# M205 Topology and Groups Notes

Based on the 2017 autumn lectures by Dr J Evans

The Author(s) has made every effort to copy down all the content on the board during lectures. The Author(s) accepts no responsibility for mistakes on the notes nor changes to the syllabus for the current year. The Author(s) highly recommends that the reader attends all lectures, making their own notes and to use this document as a reference only.

MATH M205

05-10-17 Topology and Groups

Dr Jonny Evans

Notes are on Moodle.

Will be 5 question sheets, 2 weeks (ish) for each.

Office hour:

- The letters A, B, C cannot be deformed into one another.
- No matter how you stir a cup of coffee there is a point on the surface that comes back to itself. (Brower's fixed point thm)
- Theres a point on the earth which has exactly the same temperature and barometric pressure as its antipode. (Borsuk-Ulam thm)
- You cannot untie a trefoil knot  $\not\to$ O
-  You cannot "unlink" the Borromean rings.

Develop:

The language for talking about topological spaces.

We will associate, to each topological space, a group called the "fundamental group", $\pi_1$.

A : $\mathbb{Z}$ , B : $\mathbb{Z} * \mathbb{Z}$ , C : $\{1\}$

For the example of , take $\mathbb{R}^3 \setminus$  and $\mathbb{R}^3 \setminus$ O and prove $\pi_1$ of this space is non abelian and $\pi_1$ of this space is $\mathbb{Z}$.

Can go from topology $\to$ groups but also groups $\to$ topology and prove theorems in group theory using topology.

- $PSL(2,\mathbb{Z})$ = group of Möbius transformations $\frac{az+b}{cz+d}$ $a,b,c,d \in \mathbb{Z}$, $ad-bc=1$. $PSL(2,\mathbb{Z}) = \langle X, Y \mid X^2 = Y^3 = 1 \rangle = \mathbb{Z}/_2 * \mathbb{Z}/_3$

- Nielsen - Schreier theorem:
Any subgroup of $\langle X, Y | \rangle$ (free group) is free
i.e. it has a presentation of the form $\langle A_1, A_2, \ldots | \rangle$

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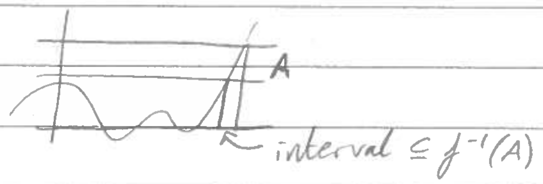
$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $\underbrace{|x-y| < \delta}_{\text{if } y \in \text{ interval of size } 2\delta \text{ around } x} \Rightarrow \underbrace{|f(x) - f(y)| < \varepsilon}_{\text{value } f(y) \text{ is within an interval of width } 2\varepsilon \text{ around } f(x)}$

We can reformulate this definition without inequalities just using the notion of an open set.

Def
A map $f: \mathbb{R} \mapsto \mathbb{R}$ is continuous at $x$ if for any open neighbourhood of $A$ of $f(x) \in \mathbb{R}$, the preimage $f^{-1}(A)$ contains an open interval around $x$.




$A = (f(x) - \varepsilon, f(x) + \varepsilon)$
interval $(x - \delta, x + \delta) \subseteq f^{-1}(A)$.

Def
A subset $U \subseteq \mathbb{R}$ is open if, $\forall x \in U$ $\exists$ open interval containing $x$ & contained in $U$.

Reformulate the definition of continuity:
Def
$f: \mathbb{R} \mapsto \mathbb{R}$ is continuous if $f^{-1}(U)$ is open whenever $U$ is open.

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Def

A topological space is a set $X$ equipped with a topology $T$, which is a collection of subsets of $X$ ("open sets") satisfying the following axioms.

- arbitrary unions of open sets are open
- finite intersections of open sets are open
- $X$ is an open set, $\phi$ is an open set

Let $U_n = (\frac{-1}{n}, \frac{1}{n})$
then $\bigcap_{n=1}^{\infty} U_n = \{0\}$ not open

Examples

1). $X = \mathbb{R}$, $T = \{$ subsets $U$ of $\mathbb{R}$ st. $\forall x \in U$ $\exists$ open interval $I$ st. $x \in I \subseteq U \}$
= set of open sets in $\mathbb{R}$

This is at topology.

2). $X = \{0, 1\}$, $T =$ set of all subsets of $X$
$= \{\phi, \{0\}, \{1\}, \{0,1\}\}$

This is a topology (unions & intersections of open sets are open).
"Discrete topology" [open $\Rightarrow$ in $T$]

3). $X = \{0, 1\}$, $T = \{\phi, \{0,1\}\}$
This is a topology
"Indiscrete topology"

What are the continuous maps $(X, T_{discrete}) \xrightarrow{F} (X, T_{indiscrete})$?
Any map is continuous. To see this, note that, for any map $F$, ~~[illegible]~~ $F^{-1}(A) \in T_{discrete}$ for any set $A \subseteq X$

What are the continuous maps $(X, T_{indiscrete}) \xrightarrow{F} (X, T_{discrete})$?

Lemma

If $F:(X, T_{indiscrete}) \mapsto (X, T_{discrete})$ is continuous then $F$ is constant.

Proof

$\forall x \in X$ the set $\{x\}$ is open in the discrete topology, $T_{discrete}$, so $F^{-1}(\{x\})$ is open in $T_{indiscrete}$ when $F$ is continuous.

$\Rightarrow F^{-1}(\{x\}) = \begin{cases} \emptyset \\ X \end{cases}$ so $F$ is constant.

$\square$

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Topological space is a set $X$, together with a <u>choice</u> of topology $T$ ($T$ is the list of all sets that we declare to be open)

$T$ is a collection of subsets of $X$ s.t.

- $X, \emptyset \in T$
- If $\mathcal{U} \subseteq T$ then $\bigcup_{u \in \mathcal{U}} u \in T$ ($T$ preserved by arbitrary unions)
- If $U, V \in T$ then $U \cap V \in T$.

A map $F : (X, T) \mapsto (X', T')$ is continuous if $\forall U \in T'$, $F^{-1}(U) \in T$.

<u>Lemma</u>

If $F : X \mapsto Y$ & $G : Y \mapsto Z$ are cts then $G \circ F : X \mapsto Z$ is cts.

<u>Proof</u>

If $U \subseteq Z$ is open then $G^{-1}(U) \subseteq Y$ is open so $F^{-1} G^{-1}(U)$ is open in $X$

" $(G \circ F)^{-1}(U)$. $\square$

<u>Def</u>

A continuous map $F : X \mapsto Y$ is a <u>homeomorphism</u> if

1). $F$ is bijective.

2). $F^{-1}$ is continuous.

e.g. $[-\pi, \pi) \mapsto S^1 \subseteq \mathbb{R}^2$, $\theta \to e^{i\theta}$

bijective, continuous but <u>not</u> a homeomorphism

$— \to O$

## Bases

**Def**

If $(X, T)$ is a topological space and $B \subseteq T$ we say $B$ is a <u>basis</u> for $T$ if every set from $T$ is a union of sets from $B$.

e.g. if $X = \mathbb{R}$ then any open set is a union of open intervals, so $B = \{\text{open intervals}\}$ is a basis for the topology.

**Lemma**

If $X$ is a set and $B$ is a collection of subsets, let $T$ be the (larger) collection of subsets obtained by taking unions of sets from $B$.

$T$ is a topology iff:

- $X \in T$ i.e. the subsets from $B$ cover the whole of $X$
- If $U, V \in B$ then $U \cap V \in T$

**Proof**

$\emptyset \in T$ because $\emptyset$ is the union of <u>no</u> sets from $B$

$X \in T$ by assumption.

$T$ is preserved under taking unions by definition.

If $U, V \in T$ then $U = \bigcup_{P \in B'} P$, $V = \bigcup_{Q \in B''} Q$

for some $B' \subseteq B$, $B'' \subseteq B$.

So $U \cap V = \bigcup_{P \in B'} \bigcup_{Q \in B''} (P \cap Q)$

$P \cap Q \in T$ so $U \cap V \in T$ as $T$ is preserved under union.

$\square$

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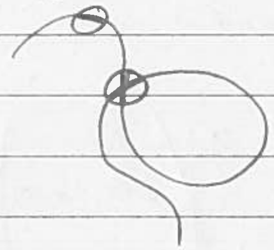
Example

If $(X, d)$ is a metric space then
$\mathcal{B} = \{B_x(r) : x \in X, r > 0\}$ is a basis for a topology ("metric topology") on $X$,
where $B_x(r) = \{y \in X \mid d(x,y) < r\}$. (exercise).

Subspaces

If $X$ is a topological space & $A \subseteq X$ is a subset, then we can define a topology on $A$ (called "the subspace topology")
- a subset $U \subseteq A$ is open iff $U = A \cap V$ for some open set $V \subseteq X$.



(exercise)

Lemma

If $X$ is a topological space, $A \subseteq X$ is a subspace then the inclusion map $i : A \hookrightarrow X$ is cts.

Proof

Let $U \subseteq X$ be an open set.
$i^{-1}(U) \subseteq A$ is $i^{-1}(U) = U \cap A$, which is open by definition of the topology. $\square$

Example

$S^1 = \{(x,y) : x^2 + y^2 = 1\} \subseteq \mathbb{R}^2$
is a subspace of $\mathbb{R}^2$
If I write my points $(x,y) \in S^1$ as $e^{i\theta} = \cos\theta + i\sin\theta$
then we see that $\cos\theta$ & $\sin\theta$ are cts. functions on $S^1$.

Proof

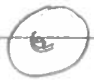
$\cos\theta$ is the composition of the inclusion map $i : S^1 \hookrightarrow \mathbb{R}^2$

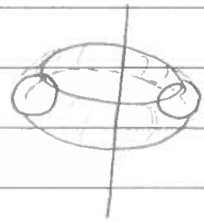
with projection map $Re : \mathbb{R}^2 \mapsto \mathbb{R}$.
$\therefore$ continuous. $\square$

Example

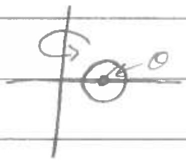
$S^n = \{ (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum x_i^2 = 1 \}$
is the $n$-dim sphere topologiesed as a subspace of $\mathbb{R}^{n+1}$

Example

 $T^2$ torus in $\mathbb{R}^3$
(genus 1)



$$\begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2+\cos\theta \\ \sin\theta \end{pmatrix}$$ $\leftarrow$ This is a parameterisation of $T^2$




 $\Sigma_2$ $\leftarrow$ surface of genus 2

 $\Sigma_g$

g holes

$\{(\cos\theta, \sin\theta, \cos\phi, \sin\phi) \in \mathbb{R}^4 : \theta, \phi \in [0, 2\pi)\} \subseteq \mathbb{R}^4$
is homeomorphic to $T^2$ but sitting in $\mathbb{R}^4$.

 is a circle when given the subspace topology.

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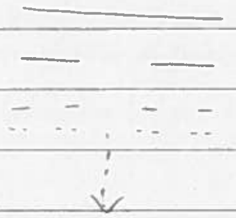
Example - Cantor Set



The set of real numbers of the form $0.a_1a_2a_3\ldots$ with $a_i \in \{0, 2\}$ in base 3

e.g. $0.22002022\ldots$

$0.2222\ldots = 1$

Not discrete.

(exercise)

e.g. $\mathbb{Q} \subseteq \mathbb{R}$ is a top. space (not discrete).

Example

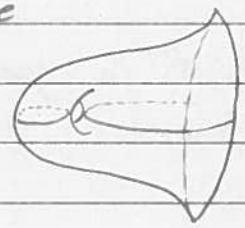
Let $P_1(x_1, \ldots, x_n), \ldots, P_k(x_1, \ldots, x_n)$ be a collection of polynomials in $n$ variables.

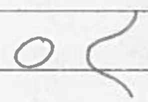
$V = \{x = (x_1, \ldots, x_n) \in \mathbb{C}^n : P_1(x) = P_2(x) = \ldots = P_k(x) = 0\}$

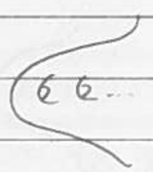
is a subspace of $\mathbb{C}^n$ ("affine variety")

e.g. $P(x,y) = y^2 - x^3 + x$

$V = \{y^2 = x^3 - x\}$ ← elliptic curve



in $\mathbb{R}^2$: 

higher powers look like 

Products

If $X_1, X_2$ are top. spaces then $X_1 \times X_2$ can be given a product topology where a basis is given by sets of the form $U_1 \times U_2$ with $U_1 \subseteq X_1$ and $U_2 \subseteq X_2$ open.

(exercise).

Sheet 1 will show that the projection maps $P_1 : X_1 \times X_2 \mapsto X_1$ and $P_2 : X_1 \times X_2 \mapsto X_2$ are cts. w.r.t. product topology.

## Examples

$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$

$T^2 = \underset{\theta}{S^1} \times \underset{\phi}{S^1}$

## Connectedness

**Def**

A topological space $X$ is disconnected if $\exists$ open sets $U, V \subseteq X$ s.t. $X = U \cup V$ and $U \cap V = \emptyset$.
Otherwise $X$ is connected.

**Theorem**

$\mathbb{R}$ is connected.

**Proof**

Suppose not, so that $\mathbb{R} = U \cup V$ with $U, V$ open and $U \cap V = \emptyset$.
Define $F: \mathbb{R} \mapsto \mathbb{R}$ by $F(x) = \begin{cases} 0 & \text{if } x \in U \\ 1 & \text{if } x \in V \end{cases}$

If $W \subseteq \mathbb{R}$ is open then $F^{-1}(W) = \begin{cases} \emptyset & \text{if } 0, 1 \notin W \\ \mathbb{R} & \text{if } 0, 1 \in W \\ U & \text{if } 0 \in W \;\; 1 \notin W \\ V & \text{if } 1 \in W \;\; 0 \notin W \end{cases}$

so $F$ is continuous.
But this contradicts the Intermediate Value thm as $F$ never takes the value $1/2$.

$\square$

**Theorem**

$\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$.

**Proof**

$\mathbb{R} \setminus \{0\}$ is disconnected (it is the union of $(-\infty, 0)$ and $(0, \infty)$).

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If $\mathbb{R}$ and $\mathbb{R}^2$ were homeomorphic then there would be a homeomorphism $F: \mathbb{R} \mapsto \mathbb{R}^2$.

Let $x = F(0) \in \mathbb{R}^2$.

$F|_{\mathbb{R}\setminus\{0\}} : \mathbb{R}\setminus\{0\} \mapsto \mathbb{R}^2 \setminus \{x\}$

is a homeomorphism. This is not possible, because $\mathbb{R}^2\setminus\{x\}$ is connected.

We'll prove this momentarily.

$\square$

Def

A space $X$ is called path-connected if $\forall x,y \in X$ $\exists$ a continuous map $\gamma: [0,1] \mapsto X$ s.t. $\gamma(0) = x$, $\gamma(1) = y$.

path from $x$ to $y$.

Lemma

$\mathbb{R}^2 \setminus \{0,0\}$ is path connected.

Proof

Let $x,y \in \mathbb{R}^2\setminus\{0,0\}$

$x(1-t) + yt = \gamma(t)$ is a path from $x$ to $y$ as long as $x$ & $y$ are not colinear.

If the line from $x$ to $y$ goes through the origin, let $p$ be the orthogonal vector to this line and use $\gamma(t) = (1-t)x + ty + p\sin(\pi t)$.

$\square$

Lemma

A path connected space is connected.

Proof

If $X$ is path connected but not connected, then we can write $X = U \cup V$, $U, V$ open & disjoint (and also non empty!)

Pick $x \in U$, $y \in V$. Path connectedness gives a path $\gamma : [0,1] \underset{\text{cts}}{\mapsto} X$ with $\gamma(0) = x$, $\gamma(1) = y$.

The sets $\gamma^{-1}(U)$ and $\gamma^{-1}(V)$ are disjoint nonempty open sets in $[0,1]$ s.t. $[0,1] = \gamma^{-1}(U) \cup \gamma^{-1}(V)$
$\Rightarrow [0,1]$ is disconnected, but this is contradicted by the intermediate value thm.

$\square$

Is $\mathbb{R}^2$ homeo. to $\mathbb{R}^3$? No – will see later.

Closed sets

Def
Let $X$ be a topological space. A ~~subset~~ subset $A \subseteq X$ is closed if its complement $X \setminus A$ is open.

Lemma
A map $F : X \mapsto Y$ is cts. iff $\forall\, A \subseteq X$ closed, $F^{-1}(A)$ is closed.

Proof
If $F^{-1}(A)$ is closed for all closed sets $A$, take an open set $U \subseteq Y$. $Y \setminus U$ is closed so $F^{-1}(Y \setminus U)$ is closed so $X \setminus F^{-1}(Y \setminus U)$ is open $= F^{-1}(U)$
$\Rightarrow F$ cts.
Conversely if $A \subseteq Y$ is closed & $F$ is cts, then $Y \setminus A$ is open, $F^{-1}(Y \setminus A)$ is open
$\Rightarrow X \setminus F^{-1}(Y \setminus A) = F^{-1}(A)$ is closed.

$\square$

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Lemma

Let $X, Y$ be top. spaces and $F: X \mapsto Y$ be a map.
If $U, V \subseteq X$ are closed subsets s.t. $X = U \cup V$,
then $F$ is cts. iff $F|_U$ and $F|_V$ are cts. w.r.t.
the subspace top. on $U$ and on $V$.

Equivalently, to define a cts. function on $X$, it suffices
to define it on $U$ and $V$, and check your definitions
agree on the overlap.

Proof

Let $D \subseteq Y$ be a closed set.
WTP: $F^{-1}(D)$ is closed.
Let $A = F|_U^{-1}(D)$ ("points in $U$ that map to $D$") $\subseteq U$
$B = F|_V^{-1}(D) \subseteq V$.
These are closed in $U$ and $V$ respectively.
By definition of the subspace topology $\exists$ closed sets
$A' \subseteq X$, $B' \subseteq X$ s.t. $A = U \cap A'$, $B = V \cap B'$.
Now observe that $U$ & $A'$ are both closed in $X$
$\Rightarrow$ $A = U \cap A'$ is closed in $X$, $B$ is also closed similarly,
so $A \cup B$ is closed in $X$.
So $A \cup B = F^{-1}(D)$ is closed $\forall$ closed $D$.
$\square$

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Limits

Compactness

Hausdorffness

Def

A sequence $x_n \in X$ in a top. space $X$ converges to $x \in X$ if $\forall$ open sets $U$ containing $x$, the sequence eventually enters $U$, i.e. $\exists N$ s.t. $\forall n \geq N$ $x_n \in U$.

Lemma

If $A$ is a closed subset in $X$ and $x_n \in A$ is a sequence, then if $x_n \to x$ we have $x \in A$.

Proof

Since $A$ is closed, $X \setminus A$ is open.

So let's assume $x \notin A$, then $x \in X \setminus A$ which is open,

so $\exists$ open set $U \subseteq X \setminus A$ s.t. $x \in U$. (e.g. $U = X \setminus A$)

By def. of convergence $\exists N$ s.t. $\forall n \geq N$, $x_n \in U \subseteq X \setminus A$ ⨳

(contradiction as $x_n \in A$ $\forall n$).

$\square$

Recall that if $B \subseteq X$ is a subset, its closure $\overline{B} \subseteq X$ is defined to be the smallest closed subset in $X$ which contains $B$.

Lemma

$\overline{B} = B \cup \{\text{all limit points}\}$

Proof

We have proved that any closed set containing $B$ (in particular $\overline{B}$) contains all limit point of sequences in $B$, so $\overline{B} \supseteq B \cup \{\text{limit points}\}$.

If $b \in \overline{B}$ is not the limit of any sequence in $B$, then

there exists an open set $U$ containing $b$ s.t. $U \cap B \neq \phi$.
Now $(X \setminus \overline{B}) \cup U$ is an open set disjoint from $B$,
so its complement is a closed set containing $B$ and
not containing $b$. So $X \setminus ((X \setminus \overline{B}) \cup U)$ is a smaller
closed set containing $B$. $\#$

$\square$

Exercise (hw 1)
If $Z$ is a top.space, $X \subseteq Z$ subspace then $X$ is
discrete with the subspace topology iff every convergent
sequence $x_n \in X$ is eventually constant.
i.e. $\exists N$ s.t. $x_n = x \ \forall n \geq N$.

Compactness

Theorem (Heine-Borel)
A subset of $\mathbb{R}$ is compact iff it is closed and bounded.
(A subset of a metric space is compact iff every sequence in
the subset has a convergent subsequence.)

Def
A top. space $X$ is compact if any cover of $X$ by open
sets admits a finite subcover.

Example
If $X$ is a compact discrete top. space, then $X$ is a finite set.

Lemma
If $F: X \mapsto Y$ is a continuous map of top. spaces and
$A \subseteq X$ is a compact subset, then $\cancel{f}(A) \subseteq Y$ is also compact.

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Proof

Pick an open cover $\mathcal{U}$ of $f(A)$ with subspace topology.

For each $U \in \mathcal{U}$ $\exists$ open set $V \subseteq Y$ s.t. $U = f(A) \cap V$.

So $\exists$ a collection of open sets $\mathcal{V}$ in $Y$ st.

$\mathcal{U} = \{ V \cap f(A) : V \in \mathcal{V} \}$

Take $\{F^{-1}(V) : V \in \mathcal{V}\}$ this is a collection of open sets in $X$. Moreover, $\{F^{-1}(V) \cap A : V \in \mathcal{V}\}$ is an open cover of $A$ w.r.t. subspace topology.

Since $A$ is compact, we can take a finite subcover $\mathcal{W}$.

Now $\{F(W) : W \in \mathcal{W}\}$ is a finite subcover of $\mathcal{U}$.

Lemma

A closed subset $A$ of a compact space $X$ is compact (w.r.t. subspace top.).

Proof

Take an open cover $\mathcal{V}$ of $A$.

By definition of the subspace topology $\exists$ a collection $\mathcal{U}$ of open sets in $X$ st. $\mathcal{V} = \{U \cap A ; U \in \mathcal{U}\}$.

The collection $\mathcal{U} \cup \{X \setminus A\}$ is an open cover of $X$.

There is a finite subcover $\mathcal{U}' \cup \{X \setminus A\}$.

Now $\mathcal{V}' = \{U \cap A : U \in \mathcal{U}'\}$ is a finite subcover of $A$. $\square$

Def (Hausdorffness)

A top. space $X$ is called Hausdorff if $\forall x, y \in X$ $(x \neq y)$ $\exists$ disjoint open sets $U, V$ $(x \in U, y \in V)$ s.t. $U \cap V = \emptyset$.

Example

The space $\{0, 1\}$ with indiscrete topology $T = \{\emptyset, \{0,1\}\}$ is not Hausdorff.

Lemma

If $X$ is Hausdorff and $x_n \in X$ is a sequence s.t. $x_n \to x$ & $x_n \to y$ then $x = y$.

Proof

If $x_n \to x$, by def for any open set $U \ni x$, $x_n$ is eventually contained in $U$.
If $x \neq y$, because $X$ is Hausdorff we can find disjoint open sets $U \ni x$, $V \ni y$.
As $x_n \to x$, $x_n \in U$ for large $n$, hence not in $V$ $(U \cap V = \emptyset)$ $\therefore x_n \not\to y$ ※

□

Lemma

If $X$ is Hausdorff and $K \subseteq X$ is compact then $K$ is closed.

Proof

WTS: $X \setminus K$ is open
Pick $y \in X \setminus K$. For any $x \in K$ $\exists\ U_x \ni x$, $V_x \ni y$ s.t. $U_x \cap V_x = \emptyset$ (Hausdorff)
$\mathcal{U} = \{U_x : x \in K\}$ is an open cover of $K$.
$K$ is compact so take a finite subcover $\{U_{x_i} : 1 \leq i \leq k\}$
Now $\bigcap_{i=1}^{k} V_{x_i}$ is an open neighbourhood of $y$ which is disjoint from $K$.
$\Rightarrow X \setminus K$ is open $\Rightarrow K$ is closed.

Theorem

If $X$ is compact and $Y$ is Hausdorff, any continuous bijection $F: X \mapsto Y$ is a homeomorphism.

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Proof

WTS: $F^{-1}: Y \mapsto X$ is continuous.

i.e. $\underbrace{(F^{-1})^{-1}(D)}_{= F(D)}$ is closed $\forall$ closed $D \subseteq X$.

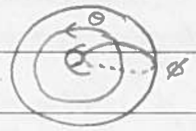
$X$ compact, $D \subseteq X$ is closed so $D$ is compact.

$F$ continuous so $F(D)$ is compact.

$Y$ is Hausdorff so $F(D) \subseteq Y$ is closed. $\square$

13-10-17 Example

$$S^1 \times S^1 \qquad\qquad T^2 \subseteq \mathbb{R}^3$$
$$(e^{i\phi}, e^{i\theta}) \longrightarrow$$




$$F(e^{i\phi}, e^{i\theta}) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2+\cos\phi \\ \sin\phi \end{pmatrix}$$

$\cos\theta = \cos(\mathrm{proj}_2(e^{i\theta}))$ is a composition of cts functions

$\vdots$ etc.

Matrix multiplication is also cts.

$\Rightarrow F$ is continuous

$F$ is bijective (can recover $(\theta, \phi)$ as angular coordinates in $\mathbb{R}^3$)

Claim

$T^2$ is Hausdorff

Proof

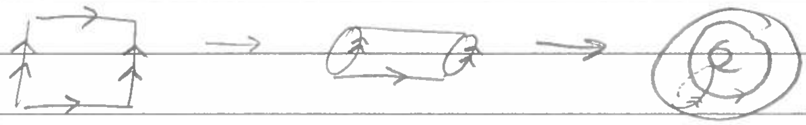
$\mathbb{R}^3$ is metric $\therefore$ Hausdorff

$T^2$ is a subspace of a Hausdorff space $\therefore$ Hausdorff. $\square$

$S^1 \times S^1$ is a product of compact spaces, therefore compact.

$\therefore F$ is a homeomorphism.

$\square$



## Quotient Topology

**Def**

Let $X$ be a topological space and $\sim$ be an equivalence relation on $X$.
Let $p: X \mapsto X/\sim$ be the quotient map to the set of equivalence classes. Then the quotient topology on $X/\sim$ is the topology whose open sets are subsets $U \subseteq X/\sim$ s.t. $p^{-1}(U)$ is open in $X$.

**Lemma**

This satisfies the axioms for a topology.

**Proof**

$X/\sim$ and $\emptyset$ are open:

$p^{-1}(\emptyset) = \emptyset$ which is open in $X$.

$p^{-1}(X/\sim) = X$ which is open in $X$. ✓

Unions of open sets are open:
Let $\mathcal{U}$ be a collection of open sets in $X/\sim$.
$\forall U \in \mathcal{U}$, $p^{-1}(U)$ is open.
Let $V = \bigcup_{U \in \mathcal{U}} U$. $p^{-1}(V) = \bigcup_{U \in \mathcal{U}} p^{-1}(U) \Rightarrow p^{-1}(V)$ is open since it is a union of open sets in $X$. ✓

Finite intersections are open:
If $U, V \subseteq X/\sim$ are open then $p^{-1}(U), p^{-1}(V)$ open in $X$
$p^{-1}(U \cap V) = p^{-1}(U) \cap p^{-1}(V)$ is open in $X \Rightarrow U \cap V$ is open. ✓ $\square$

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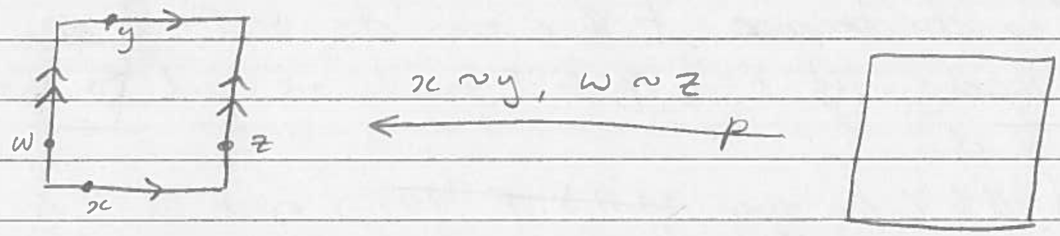
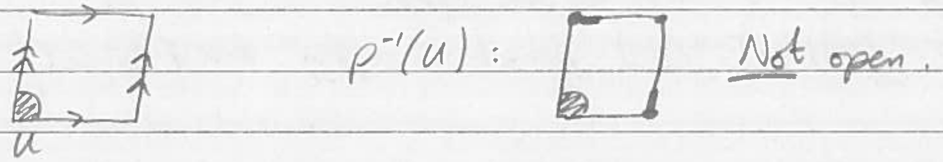
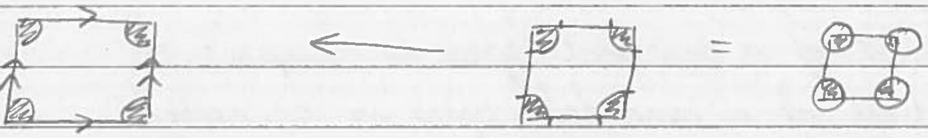
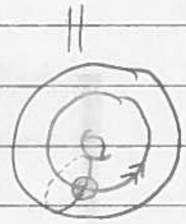
Example





This is open in the square w.r.t. the subspace topology; we just intersect 4 open balls with the square.

Lemma

Let $X$ be a top space & $\sim$ be an equivalence relation on $X$. Equip $X/\sim$ with the quotient topology. Let $p: X \mapsto X/\sim$ be the quotient map. Let $Y$ be another top. space.

Then any cts. $f^n$ $X/\sim \xrightarrow{\bar{F}} Y$ gives a cts $f^n$ $F: X \longmapsto Y$ by $F = \bar{F} \circ p$.

$$X \xrightarrow{p} X/\sim \xrightarrow{\bar{F}} Y$$

Conversely, if $F: X \mapsto Y$ is a cts map which factors as $F = \bar{F} \circ p$ for some map $\bar{F}: X/\sim \longmapsto Y$ then $\bar{F}$ is cts.

Rephrasing:

The cts $f^n$s on $X/\sim$ (set of equiv. classes) are in bijection with cts $f^n$s on $X$ which are const on each equiv class.

[function on $X$ is continuous & well defined $\Rightarrow$ function on $X/\sim$ is cts]

*Proof*

$F$ is cts because $\bar{F}$ & $p$ are cts. & $F = \bar{F} \circ p$.
Conversely, if $F = \bar{F} \circ p$ & $F$ is cts we want to show $\bar{F}$ is cts.
Let $U \subseteq Y$ be open. WTS: $\bar{F}^{-1}(U)$ is open in $X/\sim$.
We need $p^{-1}(\bar{F}^{-1}(U))$ to be open.
$p^{-1}(\bar{F}^{-1}(U)) = F^{-1}(U)$ which is open as $F$ is cts. $\square$

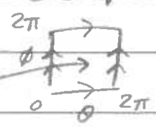
*Lemma*

- A quotient of a compact space is compact.
- A quotient of a connected space is connected.
- A quotient of a discrete space is discrete.

*Proof*

Exercise; hw1, q4.

*Example*

$X/\sim$ 

$\cong T^2$

homeomorphic.

Let $\bar{F} : X/\sim \;\mapsto T^2$ be the map

$$(\theta, \phi) \longmapsto \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2+\cos\phi \\ \sin\phi \end{pmatrix}$$

This is a continuous map because the map

$$F(\theta, \phi) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2+\cos\phi \\ \sin\phi \end{pmatrix}$$ is continuous,

and $F = \bar{F} \circ p$ because $\cos(0) = \cos(2\pi)$, $\sin(0) = \sin(2\pi)$
i.e. $\bar{F}$ is well-defined.
i.e. $F(\theta, \phi)$ only depends on the equivalence class of $(\theta, \phi)$.
$X/\sim$ is compact: its a quotient of $\square$ which is compact.

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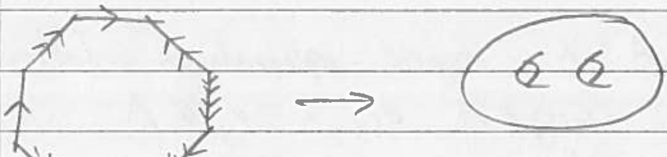
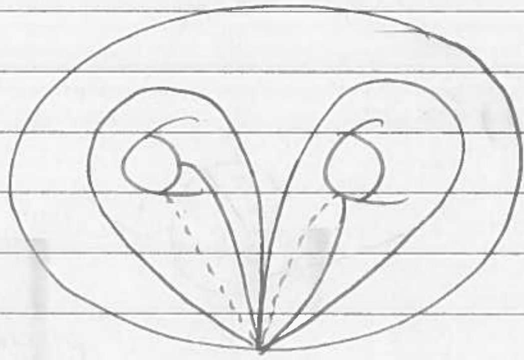
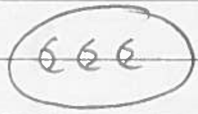
$T^2$ is Hausdorff: it's a subspace of a Hausdorff space.
$\Rightarrow$ $F$ is a homeomorphism. $\square$

<u>Example</u>



genus 2 surface in $\mathbb{R}^3$



12-gon $\longrightarrow$  genus 3 surface

<u>Example</u>

Let $D = \{z \in \mathbb{C} \mid |z| \leq 1\}$

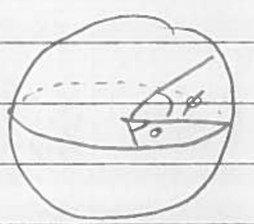
Let $\sim$ be the equivalence relation which identifies all the points on the boundary to a single point.

<u>Claim</u>

$D/\sim \; \cong S^2$

<u>Proof</u>

Write a map $F : D \mapsto S^2$
which factors through $\bar{F} : D/\sim \; \mapsto S^2$
(i.e. it's constant along the boundary of $D$)
s.t. $\bar{F}$ is a cts bijection



$F(re^{i\theta}) = \left(\theta, \pi(1-2r)/2\right)$

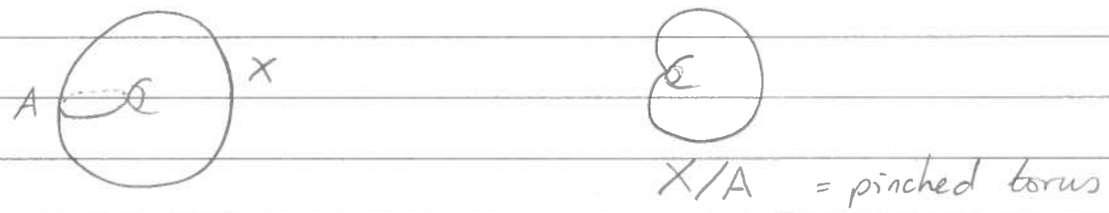
e.g. $F(0)$ = north pole
$\quad F(e^{i\theta})$ = south pole

$F$ is cts, factors as $\bar{F} \circ p$ so $\bar{F}$ is cts

$\bar{F}$ is a bijection.
$D$ is compact $\Rightarrow D/\sim$ is compact
$S^2 \subseteq \mathbb{R}^3 \Rightarrow S^2$ Hausdorff
$\Rightarrow \bar{F}$ is a homeophism.

More generally, if $A \subseteq X$, define equivalence relation $\sim_A$ s.t. $x \sim_A y$ iff $x, y \in A$ or $x = y \notin A$
$\sim_A$ "crushes $A$ to a point".
$X/\sim_A$ is written as $X/A$.

e.g. $X = S^1 \times S^1$, $A = \{x\} \times S^1$




<u>Quotient by a group action</u>

<u>Def</u>
Let $G$ be a group & $X$ a top space.
Recall that a $G$-action on $X$ is a homomorphism $\rho: G \mapsto \mathrm{Perm}(X)$.
We say that $G$ acts continuously on $X$ if $\rho(g): X \mapsto X$ is a homeomorphism $\forall g \in G$.
Given a continuous $G$-action we get an equivalence relation $x \sim y$ iff $\exists g \in G$ s.t. $\rho(g)(x) = y$ (write $g(x) = y$).

The quotient $X/\sim$ is usually written $X/G$ and it is the space of orbits of the $G$-action.

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<u>Example</u>

Let $G = \mathbb{Z}$ and $X = \mathbb{R}$.

$n \in \mathbb{Z}$ will act on $X \in \mathbb{R}$ via translations

$\rho(n) : \mathbb{R} \mapsto \mathbb{R}$, $\rho(n)(x) = x + n$.

This is a continuous action of $\mathbb{Z}$ on $\mathbb{R}$.

The quotient $\mathbb{R}/\mathbb{Z}$ is a circle $S^1$

<u>Proof</u>

$F : \mathbb{R} \mapsto S^1$, $t \mapsto e^{2\pi i t}$

this is a continuous map.

$F = \bar{F} \circ p$ because $e^{2\pi i(t+1)} = e^{2\pi i t + 2\pi i} = e^{2\pi i t}$.

i.e. $\bar{F}(t) = e^{2\pi i t}$ is well defined on $\mathbb{R}/\mathbb{Z}$

so $\bar{F} : \mathbb{R}/\mathbb{Z} \mapsto S^1$ is cts

$\bar{F}$ is bijective.

$\mathbb{R}/\mathbb{Z}$ is compact. (its also the quotient $[0,1]/\sim$ where $0 \sim 1$ and $[0,1]$ is compact)

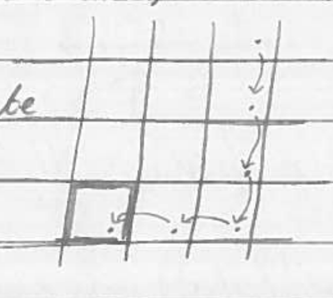
$S^1$ is Hausdorff, so $\bar{F}$ is a homeomorphism.

<u>Example</u>

Let $G = \mathbb{Z}^2$, $X = \mathbb{R}^2$

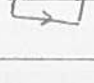
Define a $G$-action on $\mathbb{R}^2$ by letting $(a,b) \in \mathbb{Z}^2$ act on $(x,y) \in \mathbb{R}^2$ to get $(x+a, y+b) \in \mathbb{R}^2$.

The quotient is $T^2$ (in fact it's $S^1 \times S^1$).



I can use the $\mathbb{Z}^2$ action to translate any point $(x,y)$ to

$(x - \lfloor x \rfloor, y - \lfloor y \rfloor) \in [0,1) \times [0,1)$

So $\mathbb{R}^2/\mathbb{Z}^2 \cong$  $\cong T^2$

*Example*

Let $X = S^1 \subseteq \mathbb{C}$.

Let $G = \mu_n = \{ z \in \mathbb{C} : z^n = 1 \}$

$G$ acts on $X$, $\mu \in G$, $z \in X$ then $z \to \mu z$

$X/G = ?$

Every orbit has a representative in the arc between $1$ and $e^{2\pi i/n}$ so we can just quotient that arc by identifying its endpoints, the result is homeomorphic to $S^1$.

i.e. $S^1/\mu_n \cong S^1$.

*Lemma*

If $G$ acts continuously on $X$ then the quotient map $p : X \mapsto X/G$ is open. (i.e. $p(U)$ is open whenever $U \subseteq X$ is open)

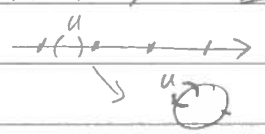
*Proof*

Let $U \subseteq X$ be an open set.

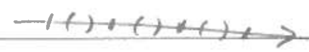
Its image, $p(U)$, comprises all equivalence classes (orbits) of points in $U$. This set is open in the quotient top. on $X/G$ iff $p^{-1}(p(U))$ is open.

The set $p^{-1}(p(U))$ consists of all points which are equivalent to points in $U$.

[e.g. $X = \mathbb{R}$, $G = \mathbb{Z}$ (acting by translation)




$p^{-1}(p(U))$ :



So $p^{-1}(p(U)) = \bigcup_{g \in G} g(U)$ homeomorphism, open $\longrightarrow$ $g(U)$ open

$\Rightarrow p^{-1}(p(U))$ is a union of open sets $\Rightarrow$ open.

$\square$

[$g$ homeo $\Rightarrow g^{-1}$ continuous. $g(U) = (g^{-1})^{-1}(U)$ open for open $U$]

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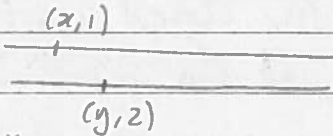
*Example*

The line with two origins!

$X = \mathbb{R} \sqcup \mathbb{R}$

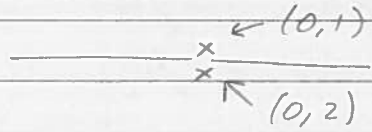



$\sim$ : $(x,1) \sim (y,1) \iff x = y$

$(x,2) \sim (y,2) \iff x = y$

$(x,1) \sim (x,2) \iff \forall x \neq 0$

$X/\sim$ :




$X/\sim$ is not Hausdorff

*Proof*

I claim that any open set containing $(0,1)$ intersects non trivially any open set containing $(0,2)$.

If $U$ is an open neighbourhood of $(0,1)$, it contains an open interval $((-\varepsilon, \varepsilon), 1)$.

If $V$ is an open neighbourhood of $(0,2)$, it contains an open interval $((-\delta, \delta), 1)$.

Now if $0 < \phi < \min(\delta, \varepsilon)$ then $(\phi, 1) \sim (\phi, 2)$ is in both neighbourhoods. $\square$

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## Attaching a Cell

Let $X$ be a top. space

Let $D$ be the closed unit disc in $\mathbb{R}^n$

Let $S = \partial D$ be the unit sphere in $\mathbb{R}^n$

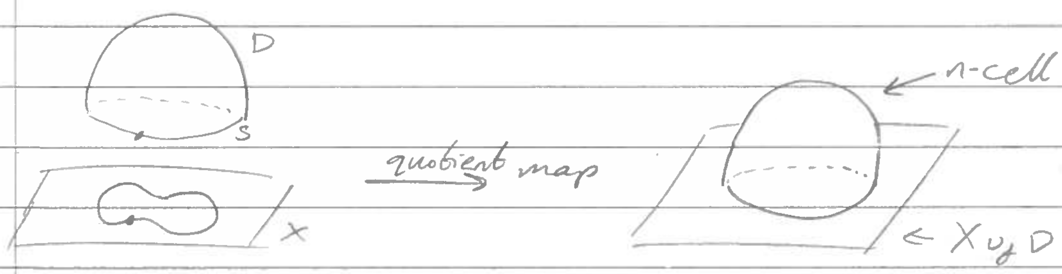
Suppose we're given a cts map $f: S \to X$

Define $X \cup_f D$ to be the quotient space $(X \sqcup D)/\sim$

where $\sim$ is the equivalence relation

$x \sim y$ if $x = y$ or $x \in S$ $y = f(x)$

or $y \in S$ $x = f(y)$.




Def

A cell complex is a top space constructed inductively by attaching cells as follows:

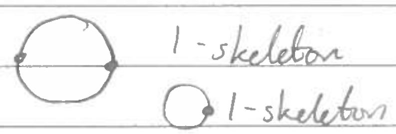
- start with a collection $X^0$ of points (discrete topology)
  "0-skeleton" 0-cells
- attach a collection of 1-cells to $X^0$ to obtain $X^1$
  "1-skeleton".
- attach a collection of 2-cells to $X^1$ to get $X^2$ "2-skeleton"

  ⋮

Examples

$S^1$ : . . 0-skeleton

OR . 0-skeleton




These are two possible cell structures on $S^1$.

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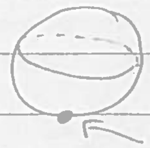
$S^2$:  $\leftarrow$ 2-cell

$\leftarrow$ 0-cell

$S^n$:  $D^n \leftarrow$ n-cell

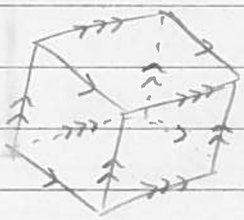
$\leftarrow$ 0-cell

$T^2$: 

one 0-cell
two 1-cells
once 2-cell (interior)

$\Sigma_2$
genus 2 surface

one 0-cell
four 1-cells
one 2-cell

$T^3$ 

Identify opposite faces to get $T^3$
one 0-cell
three 1-cells
three 2-cells
one 3-cell

<u>Real projective space $\mathbb{R}\mathbb{R}^n$</u>

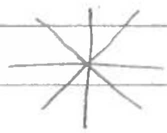
<u>Def</u>

$\mathbb{R}P^n$ is the space whose points parameterise lines in $\mathbb{R}^{n+1}$. More precisely let $\sim$ be the equivalence relation on $\mathbb{R}^{n+1} \setminus \{0\}$ defined by $x \sim y$ iff $x$ & $y$ lie on a straight line that passes through $0 \in \mathbb{R}^{n+1}$.

$\mathbb{R}P^n = \mathbb{R}^{n+1} \setminus \{0\} / \sim$ is the set of lines through $0$.

Equip $\mathbb{R}P^n$ with the quotient topology.

## Example



$\mathbb{RP}^1 = [0, \pi] / \sim \qquad 0 \sim \pi$

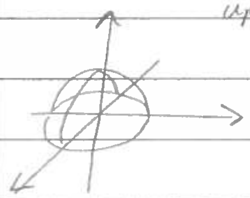
$\cong S^1$

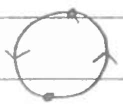
one 0-cell

one 1-cell

$\mathbb{RP}^2 = D^2 / \sim$

$\uparrow$ upper hemisphere





$\mathbb{RP}^2$

one 0-cell

one 1 cell

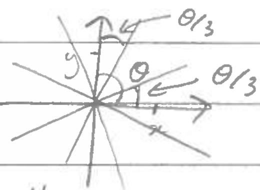
one 2 cell

## Lemma

$\mathbb{RP}^n$ is Hausdorff.

## Proof

Let $[x]$ & $[y] \in \mathbb{RP}^n$. They correspond to lines in $\mathbb{R}^{n+1}$ making an angle $\theta$




Let $U$ be the set of $[z] \in \mathbb{RP}^n$ s.t. the corresponding line in $\mathbb{R}^{n+1}$ makes angle $< \theta/3$ with $[x]$

$V$ similar for $[y]$.

$U, V$ open, $[x] \in U$, $[y] \in V$ and $U \cap V = \phi$ $\square$

## Lemma

Let $\sim_a$ be the relation on $S^n$ s.t. $x \sim_a y$ iff $x = \pm y$.

Then $S^n / \sim_a \cong \mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus \{0\}) / \sim$ (it is compact as $S^n / \sim_a$ is the quotient of a compact space $\Rightarrow$ compact).

## Proof

Take the inclusion $S^n \hookrightarrow \mathbb{R}^{n+1} \setminus \{0\}$ and compose this with the quotient map $\mathbb{R}^{n+1} \setminus \{0\} \longrightarrow \mathbb{RP}^n$.

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This gives $F : S^n \longrightarrow \mathbb{RP}^n$
Note that if $\underset{(y \sim_a x)}{y = -x}$ then $x \sim y$ ($x$ & $y$ lie on a line through $0$).

Therefore $\bar{F} : S^n/\sim_a \longrightarrow \mathbb{RP}^n$ is well-defined and continuous.
$[x]_{\sim_a} \longmapsto [x]_{\sim}$

Bijective:
$x$ & $y$ in $S^n$ lie on a line through $0$ iff $x = -y$ [injective]
and any line through the origin intersects $S^n$ at two points. [surjective]

So $\bar{F} : \underset{\text{compact}}{S^n/\sim_a} \longrightarrow \underset{\text{Hausdorff}}{\mathbb{RP}^n}$ is a continuous bijection

$\therefore$ $\bar{F}$ is a homeomorphism. □

<u>Corollary</u>
$\mathbb{RP}^n$ has a cell structure with one $k$-cell for each $k \in \{0, 1, \dots, n\}$.

<u>Proof</u> (by induction)
We have already seen this for $\mathbb{RP}^1$, $\mathbb{RP}^2$.
Claim $S^{n+1}/\sim_a$ contains $S^n/\sim_a$
pf: If $S^{n+1} = \{(x_1, \dots, x_{n+2}) : \sum x_i^2 = 1\}$
$S^n = \{(x_1, \dots, x_{n+1}, 0) : \sum x_i^2 = 1\}$
If $x \in S^n$ then $-x \in S^n$. So the inclusion $S^n \hookrightarrow S^{n+1}$ descends to a well-defined map $S^n/\sim_a \hookrightarrow S^{n+1}/\sim_a$. □
$\therefore$ $\mathbb{RP}^{n+1}$ contains a copy of $\mathbb{RP}^n$ as the "equator".
$S^{n+1} \setminus S^n = D_1 \sqcup D_2$ (northern & southern hemispheres)
$(D_1 \sqcup D_2)/\sim_a = D_1$ (any point not in the equator is either in $D_1$ or its antipode $-x$ is in $D_1$).
So $\mathbb{RP}^{n+1} = \mathbb{RP}^n \cup_f D_1$ (along an attaching map $\partial D_1 = S^n \underset{\text{quotient map}}{\longrightarrow} S^n/\sim_a$)

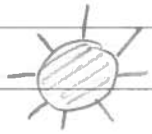
□

$\mathbb{C}P^n$ (complex analogue of $\mathbb{R}P^n$) see lecture notes.

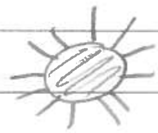
20-10-17 Homotopy equivalence

Lemma
If $X = U \cup V$ with $U$ & $V$ closed, and $F_U : U \to Y$ and $F_V : V \to Y$ are cts functions which agree on $U \cap V$ then

$$F(x) = \begin{cases} F_U(x) & x \in U \\ F_V(x) & x \in V \end{cases}$$ is cts.



not homeomorphic



but they are homotopy equivalent.

Def
Let $f_0, f_1 : X \to Y$ be cts maps.
A homotopy from $f_0$ to $f_1$ is a "continuous family of cts maps $f_t : X \to Y$ with $t \in [0,1]$.
In other words, if we define
$H : X \times [0,1] \to Y$ by $H(x,t) = f_t(x)$
then we want $H$ to be cts.
i.e. a homotopy from $f_0$ to $f_1$ is a continuous map $H : X \times [0,1] \to Y$ with $H(x,0) = f_0(x)$ and $H(x,1) = f_1(x)$.

If such a homotopy exists we will write $f_0 \simeq f_1$ ("$f_0$ is homotopic to $f_1$").

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__Example__

Let $B$ be the ball of radius 1 centred at $O$ in $\mathbb{R}^n$.

Let $f_0(x) = 0$ , $f_0 : B \to B$

$f_1(x) = x$ , $f_1 : B \to B$

Then $f_0 \simeq f_1$ via the homotopy

$H(x,t) = tx$. $H(x,0) = 0$ , $H(x,1) = x$

Therefore the identity map $f_1 : B \to B$ is homotopic to the constant map $f_0$.

__Def__

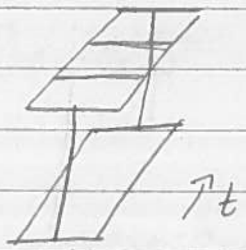
Let $X$ be a top. space, if $id_X \simeq$ constant map, then we say $X$ is contractible.

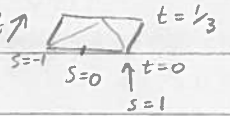
We call $H$ a "nullhomotopy" of $id_X$

__Examples__

$H: I \times [0,1] \longrightarrow I$

$H(x,0) = x$ , $H(x,1) = i$




1st segment: 


$H(s,t) = s(1-3t)$

$H(s,0) = s$ , $H(s, \frac{1}{3}) = 0$

etc for other segments

So $I$ is contractible.

$C$ is contractible.

$\cong$ : homeomorphic

$\simeq$ : homotopy equivalent

__Exercise__

$\mathbb{R}^n$ is contractible.

(follows because $\mathbb{R}^n \cong B$)

__Def__

Let $X$ and $Y$ be top. spaces. We say $X \simeq Y$ ("$X$ homotopy equivalent to $Y$") if $\exists$ continuous maps $p: X \to Y$, $q: Y \to X$ s.t. $q \circ p \simeq id_X$, $p \circ q \simeq id_Y$.

__Example__

If $X$ is contractible then $X \simeq \{x\}$.

Take $p: X \to \{x\}$ to be the constant map.

Take $q: \{x\} \to X$ to be the inclusion of a point $y \in X$ for which $id_X \simeq \left(\text{constant map } \begin{array}{c} X \to X \\ z \mapsto y \end{array} \;\forall z \in X.\right)^c$

We write $H$ for the homotopy

$$id_X \simeq c$$

$q \circ p(z) = y$ so $q \circ p = c \underset{H}{\simeq} id_X$

$p \circ q(x) = x$ so $p \circ q = id_{\{x\}}$

$\Rightarrow X \simeq \{x\}$. $\square$

__Example__

Consider the graphs $\Theta$ and $8$

They look homotopy equivalent: we're just crushing the middle edge, $A$, in the $\Theta$-graph to a point.

__Theorem__

$\Theta \simeq 8$

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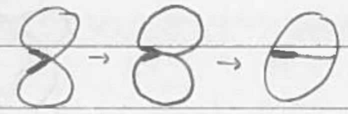
Proof

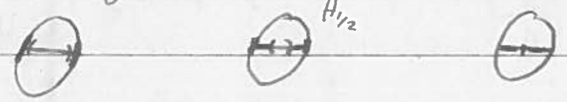
We need continuous maps $q: 8 \to \Theta$, $p: \Theta \to 8$ s.t. $p \circ q = id_\Theta$, $q \circ p = id_8$

The map $q$ is the map which contracts edge $A$ to the point at the centre of $8$.
The map $p$ is as indicated in this diagram



Key input is that
1). edge $A$ is contractible, let $h_t$ be a homotopy from $id_A$ to the constant map $c$ mapping to the centre point.
2). $h_t$ extends to a homotopy $H_t$ on $\Theta$
ie. $\exists\, H_t : \Theta \to \Theta$ s.t. $H_t|_A = h_t$

$H_0 = id_\Theta$ $\quad H_{1/2} \quad$ $H_1$



$$\begin{tikzcd} \Theta \arrow[r, "H_t"] \arrow[d, "q"'] & \Theta \arrow[d, "q"] \\ 8 \arrow[r, "\overline{H}_t"'] & 8 \end{tikzcd} \qquad \begin{tikzcd} \Theta \arrow[r, "H_1"] \arrow[d, "q"'] & \Theta \arrow[d, "q"] \\ 8 \arrow[r, "\overline{H}_1"'] \arrow[ur, "p"] & 8 \end{tikzcd}$$

This diagram commutes.

$\Rightarrow p \circ q = H_1$ , $q \circ p = \overline{H}_1$

$H_1 \simeq H_0 = id_\Theta$, $\overline{H}_1 \simeq \overline{H}_0 = id_8$

$\Rightarrow p$ & $q$ are mutually homotopy inverses to one another
$\Rightarrow \Theta \simeq 8$

$\square$

**Def**

A pair $(X, A)$ consisting of a top space $X$ and subspace $A \subseteq X$ satisfies the homotopy extension property if any homotopy $h_t : A \to A$ extends to a homotopy $H_t : X \to X$.

**Lemma**

A contractible

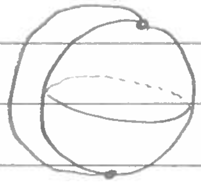
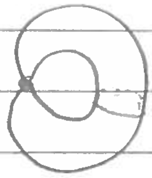
If $(X, A)$ satisfies the homotopy extension property and $A \simeq \{*\}$ then $X \simeq X/A$.

In our example $X = \Theta$, $X/A = 8$, $A =$ middle edge of $\Theta$.

**Lemma**

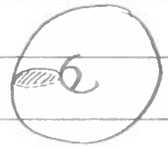
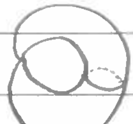
If $X$ is a cell complex and $A$ is a subcomplex then $(X, A)$ has the homotopy extension property.

**Example**

 $\simeq$ 

$\simeq$

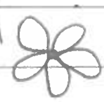
 $S^1 \vee S^2$

 $\simeq$ 

**Theorem**

A connected graph $\simeq$ a wedge of circles.

1-dimensional cell complex

 $= S^1 \vee S^1 \vee S^1 \vee S^1 \vee S^1$

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<u>Proof</u>

Let $X$ be the graph. Pick a maximal subtree $T$ in $X$, (A tree = contractible graph) containing all of the vertices of $X$.

$(X, T)$ satisfies the homotopy extension property (HEP) $\Rightarrow X \simeq X/T$.

[e.g. $X = \theta$, $T = A$
$X/T = 8$]

But $T$ contains all the vertices of $X$, so $X/T$ has only one vertex so it's a wedge of circles.

Consider the set of all subtrees.
This is partially ordered by inclusion.
If $T_1 \subseteq T_2 \subseteq \ldots$ is a chain of subtrees then $\bigcup T_i$ is an "upper bound" i.e. a subtree containing all $T_i$s.

[<u>Zorn's Lemma</u>
A partially ordered set in which all chains have upper bounds has a maximal element.]

So by Zorn's Lemma, we can find a subtree $T$ not contained in any bigger tree.

If $T$ does not contain all vertices then there is a vertex adjacent to $T$ & not contained in $T$, (using $X$ connected) so add on one more edge to get a bigger tree to get a bigger subtree. $*$ $\square$

## Fundamental group

**Def**

A path $\gamma$ in a top space $X$ is a cts map $\gamma : [0,1] \longrightarrow X$.

e.g. $\gamma(t) = (t, t^2, t^3) \in \mathbb{R}^3$ is a path ("the twisted cubic")

End points of $\gamma$ are $\gamma(0)$ and $\gamma(1)$

e.g. $\gamma(0) = (0,0,0)$, $\gamma(1) = (1,1,1)$

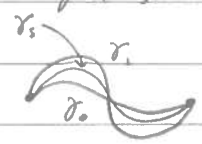
**Def**

If $X$ is a top space and $\gamma_0, \gamma_1$ are paths in $X$ with the same end points ($\gamma_0(0) = \gamma_1(0) = p$, $\gamma_0(1) = \gamma_1(1) = q$) then a homotopy rel endpoints $\gamma_0 \simeq \gamma_1$ is a cts map $H : \overset{s}{[0,1]} \times \overset{t}{[0,1]} \to X$

with $H(0,t) = \gamma_0(t)$, $H(1,t) = \gamma_1(t)$

$H(s,0) = p \ \forall s$, $H(s,1) = q \ \forall s$.

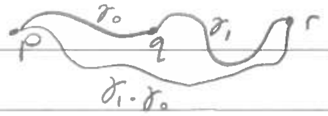
i.e. $\gamma_s(t) := H(s,t)$ is a cts path in $X$ from $p$ to $q$ $\forall s$.




**Def**

If $\gamma_0$ is a path in $X$ from $p$ to $q$ & $\gamma_1$ is a path in $X$ from $q$ to $r$, define

opp to books here compose like fns → $\gamma_1 \cdot \gamma_0$ to be the concatenation of $\gamma_0$ & $\gamma_1$




$$(\gamma_1 \cdot \gamma_0)(t) = \begin{cases} \gamma_0(2t), & t \in [0, \tfrac{1}{2}] \\ \gamma_0(2t-1), & t \in [\tfrac{1}{2}, 1] \end{cases}$$

This is continuous by the lemma (first lemma in lecture)!

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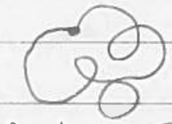
**Def**

A loop in $X$ based at $x \in X$ is a path
$\gamma : [0,1] \longrightarrow X$ with $\gamma(0) = x$, $\gamma(1) = x$.



Write $\Omega_x X$ for the set of loops based at $x$.

**Lemma**

Define $\gamma, \delta \in \Omega_x X$ to be $\gamma \sim \delta$ if there is a htpy rel endpoints from $\gamma$ to $\delta$ then this defines an equivalence relation

i.e. $\begin{cases} \gamma \simeq \delta \Rightarrow \delta \simeq \gamma \\ \gamma \simeq \gamma \\ \gamma \simeq \delta \simeq \varepsilon \Rightarrow \gamma \simeq \varepsilon \end{cases}$ [Hw 2]

**Define**

$\pi_1(X, x) := \Omega_x X / \sim$ (Fundamental group)

**Theorem**

$\pi_1(X, x)$ is a group under concatenation.

**Proof**

First NTS its well-defined.

i.e. $\gamma_0 \simeq \gamma_1$, $\gamma_0' \simeq \gamma_1'$ (via $\gamma_s$ and $\gamma_s'$ respectively)

$\Rightarrow \gamma_0 \cdot \gamma_0' \simeq \gamma_1 \cdot \gamma_1'$ via $\gamma_s \cdot \gamma_s'$. ✓

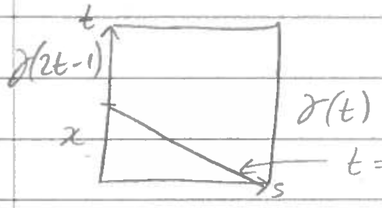
$\left.\begin{array}{l}\text{Associativity}\\ \text{Identity}\\ \text{Inverses}\end{array}\right\}$ still need to show these

Identity $\rightarrow$ constant loop at $x$, $\varepsilon(t) = x$.

$\gamma \cdot \varepsilon \simeq \gamma \simeq \varepsilon \cdot \gamma$

$\varepsilon \cdot \gamma = \begin{cases} x & t \in [0, 1/2] \\ \gamma(2t-1) & t \in [1/2, 1] \end{cases}$

$$\gamma_s(t) = \begin{cases} x, & t \leq \tfrac{1}{2}(1-s) \\ \gamma((2-s)t+s-1), & t \geq \tfrac{1}{2}(1-s). \end{cases}$$




$\gamma_0 = \gamma \cdot \varepsilon$

$\gamma_1 = \gamma$ so $\gamma \cdot \varepsilon \simeq \gamma$

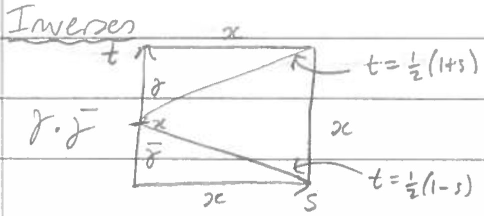
Continuous by the lemma (first in lecture)!

26-10-17 Last time: showed that the constant loop $\varepsilon(t)$ at $x \in X$ is an identity in $\pi_1(X, x)$

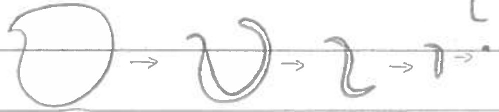
i.e. $\gamma \cdot \varepsilon \simeq \gamma$.

This time: if $\bar{\gamma}(t) = \gamma(1-t)$ is the loop that goes around $\gamma$ backwards then $\gamma \cdot \bar{\gamma} \simeq \varepsilon$.

Inverses




Homotopy from $\gamma_0$ to $\gamma_1$ is a map $H : [0,1] \times [0,1] \to X$
s.t. $H(0,t) = \gamma_0(t)$ $H(1,t) = \gamma_1(t)$
$H(s,0) = H(s,1) = x$
$\gamma_s(t) = H(s,t)$ $\gamma_0$ $\boxed{\gamma_s}$ $\gamma_1$



$$H(s,t) = \begin{cases} \bar{\gamma}(t) & t \leq \tfrac{1}{2}(1-s) \\ \bar{\gamma}(\tfrac{1}{2}(1-s)) & t \in [\tfrac{1}{2}(1-s), \tfrac{1}{2}(1+s)] \\ \bar{\gamma}(1-t) = \gamma(t) & t \geq \tfrac{1}{2}(1+s) \end{cases}$$

If $t = \frac{1}{2}(1+s)$ then $\bar{\gamma}(\frac{1}{2}(1-s)) = \bar{\gamma}(1-t) = \bar{\gamma}(1 - \frac{1}{2}(1+s))$
$= \bar{\gamma}(\frac{1}{2} - \frac{1}{2}s)$

So $H$ is continuous. $\Rightarrow \gamma \circ \bar{\gamma} \simeq \varepsilon \simeq \bar{\gamma} \circ \gamma$.

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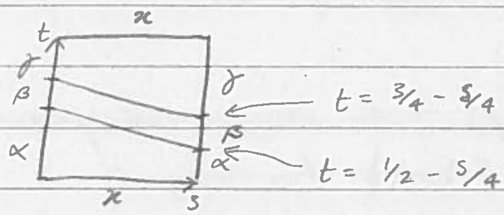
## Associativity

WTS: $(\gamma \circ \beta) \circ \alpha \simeq \gamma \circ (\beta \circ \alpha)$




$$H(s,t) = \begin{cases} \alpha(t/(\frac{1}{2} - s/4)) & t \leq 1/2 - s/4 \\ \beta(4(t - \frac{1}{2} - s/4)) & t \in [1/2 - s/4,\ 3/4 - s/4] \\ \gamma((t - 3/4 + s/4)/(\frac{1}{4} + \frac{s}{4})) & t \geq 3/4 - s/4. \end{cases}$$

$\square$

$\Rightarrow \pi_1(X, x)$ is a group.

## Example

$\pi_1(\mathbb{R}^n, 0) = \{1\}$

Proof

Given a loop $\gamma : [0,1] \longrightarrow \mathbb{R}^n$ based at $0$

we have have $\gamma \simeq \varepsilon$ $\quad (\varepsilon(t) = 0\ \forall t)$

$H(s,t) = s\gamma(t)$

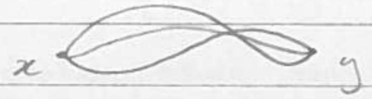
is a homotopy from $\varepsilon$ to $\gamma$. $\square$

## Def

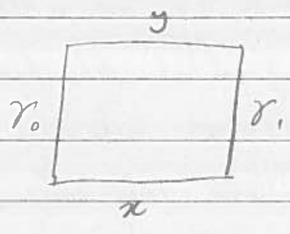
If $\pi_1(X, x) = \{1\}$ then we say $X$ is simply-connected.

## Lemma

If $\pi_1(X, x) = \{1\}$, then if $\gamma_0$ and $\gamma_1$ are paths from $x$ to $y$ then they are homotopic, $\gamma_0 \simeq \gamma_1$, through paths from $x$ to $y$.

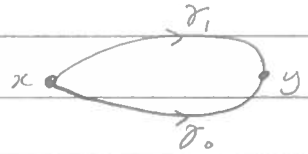



i.e.

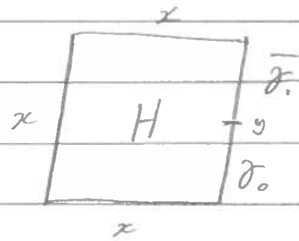


*Proof*

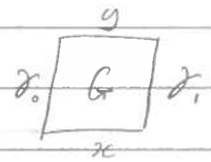
If $\gamma_0$ and $\gamma_1$ are two paths from $x$ to $y$.
Then $\bar{\gamma}_1 \cdot \gamma_0$ is a loop based at $x$.

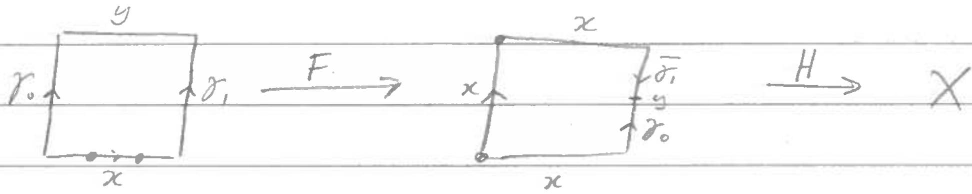



$\bar{\gamma}_1 \cdot \gamma_0 \simeq \varepsilon$ by assumption
$\Rightarrow$ we have a homotopy $H$




What we want is a homotopy



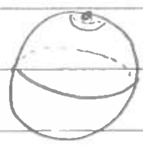

There is a continuous map $F$ which sends:
$x \to x$, $y \to y$, $\gamma_0 \to \gamma_0$ etc.
Define $G$ to be $H \circ F$. This is a homotopy from $\gamma_0$ to $\gamma_1$.

$\square$

*Example*

If $n \geq 2$ then $\pi_1(S^n) = \{1\}$

*Proof*



If we can homotope $\gamma$ so that it misses the north pole, then $\gamma \simeq \varepsilon$.
Let $B$ be a closed ball around the north pole.
$\gamma^{-1}(B)$ is closed, $\gamma^{-1}(B) \subseteq [0,1]$.
Closed subsets of $[0,1]$ are finite unions of closed intervals.
These intervals are $I_1, I_2, \ldots, I_k$ and define $\delta_i = \gamma|_{I_i}$
$\pi_1(B) = \{1\}$ so $\delta_1, \ldots, \delta_k$ are homotopic to paths in $\partial B$.

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Then $\gamma$ is homotopic to a loop which misses the open ball inside $B$. $\square$

27-10-17 Last time: $\pi_1(S^n, x) = \{1\}$, $n \geq 2$

$\pi_1(S^1, 1) = \mathbb{Z}$ (will be proved later).

Example

If $X$, $Y$ are top. spaces then $\pi_1(X \times Y, (x,y)) = \pi_1(X,x) \times \pi_1(Y,y)$ (Exercise on sheet 2).

e.g. $\pi_1(T^2) = \mathbb{Z} \times \mathbb{Z}$ since $T^2 = S^1 \times S^1$

Basepoint dependence

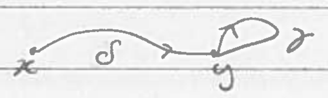
Lemma

Let $X$ be a top. space. Let $x, y \in X$ and suppose we have a continuous path $\delta$ from $x$ to $y$.
Then there is a well defined map

$F_\delta : \pi(X, y) \to \pi_1(X, x)$ which is an isomorphism

where $F_\delta([\gamma]) = [\bar{\delta} \cdot \gamma \cdot \delta]$.

$\delta$ backwards

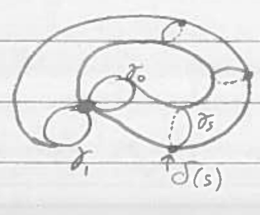



Proof

Exercise on sheet 2.

$[\ ]$ = homotopy class

If $x = y$ this is just conjugating by $[\delta]$.
So if we have a family of loops $\gamma_s$ based at $\delta(s)$ then $[\gamma_1] = [\delta]^{-1}[\gamma_0][\delta]$



Q: If there is no path from $x$ to $y$ how is $\pi_1(X,x)$ related to $\pi_1(X,y)$?
A: Not in any way.

$X =$ 


$\pi_1(X,x) = \mathbb{Z}$
$\pi_1(X,y) = \mathbb{Z}^2$

## Induced maps

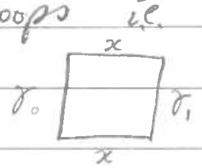
Let $X, Y$ be top. spaces, let $x \in X$, $y \in Y$ be basepoints. Let $F$ be a continuous map $X \to Y$ s.t. $F(x) = y$.
Then given a loop $\gamma$ in $X$ based at $x$, we get a loop $F \circ \gamma$ in $Y$ based at $y$.

## Lemma

This map on loops descends to a well-defined homomorphism $F_*: \pi_1(X,x) \longrightarrow \pi_1(Y,y)$
$F_*[\gamma] = [F \circ \gamma]$
Moreover, if $G: Y \longrightarrow Z$ is another map, $G(y) = z$ then $(G \circ F)_* = G_* \circ F_*$

## Proof

To show that $F_*$ is well defined, we need to check that if $\gamma_0 \simeq \gamma_1$ then $F \circ \gamma_0 \simeq F \circ \gamma_1$.
Let $H$ be a homotopy from $\gamma_0$ to $\gamma_1$
i.e. $\gamma_s(t) = H(s,t)$ is a continuous family of continuous loops i.e. $H: [0,1] \times [0,1] \to X$ is cts.




Then $F \circ H$ is a homotopy from $F \circ \gamma_0$ to $F \circ \gamma_1$. $(F \circ \gamma_s)$

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To show it's a homomorphism, we need to check that $(F \circ \gamma) \cdot (F \circ \delta) = F \circ (\gamma \cdot \delta)$

$$\big((F\circ\gamma)\cdot(F\circ\delta)\big)(t) = \begin{cases} (F\circ\delta)(t), & t\in[0,\tfrac{1}{2}] \\ (F\circ\gamma)(t), & t\in[\tfrac{1}{2},1] \end{cases}$$

$$\big(F\circ(\gamma\cdot\delta)\big)(t) = \begin{cases} F\circ\delta(t), & t\in[0,\tfrac{1}{2}] \\ F\circ\gamma(t), & t\in[\tfrac{1}{2},1] \end{cases}$$

so this follows from the definition of concatenation.

Finally, $(G\circ F)_*[\gamma] = [G\circ F\circ\gamma]$
$= G_*[F\circ\gamma]$
$= (G_*\circ F_*)[\gamma]$

since $F_*[\gamma] = [F\circ\gamma]$.

$\square$

$F_*$ is called the push forward map on $\pi_1$.

This lemma can be expressed by saying that $\pi_1$ is a 'functor' from the category of based top. spaces & cts. maps to the category of groups & homomorphisms.

Corollary

If $F: X \to Y$ is a homeomorphism then $\pi_1(X,x) \cong \pi_1(Y, F(x))$.

Proof

The map $F_*$ has an inverse $(F^{-1})_*$.

$\square$

## Lemma

Suppose $F : X \to X$ is a cts. map homotopic to $Id_X$, then $F_* : \pi_1(X, x) \to \pi_1(X, F(x))$ is an isomorphism

## Proof

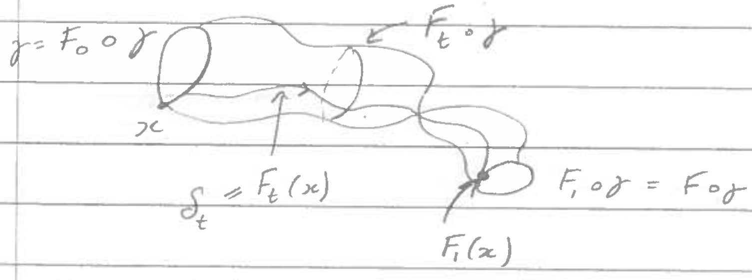
Let $F_t$ be a homotopy from $F$ to $Id_X$

i.e. $F_1 = F$ and $F_0 = Id_X$.

If $\gamma$ is a loop in $X$ based at $x$ then $F_t(x)$ is the path traced out by $x$ around the homotopy $F_t$.

i.e. you have a family of loops

$F_t \circ \gamma$ based at $F_t(x)$




From what we said earlier,

$$[F \circ \gamma] \simeq [\delta \cdot \gamma \cdot \bar{\delta}]$$

$$\|$$

so $F_*[\gamma] = [\delta \cdot \gamma \cdot \bar{\delta}]$

So $[\gamma] = [\bar{\delta} \cdot \delta \cdot \gamma \cdot \bar{\delta} \cdot \delta]$

So $F_*$ is an isomorphism. □

## Corollary

If $X$ and $Y$ are homotopy equivalent spaces via a homotopy equivalence $F : X \to Y$ then

$\pi_1(X, x) \cong \pi_1(Y, F(x))$.

## Proof

$F$ is a homotopy equiv. means $\exists\, G : Y \to X$ s.t.

$F \circ G \simeq id_Y$ and $G \circ F \simeq id_X$.

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By the previous lemma, $(F \circ G)_*$ is an isomorphism $\pi_1 Y \to \pi_1 Y$.

But $(F \circ G)_* = F_* \circ G_*$

$F_* \circ G_*$ surj. $\Rightarrow$ $F_*$ is surj.

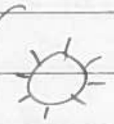
$F_* \circ G_*$ injective $\Rightarrow$ $G_*$ is injective

The same argument applied to $G \circ F$

$\Rightarrow$ $F_*$ injective and $G_*$ surjective

$\square$

So $\pi_1 X$ is an invariant of $X$, i.e. it only depends on the homotopy equivalence class of $X$.

e.g. $\pi_1 \Big($  $\Big) \cong \mathbb{Z}$

<u>Applications</u>

<u>Theorem</u> (Fund. Thm of Algebra)
Any non constant polynomial over $\mathbb{C}$ has a root.

<u>Proof</u>: Assume not.

Let $p(z) = z^n + a_{n-1} z^{n-1} + \dots + a_0$

Write down the function

$f_s : S^1 \to S^1$, $f_s(\theta) = \dfrac{p(se^{i\theta})}{|p(se^{i\theta})|}$

This makes sense because $|p(se^{i\theta})| \neq 0$.

When $s = 0$, $f_s(\theta) = \dfrac{p(0)}{|p(0)|}$ is constant.

when $s$ is large, $f_s(\theta) = \dfrac{p(e^{i\theta})}{|p(e^{i\theta})|} \sim e^{in\theta}$ $\left[\begin{array}{l} \text{e.g. } p(z) = z^2 \\ f_1(\theta) = e^{i2\theta} \end{array}\right]$

$\left[\text{i.e. } \dfrac{s^n e^{in\theta}}{s^n}\right]$

Then $f_s$ is a homotopy from a loop which is constant to a loop which represents $n \in \mathbb{Z}$ in $\pi_1(S^1, 1) = \mathbb{Z}$

※ $(n \neq 0)$

So we need to prove that for large $s$,
$f_s(\theta) \simeq e^{in\theta}$.

Write $p(z) = z^n + R(z)$

For large $\underset{|z|}{\overset{\shortparallel}{s}}$, $|z^n + tR(z)| \geq \big| |z|^n - tn\max|a_i||z|^{n-1} \big|$

$R(z) = a_{n-1}z^{n-1} + \dots + a_0$

$|R(z)| \leq n\max|a_i||z|^{n-1}$

$= |z|^{n-1} \big| \underbrace{|z| - tn\max|a_i|}_{>0} \big| > 0$ for $|z| > tn\max|a_i|$

i.e. sufficiently large $s$.

So $H_t(\theta) = \dfrac{s^n e^{in\theta} + tR(se^{i\theta})}{|s^n e^{in\theta} + tR(se^{i\theta})|}$ is well-defined

(the denominator is positive)

$t = 0$ : $H_0(\theta) = \dfrac{s^n e^{in\theta}}{s^n} = e^{in\theta}$

$t = 1$ : $H_1(\theta) = \dfrac{p(e^{i\theta})}{|p(e^{i\theta})|} = f(\theta)$

$\therefore f_1(\theta) \simeq e^{in\theta}$.

□

Application 2:

<u>Theorem</u> (Brower's fixed point theorem).
Let $F : D^2 \to D^2$ be a continuous map from $D^2 = \{(x,y) \in \mathbb{R}^2 ; x^2 + y^2 \leq 1\}$ to itself.
Then $\exists$ point $(x,y) \in D^2$ s.t. $F(x,y) = (x,y)$.

<u>Proof</u>
Assume not.
Then $\forall\, x \in D^2$, $F(x) \neq x$.

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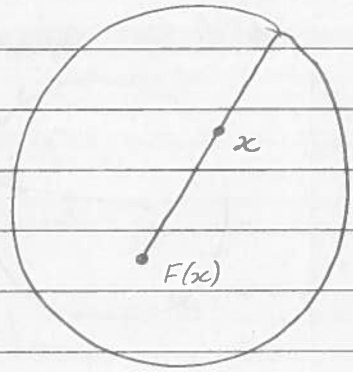
There is a unique straight line segment starting at $F(x)$ going through $x$ and ending at $\partial D^2$.




Let's call the point where the line meets the boundary, $\partial D^2$, $G(x)$. $G$ is then a function from the disc to the circle.

We will prove that $G$ is cts.

If $x \in \partial D^2$ then $x = G(x)$

Now we get a contradiction as follows.

$i : \partial D^2 \hookrightarrow D^2$ inclusion of $\partial D^2$ into $D^2$.

The composition $G \circ i$ equals $Id_{\partial D^2}$ $(G(x) = x)$.

But $(G \circ i)_* = G_* \circ i_*$

$\overset{''}{id_*} = id$

$$\pi_1(\partial D^2) \xrightarrow[i_*]{} \pi_1(D^2) \xrightarrow[G_*]{} \pi_1(\partial D^2)$$

$id_* = G_* \circ i_*$

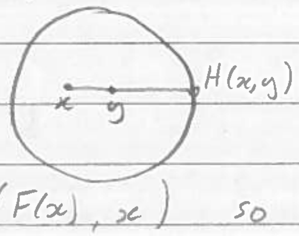
and the identity $\pi_1(\partial D^2) \to \pi_1(\partial D^2)$ factors through $i_*$

$$\pi_1(\partial D^2) = \mathbb{Z} \longrightarrow \pi_1(D^2) = \{1\} \longrightarrow \pi_1(\partial D^2) = \mathbb{Z}$$

but the identity $\mathbb{Z} \to \mathbb{Z}$ doesn't factor through the zero map. $\#$

Proof that $G$ is cts:

First define a map $H : (D^2 \times D^2) \setminus \{(x,x) : x \in D^2\} \to \partial D^2$ like this:

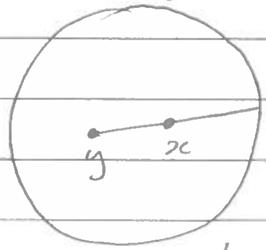



Then $G(x) = H(F(x), x)$ so $G$ is a composition of continuous functions. $\therefore$ $G$ is continuous.

To see that $H$ is cts, note that

$$H(y,x) = y + (x-y)\left(\frac{-2(x-y) + \sqrt{4((x-y)\cdot y)^2 - 4|x-y|^2(|y|^2-1)}}{2|x-y|^2}\right)$$

which is definitely cts when $x \neq y$.




$$y + t(x-y) = H(y,x)$$

where $|y + t(x-y)|^2 = 1$

this is a quadratic for $t$, take the positive root to get the required formula. $\square$

Application 3

Theorem:
The spaces $\mathbb{R}^2$ and $\mathbb{R}^3$ are not homeomorphic.

Proof
If $F : \mathbb{R}^2 \to \mathbb{R}^3$ were a homeomorphism then $\mathbb{R}^2 \setminus \{(0,0)\}$ would be homeomorphic to $\mathbb{R}^3 \setminus \{F(0,0)\}$.
But $\mathbb{R}^2 \setminus \{(0,0)\} \simeq S^1$ and $\mathbb{R}^3 \setminus \{F(0,0)\} \simeq S^2$

$\pi_1 S^1 = \mathbb{Z}$ , $\pi_1 S^2 = \{1\}$.

※

$\square$

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Today:
Statement of Van Kampen's theorem.
- Algebra
- State Theorem
- Compute examples

## Free groups & presentations

**Def**
Let $A$ be a set (alphabet).
A word of length $k$ in $A$ is an expression of the form
$\omega = a_1^{n_1} a_2^{n_2} \dots a_k^{n_k}$ where $a_i \in A$, $n_i \in \mathbb{Z}$.

A word is not reduced if either
- $a_i^0$ appears ($n_i = 0$)

or - $a_i = a_{i+1}$ for some $i$.
Otherwise $\omega$ is reduced.

Any word $\omega$ can be put into reduced form $\bar{\omega}$ by a series of steps:
either - remove an instance of $a_i^0$
or - replace $a^{n_i} a^{n_{i+1}}$ by $a^{n_i + n_{i+1}}$

The empty word is allowed (no letters).

We define a group (the free group on $A$)
$\langle A \rangle$ to be the group whose elements are reduced words in $A$ & whose product is reduced concatenation $\overline{\omega_1 \omega_2}$
e.g. $\overline{aa^{-1}} = 1$

## Lemma

$\langle A \rangle$ is a group.

## Proof

Empty word is identity.

$w = a_1^{n_1} \dots a_k^{n_k} \qquad w^{-1} = a_k^{-n_k} \dots a_1^{-n_1}$

Reduced concatenation is associative.

Observe that $\overline{\overline{w_1 w_2}} = \overline{w_1 w_2}$ (it doesn't matter which order you do the reduction in).

$$\overline{\overline{w_1 w_2} w_3} = \overline{\overline{w_1 w_2} \, \overline{w_3}}$$
$$= \overline{w_1 w_2 w_3}$$
$$= \overline{\overline{w_1} \, \overline{w_2 w_3}} = \overline{w_1 \overline{w_2 w_3}} .$$

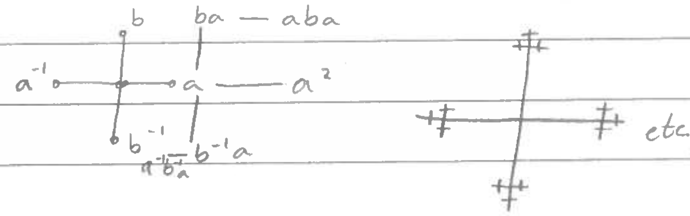
$\square$

## Examples

$\langle \{a\} \rangle = \{1, a, a^2, \dots, a^{-1}, a^{-2}, \dots\} \cong \mathbb{Z}$

$\langle \emptyset \rangle = \{1\}$

$\langle a, b \rangle = \{1, a, b, a^{-1}, b^{-1}, aba, a^2b, aba^2b^{-3}a^4, \dots\}$




## Def

Let $G$ be a group & $R \subseteq G$ a subset.

We define the normal subgroup normally generated by $R$ to be the smallest normal subgroup of $G$ containing $R$.

(Recall $H \leq G$ is normal if $\forall g \in G, \forall h \in H \ ghg^{-1} \in H$.

Recall that if $H$ is normal then $G/H$ (set of cosets) forms a group: $(g_1 H)(g_2 H) = (g_1 g_2 H)$ is well-defined.)

The notation for this is $N(R)$.

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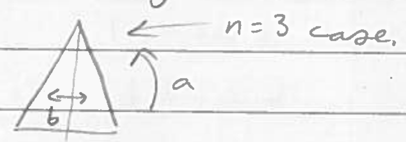
We can take the quotient group $G/N(R)$, in this quotient group the elements of $R$ are trivial.
If $G = \langle A \rangle$ we write $\langle A | R \rangle$ for the quotient $\langle A \rangle / N(R)$ and this is called a presentation of the group $\langle A \rangle / N(R)$.

__Example__

$$\langle a / a^2 \rangle \cong \quad A = \{a\}, \quad R \subseteq \langle A \rangle, \quad R = \{a^2\}$$
$$\langle a / a^2 = 1 \rangle \cong C_2 \quad \text{cyclic group of order 2.}$$

$$\langle a, b / aba^{-1}b^{-1} \rangle \cong \quad aba^{-1}b^{-1} = 1 \Rightarrow ab = ba$$
$$\langle a, b / ab = ba \rangle \cong \mathbb{Z}^2$$

$$\langle a, b / a^n = 1, b^2 = 1, bab = a^{n-1} \rangle = D_{2n}$$
This is the symmetry group of a regular $n$-gon
$a$ = rotation by $2\pi/n$
$b$ = reflection




__Lemma__
If $G$ is a group & $A$ is a subset of $G$ then there is a homomorphism $\langle A \rangle \xrightarrow{F} G$ which sends a word of length 1 $a^n$ to $F(a^n) = a^n \in G$.

__Proof__
Define $F(a_1^{n_1} \dots a_k^{n_k}) = F(a_1^{n_1}) \dots F(a_k^{n_k})$.
This is clearly a homomorphism if it is well defined.
To check $F$ is well-defined we need to show $F(w) = F(\bar{w})$.
Suppose $w_1$ is obtained from $w$ by either replacing $a^{n_i} a^{n_{i+1}}$ by $a^{n_i + n_{i+1}}$ or removing $a^0$.
$$F(a^{n_i} a^{n_{i+1}}) = F(a^{n_i}) F(a^{n_{i+1}}) = F(a)^{n_i} F(a)^{n_{i+1}} = F(a)^{n_i + n_{i+1}} = F(a^{n_i + n_{i+1}})$$
$$F(a^0) = a^0 = 1, \quad F(1) = 1 \Rightarrow F(a^0) = F(1).$$

$\square$

## Corollary

Any group has a presentation.

## Proof

Take $A = G$ then the map $F : \langle G \rangle \to G$ is a surjective homomorphism.

Take $R = \operatorname{Ker} F$, this is by definition a normal subgroup. $N(R) = R$.

So $G = \langle G \rangle / N(R) = \langle G \mid R \rangle$

$\leftrightarrow \quad G = \langle G \rangle / \operatorname{Ker} F$ (1st isom. theorem)

$\square$

## Example

$$\left.\begin{array}{l} G = C_2 = \{a, b\} \\ \langle G \rangle = \langle a, b \rangle \\ \langle G \rangle \xrightarrow{F} G \\ \quad a \longmapsto 1 \\ \quad b \longmapsto b \\ \operatorname{Ker} F = N\{a, b\} \end{array}\right\} \Rightarrow C_2 = \langle a, b \mid a = 1,\ b^2 = 1 \rangle = \langle b \mid b^2 = 1 \rangle$$

## Def

Let $A, B, C$ be groups and $f : C \to A$, $g : C \to B$ homomorphisms.

$$\left[\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "g"'] & A \\ B & \end{tikzcd}\right]$$

We define the "push out" / "amalgamated product" $A \underset{C}{*} B$ to be the group with presentation

$$\langle G_A, G_B \mid R_A, R_B, R_{amal} \rangle$$

where $R_{amal} = \{ f(c) = g(c) : c \in C \}$

$$\left[\begin{array}{l} A = \langle G_A \mid R_A \rangle \\ B = \langle G_B \mid R_B \rangle \end{array}\right]$$

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Amalgamated product of $A$ & $B$ over $C$:

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "g"'] & A \\ B & \end{tikzcd}$$

$$A \underset{C}{*} B = \langle G_A, G_B \mid R_A, R_B, R_{amal} \rangle$$

$$R_{amal} = \{ f(c) = g(c) \ \forall c \in C \}$$

$\forall c \in G_C$

Remark:

$A \underset{C}{*} B$ only depends on $A, B, C, f, g$, not on the generators / relations $G_A, G_B, R_A, R_B$.

This follows from a "universal property":

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "g"'] & A \arrow[d, "\alpha"] \arrow[ddr, bend left] & \\ B \arrow[r, "\beta"'] \arrow[drr, bend right] & G & \\ & & A \underset{C}{*} B \arrow[ul, "\exists!"'] \end{tikzcd}$$

Example

$$\begin{tikzcd} \{0\} \arrow[r, "f"] \arrow[d, "g"'] & \mathbb{Z} = \langle a \rangle \\ \mathbb{Z} = \langle b \rangle & \end{tikzcd}$$

$0 \mapsto a^0$, $0 \mapsto b^0$

$$\mathbb{Z} \underset{\{0\}}{*} \mathbb{Z} = \langle a, b \mid a^0 = b^0 \rangle$$
$$= \langle a, b \rangle$$

So we write this as $\mathbb{Z} * \mathbb{Z}$

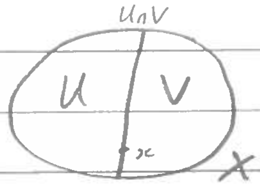
Example

$$\begin{tikzcd} \langle c \rangle = \mathbb{Z} \arrow[r, "f"] \arrow[d, "g"'] & \mathbb{Z} = \langle a \rangle \\ \{0\} & \end{tikzcd}$$

$$f(c) = a^n$$

$$\mathbb{Z}/n = \langle a \mid a^n = 1 \rangle$$

Theorem (Van Kampen) (VKT)
Let $X$ be a path-connected space.
Suppose $U, V \subseteq X$ are open subsets s.t.
$X = U \cup V$ and s.t. $U \cap V$ is path connected.
Then $\pi_1(X, x) = \pi_1(U, x) \underset{\pi_1(U \cap V, x)}{*} \pi_1(V, x)$



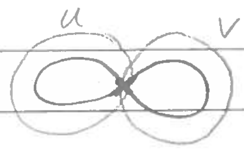

$$\begin{tikzcd} \pi_1(U \cap V, x) \arrow[r, "i_*"] \arrow[d, "j_*"'] & \pi_1(U, x) \\ \pi_1(V, x) & \end{tikzcd}$$

where $\begin{cases} i : U \cap V \to U \\ j : U \cap V \to V \end{cases}$ are the inclusions

Example

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & \mathbb{Z} \\ \mathbb{Z} & \end{tikzcd}$$

$S^1 \vee S^1$




Let $Y = S^1 \vee S^1 = \infty$
Let $U = \propto$ , $V = \reflectbox{$\propto$}$ , $U \cap V = \times$

$\pi_1(U) = \pi_1(S^1) = \mathbb{Z}$, $\pi_1(V) = \pi_1(S^1) = \mathbb{Z}$
$\pi_1(U \cap V) = \{1\}$
So by VKT, $\pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z}$

Example
$\mathbb{RP}^2$

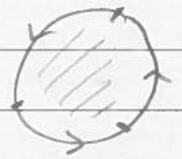

$$\text{VKT} \Rightarrow \begin{tikzcd} \mathbb{Z} = \pi_1(U \cap V) \arrow[r, "f"] \arrow[d] & \pi_1(U) = \mathbb{Z} \\ 1 = \pi_1(V) & \pi_1(\mathbb{RP}^2) \end{tikzcd}$$

$U \simeq S^1$
$V \simeq$ point
$U \cap V \simeq S^1$

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "f"] \arrow[d] & \mathbb{Z} \\ 1 & \mathbb{Z}/2 \end{tikzcd}$$

$f(U \cap V)$ wraps twice around the boundary loop
so $f(c) = a^2$
in $U \cap V$ in boundary

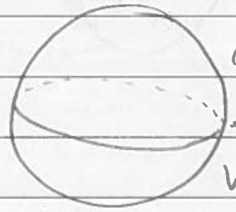
E.g.



has $\pi_1 = \mathbb{Z}/3$

(See sheet 3 for more complex example).

Example



$U \simeq \mathbb{R}^2 \qquad \pi_1(U) = 1$

$\leftarrow U \cap V \simeq S^1 \qquad \pi_1(S^1) = \mathbb{Z}$

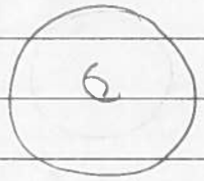
$V \simeq \mathbb{R}^2 \qquad \pi_1(V) = 1$

$$\begin{array}{ccc} \pi_1(U \cap V) & \longrightarrow & \pi_1(U) \\ \downarrow & & \\ \pi_1(V) & & \end{array} \qquad \begin{array}{ccc} \mathbb{Z} & \xrightarrow{f} & 1 \\ g\downarrow & & \\ 1 & & \end{array} \qquad 1 *_{\mathbb{Z}} 1 = 1$$

$c \in \mathbb{Z}$, $\begin{cases} f(c) = 1 \\ g(c) = 1 \end{cases} \Rightarrow f(c) = g(c) \Leftrightarrow 1 = 1.$

$\Rightarrow \pi_1(S^2) = \{1\}$.

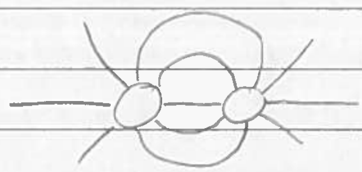
Example



$T^2 \subseteq \mathbb{R}^3 \subseteq S^3$

$U =$ solid torus inside $T^2$
$V =$ complement of $U$
$U \cap V = T^2$

V is a solid torus!

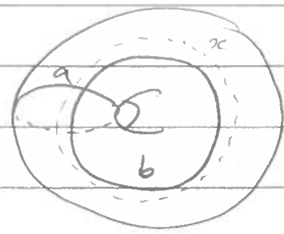


If we sit $S^3 \subseteq \mathbb{R}^4$ as the unit sphere then
$S^3 = \{(x,y,z,w) : x^2+y^2+w^2=1\}$
$T^2 = \{x^2+y^2=1,\ z^2+w^2=1\}$
The antipodal map $(x,y,z,w) \longmapsto (-x,-y,-z,-w)$ preserves $T^2$

and switches $U$ and $V$, the two components of $S^3 \setminus T^2$

$\Rightarrow U \cong V \cong S^1 \times D^2$

VKT $\Rightarrow \pi_1(S^3) = \pi_1(U) \underset{\pi_1(U \cap V)}{*} \pi_1(V)$




$\pi_1(U \cap V) = \mathbb{Z}^2$

$\pi_1(U) = \mathbb{Z}$ since $U = S^1 \times D^2 \simeq S^1$

$\pi_1(V) = \mathbb{Z}$ since $V \cong U$

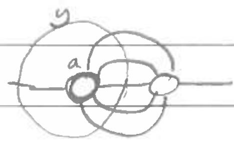
$$\begin{array}{ccc} \langle a,b \mid ab=ba \rangle & & \langle x \rangle \\ \parallel & & \\ \mathbb{Z}^2 & \xrightarrow{f} & \mathbb{Z} \\ g \downarrow & & \\ \langle y \rangle = \mathbb{Z} & & \end{array}$$

$f(a) = 1$ because $a$ bounds a disc in $U$, the solid torus.

$f(b) = x$

$g(b) = 1$ as $b$ bounds a disc in $V$

$g(a) = y$




Overall

$$\begin{array}{ccc} a \quad b \quad \mathbb{Z}^2 & \xrightarrow[\;b \mapsto x\;]{a \mapsto 1} & \mathbb{Z} = \langle x \rangle \\ \downarrow \quad \downarrow \quad \downarrow & & \\ y \quad 1 \quad \mathbb{Z} & & \\ \parallel & & \\ \langle y \rangle & & \end{array}$$

So $\pi_1(S^3) = \langle x, y \mid R_{amal} \rangle$

$R_{amal} = \{ f(c) = g(c) \ \forall c \in C \}$

$f(a) = 1 \qquad g(a) = y \qquad \Rightarrow y = 1$

$f(b) = x \qquad g(b) = 1 \qquad \Rightarrow x = 1$

$\Rightarrow \pi_1(S^3) = \langle x, y \mid x = 1 = y \rangle = \{1\}$

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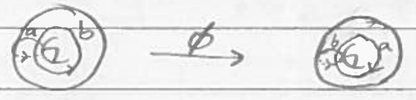
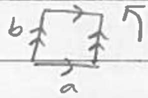
In the last example we took two solid tori $U, V$ and identified them along their boundary.




$U$ $\qquad$ $V$

$\phi(a) = b$, $\phi(b) = a^{-1}$

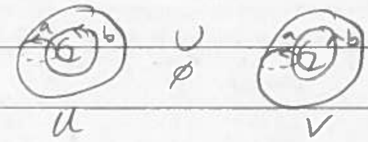



$\phi$ is a homeomorphism of $T^2$ with itself

$\phi_* : \pi_1 T^2 \longrightarrow \pi_1 T^2$

Let's try using other identification homeomorphisms $\phi$ and see what spaces we get.

e.g. $\phi = 1$




$= S^1 \times S^2$

$(U \sqcup V)/\sim \qquad x \sim y$ if $x \in \partial U$, $y \in \partial V$ and $y = \phi(x)$

$= U \cup_\phi V$

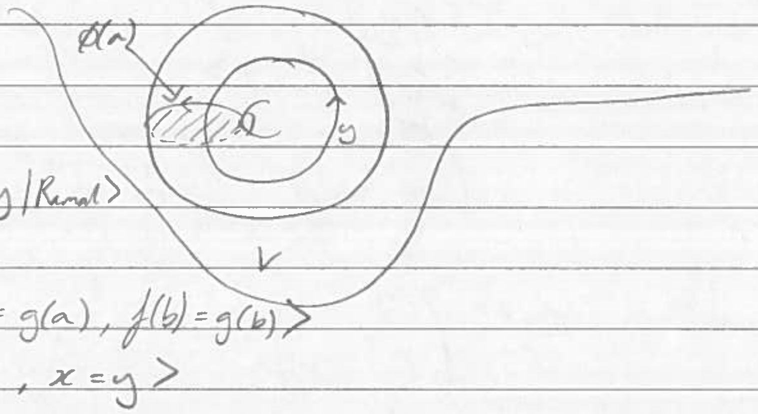
$U, V$ are solid tori i.e. $S^1 \times D^2$

$U \cap V =$ common boundary $T^2$

$$\begin{array}{ccc} \mathbb{Z}^2 = \langle a,b \mid ab=ba \rangle & & \\ \| & & \\ \pi_1(U \cap V) & \xrightarrow{f} & \pi_1(U) = \langle x \rangle \\ g \downarrow & & \| \\ \langle y \rangle = \pi_1(V) & & \mathbb{Z} \\ \| & & \\ \mathbb{Z} & & \end{array}$$

$f : \pi_1(U \cap V) \to \pi_1(U)$, $f(a) = 1$, $f(b) = x$

$g : \pi_1(U \cap V) \to \pi_1(V)$, $g(a) = 1$, $g(b) = y$




VKT

$\Rightarrow \pi_1(U \cup_\phi V) = \langle x, y \mid R_{und} \rangle$

$= \langle x, y \mid f(a) = g(a), f(b) = g(b) \rangle$

$= \langle x, y \mid 1 = 1, x = y \rangle$

$= \langle x \rangle = \mathbb{Z}$

Def

A 3-dim lens space is a space of the form $U \cup_\phi V$ for some homeo $\phi : T^2 \to T^2$ where $U$ and $V$ are solid tori.

VKT

$$\begin{array}{ccc} \mathbb{Z}^2 & \xrightarrow{f} & \mathbb{Z} \\ {\scriptstyle g}\downarrow & & \\ \mathbb{Z} & & \end{array}$$

$f(a) = 1, \; f(b) = x$

$g(a) = pr_2(\phi_*(a))$

$g(b) = pr_2(\phi_*(b))$

here $\phi_* : \pi_1(T^2) \to \pi_1(T^2)$
$$\begin{array}{ccc} \pi_1(T^2) & \to & \pi_1(T^2) \\ \| & & \| \\ \mathbb{Z}^2 & \longrightarrow & \mathbb{Z}^2 \end{array}$$

$pr_2 : \mathbb{Z}^2 \longrightarrow \mathbb{Z}$ (project onto second factor).

$\phi_* : \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2$ has a matrix $\begin{pmatrix} m & n \\ p & q \end{pmatrix}$

$\phi_*(a) = a^m b^p$

$\phi_*(b) = a^n b^q$

so $pr_2(\phi_*(a)) = y^p$, $pr_2(\phi_*(b)) = y^q$

So
$$\begin{aligned} \pi_1(U \cup_\phi V) &= \langle x, y \mid f(a) = g(a), \; f(b) = g(b) \rangle \\ &= \langle x, y \mid 1 = y^p, \; x = y^q \rangle \\ &= \langle y \mid y^p = 1 \rangle \\ &= \begin{cases} \mathbb{Z}/p & \text{if } p \neq 0 \\ \mathbb{Z} & \text{if } p = 0 \end{cases} \end{aligned}$$

For $S^3$, $\phi_* = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

For $S^1 \times S^2$, $\phi_* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

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In fact you can get any matrix in $GL(2,\mathbb{Z})$ as $\phi_*$ for some homeomorphism $T^2 \to T^2$.

($\mathbb{RP}^3$ is a lens space.)

Application!

$\pi_1$ of a cell complex.

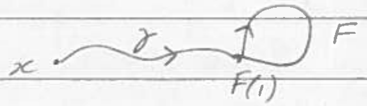
Theorem

Let $X$ be a top space, $x \in X$ a basepoint.
Let $F: S^{n-1} \to X$ be a cts map which lands in the path component of $x$.
Consider $Y = X \cup_F D^n$ (the space obtained from $X$ by attaching an $n$-cell). Then:

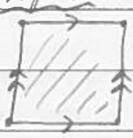
$$\pi_1(Y,x) = \begin{cases} \pi_1(X,x) * \mathbb{Z} & n=1 \\ \pi_1(X,x)/R & n=2 \\ \pi_1(X,x) & n \geq 3 \end{cases}$$

[$*$ is free product, i.e. I add a generator but no relations]

where $R$ is the normal subgroup normally generated by $F$.
More precisely $R$ is generated by the loop $\gamma^{-1}.F.\gamma$ where $\gamma$ is a path from $x$ to $F(1)$.




Example



$(\mathbb{Z} * \mathbb{Z})/R$

$\pi_1(T^2) = \langle a,b \mid b^{-1}a^{-1}ba \rangle = \langle a,b \mid ab = ba \rangle = \mathbb{Z}^2.$

This follows immediately from the theorem.
We added two 1-cells to get $a, b$ and one 2-cell whose attaching map $F$ was the loop $b^{-1}a^{-1}ba$

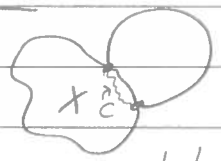

$\Sigma$ $\quad \pi_1(\Sigma) = \langle a,b,c,d \mid d^{-1}c^{-1}dc\,b^{-1}a^{-1}ba = 1 \rangle$

This generalises to any orientable surface.

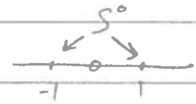
## Proof of Theorem

Assume $F(1) = x$.

$n=1$



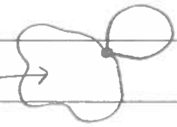

$F : S^0 \longrightarrow X$




We assumed that $F$ lands in the path component of $x$. Then there exists a 1-cell $C$ in $X$ connecting $F(1)$ and $F(-1)$.

Contract $C$ via a homotopy equivalence.

We get



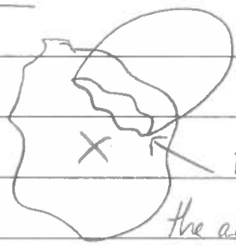
htpy equiv to $X$

so $Y = X \cup S^1$.

Use $U = X$, $V = S^1$, $U \cap V =$ point.

$\text{VKT} \Rightarrow \pi_1(Y) = \pi_1(X) * \mathbb{Z}$.

$n \geq 2$




Let $\tilde{D}$ be a concentric smaller disc.

$D^n$

$U = X \cup (D^n \setminus \tilde{D})$, $V = \tilde{D}$

$F(S^{n-1})$

the annulus $D^n \setminus \tilde{D}$ is contractible onto $X$

so $U \simeq X$

$U \cap V = S^{n-1} = \partial \tilde{D}$

VKT:

$$\begin{array}{ccc} \pi_1(S^{n-1}) & \longrightarrow & \pi_1(X) \\ \downarrow & & \\ \pi_1(\tilde{D}) & & \\ \| & & \\ \{1\} & & \end{array}$$

$n=2$ $\quad \pi_1(S^{n-1}) = \pi_1(S^1) = \mathbb{Z}$

$n \geq 3$ $\quad \pi_1(S^{n-1}) = \{1\} \Rightarrow \pi_1(Y) = \pi_1(X)$

For $n=2$:

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{1 \longmapsto [F]} & \pi_1(X) \\ \downarrow & & \\ \{1\} & & \end{array}$$

so we get $R_{\text{amal}} = \{[F] = 1\}$

So $\pi_1(Y, x) = \pi_1(X, x) / \text{normal subgroup generated by } F$. $\square$

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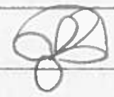
- Van Kampen's Theorem

Useful for Sheet 3 → - We saw a way to compute $\pi_1(\text{cell complex})$ (cell complex $X$)

- Say $X$ has one 0-cell.



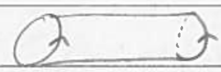
$\pi_1(X) = \langle \text{one-cells} \mid \text{2-cells} \rangle$

<u>Final application</u>: Mapping tori.

<u>Def</u>

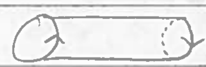
Let $X$ be a top. space and let $F: X \to X$ be a cts. map. The mapping torus $MT(X,F)$ is the space $X \times [0,1]/\sim$ where $(x,0) \sim (F(x),1)$.

e.g. $X = S^1$, $F = id : S^1 \to S^1$



$MT(X,F) = T^2$

$X = S^1$, $F(e^{i\theta}) = e^{-i\theta}$



$MT(X,F) = \text{Klein bottle}$.

<u>Theorem</u>

Let $X$ be a cell complex (write $X^n$ for the $n$-skeleton). Suppose that $X^0 = \{x\}$. Let $F: X \to X$ be a continuous cellular map (i.e. $F(X^n) \subseteq X^n$). Then $\pi_1(MT(X,F),(x,0)) = \langle \pi_1(X), \lambda \mid \text{relations in } \pi_1(X),\ \lambda g \lambda^{-1} = F_*(g)\ \forall g \in \pi_1(X) \rangle$

<u>Example</u> (Torus)

$X = S^1$, $F = id$. $\quad \pi_1(MT(X,F)) = \langle a, \lambda \mid \lambda a \lambda^{-1} = \underset{}{F_*(a)} \rangle$ ($F_*(a) = a$)

$\pi_1(X) = \langle a \rangle$ $\quad = \langle a, \lambda \mid \lambda a = a\lambda \rangle$

<u>Example</u> (Klein bottle)

$X = S^1$, $\pi_1(X) = \langle a \rangle$, $F(e^{i\theta}) = e^{-i\theta}$

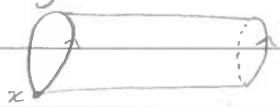
$\pi_1(MT(X,F)) = \langle a, \lambda \mid \lambda a \lambda^{-1} = a^{-1} \rangle$

$= \langle a, \lambda \mid \lambda a = a^{-1}\lambda \rangle$

Proof

We will write down a cell structure on $MT(X,F)$, and use previous result to get our presentation.

Already have a cell structure on $X$.



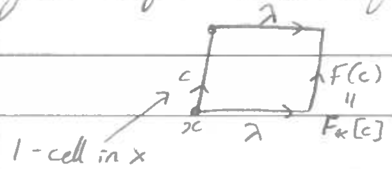

$X \times [0,1]/\sim$

The basepoint $x$ gives a new 1-cell $[x,t]$, $t \in [0,1]$.
This is a closed loop since $F$ is cellular, so $F(X^\circ) \subseteq X^\circ$. This will be $\lambda$.

More generally, if $c : [0,1]^k \to X$ is a $k$-cell then $c' = [c,t] : t \in [0,1]$ is a $(k+1)$-cell in $MT(X,F)$.

What are the 2-cells?

The 2-cells are $c'$ for $c$, a 1-cell of $X$. The 1-cells of $X$ give loops that generate $\pi_1(X)$.




So the relation we get from $c'$ is

$\lambda^{-1}(F_*(c))^{-1}\lambda c = 1$

$\Rightarrow \lambda c = F_*(c)\lambda \Rightarrow \lambda c \lambda^{-1} = F_*(c)$. $\square$

not examinable

Proof (of Van Kampen's Theorem)

We're trying to prove that if $X = U \cup V$ and $U \cap V$ is connected & $x \in U \cap V$ then $\pi_1(X) = \pi_1(U) \underset{\pi_1(U \cap V)}{*} \pi_1(V)$

$\Rightarrow \pi_1(X) = \langle \begin{smallmatrix}\text{elements of } \pi_1(U) \\ \text{elements of } \pi_1(V)\end{smallmatrix} \mid \text{relations from } \pi_1(U) \,\&\, \pi_1(V),\ R_{amal} \rangle$

$R_{amal} = \{ i_*(c) = j_*(c) : c \in \pi_1(U \cap V) \}$

$(i : U \cap V \to U,\ j : U \cap V \to V$ are the inclusions$)$.

Strategy of proof:

1). Write down a homomorphism $\phi : \underbrace{\pi_1(U) * \pi_1(V)}_{\text{(free product, no amalgamation)}} \to \pi_1(X)$

2). Prove $\phi$ is surjective.

3). Prove $\operatorname{Ker}\phi = N(R_{amal})$ (smallest normal subgroup containing $R_{amal}$)

By 1st Isom. Thm, Van Kampen Thm follows.

1). Defining $\phi$.

Let $f : U \cap V \to U$, $g : U \cap V \to V$ be the inclusion maps.

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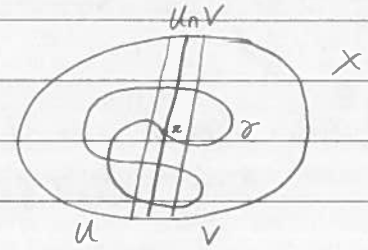
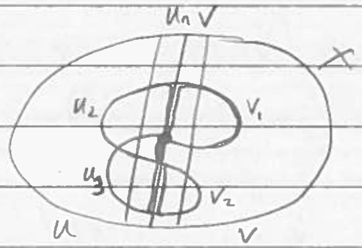
If $w \in \pi_1(U) * \pi_1(V)$ is a word $u_1 v_1 u_2 v_2 \dots u_n v_k$
then we define $\phi(u)$ to be $f_*(u_1) g_*(v_1) \dots f_*(u_k) g_*(v_k)$.
This is well defined since there are no additional relations.

2). $\phi$ is surjective

Let $\gamma \in \pi_1(X, x)$ be a loop.
$\gamma^{-1}(U)$ $\gamma^{-1}(V)$ is an open cover of the interval.
The interval is compact, so we can take a finite subcover.
i.e. the open cover is the set of connected open intervals of $\gamma^{-1}(U)$ and of $\gamma^{-1}(V)$, so so could be infinite. In the example we took four.




$\Rightarrow$ there is a finite subdivision of the interval $[0,1]$ into subintervals $[0, t_1], [t_1, t_2], \dots, [t_{n-1}, t_n]$ for $0 \le t_1 \le t_2 \le \dots \le t_n = 1$ s.t. $\gamma|_{[t_i, t_{i+1}]}$ is contained in either $U$ or $V$.

Now $U \cap V$ is path-connected by assumption so we pick paths $\delta_i : [0,1] \to U \cap V$ s.t. $\delta_i(0) = x$, $\delta_i(1) = \gamma(t_i)$.
Let $\gamma_i = \gamma|_{[t_i, t_{i+1}]}$, so $\gamma = \gamma_{n-1} \cdot \dots \cdot \gamma_0$.

$$\gamma \simeq \gamma_{n-1} \cdot \gamma_{n-2} \cdots \gamma_1 \gamma_0$$
$$\simeq (\gamma_{n-1} \delta_{n-1})(\bar{\delta}_{n-1} \gamma_{n-2} \delta_{n-2})(\bar{\delta}_{n-2} \gamma_{n-3} \cdot) \cdots (\bar{\delta}_3 \gamma_2 \delta_2)(\bar{\delta}_2 \gamma_1 \delta_1)(\bar{\delta}_1 \gamma_0)$$
(each bracket is a loop)

Now $\bar{\delta}_i \gamma_{i-1} \delta_{i-1}$ is a loop $\forall i$, completely contained in either $U$ or $V$.

$$\therefore [\gamma] = \underbrace{f_*}_{\text{or } g}(\gamma_{n-1} \delta_{n-1}) \underbrace{g_*}_{\text{or } f}(\bar{\delta}_{n-1} \gamma_{n-2} \delta_{n-2}) \cdots \underbrace{f_*}_{\text{or } g}(\bar{\delta}_1 \gamma_0)$$

$f$ or $g$ depending on whether loop is in $U$ or $V$ resp.

$\Rightarrow [\gamma] = \phi([\gamma_{n-1} \delta_{n-1}][\bar{\delta}_{n-1} \gamma_{n-2} \delta_{n-2}] \dots [\bar{\delta}_1 \gamma_0]) \in \pi_1(U) * \pi_1(V)$.
$\Rightarrow \phi$ is surjective.

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<u>Reminder</u>

$\phi : \pi_1(U) * \pi_1(V) \to \pi_1(X)$

$[u_1][v_1]\cdots[u_n][v_n] \mapsto f_*[u_1]\, g_*[v_1] \cdots f_*[u_n]\, g_*[v_n]$

$$\begin{tikzcd} U\cap V \arrow[r, "i"] \arrow[d, "j"] & U \arrow[d, "f"] \\ V \arrow[r, "g"'] & X \end{tikzcd}$$

WTS: $\operatorname{Ker}\phi$ is the normal subgroup generated by $i_*[c]\, j_*[c]^{-1}$ for $c \in \pi_1(U\cap V)$

First note that $i_*[c]\, j_*[c]^{-1} \in \ker\phi$. Why?

$\phi(i_*[c]\, j_*[c]^{-1}) = f_* i_*[c]\, g_* j_*[c]^{-1}$

$(f\circ i)_* = f_* \circ i_*$

$f\circ i = g\circ j \Rightarrow (f\circ i)_* = (g\circ j)_*$

$\Rightarrow f_* i_*[c] = g_* j_*[c]$

$\Rightarrow f_* i_*[c]\, g_* j_*[c]^{-1} = 1$

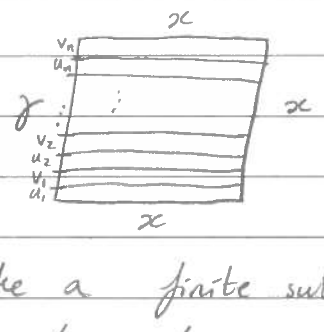
$\Rightarrow f_* i_*[c]\, g_* j_*[c]^{-1} \in \operatorname{Ker}\phi.$

So we need to show that if $w = [u_1][v_1]\cdots[u_n][v_n] \in \pi_1(U) * \pi_1(V)$ is in $\operatorname{Ker}\phi$ then it's in the normal subgroup generated by elements of the form $i_*(c)\, j_*(c)^{-1}$.

This is the same as saying that we can reduce $w$ to $1$ by making substitutions involving relations either in $\pi_1(U)$ or in $\pi_1(V)$ or $i_*(c) = j_*(c)$.

The fact that $w \in \operatorname{Ker}\phi$ means that the loop $\gamma = f(u_1)\, g(v_1) \cdots f(u_n)\, g(v_n)$ is nullhomotopic in $X$.

Let $H$ be such a homotopy, $H : [0,1]\times[0,1] \to X$




The connected components of $H^{-1}(U)$ & $H^{-1}(V)$ form an open cover of the rectangle $[0,1]\times[0,1]$.

As $[0,1]\times[0,1]$ is compact we can take a finite subcover. So we can subdivide $[0,1]\times[0,1]$ into tiny rectangles, each of which maps into $U$ or into $V$.

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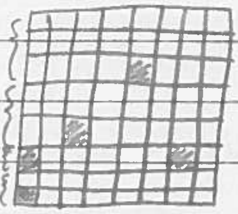
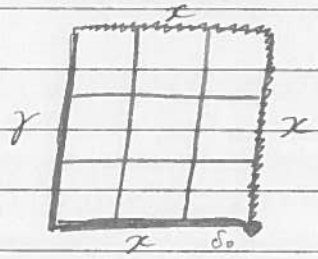
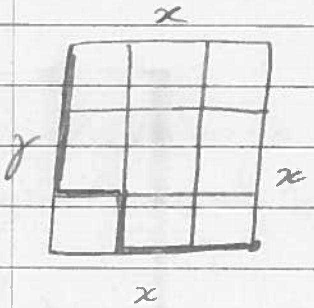


W.l.o.g. assume that this is a subdivision of the subdivision we already had.



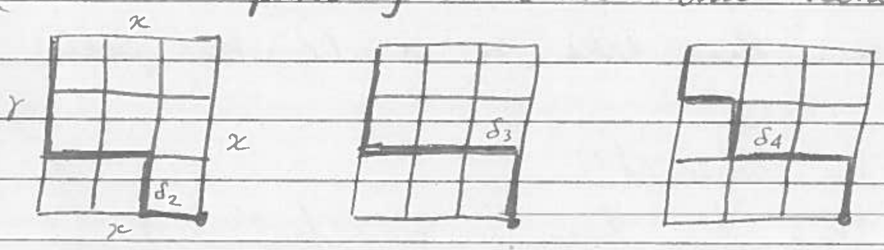

Let $\delta_0$ be the path in the rectangle ($x$ then $\gamma$).
$H \circ \delta_0$ is a path in $X$ which is homotopic to $\gamma$

Let $\delta_N$ be the other path indicated in the diagram ($x$ then $x$)
$H \circ \delta_N$ = constant path.




$\delta_1$ is the path in the diagram
$H \circ \delta_1 \simeq H \circ \delta_0$

In fact $H \circ \delta_0$ is homotopic to $H \circ \delta_1$ via a homotopy which takes place entirely in $U$ (or in $V$ depending on where this rectangle maps).



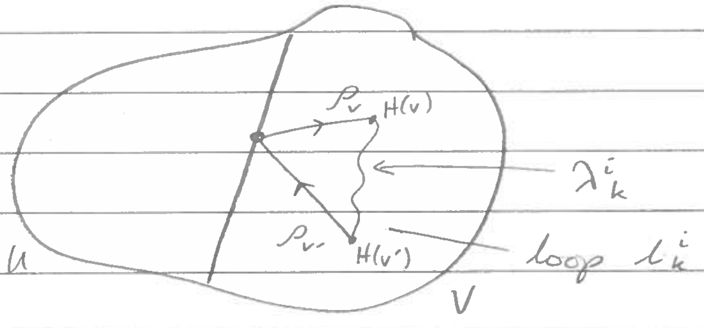

In this way we construct a family of paths $\delta_0, \delta_1, \ldots, \delta_N$ in $[0,1]^2$ s.t.
$H \circ \delta_i \simeq H \circ \delta_{i+1}$ entirely in $U$ or $V$.

$H \circ \delta_i$ is a concatenation of paths
$$\lambda_N^i \cdot \lambda_{N-1}^i \cdot \ldots \cdot \lambda_1^i$$
where $\lambda_k^i$ is $H \circ \delta_i$ restricted to the $k^{th}$ edge along $\delta_i$.
We will turn $\lambda_k^i$ into a loop as follows.
At each vertex, $v$, of the grid I will pick a path $\rho_v$

from $x$ to $H(v)$ and define a loop

$l^i_k = \overline{\rho_{v'}} \cdot \lambda^i_k \rho_v$

where $\lambda^i_k$ is $H \circ S_i$ restricted to the edge from $v$ to $v'$




Note that:

$$\lambda^i_N \cdot \lambda^i_{N-1} \cdot \ldots \cdot \lambda^i_1 \simeq \underbrace{(\lambda^i_N \rho_v)}_{l^i_N}\cancel{\phantom{.}}\underbrace{(\overline{\rho_v}\, \lambda^i_{N-1} \rho_{v'})}_{l^i_{N-1}}\cancel{\phantom{.}}\underbrace{(\overline{\rho_{v'}}\, \lambda^i_{N-2} \rho_{v''})}_{\ldots}\cancel{\phantom{.}}\underbrace{(\overline{\rho_{v''}} \ldots)(\lambda_1)}_{l^i_1}$$

$$\simeq l^i_N\, l^i_{N-1} \ldots l^i_1$$

? For each $H \circ S_i$, $\lambda^i_k$ along this edge is $V$, $U$ or $U \cap V$.

"Disambiguate" the path by choosing, for each edge in $U \cap V$, to think of it in $U$ or $V$.

Need to make sure that

- if $\lambda^i_j$ is an edge in $U \cap V$ then $\rho_v$ and $\rho_{v'}$ are both contained in $U \cap V$.
- if $H(v) = x$, then take $\rho_v = x$ (constant path).

What have we achieved?

- $w = [u_1] \ldots [v_n]$ has been replaced by first concatenating with constant loop and then subdividing each $u_i, v_i$ into loops in $U$ or in $V$.

  i.e. we have written it as

  $[l'_N][l'_{N-1}] \ldots [l'_1]$

  This has only involved subdivisions in $U$ or $V$, so only used relations in $\pi_1(U)$ or $\pi_1(V)$.

  At each step we get a word $w_i = [l^i_N] \cdots [l^i_1]$ $i = 1, \ldots, N$ where $w_{i+1}$ is obtained from $w_i$ by replacing two consecutive terms $l^i_j l^i_{j-1}$ with $l^{i+1}_j l^{i+1}_{j-1}$ (doing homotopy

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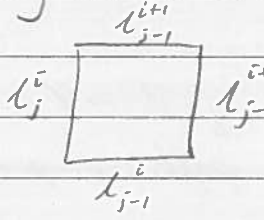
over a rectangle)

The substitution $l_j^i l_{j-1}^i \to l_j^{i+1} l_{j-1}^{i+1}$ is obtained by a relation in $U$ or $V$.

This works if $l_j^i, l_{j-1}^i, l_j^{i+1}, l_{j-1}^{i+1}$ are all contained entirely in $U$ or entirely in $V$.

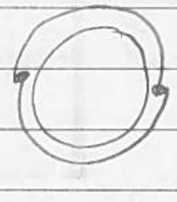



It could be that when disambiguating, I chose one or more to be in the wrong set.

I need to switch them to the correct set. I can do this since the whole rectangle is either in $U$ or in $V$.

Switching a loop in $U \cap V$ from $U$ to $V$ or $V$ to $U$ is exactly $i_*(c) \rightleftarrows j_*(c)$, $c \in \pi_1(U \cap V)$.

$\square$



$$\begin{array}{ccc} 1 = \pi_1(U \cap V) & \longrightarrow & 1 \\ \downarrow & & \\ 1 & & \end{array}$$

- Van Kampen's Thm requires $U \cap V$ connected
- Cannot use it for $\pi_1(S^1)$.

## Covering Spaces

Basic example of a covering map is the map

$p: \mathbb{R} \to S^1$, $p(t) = e^{2\pi i t}$ 

This is not an invertible map, it's $\infty$-many to 1, since all the integers $\mathbb{Z}$ map to $1 \in S^1$.

However it is locally invertible: if you make a branch cut in $\mathbb{C}$ at angle $\phi$ then we get a well defined inverse $q_\phi : S^1 \setminus e^{i\phi} \to \mathbb{R}$, $e^{i2\pi\phi} \mapsto \theta \in \left[\frac{\phi}{2\pi} - 1, \frac{\phi}{2\pi}\right)$.

e.g. standard branch of log is $q_\pi$.

$\frac{\log e^{i2\pi\theta}}{2\pi i} \in \left[-\frac{1}{2}, \frac{1}{2}\right)$ 

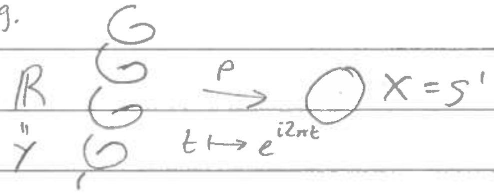
__Def.__

A continuous map $p: Y \to X$ is called a covering map if there is a collection $\mathcal{U}$ of open sets ("elementary neighbourhoods") with the following property:

$\forall x \in X \quad \exists U \in \mathcal{U}$ s.t.

$\forall y \in p^{-1}(x)$ there is a cts. map $q: U \to Y$ with $p \circ q = \mathrm{Id}_U$ s.t. $q(U)$ is the path component of $p^{-1}(U)$ containing $y$.

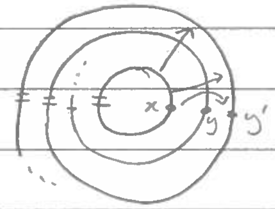
e.g.




In this example ($p: \mathbb{R} \to S^1$) the elementary neighbourhoods are $S^1 \setminus \{e^{i\phi}\}$ and the local inverses are the $q_\phi$ defined earlier.

e.g. if $x = 1$ take $U = S^1 \setminus \{-1\}$ and then, $\forall y \in \mathbb{Z}$

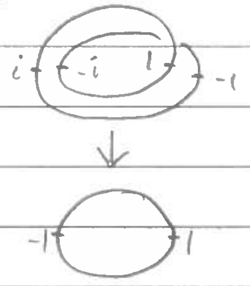
we have $q(e^{i2\pi\theta}) = \theta \in [y - \tfrac{1}{2}, y + \tfrac{1}{2})$




$q_y(x) = y$

__Example__

$p_2: S^1 \to S^1, \quad z \mapsto z^2$




Take elementary neighbourhoods $S^1 \setminus \{1\}$, $S^1 \setminus \{-1\}$

We have 2 branches of $\sqrt{\ }$ function defined on each of these, e.g. on $S^1 \setminus \{-1\}$ we have $\sqrt{e^{i\theta}} = e^{i\theta/2}$, $\sqrt{e^{i\theta}} = -e^{i\theta/2}$

$\theta \in (-\pi, \pi)$.

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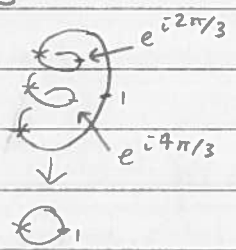
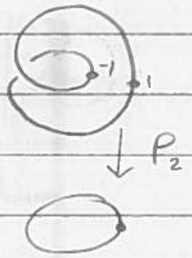
For $p_n(z) = z^n$ we have $n$ branches of $\sqrt[n]{\ }$ corresponding to the $n^{th}$ roots of unity.

$$\sqrt[n]{-1} = \begin{cases} 1 \\ e^{i2\pi/n} \\ e^{i4\pi/n} \\ \vdots \end{cases}$$

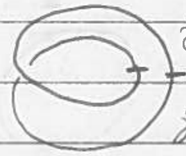
e.g. $n=3$




How to use covering maps to study $\pi_1$?

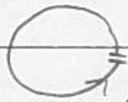



I can lift the loop $\theta \mapsto e^{i2\pi\theta}$ in $S^1$ to 2 paths $e^{i\pi\theta}$, $-e^{i\pi\theta}$ (square roots of our loop) in the covering space



$\left.\begin{array}{l} \tilde{\gamma}_1(\theta) = e^{i\pi\theta} \\ \tilde{\gamma}_2(\theta) = -e^{i\pi\theta} \end{array}\right\}$ these are paths, not loops so they define a permutation of the fibre $p^{-1}(x)$

In this case $1 \mapsto -1$, $-1 \mapsto 1$.



$\gamma(\theta) = e^{i2\pi\theta}$

This permutation is called the monodromy

$\sigma_\gamma : p^{-1}(x) \longrightarrow p^{-1}(x)$

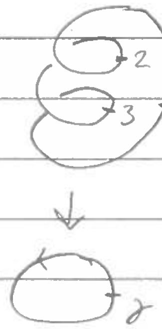
which has inverse

$\sigma_{\bar{\gamma}} = \sigma_\gamma^{-1}$.

In fact the map $\gamma \mapsto \sigma_\gamma$ is a homomorphism

$\pi_1(X) \longrightarrow \mathrm{Perm}(p^{-1}(x))$.

e.g.



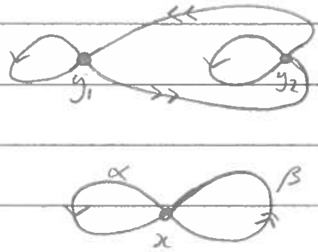

$\sigma_\gamma = (1\ 2\ 3)$

so the monodromy is a map

$\mathbb{Z} = \pi_1(S^1) \longrightarrow S_3$

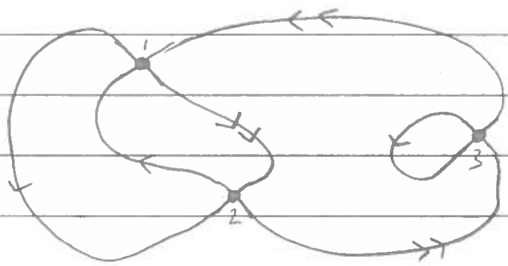
$1 \longmapsto (1\,2\,3)$

e.g. $\infty$ ?




$\sigma_\alpha = id$

$\sigma_\beta : y_1 \mapsto y_2, \quad y_2 \mapsto y_1$




$\sigma_\alpha = (1\,2)$

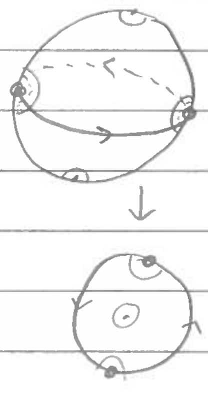
$\sigma_\beta = (1\ 2\ 3)$

$$\mathbb{Z} * \mathbb{Z} \longrightarrow S_3$$
$$\alpha \longmapsto (1\,2)$$
$$\beta \longmapsto (1\,2\,3)$$

e.g. $\pi_1(\mathbb{RP}^2) = \mathbb{Z}/2$

$S^2 \longrightarrow \mathbb{RP}^2 = S^2/\sim$, $\quad (x,y,z) \longmapsto [x,y,z]$

$(-x,-y,-z) \longmapsto [x,y,z]$



monodromy

$\mathbb{Z}/2 \longrightarrow S_2$

$1 \longmapsto (1\,2)$

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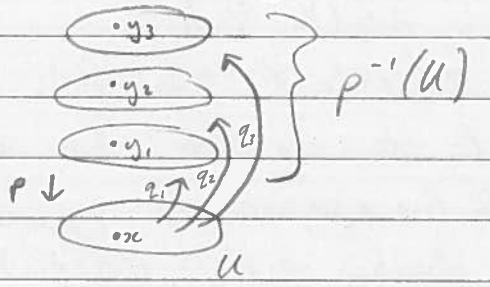
## Covering spaces

Def (reminder from last lecture)

$p: Y \to X$ is a covering map if $\forall x \in X$ $\forall y \in p^{-1}(x)$ $\exists$ subset $U \subseteq X$ and a map $q: U \to Y$ s.t. $p \circ q = id$ (i.e. $q$ is a local inverse for $p$) and s.t. the image of $q$ is the path component of $p^{-1}(U)$ containing $y$ (in particular $q(x) = y$).

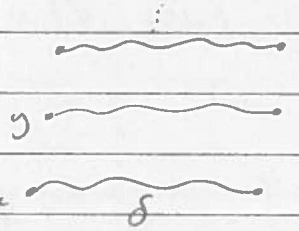



"elementary neighbourhood"

Lemma (Path-lifting lemma)

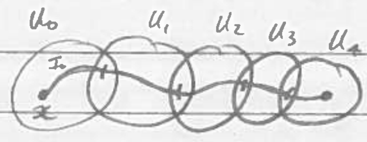
Let $p: Y \to X$ be a covering space, let $\delta: [0,1] \to X$ be a path in $X$ with $\delta(0) = x$ and let $y \in p^{-1}(x)$ ("initial condition"). Then $\exists!\ \gamma: [0,1] \to Y$ s.t. $\gamma(0) = y$ and $p \circ \gamma = \delta$.

This $\gamma$ is called a lift of $\delta$.




Proof

Let $\mathcal{U}$ be a cover of $X$ by elementary neighbourhoods. $\{\delta^{-1}(U) : U \in \mathcal{U}\}$ is a cover of $[0,1]$ by open sets $\therefore$ has a finite subcover, so we can subdivide the interval into $n$ pieces $I_0, I_1, \ldots, I_{n-1}$ where $I_i = [\frac{i}{n}, \frac{i+1}{n}]$ s.t. $\delta|_{I_m}$ lands in a fixed $U_m \in \mathcal{U}$.



We know that $\gamma(0)=y$, so let $q_0 : U_0 \to Y$ be the local inverse with $q_0(x)=y$.
Define $\gamma|_{I_0}(t) = q_0(\delta(t))$, $t \in [0, 1/n]$
This satisfies $\gamma(0) = q_0(\delta(0)) = q_0(x) = y$.
Now, to extend $\gamma$ to $I_1 = [1/n, 2/n]$, we pick $q_1 : U_1 \to Y$ to be the local inverse with $q_1(\delta(1/n)) = \gamma(1/n)$ (which we've already defined).
Then set $\gamma|_{I_1}(t) = q_1(\delta(t))$, $t \in [1/n, 2/n]$
Proceed in the same way to extend $\gamma$ to $[0,1]$.
$\gamma$ is continuous because it's cts on each $I_m$ and the different $\gamma|_{I_m}$ match at the endpoints of consecutive intervals. [By construction $p \circ \gamma(t) = p \circ \overset{\swarrow \text{some } q}{q} \circ \delta(t) = \delta(t)$
$\Rightarrow \gamma$ is a lift]
$\Rightarrow$ existence of a lift.
Uniqueness of the lift follows from the next lemma.
$\square$

<u>Lemma</u> (Uniqueness of lifts)
Let $p: Y \to X$ be a covering space, let $T$ be a non empty connected space and let $F: T \to X$ be a cts map.
If $\tilde{F}_1$ & $\tilde{F}_2 : T \to Y$ are lifts of $F$ to $Y$
(i.e. $p \circ \tilde{F}_1 = p \circ \tilde{F}_2 = F$) then $\tilde{F}_1(t) = \tilde{F}_2(t)$ $\forall t \in T$
iff $\tilde{F}_1(t) = \tilde{F}_2(t)$ for some $t \in T$.

If we apply this with $T = [0,1]$, $F = \delta$, $t = 0$ we conclude that two lifts of $\delta$ with the same initial condition agree everywhere.

<u>Proof</u>
Since $T$ is connected it's sufficent to prove that the set $S = \{t \in T : \tilde{F}_1(t) = \tilde{F}_2(t)\}$ is a). open, b). closed and c). nonempty (this is by assumption since we're assuming $\exists t \in T$ s.t. $\tilde{F}_1(t) = \tilde{F}_2(t)$).
This will imply $T = S$. (note only proving $\Leftarrow$, $\Rightarrow$ obvious).

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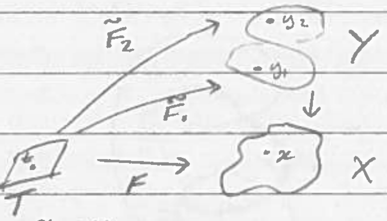
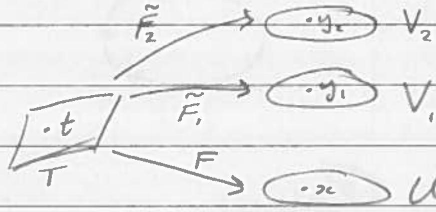
Let $t \in T$ and write $x = f(t)$.

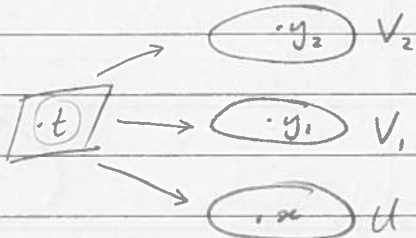



Pick an elementary neighbourhood $U$ containing $x$. Let $y_1 = \tilde{F}_1(t)$, $y_2 = \tilde{F}_2(t)$.
Let $V_1, V_2$ be the path components of $p^{-1}(U)$ containing $y_1, y_2$. $\tilde{F}_1, \tilde{F}_2$ are cts. so $\exists$ a neighbourhood $W \subseteq T$ s.t.
$\tilde{F}_1(W) \subseteq V_1$, $\tilde{F}_2(W \subseteq V_2$.




If $t \in S = \{t : \tilde{F}_1(t) = \tilde{F}_2(t)\}$ $(y_1 = y_2)$
then $V_1 = V_2$. But $p|_{V_1} : V_1 \to U$ is a bijection,
so $p\tilde{F}_1(t') = p\tilde{F}_2(t')$ means that $\tilde{F}_1(t) = \tilde{F}_2(t)$ $\forall t' \in W$
(because $\tilde{F}_1$ & $\tilde{F}_2$ both land in $V_1$).
$\Rightarrow t' \in S$ $\forall t' \in W$, so $S$ is open.

$T \setminus S$ is also open.
If $t \in T \setminus S$ then $y_1 \neq y_2$,
and $V_1 \neq V_2$. As before,
$\exists$ a neighbourhood $W \subseteq T$ of $t$
s.t. $\tilde{F}_1$ maps $W$ to $V_1$, $\tilde{F}_2$ maps $W$ to $V_2$.



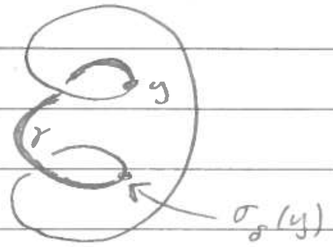

But $V_1 \cap V_2 = \emptyset$ so $\tilde{F}_1$ & $\tilde{F}_2$ do not agree on $W$.
i.e. $\forall t' \in W$ $\tilde{F}_1(t') \neq \tilde{F}_2(t')$ $\Rightarrow$ $W \not\subseteq T \setminus S$.
$\Rightarrow S$ is closed.

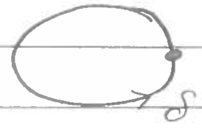
$\square$

Using this, we can now define monodromy.

Def

Let $p : Y \to X$ be a covering map, let $\delta : [0,1] \to X$ be a loop and let $y \in p^{-1}(x)$, $x = \delta(0)$.
Define $\sigma_\delta(y) = \gamma(1)$ where $\gamma$ is the unique path in $Y$ lifting $\delta$, with $\gamma(0) = y$.



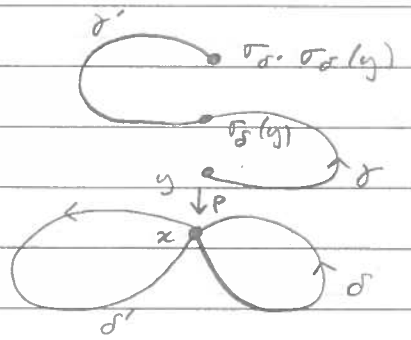




## Theorem

The map $\pi_1(X, x) \longrightarrow \mathrm{Perm}(p^{-1}(x))$, $\delta \longmapsto \sigma_\delta$
is a well-defined homomorphism.

## Proof

Homomorphism.




Let $\delta, \delta'$ be loops at $x$. Let $\gamma$ be the lift of $\delta$ starting at $y$. Let $\gamma'$ be the lift of $\delta'$ starting at $\sigma_\delta(y)$. $\gamma'(1) = \sigma_{\delta'} \cdot \sigma_\delta(y)$.

Note that the concatenation $\gamma' \cdot \gamma$ is a lift of $\delta' \cdot \delta$ starting at $y$. It ends at $\gamma'(1) = \sigma_{\delta'} \cdot \sigma_\delta(y)$ but by definition it ends at $\sigma_{\delta' \cdot \delta}(y)$

$\Rightarrow \sigma_{\delta' \cdot \delta}(y) = \sigma_{\delta'} \cdot \sigma_\delta(y)$.

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Last time:

For a covering map $p: Y \to X$, then there is a well defined group hom. <u>monodromy</u>

$\pi_1(X, x) \longrightarrow \mathrm{Perm}(p^{-1}(x))$

Have proved it's a homomorphism.
Still need to show it's well defined.

Well-definedness follows from:

<u>Homotopy lifting property</u>

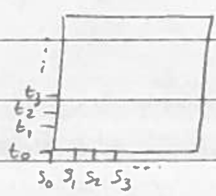
If $p: Y \to X$ is a covering map, $\delta_0 : [0,1] \to X$ is a path from $x$ to $x'$, $\delta_s$ is a homotopy of $\delta_0$ rel endpoints, and $\gamma_0 : [0,1] \to Y$ is a lift of $\delta_0$ then $\exists$ lift $\gamma_s$ of $\delta_s$ which is a homotopy of $\gamma_0$ rel endpoints.


In particular, we get
$\gamma_0(1) = \gamma_1(1)$
so the monodromy of a loop depends only on its homotopy class.

<u>Proof</u>

Let $H: [0,1] \times [0,1] \longrightarrow X$ be the homotopy $H(s, -) = \delta_s$.

Since $[0,1] \times [0,1]$ is compact, we can divide it into finitely many rectangles $R_{ij} = [s_i, s_{i+1}] \times [t_j, t_{j+1}]$, such that $\forall i,j$ $H(R_{ij}) \subset X$ is contained in some elementary neighborhood $U_{ij}$ for $p$.




For each $i,j$ we'll choose a local inverse $q_{ij}$ on $U_{ij}$ to $p$ and define our lift $\tilde{H}$ of $H$ by $\tilde{H}|_{R_{ij}} = q_{ij} \circ H|_{R_{ij}}$

Need to ensure consistency on overlaps.
We'll do this by induction on $i$.
Base case, $i=0$:
For each $j$ choose $q_{0j}$ such that $q_{0j} \circ H(0,-) = \gamma_0(-)$ on $[t_j, t_{j+1}]$.
Need to check that for each $j$ we have
$$\underbrace{q_{0j} \circ H(-, t_{j+1})}_{=\alpha} = \underbrace{q_{0j+1} \circ H(-, t_{j+1})}_{=\beta} \text{ on } [0, s_1]$$

We know that $\alpha, \beta : [0, s_1] \to Y$ are lifts of $H(-, t_{j+1})$ which satisfy $\alpha(0) = \gamma_0(t_{j+1}) = \beta(0)$.
So by uniqueness of lifts, we deduce that $\alpha = \beta$. ✓
Induction:
Suppose we've chosen $q_{ij}$ consistently for all $i<k$.
Now need to choose $q_{kj}$. Using the $q_{ij}$ we already have, we have a lift $\tilde{H}$ defined on $[0, s_k] \times [0,1]$.
Let $\gamma_{s_k} = \tilde{H}(s_k, -)$.
For each $j$, choose $q_{kj}$ such that
$q_{kj} \circ H(s_k, -) = \gamma_{s_k}(-)$ on $[t_j, t_{j+1}]$.
Proof of consistency is the same as for the base case. ✓
Now done by induction.
□

[Can choose $q_{kj}$ s.t. $q_{kj} \circ H(s_k, t_j) = \gamma_{s_k}(t_j)$.
Then by uniqueness of lifts, this is true for all $t \in [t_j, t_{j+1}]$]

The proof of Thm is completed by this □

<u>Theorem</u>
$\pi_1(S^1, 1) \cong \mathbb{Z}$ via $[\delta_m] \longleftrightarrow m$ where
$\delta_m : [0,1] \to S^1$ s.t. $t \mapsto e^{2\pi i m t}$

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Proof

Define $\varphi : \mathbb{Z} \longrightarrow \pi_1(S^1, 1)$ by $m \longmapsto [\delta_m]$.

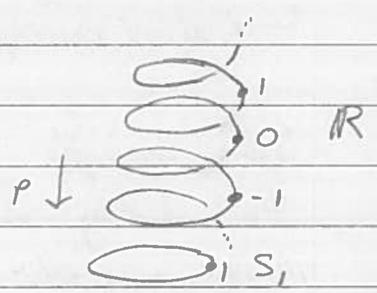
This is a homomorphism since $\delta_m \cdot \delta_n \simeq \delta_{m+n}$

$\varphi$ injective:

Have monodromy $\pi_1(S^1, 1) \longrightarrow \mathrm{Perm}(\underbrace{p^{-1}(1)}_{= \mathbb{Z}})$ where

$p : \mathbb{R} \rightarrow S^1$, $t \mapsto e^{2\pi i t}$

$[\alpha] \longmapsto \sigma_\alpha$.




We have that $\sigma_{\delta_m}(0) = m \ \forall m$

In particular, if $m \neq 0$ then $\sigma_{\delta_m}$ is not the trivial permutation, so $[\delta_m]$ must be non-trivial in $\pi_1(S^1, 1)$ $\therefore$ $\varphi$ is injective. ✓

$\varphi$ surjective:

Suppose $[\alpha] \in \pi_1(S^1, 1)$. Let $m = \sigma_\alpha(0)$

Let $\tilde{\alpha}$ be the (unique) lift of $\alpha$ to $\mathbb{R}$ which satisfies $\tilde{\alpha}(0) = 0$.

By definition of $\sigma_\alpha$, we have $\tilde{\alpha}(1) = m$.

So now have two paths $[0,1] \rightarrow \mathbb{R}$ which start at 0 and end at $m$: $\tilde{\alpha}$ and $t \mapsto mt$.

Since $\mathbb{R}$ is connected, these two paths are homotopic rel end points.

Projecting down to $S^1$ using $p$, we see that $[\alpha] = [\delta_m]$

So $[\alpha] \in \mathrm{Im}(\varphi)$. ✓

$\square$

Lemma

If $p : Y \rightarrow X$ is path-connected, then the number of points in $p^{-1}(x)$ is equal to the number of cosets of $p_* \pi_1(Y, y)$ in $\pi_1(X, x)$ for $y \in p^{-1}(x)$.

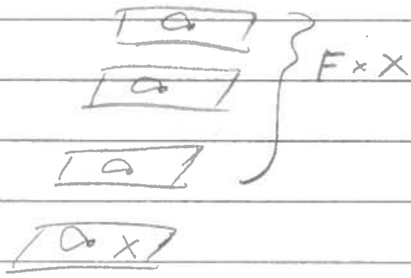
[i.e. there is a natural bijection between the two sets.]

Proof: Exercise using orbit-stabiliser.

Examples (of covering spaces)
We'll work out lots of examples, focusing on
- monodromy
- $p_* \pi_1(Y,y) \subset \pi_1(X,x)$ e.g. when is this subgroup normal?

Later, see that (connected, based) covers of (nice) $X$ are naturally parameterised by subgroups of $\pi_1(X,x)$.
↳ Galois correspondence.

Silly example
The identity map $id_X : X \to X$ is a covering map.
More generally, for any discrete space $F$, the projection $F \times X \to X$ is a covering map.



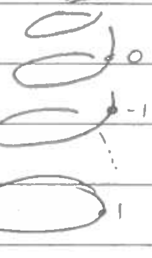

Monodromy: $\pi_1(X,x) \to Perm(F)$
$[\alpha] \mapsto id$

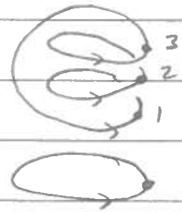
$p_* \pi_1(Y = F \times X, x) \subset \pi_1(X,x)$ is the whole of $\pi_1(X,x)$.

Circle (Covers of $S^1$):
Have seen $p : \mathbb{R} \to S^1$, $t \mapsto e^{2\pi i t}$
$p_n : S^1 \to S^1$, $z \mapsto z^n$

Monodromy: $\pi_1(S^1, 1) \cong \mathbb{Z} \longrightarrow Perm(p^{-1}(1))$ or $Perm(p_n^{-1}(1))$
$p^{-1}(1) = \mathbb{Z}$, $p_n^{-1}(1) = \{1, 2, \dots, n\}$

$\left.\begin{array}{l} 0 \mapsto 1 \\ 1 \mapsto 2 \end{array}\right\} n \to n+1$ $\longleftrightarrow$




$1 \mapsto 2 \mapsto 3 \mapsto \dots \mapsto n \mapsto 1$
i.e. cycle $(1\ 2\ \dots\ n)$

Image of $\pi_1(Y)$ in $\pi_1(X)$:
$\underbrace{p_* \pi_1(\mathbb{R}, 0)}_{=0} \subset \pi_1(S^1, 1) \cong \mathbb{Z}$ | $\underset{\cong \mathbb{Z}}{p_{n*} \pi_1(S^1, 1)} \subset \underset{\cong \mathbb{Z}}{\pi_1(S^1, 1)}$

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$$[\delta_m] \in \pi_1(S^1, 1) \xmapsto{p_{n*}} [\delta_{mn}]$$

So $p_{n*}\pi_1(S^1,1) = n\mathbb{Z} < \mathbb{Z}$

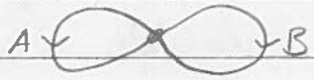
Notice: $\forall$ subgroups $H < \mathbb{Z}$ we've found a cover $p: Y \to S^1$ with $p_*\pi_1(Y) = H$.

## Covers of figure of 8

$(X, x) = \infty = S^1 \vee S^1$

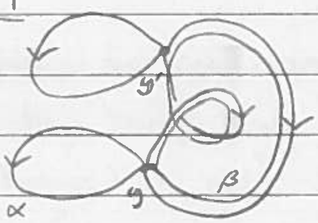
$$\begin{tikzcd} \{1\} \arrow[r] \arrow[d] & \mathbb{Z} = \langle a \rangle \\ \mathbb{Z} = \langle b \rangle & \end{tikzcd}$$

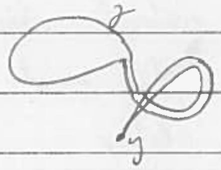
$\pi_1(X, x) = \mathbb{Z} * \mathbb{Z}$




$= \langle A, B \rangle$

## Example 1

$Y =$ 

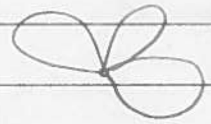

Monodromy: $\langle A, B \rangle \longrightarrow \mathrm{Perm}(\{y, y'\}) = S_2$

$A \longmapsto (y)(y') = id$

$B \longmapsto (y\,y') = (1\,2)$

$p_*\pi_1(Y, y)$:

$Y =$ 

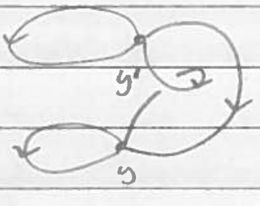

$\simeq$ 

So $\pi_1(Y, y) \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$

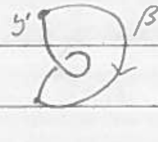
Under $p_*$: $\alpha \longmapsto A$, $\beta \longmapsto B^2$, $\gamma \longmapsto B^{-1}AB$

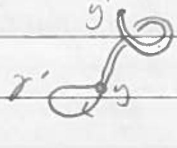
$\Rightarrow p_*\pi_1(Y, y) = \langle A, B^2, B^{-1}AB \rangle < \langle A, B \rangle$

The cover is 2:1 so this subgroup is index 2 hence normal.




Now try $y'$ as base point.

Under $p_*$: $\alpha' \mapsto A$, $\beta' \mapsto B^2$, $\gamma' \mapsto BAB^{-1}$

$p_* \pi_1(Y, y') = \langle A, B^2, BAB^{-1} \rangle < \langle A, B \rangle$.

Note: $BAB^{-1} = B^2 \cdot (B^{-1}AB) \cdot (B^2)^{-1}$

So $p_*\pi_1(Y, y) = p_*\pi_1(Y, y')$

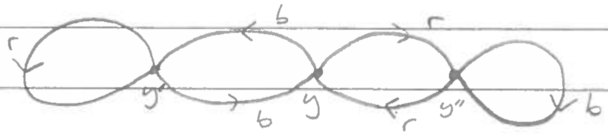
Changing the basepoint in $Y \rightsquigarrow$ conjugate $p_*\pi_1(Y)$.
In this example the subgroup was normal so didn't change under conjugation.

## Example 2

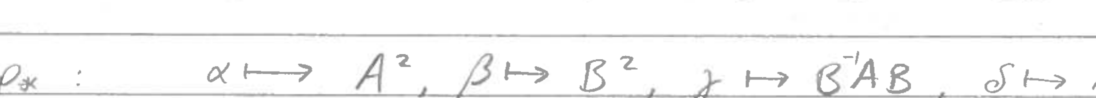
Any 4-valent directed graph whose edges are coloured red and blue, such that at each vertex it looks like 

(two in, two out of each colour)
gives a cover of $X =$ 


Each vertex of the graph maps to $x$.
Colours and directions of edges determine where they get mapped.

eg. $Y$ 


Monodromy: $A \longmapsto (y')(y\ y'')$
$B \longmapsto (y'')(y\ y')$




$p_*$: $\alpha \mapsto A^2$, $\beta \mapsto B^2$, $\gamma \mapsto B^{-1}AB$, $\delta \mapsto A^{-1}BA$

The subgroup $p_*\pi_1(Y, y)$ is not normal. It doesn't contain $A$ but it contains $B^{-1}AB$. This failure of normality reflects the fact $p_*\pi_1(Y, y')$ is a different subgroup.
The covering space has no symmetries.

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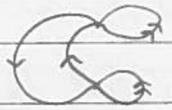
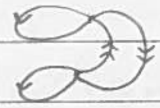
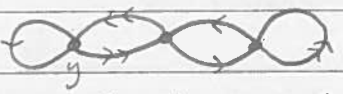
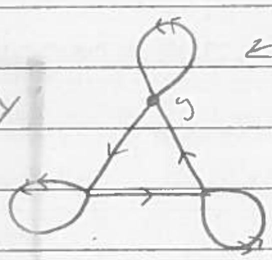
1).  2).  3). 

4).  All of the double covers of $\infty$ (2-fold symmetry)



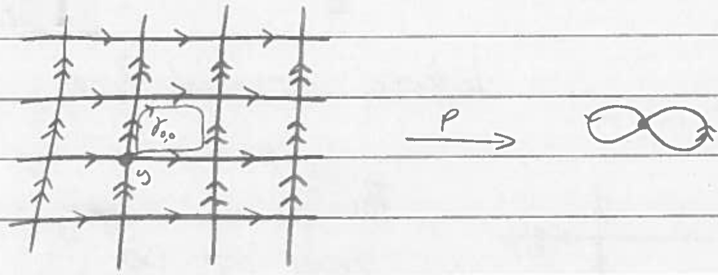

$Y$ $\leftarrow$ triple cover (no symmetries)

$p_* \pi_1(Y, y) = \langle A, B^2, B^{-1}A^2B, B^{-1}A^{-1}BAB \rangle$

$Y'$ 


$\leftarrow$ triple cover (3 symmetries)

$p_* \pi_1(Y', y) = \langle B, A^3, A^{-1}BA, ABA^2 \rangle$

$Y$ 


$\uparrow$ infinite cover

$p_* \pi_1(Y, y) = ?$

$p_*[\gamma_{0,0}] = A^{-1}B^{-1}AB \in p_*\pi_1(Y, y)$

Given a word $w$ in $A$ & $B$, I get a path $\delta_w$ by "following the instructions in the word"
(i.e. a lift of the word to a path in $Y$)
So pick the word $B^q A^p$ to get a path $\delta_{B^qA^p}$ from $(0,0)$ to $(p,q)$.
Then $\delta^{-1}_{B^qA^p} \circ \gamma_{p,q} \circ \delta_{B^qA^p}$ is a loop & these loops

generate $\pi_1(Y,y)$ (by Van Kampen) and

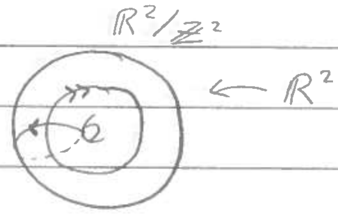
$$p_* \, \delta^{-1}_{B^qA^p} \circ \gamma_{p,q} \circ \delta_{B^qA^p} = A^{-p}B^{-q}A^{-1}B^{-1}AB\,B^qA^p.$$

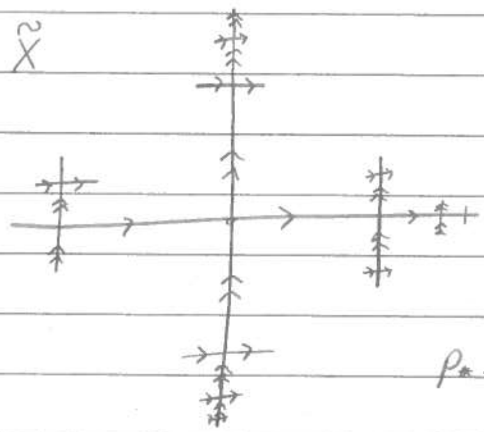
So $p_*\pi_1(Y,y)$ is generated by conjugates $w^{-1}A^{-1}B^{-1}ABw$ where $w$ runs over words in $A, B$.
i.e. $p_*\pi_1(Y,y)$ is the <u>normal</u> subgroup generated by $A^{-1}B^{-1}AB$.

The group $\mathbb{Z}^2$ acts by symmetries on $Y$

$$\pi_1(\overset{\infty}{\overset{\parallel}{X}}) \big/ p_*\pi_1(Y,y) = \langle A, B \mid A^{-1}B^{-1}AB = 1 \rangle = \mathbb{Z}^2$$

$(\pi_1(X) = \mathbb{Z} * \mathbb{Z})$




Infinite 4-valent tree

This is a simply connected cover of $\infty$

$p_*\pi_1(\tilde{X}, x) = \{1\}$.

Which subgroups of $\pi_1(X)$ arise as $p_*\pi_1(Y)$ for a covering space $p: Y \to X$?
A: All of them! (To be proved soon)

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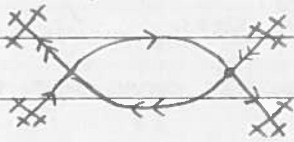
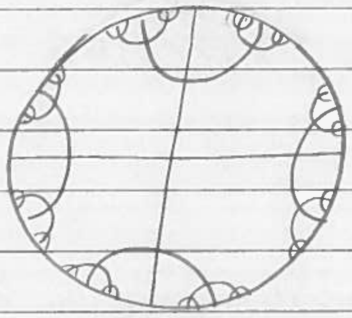
e.g. $p_* \pi_1(Y) = \langle AB \rangle$
need one generator



this is the covering space that works.



$F(z) = \dfrac{2z+1}{z+2}$

$G(z) = \dfrac{2z+i}{-iz+2}$

$\in \mathrm{Isom}(\mathbb{D}^2)$

Apply $F^{a_1}G^{b_1}F^{a_2}G^{b_2}$ to these axes & take the union to get $\tilde{X}$.

So $\tilde{X}$ has $\mathbb{Z} * \mathbb{Z}$ of symmetries.

Group Actions

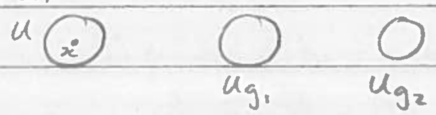
Def

A group action of $G$ on $X$ by homeomorphisms means for each $g \in G$, a homeomorphism $\varphi_g : X \to X$ s.t. $\varphi_1 = \mathrm{Id}$ and $\varphi_{gh} = \varphi_g \circ \varphi_h$.

We will write $\varphi_g(x) = xg$ (acting on the right)

Def

We say $G$ acts properly discontinuously on $X$ if $\forall x \in X$ $\exists$ open neighbourhood $U$ of $x$ s.t.
$\underbrace{Ug}_{g \text{ applied to } U} \cap U = \emptyset \quad \forall g \in G$



__Theorem__

If $G$ acts properly discontinuously on $X$ (where $X$ is a connected, locally path-connected space) then the quotient map $p: X \to X/G$ is a covering map.

__Prop__

If moreover $\pi_1(X) = \{1\}$ then $\pi_1(X/G) \cong G$.

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__Def__

A group $G$ acts properly discontinuously on $X$ if $\forall\, x \in X$ $\exists\, U$ open s.t. $x \in U$ and $U \cap Ug \neq \emptyset$ $\forall\, g \in G$ not equal to $1$.

__Theorem__

If $X$ is connected and locally path-connected, and $G$ acts properly discontinuously on $X$ then the quotient map $p: X \to X/G$ is a covering map.

Recall: __Lemma__ (*)

If a group $G$ acts on a space $X$ by homeomorphisms then the quotient map $p: X \to X/G$ is open, i.e. if $U \subseteq X$ is open then $p(U)$ is open

$$\begin{tikzcd} X \arrow[d, "p"] \\ X/G \end{tikzcd}$$

__Proof__ (of Thm)

Need to find an open cover of $X/G$ by sets $U$ & local inverses $q: U \to X$ for $p$ i.e. $p \circ q = id_U$.

Pick $[x] \in X/G$.

We need to find $U \subseteq X/G$ s.t. $[x] \in U$ & $\forall\, g \in G$ a map $q: U \to X$ s.t. $p \circ q = id$ & $q([x]) = xg$.

Since $G$ acts properly discontinuously we get an open

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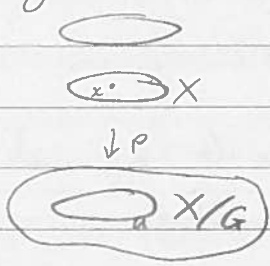
neighbourhood $V$ in $X$ of $x$ s.t. $V \ni x$ & $Vg \cap V = \emptyset$ $\forall g \neq 1$.




Since $X$ is "locally path-connected" we can assume $V$ is path-connected. We take $U = p(V)$. This is open by the lemma $(*)$

Let $q = p|_V^{-1}$.

The map $p|_V : V \to U$ is invertible (bijective)

- $U = p(V)$, so $p|_V$ is surjective
- $p|_V$ is injective because if $v_1, v_2 \in V$ s.t. $p(v_1) = p(v_2)$ then $v_1 = v_2 g$ for some $g \in G$. But $Vg \cap V = \emptyset$ unless $g = 1 \Rightarrow v_1 = v_2$

$p|_V^{-1}$ is continuous. $(p|_V^{-1})^{-1}(\underset{\uparrow \text{open set}}{A}) = p(A)$ is open by $(*)$

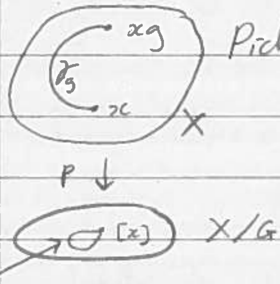
$\square$

<u>Prop</u>

If $G$ acts properly discontinuously on $X$ & $\pi_1(X) = \{1\}$ then $\pi_1(X/G, [x]) \cong G$. ($X$ path-connected)

<u>Proof</u>

Pick a basepoint $x \in X$. We will construct a map $F : G \longrightarrow \pi_1(X/G, [x])$ in the following way.



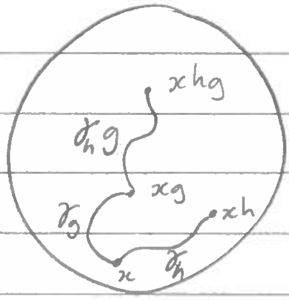

$p \circ \gamma_g$

Pick a path $\gamma_g$ from $x$ to $xg$.
Project along $p$ to get a loop $p \circ \gamma_g$ based at $[x]$.
Define $F(g) = [p \circ \gamma_g]$.

$F$ is well defined: we chose $\gamma_g$ but if we vary it inside its homotopy class then $[p \circ \gamma_g]$ is unchanged & there is only one homotopy class because $\pi_1(X) = \{1\}$, so $F(g)$ doesn't depend on the choice of $\gamma_g$.

$F$ is a homomorphism:

WTS: $F(hg) = F(h)F(g)$.




$F(hg) = \rho(\gamma_{hg})$
$F(h)F(g) = \rho(\gamma_h)\rho(\gamma_g)$

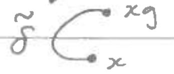
Pick $\gamma_g, \gamma_h$ from $x$ to $xg, xh$ resp.
$\underbrace{(\gamma_h g)\cdot\gamma_g}_{\gamma_{hg}}$ connects $x$ to $xhg$

$\gamma_h g \cdot \gamma_g$ is a choice of path from $x$ to $xhg$,
so $\underbrace{F(hg)}_{\rho(\gamma_{hg})} = (\rho(\gamma_h g))\cdot(\rho(\gamma_g))$

Claim: $\rho(\gamma_h g) = \rho(\gamma_h)$
This is because $\underbrace{\rho(yg)}_{[yg]} = \underbrace{\rho(y)}_{[y]}$ $\forall y \in X$

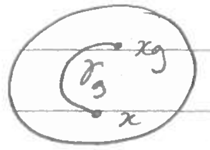
$\Rightarrow F$ is a homomorphism.

$F$ is surjective, i.e. $\forall\, \delta \in \pi_1(X/G, [x])$ $\exists\, g \in G$ s.t.
$F(g) = \delta$, i.e. s.t. $\rho(\gamma_g) = \delta$ for some path $\gamma_g$.
This follows from the path lifting lemma:
take $\gamma_g$ = the lift of $\delta$ starting at $x$.
i.e. if $\tilde{\delta}$ is the lift of $\delta$ starting at $x$




then $\tilde{\delta}(1) = xg$ for some $g \in G$,
so $F(g) = \delta$.


$F$ injective:
As $F$ is a homomorphism, we just need to check
$\mathrm{Ker}(F) = \{1\}$.
If $g \in \mathrm{Ker}\, F$ then $1 = F(g) = \rho(\gamma_g)$




A nullhomotopy of $\rho(\gamma_g)$ lifts to give
a homotopy rel endpoints from


$\gamma_g$ to a lift of the constant path at $[x]$.
A lift of a constant path must be constant, so $\gamma_g$

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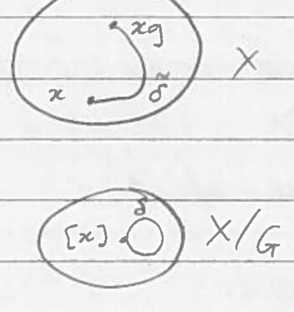
is homotopic rel end points to the constant path at $x$
$\Rightarrow \gamma_g(1) = x$ i.e. $g=1$.
$\square$




Recap of surjectivity:
Pick $\delta$ in $\pi_1(X/G)$,
lift to get a path $\tilde{\delta}$ starting at $x$, it ends at some $xg$.
Take $\tilde{\delta}$ to be $\gamma_g$, we see
$F(g) = \rho(\gamma_g) = \delta$.

<u>Lemma</u>
Suppose that $G$ acts on a metric space $(X, d)$ by isometries and that $\exists\, c > 0$ s.t. $d(x, xg) \geq c$ $\forall x \in X$ & $\forall g \neq 1$, then the action is properly discontinuously.

<u>Proof</u>
Take $U$ to be a metric ball of radius $r < \frac{c}{2}$ around $x$, then $U \cap Ug = \emptyset$ $\forall g \neq 1$.
$\square$

<u>Example</u>
$X = \mathbb{R}$, $G = \mathbb{Z}$
"$xg$" $= x+g$
Properly discontinuous, take $c=1$ in the lemma.
$d(x, x+g) \geq |g| \geq 1$ ✓
$\Rightarrow \pi_1(S^1) = \pi_1(\mathbb{R}/\mathbb{Z}) = \mathbb{Z}$

<u>Example</u>
$X = \mathbb{R}^2$, $G = \mathbb{Z}^2$
$(x,y) \longmapsto (x+m, y+n)$ $\quad (m,n) \in \mathbb{Z}^2$
$d((x,y), (x+m, y+n)) = \sqrt{m^2+n^2} \geq \max(|m|, |n|)$
$\geq 1$ if $(m,n) \neq (0,0)$

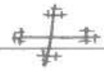
so the lemma applies

$\Rightarrow \pi_1(T^2) = \pi_1(\mathbb{R}^2/\mathbb{Z}^2) \cong \mathbb{Z}^2$

Remark

Given $X$ and a simply connected cover $\tilde{X} \to X$, we call $\tilde{X}$ the "universal cover" of $X$ (because $\tilde{X}$ covers every other covering space of $X$).

We've just seen $\tilde{S}^1 = \mathbb{R}$, $\tilde{T}^2\ \mathbb{R}^2$

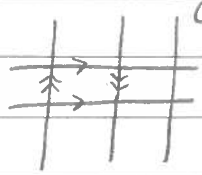
Last time we saw $\tilde{\infty} =$ 

Example

Let $G$ be the following group of isometries of $\mathbb{R}^2$:

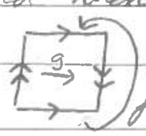
$f : \mathbb{R}^2 \to \mathbb{R}^2$, $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x+1 \\ 1-y \end{pmatrix}$ $\qquad g: \mathbb{R}^2 \to \mathbb{R}^2$, $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y+1 \end{pmatrix}$

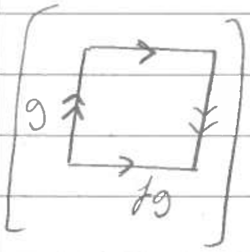
$G$ is the subgroup of $\mathrm{Isom}\,\mathbb{R}^2$ generated by $f$ and $g$



We see that $\mathbb{R}^2/G$ will be the Klein bottle.

You can translate any $(x,y)$ using $f$ to get $x \in [0,1]$ and then using $g$ to get $y \in [0,1]$, so

$\mathbb{R}^2/G =$  with some identifications $\Rightarrow$ Klein bottle.



Claim:

$fg = g^{-1}f$

$\left(\begin{pmatrix} x \\ y \end{pmatrix} f\right) g = \begin{pmatrix} x+1 \\ 1-y \end{pmatrix} g = \begin{pmatrix} x+1 \\ 2-y \end{pmatrix}$, $\left(\begin{pmatrix} x \\ y \end{pmatrix} g^{-1}\right) f = \begin{pmatrix} x \\ y-1 \end{pmatrix} f = \begin{pmatrix} x+1 \\ 2-y \end{pmatrix}$

$\Rightarrow fg = g^{-1}f$

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So our group has 2 generators $f$ & $g$ and at least 1 relation $fg = g^{-1}f$.
In fact, we need no further relations.
To see that, note that we can use $fg = g^{-1}f$ (equiv. $gf = fg^{-1}$) to rewrite any word of $f$'s and $g$'s as $f^n g^m$.

Exercise:

$$(x,y) f^n g^m = \left( x+n, \begin{cases} y+m & n \text{ even} \\ m+1-y & n \text{ odd} \end{cases} \right)$$

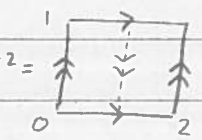
If $(n,m) \neq (0,0)$, then $f^n g^m \neq id$, so no more relations appear.

$\Rightarrow \pi_1(\text{Klein bottle}) = \langle f, g \mid fg = g^{-1}f \rangle$
(Provided we can check proper discontinuity)

Proof (of proper discontinuity)

$$d\left((x,y)f^n g^m, (x,y)\right) = \begin{cases} n^2+m^2 & n \text{ even} \\ n^2 + (1+m-2y)^2 & n \text{ odd} \end{cases}$$
$$\geq 1 \qquad \square$$

In fact, the torus itself covers the Klein bottle.

$T^2 =$ 

$\mathbb{Z}/2$ - action on $T^2$ via $f$
quotient is Klein bottle

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "\text{universal cover}"'] \arrow[rr, bend right, "\text{universal cover}"'] & T^2 \arrow[r, "2\text{-to-}1"] & K \end{tikzcd}$$

Example

$\mathbb{RP}^2$ has a double cover by $S^2$

Proof

Consider $\mathbb{Z}/2$ acting on $S^2$ sending $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in S^2$ to $\begin{pmatrix} -x \\ -y \\ -z \end{pmatrix}$

[Could also choose distance to be along great circle]

This action is properly discontinuously.

$d\left(\begin{pmatrix}x\\y\\z\end{pmatrix}, \begin{pmatrix}-x\\-y\\-z\end{pmatrix}\right) = \text{diameter of } S^2 = 2$ for unit sphere
$\geq 2$

so we can apply the lemma

The quotient map $p: S^2 \to S^2/(\mathbb{Z}/2)$ is a covering map. $S^2/(\mathbb{Z}/2) = \mathbb{RP}^2$ $\square$

$\Rightarrow \pi_1(\mathbb{RP}^2) = \mathbb{Z}/2$

Theorem (Borsuk-Ulam Thm)
If $f: S^2 \to \mathbb{R}^2$ is a cts map then $\exists x \in S^2$ st. $f(x) = f(-x)$.

i.e. there are 2 antipodal points on Earth st. they have the same temp & barrometric pressure.

Proof
Assume this is not the case, i.e. that we have a map $f: S^2 \to \mathbb{R}^2$ st. $f(x) \neq f(-x)$ $\forall x \in S^2$.
Define $g(x) = f(x) - f(-x) \in \mathbb{R}^2 \setminus \{0\}$.
$g: S^2 \to \mathbb{R}^2 \setminus \{(0,0)\}$.

The map $g$ satisfies $g(-x) = -g(x)$ so it descends to a map $\bar{g}: \mathbb{RP}^2 \to (\mathbb{R}^2 \setminus \{(0,0)\})/\mathbb{Z}/2$
$[x] \longmapsto [g(x)]$
acting by mult. by $-1$.

So we get a diagram of maps and spaces:

$$\begin{array}{cccccc}
\pi_1 = \{1\} & S^2 & \xrightarrow{g} & \mathbb{R}^2 \setminus \{(0,0)\} \simeq S^1 & \pi_1 = \mathbb{Z} \\
& p \downarrow & & \downarrow p' & \\
\pi_1 = \mathbb{Z}/2 & \mathbb{RP}^2 & \xrightarrow[\bar{g}]{} & (\mathbb{R}^2 \setminus \{(0,0)\})/\mathbb{Z}/2 \simeq S^1 & \pi_1 = \mathbb{Z}
\end{array}$$

($\bar{g}$ is continuous by definition of the quotient topology)

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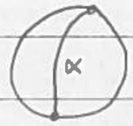
$$\Rightarrow \begin{array}{ccc} 1 & \xrightarrow{g_*} & \mathbb{Z} \\ {\scriptstyle p_*}\downarrow & & \downarrow{\scriptstyle p'_*} \\ \mathbb{Z}/2 & \xrightarrow[\bar{g}_*]{} & \mathbb{Z} \end{array} \qquad \begin{array}{c} 1 \\ \downarrow \\ 2 \end{array}$$

loop going once around $S^1$ goes to one that goes twice around

$\bar{g}_*(1) = 0$ as $\bar{g}_*(2 \times 1) = 2 \times \bar{g}_*(1) = 0$

Let $\alpha$ be a path on $S^2$ from the North to South pole.




This projects to the loop $1 \in \pi_1(\mathbb{RP}^2)$

The monodromy $\sigma_1$ switches North & South poles (non trivial!)

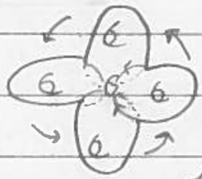
Therefore $\bar{g}_* p_* \alpha$ has non trivial monodromy.

But $\bar{g}_*$ = trivial, so $\bar{g}_* p_*(\alpha) = 0 \in \mathbb{Z}$

so has trivial monodromy. $\#$ $\square$

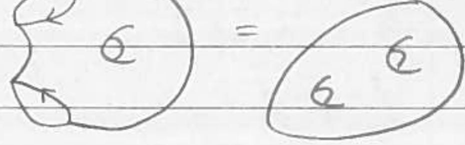
$\Rightarrow$ $g$ must hit $(0,0)$.

<u>Example</u> Covers of surfaces by surfaces.



genus 5 surface with $\mathbb{Z}/4$ action

The quotient is



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## Lifting criterion

$$\begin{tikzcd} & (Y,y) \arrow[d, "p"] \\ (T,t) \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"] & (X,x) \end{tikzcd}$$

covering space ($f(t) = x$)

"Test space"

path-connected
locally path connected

When is there a map $\tilde{f}: T \to Y$ s.t. $p \circ \tilde{f} = f$ ("lift of $f$") & s.t. $\tilde{f}(t) = y$ $(p(y) = x)$?

**Theorem**
There exists a lift $\tilde{f}$ with $\tilde{f}(t) = y$ iff $f_*(\pi_1(T,t)) \subseteq p_*(\pi_1(Y,y))$.

e.g. path-lifting / homotopy-lifting follow because in those cases $\pi_1(T,t) = \{1\}$, so the criterion is automatically satisfied.

**Proof**
[$\Rightarrow$ / only if]

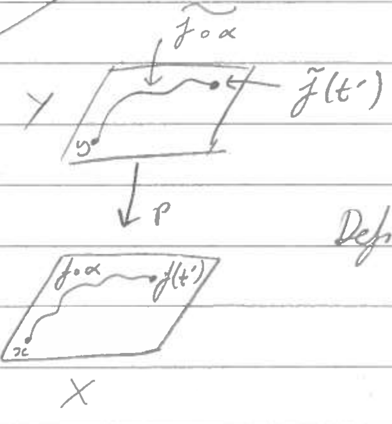
$f = p \circ \tilde{f}$

$f_* = p_* \circ \tilde{f}_*$

$\tilde{f}_*(\pi_1(T,t)) \subseteq \pi_1(Y,y)$

$\underbrace{p_* \tilde{f}_*}_{= f_*} \pi_1(T,t) \subseteq p_* \pi_1(Y,y)$

[$\Leftarrow$ / if]
Construction of $\tilde{f}$.
Use path-lifting!


Define $\tilde{f}(t') = \widetilde{f \circ \alpha}(1)$

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Path-lifting gives a lift of $f \circ \alpha$ for any path $\alpha$ in $T$

$\tilde{f}$ is well-defined: uses algebraic criterion

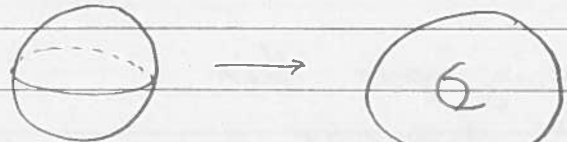
$f_* \pi_1(T,t) \subseteq p_* \pi_1(Y,y)$

$\tilde{f}$ is continuous: uses local path-connectedness of $T$

Watch video to see why!

$\square$

<u>Corollary</u>



Any continuous map $f : S^2 \to T^2$ is homotopic to the constant map. (i.e. $\pi_2(T^2) = \text{trivial}$)

homotopy classes of maps $S^2 \to T^2$

<u>Proof</u>

$$\begin{tikzcd} & \mathbb{R}^2 \arrow[d, "p"] \\ S^2 \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & T^2 \end{tikzcd}$$

covering space

$\pi_1 S^2 = \{1\}$

$\Rightarrow f_* \pi_1(S^2) = \{1\} \subseteq p_* \pi_1(\mathbb{R}^2)$

$\Rightarrow \exists \, \tilde{f} : S^2 \to \mathbb{R}^2$

$\mathbb{R}^2$ is contractible so $\tilde{f}$ is nullhomotopic,

$\tilde{f}_t$ s.t. $\tilde{f}_0 = \tilde{f}$, $\tilde{f}_1 = \text{const}$

$\Rightarrow p \circ \tilde{f}_t = f_t$ is a nullhomotopy of $f$

i.e. $f_1 = \text{const}$, $f_0 = f$.

$\square$

## The Galois theory of covering spaces

$$\begin{tikzcd}[column sep=small] & Y_3 \arrow[dl] \arrow[d] \arrow[dr] & \\ Y_1 \arrow[dr] & X & Y_2 \arrow[dl] \\ & & \end{tikzcd} \xrightarrow{\pi_1} \begin{tikzcd}[column sep=small] & \pi_1(Y_3) \arrow[dl] \arrow[d] \arrow[dr] & \\ \pi_1(Y_1) \arrow[dr] & \pi_1(X) & \pi_1(Y_2) \arrow[dl] \\ & & \end{tikzcd}$$

Galois Correspondence

### Lemma

If $p: Y \to X$ is a covering space then $p_*: \pi_1(Y, y) \longrightarrow \pi_1(X, p(y))$ is injective.

[e.g  ]

### Proof

Suppose $\gamma \in \operatorname{Ker} p_*$.

Then $\delta = p \circ \gamma$ is nullhomotopic in $X$.

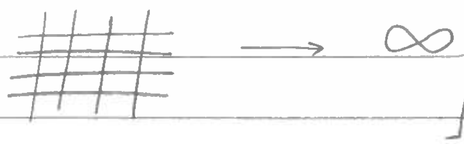
Let $\delta_t$ be a nullhomotopy of $\delta$ in $X$.

Homotopy lifting $\Rightarrow$ $\delta_t$ lifts to a nullhomotopy $\gamma_t$ of $\gamma$

so $\gamma = 1$

$\Rightarrow$ $\operatorname{Ker} p_* = \{1\}$

$\square$

[  ]

### Corollary

The free group $\mathbb{Z} * \mathbb{Z}$ contains the free group $\mathbb{Z} * \mathbb{Z} * \mathbb{Z}$, $\mathbb{Z}^{*\infty}$ all possible free groups.

### Proof

 is a covering space, $\pi_1(Y) = \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$

 is a covering space with $\pi_1 = \mathbb{Z}^{*\infty}$

Result follows from lemma.

$\square$

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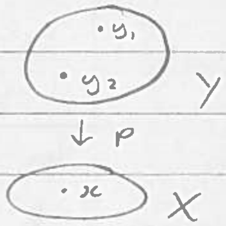
How does this depend on $y$?

$$\pi_1(Y, y) \xrightarrow{p_*} \pi_1(X, p(y))$$



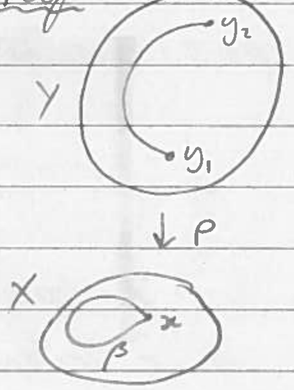

__Lemma__

Let $p: Y \to X$ be a covering space.
Let $x \in X$, $y_1, y_2 \in Y$ s.t. $p(y_1) = p(y_2) = x$
Then, if $\alpha$ is a path in $Y$ from $y_1$ to $y_2$ and $\beta = p \circ \alpha$ is its projection, we have

$$[\beta]\, p_* \pi_1(Y, y_1)\, [\beta]^{-1} = p_* \pi_1(Y, y_2)$$

$\uparrow$ back to $y_2$ $\qquad$ $\uparrow$ $\beta^{-1}$ from $y_2$ to $y_1$

__Proof__

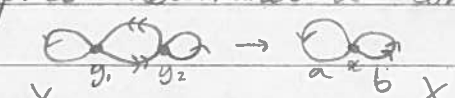



$\beta$ is a loop, as $\alpha$ sends $y_1$ to $y_2$ and $p(y_1) = p(y_2)$.
The usual basepoint changing map $F_{\alpha^{-1}}(\gamma) = \alpha \cdot \gamma \cdot \alpha^{-1}$ projects via $p$ to an isomorphism.

$$p_* \pi_1(Y, y_1) \longrightarrow p_* \pi_1(Y, y_2)$$
$$[p \circ \gamma] \longmapsto (p \circ \alpha)\cdot(p \circ \gamma)\cdot(p \circ \alpha)^{-1}$$
$$= [\beta]\cdot(p \circ \gamma)\cdot[\beta^{-1}].$$

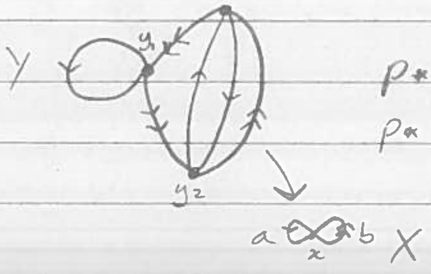
$\square$

A single covering space $p: Y \to X$ gives many subgroups $p_* \pi_1(Y, y) \leq \pi_1(X, x)$, one for each $y \in p^{-1}(x)$, but these subgroups are all conjugate, so a covering space determines a conjugacy class of subgroups in $\pi_1(X, x)$.




since it is a normal subgroup

$$p_* \pi_1(Y, y_1) = \langle a, bab, b^2 \rangle = p_* \pi_1(Y, y_2)$$




$$p_* \pi_1(Y, y_1) = \langle a, bab, ba^{-1}b, b^3 \rangle$$
$$p_* \pi_1(Y, y_2) = \langle ab, a^2, b^2a, bab^{-1} \rangle$$

Conjugate but not equal.

Conjugate by $b^{-1}$:

$$\left.\begin{aligned} a &\mapsto b^{-1}ab \\ bab &\mapsto b^{-1}babb = ab^2 \\ ba^{-1}b &\mapsto b^{-1}ba^{-1}bb = a^{-1}b^2 \\ b^3 &\mapsto b^3 \end{aligned}\right\} \text{Should generate } p_*\pi_1(Y,y_2)$$

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$$\begin{tikzcd} & (Y,y) \arrow[d, "p"] \\ (T,t) \arrow[r, "f"'] \arrow[ur, dashed, "\tilde{f}"] & (X,x) \end{tikzcd}$$

Lifting Criterion:

$\exists!$ lift with $\tilde{f}(t) = y$ iff $f_*\pi_1(X,x) \subseteq p_*\pi_1(Y,y)$

Changing basepoint: $\beta p_*\pi_1(Y,y)\beta^{-1} = p_*\pi_1(Y, \sigma_\beta(y))$.

<u>Covering transformations</u>

$$\begin{tikzcd} Y_1 \arrow[rr, "F"] \arrow[dr, "p_1"'] & & Y_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

<u>Def</u>
Given covering spaces $p_1 : Y_1 \to X$, $p_2 : Y_2 \to X$
a covering transformation is a continuous map $F : Y_1 \to Y_2$ s.t. $p_2 \circ F = p_1$

<u>Def</u>
A covering transformation $F$ is a <u>covering isomorphism</u> if $F$ is a homeomorphism, $p_2 = p_1 \circ F^{-1}$.
When this happens, $F^{-1}$ is also a covering transformation.

<u>Def</u>
Given $p: Y \to X$, the group of covering transformations $Y \to Y$ is called the deck group $\mathrm{Deck}(Y,p)$, its elements are called deck transformations of $Y$.

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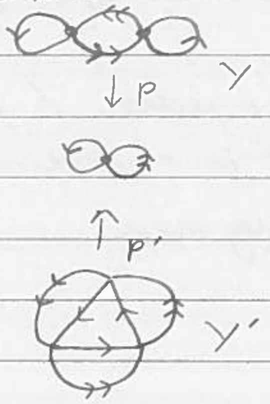
**Examples**



$\mathrm{Deck}(Y,p) = \{\mathrm{Id},\ 180^\circ \text{ rotation}\} \cong \mathbb{Z}/2$

$\mathrm{Deck}(Y',p') = \{\mathrm{Id},\ 120^\circ,\ 240^\circ\}$

**Lemma**

$$\begin{array}{ccc} y_1 \in Y_1 & & Y_2 \ni y_2 \\ & p_1 \searrow \ \swarrow p_2 & \\ & x \in X & \end{array} \qquad p_1(y_1) = p_2(y_2) = x$$

$\exists$ iff... $\qquad \exists!$ covering transformation $F: Y_1 \to Y_2$

$\exists \Rightarrow !$ $\qquad$ s.t. $F(y_1) = y_2$ iff $(p_1)_* \pi_1(Y_1, y_1) \subseteq (p_2)_* \pi_1(Y_2, y_2)$

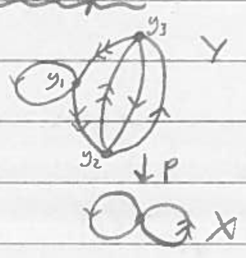
**Proof**

Apply the lifting criterion with $T = Y_1$, $t = y_1$, $Y = Y_2$, $y = y_2$, $f = p_1$, $p = p_2$, $\tilde{f} = F$. $\square$

Consequence: If $Y_1 = Y_2 = Y$ then $\exists!$ deck transformation $F: Y \to Y$ with $F(y) = y'$ for $y, y' \in p^{-1}(x)$ iff $p_* \pi_1(Y,y) = p_* \pi_1(Y,y')$.

$\left\{ \begin{array}{l} \subseteq \Rightarrow \text{covering transformation exists} \\ \supseteq \Rightarrow \text{so does its inverse.} \end{array} \right.$

**Example**



$\mathrm{Deck}(Y,p) = \{\mathrm{Id}\}$

$y_1$ is distinguished from $y_2$ & $y_3$ as it's the only vertex with a loop attached so any deck transformation $F$ sends $y_1$ to $y_1$ i.e. $F(y_1) = y_1$. The identity also sends $y_1$ to $y_1$ so by uniqueness in Lemma $F = id$.

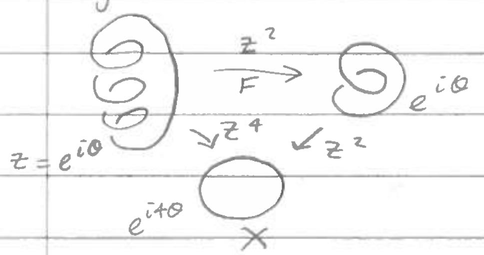
**Lemma** (covering transformations are covering maps)

$$\begin{tikzcd} Y_1 \arrow[rr, "F"] \arrow[dr, "p_1"'] & & Y_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

$\leftarrow$ Covering spaces

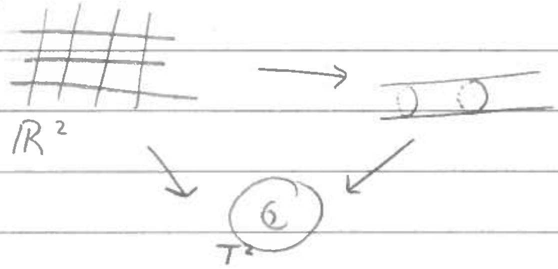
Suppose $Y_2$ is path-connected, $X$ is locally path connected. Suppose $F$ is a covering transformation, then $F$ is a covering map.

e.g.




$F$ is a covering transformation as $(z^2)^2 = z^4$

e.g.

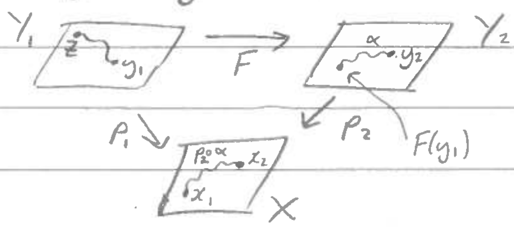



**Proof**

1). $F$ is surjective

2). Local inverses exist

1). Surjectivity.




Pick $y_2 \in Y_2$, set $x_2 = p_2(y_2)$

Pick $y_1 \in Y_1$, set $x_1 = p_1(y_1)$

Pick a path $\alpha$ in $Y_2$ from $F(y_1)$ to $y_2$.

Project to get $p_2 \circ \alpha$ in $X$ from $x_1$ to $x_2$.

Lift to get $\widetilde{p_2 \circ \alpha}$ in $Y_1$ from $y_1$ to $z = \widetilde{p_2 \circ \alpha}(1)$.

Note that $F(z) = y_2$.

Proof $F$ is a covering transformation so $p_2 \circ F = p_1$

$F \circ (\widetilde{p_2 \circ \alpha})$ is therefore a lift of $p_2 \circ \alpha$

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$$p_2 \circ F \circ \widetilde{p_2 \circ \alpha} = p_1 \circ \widetilde{p_2 \circ \alpha} = p_2 \circ \alpha$$

but $\alpha$ is already a lift of $p_2 \circ \alpha$

so by uniqueness of path-lifting

$\Rightarrow F \circ \widetilde{p_2 \circ \alpha} = \alpha$

$\Rightarrow F \circ \widetilde{p_2 \circ \alpha}(1) = y_2$

$\Rightarrow F(z) = y_2$

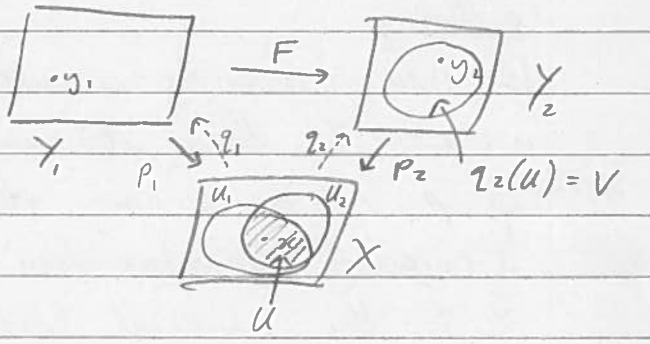
$\Rightarrow F$ is surjective $\square$

2). Find local inverses for $F$.

$$\begin{tikzcd} Y_1 \ni y_1 \arrow[r, "F"] \arrow[dr, "p_1"'] & Y_2 \ni y_2 \arrow[d, "p_2"] \\ & X \end{tikzcd}$$

Pick $y_2 \in Y_2$ & $y_1 \in Y_1$ s.t. $F(y_1) = y_2$

Need to find open neighbourhood $V$ of $y_2$ and local inverse $q : V \to Y_1$ s.t. $F \circ q = Id_V$ & $q(y_2) = y_1$.




Let $U_1 \subseteq X$ be an elementary nbhd of $x$ for $p_1$ & $q_1 : U_1 \to Y_1$ be a local inverse for $p_1$ with $q_1(x) = y_1$.

Let $U_2$, $q_2$ be similar for $p_2$.

Let $U = U_1 \cap U_2$. Since $X$ is locally path-connected I can assume $U$ is path-connected.

Set $V = q_2(U)$ and $q = q_1 \circ p_2 : V \to U \to Y_1$

Claim: $q \circ F = Id$

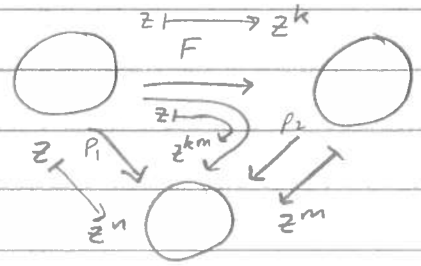
Proof

$$q \circ F = q_1 \circ p_2 \circ F \overset{\text{since diagram commutes}}{=} q_1 \circ p_1 = id$$

$\square$

$\square$

Note: need covering maps to be surjective.




If $\exists F$ does it imply $n \geq m$?

$\Rightarrow n = km \qquad \Rightarrow$ Yes!

$F_* : \mathbb{Z} \to \mathbb{Z}$ is an inclusion, $i \mapsto ki$ for some $k$

$(p_2)_*(i) = mi$ , $(p_1)_*(i) = ni$

$\Rightarrow p_2 \circ F = p_1$

$\Rightarrow \underbrace{((p_2)_* \circ F_*)(i)}_{= kmi} = (p_1)_*(i) = ni$

$\Rightarrow km = n$ for some $k \in \mathbb{Z}$, $k \neq 0$

$\Rightarrow |n| \geq |m|$

<u>Corollary</u>

If there is a simply-connected covering space $u : \tilde{X} \to X$ then it's unique up to isomorphism and if $p : Y \to X$ is any other covering space of $X$ then $\exists$ covering transformation $F : \tilde{X} \to Y$ s.t. $u = p \circ F$. $\tilde{X}$ is the universal cover.

<u>Proof</u>

Suppose $\tilde{X}_1$ and $\tilde{X}_2$ are both simply connected covers.

Then $(u_1)_* \pi_1(\tilde{X}_1) = \{1\} = (u_2)_* \pi_1(\tilde{X}_2)$

$\Rightarrow \exists F : \tilde{X}_1 \to \tilde{X}_2$ & $F^{-1}$ covering transformations

$\Rightarrow \tilde{X}_1 \cong \tilde{X}_2$

$$\begin{tikzcd} \tilde{X} \arrow[rr, "F"] \arrow[dr, "u"'] & & Y \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

$u_* \pi_1(\tilde{X}) = \{1\} \subseteq p_* \pi_1(Y)$

$\Rightarrow \exists$ covering transformation $F : \tilde{X} \to Y$

$\Rightarrow u = p \circ F$.

$\square$

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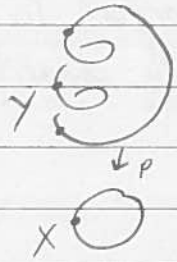
## Examples




$z \in S^1 \mapsto z^3$

The deck group is $\mathbb{Z}/3 = \frac{\mathbb{Z}}{3\mathbb{Z}} = \frac{\pi_1 X}{p_* \pi_1 Y}$

$id : z \longmapsto z$

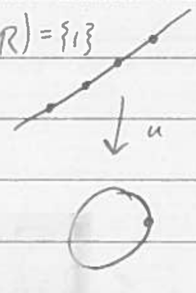
$\mu_1 : z \longmapsto z e^{i 2\pi/3}$

$\mu_2 : z \longmapsto z e^{i 4\pi/3}$

$\pi_1(Y) = \mathbb{Z} \to 3\mathbb{Z}$

For $z \longmapsto z^n$ we get $\mathbb{Z}/n$

$\pi_1(\mathbb{R}) = \{1\}$ $\quad \mathbb{R} = \tilde{S}^1$




$\mathrm{Deck}(\mathbb{R}, u) = \mathbb{Z}$

$u(\theta) = e^{i2\pi\theta}$

deck transformations are $\theta \mapsto \theta + n$, $n \in \mathbb{Z}$

$u_* \pi_1(\mathbb{R}) = \{1\}$ $\quad \mathrm{Deck}(\mathbb{R}, u) = \mathbb{Z} = \frac{\pi_1 X}{u_* \pi_1(\mathbb{R})}$

We would like to conjecture that

$$\mathrm{Deck}(Y, p) \cong \frac{\pi_1 X}{p_* \pi_1(Y)}$$

In general this doesn't even make sense (unless $p_* \pi_1(Y)$ is normal)

### Theorem

$\mathrm{Deck}(Y, p) \cong N_H / H$ where $H = p_* \pi_1(Y)$, $N_H$ is the biggest subgroup of $\pi_1 X$ in which $H$ is normal, the "normaliser of $H$"

### Normal covers

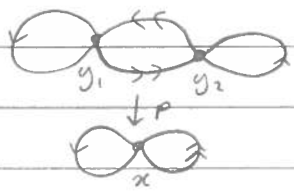
**Def**

A covering space is called normal (or Galois / regular) if $\mathrm{Deck}(Y, p)$ acts transitively on $p^{-1}(x)$ for some $x \in X$.

## Theorem

$$\begin{array}{c} Y \\ \downarrow p \\ X \end{array}$$ is normal iff $p_* \pi_1(Y, y)$ is normal in $\pi_1(X)$ for some $y \in Y$. ($Y$ is path connected)

## Example



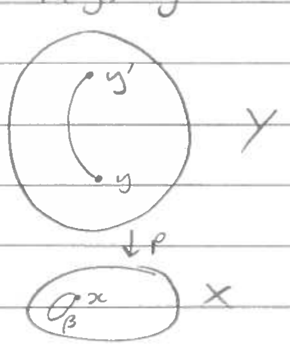

$\nwarrow$ is normal

## Proof

[$\Leftarrow$]

Suppose $p_* \pi_1(Y, y)$ is normal.

Given $y, y' \in p^{-1}(x)$, I need to find $F \in \mathrm{Deck}(Y, p)$ s.t. $F(y) = y'$.




Pick a path $\alpha$ in $Y$ from $Y$ to $Y'$

Project to get $\beta = p \circ \alpha$, a loop in $X$

Note that $y' = \sigma_\beta(y)$

$$\beta\, p_* \pi_1(Y, y) \beta^{-1} = p_* \pi_1(Y, \underbrace{\sigma_\beta(y)}_{\text{''} y'})$$

$$\overset{\shortparallel}{p_* \pi_1(Y, y)} \text{ by normality}$$

$\Rightarrow \exists$ covering transformation $F$ as required.

[$\Rightarrow$]

Assume $\mathrm{Deck}(Y, p)$ acts transitively.

WTP: $p_* \pi_1(Y, y)$ is normal

i.e. $\beta\, p_* \pi_1(Y, y) \beta^{-1} = p_* \pi_1(Y, y) \quad \forall \beta \in \pi_1 X$

So take $\beta \in \pi_1 X$. We know $\beta\, p_* \pi_1(Y, y) \beta^{-1} = p_* \pi_1(Y, \sigma_\beta(y))$

By assumption $\exists F \in \mathrm{Deck}(Y, p)$ s.t. $F(y) = \sigma_\beta(y)$.

$F \in \mathrm{Deck}(Y, p) \Rightarrow p = p \circ F$

$$\begin{tikzcd} Y \arrow[rr, "F"] \arrow[dr, "p"'] & & Y \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

$p_* \pi_1(Y, y) = p_* F_* \pi_1(Y, y)$

$F_* : \pi_1(Y, y) \xrightarrow{\cong} \pi_1(Y, \sigma_\beta(y)) = F_* \pi_1(Y, y)$

$\Rightarrow p_* \pi_1(Y, y) = p_* \pi_1(Y, \sigma_\beta(y)) = \beta\, p_* \pi_1(Y, y) \beta^{-1} \Rightarrow p_* \pi_1(Y, y)$ is normal.

$\square$

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**Theorem**

In general $Deck(Y,p) \cong N_H / H$

where $H = p_* \pi_1(Y,y)$

**Proof**

See notes.

**Corollary**

If $p_* \pi_1(Y,y)$ is normal then $Deck(Y,p) = \dfrac{\pi_1(X,x)}{p_* \pi_1(Y,y)}$

(normaliser of a normal subgroup is the whole group)

**Corollary**

$Deck(\tilde{X},u) \cong \pi_1(X)$

**Proof**

$u_* \pi_1(\tilde{X})$ is trivial so definitely normal

$\Rightarrow Deck(\tilde{X},u) = \dfrac{\pi_1(X)}{\{1\}} = \pi_1(X).$

$\square$

What we will prove is that $Deck(Y,p)$ acts properly discontinuously $\quad \tilde{X}/Deck(\tilde{X},u) = X$

Let $\Gamma \leq \pi_1(X)$ be a subgroup of $\pi_1(X)$

$\Rightarrow \Gamma \leq Deck(\tilde{X},u)$

$\Rightarrow \Gamma$ acts properly discontinuously on $\tilde{X}$ via Deck transformations

$Y = \tilde{X}/\Gamma$ is then a covering space of $X$, $\pi_1(Y) = \Gamma$

$\Rightarrow p: Y \to X \quad , \quad p_* \pi_1 Y = \Gamma$

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Video 1

## Galois correspondence for covering spaces

Recall:

Given a covering space $p: Y \to X$, $y \in p^{-1}(x)$, we get a subgroup $p_* \pi_1(Y, y) \subseteq \pi_1(X, x)$.

**Theorem**

Assume that $X$ is a path-connected, locally path-connected space and that there exists a simply-connected universal cover $u: \tilde{X} \to X$. Then for any subgroup $H \subseteq \pi_1(X, x)$ there exists a covering space $p: Y \to X$ and a point $y \in p^{-1}(x)$ such that $p_* \pi_1(Y, y) = H$

**Proof**

$\mathrm{Deck}(\tilde{X}, u) = \pi_1(X, x)$.

1). Deck group action is properly discontinuous.

2). $Y = \tilde{X}/H$ $\quad \pi_1(Y) = H$ since the group action is properly discont.

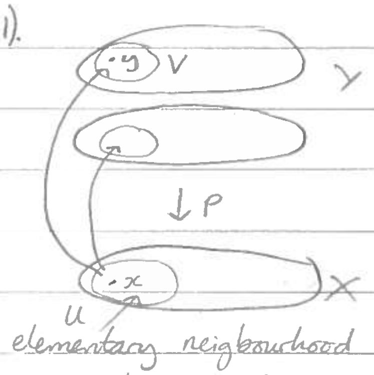
3). $p: Y \to X$ is a covering map.

$y \in \tilde{X}$, $[y]_H \in Y$, $[y]_X \in X$

$p([y]_H) = [y]_X$    $[y]_H$: H-orbit, $[y]_X$: entire $\pi_1(X)$-orbit

$\Rightarrow p_* \pi_1(Y) = H$.

1).




$g \in \mathrm{Deck}(Y, p)$, $g \neq 1$

$$V_g \cap V = \begin{cases} \phi & \checkmark \\ V_g = V \end{cases}$$

$yg = y \Rightarrow g = 1$ (by uniqueness of transformations)

$\Rightarrow$ the deck group acts properly discontinuously #

(for all covering spaces, in particular the universal cover)

2). follows from 1.

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3). Suppose $G$ acts properly discontinuously on $Z$ and that $H$ is a subgroup of $G$.
Then the quotient map $p: Z/H \to Z/G$ is a covering map.
[In our case $Z = \tilde{X}$, $G = \pi_1(X)$, $Z/H = \tilde{X}/H = Y$,
$Z/G = \tilde{X}/\pi_1(X) = X$]

$[z]_H \in Z/H$

$[z]_G \in Z/G$

$p([z]_H) = [z]_G$

$[z]_H = [z']_H \Rightarrow \exists h$ s.t. $z = z'h$, $h \in H \subseteq G \Rightarrow h \in G$

$\Rightarrow [z]_G = [z']_G \Rightarrow$ well definedness of map

$Z \longrightarrow Z/G$, $z \mapsto [z]_G$ continuous by def$^n$ of quotient topology on $Z/G$

$Z \longrightarrow Z/H$, $z \mapsto [z]_H$ continuous similarly

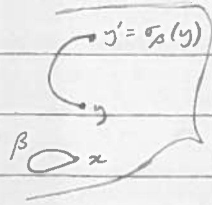
$$\begin{tikzcd} Z \arrow[r] \arrow[d] & Z/G \\ Z/H \arrow[ur, "p"'] & \end{tikzcd}$$

$p$ is continuous since the map $Z \to Z/G$ is continuous and factors through $Z/H$.

Since $G$ acts properly discontinuously on $Z$, $Z \to Z/G$ is a covering map, so we have local inverses $q: Z/G \to Z$.
$\Rightarrow$ local inverses of $p$ are $(Z \to Z/H) \circ q$
$\Rightarrow p$ is a covering map.

$\square$

Video 2



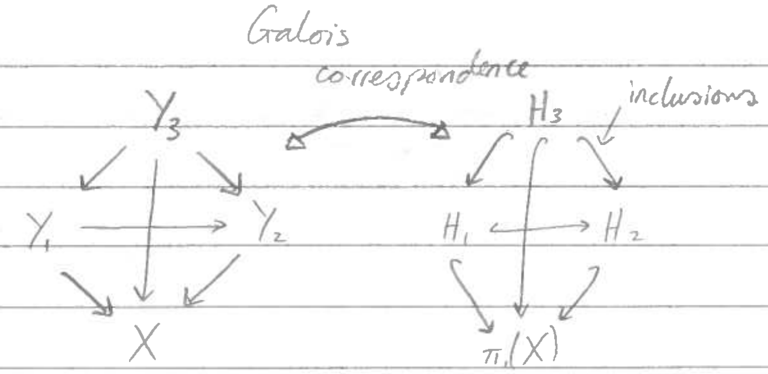

1). Each covering space $p: Y \to X$ and point $y \in p^{-1}(x)$ gives us a subgroup $p_* \pi_1(Y, y) \subseteq \pi_1(X, x)$. Every subgroup arises this way.

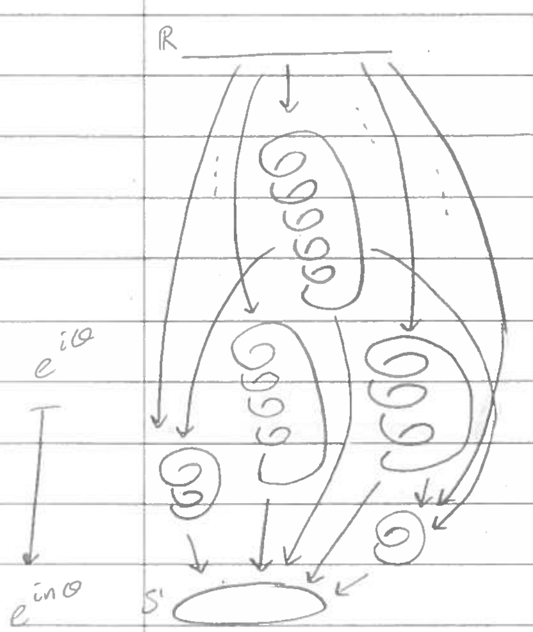
2). $\beta p_* \pi_1(Y, y) \beta^{-1} = p_* \pi_1(Y, \underset{\text{monodromy}}{\sigma_\beta}(y))$ ($\beta$ homotopy class of loops in $X$)

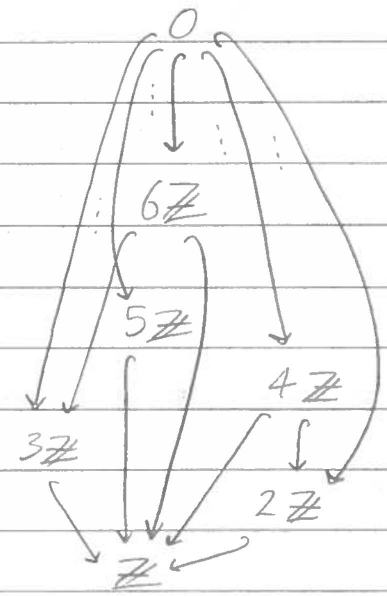
3). There is a covering transformation $F: Y_1 \to Y_2$ such that $F(y_1) = y_2$ iff $(p_1)_* \pi_1(Y_1, y_1) \subseteq (p_2)_* \pi_1(Y_2, y_2)$

4). If $p_* \pi_1(Y, y) = H$, $\mathrm{Deck}(Y, p) \cong N_H/H$
$N_H$ = largest subgroup in $\pi_1 X$ in which $H$ is normal

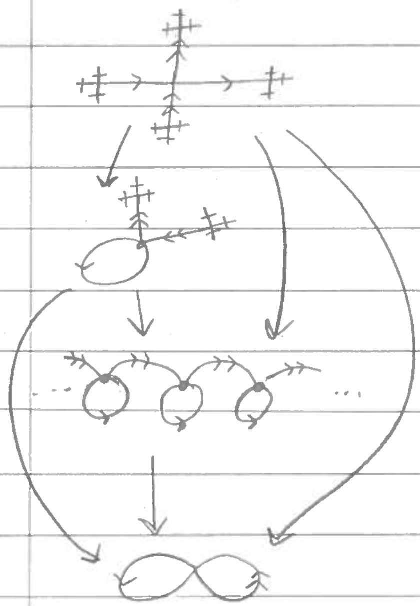
(point 3)

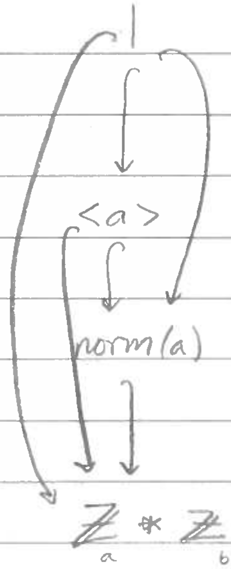



$n\mathbb{Z}$






Here norm(a) is the subgroup generated by a and all its conjugates

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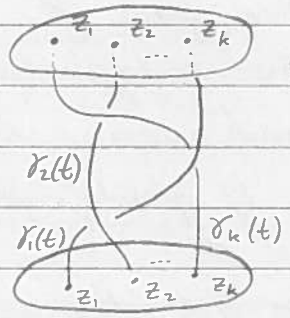
Video 1

Braids Introduction

Fix a collection of $k$ points $z_1, \dots, z_k$ in $\mathbb{C}$. A $k$-strand braid $F$ is a collection of $k$ continuous maps $F_1, \dots, F_k : [0,1] \to \mathbb{C}$ such that

$t=1$




$t=0$

$$\gamma_i(t) = (F_i(t), t)$$
$$F_i(t) \neq F_j(t), \quad i \neq j$$

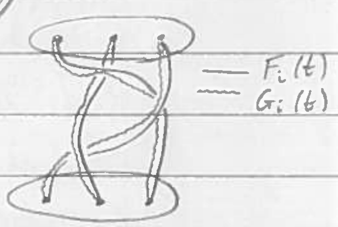
[The $\gamma_i$ are pairwise disjoint paths in $\mathbb{C} \times [0,1]$]

$$F_i(0) = z_i, \quad F_i(1) = z_{s(i)} \quad (s \text{ a permutation})$$
$$s = (1\,3)$$

Since $[0,1]$ is compact and the image of a compact set under a cts map is compact, the images of $F_k$ are contained in some compact set in the plane.

Equivalence of braids

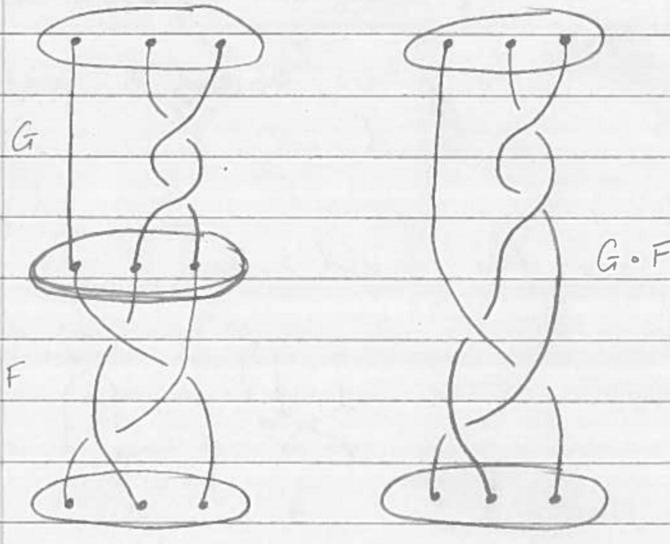
Two braids $(F_i(t), t)$, $(G_i(t), t)$ are equivalent if there are homotopies $H_i(s,t)$ s.t.
$H_i(0,t) = F_i(t)$, $H_i(1,t) = G_i(t)$, and s.t. for each $s$, $H_i(s,t)$ defines a braid.




[We can homotope everything to assume our braids are contained in $D \times [0,1]$]

Group Law

$t=1$




$t=0$

[Do F first, then G.]

Theorem
The set of equivalence classes of $n$-strand braids form a group under this stacking product.
Proof: Exercise.

$$(G \cdot F)_i(t) = \begin{cases} F_i(2t), & t \in [0, \tfrac{1}{2}] \\ G_{s(i)}(2t-1), & t \in [\tfrac{1}{2}, 1] \end{cases}$$

## Configuration space

A braid is a path in the configuration space of points in the plane starting with some configuration $z_1, \dots, z_k$ which moves the points around until they come back to the same configuration of points, possibly with a permutation.
(As we move up the braid slice by slice the points move around.)

ordered configuration of $k$ points in the plane

$$OC_k = \{(x_1, \dots, x_k) : x_i \neq x_j,\ i \neq j\} \qquad OC_k \subseteq \mathbb{C}^k$$

$$UC_k = OC_k / S_k$$

$S_k$: permutation group

$(x_1, \dots, x_k) \in OC_k$ is an ordered configuration of $k$ points in the disc.

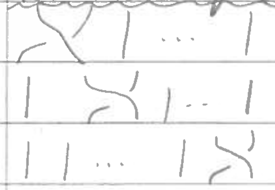
$UC_k$: unordered configuration of $k$ points in the plane

$\underbrace{(x_1, \dots, x_n)s = (x_{s(1)}, \dots, x_{s(n)})}_{\text{the action of } S_n \text{ on } OC_n}$

**Theorem**
$k$-strand braid group $\cong \pi_1(UC_k, (z_1, \dots, z_k))$

*Proof*: A braid is a collection of paths $F_i$ which defines a loop $[F_1(t), \dots, F_k(t)]$ in $UC_k$ based at $(z_1, \dots, z_k)$ and conversely. By def$^n$ a homotopy of braids gives a homotopy of loops in $UC_k$. Stacking braids corresponds to concatenating loops. □

### Presentation of the braid group



$\sigma_1$

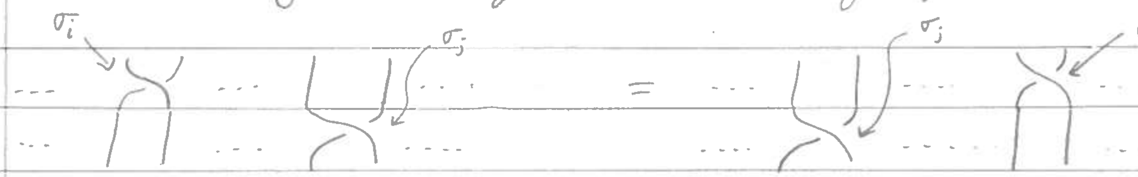
$\sigma_2$

$\vdots$

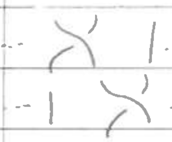
$\sigma_{k-1}$

$\sigma_i$ moves strand $i$ behind strand $i+1$.

The $\sigma_i$ are generators for the braid group



$$\sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i-j| \geq 2$$



$\sigma_i$

$\sigma_{i+1}$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

These generators and relations are sufficient to generate the whole braid group.

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Video 2

The Artin action

The braid group on $n$ strands is the fundamental group of the unordered configuration space $UC_n$. This group acts on the the group $\mathbb{Z}^{*n}$ via automorphisms; this action is called the Artin action of the braid group on the free group ($F_n = \mathbb{Z}^{*n}$).

$B_n = \pi_1(UC_n, z)$

$F_n = \pi_1(\mathbb{R}^2 \setminus z)$

Monodromy

Inside $\mathbb{C} \times UC_n$ we have a tautological subspace

$T_n = \{(x, c) : x \in c\}$

which meets the disc $\mathbb{C} \times \{c\}$ over a configuration $c \in UC_n$ in the $n$ points defining the configuration $c$.

Consider the complement of $T_n$, that is

$U_n := (\mathbb{C} \times UC_n) \setminus T_n$.

This is called the universal family over configuration space. The space $U_n$ has a natural projection

$p : U_n \to UC_n$ whose fibre $F_c = p^{-1}(c)$ over $c$ is precisely the plane punctured along the configuration $c$.

The universal family has a nice property: while it is not a covering space of the configuration space, it is a fibration over the configuration space.

This means that if $F : X \times [0,1] \to UC_n$ is a map and $F_0 : X \to U_n$ is a lift of $F|_{X \times \{0\}}$ then there exists a (not necessarily unique) lift $\tilde{F} : X \times [0,1] \to U_n$ of $F$.

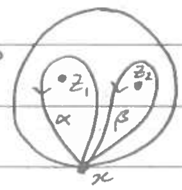
In particular, given a path $\gamma$ in $UC_n$, we get a monodromy map from the fibre $F_{\gamma(0)}$ to the fibre $F_{\gamma(1)}$, which is well defined up to homotopy.

In particular, we get an action of $\pi_1(UC_n, c)$ on $\pi_1(F_c, [z_1, \dots, z_n]) \cong \mathbb{Z}^{*n}$.
This is called the Artin action of the braid group on the free group.

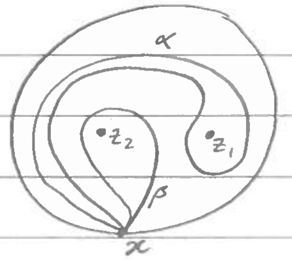
Explicit action

Rather than proving the fibration property of of the universal family, let us see how some explicit braids act in the Artin action.

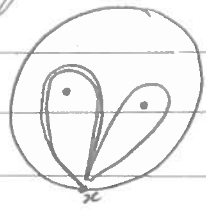
Let $n = 2$ and consider the elementary braid $\sigma_1$ .
The fundamental group of $\mathbb{C} \setminus \{z_1, z_2\}$ is $F_2 = \mathbb{Z} * \mathbb{Z}$, with generators $\alpha, \beta$.




The braid $\sigma_1$ moves the points $z_1, z_2$ around one another until they switch places:




We can see that $\sigma_1(\beta) = \alpha$ and, after a homotopy, we see that $\sigma_1(\alpha) = \alpha\beta\alpha^{-1}$;




Therefore we have

$\sigma_1(\alpha) = \alpha\beta\alpha^{-1}, \quad \sigma_1(\beta) = \alpha$

What about when we add more strands?
In fact, if we just focus on elementary braids (which generate the braid group), we already know the answer:
an elementary braid only affects two of the strands, and we can take the monodromy to be the identity away

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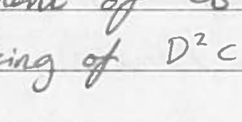
from these two strands. In other words, the action of $\sigma_i$ on $\alpha_1, \dots, \alpha_n$ is

$$\alpha_1 \longmapsto \alpha_1$$
$$\alpha_2 \longmapsto \alpha_2$$
$$\vdots \qquad \vdots$$
$$\alpha_i \longmapsto \alpha_i \alpha_{i+1} \alpha_i^{-1}$$
$$\alpha_{i+1} \longmapsto \alpha_i$$
$$\vdots \qquad \vdots$$
$$\alpha_n \longmapsto \alpha_n.$$

Video 3

## The Wirtinger Presentation

Let $B$ be an $n$-strand braid inside $D^2 \times [0,1]$. If we take the quotient space $D^2 \times S^1 = (D^2 \times [0,1]) / \sim$, $(x,0) \sim (x,1)$, then the braid closes up to become a collection of embedded circles $C_B$ in $D^2 \times S^1$ (because the component paths $B_k(t)$ start and end in the set of points $z_1, \dots, z_n$).
This is called the braid closure $C_B$ of $B$.

e.g. the braid closure of the 2-strand braid $\sigma_1^{\,3}$ is the trefoil knot: 

**Lemma**
Let $X_B = (D^2 \times S^1) \setminus C_B$ denote the complement of $C_B \subset D^2 \times S^1$.
Let $x = [1, 0] \in (D^2 \times [0,1])/\sim$ (we are thinking of $D^2 \subset \mathbb{C}$, so $1 \in D^2$ makes sense). We have
$\pi_1(X_B, x) = \langle \alpha_1, \dots, \alpha_n, g \mid g \alpha_k g^{-1} = B(\alpha_k) \text{ for } k = 1, \dots, n \rangle$.

Here $g$ is is the loop $x \times S^1$ and, for $k \in \{1, \dots, n\}$, $\alpha_k$ is the element of $\pi_1(D^2 \setminus \{z_1, \dots, z_n\})$ given by the loop as shown 

, and $B(\alpha_k)$ denotes the Artin action of $B$ on

$\alpha_k \in \pi_1(D^2 \setminus \{z_1, \dots, z_n\}) \cong \mathbb{Z}^{*n}$

Proof

The space $X_B$ is the mapping torus of the homeomorphism $\mathrm{Art}_B : D^2 \setminus \{z_1, \dots, z_n\} \longrightarrow D^2 \setminus \{z_1, \dots, z_n\}$, so the lemma follows from the result we proved earlier which gave a presentation for the fundamental group of a mapping torus. $\square$

Theorem

If we embed $D^2 \times S^1$ as the standard solid torus in $\mathbb{R}^3$ then the complement of the braid closure $C_B \subset \mathbb{R}^3$ has

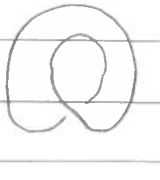
$\pi_1(\mathbb{R}^3 \setminus C_B) = \langle \alpha_1, \dots, \alpha_k \mid \alpha_k = B(\alpha_k),\ k = 1, \dots, n \rangle$

where $B(\alpha_k)$ is the Artin action of $B$ on the free group $\langle \alpha_1, \dots, \alpha_k \rangle$

Proof

(A homotopy retract of) the complement $\mathbb{R}^3 \setminus C_B$ is obtained from $X_B$ by attaching a 2-cell along the circle $x \times S^1$, which adds the relation $x = 1$ to the presentation from the lemma, geilding the desired presentation. $\square$

This allows us to compute the fundamental group of any knot complement since any knot is isotopic to a braid closure (see Alexander 1923 "A lemma on a system of knotted curves").



Example

Consider the 2 strand braid $\sigma_1$ (whose braid closure is the unknot). We have $\sigma_1(\alpha) = \alpha\beta\alpha^{-1}$, $\sigma_1(\beta) = \alpha$, so the Wirtinger presentation is $\langle \alpha, \beta \mid \alpha = \alpha\beta\alpha^{-1}, \beta = \alpha \rangle$. We can simplify this to just get $\langle \alpha \rangle$, so the fundamental group is $\mathbb{Z}$.