_i) = \sum_{i=1}^{n} b(u_i \otimes v_i)$$

$$= b\left(\sum_{i=1}^{n} u_i \otimes v_i\right) = b(x)$$

since $b$ is $\mathbb{C}$-linear.
$\Rightarrow b \in Hom_{\mathbb{C}[G]}(V \otimes V, \mathbb{C})$. $\square$

Recall $V \otimes V = S^2V \oplus \Lambda^2 V$ as $\mathbb{C}[G]$-modules
where $S^2V = \{x \in V \otimes V \text{ s.t. } \sigma(x) = x\}$
$\Lambda^2 V = \{x \in V \otimes V \text{ s.t. } \sigma(x) = -x\}$ where $\sigma : V \otimes V \to V \otimes V$, $u \otimes v \mapsto v \otimes u$.

**Lemma**
We have an isomorphism of $\mathbb{C}$-vector spaces
$$Hom_{\mathbb{C}[G]}(V \otimes V, \mathbb{C}) = Hom_{\mathbb{C}[G]}(S^2V, \mathbb{C}) \oplus Hom_{\mathbb{C}[G]}(\Lambda^2 V, \mathbb{C})$$

$$\phi \longmapsto (\phi|_{S^2V}, \phi|_{\Lambda^2 V})$$

$$\phi \longleftarrow (\alpha, \beta)$$

$\phi(x+y) = \alpha(x) + \beta(y)$
where $x \in S^2V$, $y \in \Lambda^2 V$.

**Remark**
We can identify $Hom_{\mathbb{C}[G]}(S^2V, \mathbb{C})$ with
$\{\phi \in Hom_{\mathbb{C}[G]}(V \otimes V, \mathbb{C}) \text{ s.t. } \phi|_{\Lambda^2 V} = 0\}$
and we can identify $Hom_{\mathbb{C}[G]}(\Lambda^2 V, \mathbb{C})$ with
$\{\phi \in Hom_{\mathbb{C}[G]}(V \otimes V, \mathbb{C}) \text{ s.t. } \phi|_{S^2V} = 0\}$

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Define the $\mathbb{C}$-vector spaces

$$\mathrm{Bil}_G(V,\mathbb{C})_S := \{\beta \in \mathrm{Bil}_G(V,\mathbb{C}) : \beta \text{ is symmetric}\} \quad (\beta(u,v)=\beta(v,u))$$

$$\mathrm{Bil}_G(V,\mathbb{C})_\Lambda := \{\beta \in \mathrm{Bil}_G(V,\mathbb{C}) : \beta \text{ is skew-symmetric}\} \quad (\beta(u,v)=-\beta(v,u))$$

Lemma

We have $\mathrm{Bil}_G(V,\mathbb{C}) = \mathrm{Bil}_G(V,\mathbb{C})_S \oplus \mathrm{Bil}_G(V,\mathbb{C})_\Lambda$

Proof

Given $\beta \in \mathrm{Bil}_G(V,\mathbb{C})$, define $\beta' \in \mathrm{Bil}_G(V,\mathbb{C})$ by $\beta'(u,v)=\beta(v,u)$.

Then $\beta = \underbrace{\frac{\beta+\beta'}{2}}_{\in \mathrm{Bil}_G(V,\mathbb{C})_S} + \underbrace{\frac{\beta-\beta'}{2}}_{\in \mathrm{Bil}_G(V,\mathbb{C})_\Lambda}$

Moreover if $\beta \in \mathrm{Bil}_G(V,\mathbb{C})_S \cap \mathrm{Bil}_G(V,\mathbb{C})_\Lambda$, then

$$\beta(v,u) \underset{\text{symmetric}}{=} \beta(u,v) \underset{\text{skew-symmetric}}{=} -\beta(v,u)$$

$\Rightarrow \beta(v,u) = -\beta(v,u) \quad \forall u,v \in V \Rightarrow \beta(v,u)=0 \quad \forall u,v \in V$

$\Rightarrow \beta = 0$

$\square$

Lemma

The bijection $\varphi : \mathrm{Hom}_{\mathbb{C}[G]}(V\otimes V, \mathbb{C}) \xrightarrow{\sim} \mathrm{Bil}_G(V,\mathbb{C})$

gives us isomorphisms

$$\varphi_S : \mathrm{Hom}_{\mathbb{C}[G]}(S^2V, \mathbb{C}) \longrightarrow \mathrm{Bil}_G(V,\mathbb{C})_S$$

$$\varphi_\Lambda : \mathrm{Hom}_{\mathbb{C}[G]}(\Lambda^2V, \mathbb{C}) \longrightarrow \mathrm{Bil}_G(V,\mathbb{C})_\Lambda.$$

Proof

For $\varphi_S$: (proof for $\varphi_\Lambda$ is similar).

Recall that $\mathrm{Hom}_{\mathbb{C}[G]}(S^2V,\mathbb{C}) = \{\phi \in \mathrm{Hom}_{\mathbb{C}[G]}(V\otimes V,\mathbb{C}) : \phi|_{\Lambda^2V} = 0\}$

Let $\phi \in \mathrm{Hom}_{\mathbb{C}[G]}(V\otimes V,\mathbb{C})$ and let $\beta = \varphi(\phi)$.

Then $\phi|_{\Lambda^2V} = 0 \Leftrightarrow \phi(u\otimes v - v\otimes u) = 0 \quad \forall u,v \in V$

(since elements of this form span $\Lambda^2V$).

$\Leftrightarrow \phi(u\otimes v) - \phi(v\otimes u) \quad \forall u,v \in V$

$\Leftrightarrow \beta(u,v) = \beta(v,u) \quad \forall u,v \in V \quad \Leftrightarrow \beta \in \mathrm{Bil}_G(V,\mathbb{C})_S$

$\square$

We can count $\dim_{\mathbb{C}} \operatorname{Hom}_{\mathbb{C}[G]}(S^2V, \mathbb{C})$ and $\dim_{\mathbb{C}} \operatorname{Hom}_{\mathbb{C}[G]}(\Lambda^2 V, \mathbb{C})$ using character theory.

For a finite group $G$, let $e_G = \frac{1}{|G|} \sum_{g \in G} g \in \mathbb{C}[G]$

For finitely generated $\mathbb{C}[G]$-modules $U, V, W$ with charaters $\chi_U, \chi_V, \chi_W$ we have

- $W^G := e_G W$ and $\dim_{\mathbb{C}} W^G = Tr_W(e_G)$ $\quad e_G : W \to W,\ w \mapsto e_G w$
$$= \frac{1}{|G|} \sum_{g \in G} \chi_W(g)$$

- $\operatorname{Hom}_{\mathbb{C}[G]}(U, V) = \operatorname{Hom}_{\mathbb{C}}(U, V)^G$

- $\operatorname{Hom}_{\mathbb{C}}(U, V) \cong U^* \otimes V$ has character $\overline{\chi_U} \chi_V$

$\Rightarrow \dim_{\mathbb{C}}(\operatorname{Hom}_{\mathbb{C}[G]}(U, V)) = \langle \chi_U, \chi_V \rangle_G$

It follows that we have:

**Theorem**

Let $\mathbb{1}$ be the trivial character, then

(i) $\dim_{\mathbb{C}} Bil_G(V, \mathbb{C})_S = \dim_{\mathbb{C}} \operatorname{Hom}_{\mathbb{C}[G]}(S^2 V, \mathbb{C})$
$$= \langle \mathbb{1}, \chi_{S^2} \rangle_G$$

(ii) $\dim_{\mathbb{C}} Bil_G(V, \mathbb{C})_\Lambda = \dim_{\mathbb{C}} \operatorname{Hom}_{\mathbb{C}[G]}(\Lambda^2 V, \mathbb{C})$
$$= \langle \mathbb{1}, \chi_{\Lambda^2} \rangle_G$$

□

Suppose that $V$ is a simple $\mathbb{C}[G]$-module with character $\chi$.

By the theorem at the end of last lecture, $\chi$ is realisable over $\mathbb{R} \Leftrightarrow Bil_G(V, \mathbb{C})_S \neq 0$
$$\Leftrightarrow \langle \mathbb{1}, \chi_{S^2} \rangle_G \geq 1$$

Note that $\chi^2 = \chi_{S^2} + \chi_{\Lambda^2}$ and if $\chi$ is irreducible then
$$\underset{\text{formula for } \langle , \rangle}{\langle \mathbb{1}, \chi^2 \rangle_G =} \underset{\text{both irreducible characters}}{\langle \overline{\chi}, \chi \rangle} = \begin{cases} 1, & \chi = \overline{\chi} \quad \text{i.e. } \chi \text{ real valued} \\ 0, & \chi \neq \overline{\chi} \end{cases}$$

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Let $\langle \mathbb{1}, \chi_{S^2} \rangle = a \in \mathbb{Z}_{\geq 0}$

and $\langle \mathbb{1}, \chi_{\Lambda^2} \rangle = b \in \mathbb{Z}_{\geq 0}$

Then assuming $\chi$ is irreducible,

$$a+b = \begin{cases} 1 & \text{if } \chi \text{ is real valued} \\ 0 & \text{otherwise} \end{cases}$$

If $\chi$ is real valued either $a=1, b=0$ or $a=0, b=1$

Define the (Frobenius-) Schur indicator function of an irreducible complex character as:

$$\iota(\chi) = \begin{cases} 1 & \text{if } \langle \mathbb{1}, \chi_{S^2} \rangle = 1, \ \langle \mathbb{1}, \chi_{\Lambda^2} \rangle = 0 \\ 0 & \text{if } \langle \mathbb{1}, \chi_{S^2} \rangle = \langle \mathbb{1}, \chi_{\Lambda^2} \rangle = 0 \quad \leftarrow \chi \text{ not real valued} \\ -1 & \text{if } \langle \mathbb{1}, \chi_{S^2} \rangle = 0, \ \langle \mathbb{1}, \chi_{\Lambda^2} \rangle = 1 \end{cases}$$

Then we have

$$\iota(\chi) = \begin{cases} 1 & \text{if } \chi \text{ is realisable over } \mathbb{R} \\ 0 & \text{if } \chi \text{ not real valued} \\ -1 & \text{if } \chi \text{ is real valued but not realisable over } \mathbb{R} \end{cases}$$

Note that

$$\iota(\chi) = \langle \mathbb{1}, \chi_{S^2} \rangle - \langle \mathbb{1}, \chi_{\Lambda^2} \rangle$$
$$= \langle \mathbb{1}, \chi_{S^2} - \chi_{\Lambda^2} \rangle$$

Recall $\chi_{S^2}(g) = \frac{1}{2}(\chi(g)^2 + \chi(g^2))$

$\chi_{\Lambda^2}(g) = \frac{1}{2}(\chi(g)^2 - \chi(g^2))$

so $(\chi_{S^2} - \chi_{\Lambda^2})(g) = \chi(g^2)$

$$\Rightarrow \iota(\chi) = \langle \mathbb{1}, \chi_{S^2} - \chi_{\Lambda^2} \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\chi(g^2)}$$

$$= \frac{1}{|G|} \sum_{g \in G} \chi(g^{-2}) = \frac{1}{|G|} \sum_{g \in G} \chi(g^2)$$

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Recall that if $G$ is a finite group and $\chi$ is an irreducible character, then

$$\iota(\chi) = \begin{cases} 1 & , \chi \text{ is realisable over } \mathbb{R} \\ 0 & , \chi \text{ is not real valued} \\ -1 & , \chi \text{ is real valued but not realisable over } \mathbb{R} \end{cases}$$

We have $\iota(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g^2)$

Definition

Suppose $G$ is a finite group and $\chi$ is an irreducible character, which is real valued.
If $\iota(\chi) = 1$ then we say $\chi$ is orthogonal.
If $\iota(\chi) = -1$ then we say $\chi$ is symplectic.

Example

| sizes | 1 | 1 | 2 | 2 | 2 |
|---|---|---|---|---|---|
| $Q_8$ reps | $e$ | $x^2$ | $x$ | $y$ | $xy$ |
| $\chi_1$ | 1 | 1 | 1 | 1 | 1 |
| $\chi_2$ | 1 | 1 | 1 | -1 | -1 |
| $\chi_3$ | 1 | 1 | -1 | 1 | -1 |
| $\chi_4$ | 1 | 1 | -1 | -1 | 1 |
| $\chi_5$ | 2 | -2 | 0 | 0 | 0 |

| sizes | 1 | 1 | 2 | 2 | 2 |
|---|---|---|---|---|---|
| $g$ | $e$ | $x^2$ | $x$ | $y$ | $xy$ |
| $g^2$ | $e$ | $e$ | $x^2$ | $x^2$ | $x^2$ |

$\Rightarrow \iota(\chi_5) = \frac{1}{8}(2 + 2 + 2(-2) + 2(-2) + 2(-2)) = 1$

So $\chi_5$ is symplectic.

Lemma

If $\chi$ is a linear character, $\iota(\chi) = \begin{cases} 1 & \text{if } \chi \text{ is real valued} \\ 0 & \text{otherwise} \end{cases}$

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Proof

$\iota(\chi) = \frac{1}{|G|} \sum \chi(g^2) = \frac{1}{|G|} \sum \chi(g)\chi(g)$

$$= \langle \chi, \bar{\chi} \rangle_G = \begin{cases} 1 & \text{if } \chi = \bar{\chi} \\ 0 & \text{otherwise} \end{cases} \quad \square$$

Remark*

If $\chi$ is an irreducible symplectic character with $\mathbb{C}[G]$-module $V$, then $V$ has a ($G$-invarient) skew-symmetric bilinear form.

This bilinear form is non-degenerate because $V$ is a simple $\mathbb{C}[G]$-module.

If $W$ is a $\mathbb{C}$-vector space of dimension $d$ with a non-degenerate skew-symmetric bilinear form, then $d$ is even.

In particular, $\iota(\chi) = -1$ only when $\chi(e) \in 2\mathbb{Z}$.

Let $G$ be a finite group and define $r_2 : G \to \mathbb{Z}$ to be $r_2(g) = \#\{h \in G : h^2 = g\}$ (ie. the size of the set)

We call an element an involution if $h^2 = e$, so $r_2(e)$ counts the number of involutions in $G$.

Lemma

$r_2$ is a class function.

Proof

If $g' = x^{-1}gx$ then if $h \in G$ we have

$h^2 = g \Leftrightarrow (x^{-1}hx)^2 = x^{-1}h^2x = x^{-1}gx = g'$

Hence $h \mapsto x^{-1}hx$ is a bijection between

$\{h \in G : h^2 = g\} \longrightarrow \{h' \in G : h'^2 = g'\} \Rightarrow r_2(g) = r_2(g')$

$\square$

## Theorem (Frobenius - Schur count of involutions)

$$r_2(g) = \sum_{\chi \text{ irreducible}} \iota(\chi)\chi(g)$$

## Proof

Since $r_2$ is a class function, we have

$$r_2 = \sum_{\chi \text{ irred.}} a_\chi \chi \quad \text{for some } a_\chi \in \mathbb{C}.$$

Then $a_\chi = \langle r_2, \chi \rangle_G = \frac{1}{|G|} \sum_{g \in G} r_2(g)\overline{\chi(g)}$

$$= \frac{1}{|G|} \sum_{g \in G} \sum_{\substack{h \in G \\ h^2 = g}} \overline{\chi(h^2)}$$

$$= \frac{1}{|G|} \sum_{h \in G} \overline{\chi(h^2)} = \overline{\iota(\chi)} = \iota(\chi) \quad (\text{since } \iota(\chi) \in \mathbb{R})$$

$\square$

## Corollary

Suppose that $G$ has no symplectic irreducible characters, then $r_2(e) \geq r_2(g) \quad \forall g \in G$.

## Proof

By assumption, $\iota(\chi) = 0$ or $1$ $\forall$ irreducible characters.

$$r_2(g) = |r_2(g)| = \left| \sum_{\chi \text{ irred}} \iota(\chi)\chi(g) \right|$$

$$|\chi(g)| \leq \chi(e) = \dim \chi$$

$$\leq \sum_{\chi \text{ irred}} \iota(\chi)|\chi(g)| \leq \sum_{\chi \text{ irred}} \iota(\chi)\chi(e) = r_2(e)$$

$\square$

## Artin-Wedderburn decomposition of $\mathbb{R}[G]$

Let $G$ be a finite group.

Recall that
$$\mathbb{R}[G] \cong M_{n_1}(\mathbb{R}) \times \dots \times M_{n_a}(\mathbb{R})$$
$$\times M_{p_1}(\mathbb{C}) \times \dots \times M_{p_b}(\mathbb{C})$$
$$\times M_{q_1}(\mathbb{H}) \times \dots \times M_{q_c}(\mathbb{H}) \qquad a, b, c \geq 0,\ n_i, p_i, q_i > 0.$$

Then $a+b+c$ is the number of irreducible reps of $G$ over $\mathbb{R}$ up to isomorphism.

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$a = $ # irreducible reps up to iso with $End_{\mathbb{R}[G]} = \mathbb{R}$

$b = $ " " " " " " " $End_{\mathbb{R}[G]} = \mathbb{C}$

$c = $ " " " " " " " $End_{\mathbb{R}[G]} = \mathbb{H}$

The dimensions of the irreducible reps are

$\dim_{\mathbb{R}}(\mathbb{R}^{n_i}) = n_i$, $\dim_{\mathbb{R}}(\mathbb{C}^{p_i}) = 2p_i$, $\dim_{\mathbb{R}}(\mathbb{H}^{q_i}) = 4q_i$

Taking $\dim_{\mathbb{R}}(LHS) = \dim_{\mathbb{R}}(RHS)$

$\Rightarrow |G| = (n_1^2 + \dots + n_a^2) + 2(p_1^2 + \dots + p_b^2) + 4(q_1^2 + \dots + q_c^2)$

Taking $\dim_{\mathbb{R}}(Z(LHS)) = \dim_{\mathbb{R}}(Z(RHS))$

# conj classes of $G = a + 2b + c$

$$\left[\begin{array}{l} Z(M_n(\mathbb{R})) = \mathbb{R} \\ Z(M_n(\mathbb{C})) = \mathbb{C} \\ Z(M_n(\mathbb{H})) = \mathbb{R} \quad (Z(\mathbb{H}) = \mathbb{R}) \end{array}\right]$$

**Theorem**

Let $G$ be a finite group and suppose its irreducible characters are $\chi_1, \dots, \chi_a, \psi_1, \overline{\psi_1}, \dots, \psi_b, \overline{\psi_b}, \varphi_1, \dots, \varphi_c$ where $\iota(\chi_i) = 1$, $\iota(\psi_i) = \iota(\overline{\psi_i}) = 0$, $\iota(\varphi_i) = -1$.

Let $\chi_i(e) = n_i$, $\psi_i(e) = p_i$, $\varphi_i(e) = 2q_i$ $\quad (q \in \frac{1}{2}\cancel{\mathbb{Z}})$

Then the real irreducible representations of $G$ have characters $\chi_1, \dots, \chi_a, \psi_1 + \overline{\psi_1}, \dots, \psi_b + \overline{\psi_b}, 2\varphi_1, \dots, 2\varphi_c$

and $\mathbb{R}[G] = M_{n_1}(\mathbb{R}) \times \dots \times M_{n_a}(\mathbb{R}) \times M_{p_1}(\mathbb{C}) \times \dots \times M_{p_b}(\mathbb{C}) \times M_{q_1}(\mathbb{H}) \times \dots \times M_{q_c}(\mathbb{H})$.

**Proof**

Note that $n_1^2 + \dots + n_a^2 + 2(p_1^2 + \dots + p_b^2) + 4(q_1^2 + \dots + q_c^2) = |G|$

- $\chi_i$ are realisable over $\mathbb{R}$ so $\exists$ simple $\mathbb{R}[G]$-module $U_i$ with character $\chi_i$. Also $1 = \langle \chi_i, \chi_i \rangle_G = \dim_{\mathbb{R}}(End_{\mathbb{R}[G]}(U_i))$

  $\Rightarrow End_{\mathbb{R}[G]}(U_i) = \mathbb{R}$

(Recall that if $V$ is a simple $\mathbb{C}[G]$-module and $V_{\mathbb{R}}$ is not a simple $\mathbb{R}[G]$-module then $\chi_V$ is realisable over $\mathbb{R}$.)

- Let $M_i$ be a $\mathbb{C}[G]$-module with character $\psi_i$ ($\psi_i$ not realisable)

  Then $V_i := (M_i)_{\mathbb{R}}$ is a simple $\mathbb{R}[G]$-module with character $\psi_i + \overline{\psi_i}$.

$2 = \langle \psi_i + \overline{\psi_i}, \psi_i + \overline{\psi_i} \rangle = \dim_{\mathbb{R}}(\mathrm{End}_{\mathbb{R}[G]}(V_i))$

$\Rightarrow \mathrm{End}_{\mathbb{R}[G]}(V_i) = \mathbb{C}$

Also $\dim_{\mathbb{R}}(V_i) = 2\dim_{\mathbb{C}}(M_i) = 2p_i$

• $\varphi_i$ is not realisable over $\mathbb{R}$, so $\exists$ simple $\mathbb{R}[G]$-module $W_i$ with character $\varphi_i + \overline{\varphi_i} = 2\varphi_i$

Note that $\langle 2\varphi_i, 2\varphi_i \rangle = 4 \Rightarrow \mathrm{End}_{\mathbb{R}[G]}(W_i) = \mathbb{H}$

$\dim_{\mathbb{R}}(W_i) = 2\varphi_i(e) = 4q_i$ and $W_i \cong \mathbb{H}^{d_i}$ for some $d_i$

So $\dim_{\mathbb{R}} W_i \in 4\mathbb{Z} \Rightarrow 4q_i \in 4\mathbb{Z}$

$\Rightarrow q_i \in \mathbb{Z}$

These characters are all distinct by considering $\langle \chi, \psi \rangle_G$

So $\mathbb{R}[G]$ has characters:

• $\chi_1, \ldots, \chi_a$, $\mathrm{End}_{\mathbb{R}[G]}(\ ) = \mathbb{R}$, $\dim_{\mathbb{R}}(\ ) = n_1, \ldots, n_a$

• $\psi_1 + \overline{\psi_1}, \ldots, \psi_b + \overline{\psi_b}$, $\mathrm{End}_{\mathbb{R}[G]}(\ ) = \mathbb{C}$, $\dim_{\mathbb{R}}(\ ) = 2p_1, \ldots, 2p_b$

• $2\varphi_1, \ldots, 2\varphi_c$, $\mathrm{End}_{\mathbb{R}[G]}(\ ) = \mathbb{H}$, $\dim_{\mathbb{R}}(\ ) = 4q_1, \ldots, 4q_c$

So $\mathbb{R}[G] \cong M_{n_1}(\mathbb{R}) \times \ldots \times M_{n_a}(\mathbb{R}) \times M_{p_1}(\mathbb{C}) \times \ldots \times M_{p_b}(\mathbb{C}) \times \ldots \times M_{q_1}(\mathbb{H}) \times \ldots \times M_{q_c}(\mathbb{H}) \times Q$

where $Q = \prod_{i=1}^{k} M_{n_i}(D_i)$, $D_i \cong \mathbb{R}, \mathbb{C}, \mathbb{H}$

but since (taking $\dim_{\mathbb{R}}(LHS) = \dim_{\mathbb{R}}(RHS)$)

$|G| = \sum n_i^2 + 2\sum p_i^2 + 4\sum q_i^2 + \dim_{\mathbb{R}}(Q)$

but $|G| = \sum n_i^2 + 2\sum p_i^2 + 4\sum q_i^2 \Rightarrow \dim_{\mathbb{R}}(Q) = 0$

$\square$

$$\left[\begin{array}{lll} \mathbb{C}[G] & \iota(\chi) & \\ \big\downarrow & 1 & M_n(\mathbb{C}) \rightsquigarrow M_n(\mathbb{R}) \\ & 0 & M_n(\mathbb{C}) \times M_n(\mathbb{C}) \rightsquigarrow M_n(\mathbb{C}) \\ & -1 & M_{2n}(\mathbb{C}) \rightsquigarrow M_n(\mathbb{H}) \\ \mathbb{R}[G] & & \end{array}\right]$$

<u>Example</u>

$G = C_n = \langle x \rangle$, where $n$ is odd

The irred. characters of $G$ are $\chi_0, \ldots, \chi_{n-1}$, $\chi_j(x) = \exp(\frac{2\pi i}{n} j)$

Since $n$ is odd the only real valued character is $\chi_0$.

$\chi_1, \ldots, \chi_{n-1}$ are not real-valued, $\iota(\chi) = 0$.

$\Rightarrow \mathbb{R}[C_n] = \mathbb{R} \times \underbrace{\mathbb{C} \times \ldots \times \mathbb{C}}_{\frac{n-1}{2}}$