M205 Topology and Groups Notes

Based on the 2016 spring lectures by Dr L Louder

Topology and Groups 11th Jan

Books:

Pointset topology - mumkres

Intro - massey good for covering spaces.

Crashcourse in pointset topology

Review of $\mathbb{R}^n$

Definition: A subset $U \subseteq \mathbb{R}^n$ is open if $\forall x \in U$, $\exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \subseteq U$, where $B_\varepsilon(x) = \{y : d(x,y) < \varepsilon\}$

Definition A map $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is continuous at $x \in \mathbb{R}^n$ if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. if $d(x,y) < \delta$ then $d(f(x), f(y)) < \varepsilon$.

f is continuous if it is continuous $\forall x \in \mathbb{R}$

Same definitions for $f: X \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$

Collection of all open subsets of $\mathbb{R}^n$ is called the metric topology

Features:

1) $\mathbb{R}^n$ is open
2) $\emptyset$ is open
3) Arbitrary unions of open sets are open
4) Finite intersections of open sets are open

Exercise: $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is continuous iff forall open sets in $\mathbb{R}^m$, $U \subseteq \mathbb{R}^m$, $f^{-1}(U)$ is open

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Definition: A basis for a topology τ on X is a family of open sets $\mathcal{B}$ s.t. any element of τ is a union of elements of $\mathcal{B}$

E.g. $\mathbb{R}^n$, metric topology, $\mathcal{B} = \{B_\varepsilon(x)\}$

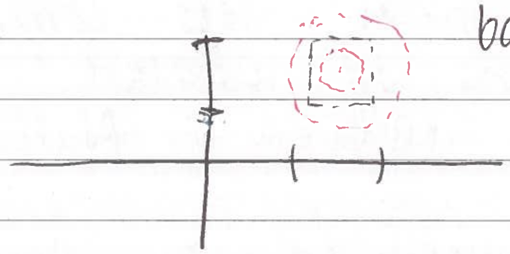
Note: A basis "generates" a topology.

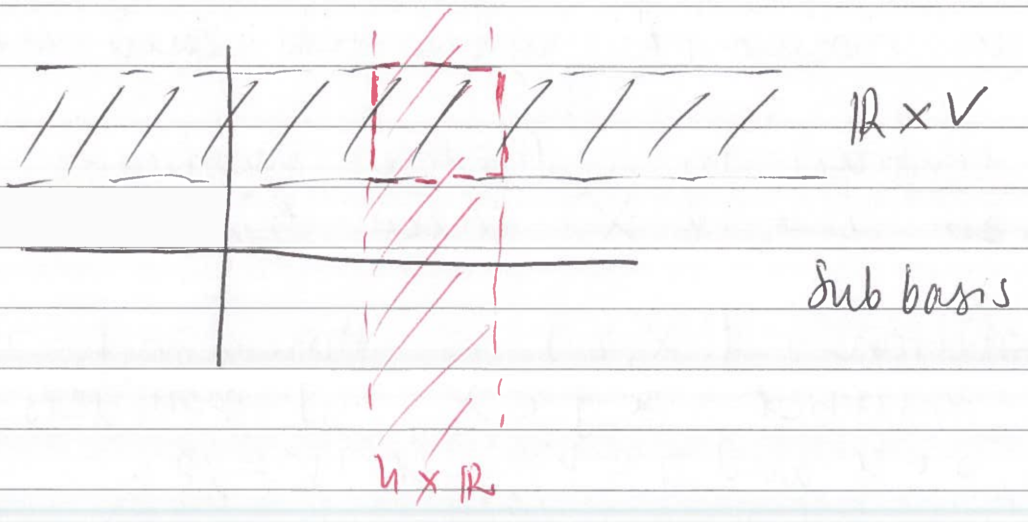
Definition: A subbasis for a topology τ is a collection of open sets S s.t. the collection of finite intersections of elements of S forms a basis

Example

$\mathbb{R}^n$, $\{B_\varepsilon(x)\}$ and $\{U_1 \times \dots \times U_n \mid U_i$ open interval in $\mathbb{R}\}$

are basis. Subbasis of $\mathbb{R}^n$ is
$\{\mathbb{R} \times \dots \times \mathbb{R} \times U_i \times \mathbb{R} \times \dots \times \mathbb{R}\} \; U_i \subseteq \mathbb{R}$ open, $1 \le i \le n\}$





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Definition ✳ A map $f:(X,\tau) \longrightarrow (Y,\sigma)$ is continuous if $\forall\, U \subseteq Y$ open, $f^{-1}(U)$ is open in X.

Example: $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is ε-δ continuous iff f is continuous w.r.t. definition ✳

New spaces from old spaces

Definition (Subspace topology)
Given (X,τ) and $Y \subseteq X$ the subspace topology on Y is $\sigma = \{U \cap Y \mid U \in \tau\}$

Note: The subspace topology on Y is the smallest topology on Y s.t. the inclusion map $i: Y \longrightarrow X$ is continuous i.e. $i^{-1}(U) = U \cap Y$.

Example: $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $m<n$ differentiable $y \in \mathbb{R}^m$ a regular value $M = f^{-1}(y) \subseteq \mathbb{R}^n$ is a manifold, give M the subspace topology which agrees with metric topology

Q: What does M look like?

$$S^n = \{ (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum x_i^2 = 1 \}$$

Definition (X,τ), (Y,σ) are topological spaces. The product topology on $X \times Y$ is the smallest topology s.t. both projections $\pi_X : X \times Y \longrightarrow X$ and $\pi_Y : X \times Y \longrightarrow Y$ are continuous.

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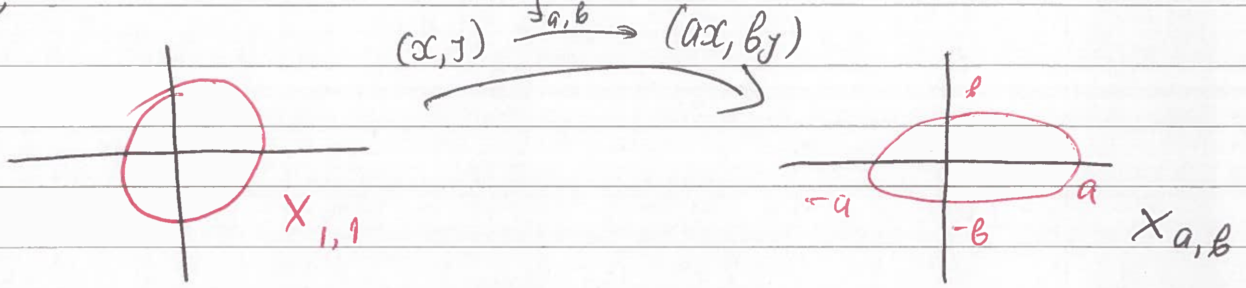
2) f is bijective

3) f^{-1} is continuous.

Example : $\{(x,y) \mid (\frac{x}{a})^2 + (\frac{y}{b})^2 = 1\} = X_{a,b}$

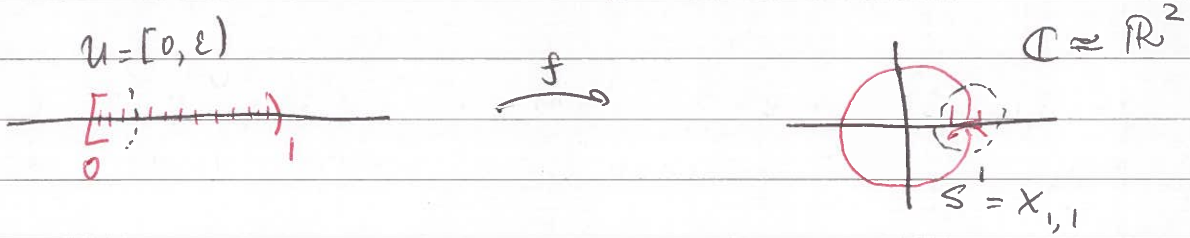
$a, b > 0$



$f_{a,b} : X_{1,1} \longrightarrow X_{a,b}$ is a homeomorphism

Example $f : [0,1) \longrightarrow S^1$ not homeo.

$t \longrightarrow e^{2\pi i t}$



f is continuous and bijective but f^{-1} is not continuous. $\equiv \exists U \in [0,1)$ open s.t. $f(U)$ is not open

Observation : Any neighbourhood of $(1,0)$ contains a point on S^1 below the x-axis $f([0,\varepsilon))$ is in the upper half so it can't be open since any open set containing 0 in $\mathbb{R}^2$ has points in the lower half of $\mathbb{C}$.

Remark : We have shown that f is not a homeo, but we haven't shown that there is no homeo between them.

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Compactness

This is the analogue of finiteness.

Definition: Let X be a topological space. A family $\mathcal{U}$ of open subsets of X is an open cover if $X = \bigcup_{U \in \mathcal{U}} U$.

If $Y \subseteq X$ say that a family $\mathcal{U}$ of open subsets of X is an open cover if $Y \subseteq \bigcup_{U \in \mathcal{U}} U$.

If $\mathcal{U}$ is an open cover of X then $\mathcal{V} \subseteq \mathcal{U}$ is a subcover if $X \subseteq \bigcup_{U \in \mathcal{V}} U$

Definition: X is compact if every open cover of X has a finite subcover.

Note: $Y \subseteq X$, and $\mathcal{U}$ is an open cover of Y. $\mathcal{U} = \{$ open subsets of $Y \}$

if Y is compact $\Rightarrow \exists$ finite subcover

Sometimes cover Y by open subsets of X, $\rightsquigarrow$ find a finite subcover.

Note: open in Y isn't necessarily open in X but it is relatively open.

if U open in $Y \Rightarrow \exists\, U' \subseteq X$ s.t $U = X \cap U'$

Example X with the finite complement topology is always compact

$\mathcal{U} = \{U_i\}_{i \in I} \Rightarrow U_0 = X \setminus F$, where F is finite

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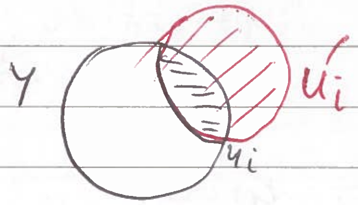
$V = \bigcap_{i=1}^{n} V_{y_i} \ni x_0$ is open, disjoint from $U_{y_1} \cup \dots \cup U_{y_n}$ $\Rightarrow x_0 \in V$, $V \cap Y = \emptyset$

$\Rightarrow$ Y is closed since it doesn't depend on the point $x_0 \in X \setminus Y$ ■

Lemma: X is compact, Y closed, $Y \subseteq X$, then Y is compact.

Proof: $\mathcal{U}$ open cover of Y, $\mathcal{U} = \{U_i \subseteq Y\}_{i \in I}$

let $\mathcal{U}' = \{U_i' \subseteq X\}_{i \in I}$ s.t. $U_i = Y \cap U_i'$



$\mathcal{U}' \cup \{\underbrace{X \setminus Y}_{\text{open}}\}$ is an open cover of X

and it has a finite subcover by compactness of X $\Rightarrow \exists\ i_1, \dots, i_n$ s.t. $\{U_{i_1}', \dots, U_{i_n}', X \setminus Y\}$ covers X $\Rightarrow \{U_{i_1}, \dots, U_{i_n}\} \subseteq \mathcal{U}$

$U_{i_j} = U_{i_j}' \cap Y$ is a finite subcover. ■

Heine - Borel Thm

$X \subseteq \mathbb{R}^n$ compact iff X is closed & bdd

$\Rightarrow$ we have done in example ✻

$\Leftarrow$ By rescalling we can assume $X \subseteq [0,1]^n$

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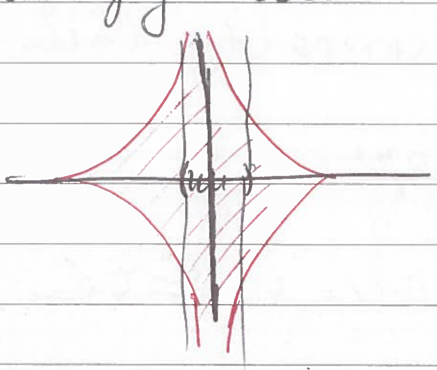

$\mathbb{R}^2$, $\{0\} \times \mathbb{R}$

$\mathcal{U} = \{(x,y) \mid |x| < e^{-|y|}\}$

$\{0\} \times \mathbb{R} \subseteq \mathcal{U}$

The tube lemma fails because $\mathbb{R}$ is not compact.

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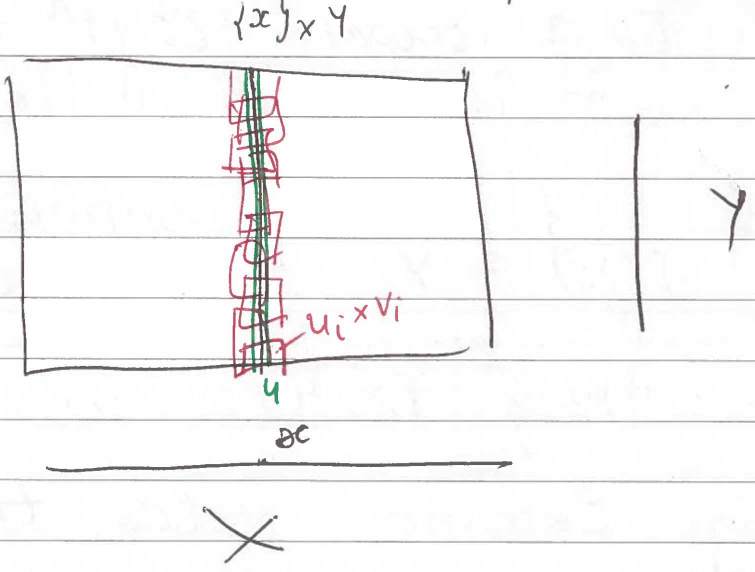
Lemma X arbitrary, Y compact, $\mathcal{U}_x = \{U_i \times V_i\}_{i \in I}$ open cover of $\{x\} \times Y$ $x \in X$ $x \in U_i$. Then $\exists\, \mathcal{U}_x' \subseteq \mathcal{U}_x$ finite subcover and $U_x \subseteq X$ s.t. $U_x \times Y \subseteq \bigcup_{W \in \mathcal{U}_x'} W$.

Proof: $\{V_i\}$ is an open cover of Y



Y compact $\Rightarrow \exists\, i_1, \dots, i_n$ s.t. $\{V_{i_1}, \dots, V_{i_n}\}$ is an open cover of Y. Then $\{U_{i_1} \times V_{i_1}, \dots, U_{i_n} \times V_{i_n}\}$ is an open cover of $\{x\} \times Y$.

Let $\mathcal{U} = \bigcap_{j=1,\dots,n} U_{i_j} \quad \bigcup_{W \in \mathcal{U}} W \supseteq U \times Y$ ∎

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Ex. 1 $\{0,1\}$ discrete topology not connected

Ex 2 $[0,1]$ standard topology, connected

Definition : • X is disconnected if $\exists\, U, V \overset{\text{open}}{\subseteq} X$ s.t. $U \neq \emptyset \neq V$ and $U \cap V = \emptyset$, $U \cup V = X$

in Ex 1 take $U = \{0\}$ and $V = \{1\}$

• X is connected if it is not disconnected.

Definition (X_1, τ_1) and (X_2, τ_2) be topological spaces, then the disjoint union of X_1 and X_2, $X_1 \sqcup X_2$ is topologised that U is open in $X_1 \sqcup X_2$ iff $U \cap X_1$ and $U \cap X_2$ are open

Remark : X_1 and $X_2 \subseteq X_1 \sqcup X_2$ disconnect $X_1 \sqcup X_2$

Remark : $(X_i, \tau_i)_{i \in I}$ denote the disjoint union ~~on~~ by $\coprod X_i$

Ex $X_i \sim$ point $\coprod_{i \in I} X_i$ has discrete topology

Lemma X connected, $f : X \to Y$ continuous and $f(X) \subseteq Y$, is connected

Proof homework

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Intermediate Value Thm

X connected $f: X \longrightarrow \mathbb{R}$ continuous and $f(a) < c < f(b)$ then $\exists\, d \in X$ s.t. $f(d) = c$

Proof: $f(X)$ is in interval
Use lemma about f(connected) is connected. ∎

Path Connectedness

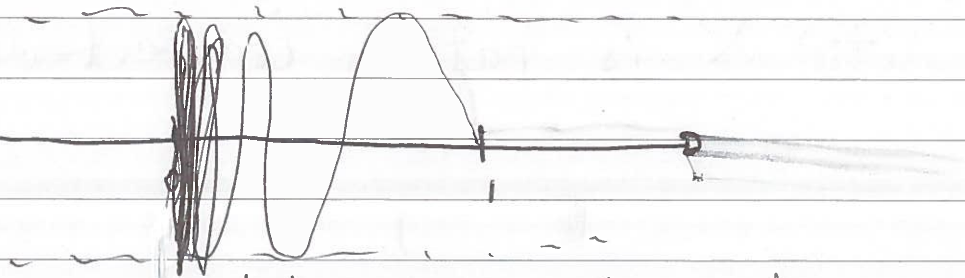
Definition: X is path connected if $\forall x, y \in X$ $\exists$ continuous map $f: [0,1] \longrightarrow X$ s.t. $f(0) = x$ and $f(1) = y$

Lemma X path connected then X is connected.

Proof: Suppose not i.e. U and V disconnect X s.t. $x \in U$ and $y \in V$ and $f: [0,1] \longrightarrow X$ is a path from x to y. then $f^{-1}(U)$ and $f^{-1}(V)$ disconnect $[0,1]$ ✗ ∎

Example Topologists sine curve.

$$\{(0,0)\} \cup \{(x, \sin\tfrac{1}{x}) \mid x \in (0,1]\}$$



is connected but not path connected

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Quotient Spaces

Question : $f : X \longrightarrow Y$, X set , Y topological space. How can we topologise X so that f is continuous?

Ans :
- Silly option - give X discrete topology
- Define $\tau = \{ f^{-1}(U) \mid U \subseteq Y \text{ open} \}$
  check axioms.

This is the smallest topology making f continuous.

Reverse. $f : X \twoheadrightarrow Y$, X topological space Y set. How can we topologise Y? to make f continuous?

1. Y indiscrete topology $\{ \emptyset, Y \}$
2. $\sigma = \{ U \subseteq Y \mid f^{-1}(U) \text{ open} \}$
   check axioms.

Definition : σ is called the quotient topology on Y

Definition $f : X \twoheadrightarrow Y$ continuous is a quotient map if $f^{-1}(U)$ open iff U is open.

Note: Y has the quotient topology. This is the largest topology making f continuous.

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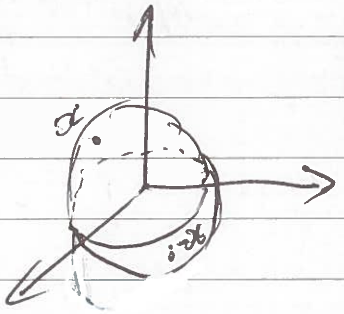
Example $\mathbb{R}P^n$, real projective spaces

$$S^n \subseteq \mathbb{R}^{n+1}, \quad S^n = \{(x_1, \dots, x_{n+1}) \mid \sum x_i^2 = 1\}$$

Define an equivalence relation by $x \sim -x$.



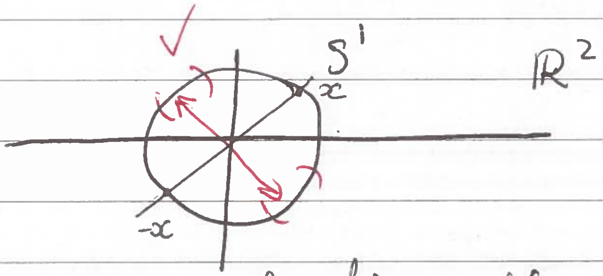
As a set $\mathbb{R}P^n = \{[x]_\sim\}$ pairs of antipodal pts. which we topologise by the quotient topology.

$n = 0$

$$S^0 = \{-1, 1\} \xrightarrow{f} \mathbb{R}P^0 = \{[-1, 1]\}$$

$f^{-1}([-1,1]) = \{-1, 1\}$ so $\mathbb{R}P^0$ has the discrete topology.

$n = 1$



$\xrightarrow{f} \mathbb{R}P^1 = \{[x, -x]\}$

$\cong ?$ $S^1 = [0,1]/_{0 \sim 1}$

Space of lines through the origin in $\mathbb{R}$

An open set $\mathcal{U} \in \mathbb{R}P^n$ is a set s.t. $f^{-1}(\mathcal{U})$ is open, $\mathcal{U} = \{[x, -x] \in \text{some collection}\}$

$$f^{-1}(\mathcal{U}) = \{x, -x \mid \text{s.t. } [x] \in \mathcal{U}\}$$

Every open $\mathcal{U} \subseteq \mathbb{R}P^1 \xleftrightarrow{\text{bijective}}$ with some open set $V \subset S^1$ s.t. $V = -V$

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More on quotient maps:

Definition: A map $q : X \twoheadrightarrow Y$, X space, Y subspace is a quotient map if (U) open iff U is open.

Proposition

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, two heads, "q"'] & Z \\ & Y \arrow[u, "h"'] \end{tikzcd}$$

q quotient and $h \circ q = f$

Then h is continuous iff f continuous

Proof:

$\underline{\Rightarrow}$ compositions of continuous maps are continuous

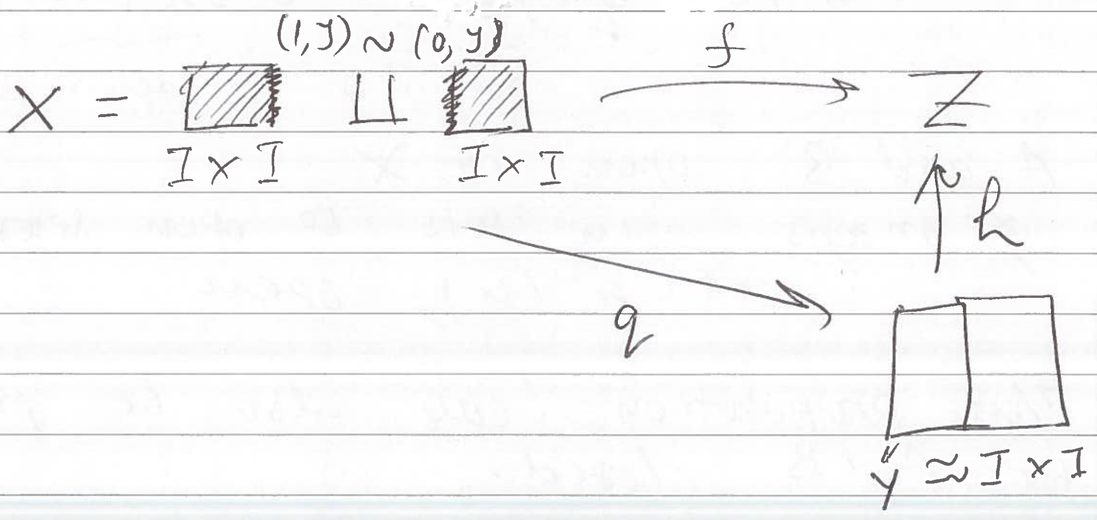
$\underline{\Leftarrow}$ Show h is continuous if f continuous

$U \subseteq Z$ open, $f^{-1}(U) = q^{-1}(h^{-1}(U))$

f continuous $\Rightarrow f^{-1}(U)$ is open and

q quotient $\Rightarrow h^{-1}(U)$ open ∎

Ex Columns continuous means



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Definition : A map $f: X \longrightarrow Y$ is **open** if $\mathcal{U}$ open $\Rightarrow f(\mathcal{U})$ open . A map $f: X \longrightarrow Y$ is **closed** if C closed $\Rightarrow f(C)$ closed

Proposition $f: X \longrightarrow Y$ continuous s.t. open or closed then f is a quotient map

Saturated Open sets

X space , $\sim$ an equivalence relation on X

Definition. An open set $\mathcal{U} \subseteq X$ is called **saturated** if $U \cap [x] \neq \emptyset \Rightarrow [x] \subseteq X$

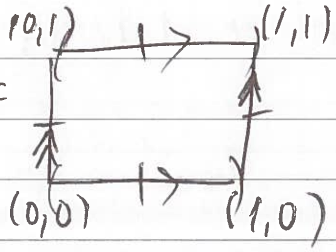
$$X/\sim \; = \{ [x] \mid [x] \text{ is an equivalence class} \}$$

We give this the quotient topology.

There is a bijection between open sets in $X/\sim$ and saturated open sets in $X \Leftrightarrow q: X \longrightarrow X/\sim$ is a **quotient topology**

Ex

$X =$



$= I \times I$

Let $\sim$ be generated by $(x,0) \sim (x,1)$ and $(0,y) \sim (1,y)$

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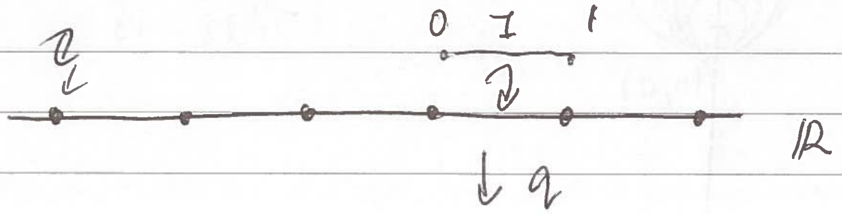
Construction

Let X be a space, $Y \subseteq X$ a subspace $f: Y \longrightarrow X$ be a continuous map and let $\sim$ be the equivalence relation generated by $Y \sim f(y)$.

Let $X \cup_f = X/\sim$ with the quotient topology

Ex: $X = \mathbb{R}$, $Y = \mathbb{R}$ $f: Y \longrightarrow X$ $f(x) = x + 1$

then $x \sim y$ iff $x - y \in \mathbb{Z}$



$$I \xrightarrow{q'} \mathbb{R} \cup_f = \mathbb{R}/\mathbb{Z}$$

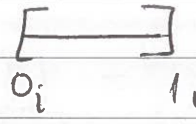
except at 0 and 1

$q': I \longrightarrow \mathbb{R}/\sim$ is a quotient map

$$\Rightarrow \underset{\underset{S^1}{\parallel}}{I/_{0\sim 1}} \cong \mathbb{R}/\mathbb{Z}$$

Ex: $X = \left(\coprod_{i \in \mathbb{Z}} X_i\right) \amalg \{*\}$ $X_i \cong I$

$Y = \{0_i, 1_i\}$ $X_i =$



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$$X^{(n)} = \left(X^{(n-1)} \amalg \coprod_{i \in I_n} D_i^n \right) \cup_{\varphi_n}$$

where $\varphi_n : \coprod_{i \in I_n} \partial D_i^n \longrightarrow X^{(n-1)}$

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CW complexes

X is a CW complex if $X = \bigcup_{n\in\mathbb{N}} X^{(n)}$,

$X^{(n)}$ is n-skeleton s.t. $X^{(0)}$ = discrete set of pts

$X^{(n)} = \left(X^{(n-1)} \amalg \left(\coprod_{\alpha\in J} D_\alpha^n \right) \right) / \sim$

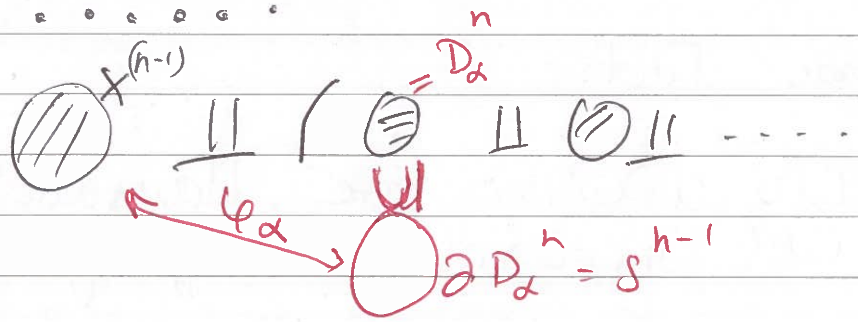
for each α, $\exists\ \varphi_\alpha : \underbrace{\partial D_\alpha^n}_{S^{n-1}} \longrightarrow X^{(n-1)}$

Continuous. For each $x \in \partial D_\alpha^n$, $x \sim \varphi_\alpha(x)$

$\mathcal{U}$ open in X iff $\mathcal{U} \sim X^{(n)}$ open for each n

$X^{(0)} = \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$

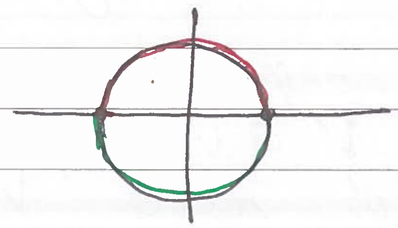
$X^{(n)} =$



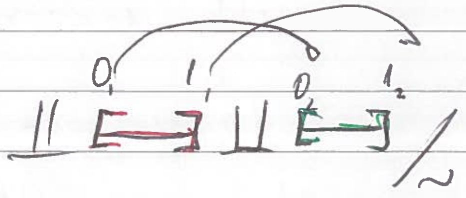
Example $S^n \subseteq \mathbb{R}^{n+1}$

$S^0 =$

$S^1 =$



$= S^0 \amalg$



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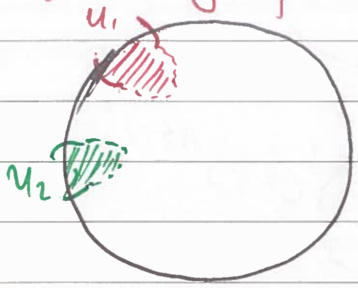
Proof by picture of lemma 2



$S^n \subseteq D^{n+1}$

$V_i = (1-\varepsilon, 1] \times U_i \subseteq D^n$

Proof of lemma 1.

$$X^{(n+1)} = X^n \amalg D^{n+1} / x \sim \varphi(x)$$

$\varphi : \partial D^{n+1} \longrightarrow X^{(n)}$ continuous

$U_1, U_2 \subseteq X^{(n)}$, disjoint, open

$\varphi^{-1}(U_1), \varphi^{-1}(U_2)$ disjoint open subsets of $\partial D^{n+1} = S^n$

by the lemma 2 . we can find

W_1 and W_2 open and disjoint subsets of D^{n+1} s.t. $W_i \cap S^n = \varphi^{-1}(U_i)$.

Now $U_1 \cup W_1 \subseteq X^{(n)} \amalg D^{n+1}$

$\underbrace{U_2 \cup W_2} \subseteq X^{(n)} \amalg D^{n+1}$

open, are unions of equivalence classes and disjoint

$\Rightarrow$ decend to open disjoint V_1 and $V_2 \subseteq X^{(n+1)}$

Topology and Groups 25th Jan

Given two spaces X and Y :

1. How to show that $X \not\simeq Y$? (Alg. Topology)
2. Given a group G, build space X s.t. $\pi_1(X) = G$, study the group G by way of the topology of X

Homotopy

$f_0, f_1 : X \longrightarrow Y$ continuous maps are homotopic if $\exists$ $F : X \times I \longrightarrow Y$ continuous s.t. $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$.

Write $F(x,t) = f_t(x)$

$F(x,t)$ is a 1-parameter family of maps from X to Y

Ex : $f : X \longrightarrow \mathbb{R}^n$ is homotopic to the continuous map $0 : X \longrightarrow 0 \in \mathbb{R}^n$

$$F(x,t) = t \cdot f(x) , \quad F(x,0) = 0$$
$$F(x,1) = f(x)$$

Ex : $F(x,t) : \mathbb{R}^n \setminus \{0\} \longrightarrow S^{n-1}$

$$F(x,t) = \frac{x}{\|x\|^t}$$

$$F(x,0) = x = id(x) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

$$F(x,1) = \frac{x}{\|x\|} = r(x) : \mathbb{R}^n \longrightarrow S^{n-1}$$

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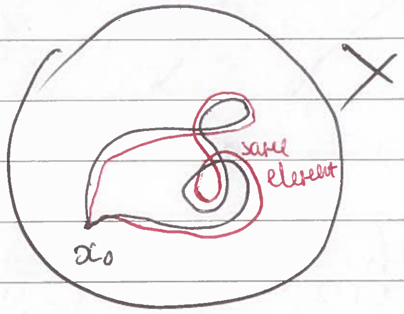
Ex: $\mathcal{C} = \{\gamma : [0,1] \longrightarrow X \mid \gamma(0) = x_0, \gamma(1) = x_1\}$
with the compact open topology
F is a path in $\mathcal{C}$ from γ_0 to γ_1.

Definition The fundamental group of X based at $x_0 \in X$ is the set of path homotopy classes of paths beginning and ending at x_0. Denote it by $\pi_1(X, x_0)$



We need to:

1. Define the group law
2. Show it's well-defined
3. $X \xrightarrow{f} Y$, $f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, f(x_0))$
and if X and Y are homotopy equiv.
$X \underset{f}{\overset{g}{\rightleftarrows}} Y$, then f_* is an iso.

Notation: α is a path based at x_0, then $[\alpha] \in \pi_1(X, x_0)$ for the associated element of $\pi_1(X, x_0)$

The group law is concatenation
First of paths:
If $\alpha : [0,1] \to X$ and $\beta : [0,1] \to X$ paths with $\alpha(1) = \beta(0)$ then

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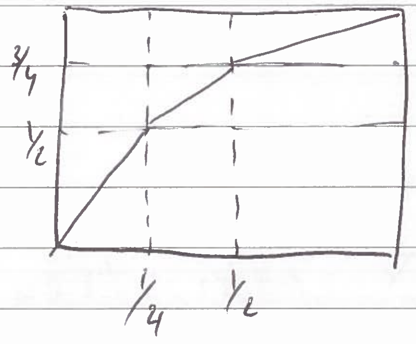
1. id ⟸ constant path
2. inverse ⟸ the same path backwards
3. associativity ⟸ we use reparametrisation

Lemma: $f_0, f_1 : I \longrightarrow I$ continuous s.t. $f_0(0) = f_1(0)$ and $f_0(1) = f_1(1)$, then $f_0 \sim f_1$

"straight line homotopy"

$$F(x,t) = (1-t) f_0(x) + t f_1(x)$$
$$F(x, 0) = f_0(x)$$
$$F(x, 1) = f_1(x)$$

I) Associativity:



graph of f

We want $(\alpha \cdot \beta) \cdot \gamma) \sim \alpha \cdot (\beta \cdot \gamma)$

$$\alpha \cdot (\beta \cdot \gamma) \circ f = (\alpha \cdot \beta) \cdot \gamma$$

But $f \sim \mathbb{1}d$ by straight line homotopy

$$\Rightarrow (\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma) \circ f \sim \alpha \cdot (\beta \cdot \gamma) \circ 1$$
$$= \alpha \cdot (\beta \cdot \gamma)$$

$$\Rightarrow (\alpha \cdot \beta) \cdot \gamma \sim \alpha \cdot (\beta \cdot \gamma)$$

II) Identity

: constant path at x_0 is denoted by X_0

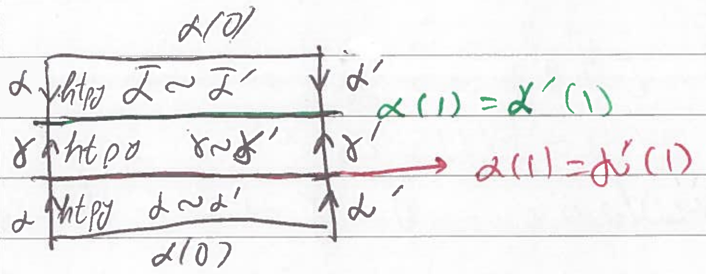
Need $X_0 \cdot \alpha \sim \alpha \sim \alpha \cdot X_0$

Topology and Groups

25th Jan

If $\alpha \sim \alpha'$ and $\gamma \sim \gamma'$

then $\alpha \cdot \gamma \cdot \bar{\alpha} \sim \alpha' \cdot \gamma' \cdot \bar{\alpha}'$



α_* is a homomorphism

$$\alpha_*([\gamma_1])\,\alpha_*([\gamma_2]) = \alpha_*([\gamma_1 \gamma_2])$$

$$\alpha \cdot \gamma_1 \bar{\alpha} \cdot \alpha \cdot \gamma_2 \cdot \bar{\alpha} \simeq \alpha \cdot \gamma_1 \cdot \gamma_2 \cdot \bar{\alpha}$$

Inverse of α_* is $(\bar{\alpha})_* : \pi_1(X, \alpha(0)) \to \pi_1(X, \alpha$

$\alpha_* \circ (\bar{\alpha})_* ([\gamma]) = [\alpha \cdot \bar{\alpha} \cdot \gamma \cdot \alpha \cdot \bar{\alpha}] = [\gamma]$ α_* onto

$(\bar{\alpha})_* \circ \alpha_*([\gamma]) = [\bar{\alpha} \cdot \alpha \cdot \gamma \cdot \bar{\alpha} \cdot \alpha] = [\gamma]$

α_* into

$\Rightarrow \alpha_*$ is an isomorphism ■

<u>Ex</u>: $\pi_1(\mathbb{R}^n, 0) = 1$

$$F : \mathbb{R}^n \times I \longrightarrow \mathbb{R}^n$$
$$(x, t) \longrightarrow xt$$

Fixes 0, $[\gamma] \in \pi_1(\mathbb{R}^n, 0)$

$F(\gamma(s), t)$ is a homotopy

Topology and Groups

28th Jan

<u>Last time</u>:

1. $\pi_1(X, x_0)$, $x_0 \in X$
   $\{$equiv. classes of loops based at $x_0\}$

2. $\alpha : [0,1] \longrightarrow X$
   $\exists$ iso $\alpha_* : \pi_1(X, \alpha(1)) \longrightarrow \pi_1(X, \alpha(0))$
   given by $\alpha_*([\gamma]) = [\alpha \cdot \gamma \cdot \bar{\alpha}]$

3. $[\gamma] = x_0 \equiv$

$$\begin{tikzcd} {[\text{---}]} \arrow[r, "\gamma"] \arrow[dr] & X \ni x_0 \\ & S^1 \arrow[u] \\ & \text{in } D^2 \arrow[uu] \end{tikzcd}$$

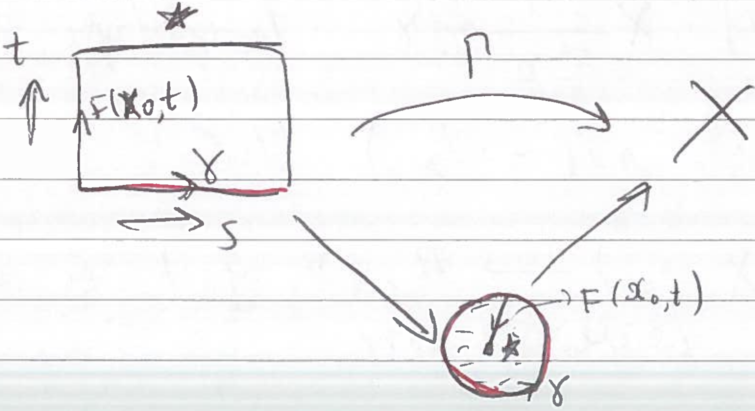
trivial iff bound a disk

Definition: X is contractible if $\exists F : X \times I \longrightarrow X$
$F(-,0) = id_X(-)$
$F(-,1) = * \in X$

Example: $\mathbb{R}^n$, $F(x,t) = (1-t).x$

Lemma: X is contractable then $\pi_1(X, x_0) = 1$

Proof: Let F be a homotopy between id_X and $*$=constant map. s.t. $\gamma \in [\gamma] \in \pi_1(X, x_0)$. Define
$\Gamma(s,t) = F(\gamma(s), t)$
$\Gamma : I \times I \longrightarrow X$



This digram commutes
γ bounds a disk
$\Rightarrow [\gamma] = x_0$

Topology and Groups

28th Jan

Lemma: $X \underset{f_1}{\overset{f_0}{\rightrightarrows}} Y$, f_0 and f_1 are continuous homotopic maps by $F(-,0) = f_0(-)$ and $F(-,1) = f_1(-)$. Let $\alpha(t) = F(x_0, t)$. Then

$$\alpha_* \circ f_{1*} = f_{0*}$$

Proof: Need to show that

$$\alpha_* \circ (f_1)_* ([\gamma]) = (f_0)_* ([\gamma])$$

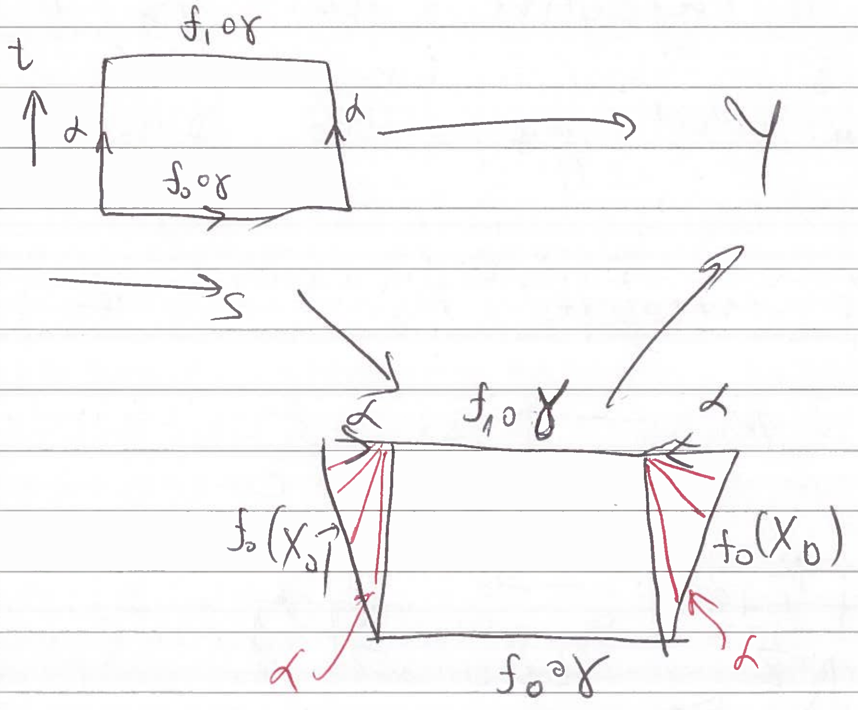
$$\Downarrow$$

$\alpha \cdot (f_1 \circ \gamma) \cdot \bar{\alpha}$ to show this is homotopic to $f_0 \circ \gamma$

Consider $F(\gamma(s), t)$

$F(\gamma(s), 0) = f_0 \circ \gamma$

$F(\gamma(s), 1) = f_1 \circ \gamma$



$$\Rightarrow \alpha \cdot (f_1 \circ \gamma) \cdot \bar{\alpha} \sim f_0 \circ \gamma$$ ∎

Topology and Groups

28th Jan

Claim $\exists\ \tilde{\gamma} : [0,1] \to \mathbb{R}$ (lift of γ)
s.t. $e^{2\pi i \tilde{\gamma}(t)} = \gamma(t)$ s.t.
$\tilde{\gamma}(0), \tilde{\gamma}(1) \in \mathbb{Z}$
$w(\gamma) = \tilde{\gamma}(1) - \tilde{\gamma}(0)$ called the winding number.

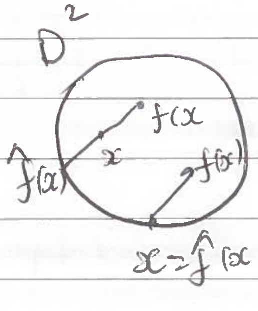
1. If $\gamma' \sim \gamma$ then they have the same winding number i.e. $w(\gamma') = w(\gamma)$

2. $w : \pi_1(S^1, 1) \longrightarrow \mathbb{Z}$
$\hat{w}([\gamma]) = w(\gamma)$ is an iso

We will prove if X is "nice", $\tilde{X}$-universal cover of X, $\tilde{X} \circlearrowleft G$, $\tilde{X}/G = X$ and $\pi_1(X) = G$

Note $\mathbb{R} \to S^1$
$\mathbb{Z} \circlearrowleft \mathbb{R}$ translations, $\mathbb{R}/\mathbb{Z} = S^1$, $\pi_1(S^1) = \mathbb{Z}$

Application:
Brouwer's Fixed Point Thm
$f : D^2 \longrightarrow D^2$ closed disk, f cont.
Then $\exists\ x$ s.t. $f(x) = x$



Suppose not

$$\begin{tikzcd} S^1 \arrow[r, hook, "i"] \arrow[rr, bend right, "id_{S^1}"'] & D^2 \arrow[r, "\hat{f}"] & S^1 \end{tikzcd}$$

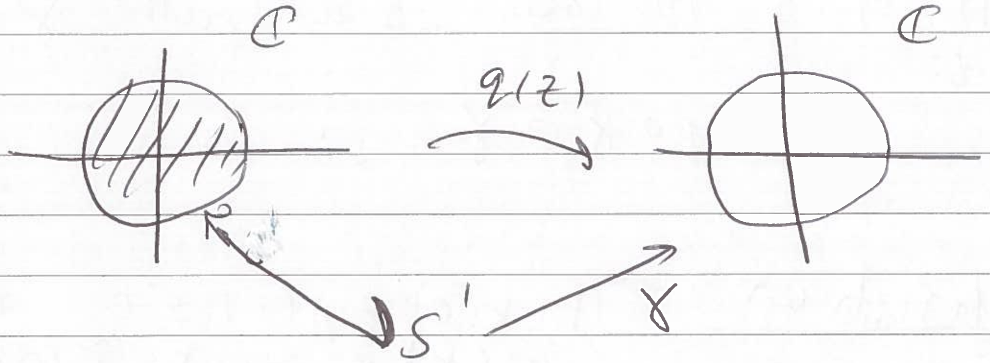
$(\hat{f})_* \circ i_* = (id_{S^1})_*$

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "(i)_*"] \arrow[rr, bend right, "(id_{S^1})_*"'] & \pi_1(D^2) \arrow[r, "\hat{f}_*"] & \mathbb{Z} \end{tikzcd}$$

Topology and Groups

1st Feb

$F(s,t)$ is a homotopy between γ and γ' in $\mathbb{C}\setminus\{0\}$. If $q(z)$ doesn't have a root then $q(^{U}) \subseteq \mathbb{C}\setminus\{0\} \Rightarrow \gamma$ bounds a disk

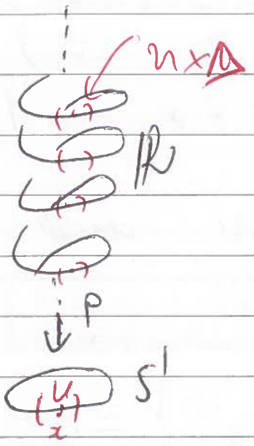


γ bounds a disk $\Rightarrow \gamma \sim \gamma'$ in $\mathbb{C}\setminus\{0\}$
$\Rightarrow [\gamma'] = 0 = n \in \pi_1(\mathbb{C}) \setminus \{0\}, 1)$
$\cong \pi_1(S^1, 1)$

$\Rightarrow n = 0$ ⨳ ∎

Goal $\pi_1(S^1, 1) = \mathbb{Z}$

Map $p : \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z} \cong S^1$
$t \longmapsto e^{2\pi i/t}$



p has "covering map property"
For every $x \in S^1$ there exist an open neighbourhood $U \ni x$ s.t. $p^{-1}(U) \cong U \times \Delta$, where Δ is a discrete set.

$$\begin{tikzcd}
p^{-1}(U) \arrow[rr, phantom, "\cong"] \arrow[dr, "p"'] & & U \times \Delta \arrow[dl, "proj"] \\
& U &
\end{tikzcd}$$

U - covering neighbourhood

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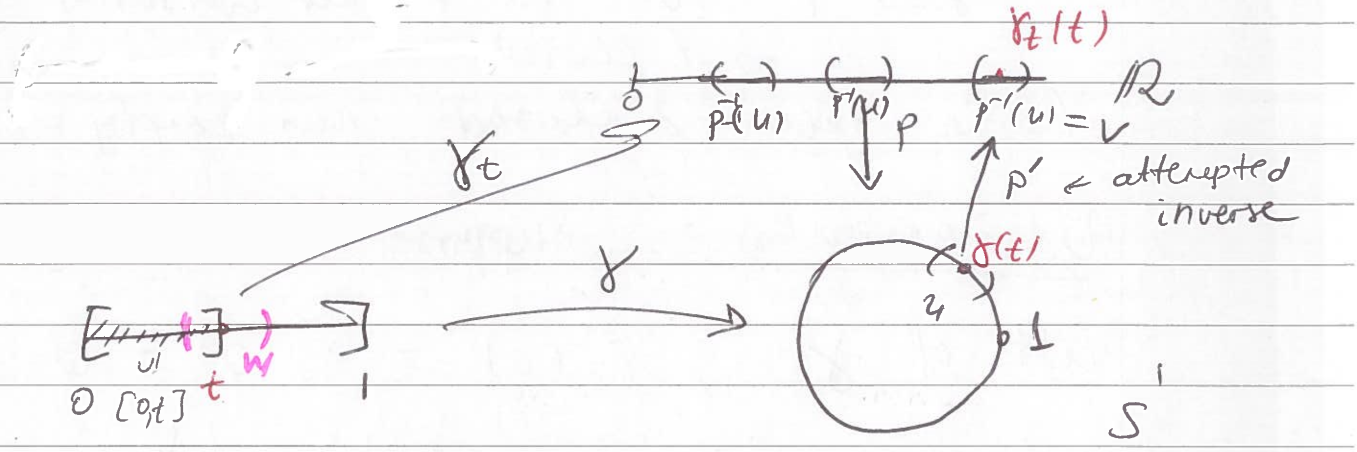
$\mathcal{U}$ be a covering neighbourhood of $\gamma(t)$, ($p^{-1}(\mathcal{U}) \cong \mathcal{U} \times \Delta$). Let V be the component of $\mathcal{U} \times \Delta$ which contains $\gamma_t(t)$



Let $p' : \mathcal{U} \to V$ be a homeomorphism s.t. $p \circ p' = id_{\mathcal{U}}$

Now let $W \subseteq \gamma^{-1}(\mathcal{U})$ open, connected with $t \in W$. Now define

$$\alpha = \begin{cases} \gamma_t & \text{on } [0,t] \\ p' \circ \gamma & \text{on } [t,1] \cap W \end{cases}$$

$$p \circ \alpha = \gamma \big|_{[0,t] \cup ([t,1] \cap W)}$$

$\Rightarrow T$ contains a neighbourhood of t

3. T closed: Suppose not. Then $T = [0, \tau)$ choose a covering neighbourhood of τ

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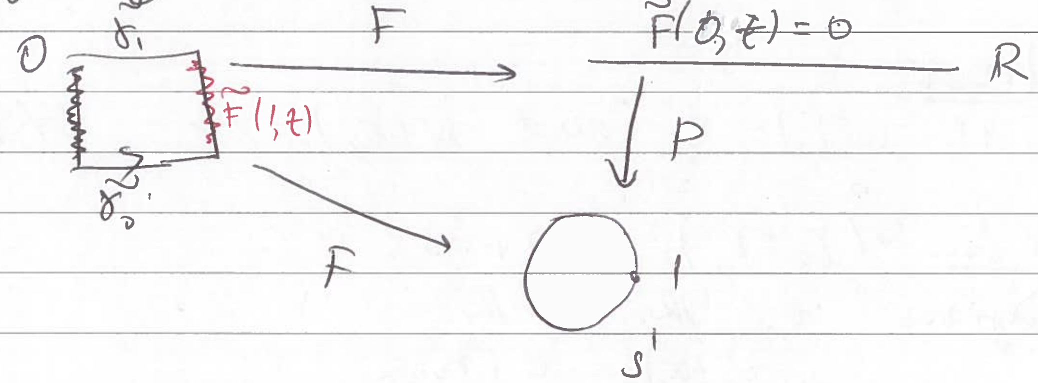
Lemma : The winding number is well-defined

Proof: γ_0 and $\gamma_1 : [0,1] \longrightarrow S^1$ closed loops based at 1, F a homotopy between them i.e. from γ_0 to γ_1.

Let $\tilde{F}$ be the lift of the homotopy F to R given by the Homotopy Lifting Lemma s.t. $\tilde{F}(0,t) = 0$



Observe that $p \circ \tilde{F}(1,t) = F(1,t) = 1$

$\Rightarrow \tilde{F}(1,t) \leq p^{-1}(1) = \mathbb{Z} \subseteq \mathbb{R}$

Now Consider $\{(1,t) \in [0,1] \times [0,1]\} \cong I$ is connected

$\Rightarrow \tilde{F}(1,t)$ is constant.

But $\tilde{\gamma}_0(1) = \tilde{F}(1,0)$ since const.

$\tilde{\gamma}_1(1) = \tilde{F}(1,1)$

$\tilde{F}(1,0) = \tilde{F}(1,1)$

$\Rightarrow \tilde{\gamma}_1(1) = \tilde{\gamma}_0(1)$

$\Rightarrow w(\gamma_0) = w(\gamma_1)$ ∎

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Surjectivity : $n \in \mathbb{Z}$

$t \in [0,1]$

$$\begin{tikzcd} t \arrow[r, mapsto, "\times n"] \arrow[rr, bend right, "p\circ(\cancel{x}n)=\gamma"'] & n.t \in \mathbb{R} \arrow[r, "proj"] & S^1 \end{tikzcd}$$

then $w(\gamma) = n$ $\Rightarrow$ surj.

$\blacksquare$

Remarks

1. Path-lifting lemma
   Homotopy-lifting lemma
   only use $\exists$ of covering neighbourhoods
2. $t_n : \mathbb{R} \to \mathbb{R}$

$$\begin{tikzcd} \mathbb{R} \arrow[rr, "t_n"] \arrow[dr, "p"'] & & \mathbb{R} \arrow[dl, "p"] \\ & S^1 & \end{tikzcd}$$

$$p \circ t_n = p$$

There is a map

$\mathbb{Z} \longrightarrow \{$homeomorphisms of $\mathbb{R}\}$

$n \longrightarrow t_n$

This map is a group homomorphism gives an action of $\mathbb{Z} \circlearrowright \mathbb{R}$. And the quotient map $\mathbb{R} \xrightarrow{p} \mathbb{R}/\mathbb{Z} = S^1$

Covering Space Theory:

Definition : X is path connected topological space and Y is a topological space with map $p: Y \longrightarrow X$ s.t. $\forall x \in X$ there is a neighbourhood U of X and a homeomorphism $p^{-1}(U) \longrightarrow U \times \Delta$ s.t. the obvious diagram commutes.

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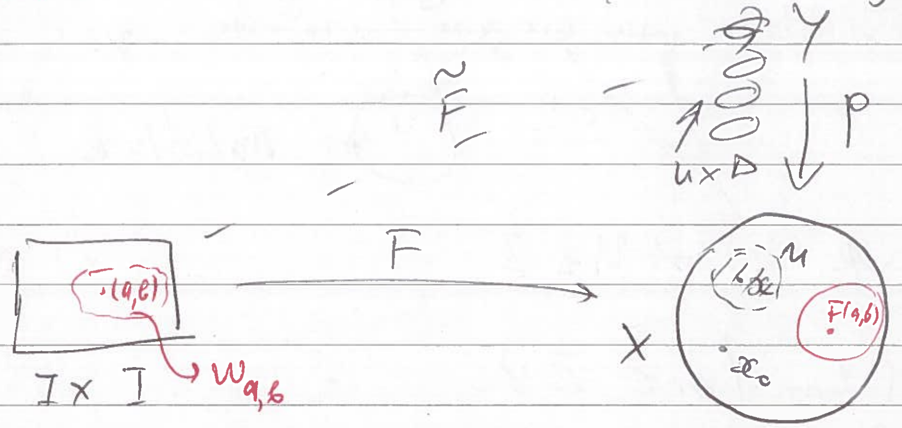
Proof: Similar to path-lifting lemma



For each $x \in X$, let $\mathcal{U}_x$ be a covering neighbourhood of X. For each $(a,b) \in [0,1] \times [0,1]$ let $w_{a,b} \subseteq F^{-1}(\mathcal{U}_{F(a,b)})$ be a connected neighbourhood of (a,b) contained in $F^{-1}(\mathcal{U}_{F(a,b)})$.

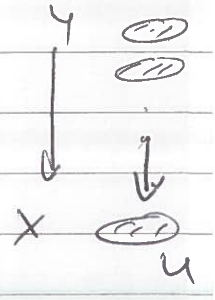
$W = \{W_{a,b}\}$ is an open cover of the square $[0,1] \times [0,1]$. Let ε be the Lebesgue # of W

Lebesgue # lemma: $\mathcal{U}$ open cover of $[0,1]$ then $\exists\ \varepsilon > 0$ s.t. $\forall x \in [0,1]\ \exists\ \mathcal{U} \in \mathcal{U}$ s.t. $B_\varepsilon(x) \subseteq \mathcal{U}$.

Recall:

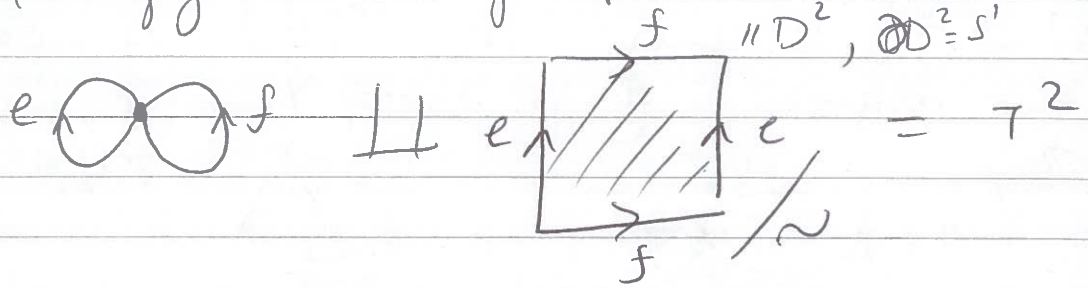
Definition: A map $p: \cancel{X} \longrightarrow X$ is a covering map if $\forall x \in X\ \exists\ x \in U$ open

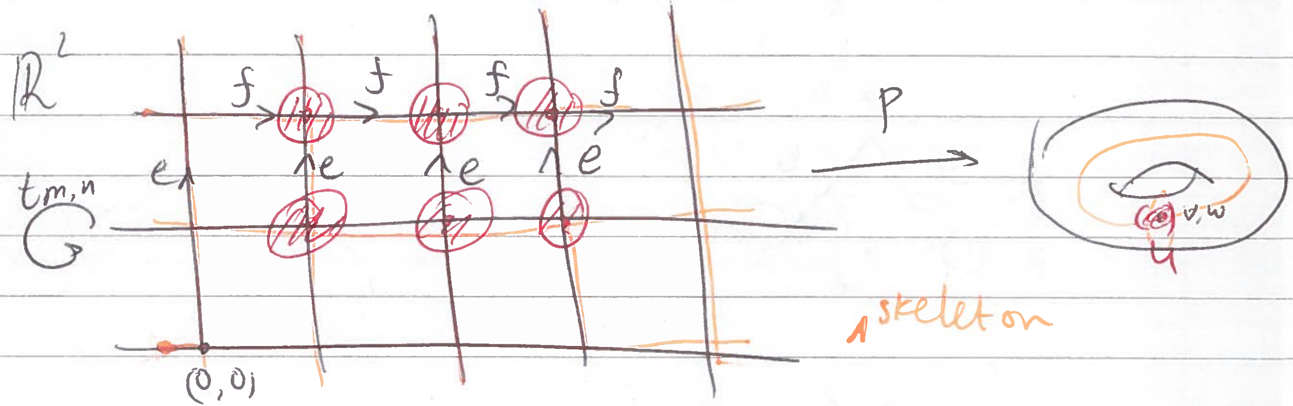
$$\begin{tikzcd}
p^{-1}(U) \arrow[rr, "\cong"] \arrow[dr, "p"'] & & U \times \Delta \arrow[dl, "proj"] \\
& U &
\end{tikzcd}$$



Topology and groups

4th Feb



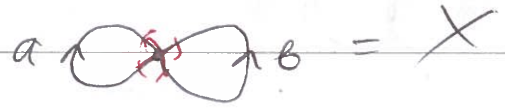


$p^{-1}(U) \cong U \times \mathbb{Z}^2$

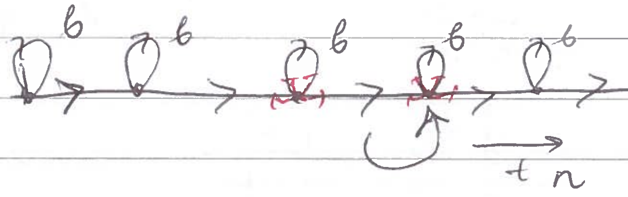
$\pi_1(S^1 \times S^1) = \mathbb{Z}^2 = \{ t_{m,n}(x,y) \longmapsto (x+m, y+n) \} = \mathbb{Z}^2$

$p \circ t_{m,n} = p$

Examples of Friezes

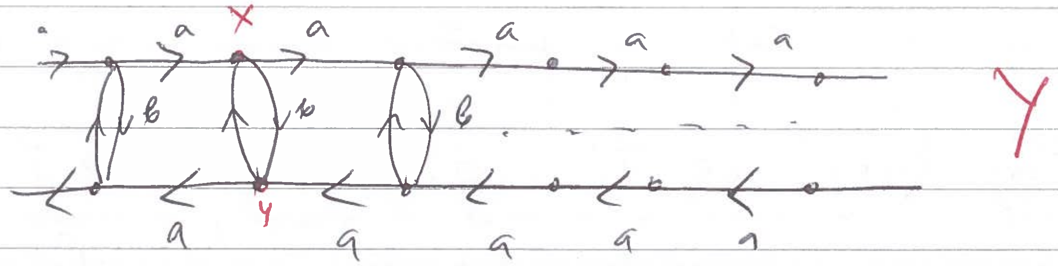


1)



$\{t_m\} = \mathbb{Z}$

2)



translations

$t : Y \longrightarrow Y$

$t(x) = y$

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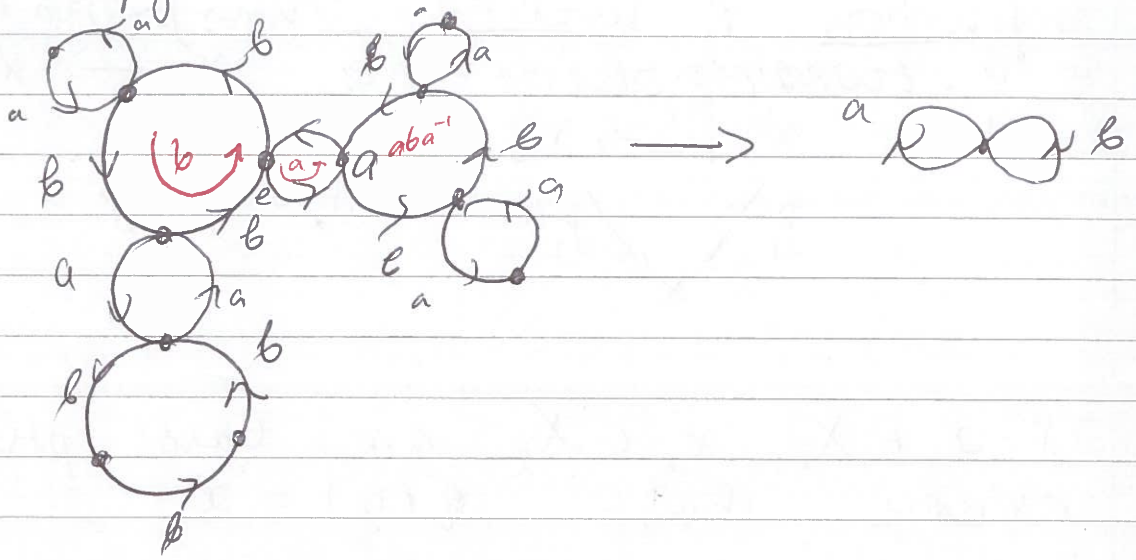
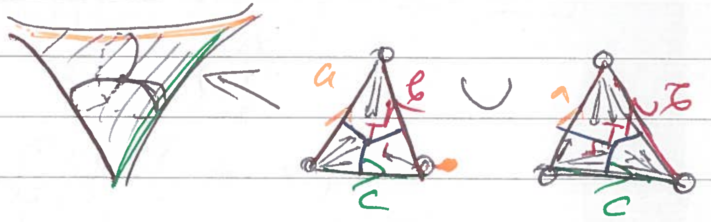
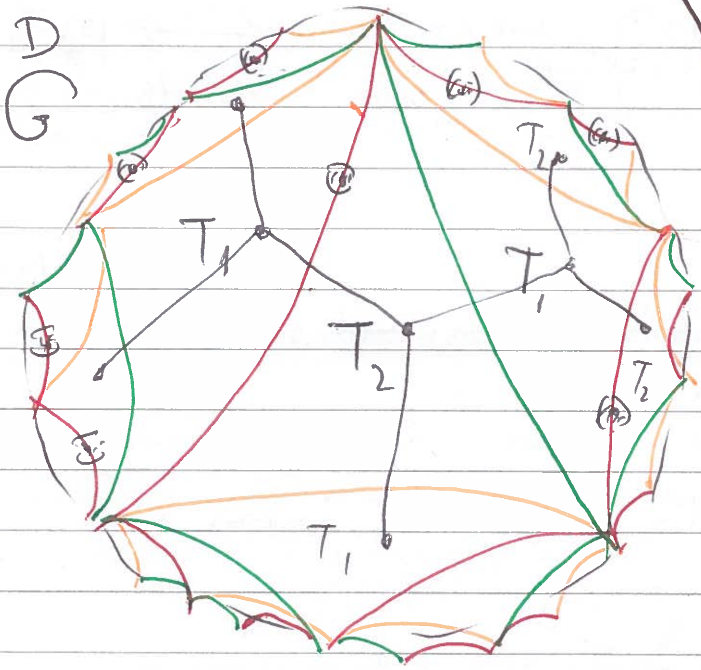
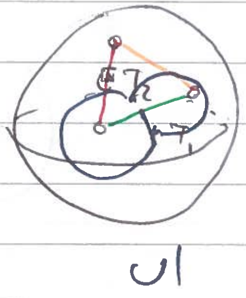


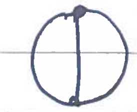
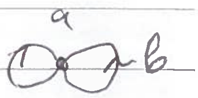
The group of symmetries ie. Decks group is $PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z})/\{\pm I\}$

$X = S^2 \setminus \{3 \text{ points}\}$







$T/D =$  $\cong$ 

Tree T

Farey graph

Topology and Groups

8th Feb

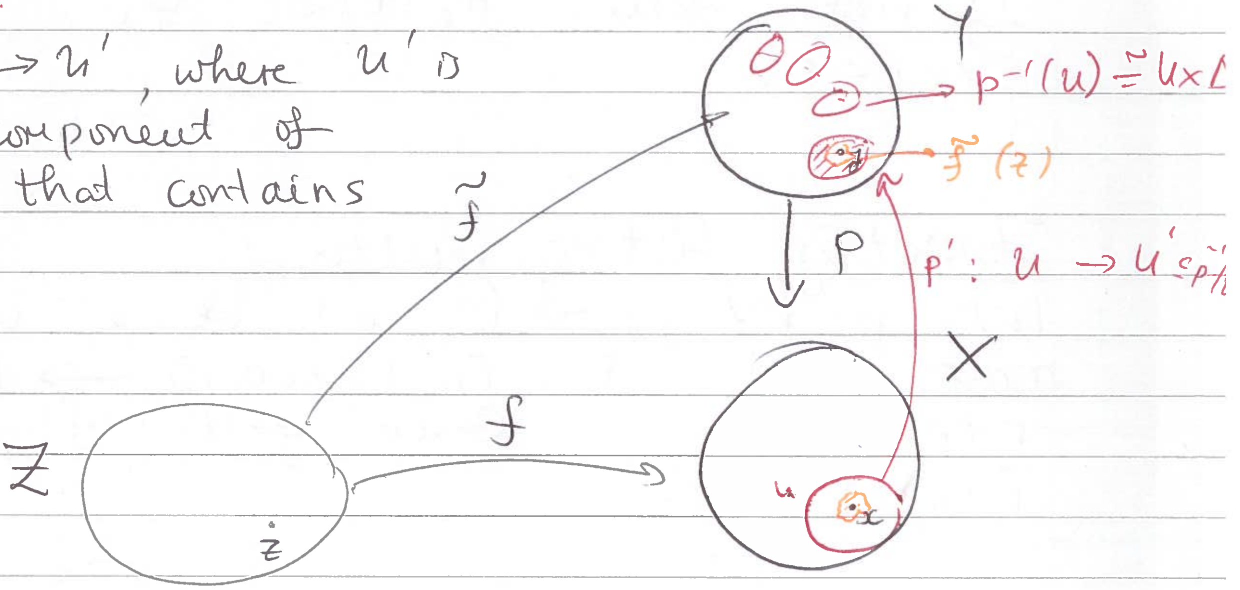
Basic Lifting lemma

$$\begin{tikzcd} & (Y,y) \arrow[d, "p"] \\ (Z,z) \arrow[ur, "\exists! \tilde{f}"] \arrow[r, "f"'] & (X,x) \end{tikzcd}$$

Suppose $f(Z) \subseteq U$ open covering neighbourh and Z is connected. Then $\exists! \tilde{f}$ s.t. $\tilde{f}(z) = y$

Proof:

$p' : U \to U'$, where U' is the component of $p^{-1}(U)$ that contains y



We can define $\tilde{f} = p' \circ f$. Z is connected $\Rightarrow \tilde{f}$ is unique ∎

Lebesgue number lemma: X compact metric space, $\{U_1, \dots, U_n\}$ finite open cover. Then $\exists\, \varepsilon > 0$ s.t. $\forall x \in X$, $B_\varepsilon(x) \subseteq U_i$ for some i

Proof: For each i let $d_i : X \to \mathbb{R}$ be the function $\inf_{y \in X \setminus U_i} d(x, y)$

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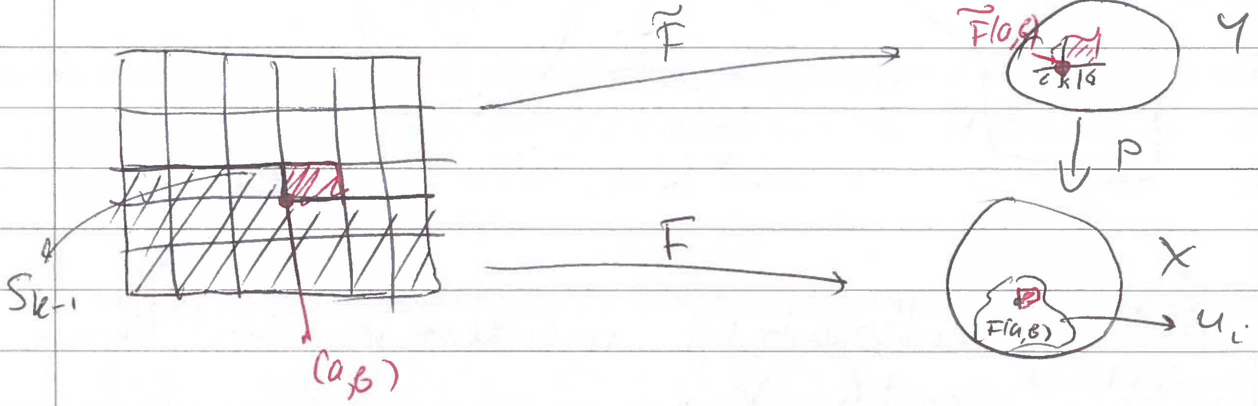
Proof: The collection of $F^{-1}(U)$ / U is a covering neighbourhood of x is an open cover of $[0,1]\times[0,1]$, which is compact $\Rightarrow \exists$ finite subcover $\{W_1, \dots, W_n\}$ with $W_i = F^{-1}(U_i)$, U_i is some covering neighbourhood.

By the Lebesgue number lemma $\exists$ subdivision of $[0,1]\times[0,1]$ into subsquares $S_1, \dots, S_e$, s.t. $S_j \subseteq W_i$ for some i. Diameter of each square $< \frac{\varepsilon}{10}$. Order them S_i from left to right, bottom to top.

Define $\tilde{F}$ inductively: On each square use the basic lifting lemma to produce a lift.

Suppose $\tilde{F}$ has been defined on $S_1 \cup .. \cup S_{k-1}$

want to define $\tilde{F}$ on $S_1 \cup .. \cup S_{k-1} \cup S_k$



Define $\tilde{F}|_{S_k}$ to be the lift of $F|_{S_k}$ to Y given by basic lifting lemma.

Topology and Groups

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So far: $(Y, y_0) \xrightarrow{p} (X, x_0)$, p is a cover map. Y, X path connected.

$$\underset{\text{bijection}}{(Y, y_0) \longmapsto} p_*(\pi_1(Y, y_0)) < \pi_1(X, x_0)$$

We show $(Y', y_1) \xrightarrow{p'} (X, x_0)$

If $p_*(\pi_1(Y', y_1)) = p_*(\pi_1(Y, y_0)) \Rightarrow$

$$(Y, y_1) \cong (Y, y_0)$$

<u>Surjectivity</u>: Given $H < \pi_1(X, x_0)$ need to build a covering space $(Y_H, y_H) \xrightarrow{p} (X, x_0)$ s.t.

$$p_*(\pi_1(Y_H, y_H)) = H$$

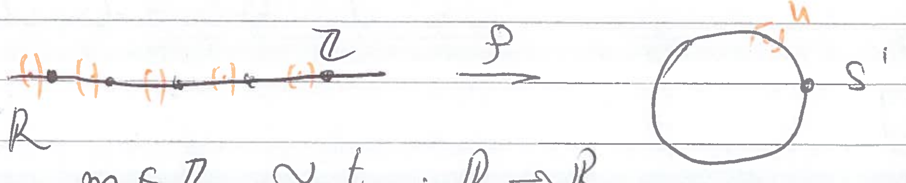
If $H \lhd \pi_1(X, x_0)$ then

$$\underset{\substack{\| \\ \text{Deck group}}}{\{\text{covering transformations of } (Y_H, y_H)\}} \cong \pi_1(X, x_0)/H$$

If $H = 1$ then Deck group $\cong \pi_1(X, x_0)$

Definition The covering space associated to 1 is called the "universal cover"

Example



$m \in \mathbb{Z} \sim t_m : \mathbb{R} \to \mathbb{R}$

$t_m(x) \to x + m$

Topology and Groups

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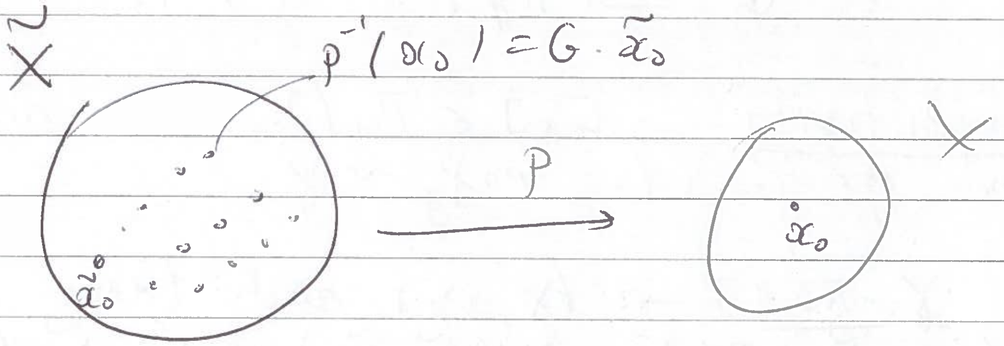
Proposition : If $G \curvearrowright \tilde{X}$ f.d. $\tilde{X}$ path connected and simply connected then $\pi_1(\tilde{X}/G) \cong G$

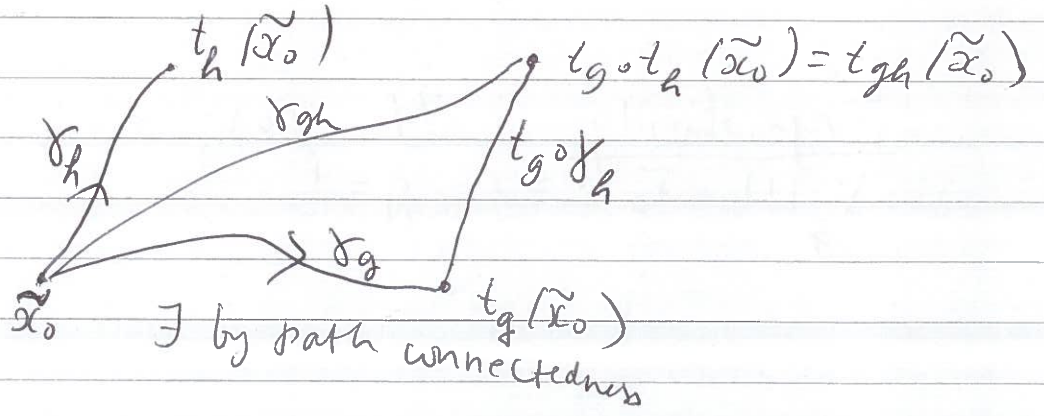
Proof : $p : \tilde{X} \to \tilde{X}/G = X$. Chose $x_0 \in X$, $\tilde{x}_0 \in p^{-1}(x_0)$



For each $g \in G$, let $t_g : \tilde{X} \to \tilde{X}$ is an associative map and let $\gamma_g : [0,1] \to \tilde{X}$ be a path s.t. $\gamma_g(0) = \tilde{x}_0$ and $\gamma_g(1) = t_g(\tilde{x}_0)$

Define : $G \to \pi_1(X, x_0)$ by
$g \to p \circ \gamma_g$

Picture :



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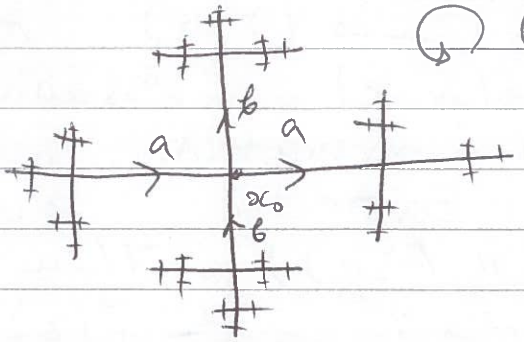
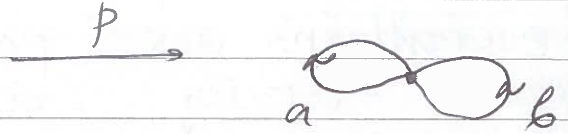
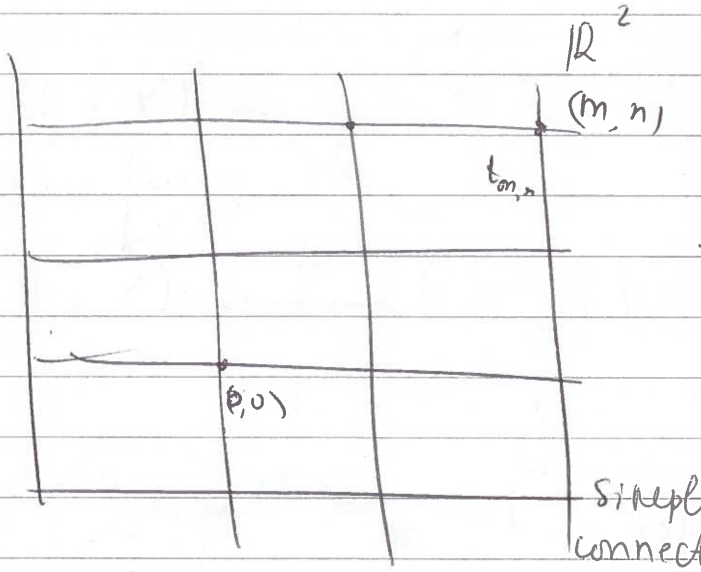


$\circlearrowright$ $G = \langle \text{Deck transf.} \rangle = F_2 = \langle a, b \rangle$





$\mathbb{Z}^2 \circlearrowright \mathbb{R}^2$ is f.d.

$$\begin{tikzcd} \{t_{m,n}\} \arrow[r, "\cong"] \arrow[d, equal] & \pi_1(T, x_0) \\ \mathbb{Z}^2 & \end{tikzcd}$$

Example $S^2 \longrightarrow \mathbb{R}P^2 = S^2/_{x \sim -x}$

$f : S^2 \to S^2 \qquad x \to -x$

$id_{S^2} : S^2 \longrightarrow S^2 \qquad x \to x$

$\langle f, id_{S^2} \rangle = \mathbb{Z}/2\mathbb{Z} = G$

$\pi_1(S^2) = 1$, $G \circlearrowright S^2$ is f.d.

$\Rightarrow \pi_1(\mathbb{R}P^2) = \mathbb{Z}/2\mathbb{Z}$ true $\forall n \geq 2$

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Continuity

$\tilde{f}$ is defined locally by a local inverse of p.

Pick a covering neighbourhood U of $f(z)$. Then look at $U' \subset Y$, $U' \subseteq U \times \Delta$

And look at $f^{-1}(U)$. So $\exists$ a path connected neighbourhood V of z contained in $f^{-1}(U)$. Then $V \subseteq \tilde{f}^{-1}(U')$

so $\tilde{f} = p' \circ f$ on V, where $p' : U \rightarrow U'$
U' contains y. i.e. the component of U' that contains y.

What if we have a different path
$\gamma' : [0,1] \rightarrow Z$ s.t. $\gamma'(0) = z_0$
$\gamma'(1) = z$

$\gamma \circ \bar{\gamma}'$ is a closed path in Z $\Rightarrow$
$[\gamma \circ \bar{\gamma}'] \in \pi_1(Z, z_0)$

$f_*([\gamma \cdot \bar{\gamma}']) \in p_*(\pi_1(Y, y_0))$

$\Rightarrow$ $(f \circ \gamma) \cdot f \circ \bar{\gamma}'$ lifts to a closed path in Y

By uniqueness of paths $\widetilde{f \circ \gamma'}$ should end up at the same point y as $\widetilde{f \circ \gamma}$.

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$$\begin{tikzcd}
 & (X_1, x_1) \arrow[d, "p_1"] \\
(X_1, x_1) \arrow[r, "p_1"] \arrow[ur, bend left, "\widetilde{id}"] \arrow[ur, "\tilde{p}_2 \circ \tilde{p}_1"'] & (X, x_0)
\end{tikzcd}$$

By uniqueness of lifts $\tilde{p}_2 \circ \tilde{p}_1 = id_{X_1}$

Simetrically $\tilde{p}_1 \circ \tilde{p}_2 = id_{X_2}$

$\Rightarrow$ $\tilde{p}_1$ & $\tilde{p}_2$ are homeomorphisms

$$\begin{tikzcd}
(X_1, x_1) \arrow[rr, "\tilde{p}_1"] \arrow[dr, "p_1"'] & & (X_2, x_2) \arrow[ll, bend left, "\tilde{p}_2"] \arrow[dl, "p_2"] \\
 & (X, x_0) &
\end{tikzcd}$$

Two connected covering spaces corresponding to the same subgroup of $\pi_1(X, x_0)$ are equivalent.

The subgroup determines the covering space.

2. (X, x_0) path and locally path connected. Assume X has a universal cover $\tilde{X}$. i.e.
$$\begin{tikzcd}
\tilde{X} \arrow[d, "p"] \\
X
\end{tikzcd}$$
, $\tilde{X}$ simply connected, connected

Recall Deck transformation is a homo.

$$\begin{tikzcd}
\tilde{X} \arrow[rr, "g"] \arrow[dr, "p"'] & & \tilde{X} \arrow[dl, "p"] \\
 & X &
\end{tikzcd}$$

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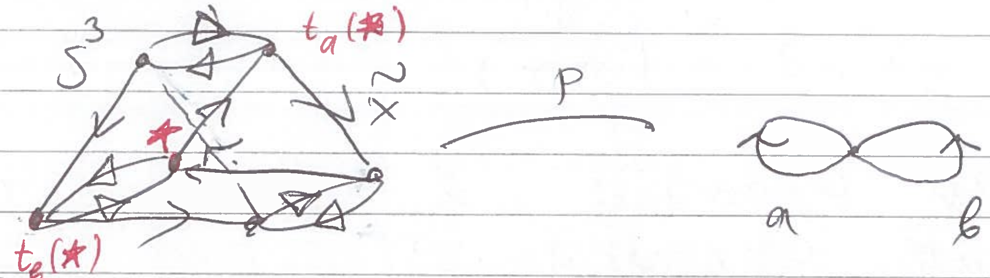
Illustrative examples



Deck group acts transitively

Symmetry : t_a clockwise rotation

t_b order 2

$t_a \circ t_b(*)$ = endpoints of the path ab

$p_*(\pi_1(\tilde{X})) \triangleleft \pi_1(\infty)$

Deck group $\cong \pi_1(\infty)/H$

$S^3 = \langle a, b \mid a^3, b^2, (ab)^2 \rangle$.

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Classification Theorem

X "nice". We want to provide a bijection

connected covering spaces$/\sim$ $\longleftrightarrow$ subgroups of $\pi_1(X)$

Deck group $\longleftrightarrow$ $H < \pi_1(X)$

$D.G = N(H)/H$

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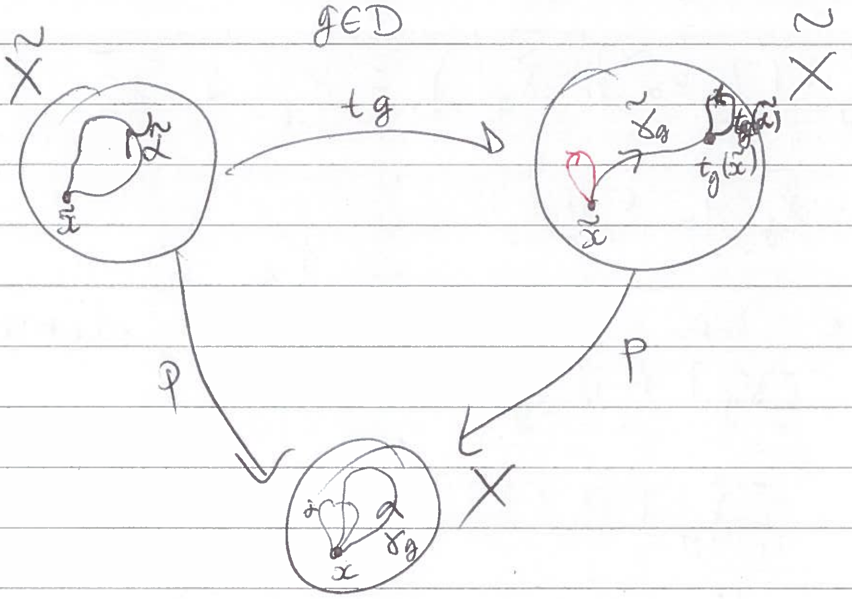
I. Given an element of D we want to produce an element of $N(P_*(\pi_1(\tilde{X}, \tilde{x})))$ $\overline{P_*(\pi_1(\tilde{X},\tilde{x}))}$

let $H = p_*\pi_1(\tilde{X}, \tilde{x}))$



Define $\tilde{\gamma}_g$ to be a path $\tilde{x}$ to $t_g(\tilde{x})$

and $\gamma_g = p \circ \tilde{\gamma}_g$, γ_g is a closed path in X

Since $p \circ t_g(\tilde{x}) = p(\tilde{x})$ for the diagram to commute so the start and end point of $\tilde{\gamma}_g$ becomes one in γ_g.

Need to show $[\gamma_g] \in N(H)$

Obsepse: 1. γ_g is only undefined up to prepending by elements of H.

Consider an element $[\alpha] \in H$, and a lift $\tilde{\alpha}$ to $\tilde{X}$ starting and ending at $\tilde{x}$. Then $t_g \circ \tilde{\alpha}$ is a closed

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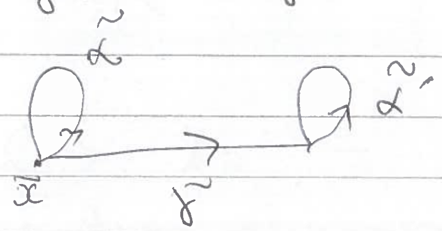
choose $[\gamma] \in N(H)$, $[\alpha] \in H$. Then
$[\gamma][\alpha][\gamma]^{-1} \in H$



$\gamma \alpha' \bar{\gamma}$ lifts to a closed path $\tilde{\gamma}\tilde{\alpha}'\overline{\tilde{\gamma}}$
where $\tilde{\alpha}'$ starts at $\tilde{\gamma}(1)$

$\Rightarrow$ α lifts to a closed path $\tilde{\alpha}'$ based at $\tilde{\gamma}(1)$ $\Rightarrow$ $p_*(\pi_1(\tilde{X}, \tilde{\gamma}(1))) < p_*(\pi_1(\tilde{X}, \tilde{x})) = H$

Likewise $p_*(\pi_1(\tilde{X}, \tilde{x})) < p_*(\pi_1(\tilde{X}, \tilde{\gamma}(1)))$

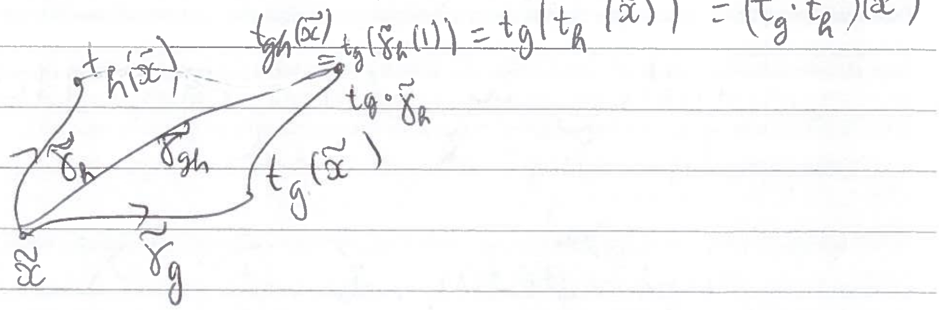
By the lifting lemma

$\exists\, p : (\tilde{X}, \tilde{x}) \longrightarrow (\tilde{X}, \tilde{\gamma}(1))$ which is

a deck transformation. The only ambiguity is up to an element of H

$\underline{\text{III}}$ Deck group $\longrightarrow N(H)/H$

In $\tilde{X}$:
$g, h \in D$



$\tilde{\gamma}_g \cdot (t_g \circ \tilde{\gamma}_h) \cdot \overline{\tilde{\gamma}_{gh}}$ loop based at $\tilde{x}$

$p \circ (\tilde{\gamma}_g \cdot (t_g \circ \tilde{\gamma}_h) \cdot \overline{\tilde{\gamma}_{gh}}) = \gamma_g \cdot \gamma_h \cdot \overline{\gamma_{gh}}$
$[\gamma_g \cdot \gamma_h \cdot \overline{\gamma_{gh}}] = [\gamma_g][\gamma_h][\gamma_{gh}]^{-1} \in H$

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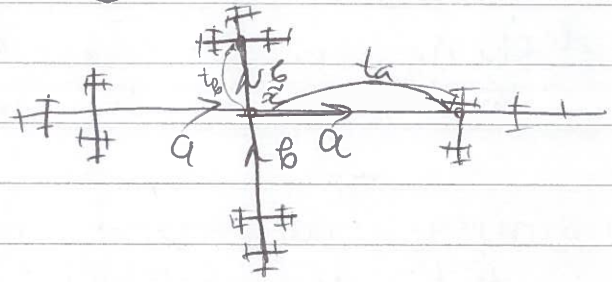
$\tilde{X}$: Universal cover



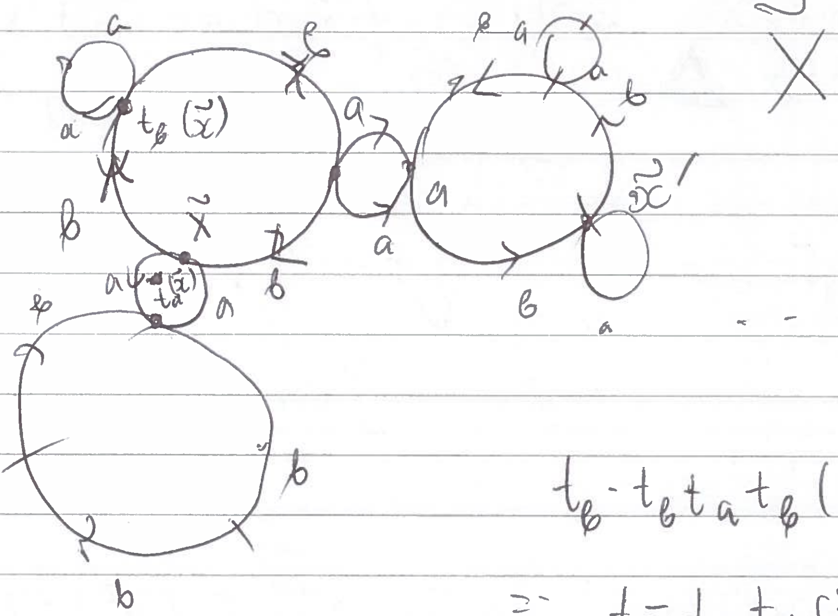
$\pi_1(\tilde{X}, \tilde{x}) = 1$

$\tilde{X}$ is path connected and locally path connected

$D = N(1)/1 =$
$= \pi_1(\infty)$ a b

$D = \langle t_a, t_b \rangle = F_2$

D is generated by t_a and t_b. Elements of $\pi_1(a \infty b)$ $\xleftrightarrow{\text{Bijection}}$ vertices in $\tilde{X}$ and every element in $\pi_1(\infty)$ can be associated in a unique way with a sequence of letters in the alphabet $a, b, \bar{a}, \bar{b}$



$t_b \cdot t_b t_a t_b(\tilde{x}) = \tilde{x}'$

$\equiv t_{\bar{b}} t_a t_b(\tilde{x}) = \tilde{x}'$

Observations: $t_b^3 = 1$, $t_a^2 = 1$

$t_b t_b t_a t_b = t_{\bar{b}} \underbrace{t_b t_b t_b}_{id} t_a t_b = t_{\bar{b}} t_a t_b$

D is generated by t_a and t_b and the relations $(t_a)^2 = id$ and $(t_b)^3 = id$

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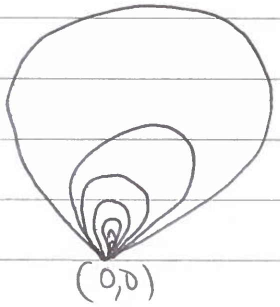
$\Rightarrow$ If X has a universal cover then $\forall x \in X$ $\exists\, \underset{\text{path connected}}{\mathcal{U}} \ni x$ s.t. $\pi_1(\mathcal{U}) \longrightarrow \pi_1(X)$ is trivial.

The above means semi-locally simply connected.

The 3 conditions are necessary but also sufficient.

Example

X



$= \bigcup_{n \in \mathbb{N}}$ circles of radius $\frac{1}{n}$ center at $(0, \frac{1}{n})$

$\tilde{X}$ does not have a~~n~~ Universal cover.
$\therefore$ Any neighbourhood $\mathcal{U}$ of $(0,0)$ has the feature that $\pi_1(\mathcal{U}) \twoheadrightarrow \mathbb{Z}$

There is a bijection

$$\begin{tikzcd} \tilde{X}_H \arrow[d] \arrow[r, leftrightarrow] & H < \pi_1(X) \\ X/\sim & \end{tikzcd}$$

If X has a universal cover

$$\begin{tikzcd} G \curvearrowright & \tilde{X} \arrow[d] \\ & X \end{tikzcd}$$

$G = \pi_1(X)$ is the deck group. The deck group~~s~~ acts freely and discontinuously. In particular if $H < G$ it also acts freely and discontinuously

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Normalizers

$$\begin{tikzcd}\tilde{x} \in \tilde{X} \arrow[d, "p"] \\ x \in X\end{tikzcd}$$

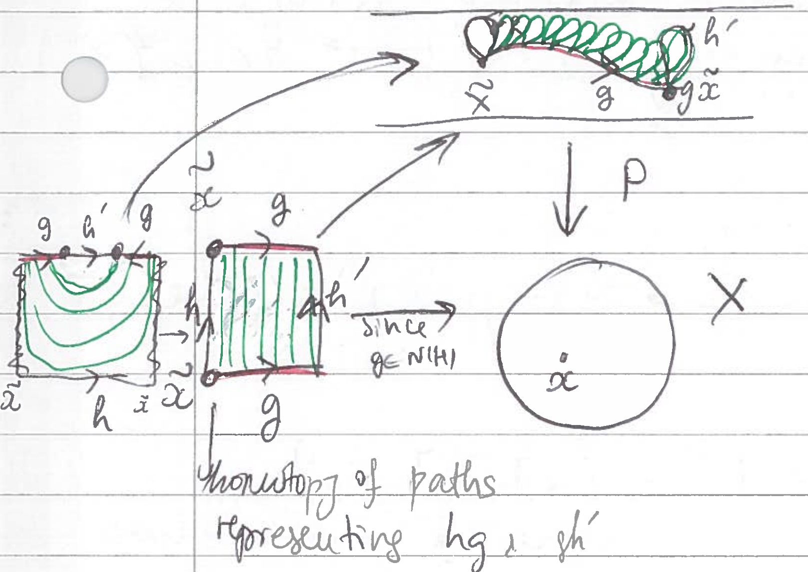
$H = p_*(\pi_1(\tilde{X}, \tilde{x})) < G = \pi_1(X, x)$

$H \triangleleft N(H)$

for $g \in N(H)$, $gHg^{-1} = H$

$gH = Hg$

Given $h \in H$ $\exists h' \in H$ s.t. $gh' = hg$

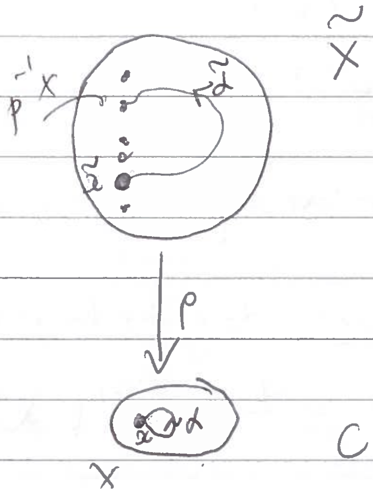


$\tilde{X} \circlearrowleft N(H)/H$

This picture doesn't depend on h

Cosets

$$\begin{tikzcd}\tilde{x} \in \tilde{X} \arrow[d, "p"] & H \\ x \in X & G \arrow[u, phantom, "\wedge"]\end{tikzcd}$$



Given $g \in G$
Represent $g = [\alpha]$
and take the lift of α
$\tilde{\alpha}$ s.t. $\tilde{\alpha}(0) = \tilde{x}$
$c: G \to p^{-1}(x)$ by
$g \to \tilde{\alpha}(1)$

1. well defined? If $\alpha \sim \alpha'$ ~~and~~ then $\tilde{\alpha}(1) = \tilde{\alpha}'(1)$
by homotopy lifting lemma

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Theorem: The degree of p (# of sheets) $= [G : H]$

Corollary: If $\tilde{X} \xrightarrow{p} X$, $\pi_1(\tilde{X}) = 1$
Then $|\pi_1(X)| = |p^{-1}(x)|$

Products

A, B, C topological spaces

$$\begin{tikzcd} A \times B \arrow[r, "p_B"] \arrow[d, "p_A"'] & B \\ A & C \arrow[l, "f"] \arrow[u, "g"'] \arrow[ul, "\exists! h"'] \end{tikzcd}$$

Given $f : C \to A$
$g : C \to B$
are continuous maps
then $\exists!$ continuous map $h : C \to A \times B$
s.t. $p_A h = f$ and
$p_B h = g$

The product topology $A \times B$ was designed so that the above statement is true.

$$h(x) = (f(x), g(x))$$

Why is h continuous?

Need to check on sets of the form $U \times B$, $A \times V$ where $U \subseteq A$, $V \subseteq B$ open.

$h^{-1}(U \times B) = f^{-1}(U)$ which is open ■

$= \{x \mid (f(x), g(x)) \in U \times B\} = \{x \mid f(x) \in U\}$

Pullbacks

Given $f : A \to C$ and $g : B \to C$
continuous

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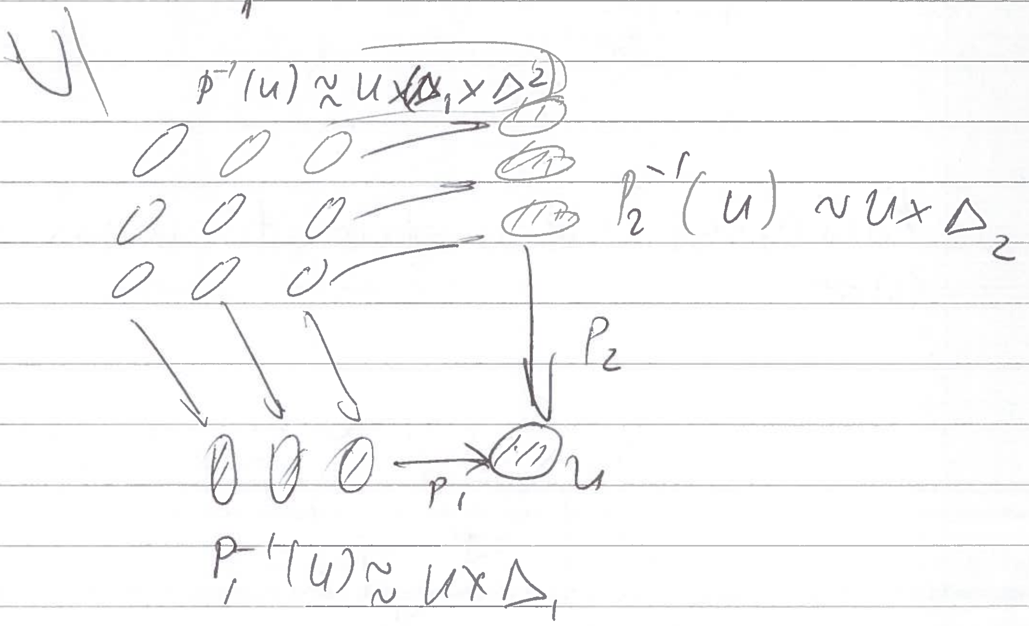
$$\begin{tikzcd}
U \simeq U \times_U U \subseteq U \times U \arrow[r] \arrow[d] & U \arrow[d] \\
U \arrow[r] & Y
\end{tikzcd}$$

Apply to Covering Spaces

$$\begin{tikzcd}
\tilde{X}_1 \times_X \tilde{X}_2 \arrow[r] \arrow[d] \arrow[dr, "p"] & X_2 \arrow[d, "p_2"] \\
X_1 \arrow[r, "p_1"'] & X
\end{tikzcd}$$



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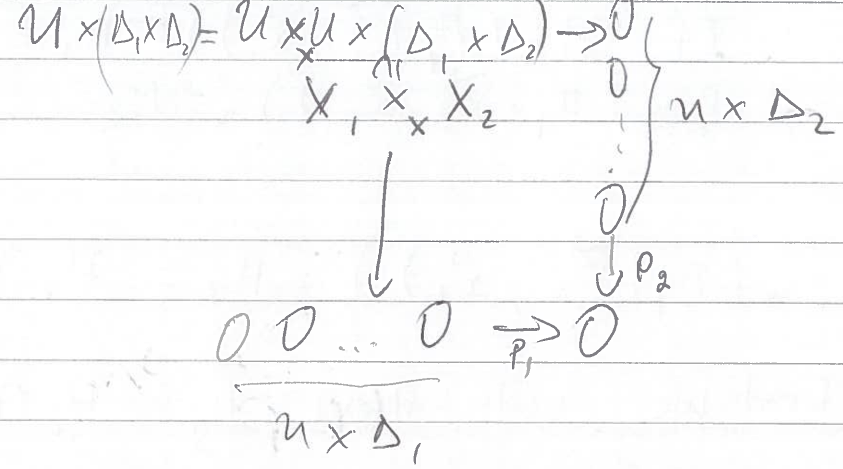
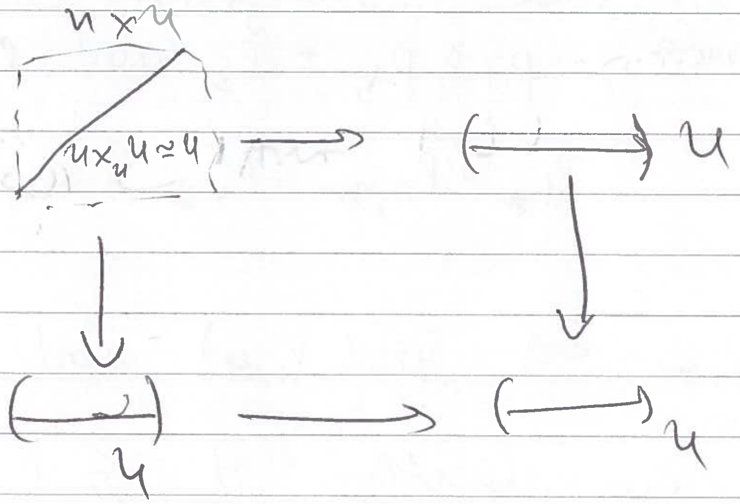


E.g



$$\begin{tikzcd}
\tilde{x}_3=(\tilde{x}_1,\tilde{x}_2)\in\tilde{X}_3 \arrow[r, "p_{x_2}"] \arrow[d, "p_{x_1}"] \arrow[dr, dashed, "p_3"] & \tilde{X}_2\ni\tilde{x}_2 \arrow[d, "p_2"] \\
\tilde{x}_1\in\tilde{X}_1 \arrow[r, "p_1"'] & X\ni x
\end{tikzcd}$$

$(*)$

where $\tilde{X}_3$ is the connected component of $\tilde{X}_1\times_X\tilde{X}_2$ containing $(\tilde{x}_1,\tilde{x}_2)$

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such that $p_{\tilde{x}_i} \circ \tilde{\gamma}_3 = \tilde{\gamma}_i$

$$\tilde{\gamma}_3(0) = (\tilde{\gamma}_1(0), \tilde{\gamma}_2(0)) = (\tilde{x}_1, \tilde{x}_2) = \tilde{x}_3$$

and $\tilde{\gamma}_3(1) = (\tilde{\gamma}_1(1), \tilde{\gamma}_2(1)) = (\tilde{x}_1, \tilde{x}_2) = \tilde{x}_3$

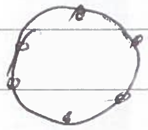
$$\Rightarrow [\gamma] \in P_{3*}(\tilde{X}_3, \tilde{x}_3) = H_3$$

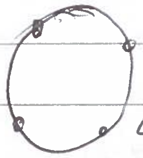
$$\Rightarrow H_1 \cap H_2 \subseteq H_3$$

$$\Rightarrow H_1 \cap H_2 = H_3$$ ■

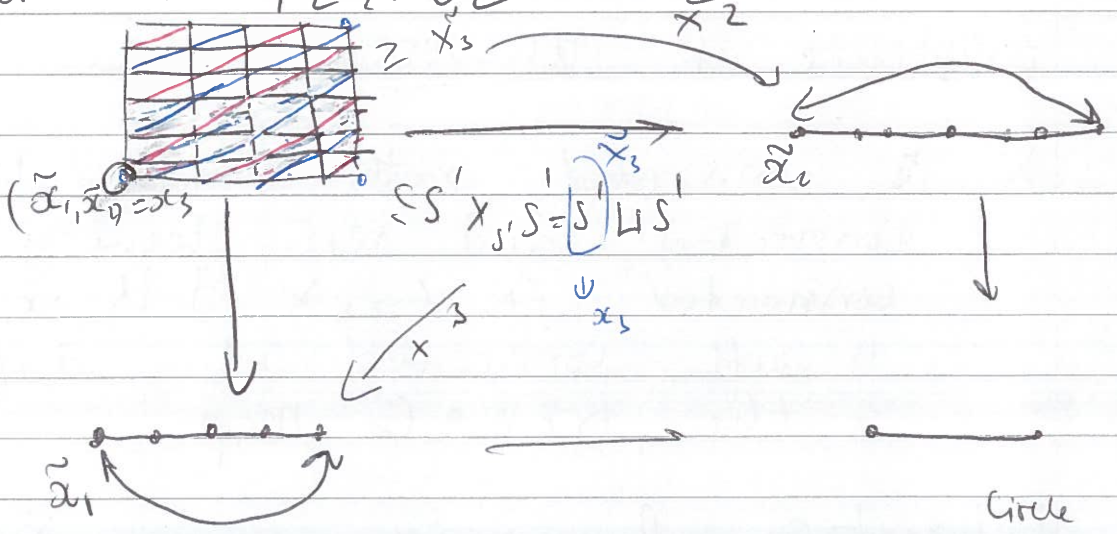
Example :

$$\begin{tikzcd}
 & S^1 \times_{S^1} S^1 \arrow[r] \arrow[d] \arrow[dr, "\times 24"] & S^1 \arrow[d, "\times 6"] & 6\mathbb{Z} \arrow[l] \\
 & S^1 \arrow[r, "\times 4"'] & S^1 & \\
4\mathbb{Z} \arrow[ur] & & & \mathbb{Z} \arrow[ul]
\end{tikzcd}$$





What is $4\mathbb{Z} \cap 6\mathbb{Z} = 12\mathbb{Z}$



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$$\begin{tikzcd}
\tilde{X} \arrow[d, "p"'] \arrow[dr] & \\
 & \tilde{X}/H \\
(X,x) \arrow[ur, "p_H"'] &
\end{tikzcd}$$

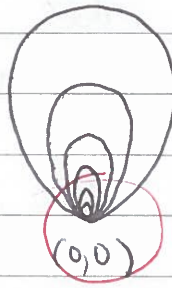
$\pi_1(X,x) = G \geq H$

$G \curvearrowright \tilde{X}$ by deck transform

$H < G$

$\tilde{X}$ covers every other covering space.

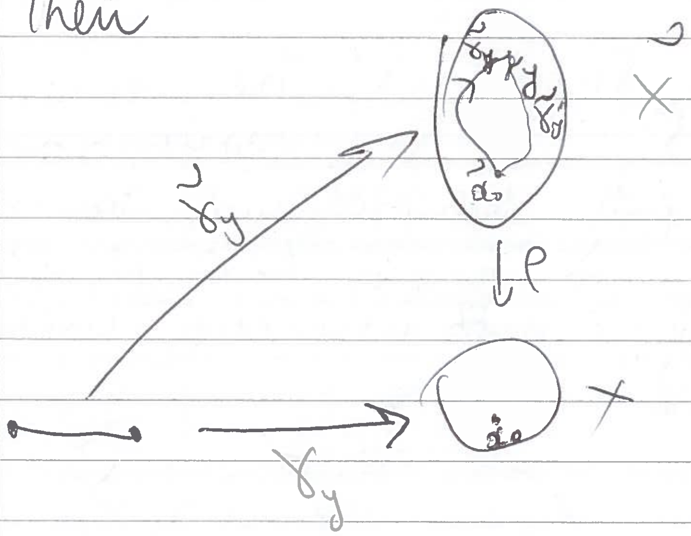
Example



Any neighbourhood of (0,0) doesn't induce the trivial map $\pi_1(U) \to \pi_1(X$

So no universal cover exists.

Suppose X has universal cover $\tilde{X}$.
Then



Let $y \in \tilde{X}$. Construct $\tilde{\gamma}_y$ a path in $\tilde{X}$ from $\tilde{x}_0$ to y.

There is a map from paths in $\tilde{x}$ starting

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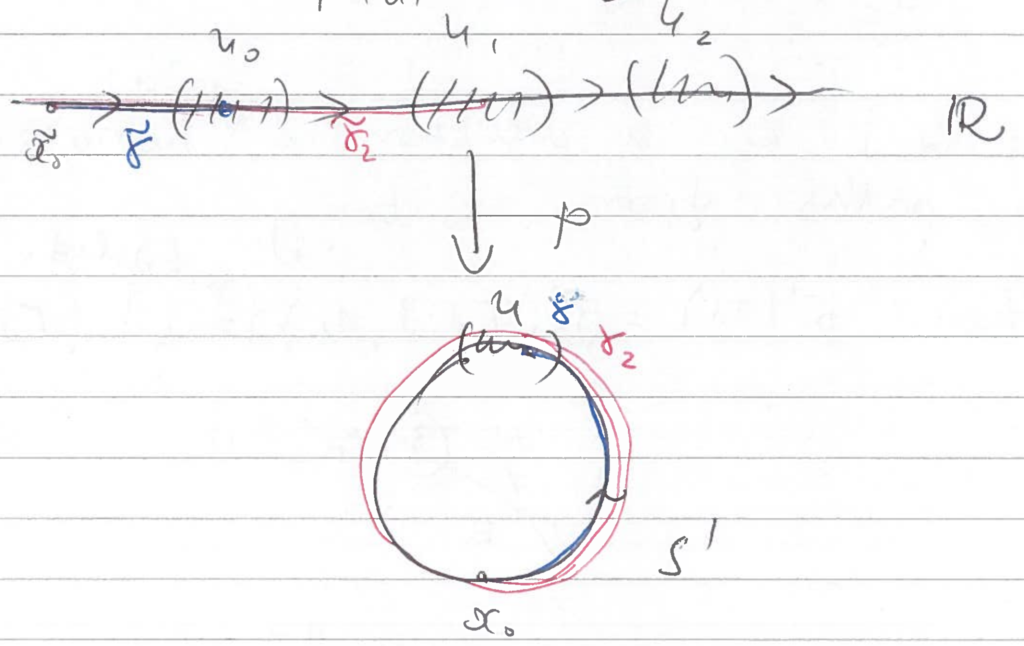
In $\tilde{X}$, we have a basis ~~of the~~ open sets of the form $U \times \{\delta\}$, $\delta \in \Delta$

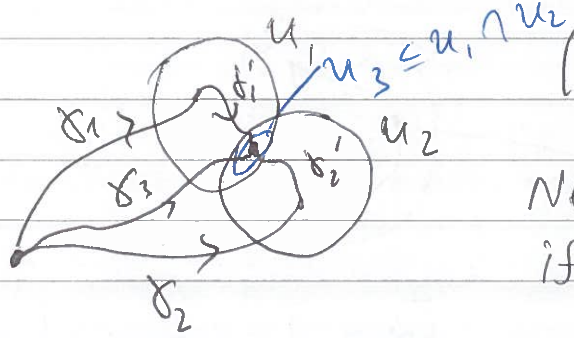
Define $([\gamma], U) = \{[\beta] \in X \mid \exists$ path γ' in U s.t. $\beta \sim \gamma \cdot \gamma'\}$

Take $\{([\gamma], U)\}$ to be a basis for a topology on $\tilde{X}$.

$p^{-1}(U) \cong U \times \mathbb{Z}$ ← discrete set



Why is $\{([\gamma], U)\}$ a basis of open sets?



$([\gamma_1], U_1) \cap ([\gamma_2], U_2) \neq \emptyset$

Need to show if $[\gamma_3] \in ([\gamma_1], U_1) \cap ([\gamma_2], U_2)$

Then $\exists$ U_3 of $\gamma_3(1)$ s.t. $([\gamma_3], U_3) \subseteq ([\gamma_1], U_1) \cap ([\gamma_2], U_2)$

So it forms a topology.

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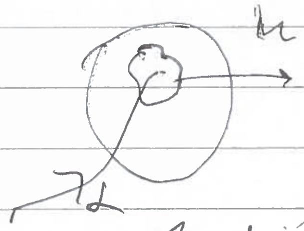
$\Rightarrow \alpha' \sim \alpha''$

$\Rightarrow \alpha_i \cdot \alpha' \sim \alpha_i \cdot \alpha''$

$\Rightarrow [\alpha_i \alpha'] = [\alpha_i \alpha''] \in ([\alpha_i], U)$

$([\alpha_i], U) \longrightarrow U$ continuous bijection

Why is the inverse continuous



$W \subseteq U$ $\pi_1(W) \rightarrow \pi_1(X)$ trivial

$p([\alpha], W) = W$

$\Rightarrow p$ is open

And since it is also cont. bijection

$\Rightarrow p|_{([\alpha], U)}$ is a homeo.

$\Rightarrow$ it is a covering map ▨

2. Why is $\tilde{X}$ path connected?

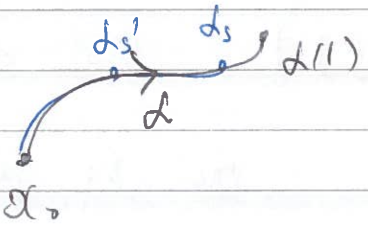
$[\alpha] \in \tilde{X}$, $\alpha(0) = x_0$ want a path from $[\alpha]$ to $[x_0]$

Define $\alpha_s = (t \longrightarrow \alpha(st))$

$\alpha_0 =$ constant path $= x_0$

$\alpha_1 = \alpha$

path : $s \longrightarrow [\alpha_s]$



$s' < s$

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i.e. $[\alpha] = [x_0] \Rightarrow \pi_1(\tilde{X}) = 1$ ■

Free Groups and graphs

Definition: S set, a free group on S is a group F with a map $\varphi : S \longrightarrow F$ s.t. if $\psi : S \longrightarrow H$ - group then $\exists!$ $h : F \longrightarrow H$ s.t.

$$\begin{tikzcd} & F \arrow[d, dashed, "!h"] \\ S \arrow[ur, "\varphi"] \arrow[r, "\psi"'] & H \end{tikzcd}$$

$$\mathrm{Hom}_{set}(S, H^{set}) \cong \mathrm{Hom}_{Group}(\underset{\cong F_S}{F}, H)$$

Lemma If F' is free on S, $\varphi' : S \longrightarrow F'$ Then $\exists!$ $h : F \longrightarrow F'$ s.t. $h \circ \varphi = \varphi'$

Proof:

$$\begin{tikzcd} & F \arrow[dd, shift left, "h"] \\ S \arrow[ur, "\varphi"] \arrow[dr, "\varphi'"'] & \\ & F' \arrow[uu, shift left, "h'"] \end{tikzcd} \rightsquigarrow \begin{tikzcd} & F \arrow[dd, bend right, "id_F"'] \arrow[dd, bend left, "h' \circ h"] \\ S \arrow[ur, "\varphi"] \arrow[dr, "\varphi"'] & \\ & F \end{tikzcd}$$

Uniqueness $\Rightarrow$ $id_F = h' \circ h$ similarly
$id_F = h \circ h'$

$\Rightarrow h$ is an isomorphism ■

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These arcs only connect the top to the bottom

There is a map $\Phi : S^* \longrightarrow F = \{$ reduced words $\}$
$\in \phi$

Group law on F: given w, w' reduced words group law is concatanate w & w' and reduce. $w \cdot w'$ is $\overline{ww'}$

Identity element: $\emptyset$

Inverses: Read a word in reverse and reverse the signs
$(a_1 a_2 a_1 a_2^{-1})^{-1} = a_2 a_1^{-1} a_2^{-1} a_1^{-1}$

The lemma $\Rightarrow$ multiplication is well defined and associative

$$\begin{tikzcd}
\overline{w_1 (\overline{w_2 w_3})} & \text{by the } \cancel{\overline{\text{is}}} \text{ lemma} & \overline{(\overline{w_1 w_2}) w_3} \\
& w_1 w_2 w_3 \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

$\varphi : S \longrightarrow F$ $\quad a_i \longrightarrow$ word with one letter a_i

$$\begin{tikzcd}
S \arrow[r, "\varphi"] \arrow[dr, "\psi"'] & F \arrow[d, "h"] \\
& H
\end{tikzcd}$$

$h(w(a_i^{\pm})) = w(\psi(a_i^{\mp}))$

$a_i \longrightarrow h(a_i)$

$a_i^{-1} \longrightarrow h(a_i^{-1})$

$h(ww') \overset{?}{=} h(w) h(w')$

Uniqueness follows from everything is completely determined by $\psi(a_i)$

$$L: \mathrm{Sym}^p(V) \otimes \mathrm{Sym}^q(V) \longrightarrow \mathrm{Sym}^{p+q} V$$

$$L\left(R^{\otimes(p+q)}(q)\ \underset{\overset{\downarrow}{x_1 \cdots x_p \otimes x_{p+1} \cdots x_q)}}{v} \right) = R^{\otimes(p+q)}(g) L(v)$$

$$L\left(R^{\otimes p}(x_1 \cdots x_p) \otimes R^{\otimes q}(x_{p+1} \cdots x_{q+p}) \right) =$$

$$= L\left(Rx_1 \cdots Rx_p \otimes Rx_{p+1} \cdots Rq_{q+p} \right) =$$

$$= Rx_1 \cdots Rx_{q+p} = R$$

RHS $R^{\otimes(p+q)}(g) L(v) =$

$$= R^{\otimes(p+q)}(g)(x_1 \cdots x_{p+q}) =$$

$$= Rx_1 \cdots Rx_{p+q}$$

Topology and Groups

10th Mar

Free groups

S, $F_S = \{$ redused words in $S^{\pm}\}$

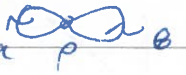
$$\begin{tikzcd} & F_S \arrow[d, "h"] \\ S \arrow[ur, "\varphi"] \arrow[r, "\psi"] & H \end{tikzcd}$$

$w(a_i^{\pm}) \longmapsto w(\psi(a_i)^{\pm})$

G : $F_G \xrightarrow{h} G$ (e.g. $GL_n(\mathbb{R})$)

$g \longrightarrow g$

$K_1 = Ker(h)$ encodes all relations in G.

Goals: $F := F_{\{a,b\}} = \pi_1($  $, p)$ free on $\{a, b\}$

$K < F$. and there is a cover $\tilde{R}$ of

that corresponds to K.

if $G = D_6 = \langle a, b / a^2, b^2, (ab)^6 \}$

In the cover we have  

$K \lhd F \Rightarrow$ the deck gr

$F/_K \cong D_6$

Graphs :

1-d. C-W complex

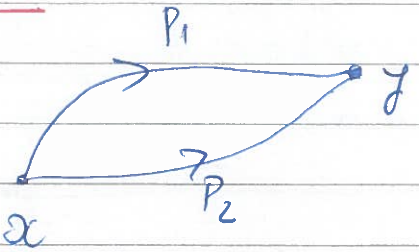
$X = X^{(0)} \amalg I/\sim$

Lemma: Graph is locally contractible.

Topology and Groups

10th Mar.

Proof of 1:



consider $p_1 \bar{p_2}$ if it is reduced then we have ✗ since we are in a tree.

Hence $p_1 \bar{p_2}$ is not reduced

let $p_1 = p_1' . e$ and $p_2 = p_2' e$

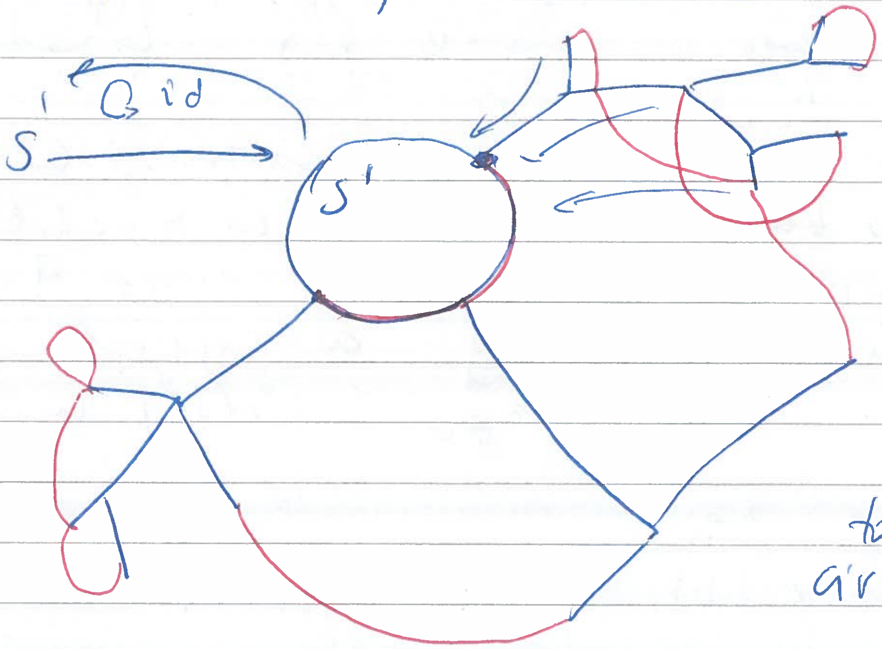
$$\Rightarrow p_1 \bar{p_2} = p_1' e . \bar{e} \bar{p_2'} = p_1' \bar{p_2'}$$

∎

* from 3: a tree is contractible & contr. + path conn. ⇒ simply connect

Not tree ⇒ not simply connect.
If you are not a tree then you contain an embedded circle. S^1

$X \geq S^1$, $X = S^1 \cup \{T\} \cup$ edges



$T_i \cap T_j = \phi$
No more loops.

Mapping all pts on tree to the point of the circle and edges – to edges on the circle connecting the points $\Rightarrow \pi_1(X) \neq 1$

Topology and Groups

10th Mar

Let w_g = reduced edge path from $\tilde{b}$ to $\tau_g(\tilde{b})$

Think of w_g as a reduced word in $\{a_1^{\pm}, \ldots, a_n^{\pm}\}$

Since for every reduced edge paths $\exists$ reduced words and every element of $\pi_1(R_n\ b)$ ~~is~~ can be written in terms of reduced edge paths Ⓔ

Topology and Groups   17th Mar

Example $abbabb$

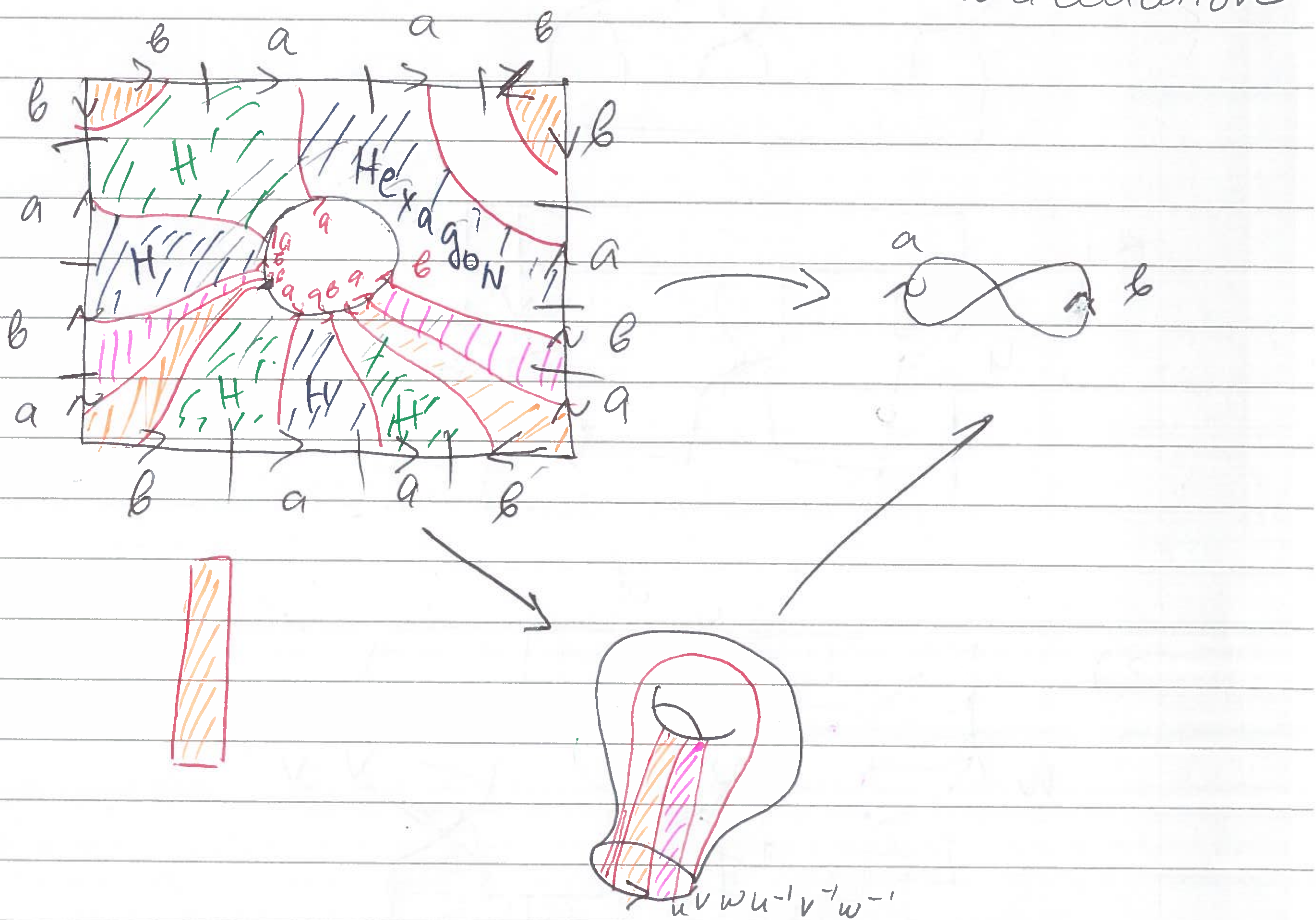
$bwb^{-1} = ba\,bba\,b$

Example $[abab^{-1}, baab^{-1}] = \gamma$

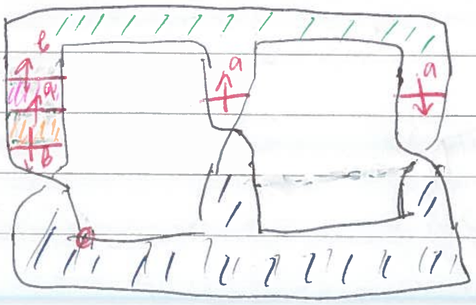
$abab^{-1}baab^{-1}ba^{-1}b^{-1}a^{-1}ba^{-1}a^{-1}b^{-1} =$

$= abaab^{-1}a^{-1}ba^{-1}a^{-1}b^{-1}$

$b^{-1}\gamma b = \underbrace{b^{-1}ab}_{u}\underbrace{a}_{v}\underbrace{a}_{w}\underbrace{b^{-1}a^{-1}b}_{u^{-1}}\underbrace{a^{-1}}_{v^{-1}}\underbrace{a^{-1}}_{w^{-1}} = uvwu^{-1}v^{-1}w^{-1}$ without cancellation



↑ Torus with boundary   Torus with boundary



$\underbrace{b^{-1}ab}_{u}\underbrace{a}_{v}\underbrace{a}_{w}\underbrace{b^{-1}a^{-1}b}_{u^{-1}}\underbrace{a^{-1}}_{v^{-1}}\underbrace{a^{-1}}_{w^{-1}}$

$[uv, wu^{-1}]$

Topology and Groups

17th Mar.

Lyndon's Theorem

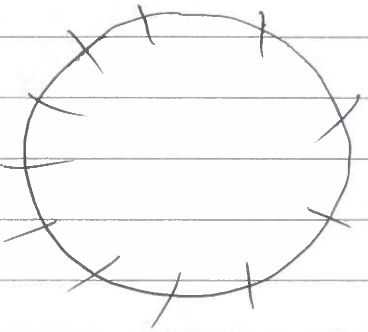
A commutator in a free group is not a proper power

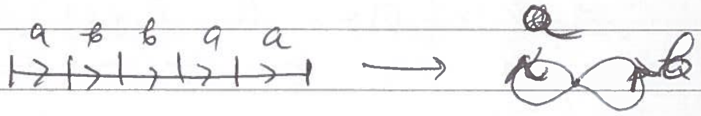
Definition: A commutator is an element of the form $[u,v]=uvu^{-1}v^{-1}$, $u,v\in F=\langle a,b\rangle$

Definition: A cyclically reduced word is a reduced word which doesn't look like

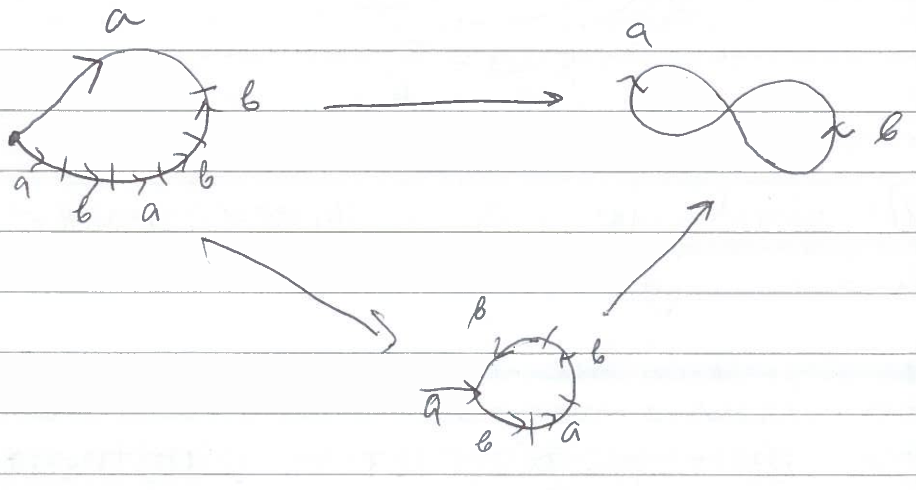
$\genfrac{}{}{0pt}{}{a}{b}$______ $\genfrac{}{}{0pt}{}{a^{-1}}{b^{-1}}$ or $\genfrac{}{}{0pt}{}{a^{-1}}{b^{-1}}$______ $\genfrac{}{}{0pt}{}{a}{b}$

A cyclically reduced word looks like a circle





Example $ababba^{-1} \sim babb$



Topology and Groups 21st Mar.

Lyndon's Theorem

$[p,q] \neq z^k$, $k > 1$ for $p,q \in \mathbb{F}$

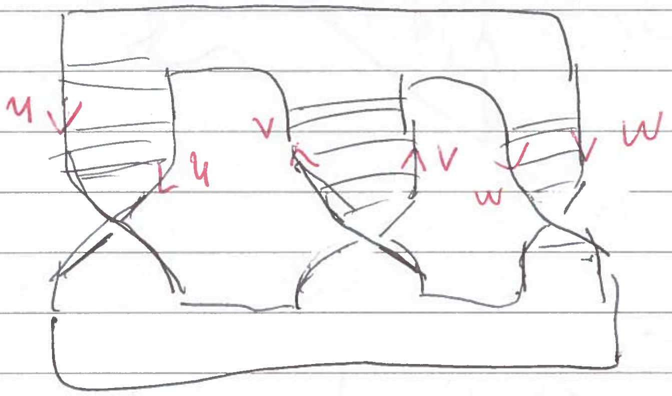
Wick's forms

$[p,q]$ "=" $uvu^{-1}v^{-1}$ conjugates to a word of this form

or "=" $uvwu^{-1}v^{-1}w^{-1}$

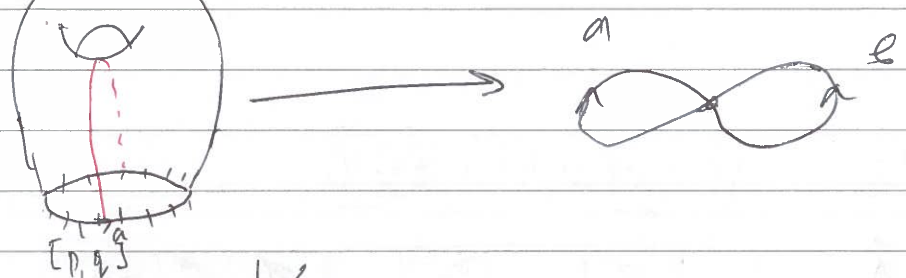
or "=" $udvd^{-1}u^{-1}dv^{-1}d^{-1}$

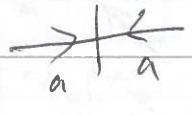
As a reduced product



Torus

u, v, w pieces





If we have a word $\gamma \in \mathbb{F}$ cyclically reduced $(a \times a^{-1})$

$$\begin{tikzcd} P_\gamma = \{(x,y) \in S^1 \times S^1 \mid \gamma(x) = \gamma(y)\} \arrow[r] \arrow[d] \arrow[loop above, "i"] & S^1 \arrow[d, "\gamma"] \\ S^1 \arrow[r, "\gamma"] & \infty \end{tikzcd}$$

i exchanges the factors.

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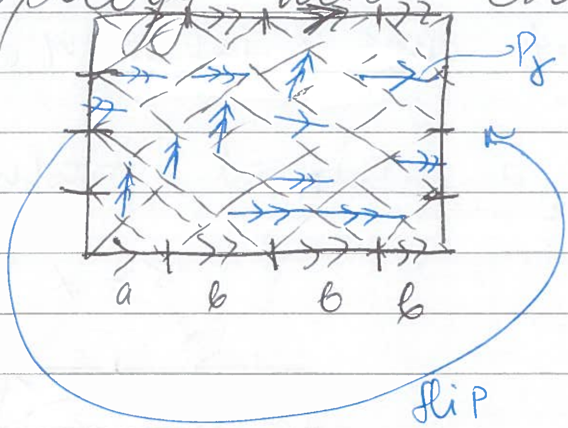
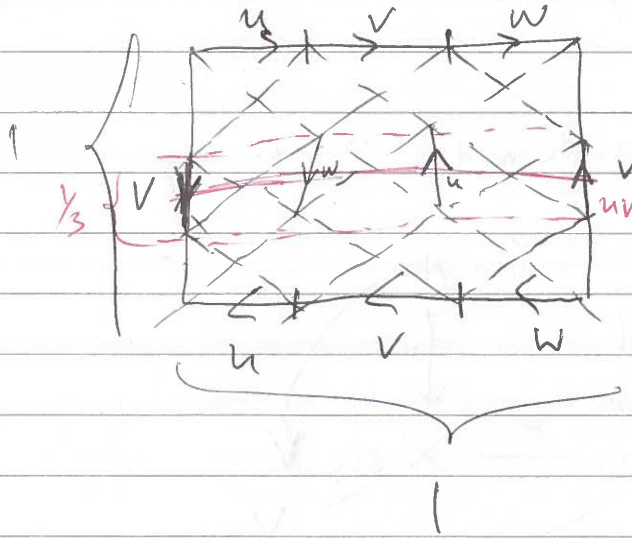


Observation: Never see a TEE

Cancellation i.e. γ is not reduced

Apply to Wicks's form of a commutator

$uvwu^{-1}v^{-1}w^{-1}$ as a reduced product



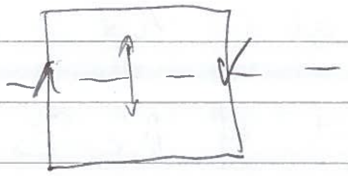
lengths are not accurate

P_γ contains a subset that looks like

$uvwu^{-1}v^{-1}w^{-1} = \alpha^2$
$\Rightarrow \exists$ TEE ✱
If $uvwu^{-1}v^{-1}w^{-1} = \alpha^3$
$\Rightarrow \exists$ TEE ✱

Observe: 1) $|u| + |v| + |w| = 1$

2) Symmetric with arrow flips through the horizontal axis.



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Ex : Top. spaces: $X = A \cup_C B$, $A \cap B = C$

$$\begin{tikzcd}
C \arrow[r, "\alpha"] \arrow[d, "\beta"'] & A \arrow[d, "\alpha'"] \arrow[ddr, bend left, "f"] & \\
B \arrow[r, "\beta'"] \arrow[drr, bend right, "g"'] & X \arrow[dr, dashed, "\exists! h"] & \\
& & Y
\end{tikzcd}$$

$\exists!$ continuous map

$h : X \longrightarrow Y$

s.t. $h\beta' = h\alpha'$

Ex : Groups

$$\begin{tikzcd}
1 \arrow[r] \arrow[d] & \mathbb{Z} = \langle a \rangle \arrow[d] \arrow[ddr, bend left, "a \to f(a)"] & \\
\mathbb{Z} = \langle b \rangle \arrow[r] \arrow[drr, bend right] \arrow[drr, bend right=40, "b"', "g(b)"] & F_{\{a,b\}} = \langle a, b \rangle \arrow[dr, "\exists! h"] & \\
& & G
\end{tikzcd}$$

Q: When do they exist? tricky
Are they unique? easy

Proof of uniqueness Suppose X, X' are a pushout

$$\begin{tikzcd}
C \arrow[r] \arrow[d] & A \arrow[d] \arrow[ddr, bend left] & \\
B \arrow[r] \arrow[drr, bend right] & X \arrow[dr, shift left, "\exists! h"] & \\
& & X' \arrow[ul, shift left, "h'"]
\end{tikzcd}$$

since X is pushout

since X' is pushout

Topology and Groups 21st Mar.

Lemma: Any word is equivalent to a unique reduced word.

$G * H = \{$ reduced words in $G \cup H \}$

$\overset{\Downarrow}{w} = g_1 h_1 g_2 h_2 \cdots g_n h_n$ with

$g_i \neq 1$ for $i > 1$ i.e. g_1 can be $=1$

$h_i \neq 1$ for $i < n$ or ~~h_n~~ h_n can be $= 1$

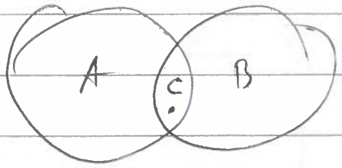
$$\begin{tikzcd}
1 \arrow[r] \arrow[d] & G \arrow[d] \arrow[ddr, bend left, "f"] & \\
H \arrow[r] \arrow[drr, bend right, "g"'] & G * H \ \exists h! \arrow[dr] & \\
& & L
\end{tikzcd}$$

$w = g_1 h_1 \cdots g_n h_n$

Define h by $h(w) = f(g_1) g(h_1) \cdots f(g_n) g(h_n)$

Van Kampen

$X = A \cup_C B$ and $A \cap B = C$, A, B, C are path connected & open.



Then the diagram

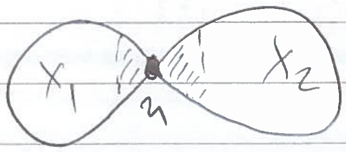
$$\begin{tikzcd}
\pi_1(C) \arrow[r] \arrow[d] & \pi_1(A) \arrow[d] \\
\pi_1(B) \arrow[r] & \pi_1(X)
\end{tikzcd}$$

is a pushout.

$X = X_1 \cup_b X_2$ connected complexes

b has a contractible neighbourhood $\mathcal{U}$.

$A = X_1 \cup \mathcal{U}$ ← open



Topology and Groups 21st Mar.

Definition: $R \subseteq G$ then $\boxed{\langle\langle R \rangle\rangle = \bigcap_{H \trianglelefteq G, R \subseteq H} H}$

The normal closure

Definition: A presentation of a f.g group G is: 1) surj map $\mathbb{F}_{\{a_1,\dots,a_n\}} \xrightarrow{f} G$

2) A list of elements $\{r_1, \dots, r_n, \dots\} = R$ such that $\langle\langle R \rangle\rangle = \ker(f)$

write $\langle a_1, \dots, a_n \mid r_1, \dots, \rangle$

Definition: G is finitely presented if $\exists$ $G \cong \langle a_1, \dots, a_n \mid r_1, \dots, r_m \rangle$

Theorem G is finitely presented iff $G = \pi_1(X)$ where X is 2-dim compact CW complex.

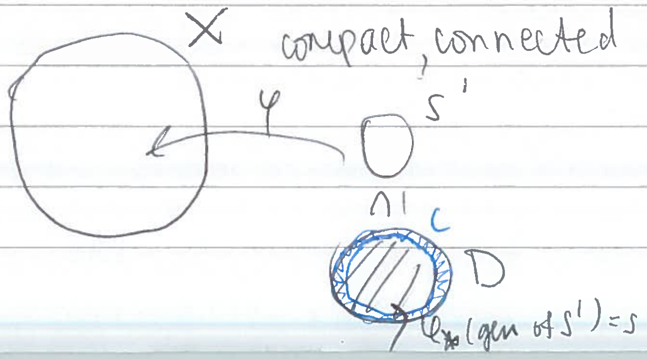
Suppose $G = \langle a_1, \dots, a_n \mid r_1, \dots, r_m \rangle$ and $s \in G$. Then:

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[d, bend right] & G \arrow[d, bend left] \\ 1 \arrow[r] & G/\langle\langle s \rangle\rangle \end{tikzcd}$$

gen $\longmapsto$ s

This diagram is a pushout.

G finite presentable then find a compact 2 complex with $\pi_1(X) = G$. X is compact connected CW complex, show that $\pi_1(X)$ is finitely presented



$\varphi : S^1 \longrightarrow X^{(1)}$

$X = X' \cup_\varphi D$

Topology and Groups
Non - Examinable

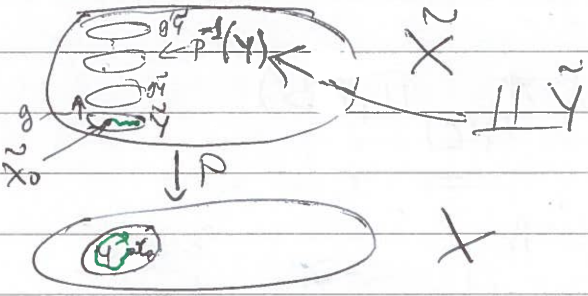
24th Mar

Bass - Serre Theory

$x_0 \in Y \subseteq X$ connected, path connected, Y open

$\pi_1(Y, x_0) \hookrightarrow \pi_1(X, x_0)$ inj.

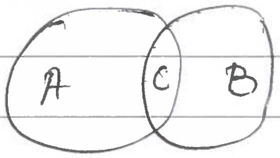
X has universal cover $\tilde{X}$.



Observation: $\mathrm{Stab}(\tilde{Y}) = \pi_1(Y) < \pi_1(X)$

$\mathrm{Stab}(g\tilde{Y}) = g\,\pi_1(Y)\,g^{-1}$ g is a Deck Transf.

SVK:



$X = A \cup_C B$

C path connected

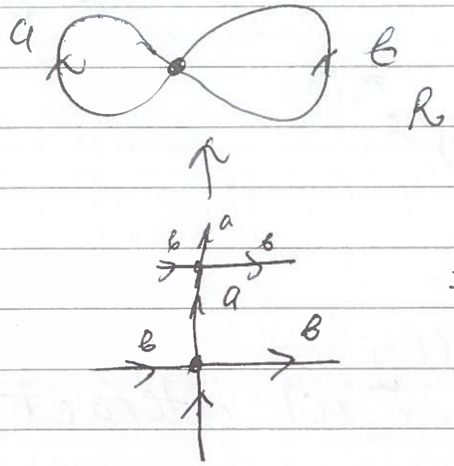
$$\begin{tikzcd}
\pi_1(C) \arrow[r] \arrow[d] & \pi_1(A) \arrow[d] \\
\pi_1(B) \arrow[r] & \pi_1(X)
\end{tikzcd}$$

push out.

Assumption: $\pi_1(C) \hookrightarrow \pi_1(A)$ injective
$\pi_1(C) \hookrightarrow \pi_1(B)$

Topology and Groups

24th Mar.



$Z_i \subseteq Z_{i+1}$

$\tilde{R} = \bigcup Z_i$

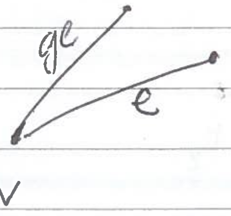
There is a tree T:
vertices of T are
1. translates of $\tilde{A}$
2. translates of $\tilde{B}$

Edges of T:
put an edge between $g\tilde{A}$ and $h\tilde{B}$ if they're connected by a copy of $\tilde{C} \times I$

There is a G-equivariant map
$X' \longrightarrow T$

In $\tilde{X}$, stab of $g\tilde{A}$ is $g\pi_1(A)g^{-1}$
$\longleftrightarrow$ vertex stabilizers are conjugates of $\pi_1(A)$ & $\pi_1(B)$.

Edge stabilizers of T are conjugates of $\pi_1(C)$



$\text{stab}(e) \hookrightarrow \text{stab}(v)$

Topology and Groups

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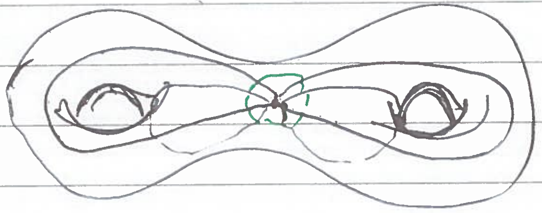
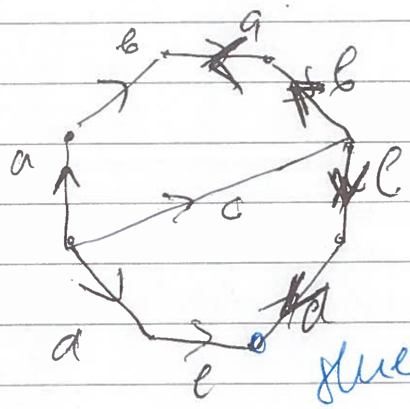
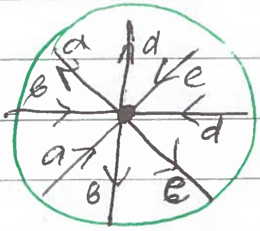


= ? open disk





glue 8 octagons around a vertex
a vertex