M206 Lie Groups and Lie Algebras Notes

Based on the 2016 spring lectures by Dr J Evans

Lie Groups and Lie Algebras - 11th Jan

Books:
- Lectures on Lie Groups and Lie Algebras
  Segal, Carter, McDonald
- Rep theory - Fulton, Harris - Springer
  for 2nd half of the course
- Lectures on Lie Groups - Adams
  for 1st half of the course

1. Introduction

1.1. What is a representation?

Definition: A representation of a group G on a vectorspace V over a field $\mathbb{K}$ is an assignment of a $\mathbb{K}$-linear map $\rho(g): V \longrightarrow V$ for each $g \in G$ s.t:

- $\rho(1) = 1$
- $\rho(gh) = \rho(g)\rho(h), \quad \forall g, h \in G$

Equivalently:

1) If you pick a basis of V then each $\rho(g)$ becomes a matrix. In this definition we can replace "$\rho(g): V \longrightarrow V$ is a linear map" by "$\rho(g) \in GL(V)$".

2) A representation is a homomorphism $\rho: G \longrightarrow GL(V)$ s.t. $g \to \rho(g)$

3) A representation is an action $\tilde{\rho}: G \times V \longrightarrow V$ of G on V s.t. $\forall g$, $\tilde{\rho}(g, -)$ is a $\mathbb{K}$-linear map

1.2. Why do we care?

Motivating example (binary quadratic forms / $\mathbb{C}$)

$$ax^2 + bxy + cy^2 = (x, y)\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} =$$

$$= \underline{x}^T M \underline{x}$$

So $V := \{\text{binary quadratic forms}\} \leftrightarrow \left\{\begin{array}{l} 2\times 2\ \mathbb{C}\text{-matrices that} \\ \text{are symmetric} \end{array}\right\}$

Question: When are two binary quadratic forms related by a change of coords? Equivalently when are two conic curves in $\mathbb{C}^2$ related by coordinate transformation? $\{ax^2 + bxy + cy^2 = 0\} \subseteq \mathbb{C}^2$

e.g. Let $G = SL(2, \mathbb{C}) = \{2\times 2\ \mathbb{C}$ matrices $\det = 1\}$ Change coordinates using $S \in G$. Then the matrix of our quadratic form in the new coordinates is $M' = S^T M S$ This gives us a representation of G on V.

Observe that $\det(M') = \det(S^T M S) = \det(M)$ since det is a homomorphism & $\det(S^T) = \det(S) = 1$.

Define $\Delta := \det(M)$ then this is an invariant of M under the action of G. , $M = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \Rightarrow \Delta = ac - \frac{b^2}{4} = -\frac{1}{4}(b^2 - 4ac)$

Further, Δ is "discriminant" of the quadratic form.

Lie Groups and Lie Algebras - 11^{th} Jan

<u>Step 1</u> : Diagonalise M (possible because M is symmetric) via the action of G i.e. $M \rightsquigarrow S^T M S$, S is orthogonal $S^T = S^{-1}$ i.e. wlog $M = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

<u>Step 2</u> Use action of $S = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$

$$M' = S^T M S = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} =$$

$$= \begin{pmatrix} \lambda^2 \lambda_1 & 0 \\ 0 & \lambda_2 \lambda^{-2} \end{pmatrix}$$

<u>Case 1</u> $\lambda_1 = \lambda_2 = 0$, $M = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

<u>Case 2</u> λ_1 or $\lambda_2 = 0$ but not both
wlog assume $\lambda_1 = 0$

Consider $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in G = SL(2, \mathbb{C})$

$$S^T M S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{pmatrix}$$

Then we use $S = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ as in <u>Step 2</u>

to assume $\lambda_2 = 1$

<u>Case 3</u> $\lambda_1, \lambda_2 \neq 0$ Use $S = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ to get $\lambda_2 = 1$

Conclusion: By a change of coordinates in group $G = SL(2, \mathbb{C})$ we can transform any binary quadratic form into one of

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} ? & 0 \\ 0 & 1 \end{pmatrix}$$

? must be Δ because Δ is invariant under the G-action and $\det \begin{pmatrix} ? & 0 \\ 0 & 1 \end{pmatrix} = ? = \Delta$

Theorem: Any binary quadratic form over $\mathbb{C}$ can be put into the form 0 or $\Delta x^2 + y^2$, by the action of $G = SL(2, \mathbb{C})$ coordinate transformations.

i.e. Δ is almost a complete invariant.
i.e. allows us to distinguish any two non-degenerate quadratic forms.

Therefore, we should be interested in invariants.

In terms of rep theory, what is Δ?

We have $V = \{\text{binary quadratic forms}\} = a, b, c$
and $Sym^2 V = \{\text{quadratic poly in } a, b, c\} =$
$= \{Aa^2 + Bb^2 + Cc^2 + Dab + Ebc + Fac\}$
Note $\Delta \in Sym^2 V$. This $Sym^2 V$ is 6-dimensional vector space and it is also a representation of G.

The fact that $\Delta \in Sym^2 V$ is invariant means that it defines a trivial 1-dim subrep.

Lie Groups and Lie Algebras - 11^{th} Jan

Moral: We need to find (given a repr) subrepresentations / invariant subspaces.

The goal of Rep Theory is to decompose complicated reps into irreducible reps.

1.3 Smoothness

Recall a rep is equivalent to a homomorph. $G \longrightarrow GL(V)$.

Example: Take $G = (\mathbb{R}, +)$. What are the homomorphisms $\mathbb{R} \longrightarrow \mathbb{R}$?

Consider $\mathbb{R}$ as vector space over $\mathbb{Q}$. Pick a basis (using Axiom of choice) $= A \leq \mathbb{R}$ $(A \longleftrightarrow \mathbb{R}/\mathbb{Q})$. Pick a function $\lambda : A \longrightarrow \mathbb{R}$ and define $\sum\limits_{\substack{a \in A \\ \text{finite sum}}} c_a a \longmapsto \sum c_a \lambda(a) \cdot a$

All homomorphisms $\mathbb{R} \to \mathbb{R}$ have this form $f : \mathbb{R} \to \mathbb{R} \Rightarrow f(n) = nf(1)$ & $q\, f(p/q) = f(p) = pf(1)$

$\mathbb{R}$ is better than just a group, it's a lie group i.e. it has a coordinate s.t. addition is a diff. function:
$(x, y) \longrightarrow x + y$ is diff. in x & y
Moreover, inversion: $x \longrightarrow -x$ is also diff. map.

Lemma: The differentiable homomorphisms $\mathbb{R} \to \mathbb{R}$ are precisely $x \to \lambda x$, $\lambda \in \mathbb{R}$

Proof: Let $f: \mathbb{R} \to \mathbb{R}$ be a diff. homo

$$f(x+y) = f(x) + f(y).$$

Differentiate w.r.t. x

$$LHS = \frac{\partial}{\partial x}\bigg|_{x=0} f(x+y) = f'(x+y)\Big|_{x=0} = f'(y)$$

$$RHS = \frac{\partial}{\partial x}\bigg|_{x=0} f(x)+f(y) = f'(0)$$

$$\Rightarrow f'(y) = f'(0) \quad \forall y$$

$$\Rightarrow f' = \text{const} = \lambda$$

$$\Rightarrow f(x) = \lambda x + c \quad \text{but } f(0) = 0$$

$$\Rightarrow f(x) = \lambda x \quad \text{since } c = 0 \quad \blacksquare$$

2. Exponential map

2.1. The matrix exponential

The simplest Lie group is $U(1) = \{z \in \mathbb{C} : |z| = 1\}$
Any $z \in U(1)$ can be written as
$z = \cos(\theta) + i\sin(\theta) = e^{i\theta}$
i.e. every unit complex number = exp of purely imaginary number.

Definition: The exponential of a matrix A is $\exp(A) = 1 + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3$

Lie Groups and Lie Algebras 11^{th} – Jan

Example $A = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}$. Compute $\exp(A)$

$$A^2 = \begin{pmatrix} -\theta^2 & 0 \\ 0 & -\theta^2 \end{pmatrix} = -\theta^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -\theta^2 . Id$$

$$A^3 = -\theta^2 \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} = \begin{pmatrix} 0 & \theta^3 \\ -\theta^3 & 0 \end{pmatrix}$$

$$\exp(A) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} \frac{1}{2} \begin{pmatrix} -\theta^2 & 0 \\ 0 & -\theta^2 \end{pmatrix} + $$
$$+ \frac{1}{3!} \begin{pmatrix} 0 & \theta^3 \\ -\theta^3 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 + 0 - \theta^2/2 \ldots & -\theta + \frac{1}{3!}\theta^3 \ldots \\ \theta - \frac{1}{3!}\theta^3 \ldots & 1 - \theta^2 \ldots \end{pmatrix} =$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$ Rotation matrix

2.2 Convergence

We can talk about convergence in any metric space / normed space.
For matrices we use "operator norm"
$\|A\| = \inf \{ C \in \mathbb{R} : |Av| \leq C |v|, \ \forall v \in \mathbb{R}^n \}$

Definition: A sequence of matrices A_k converges to A if $\|A - A_k\| \longrightarrow 0$

Properties of $\|\cdot\|$:

1) $\|A\| = 0 \Rightarrow A = 0$
2) $\|A+B\| \leq \|A\| + \|B\|$
3) $\|AB\| \leq \|A\| \cdot \|B\|$
4) $\|A\| \geq 0$
5) Cauchy sequences of matrices converge

Lemma: The exp power series converges absolutely i.e. $\exists$ K s.t. $\forall \varepsilon > 0$, $\exists N$ s.t.

$$\left| \sum_{n=0}^{M} \frac{1}{n!} \|A^n\| - K \right| < \varepsilon \quad , \forall \ M \geq N$$

Proof: $\sum_{n=0} \frac{1}{n!} \|A^n\| \leq \sum_{n=0} \frac{1}{n!} \|A\|^n \leq \exp(\|A\|)$

Therefore $\sum \frac{1}{n!} \|A^n\|$ is bdd above and monotone increasing $\Rightarrow$ this sequence converges ∎

Corollary: $\exp A$ converges i.e. $\exists \exp(A)$ s.t. $\left\| \sum \frac{1}{n!} A^n - \exp A \right\| \to 0$ as $N \to \infty$

In fact any absolutely convergent series converges.

Proof: Aim is to show $\sum \frac{1}{n!} A^n$ is Cauchy

$$\left\| \sum_{n=0}^{N} \frac{1}{n!} A^n - \sum_{m=0}^{M} \frac{1}{m!} A^m \right\| = \left\| \sum_{n=N+1}^{M} \frac{1}{n!} A^n \right\| \leq \sum_{n=N+1}^{M} \frac{1}{n!} \|A^n\| \to 0$$

since it is the tail of convergent series

Lie Groups and Lie Algebras 11th Jan

$\sum \frac{1}{n!} \|A^n\|$ => $\sum \frac{1}{n!} A^n$ is Cauchy and by property 5) it converges ∎

Remark: If we apply Weierstrass M test, we get that $F_N(A) = \sum_{n=0}^{N} \frac{1}{n!} A^n$ converges uniformly to $\exp A$ and the same is true for partial derivatives (w.r.t. the matrix entries)
=> $\exp A$ is diff. w.r.t. matrix entries

Corollary The function $t \longrightarrow \exp(tA)$ is differentiable and its derivative is
$$\frac{d}{dt}(\exp tA) = A\exp(tA) = \exp(tA).A$$

Proof: Because of uniform convergence we can differentiate term by term
$$\frac{d}{dt}\left(\sum_n \frac{1}{n!} t^n A^n\right) = \sum \frac{n}{n!} t^{n-1} A^n = \left(\sum \frac{1}{(n-1)!} t^{n-1} A^{n-1}\right).A$$

$$= \exp(tA) . A$$ ∎

Corollary (Cauchy Product formula)
$$\exp(A).\exp(B) = \sum_{k=0}^{\infty} \sum_{i+j=k} \frac{1}{i!\,j!} A^i B^j$$

This result follows from absolute convergence.

Corollary:

a) $\exp(-A) \cdot \exp(A) = I$

b) if $AB = BA$, then $\exp(A)\exp(B) = \exp(B)\exp(A)$

c) if $AB = BA$ then $\exp(A) \cdot \exp(B) = \exp(A+B)$

Proof:

a) $\exp(-A)\exp(A) = \sum_{K=0}^{\infty} \sum_{i+j=K} \frac{1}{i!\,j!} (-A)^i (A)^j =$

$= \sum_{K=0}^{\infty} \frac{1}{K!} (-A+A)^K = I + \sum_{K=1}^{\infty} \frac{1}{K!} (-A+A)^K =$

$= I$

b) $\exp(A)\exp(B) = \sum_{K=0}^{\infty} \sum_{i+j=K} \frac{1}{i!\,j!} A^i B^j \overset{\text{relable}}{\underset{i,j}{=}}$

$= \sum_{K=0}^{\infty} \sum_{i+j=K} \frac{1}{i!\,j!} B^i A^j = \exp(B)\exp(A)$

c) $\exp(A)\exp(B) = \sum_{K=0}^{\infty} \sum_{i+j=K} \frac{1}{i!\,j!} A^i B^j =$

$= \sum_{K=0}^{\infty} \frac{1}{K!} (A+B)^K = \exp(A+B)$ ∎

Lie Groups and Lie Algebras 14th Jan

2.3 $U(n)$

Recall $A \in U(n)$ if $A^+ A = 1$, $+$ denotes Hermitian tanspose, namely $A^+ = \overline{A^T}$, $A \in \mathbb{C}_{n\times n}$ matrix.

Definition - A matrix B is skew-Hermitian if $A^+ = -A$
- A matrix B is Hermition if $A^+ = A$

Lemma: B is skew-Hermitian iff $\exp(tB) \in U(n)$ $\forall\, t \in \mathbb{R}$

Special case $(n=1)$ $e^z \in U(1) \Leftrightarrow z \in i\mathbb{R} \Leftrightarrow \bar{z} = -z$

Proof: Assume $B^+ = -B$, then $[\exp(tB)]^+$

$$[\exp(tB)]^+ = \left[\sum \frac{1}{n!} t^n B^n\right]^+ = \sum \frac{1}{n!} t^n (B^+)^n =$$

$$= \sum \frac{1}{n!} t^n (-B)^n = \exp(-tB)$$

We saw that $\exp X \exp(-X) = Id$ so $\exp(-X) = (\exp X)^{-1}$. Thus

$$\exp(tB)^+ \cdot \exp(tB) = 1 \Rightarrow \exp(tB) \in U(n)$$

Conversely if $\exp(tB) \in U(n)$ $\forall t$, then

$$\underset{\exp(tB^+)}{\underset{\parallel}{[\exp(tB)]^+}} = \exp(tB)^{-1} = \exp(-tB)$$

Differentiate w.r.t t (because it is a Lie group)

$$B^{+}\exp(tB^{+}) = -B\exp(-tB)$$

let $t=0 \Rightarrow B^{+}=-B$ ■

Definition: If $G \subseteq GL(n,\mathbb{R})$ is a subgroup we define the Lie algebra

$$\mathfrak{g} = \{B : \exp(tB) \in G \ \forall t \in \mathbb{R}\} = \text{Lie } G$$

Corollary The Lie algebra of $U(n)$ is the space of skew Hermitian matrices.
We write $u(n)$ for $\{B : B^{+}=-B\}$
$u(1) = \text{Lie } U(1) = i\mathbb{R}$

Lemma: $\text{Lie}\, GL(n,\mathbb{R})$ = all $n\times n$ matrices $= gl(n,\mathbb{R})$

Proof: $\text{Lie } GL(n,\mathbb{R}) \subseteq gl(n,\mathbb{R})$ by definition

But if $A \in gl(n,\mathbb{R})$ then $\exp(tA)^{-1} = \exp(-tA)$
So $\exp(tA) \in GL(n,\mathbb{R})\ \forall t$ ■

2.4 SU

Example $SU(2) = \{A \in U(2) : \det A = 1\}$

Claim $su(2) = \text{Lie } SU(2) = \left\{ \begin{pmatrix} ix & y+iz \\ -y+iz & -ix \end{pmatrix} : (x,y,z) \in \mathbb{R}^3 \right\}$

Lie Groups and Lie Algebras 14^{th} Jan

$M_v^+ = -M_v$ but $tr\, M_v = 0$

Lemma $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2)$ then

$$A = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \text{ with } |a|^2 + |b|^2 = 1$$

Proof: if $A \in SU(2) \Rightarrow A^+ = A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$\det(A) = 1$

and $A^+ = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = A^{-1}$

$\Rightarrow \bar{a} = d$
and $\bar{b} = -c$

$\det(A) = a \cdot \bar{a} + b \cdot \bar{b} = 1$
$\Rightarrow |a|^2 + |b|^2 = 1$ ∎

$\Rightarrow$ $SU(2)$ is 3-dim sphere in $\mathbb{R}^4$

$a = x_1 + i x_2$ $b = x_3 + i x_4$

$|a|^2 + |b|^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$
3-dim sphere in $\mathbb{R}^4$

In sheet 1, we will see that

$\exp(\theta M_n)$, n is a unit vector
$n = (x, y, z)^T$, $|n| = 1$,

$M_n = \begin{pmatrix} ix & y+iz \\ -y+iz & -ix \end{pmatrix}$

$$\exp(\theta M_u) = \begin{pmatrix} \cos\theta & 0 \\ 0 & \cos\theta \end{pmatrix} + \sin\theta\, M_u$$

$$= \begin{pmatrix} \cos\theta + ix\sin\theta & (y+iz)\sin\theta \\ (-y+iz)\sin\theta & \cos\theta - iz\sin\theta \end{pmatrix}$$

Given $A = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in SU(2)$, $\exists\, \theta, u$ s.t.

$A = \exp(\theta M_u)$

Proof : $\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} = \begin{pmatrix} \cos\theta + ix\sin\theta & y\sin\theta + iz\sin\theta \\ -y\sin\theta + iz\sin\theta & \cos\theta - iz\sin\theta \end{pmatrix}$

take $\cos\theta = x_1$ since $|a|^2 \le 1 \Rightarrow |x_1| \le 1$
then $\cos^2\theta + \sin^2\theta = 1$
and $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$
$\Rightarrow \sin^2\theta = x_2^2 + x_3^2 + x_4^2$ determines $\sin\theta$ up to sign

Take $u = \begin{pmatrix} x_2/\sin\theta \\ x_3/\sin\theta \\ x_4/\sin\theta \end{pmatrix}$ This is now a unit vector

we have $\left.\begin{matrix} \cos\theta \\ \sin\theta \\ u \end{matrix}\right\} \Rightarrow$ we have θ ∎

Corollary $\exp : su(2) \to SU(2)$ is surjective

Lie Groups and Lie Algebras 14th Jan

3. Local logarithm

3.1 Calculus of several variables

Definition: suppose $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ are open sets, $f: U \to V$. is a map.

We say f is smooth if all possible partial derivatives $\frac{\partial^k f_j}{\partial x_{i_1} \dots \partial x_{i_k}}$ exist.

In this situation, define, for each $p \in U$

$$\boxed{d_p f = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_m} \end{pmatrix} : \mathbb{R}^m \longrightarrow \mathbb{R}^n}$$

linear map called the derivative of f

This is the best linear approx. to f at p. i.e. $f(p+tv) = f(p) + t d_p f(v) + O(t^2)$
Taylor series

Example $F: \mathbb{R} \to \mathbb{R}$
$x \to x^2$
$F(x+t) = (x+t)^2) = x^2 + 2tx + t^2$
$\Rightarrow d_p F(v) = 2xv$

Example $H = \{A : A^+ = A\}$ Consider
$F: GL(n, \mathbb{C}) \longrightarrow H$

$F(A) = A^+ A$ Note $(A^+A)^+ = A^+A$

$$F(A+B) = (A+B)^t (A+B) =$$

$$= A^tA + B^tA + A^tB + B^tB$$

$$= F(A) + B^tA + A^tB + O(B^2)$$

So $d_A F(B) = B^tA + A^tB$ this is linear in B

Lemma: If $U_1 \subseteq \mathbb{R}^{n_1}$, $U_2 \subseteq \mathbb{R}^{n_2}$, $U_3 \subseteq \mathbb{R}^{n_3}$ are open and $F: U_1 \to U_2$, $G: U_2 \to U_3$ then

$d_p(G \circ F) = (d_{F(p)} G)(d_p F)$ matrix product

Chain Rule

Theorem (Inverse function thm)

If $F: U \to V$ is a smooth map and $p \in U$ is s.t. $d_p F: \mathbb{R}^m \to \mathbb{R}^n$ is invertible then $\exists$ $F(p) \in V' \subseteq V$ and $U' \ni p$

and map $F^{-1}: V' \to U'$ s.t. $F^{-1} \circ F = Id$

Example $F(x) = x^2$

$d_x F(t) = 2xt$ invertible if $x \neq 0$

so we get $\sqrt{\ }$ defined on any open set in $\mathbb{R}$ not containing 0.

Lie Groups and Lie Algebras 14^{th} Jan

Theorem: Let $\exp : gl(n,\mathbb{R}) \to GL(n,\mathbb{R})$
$\exists\ \mathcal{U} \to 1$ in $GL(n,\mathbb{R})$ and $V \ni 0 \in gl(n,\mathbb{R})$
and $\log : V \to \mathcal{U}$ s.t. $\exp(\log x) = x$ and
$\log(\exp(x)) = x$.

Proof: To apply the inverse function thm, need to show that $d_0 \exp$ is invertible

$$\exp(A) = I + A + \frac{1}{2}A^2 + \cdots$$

$$= \exp(0) + id(A) + O(A^2)$$

$\Rightarrow \quad d_0\exp = id$ and id is invertible ∎

18^{th} Jan

3.2. Local logarithm

What we proved

Theorem $\exists\ \underset{\text{open}}{\mathcal{U}, V}$ s.t. $0 \in \mathcal{U}$, $1 \in V$.

$$0 \in \underset{\text{all matrices}}{gl(n,\mathbb{R})} \xrightarrow{\exp} 1 \in GL(n,\mathbb{R})$$

exp is local diffeomorphism : $\mathcal{U} \to V$
i.e. it's a bijection with smooth inverse, called log.

More explicitly:

3.3 Baker - Campbell - Hausdorff Formula

<u>lemma</u>: Log has a power series expansion around the identity 1

$$\boxed{\log(1+X) = X - \frac{1}{2}X^2 + \frac{1}{3}X^3 - \dots}$$

<u>Proof</u>: $\exp X = 1 + X + \frac{X^2}{2!} + \dots$

$\log(1+X) = b_1 X + b_2 X^2 + \dots$

$$\log(\exp X) = \log\left(1 + X + \frac{X^2}{2!} + \dots\right) =$$

$$= b_1\left(X + \frac{X^2}{2!} + \dots\right) + b_2\left(X + \frac{X^2}{2!} + \dots\right)^2 + \dots$$

$$= b_1 X + \left(\frac{b_1}{2!} + b_2\right)X^2 + \dots =$$

$$= X$$

$$\Rightarrow b_1 = 1 \qquad b_2 = -\frac{b_1}{2!} = -\frac{1}{2}, \dots$$

This recursion relation for b_i is the same if X is a number or a Matrix, so we get the same Taylor series

We saw: $\exp A \cdot \exp B = \exp(A+B)$ if A & B commute

Lie Groups and Lie Algebras 18th Jan

$$\log(\exp A \cdot \exp B) = \log\left(\left(1 + A + \frac{A^2}{2!} + \cdots\right)\left(1 + B + \frac{B^2}{2!} + \cdots\right)\right)$$

$$= \log\left(1 + A + B + \frac{A^2}{2!} + AB + \frac{B^2}{2!} + \cdots\right)$$

$$= \left(A + B + \frac{A^2}{2!} + AB + \frac{B^2}{2!} + \cdots\right) - \frac{1}{2}\left(A + B + \frac{A^2}{2!} + AB + \frac{B^2}{2!} + \cdots\right)^2 + \cdots$$

$$= A + B + \frac{A^2}{2} + AB + \frac{B^2}{2} - \frac{1}{2}(A+B)^2 + \cdots =$$

$$= A + B + \frac{A^2}{2} + AB + \frac{B^2}{2} - \frac{1}{2}A^2 - \frac{1}{2}AB - \frac{BA}{2} - \frac{B^2}{2} + \cdots$$

$$= A + B + \frac{1}{2}(AB - BA) + \cdots$$

Lemma The next term (cubic term) is

$$\frac{1}{12}[A,[A,B]] - [B,[A,B]]$$

where $[A,B] = AB - BA$

Baker – Campbell – Hausdorff Formula

$$\log(\exp A \cdot \exp B) = \sum_{n>0} \frac{(-1)^{n+1}}{n} \sum_{\substack{r_i + s_i > 0 \\ 1 \le i \le n}} \frac{\left(\sum_{i=1}^{n} (r_i + s_i)\right)^{-1} ad_A^{r_1} ad_B^{s_1} \cdots ad_A^{r_{n-1}} ad_B^{s_{n-1}} K_{r_n s_n}}{r_1! s_1! \dots r_n! s_n!}$$

$$\text{where } K_{r_n s_n} = \begin{cases} ad_A^{r_n} B & s_n = 1 \\ A & r_n = 1 \text{ and } s_n = 0 \\ 0 & \text{otherwise} \end{cases}$$

$ad_X Y = [X, Y]$, $ad_X^2 Y = [X,[X,Y]]$ etc..

Remark: The particular formula is not so important, but observe that $\log(\exp A \cdot \exp B)$ can be given as a power series in terms of iterated commutator brackets of A and B.

So knowing $gl(n, \mathbb{R})$ and $[,]$ is enough to recover the group structure of $GL(n, \mathbb{R})$

The space $gl(n, \mathbb{R})$ with the operation $[,] : gl(n, \mathbb{R}) \times gl(n, \mathbb{R}) \to gl(n, \mathbb{R})$ is a Lie algebra

3.4 Lie algebras

Definition: Let $\mathbb{K}$ be a field ($\mathrm{char}\,\mathbb{K} \neq 2$) and $\mathfrak{g}$ be a vector space over $\mathbb{K}$. A Lie algebra structure of $\mathfrak{g}$ is a bilinear operation $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ s.t.

- $[A, B] = -[B, A]$ $\quad ([A, A] = 0 \ \forall A)$
- $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$

Equivalently $ad_X ad_Y Z - ad_{[X,Y]} Z - ad_Y ad_X Z = 0$

$(ad_X ad_Y - ad_Y ad_X)(Z) = ad_{[X,Y]} Z$ - Jacobi identity

Bilinearity means $[\lambda A + \mu B, C] = \lambda [A, C] + \mu [B, C]$

Recall, we defined the Lie algebra of a subgroup $G \subseteq GL(n, \mathbb{R})$ to be

$$\mathfrak{g} = \{ X \in gl(n, \mathbb{R}) : \exp(tX) \in G \ \forall t \}$$

Lie groups and Lie algebras 18th Jan

This is a Lie algebra under taking commutator i.e. this is a lie sub alg. of $gl(n,\mathbb{R})$

4. Matrix groups

Definition: A matrix group is a subgroup G of $GL(n,\mathbb{R})$ which is closed topologically w.r.t. the operator norm
i.e. if $g_i \in G$ is a sequence of matrices s.t. $g_i \longrightarrow g$ for some $g \in GL(n,\mathbb{R})$ then $g \in G$

Lemma: Given subgroup $G \leq GL(n,\mathbb{R})$ define
$\overline{G} = \{g \in GL(n,\mathbb{R})$ s.t. $\exists\, g_i \in G$ s.t. $g_i \to g\}$
This is a closed subgroup of $GL(n,\mathbb{R})$

Proof: • Suppose $1 \in G \Rightarrow g_i = 1\ \forall_i$ converges to $1 \Rightarrow 1 \in \overline{G}$

• Let $g, h \in \overline{G}$ then $\exists$ seq. $g_i \longrightarrow g$ and $h_i \to h$, $g_i, h_i \in G$, Then
Then

$$\|g_ih_i - gh\| = \|g_ih_i - gh_i + gh_i - gh\|.$$

$$\leq \|g_ih_i - gh_i\| + \|gh_i - gh\|$$

$$\leq \underbrace{\|g_i - g\|}_{\downarrow 0}.\underbrace{\|h_i\|}_{\downarrow \|h\|} + \|g\|.\underbrace{\|h_i - h\|}_{\downarrow 0}$$

$$\longrightarrow 0$$

$\Rightarrow g_ih_i \to gh \Rightarrow gh \in \overline{G}$

• Given $g \in \overline{G}$, $\exists\, g_i \to g$

Want $g_i^{-1} g \to 1$

$$\|g_i^{-1} g - 1\| = \|g_i^{-1} g - g_i^{-1} g_i\|$$

$$= \underbrace{\|g_i^{-1}\|}_{bdd} \underbrace{\|g - g_i\|}_{\downarrow 0}$$

$$\to 0 \qquad \Rightarrow \qquad g^{-1} \in \overline{G}$$

$\|g_i^{-1}\| = \|g_i\|^{-1} \Rightarrow \|g\|^{-1}$ so bdd.

$\mathbb{R}$ $x \to x^{-1}$ is continuous map $GL(n, \mathbb{R}) \to GL(n, \mathbb{R})$
and g_i converges $\Rightarrow$ g_i^{-1} converges

Lemma : If G is abelian then so is $\overline{G}$.

Proof : $g, h \in \overline{G}$, $\exists\, g_i, h_i \in G$ s.t. $g_i \to g$, $h_i \to h$

$g_i h_i \to gh$ (in proof of lemma before)
$h_i g_i \to hg$
since g_i, h_i commute since G is abelian

By uniqueness of limits $\Rightarrow gh = hg$
$\Rightarrow \overline{G}$ is abelian ∎

Remark : We will also use the fact that matrix multiplication is continuous w.r.t operator norm.

Example . Let Q be an $n \times n$ matrix and define $G = O(Q) := \{A : A^T Q A = Q\}$ - orthogonal group of Q

e.g. if $Q = 1$ $O(Q) = \{A : A^T A = 1\} = O(n)$

Claim $O(Q)$ is a matrix group

Lie Groups and Lie Algebras 18th Jan

Proof: Let $A_i \in O(Q)$ be a sequence s.t. $A_i \to A$
want to show $A \in O(Q)$

We know that $A \to A^T Q A$ is continuous w.r.t. operator norm i.e.

$$\| (A+B)^T Q (A+B) - A^T Q A \| =$$

$$\| A^T Q A + B^T Q A + A^T Q B + B^T Q B - A^T Q A \|$$

$$\leq \|B\| \left(\|QA\| + \|A^T Q\| + \|QB\| \right)$$

for $\|B\| < \delta$ this is $\leq \delta 3 \|A\| \|Q\| = \varepsilon$

$$\delta = \frac{\varepsilon}{3 \|A\| \|Q\|}$$

so we get continuity

$A_i \to A$ $\quad A \to A^T Q A$ continuous $\Rightarrow$
$A_i^T Q A_i \to A^T Q A$
because $A_i \in O(Q)$, $A_i^T Q A_i = Q$

$$\Rightarrow A^T Q A = Q \Rightarrow A \in O(Q) \quad \blacksquare$$

This means that $O(Q)$ is a matrix group.

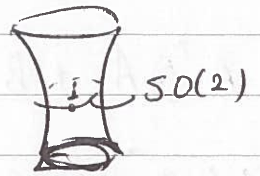
E.g. $Q = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}_{n \times n} \Rightarrow O(1, n-1)$ - Lorentz group

$O(1,1)$ is A s.t. $A^T \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

e.g. $A = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$

This looks a lot like the matrix
$\begin{pmatrix} \cos & -\sin \\ \sin & \cos \end{pmatrix} \in O(2)$

Indeed both lie in $O(2, \mathbb{C})$ looks like



Lie algebra of $O(Q)$ is
$\{ B : B^T Q + QB = 0 \}$

e.g. $Q = 1$ $\qquad B^T + B = 0 \Rightarrow B^T = -B$

e.g. $Q = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ $\qquad B^T \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} B = 0$

let $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -a & c \\ -b & d \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -a & -b \\ c & d \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -2a & c-b \\ c-b & 2d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\Rightarrow a = 0, \; d = 0, \; c = b$

Lie groups and Lie algebras 18th Jan

$\Rightarrow$ lie alg of $O(1,1)$ is $\left\{\begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix} : b \in \mathbb{R}\right\}$

$$\exp\begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix} + \frac{1}{2}\left\{t^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right. +$$

$$+ \frac{1}{3!} t^3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 + \frac{1}{2!}t^2 + \frac{1}{4!}t^4 + \dots & t + \frac{1}{3!}t^3 + \dots \\ t + \frac{1}{3!}t^3 + \dots & 1 + \frac{1}{2}t^2 + \dots \end{pmatrix} =$$

$$= \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$$

Example

$$Q = \left(\begin{array}{cc|cc|c} 0 & 1 & & 0 & 0 \\ -1 & 0 & & & \\ \hline & 0 & 0 & 1 & 0 \\ & & -1 & 0 & \\ \hline & 0 & & 0 & \ddots \end{array}\right)$$

$O(Q) = Sp(2n, \mathbb{R})$ simplectic group.

Example $GL(n, \mathbb{C}) \subseteq GL(2n, \mathbb{R})$

Given a complex matrix, replace each entry $a+ib$ with 2×2 matrix $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

This gives me an embedding of $GL(n, \mathbb{C})$ into $GL(2n, \mathbb{R})$ whose image comprises matrices A s.t. $AJ = JA$, where J is Q in the previous example.

1×1 case
$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{J} = \begin{pmatrix} b & a \\ -a & b \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} b & a \\ -a & b \end{pmatrix}$$

so they do commute with J

Moreover if $AJ = JA \Rightarrow A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

$$a+ib \rightsquigarrow \underset{A}{\begin{pmatrix} a & -b \\ b & a \end{pmatrix}} \sim AJ = \begin{pmatrix} b & a \\ -a & b \end{pmatrix}$$

$$b - ia = (-i)(a+ib)$$

$\mathbb{C}$	$\mathbb{R}$
$a+ib \to -i(a+ib)$	$A \to AJ$
$(a+ib)i = i(a+ib)$	$\Leftrightarrow AJ = JA$

Now, Hermitian conjugate is just transposition

$$\underset{\substack{\uparrow \\ \text{~~$n\times n$ complex~~} \\ \text{matrix}}}{A^{+}}_{\in GL(n,\mathbb{C})} \longrightarrow \underset{\in GL(2n,\mathbb{R})}{B^{T}}$$

$$\underset{\in GL(n,\mathbb{C})}{A} \longrightarrow \underset{\in GL(2n,\mathbb{R})}{B}$$

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}^{T} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

$$a+ib \qquad\qquad = a - ib = \overline{(a+ib)}$$

Lie groups and Lie algebras 18th Jan

4.3 The Lie algebra

Recall $\mathfrak{g} = \{ X$ s.t. $\exp(tX) \in G \ \forall t \in \mathbb{R} \} \subseteq \mathfrak{gl}(n,\mathbb{R})$
$G \subseteq GL(n,\mathbb{R})$

Goal : to prove that $\mathfrak{g}$ is a vector subspace of $\mathfrak{gl}(n,\mathbb{R})$.

In fact, we will be able to identify geometrically as the tangent space of G at $1 \in G$

Lemma : Let $H \subseteq GL(n,\mathbb{R})$ be a matrix group. If $h_m \in \mathfrak{gl}(n,\mathbb{R})$ is a sequence such that $\exp(h_m) \in H$, $h_m \to 0$ and $\frac{h_m}{|h_m|} \to v$ for some $v \in \mathfrak{gl}(n,\mathbb{R})$ - then $\exp(tv) \in H \ \forall t \in \mathbb{R}$

Comment : Could be that $\exists w \in \mathfrak{gl}(n,\mathbb{R})$ s.t. $\exp(w) \in H$ but $w \notin$ Lie H

Proof : We want to show that $\exp(tv) \in H$ $\forall t$, so fix t. Let m_n be the largest integer less than $\frac{t}{|h_n|}$. Since $h_n \to 0$, $m_n \to \infty$

Then
$$\frac{t}{|h_n|} - 1 \leq m_n \leq \frac{t}{|h_n|}$$

$$\Rightarrow \underbrace{t - |h_n|}_{\to t} \leq m_n |h_n| \leq t$$

$\exp(h_n) \in H$; $\exp(m_n h_n) = (\exp(h_n))^{m_n} \in H$

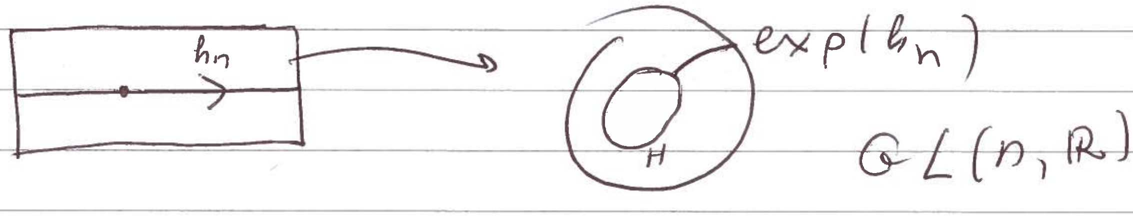
$$\exp(m_n h_n) = \exp(\underbrace{m_n |h_n|}_{\to t} \underbrace{\frac{h_n}{|h_n|}}_{\to v}) \to \exp(tv)$$

$\Rightarrow \exp(tv) \in H$ since H is a Matrix group i.e. closed. $\blacksquare$

Lie Groups and Lie Algebra $\quad 21^{st}$ Jan

Recall: lemma * Let $H \subseteq GL(n, \mathbb{R})$ be a matrix. If $h_n \in H$ is a sequence sequence of matrices in $gl(n, \mathbb{R})$ s.t. $\exp(h_n) \in H$ s.t. $\frac{h_n}{|h_n|} \to V$ and $h_n \to 0$

then $\exp(tv) \in H \quad \forall v$



Theorem

If $G \subseteq GL(n, \mathbb{R})$ is a matrix group then $\mathfrak{g} = \{ v \in gl(n, \mathbb{R}) : \exp(tv) \in G \ \forall t \}$ is a vector subspace of $gl(n, \mathbb{R})$

Proof:

- $v \in \mathfrak{g} \Rightarrow \lambda v \in \mathfrak{g} \quad \forall \lambda \in \mathbb{R}$

We don't need to check anything here as $\exp(tv) \in G \ \forall t$. So true for any rescalling

$$v \in \mathfrak{g} \Leftrightarrow \exp(tv) \in G \Leftrightarrow \exp\left(\tfrac{t}{\lambda} \lambda v\right) \in G$$
$$\Leftrightarrow \exp(t/\lambda v) \in G$$

- $w_1, w_2 \in \mathfrak{g} \Rightarrow w_1 + w_2 \in \mathfrak{g}$

Assume $w_1, w_2 \in \mathfrak{g}$ then $\exp(tw_1)\exp(tw_2) \in G$ $\forall t \quad \therefore \underbrace{\exp(tw_1)\exp(tw_2)}_{=\gamma(t)} \in G$

For t sufficiently small, $\gamma(t)$ is closed to 1
Recall: exp is invertible on a small neighbourhood of 1 i.e. $\exists$

$\exists \ \log : \underset{\text{neighbourhood of } 1 \in GL(n,\mathbb{R})}{V} \longrightarrow \underset{\text{neighbourhood of } 0 \in gl(n,\mathbb{R})}{\mathcal{U}}$

$\Rightarrow$ when t small $\gamma(t) = \exp f(t)$ for some $f(t) \in gl(n, \mathbb{R})$

In fact, Baker-Campbell-Hausdorff formula gives

$$\exp(tw_1)\exp(tw_2) = \exp(\underbrace{t(w_1+w_2) + O(t^2)}_{f(t)})$$

Take $h_n = f\left(\frac{1}{n}\right)$

$\exp(h_n) \in G$ by construction

$h_n \to 0$ is true because $\frac{1}{n} \to 0$, $f(0) = 0$

$\frac{h_n}{|h_n|} \longrightarrow v$ for some v

$$\lim_{n\to\infty} \frac{h_n}{|h_n|} = \lim_{t\to 0} \frac{f(t)}{|f(t)|} = \lim_{t\to 0} \frac{t(w_1+w_2) + O(t^2)}{|t(w_1+w_2) + O(t^2)|}$$

By L'hospital : $= \dfrac{w_1+w_2}{|w_1+w_2|}$

Using lemma (*) $\Rightarrow \exp\left(\frac{t(w_1+w_2)}{|w_1+w_2|}\right) \in G$

$\forall \ \frac{w_1+w_2}{|w_1+w_2|}$, $\forall t$

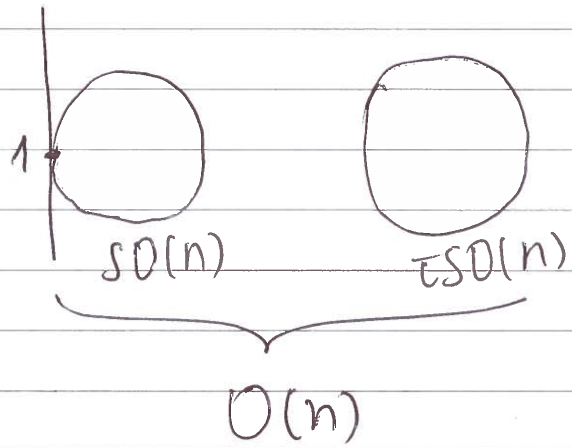
$\Rightarrow \exp(t(w_1+w_2)) \in G \ \forall t \Rightarrow w_1+w_2 \in \mathfrak{g}$ ■

Lie Groups and Lie Algebras 21st Jan

Example

- $G = U(n)$ $\mathfrak{g} = u(n) = \{B : B^+ = -B\}$ skew-Hermitian
- $G = SL(n, \mathbb{K}) = \{A : \det(A) = 1\}$
  $\mathfrak{gl}(n, \mathbb{K}) = \{B : Tr(B) = 0\} = sl(n, \mathbb{K})$
  follows from the fact that
  $\det(\exp A) = \exp Tr(A)$
- $G = SU(n) = SL(n, \mathbb{C}) \cap U(n)$
  $\mathfrak{g} = su(n) = u(n) \cap sl(n, \mathbb{C})$
- $G = O(n) = \{A^T A = 1\}$
  $o(n) = \{B : B^T = -B\}$ never see this
- $G = SO(n) = \{A^T A = 1, \det A = 1\}$
  $so(n) = o(n)$ $Tr B^T = Tr B = 0$ if $B^T = B$

$O(n) = SO(n) \cup \tau SO(n)$, $\tau = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$



4.4 Exponential charts

We saw that exp gives a bijection
$U \to V$ $U \ni 0$ in $gl(n, \mathbb{R})$, $V \ni 1$ in $Gl(n, \mathbb{R})$

This is also true for $\exp : \mathfrak{g} \to G$

Definition: If G is matrix group with Lie algebra $\mathfrak{g}$ then an **exponential chart** is a pair

$$\begin{array}{ccc} 0 & & 1 \\ \cap & & \cap \\ B & \xrightarrow{\exp} & C \\ \cap & & \cap \\ \mathfrak{g} & & G \end{array}$$ is a diffeomorphism

Theorem

$\exp\log : \mathfrak{g} \to G$ and if U and V are neighbourhoods of 0 and 1 in $gl(n,\mathbb{R})$ and $Gl(n,\mathbb{R})$ then $\mathfrak{g} \cap U$ and $G \cap V$ form an exponential chart

Proof

We want to prove that

$\exp|_{\mathfrak{g}\cap U} : \mathfrak{g} \cap U \to G \cap V$ is a bijection

$\exp : U \to V$ is injective so a restriction of it is also injective i.e.

$\exp|_{\mathfrak{g}\cap U} : \mathfrak{g} \cap U \to G \cap V$ is injective

$\Rightarrow$ only need to show it's surjective

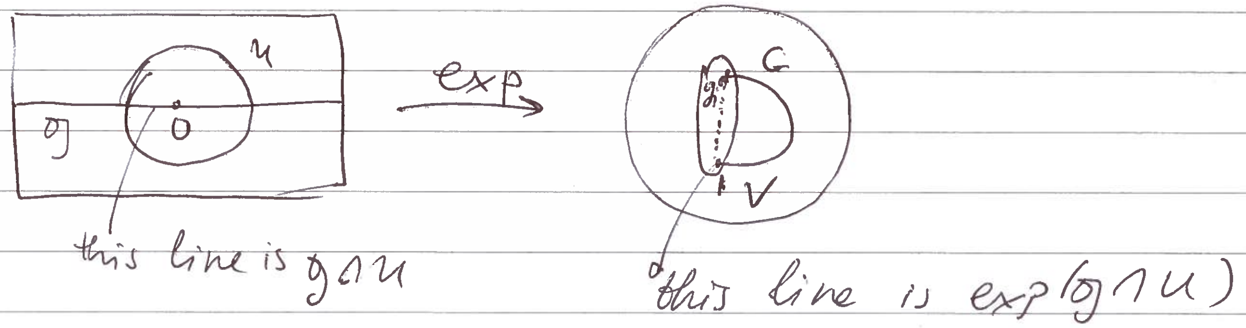
We are restricting the domain so surj. could fail. None the less, after possibly shrinking U and V we can assume $\exp|_{\mathfrak{g}\cap U}$ is surj onto $G \cap V$

Proof by contradiction: Assume surj. fails $\forall$ choices U, V then for a choice

Lie Groups and Lie Algebras — 21st Jan

of $\mathcal{U}, V$ $\exists g \in V \cap G$ s.t. $g \neq \exp v$ for any $v \in \mathcal{U} \cap \mathfrak{g}$

In particular, shrinking V, we get a sequence g_i : s.t $g_i \notin \mathrm{Im}(\exp_{\mathfrak{g} \cap \mathcal{U}})$ and s.t. $g_i \to 1$



Lemma

Suppose $\mathfrak{gl}(n, \mathbb{R}) = W_1 \oplus W_2$ for some subspace W_1 and W_2. Then there exists neighbourhoods $\mathcal{U}$ s.t. $0 \in W_1 \oplus W_2$ and V s.t. $1 \in GL(n, \mathbb{R})$ s.t.

$$\mathcal{U} \longrightarrow V$$
$$(w_1, w_2) \longmapsto \exp w_1 \exp w_2 \quad \text{is a diffeo}$$

ex on sheet 2

In our case, take $W_1 = \mathfrak{g}$, W_2 be any complement

Lemma says any point in $GL(n, \mathbb{R})$ near the identity has the form $\exp v \exp w$ for som $v \in \mathfrak{g}$, $w \in W_2$.

Eventually, g_i lives near 1 so has this form

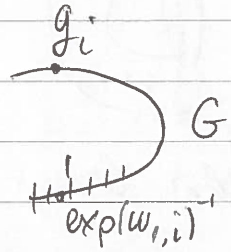
$$g_i = \exp(w_{1,i}) \exp(w_{2,i})$$
$$w_{1,i} \in \mathfrak{g} \quad , \quad w_{2,i} \in W_2$$
near 0

But by shrinking $\mathcal{U}$, we can assume $w_{1,i} \in \mathcal{U} \cap \mathfrak{g}$ and we know by hypothesis that $g_i \neq \exp v$ for any $v \in \mathcal{U} \cap \mathfrak{g}$

$$\Rightarrow \exp(w_{2,i}) \neq 1$$

So we get a sequence $w_{2,i} \in W_2 \setminus \{0\}$ s.t

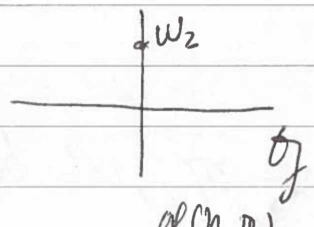
$$\exp(w_{2,i}) = \underset{\substack{\Uparrow \\ G}}{\exp(w_{1,i})^{-1}} \underset{\substack{\Uparrow \\ G}}{g_i}$$



By applying lemma ☆ we deduce that

$\exp\left(\frac{t w_{2,i}}{|w_{2,i}|}\right) \in G \;\forall t \Rightarrow w_{2,i} \in \mathfrak{g}$ by definition

But $w_{2,i} \in W_2 \setminus \{0\} \Rightarrow w_{2,i} \notin \mathfrak{g}$ ※



$\therefore \exp|_{\mathfrak{g} \cap \mathcal{U}} : \mathfrak{g} \cap \mathcal{U} \to G \cap V$ is surj. and thus gives an exponential chart for G

This gives coordinates on our group near 1. Any given 1 can be written as $\exp(v)$ for a unique $v \in \mathfrak{g}$ living near 0!

i.e. $\exists$ local log function for any matrix group.

Lie Groups and Lie Algebras 21st Jan

Remark: In fact one obtains coordinate charts near any point $g \in G$ as follows

Let B, C be an exponential chart near 1 then

$$B \longrightarrow gC$$
$$v \longrightarrow g\exp v$$

gives a parametrisation a neighbourhood of g by a ball in the lie algebra

Lie Groups and Lie Algebras 25th Jan

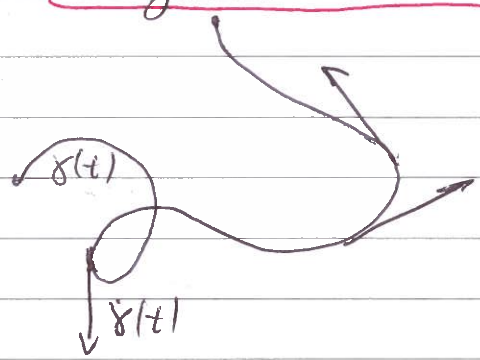
Last time: If G is a matrix group then
* $\mathfrak{g} = \{v : \exp(tv) \in G\}$ is a v. space

* $\exists$ exponential chart $0 \in B \subseteq \mathfrak{g}$, $1 \in C \subseteq G$ s.t. $\exp : B \to C$ is a diffeo.

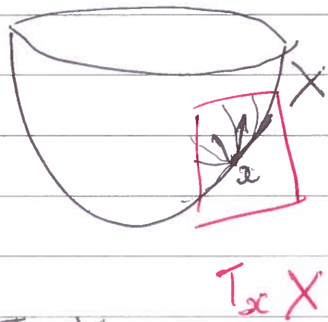
This time: WTS:
* $\mathfrak{g} = T_1 G$

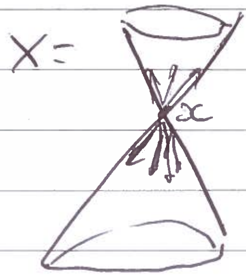
* $X, Y \in \mathfrak{g} \Rightarrow [X, Y] \in \mathfrak{g}$

Definition 1 Let $\gamma = (\gamma_1, \dots, \gamma_n) : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ is a continuously differentiable path then the tangent vector to γ is $\dot{\gamma}(t) = \left(\frac{d\gamma_1}{dt}(t), \dots, \frac{d\gamma_n}{dt}(t)\right)$



Definition 2 If $X \subseteq \mathbb{R}^n$ is a subset then the tangent cone at $x \in X$ is the set of all vectors v s.t. $v = \dot{\gamma}(0)$ for some path $\gamma : (-\varepsilon, \varepsilon) \to X$ s.t. $\gamma(0) = x$





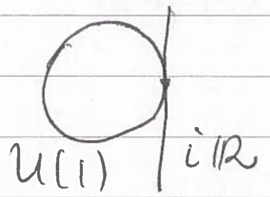
$T_x X$ = the cone X

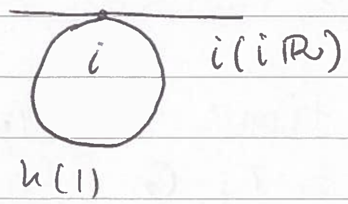
If $T_x X$ is a subspace of $\mathbb{R}^n$, we say the tangent space at x

$$G \subseteq GL(n,\mathbb{R}) \subseteq gl(n,\mathbb{R}) = \mathbb{R}^{n^2}$$

Proposition : $\mathfrak{g} = T_1 G$, Moreover $g\mathfrak{g} = T_g G$

e.g





Proof : If $v \in \mathfrak{g}$ then $gv \in T_g G$:

Need to find a path $\gamma(t)$ s.t. $\gamma(t) \in G$
$\gamma(0) = g$ and $\dot{\gamma}(0) = gv$.

Set $\gamma(t) = g\exp(tv) \in G$

$$\gamma(0) = g$$

$$\dot{\gamma}(0) = gv\exp(tv)|_{t=0} = gv$$

$$\Rightarrow \quad g\mathfrak{g} \subseteq T_g G$$

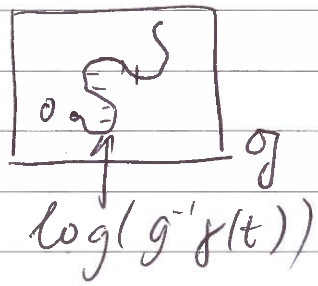
Next WTS $T_g G \subseteq g\mathfrak{g}$

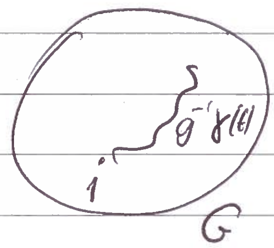
Conversely, suppose γ is a path in G
with $\gamma(0) = g$, $\dot{\gamma}(0)$ is the tangent
vector . WTS $g^{-1}\dot{\gamma}(0) \in \mathfrak{g}$
Taylor expansion

$$g^{-1}\gamma(t) = 1 + t\cdot g^{-1}\dot{\gamma}(0) + O(t^2)$$

$$\log(g^{-1}\gamma(t)) = t g^{-1}\dot{\gamma}(0) + O(t^2)$$

Lie Groups and Lie Algebras 25th Jan





Take $h_n = \log g^{-1}\gamma\left(\frac{1}{n}\right)$ $h_n \to 0$ as $n \to \infty$

$\exp(h_n) = g^{-1}\gamma\left(\frac{1}{n}\right) \in G$

$$\lim_{n\to\infty} \frac{h_n}{|h_n|} = \lim_{n\to\infty} \frac{1 + h_n - 1}{|h_n|} = \lim_{n\to\infty} \frac{1 + h_n + \frac{h_n^2}{2} + \dots - 1}{|h_n|}$$

Note $\frac{h_n^2}{2!} + \dots = O\left(\frac{1}{n^2}\right)$ so $\to 0$

$$= \lim_{n\to\infty} \frac{g^{-1}\gamma\left(\frac{1}{n}\right) - g^{-1}\gamma(0) + O\left(\frac{1}{n^2}\right)}{\frac{1}{n}\left|g^{-1}\dot{\gamma}(0)\right| + O\left(\frac{1}{n^2}\right)} =$$

$$\lim_{n\to\infty} \frac{h_n}{|h_n|} = \lim_{n\to\infty}\left(\frac{g^{-1}\gamma\left(\frac{1}{n}\right) - g^{-1}\gamma(0)}{1/n}\right)\frac{1}{\left|g^{-1}\dot{\gamma}(0)\right|} =$$

$$= \frac{g^{-1}\dot{\gamma}(0)}{\left|g^{-1}\dot{\gamma}(0)\right|}$$

$\Rightarrow$ $g^{-1}\dot{\gamma}(0) \in \mathfrak{g}$ by lemma ✱

$\Rightarrow T_g G \subseteq g\mathfrak{g}$

$\Rightarrow T_g G = g\mathfrak{g}$ ∎

Lemma: If $F: \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is continuously differentiable map so that $F(p+v) = F(p) + d_pF(v) + O(v^2)$ then let $F(p) = q$, if $d_pF: \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is surjective then $T_p F^{-1}(q) = \ker(d_pF)$

with $d_pF = \begin{pmatrix} \partial F_1/\partial x_1 & \cdots & \partial F_1/\partial x_m \\ \partial F_n/\partial x_1 & & \partial F_n/\partial x_m \end{pmatrix}$

Example: Let $H = \{v \in gl(n, \mathbb{C}) : v^+ = v\}$
Hermitian matrices

and consider the map

$$F : gl(n, \mathbb{C}) \longrightarrow H$$
$$v \longrightarrow v^+ v$$
$$(v^+v)^+ = v^+v \in H$$

$$F^{-1}(1) = \{v : v^+v = 1\} = U(n)$$

$T_1 U_n = \ker(d_1 F)$ if $d_1 F$ is surjective

$$\begin{aligned} F(1+B) &= (1+B^+)(1+B) = \\ &= 1 + B^+ + B + B^+B \\ &= F(1) + d_1 F(B) + O(B^2) \end{aligned}$$

$$\Rightarrow d_1 F(B) = B^+ + B \in H$$

Need to check this is surjective
i.e. $\forall\ C \in H$, $\exists\ B$ s.t. $d_1 F(B) = B^+ + B = C$

Take
$B = \frac{C}{2}$ then $d_1 F\left(\frac{C}{2}\right) = \left(\frac{C}{2}\right)^+ + \frac{C}{2} = C$

since $C \in H$

Lie Groups and Lie Algebras 25th Jan

$\Rightarrow T_1 U(n) = \ker(d_1 F) = \{B : B^+ + B = 0\}$
$= u(n)$ skew-Hermitian matrices

Remark: Use this when asked to compute $T_1 G$ for some G.

4.6 Lie Bracket

Lemma: If G is a matrix group with ~~Lie~~ algebra $\mathfrak{g}$ then $X, Y \in \mathfrak{g}$
$\Rightarrow [X, Y] = XY - YX \in \mathfrak{g}$

Proof: All we need to do is to write down a path $\gamma(t) \in G$ s.t. $\dot{\gamma}(0) = [X, Y]$ and $\gamma(0) = 1$

Define $C_{s,t} = \exp(sX)\exp(tY)\exp(-sX)\exp(-tY) \in G$
depends on two parameters s, t so it is not exactly a path yet.
Using Baker-Campbell-Hausdorff:
$$\exp(sX)\exp(tY) = \exp\left(sX + tY + \frac{1}{2}st[X,Y] + \right.$$
$$\left. + \frac{1}{12}\left(s^2t[X,[X,Y]] - t^2s[Y,[X,Y]]\right) + O(s^2t^2)\right)$$

$\Rightarrow$
$$C_{s,t} = \exp\left(sX + tY - sX - tY + \frac{1}{2}st[X,Y] + \frac{1}{2}st[X,Y] + \right.$$
$$\left. + (O(t) + O(s))st\right) =$$

$$= \exp\left(st[X,Y] + (O(t)+O(s))\, st\right)$$

Set $s = t = \sqrt{u}$

$$\Rightarrow \gamma(u) = C_{\sqrt{u},\sqrt{u}} = \exp\left(u[X,Y] + O(u^{3/2})\right)$$

$$\dot{\gamma}(0) = [X,Y]$$

$$\Rightarrow [X,Y] \in \mathfrak{g} \qquad \blacksquare$$

Theorem If G is a matrix group then $\mathfrak{g} = \{v : \exp(tv) \in G \ \forall t\}$ is a vector space, closed under the Lie bracket. In fact, $\mathfrak{g} = T_1 G$ and $\exp : \underset{\mathfrak{g}}{\overset{0 \in}{B}} \rightarrow \underset{G}{\overset{\ni 1}{C}}$ is a diffeo for some B, C.

5. Smooth homomorphisms

5.1 Smoothness in exponential charts

Let G_1, G_2 be matrix groups $\mathfrak{g}_1$, $\mathfrak{g}_2$ are the linear algebras

$F : G_1 \longrightarrow G_2$ homos

$$\left.\begin{aligned} \exp &: B_1 \longrightarrow C_1 \\ \exp &: B_2 \longrightarrow C_2 \end{aligned}\right\} \text{ exponential charts on } G_1 \text{ and } G_2$$

Lie Groups and Lie Algebras

Definition: Assume $F(C_1) \subseteq C_2$ (always possible after shrinking C_1, C_2)

F is smooth if $f = \log \circ F \circ \exp$ is smooth as a map between V. spaces $\mathfrak{g}_1$ & $\mathfrak{g}_2$ i.e. all partial derivatives exist.

$$\begin{array}{ccc}
\mathfrak{g}_1 \supseteq B_1 & \xrightarrow{f} & B_2 \subseteq \mathfrak{g}_2 \\
\exp \downarrow & & \uparrow \log \\
C_1 & \xrightarrow{F} & C_2 \\
\cap & & \cap \\
G_1 & & G_2
\end{array}$$

Note: $\exp(f(X)) = F(\exp X)$ if $X \in B_1$

Example: On Sheet 1, there was an example of a homo $\underset{= S^3}{SU(2)} \xrightarrow{R} \underset{= \mathbb{RP}^3}{SO(3)}$

$\underset{\underset{su(2)}{\uparrow}}{\exp(\theta M_u)}$ acts by rotation about u by 2θ

and $\exp(\theta K_u)$ rotation by θ around u

$$K_u = \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}$$

$R(\exp(\theta M_u)) = \exp(2\theta K_u)$
in an exponential chart

$r = \log \circ R \circ \exp$ has the form
$r(M_u) = 2K_u$ i.e.

$$r\begin{pmatrix} ix & y+iz \\ -y+iz & -ix \end{pmatrix} = \begin{pmatrix} 0 & -2z & 2y \\ 2z & 0 & -2x \\ -2y & 2x & 0 \end{pmatrix}$$

this is a linear map.

This is a linear map encoding a lot of information about rotations. This is a huge saving and enables us to use Linear algebra.

We will see that any smooth homomorphism of matrix group gives linear map in exponential coordinates.

5.2 One parameter subgroups

$(\mathbb{R},+)$ is realised as a matrix group

$$\left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, x \in \mathbb{R} \right\} \cong \mathbb{R}$$

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & (x+y) \\ 0 & 1 \end{pmatrix}$$

Definition A one parameter subgroup of a matrix group G is a smooth homomorphism $\phi : \mathbb{R} \to G$

Example: Pick $X \in \mathfrak{g}$ then

$t \xrightarrow{\phi} \exp(tX)$ is a homeomorphism $(\mathbb{R},+) \longrightarrow G$ i.e. it's a 1 param. subgroup

Lie Groups and Lie Algebras 25th Jan

Alternatively, define $\psi(s) = \phi(s+t)$ and $\theta(s) = \phi(s)\phi(t)$. We want to show $\psi(s) = \theta(s)$

We will show they both solve a fixed ODE with the same initial condition.

$$\frac{d\psi}{ds} = \dot{\phi}(s+t) = X.\phi(s+t) = X\psi(s).$$

and

$$\frac{d\theta}{ds} = \dot{\phi}(s)\phi(t) = X\phi(s)\phi(t) = X\theta(s)$$

Using $\frac{d}{ds}\exp(sX) = X\exp(sX)$

Set $s=0$ $\quad \dot{\psi}(s) = X\psi(s)$

$\dot{\theta}(s) = X\theta(s)$

So both solve same ODE and

$\psi(0) = \phi(t)$ and $\theta(0) = \phi(t)$

$\Rightarrow \psi(s) = \theta(s) \quad \forall s \ \forall t$

$\Rightarrow \phi(s+t) = \phi(s)\phi(t)$

$\Rightarrow \phi$ is a homomorphism

Lemma: Any one parameter subgroup $\phi : \mathbb{R} \to G$ has the form $\phi(t) = \exp(tX)$

Proof: Being a homomorphism $\Rightarrow$

$\phi(0) = 1$ and $\phi(s+t) = \phi(s)\phi(t)$

Differentiate $\phi(s+t) = \phi(s)\,\phi(t)$ w.r.t. s

$\Rightarrow \quad \dot{\phi}(s+t) = \dot{\phi}(s)\,\phi(t)$

set $s=0 \Rightarrow \dot{\phi}(t) = \underbrace{\dot{\phi}(0)}_{X}\,\phi(t)$

define $X = \dot{\phi}(0)$

Then $\phi(0) = 1$ and $\dot{\phi}(t) = X\,\phi(t)$

but $\exp(0) = 1$ and $\frac{d}{dt}(\exp tX) = X\exp(tX)$

so $\phi(t)$ and $\exp(tX)$ satisfy the same DE and same initial condition so they agree $\Rightarrow \phi(t) = \exp(tX)$ ∎

5.3 Linearity in exp chart

Theorem: If $F : G_1 \to G_2$ is a smooth homomorphism and $\mathfrak{g}_i$ Lie alg $i=1,2$ $\exp : B_i \to C_i$ are exp. charts then $f = \log \circ F \circ \exp$ is a linear map
$$f : B_1 \longrightarrow B_2$$

Proof: $\forall X \in \mathfrak{g}_1$, $\exp(tX)$ is a one parameter subgroup $\phi(:\mathbb{R} \to G_1)$ and since $F : G_1 \to G_2$ is smooth homo, then

$F \circ \phi$ is a smooth homomorphism

Lie Groups and Lie Algebras 25th Jan

i.e. $F(\exp(tX))$ is a one parameter subgroup in G_2

therefore $\exists\, Y \in \mathfrak{g}_2$ s.t. $F(\exp(tX)) = \exp(tY)$ by lemma

Since $\exp(fX) = F(\exp X)$ true $\forall X \in B_1$

and since $tX \in B_1$ for sufficiently small t. we get that

$$\exp(f(tX)) = F(\exp(tX)) = \exp(tY)$$

for small t. Take logs

$$f(tX) = tY$$

Take the derivative at 0

$$d_0 f(X) = Y$$

$$f(tX) = 0 + t d_0 f(X) + O(t^2) = tY$$

$$\Rightarrow f = d_0 f \Rightarrow f \text{ is linear} \quad \blacksquare$$

Remark: f is only defined on B_1 the domain of the exponential chart but $d_0 f$ is defined on the whole Lie algebra. i.e. on $\mathfrak{g}_1$.
We might as well define $F_* : \mathfrak{g}_1 \to \mathfrak{g}_2$ to be the linear map $d_0 f$
"induced map on lie alg." or "linearisation of F at id"

or "Pushforward map of tangent spaces".

Proposition $F(\exp X) = \exp(F_* X)$ is true $\forall X \in \mathfrak{g}_1$

Proof: We saw this in an exp. chart we need to extend to all of $\mathfrak{g}_1$

$F(\exp tX)$ is a one parameter subgroup
$\Rightarrow F(\exp tX) = \exp(tY)$ (1) for some Y
Moreover for small t
$$F(\exp tX) = \exp(tF_*(X))$$
because $tx \in B_1$

$$\therefore \exp(tY) = \exp(tF_*(X))$$

$$\Rightarrow Y = F_*(X) \text{ for small } t$$
& taking logs

(1) $\Rightarrow F(\exp X) = \exp(F_* X)$

Lie Groups and Lie Algebras 28th Jan

Just proved: If G_1, G_2 are matrix groups and $F: G_1 \longrightarrow G_2$ is a smooth homomorphism then $\exists$ a linear map $F_*: \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$ s.t.

$$\exp(F_* X) = F(\exp X) \quad *$$

Which linear maps occur this way?

Maybe F_* should preserve the Lie bracket

B.C.H Formula told us that $\exp X \exp Y$ can be expressed purely in terms of the brackets.

Section 5.4 Lie alg. homomorphisms

Definition: A homomorphism of Lie algebras is a linear map $\mathfrak{g} \xrightarrow{f} \mathfrak{h}$ s.t.
$f[X,Y] = [fX, fY] \quad \forall X, Y \in \mathfrak{g}$

Theorem: If F_* is the linearisation of A smooth homo F then F_* is a Lie algebra homo.

Proof: $\exp(tX)\exp(tY)\exp(-tX)\exp(-tY)$
$= \exp(t^2[X,Y] + O(t^3))$

Apply F: since F is homo:
$F(\exp tX) F(\exp tY) F(\exp(-tX)) F(\exp(-tY))$
$= F(\exp(t^2[X,Y] + O(t^3)))$

Using $*$ we get
$\exp(tF_*X)\exp(tF_*Y)\exp(-tF_*X)\exp(-tF_*Y) =$
$= \exp(t^2 F_*[X,Y] + O(t^3))$

By B.C.H formula tells us that
LHS $= \exp(t^2[F_* X, F_* Y] + O(t^3))$

$\Rightarrow \exp(t^2 F_*[X,Y] + O(t^3)) =$
$= \exp(t^2[F_* X, F_* Y] + O(t^3))$

$u = t^2$
$\Rightarrow \exp(u F_*[X,Y] + O(u^2)) =$
$= \exp(u[F_* X, F_* Y] + O(u^2))$

Diff w.r.t. u
$\left.\frac{d}{du}\right|_{u=0} \rightsquigarrow F_*[X,Y] = [F_* X, F_* Y]$

$\Rightarrow F_*$ is a Lie algebra homomorphism. ∎

6. Lie's Theorem

Does every Lie algebra homomorphism arise as F_* for some map $F: G_1 \Rightarrow G_2$?

Counter Example: Consider $G = U(1)$
$u(1) = i\mathbb{R}$, $[X,Y] = 0$ because 1 by 1 complex matrices commute. Equivalently $U(1)$ is abelian. So $e^{i\theta}e^{i\varphi} = e^{i(\theta+\varphi)}$
No correction term in B.C.H.
$\Rightarrow [\,,\,] = 0$

We say it is an abelian Lie Algebra. Because $[X,Y] = 0 \ \forall \ X, Y$

Lie Groups and Lie Algebras 28th Jan

So Lie algebra homos
$$u(1) \xrightarrow{f} u(1)$$
are linear maps $\mathbb{R} \to \mathbb{R}$ s.t.
$$f\underbrace{[X,Y]}_{0} = \underbrace{[fX, fY]}_{0}$$
This holds $\forall$ linear maps f.

The linear maps $\mathbb{R} \to \mathbb{R}$ are all of the form $x \to \lambda x$ for some λ

For which $\lambda \in \mathbb{R}$ does there exist a smooth homo. $U(1) \xrightarrow{F} U(1)$ s.t.
$$F_* ix = \lambda ix \ ?$$

$$\exp(F_* ix) = F(\exp ix)$$
$$\exp(i\lambda x) = F(\exp ix)$$

e.g. is $e^{ix} \to e^{ix/2}$ a well defined map

<u>No</u>, because $e^{i2\pi} \to e^{i\pi}$
$$1 \to -1$$
but $e^{i} \to e^{i0}$
$$1 \to 1$$

So $F(e^{ix}) = e^{i\lambda x}$ defines a map iff $\lambda \in \mathbb{Z}$

$\therefore$ homos $F: U(1) \to U(1)$ Are precisely $e^{i\theta} \to e^{in\theta}$, $n \in \mathbb{Z}$

This is a classification of homos $U(1) \to U(1)$

Theorem: If G_1, G_2 are path-connected matrix groups with Lie Algebras $\mathfrak{g}_1, \mathfrak{g}_2$ respectively, suppose that $\mathfrak{g}_1$ is simply connected then for every Lie algebra homos $f: \mathfrak{g}_1 \to \mathfrak{g}_2$ $\exists$ a smooth homomorphism $F: G_1 \to G_2$ with $F(\exp X) = \exp(fX)$

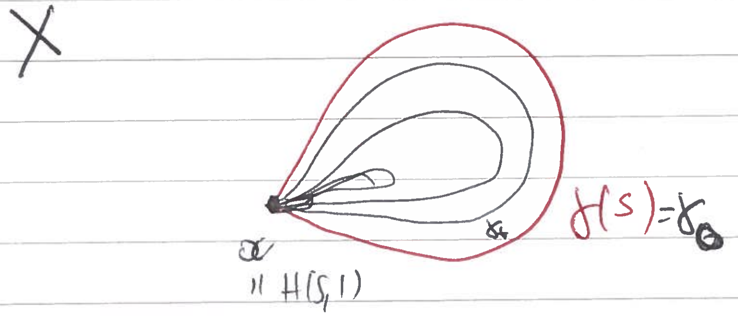
Definition: A space (matrix group) X is path connected if $\forall\, x, y \in X$ $\exists\, \gamma: [0,1] \to X$ continuous (smooth) map s.t. $\gamma(0) = x$ & $\gamma(1) = y$

Remark: The Stone-Weierstrass approx thm $\Rightarrow$ If x, y connected by continuous path then also connected by a smooth path.

Definition: A space X is simply-connected if for any continuous/smooth map $\gamma: [0,1] \to X$ with $\gamma(0) = \gamma(1) = x$ $\exists$ a null-homotopy

$$H : \underset{s}{[0,1]} \times \underset{t}{[0,1]} \longrightarrow X \quad \text{continuous.}$$

$$H(0,t) = H(1,t) = x$$
$$H(s,1) = x$$
$$H(s,0) = \gamma(s)$$



Lie Groups and Lie Algebras

28th Jan

S^n is simply connected

Proof: Any {continuous, smooth} path can be {homotoped so that it} misses a pt in S^n

Let N be this point. $N \notin \gamma([0,1])$
$S^n \setminus \{N\} \cong \mathbb{R}^n$
Now work in $\mathbb{R}^n$ and set $H(s,t) = (1-t)\gamma(s)$ ∎

$GL^+(n,\mathbb{R}) \simeq SO(n)$
$\pi_1 SO(n) = \mathbb{Z}/2$ so not simply connected

$SL(n,\mathbb{R}) \simeq SO(n)$ not simply connected

$U(n)$ not simply connected

$$\begin{array}{cc} U(n) & \pi_1(U(n)) = \mathbb{Z} \\ \big\downarrow \det & \big\downarrow \det \\ U(1) & \pi_1(U(1)) = \mathbb{Z} \end{array}$$

$SU(n)$ is simply connected.
$SU(2) \cong S^3$ is simply connected.

$$\begin{array}{ccc} SU(n-1) & \hookrightarrow & SU(n) \\ & & \big\downarrow \\ & & S^{2n-1} \end{array}$$

We can show $SU(n)$ is simply connected by induction using fibre-bundles structure

For any (reasonably nice) space X, there exists a space $\tilde{X}$ and map $\tilde{X} \to X$ called a covering map s.t. $\tilde{X}$ is simply connected.
universal cover.
If X is a matrix group then $\tilde{X}$ is a Lie group.

We saw on Sheet 1 a homo 2 − 1 from $SU(2)$ to $SO(3)$. $SU(2)$ is simply connected and $SO(3)$ isn't $\pi_1(SO(3)) \cong \mathbb{Z}/2$

Sheet 4 is a guided proof of Lie's thm.

Theorem (Lie)

- $\forall$ Lie Alg $\mathfrak{g}$ $\exists!$ simply connected Lie group G with $Lie(G) = \mathfrak{g}$.

- If $\mathfrak{h} \subseteq \mathfrak{g}$ is Lie subalg. then $\exists$ $H \subseteq G$ is a Lie subgroup.

- If $f : \mathfrak{g} \to \mathfrak{h}$ is a homo of Lie alg. then $\exists!$ smooth homo of Lie groups (simply connected) $G \xrightarrow{F} H$ with $F_* = f$

1st Feb

7. Representation theory of matrix groups and their Lie algebras

Definition: A representation now means a smooth homomorphism $G \longrightarrow GL(V)$ where V is a vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$

Examples: • Zero representation $V = 0$, $GL(V) = \{1\}$

• Trivial representation: given any v.s. V, the trivial rep of G on V is
$R : G \to GL(V)$, $R(g) = 1$ $\forall g \in G$

Lie Groups and Lie Algebras 1st Feb

- $G = U(n)$ "standard representation"
  $V = \mathbb{C}^n$
  $U(n) \xrightarrow{\text{inclusion}} GL(n, \mathbb{C})$ (by def)
  is a rep of $U(n)$ on $\mathbb{C}^n$

- $G = SO(n)$ $SO(n) \hookrightarrow GL(n, \mathbb{R})$
  $V = \mathbb{R}^n$ standard rep.

- Adjoint representation : For any matrix group G, let $V = \mathfrak{g}$ and define the adjoint rep as follows

$$G \xrightarrow{Ad} GL(\mathfrak{g})$$
$$g \longrightarrow Ad_g \text{, where } Ad_g : \mathfrak{g} \longrightarrow \mathfrak{g}$$
$$v \longrightarrow gvg^{-1}$$

i.e. $\boxed{Ad_g(v) = gvg^{-1}}$

E.g. if G is abelian, then $gvg^{-1} = v$ because g and v commute.
In this case the adjoint rep $\cong$ trivial rep.

Subexample : $G = SU(2)$, $\mathfrak{g} = su(2) \cong \mathbb{R}^3$.
In sheet 1, we saw that
$Ad : SU(2) \longrightarrow GL(su(2)) = GL(3, \mathbb{R})$
actually lands in the subgroup of rotations $SO(3) \subseteq GL(3, \mathbb{R})$.
Explicitly we saw : $\exp(\theta M_u) = \exp(2\theta K_u)$
where $M_u = \begin{pmatrix} ix & y+iz \\ -y+iz & -ix \end{pmatrix}$, $K_u = \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}$

$u = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Definition: Let $R_1 : G \to GL(V_1)$ be
$R_2 : G \to GL(V_2)$
two reps over $\mathbb{K}$ of G a **morphism of representations** is a $\mathbb{K}$-linear map $L : V_1 \to V_2$ s.t. $L(R_1(g)v) = R_2(g)Lv$ $\forall g \in G$, $v \in V_1$

Remark we can think of it as $L R_1 L^{-1} = L$, this holds if L is invertible, however this doesn't need to be the case

If L is invertible in addition then we say it is an **isomorphism of reps**. Another word is "intertwiner".

Another way of representing it is

$$\begin{tikzcd} V_1 \arrow[r, "L"] \arrow[d, "R_1(g)"'] & V_2 \arrow[d, "R_2(g)"] \\ V_1 \arrow[r, "L"] & V_2 \end{tikzcd}$$ commutes $\forall g$

Example: Let V be the standard rep $\overset{R_1}{\vee}$ $\mathbb{R}^3$ of $SO(3)$, let W be the adjoint rep $\underset{R_2}{}$ $so(3)$ of $SO(3)$

$so(3) = \{$antisymmetric 3×3 matrices$\}$ i.e.

$$K_v = \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}, \quad v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

So adjoint rep is $\mathbb{R}$ 3-dim

Lie Groups and Lie Algebras 1st Feb

The map $L(v) = K_v$ is a morphism of representations.

WTS

$$L R_1(g) v = R_2(g) L v = Ad_g(Lv) =$$
$$= Ad_g(K_v) =$$
$$= g K_v g^{-1}$$

$$L R_1(g) v = L(gv) = K_{gv}$$

To show $K_{gv} = g K_v g^{-1}$

Let $g = \exp(\theta K_u)$, $\theta \in \mathbb{R}$, $|u| = 1$

Compute $\exp(\theta K_u) K_v \exp(-\theta K_u) = K_{\exp(\theta K_u) v}$

$\exp(\theta K_u) = 1 + K_u \sin\theta + (1-\cos\theta) K_u^2$ (sheet 1)

$[K_u, K_v] = K_{u \times v}$

$K_u K_v K_u = -(u \cdot v) K_u$

$$\exp(\theta K_u) K_v \exp(-\theta K_u) = K_v \cos\theta + K_{u\times v} \sin\theta + (1-\cos\theta)\, u \cdot v\, K_u = K_{\exp(\theta K_u) v}$$

$\Rightarrow L$ is a morphism of reps. Furthermore it is an iso since L is invertible.

7.2. Subrepresentations, irreducibility

Suppose $R: G \to GL(V)$ is a representation

Definition: A subrepresentation is a subspace $W \subseteq V$ s.t. $R(g)w \in W$, $\forall w \in W, g \in G$

Let's write $\mathrm{Rest}_W R : G \to GL(W)$ for

$$(\mathrm{Rest}_W R)g = R(g)|_W : W \to W$$

If we pick a basis for W extend to a basis for V, then in terms of this basis

$$R(g) = \begin{array}{c|cc} & W & W' \\ \hline W & \mathrm{Rest}_W R(g) & ? \\ W' & 0 & ? \end{array}$$

Example: $U(n)$ acts on $gl(n,\mathbb{C})$ by conjugation, this gives a rep of $U(n)$

$$R: U(n) \to GL(gl(n,\mathbb{C}))$$
$$R(g)m = gmg^{-1}$$

If $m \in u(n)$ (i.e. $m^+ = -m$) then

$$(gmg^{-1})^+ = (g^{-1})^+ m^+ g^+ = -gmg^{-1} \Rightarrow gmg^{-1} \in u(n)$$

since $g \in U(n)$: $g^{-1} = g^+$

$\Rightarrow u(n) \subseteq gl(n,\mathbb{C})$ is a subrepresentation.

Lie Groups and Lie Algebras 1^{st} Feb

Here's another : $su(n) \subseteq u(n)$, $su(n) = \{m : m^{\dagger} = -m, Tr(m) = 0\}$

let $m \in su(n)$ $\quad Tr(gmg^{-1}) = Tr(g^{-1}gm) = Tr(m)$

$\Rightarrow$ $su(n)$ is also a subrepresentation of $Ad_g : u(n) \longrightarrow GL(\underset{u(n)}{\underset{\shortparallel}{gl(n, \mathbb{C})}})$

Definition : If $R : G \longrightarrow GL(V)$ is a representation, then there are two obvious reps :

* $0 \subseteq V$ zero rep and
* $V \subseteq V$

A rep is called irreducible if there are no other subreps. i.e. no proper subreps.

7.3. New reps from old

Direct sums : Given two reps $R_1 : G \to GL(V_1)$ and $R_2 : G \longrightarrow GL(V_2)$ we define the direct sum to be the following rep on the v.s. $V = V_1 \oplus V_2$

Remember $v \in V_1 \oplus V_2$ $\quad v = (v_1, v_2)$ with $v_i \in V_i$

$$(R_1 \oplus R_2)(g)\,(v_1, v_2) = (R_1(g)v_1, R_2(g)v_2)$$

In other words, as matrices

$$(R_1 \oplus R_2)(g) = \left(\begin{array}{c|c} R_1(g) & 0 \\ \hline 0 & R_2(g) \end{array}\right)$$

This is never irreducible unless $V_i = 0$ because V_1, V_2 are proper subreps.

Question: if $R : G \to GL(V)$ is now irreducible rep, is it $\cong R_1 \oplus R_2$ for some R_1, R_2?

Example: Take $G = (\mathbb{R}, +)$

$$\mathbb{R} \longrightarrow GL(2, \mathbb{R})$$
$$x \longrightarrow \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

Note that vectors $\begin{pmatrix} a \\ 0 \end{pmatrix}$ are fixed

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix}$$

So this subspace is a subrep. There is no complementary subrep.
Suppose there were, spanned by $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ with $b_2 \neq 0$
As this is supposed to be subrep

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \lambda \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \quad \text{for some } \lambda(x)$$

$$\left| \begin{array}{l} b_1 + x b_2 = \lambda b_1 \\ b_2 = \lambda b_2 \end{array} \right. \Rightarrow \lambda(x) = 1$$

$$\Rightarrow b_1 + b_2 x = b_1 \qquad \forall x$$

$$\Rightarrow b_2 = 0 \quad \text{※}$$

Remark: Will see if G is compact then every

Lie Groups and Lie Algebras 1^{st} Feb

rep. decomposes as a direct sum of irreducible reps called "complete reducibility"

Compact $\Leftrightarrow$ if $G = \bigcup_{i \in I} U_i$, U_i open, then

$\exists$ finite subset $J \subseteq I$ s.t. $G = \bigcup_{j \in J} U_j$

$\Leftrightarrow$ Every bdd seq. has a convergent subseq.

$\Leftrightarrow$ G closed and bdd set.

because G is a matrix groups

e.g. $SL(2, \mathbb{R})$ is unbdd since $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \to \infty$ as $\lambda \to \infty$ i.e. not compact

or $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \to \infty$ as $x \to \infty$

e.g. $SU(2)$ is bdd (it's a 3-sphere)

i.e. compact.

Unitarity

Recall that a Hermitian inner product on a $\mathbb{C}$ v.s is a map $\langle , \rangle : V \times V \to \mathbb{C}$

s.t. $\langle \mu u + \lambda v, w \rangle = \bar{\mu} \langle u, w \rangle + \bar{\lambda} \langle v, w \rangle$

and $\langle u, \mu v + \lambda w \rangle = \mu \langle u, v \rangle + \lambda \langle u, w \rangle$

and $\overline{\langle w, v \rangle} = \langle v, w \rangle$, $\langle v, v \rangle \geq 0$

with equality iff $v = 0$.

e.g. $<v,w> = \sum \bar{v}_i w_i$, $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$, $w = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$

If $g \in GL(n, \mathbb{C})$ s.t. $<gv, gw> = <v,w> \ \forall v,w$

then $g \in U(n)$ and conversely.
unitary group

Definition: A unitary rep of a group G is a rep R on V s.t. $\exists$ an invariant Hermitian inner product i.e. $\exists <,> : V \times V \to \mathbb{C}$ s.t. $< R(g)v, R(g)w > = <v, w>$

$\Leftrightarrow$ $R : G \to GL(V)$ lands in the unitary group for this particular inner product.

Each Hermitian inner product is preserved by a subgroup $U_{<,>}$ of $GL(n, \mathbb{C})$ but because I can change coordinates with an element of $GL(n, \mathbb{C})$ to turn a given inner product into the standard one, so these are all conjugates

i.e. it doesn't matter which inner product we use we'll always just write $U(n)$ for it's unitary group.

Lie Groups and Lie Algebras 1^{st} Feb

Lemma: For any $\mathbb{C}$-rep of a finite group $\exists$ invariant Hermitian inner product $\therefore$ the rep is unitary.

Proof: Let $\langle , \rangle'$ be any Hermitian inner product (not necessarily invariant)

$$\langle v, w \rangle = \frac{1}{|G|} \sum_{g \in G} \langle R(g) v, R(g) w \rangle' \text{ is}$$

invariant

Why? $\langle R(h)v, R(h)w \rangle = \frac{1}{|G|} \sum_{g \in G} \langle R(g)R(h)v, R_g R(h) w\rangle' =$

$$= \frac{1}{|G|} \sum_{g \in G} \langle R(gh) v, R(gh) w \rangle' \quad \text{Relable } gh = k$$

$$= \frac{1}{|G|} \sum_{k \in G} \langle R(k) v, R(k) w \rangle' = \langle v, w \rangle$$

■

Proposition (Weyl unitarian trick)

$\forall$ $\mathbb{C}$-rep $R: G \longrightarrow GL(V)$ of a compact group $\exists$ invariant Hermition inner product.

Proof: Let $\langle , \rangle'$ be any Hermitian inner product then

$$\langle v, w \rangle = \frac{\int_G \langle R(g) v, R(g) w \rangle dg}{\int_G dg}$$

The same argument works provided we can define

$$\int_G f(g)\, dg \text{ for } f: G \to \mathbb{C}$$

For invariance proof, we need

$$\int_G f(gh)\, dg = \int f(g)\, dg \qquad (*)$$

set $gh = k \Rightarrow \int_G f(gh)\, dg = \int_G f(k)\, (d k h)$

So for $(*)$ to hold we write the measure of integration / volume form to be invariant. Such an integral exists, called Haar integral if G is compact and $\int_G dg < \infty$.

For $G = S^1 = U(1)$

$$\int_0^{2\pi} f(\theta)\, d\theta$$

Need to check : $\theta \to \theta + \varphi$ for some φ

change of coords using group act

$d(\theta+\varphi) = d\theta$ as φ is just a const.

$$\int f(\theta+\varphi)\, d\theta = \int f(\theta)\, d\theta \quad \forall \varphi.$$

Lie Groups and Lie Algebras 1^{st} Feb

Remarks: If $G = SU(2) \cong S^3$ the standard volume form on S^3 is a Haar measure.

Theorem: Any rep of a compact group is completely reducible i.e. if V is such a rep of a compact group and $W \subseteq V$ is a subrep. then $\exists$ $W' \subseteq V$ s.t W' is a subrep and $V = W \oplus W'$.

Proof: We know there exists invariant Hermitian inner product. $\langle \cdot , \cdot \rangle$
i.e. $\langle R(g)v, R(g)w \rangle = \langle v, w \rangle$

So take W' to be $W' = \{ v \in V \text{ s.t. } \langle v, w \rangle = 0 \ \forall w \in W \}$

This satisfies $V = W \oplus W'$ moreover, if $v \in W'$ then $\langle R(g)v, w \rangle$ $\overset{*}{=}$
$= \langle R(g^{-1})R(g)v, R(g^{-1})w \rangle =$

$= \langle v, \underbrace{R(g^{-1})w}_{\in W} \rangle = 0$ since $\langle v, w \rangle = 0$ $v^* \in W'$
as w is a subrep

$\Rightarrow R(g)v \in W'$ ∎

Corollary: Any Greps a compact G splits as $V_1 \oplus \dots \oplus V_N$ where each V_i is irreducible

Proof: By induction using the thm

Duals : If V is a finite-dimensional vector space over a field $\mathbb{K}$ then V^* is the v-space of $\mathbb{K}$-linear maps $f : V \xrightarrow{f} \mathbb{K}$ (same dim as V)

$(V^{**} \cong V)$

Suppose $R : G \to GL(V)$ is a rep. How do we get a rep: $R^* : G \to GL(V^*)$

Set $R^*(g) \underset{\in V^*}{f} \in V^*$

$$(R^*(g)f)(v) = f(R(g^{-1})v)$$

Check R^* is a rep.

$$(R^*(g)(R^*(h)f))v = (R^*(h)f)(R(g^{-1})v) =$$
$$= f(R(h^{-1})R(g^{-1})v) =$$
$$= f(R(h^{-1}g^{-1})v) =$$
$$= f(R((gh)^{-1})v) =$$
$$= (R^*(gh)f)v$$

Let's pick a basis so V = column vectors
V^* = row vectors

$$V^* \times V \longrightarrow K$$
$$(f, v) \longrightarrow f(v) \qquad (f_1, \dots, f_n)\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \sum_i f_i v_i$$

Lie Groups and Lie Alg. 1^{st} Feb

$$\begin{pmatrix} & & \\ & M & \\ & & \end{pmatrix}_{n \times n} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \quad \text{representation}$$

$$(f_1, \dots, f_n) \begin{pmatrix} & & \\ & M & \\ & & \end{pmatrix}^{-1} \quad \text{dual rep.}$$

$$f N^{-1} M^{-1} = f(MN)^{-1}$$

4^{th} Feb

Given a rep $R: G \to GL(V)$ we constructed
$R^*: G \to GL(V^*)$
$$R^*(g) f = f \circ R(g^{-1})$$

Think of V^* as row vectors and the dual rep is matrices acting on the right.

Tensor Product

Given two vector spaces V, W over $\mathbb{K}$ form $V \otimes W$ as follows. Let $e_1, \dots, e_m$ be a basis for V let $f_1, \dots, f_n$ be a basis for W. Then $e_i \otimes f_i$ (symbols) form a basis for $V \otimes W$. So $\dim(V \otimes W) = m \times n = \dim V \cdot \dim W$

<u>Lemma</u>: The bilinear map $\varphi: V \times W \to V \otimes W$
$\varphi(\sum v_i e_i, \sum w_j f_j) = \sum (v_i w_j)(e_i \otimes f_j)$
has the following universal property

for any bilinear map $h: V \times W \longrightarrow X$
$\exists!\ h': V \otimes W \longrightarrow X$ s.t. $h = h' \circ \psi$

$$\begin{tikzcd} V \times W \arrow[r, "\psi"] \arrow[dr, "h"'] & V \otimes W \arrow[d, dashed, "h'"] \\ & X \end{tikzcd}$$

Proof: Let $\langle g_k \rangle$ be a basis for X
$h(e_i, f_j) = \sum A_{ijk} g_k$ for some

uniquely defined A_{ijk}

Then $h(\sum v_i e_i, \sum w_j f_j) \overset{\text{Bilinearity}}{=}$

$$\sum v_i w_j h(e_i, f_j) = \sum_{i,j,k} v_i w_j A_{ijk} g_k$$

define $h'(e_i \otimes f_j)$ to be $A_{ijk} g_k$

Then $h = h' \circ \psi$ ∎

To see uniqueness note that any bilinear map h'' satisfies

$$h''(\sum v_i e_i, \sum w_j f_j) = \sum B_{ijk} v_i w_j g_k \text{ for}$$

some coeff. B_{ijk}

Because if h'' is bilinear then it is determined by $h''(e_i, f_j) = \sum B_{ijk} g_k$

Lie Groups and Lie Algebras 4th Feb

Corollary: Suppose there were another v.s. M and map $m: V \times W \longrightarrow M$ with this universal property. Then $M \cong V \otimes W$ canonically

Proof:

$$\begin{tikzcd}
V \times W \arrow[r, "\psi"] \arrow[dr, "m"] \arrow[ddr, "\psi"'] & V \otimes W \arrow[d, dashed, "\exists!\, m'"] \\
& M \arrow[d, dashed, "\exists!\, \psi'"] \\
& V \otimes W
\end{tikzcd}$$

By Univ. Prop. of ψ

by Univ. Prop of m

$$\psi = \psi' \circ m' \circ \psi$$

But the previous lemma told us there is unique factorisation of any bilinear map has $h' \circ \psi$ and ψ is bilinear $\therefore$ $\psi = \psi' \circ m' \circ \psi$ But so is

$$\psi = id \circ \psi$$

$\Rightarrow$ $\psi' \circ m' = id$ & $m' \circ \psi' = id$

$\Rightarrow$ ψ' and m' are inverses and hence iso. ∎

$\Rightarrow$ No dependence up to isomorphism on the basis.

Definition

If $R_1 : G \longrightarrow GL(V_1)$
$R_2 : G \longrightarrow GL(V_2)$

are reps then define $R_1 \otimes R_2 : G \to GL(V_1 \otimes V_2)$ by

$$\boxed{(R_1 \otimes R_2)(g)(v_1 \otimes v_2) = (R_1(g) v_1) \otimes (R_2(g) v_2)}$$

Warning! Not every element of $V \otimes W$ has the form $a \otimes b$, $a \in V$, $b \in W$

e.g. $\mathbb{R}^4 = V = W$

$$V \otimes W \ni e_1 \otimes e_2 + e_3 \otimes e_4$$

This is not of the form $a \otimes b$ for any a, b

Tensors of the form $a \otimes b$ are called **pure tensors**

e.g: $(e_1 + 2e_2) \otimes (3e_3) = 3e_1 \otimes e_3 + 6e_2 \otimes e_3$
This is pure tensor because it factorises as $a \otimes b$

Pure tensors form a subvariety of $V \otimes W$, cut out by "Plucker relations"

Because $(R_1 \otimes R_2)(g)$ is a linear map, I only need to define it on a basis and $e_i \otimes f_j$ is (by def.) a basis consisting of pure tensors.
So often, I will define/prove things only on pure tensors, and extend to other tensors by linearity.

Hom spaces: If V, W are v.s., R and S are G-reps on V, W then

$$\mathrm{Hom}(V, W) = \{f : V \to W \text{ linear}\}$$
$$\cong$$
$$V^* \otimes W$$

$\eta = e_i^* \otimes f_j$, $\quad \eta(e_i) = f_j$

corresponds to
$$\begin{array}{cc} & \quad j^{th} \\ i^{th} & \begin{pmatrix} 0 & \cdots & 0 \\ & 1 & \\ 0 & \cdots & 0 \end{pmatrix} \end{array}$$

Lie Groups and Lie Algebras $\quad 4^{th}$ Feb

So $T = R^* \otimes S : G \to GL(\mathrm{Hom}(V,W))$
$\qquad\qquad\qquad\qquad\qquad\qquad \ni T(g)$

$$\boxed{(T(g)\underset{\in \mathrm{Hom}(V,W)}{F})(\underset{v}{v}) = S(g)(F(R(g^{-1})v))}$$

$$\begin{array}{ccc} V & \xrightarrow{F} & W \\ {\scriptstyle R(g)}\downarrow & & \downarrow{\scriptstyle S(g)} \\ V & \longrightarrow & W \end{array}$$

$S(g) \circ F \circ R(g^{-1})$
$\quad =$
$T(g)F$

Symmetric powers $\qquad$ char $\mathbb{K} = 0$

Definition: Consider the action of S_n on $V^{\otimes n} = \underbrace{V \otimes V \otimes \cdots \otimes V}_{n \text{ times}}$ Given by $\sigma \in S_n$

acting by $\sigma(v_1 \otimes \cdots \otimes v_n) = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$

e.g $\sigma = (12) \qquad V^{\otimes 2} \quad \sigma(v \otimes w) = w \otimes v$

$\sigma(e_1 \otimes e_1 \otimes e_2) = e_1 \otimes e_1 \otimes e_2$

just swaps the first two vectors

Define $\mathrm{Sym}^n V$ to be the subspace of V

$\mathrm{Sym}^n(V) = \{ v \in V^{\otimes n} \text{ s.t. } \sigma(v) = v \ \forall \sigma \}$

Remark $\quad e_1 \otimes e_1 \otimes e_2 \notin \mathrm{Sym}^n(V)$ since $\tau = (13)$

$\tau(e_1 \otimes e_1 \otimes e_2) \neq e_1 \otimes e_1 \otimes e_2$

Define the averaging map $Av: V^{\otimes n} \to V^{\otimes n}$ by $v \longrightarrow Av(v) \stackrel{\Delta}{=} \frac{1}{n!} \sum \sigma(v)$

<u>Claim</u> : $Av(v) \in Sym^n(V)$

If $v \in Sym^n V$ then $Av(v) = v$, because $\sigma(v) = v \ \forall \sigma \Rightarrow Av(v) = \frac{1}{n!} \sum_{\sigma \in S_n} v = v$

$\Rightarrow Av: V^{\otimes n} \longrightarrow Sym^n V$ is surj

<u>Proof of Claim</u> : $\tau(Av(v)) = \tau\left(\frac{1}{n!} \sum \sigma(v)\right) =$

$$= \frac{1}{n!} \sum_{\sigma} \underbrace{\tau\sigma}_{\gamma}(v) =$$

$$= \frac{1}{n!} \sum_{\sigma} \gamma(v) = Av(v) \quad \blacksquare$$

<u>Proposition</u> : $Av: V^{\otimes n} \longrightarrow V^{\otimes n}$ is a morphism of reps.

<u>Remark</u> : On sheet 5 we'll prove that Im & Ker of a morphism of reps are subreps

$\Rightarrow Sym^n V \subseteq V^{\otimes n}$ is subrep. because $Sym^n V = Im(Av)$.

<u>Proof of Proposition</u> :

WTS

$$Av(R^{\otimes n}(g)\, v) = R^{\otimes n}(g)\, Av(v)$$

Assume $v = v_1 \otimes \cdots \otimes v_n$ which is pure wlog.

Lie Groups and Lie Algebras $\quad 4^{th}$ Feb.

$$R^{\otimes n}(g)(v_1 \otimes \dots \otimes v_n) =$$

$$= R(g)v_1 \otimes \dots \otimes R(g)v_n$$

$$Av(R^{\otimes n}(g)(v_1 \otimes \dots \otimes v_n)) =$$

$$= \frac{1}{n!} \sum_{\sigma \in S_n} R(g)v_{\sigma(1)} \otimes \dots \otimes R(g)v_{\sigma(n)} =$$

$$= R^{\otimes n}(g)\left(\frac{1}{n!} \sum_{\sigma \in S_n} v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(n)}\right) =$$

$$= R^{\otimes n}(g)(Av(v_1 \otimes \dots \otimes v_n)) \quad \blacksquare$$

This defines the symmetric power $Sym^n V$ of a rep V.

Write $v_1 v_2 \dots v_n$ for $Av(v_1 \otimes \dots \otimes v_n)$ and can think of $Sym^n V$ as homogeneous polyn. of deg n.

e.g. $V = \mathbb{R}^2$ $\quad x, y$ basis of V^*

$$\left.\begin{matrix} x^2 \\ xy \\ y^2 \end{matrix}\right\} \text{ form a basis of } Sym^2(V^*)$$

Now they're really poly functions on V.

8th Feb

Last time we defined $Sym^n V$ for a G-rep V as the subrep of $V^{\otimes n}$ consisting of the tensors in the image of the averaging map $Av(v_1 \otimes v_2 \otimes \dots \otimes v_n) =$
$$= \frac{1}{n!} \sum_{\sigma \in S_n} v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(n)}$$

Also think of $v \in Sym^n V$ as fixed vectors of Av, i.e. $Av(v) = v$

Exterior powers

Given a G-rep V, define the alternating map
$$Alt : V^{\otimes n} \longrightarrow V^{\otimes n}$$
$$\boxed{Alt(v_1 \otimes \dots \otimes v_n) = \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^{\sigma} v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(n)}}$$

where $(-1)^{\sigma}$ denotes the sign of $\sigma \in S_n$, i.e. $+1$ if even permutation & -1 if odd permutation.

Define $\Lambda^n V = Image(Alt)$

e.g. $e_1 \otimes e_1$ in $\mathbb{R}^2$

$$Alt(e_1 \otimes e_1) = \frac{1}{2}(e_1 \otimes e_1 - e_1 \otimes e_1) = 0$$

$$Alt(e_1 \otimes e_2) = \frac{1}{2}(e_1 \otimes e_2 - e_2 \otimes e_1)$$

Lie Groups and Lie Algebras 8th Feb-

Tensors in $\Lambda^n V$ are called n-forms. They switch sign if you apply an odd permutation to all factors

$\underline{Alt}(e_1 \otimes e_2 \otimes e_3) = \frac{1}{6}(e_1 \otimes e_2 \otimes e_3 + e_2 \otimes e_3 \otimes e_1 +$

$+ e_3 \otimes e_1 \otimes e_2 - e_1 \otimes e_3 \otimes e_2 -$

$- e_2 \otimes e_1 \otimes e_3 - e_3 \otimes e_2 \otimes e_1)$

$\Lambda^n V$ is a rep, in fact a subrep of $V^{\otimes n}$ because it's the image of Alt and Alt is a morphism of reps. The proof that Alt is a morphism is the same as for Av but with some signs $(-1)^\sigma$ everywhere.

If $\dim V = r$, what is $\dim \Lambda^k V = \binom{r}{k}$?

If $e_1, \dots, e_r$ is a basis for V. Pick k of them and consider $e_{i_1} \wedge \dots \wedge e_{i_k} =$
$= Alt(e_{i_1} \otimes \dots \otimes e_{i_k})$

Note if $e_{i_a} = e_{i_b}$ then $e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_k} = 0$

If $e_{i_1} \wedge \dots \wedge e_{i_k}$ and $e_{j_1} \wedge \dots \wedge e_{j_k}$ are reorderings

then let σ be the permutation $i_1 \Rightarrow j_1$, $i_k \rightarrow j_k$ then $e_{i_1} \wedge \dots \wedge e_{i_k} = (-1)^\sigma e_{j_1} \wedge \dots \wedge e_{j_k}$

These are linearly dependent and

for picking a basis we can assure $i_1 < i_2 < \cdots < i_k$

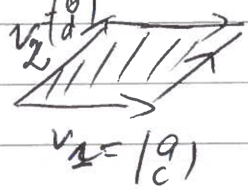
So e.g $\Lambda^2 \mathbb{R}^3$ has a basis
$e_1 \wedge e_2,\ e_2 \wedge e_3,\ e_1 \wedge e_3$

So we see $\dim \Lambda^k V = \binom{r}{k}$. In particular
$\dim \Lambda^r V = 1$ spanned by $e_1 \wedge \cdots \wedge e_r$
'' volume form.

Given n vectors in $\mathbb{R}^n$ $v_1, \dots, v_n$

$v_1 \wedge \cdots \wedge v_n$ is an element of $\Lambda^n \mathbb{R}^n$ which is 1-dim, it is a multiple of $e_1 \wedge \cdots \wedge e_n$, where $e_1, \dots, e_n$ are standard basis.
It turns out that $v_1 \wedge \cdots \wedge v_n = \det(v_1 | v_2 | \dots | v_n |)$
$e_1 \wedge e_2 \cdots \wedge e_n$

$\det(v_1 | v_2 | \dots | v_n |) =$ volume of parallelopiped spanned by $v_1, \dots, v_n$



Area $= \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ That's why it's called a volume form.

8. Reps of Lie algebras

Definition: Let $\mathbb{K}$ be a field. A rep of a Lie algebra $\mathfrak{g}$ is a Lie algebra homo. $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ for some $\mathbb{K}$ v.s. V.

Lie Groups and Lie Algebras 8^{th} Feb

Note: You can have $\mathbb{C}$-rep of $\mathbb{R}$-Lie alg. because $\mathfrak{gl}(n,\mathbb{C}) \subseteq \mathfrak{gl}(2n,\mathbb{R})$ is also a real lie alg.

For a Lie alg. homo we require $\rho[X,Y] = [\rho(X), \rho(Y)]$ and ρ is a linear map

e.g. $\mathfrak{su}(n) \subseteq \mathfrak{gl}(n,\mathbb{C})$ so the inclusion map is a rep "standard rep"
$\mathfrak{so}(n) \subseteq \mathfrak{gl}(n,\mathbb{R})$

e.g The adjoint rep of a Lie alg. $\mathfrak{g}$ is $ad : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$
$$X \longmapsto ad_X$$

$$ad_X Y = [X, Y]$$

The condition that ad is a rep.
$ad_{[X,Y]} Z = [ad_X, ad_Y] Z$ equivalent to

$ad_{[X,Y]} Z = ad_X ad_Y Z - ad_Y ad_X Z$ this is called the Jacobi identity.

How is this related to $Ad : G \to GL(\mathfrak{g})$?

$Ad_g Y = gYg^{-1}$ and

$ad_X Y = [X, Y]$

Let $g = \exp(tX)$ and consider

$$Ad_{\exp(tX)} Y = \exp(tX) Y \exp(-tX)$$

diff. w.r.t. t

$$\frac{d}{dt}\Big|_{t=0}\left(Ad_{\exp(tX)} Y\right) = \frac{d}{dt}\Big|_{t=0} \exp(tX)\, Y \exp(-tX) =$$

$$= XY - YX =$$

$$= [X, Y] = ad_X Y$$

We saw earlier that for every smooth rep $F: G \to GL(V)$. There is a Lie algebra rep $F_*: \mathfrak{g} \to \mathfrak{gl}(V)$ s.t.

$$F(\exp X) = \exp(F_* X)$$

So if $F = Ad$, $F_* = ad$

What are the Lie alg. reps corresponding to 1) direct sum, 2) dual, 3) tensor product, 4) symmetric powers and 5) external powers?

1. If R_1, R_2 are reps of G on V_1, V_2 and ρ_1, ρ_2 are $(R_1)_*, (R_2)_*$ respectively

then (recall

$$(R_1 \oplus R_2)(\exp(tX))(v_1 \oplus v_2) =$$
$$= (R_1(\exp tX)\, v_1) \oplus (R_2(\exp tX)\, v_2) \Rightarrow$$

$\frac{d}{dt}\Big|_{t=0}$ gives $(\rho_1 \oplus \rho_2)(X)(v_1 \oplus v_2) = (\rho_1(X) v_1) \oplus (\rho_2(X) v_2)$

Lie Groups and Lie Algebras 8^{th} Feb

So we define $\rho_1 \oplus \rho_2 : \mathfrak{g} \to \mathfrak{gl}(V_1 \oplus V_2)$
by $(\rho_1 \oplus \rho_2)(X)(v_1 \oplus v_2) = (\rho_1(X) v_1) \oplus (\rho_2(X) v_2)$

2. Duals $R : G \to GL(V)$
$$(R^*(g) f) v = f(R(g^{-1}) v)$$

Now let $g = \exp tX$

$$\Rightarrow (R^*(\exp tX) f) v = f(R(\exp(-tX)) v)$$

$\left.\frac{d}{dt}\right|_{t=0}$: $(\rho^*(X) f) v = f(-\rho(X) v) =$
$= -f(\rho(X) v)$

Since $\left.\frac{d}{dt}\right|_{t=0} R(\exp tX) = \rho(X)$

and f is a linear map with const. coeff $\frac{d}{dt} f(s(t)) = f\left(\frac{d}{dt} s(t)\right)$

3. Tensor products

$(R_1 \otimes R_2)(\exp tX)(v_1 \otimes v_2) =$
$= (R_1(\exp(tX)) v_1) \otimes (R_2(\exp tX) v_2)$

$\left.\frac{d}{dt}\right|_{t=0} R_1(\exp tX) = \rho_1(X)$ & $\left.\frac{d}{dt}\right|_{t=0} R_2(\exp tX) = \rho_2(X)$

Use Liebnitz Rule to differentiate

$\left.\frac{d}{dt}\right|_{t=0}$ we get $(\rho_1(X) v_1) \otimes v_2 + v_1 \otimes (\rho_2(X) v_2)$

Proof: $R_1(\exp(tX))$ is the matrix $A_{ij}(t)$ and $R_2(\exp(tX))$ is $B_{\alpha\beta}(t)$.

$V_1 = \langle e_1, \dots, e_n \rangle$ and $V_2 = \langle f_1, \dots \rangle$

Then $(R_1 \otimes R_2)(\exp tX)(e_i \otimes f_\alpha) = (*)$

$$= \left(\sum A_{ij}(t) e_j\right) \otimes \left(\sum B_{\alpha\beta}(t) f_\beta\right) =$$

$$= \sum\sum A_{ij}(t) B_{\alpha\beta}(t)(e_j \otimes f_\beta)$$

$$\frac{d}{dt}\Big|_{t=0} (*) = \sum\sum\left(\frac{dA_{ij}(t)}{dt}\Big|_{t=0} \delta_{\alpha\beta} + \delta_{ij}\frac{d}{dt}\Big|_{t=0} B_{\alpha\beta}(t)\right)(e_j \otimes f_\beta)$$

$$= (\rho_1(X)e_i) \otimes f_\alpha + e_i \otimes (\rho_2(X) f_\alpha)$$

$\blacksquare$

Some Examples of Sym^n & Λ^n for Lie alg. reps

Take $\mathfrak{sl}(2,\mathbb{C})$ on $\mathbb{C}^2$ (standard rep)

$\mathfrak{sl}(2,\mathbb{C}) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a+d=0\right\}$ $\mathbb{C}$ - 3 dim

$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

is a basis for $\mathfrak{sl}(2,\mathbb{C})$

If e_1, e_2 are a basis for $\mathbb{C}^2$ then

$Xe_1 = 0$, $Xe_2 = e_1$

$Ye_1 = e_2$, $Ye_2 = 0$

$He_1 = e_1$, $He_2 = -e_2$

Lie Groups and Lie Algebras 8^{th} Feb

Take $Sym^2 \mathbb{C}^2$. This has basis
$e_1 \otimes e_1$, $e_2 \otimes e_2$ and $\frac{1}{2}(e_1 \otimes e_2 + e_2 \otimes e_1) =$
$\overset{\shortparallel}{e_1^2}$ $\overset{\shortparallel}{e_2^2}$
$= e_1 . e_2$

$$(Sym^2 H)(e_1 \otimes e_1) = He_1 \otimes e_1 + e_1 \otimes He_1 =$$
$$= 2e_1 \otimes e_1$$

$$(Sym^2 H)\left[\frac{1}{2}(e_1 \otimes e_2 + e_2 \otimes e_1)\right] =$$

$$= \frac{1}{2}\left(He_1 \otimes e_2 + e_1 \otimes He_2 + He_2 \otimes e_1 + e_2 \otimes He_1\right)$$

$$= \frac{1}{2}\left(e_1 \otimes e_2 - e_1 \otimes e_2 - e_2 \otimes e_1 + e_2 \otimes e_1\right)$$

$$= 0$$

$$(Sym^2 H)(e_2 \otimes e_2) = -2e_2 \otimes e_2$$

$$(Sym^2 X)(e_1 \otimes e_1) = 0$$
$$(Sym^2 X)\left(\frac{1}{2}(e_1 \otimes e_2 + e_2 \otimes e_1)\right) =$$

$$= \frac{1}{2}(e_1 \otimes e_1 + e_1 \otimes e_1) = e_1 \otimes e_1$$

$$(Sym^2 X)(e_2 \otimes e_2) = 2\left(\frac{1}{2}\left[e_1 \otimes e_2 + e_2 \otimes e_1\right]\right)$$

$$\mathrm{Sym}^2 Y\,(e_1 \otimes e_1) = 2\cdot\left[\tfrac{1}{2}\, e_2 \otimes e_1 + e_1 \otimes e_2\right]$$

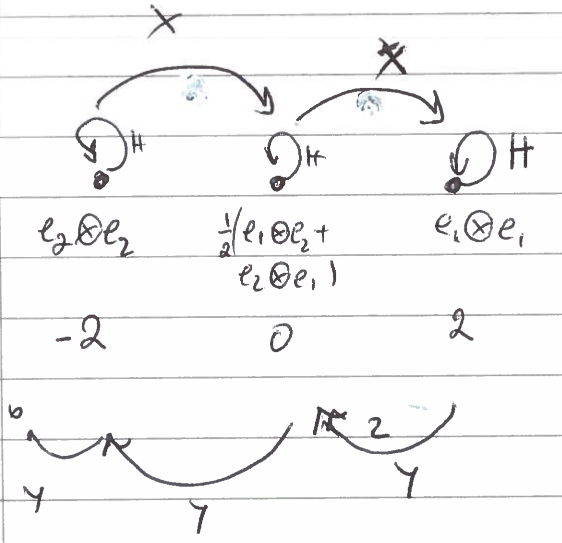
$$\mathrm{Sym}^2 Y\left(\tfrac{1}{2}\left[(e_1 \otimes e_2) + e_2 \otimes e_1\right]\right) =$$

$$= e_2 \otimes e_2$$

$$\mathrm{Sym}^2 Y\,(e_2 \otimes e_2) = 0$$

$$[X, Y] = XY - YX = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = H$$



<u>Example</u>: $\mathbb{C}^2$ = st. rep of $\mathfrak{sl}(2, \mathbb{C})$

$\Lambda^2 \mathbb{C}^2$ is 1 dim over $\mathbb{C}$

and is spanned by $e_1 \wedge e_2$

$e_1 \wedge e_2 = \frac{1}{2}(e_1 \otimes e_2 - e_2 \otimes e_1)$

$$(\Lambda^2 H)(e_1 \wedge e_2) = e_1 \wedge e_2 - e_1 \wedge e_2 = 0$$

$$(\Lambda^2 X)(e_1 \wedge e_2) = 0 + e_1 \wedge e_1 = 0$$

The trivial rep.

$$(\Lambda^2 Y)(e_1 \wedge e_2) = 0 + e_2 \wedge e_2 = 0$$

Lie Groups and Lie Algebras 8th Feb

This is the trivial 1-dim rep where "trivial" for Lie alg. reps means $\rho(x) = 0 \ \forall x \in \mathfrak{g}$ because then $\exp(\rho(x)) = 1$.

8.3 Complexification

Definition: If $\mathfrak{g}$ is a Lie alg / $\mathbb{R}$ then define $\mathfrak{g}_{\mathbb{C}}$ to be the Lie algebra

$\mathfrak{g}_{\mathbb{C}} = \{ v + iw : v, w \in \mathfrak{g} \} = \mathfrak{g} \oplus \mathfrak{g}$

Where the bracket is $[v+iw, a+ib] =$

$= [v, a] - [w, b] + i([w, a] + [v, b])$

This is now a complex Lie algebra.

e.g. $\mathfrak{gl}(n, \mathbb{R})_{\mathbb{C}} \cong \mathfrak{gl}(n, \mathbb{C})$

$\mathrm{Re}(M), \mathrm{Im}(M)$ $\quad M$

$\mathfrak{sl}(n, \mathbb{R})_{\mathbb{C}} \cong \mathfrak{sl}(n, \mathbb{C})$

$\mathfrak{u}(n) = \{ A : A^+ = -A \}$ skew-Hermitian

If $A \in \mathfrak{u}(n)$, $(iA)^+ = (-i) A^+ = iA$

$\Rightarrow$ iA is Hermitian

If $A \in \mathfrak{gl}(n, \mathbb{C})$ then $A = \underbrace{\frac{1}{2}(A + A^+)}_{\substack{\text{Hermitian} \\ \in i\mathfrak{u}(n)}} + \underbrace{\frac{1}{2}(A - A^+)}_{\substack{\text{skew Hermitian} \\ \in \mathfrak{u}(n)}}$

$\Rightarrow \quad u(n)_{\mathbb{C}} = u(n) \oplus i\,u(n) = \mathfrak{gl}(n, \mathbb{C})$

Similarly $\quad su(n)_{\mathbb{C}} \cong sl(n, \mathbb{C})$

Lemma: There is a 1-1 correspondence between

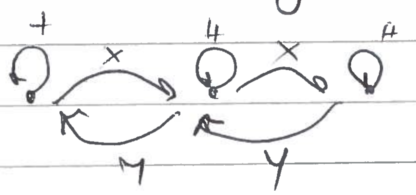
$$\left\{ \begin{array}{c} \mathbb{R}\text{-linear} \\ \text{Lie algebra homo.} \\ \mathfrak{g} \to \mathfrak{gl}(n, \mathbb{C}) \end{array} \right\} \underset{\tilde{f}|_{\mathfrak{g}} \leftarrow \tilde{f}}{\overset{f \to \tilde{f}}{\longleftrightarrow}} \left\{ \begin{array}{c} \mathbb{C}\text{-linear} \\ \text{Lie alg. homo} \\ \mathfrak{g}_{\mathbb{C}} \to \mathfrak{gl}(n, \mathbb{C}) \end{array} \right\}$$

Proof: If $f : \mathfrak{g} \to \mathfrak{gl}(n, \mathbb{C})$ is an $\mathbb{R}$-linear Lie-alg. homo then $\tilde{f}(v + iw) = f(v) + i f(w)$
$\tilde{f} : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{gl}(n, \mathbb{C})$ ∎

Corollary: If $\mathfrak{g}_{\mathbb{C}} \cong \mathfrak{h}_{\mathbb{C}}$ then there is a 1-to-1 corresp. between $\mathbb{R}$-linear reps $\mathfrak{g} \to \mathfrak{gl}(n, \mathbb{C})$ and $\mathbb{R}$-linear reps $\mathfrak{h} \to \mathfrak{gl}(n, \mathbb{C})$.

e.g. $\left.\begin{array}{l} \mathfrak{g} = su(n) \\ \mathfrak{h} = sl(n, \mathbb{R}) \end{array}\right\}$ these have the same rep theory if we take $\mathbb{C}$-reps, i.e. $\mathbb{R}$-linear maps to $\mathfrak{gl}(n, \mathbb{C})$

We've already seen



picture for the structure of an $sl(2, \mathbb{C})$ rep. But X, Y, H are all real

Lie Groups and Lie Algebras 8^{th} Feb.

so this comes as $\tilde{f}$ for a rep f of $\mathfrak{sl}(2,\mathbb{R})$. By the corollary it's also a rep of $\mathfrak{su}(2)$, but $X, Y, H \notin \mathfrak{su}(2)$

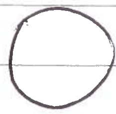
However $\mathfrak{su}(2)$ is "better" because it's the Lie alg of $SL(2)$ which is compact while $SL(2,\mathbb{R})$ is not compact.

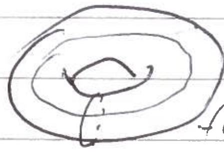
This allows us to pass between reps of a compact group where we have complete reducibility and a noncompact group where we have a nice basis for Lie alg.

11^{th} Jan

9. Representations of tori

Definition: The n-torus is the group $\underbrace{U(1) \times \cdots \times U(1)}_{n \text{ times}} = T^n$

Geometrically $U(1)$ is  in $\mathbb{C}$

$U(1) \times U(1)$  T^2

$(e^{i\theta}, e^{i\varphi})$

9.1. Reps of $U(1)$

We saw earlier that all smooth homomorphism $U(1) \longrightarrow U(1)$ had the form $F_k : e^{i\theta} \to e^{ik\theta}$, $k \in \mathbb{Z}$

Lemma (Schur's lemma)

If $F: V \to W$ is a morphism of complex reps (irreducible) then either F is an iso or it is the 0-map

Proof: $Ker(f)$ and $Im(f)$ are subreps of V and W respectively
V, W irred $\Rightarrow$ Either $Ker(f) = 0$
or $Ker(f) = V$
Similarly either $Im(f) = W$ or 0

If $Ker(0) \Rightarrow f$ is ~~inj~~, if $Im(f) = 0 \Rightarrow f = 0$
If $Im(W) =$ surj and if $Ker(f) = V \Rightarrow f = 0$

$\Rightarrow f$ is iso or $f = 0$ ∎

Corollary: If $R: U(1) \to GL(V)$, which is a complex irreducible finite dim. rep and $L \in GL(V)$ is a linear map which commutes with $R(e^{i\theta})$ $\forall \theta$ then $L = \lambda I$.

Proof: Consider the map $L - \lambda I = F$
F is a morphism of reps $V \to V$
$F(R e^{i\theta}) v) = R(e^{i\theta}) F(v)$ holds
because $L R(e^{i\theta}) = R e^{i\theta} L$ $\forall \theta$

Because L is a morphis of reps and V is irreducible $\Rightarrow F$ is isomorphism or F is 0.

Lie Groups and Lie Algebras 11th Feb.

Because $\mathbb{C}$ is alg. closed L has an eigenvalue $\Rightarrow$ for some λ $\exists$ v s.t. $(L-\lambda I)v = 0$ $\Rightarrow$ $L-\lambda I$ is not an iso $\Rightarrow$ $L-\lambda I = 0$ $\Rightarrow$ $L = \lambda I$ ∎

Corollary If $R: U(1) \to GL(V)$ is an $\mathbb{C}$-irred. rep of $U(1)$ then $\dim_{\mathbb{C}}(V) = 1$

Proof: $\forall$ ϕ $R(e^{i\phi})$ commutes with $R(e^{i\theta})$ $\forall$ θ because $U(1)$ is abelian and R is a homeomorphism.

Let $L = R(e^{i\phi})$ and apply previous corollary then $R(e^{i\phi}) = \lambda(\phi)I$ for some scalar $\lambda(\phi) \in \mathbb{C}$. Then $\forall v \in V, v \neq 0$ $R(e^{i\phi})v = \lambda(\phi)v$ the subspace $\mathbb{C}v$ spanned by v in V is a subrep but V is irreducible $\Rightarrow$ $V = \mathbb{C}v$ for some $v \neq 0$ $\Rightarrow$ $\dim_{\mathbb{C}} V = 1$ ∎

Lemma: Let $R: U(1) \to GL(1, \mathbb{C})$ be a 1-dim complex rep of $U(1)$. Then $R = F_n$, for some $n \in \mathbb{Z}$ where $F_n(e^{i\theta}) = e^{in\theta} \in GL(1, \mathbb{C}) = \mathbb{C}^{\times}$

Proof: We first show that $\mathrm{Im}(R) \subseteq U(1) \subseteq \mathbb{C}^{\times}$ Then it will follow from our classification of homomorphism $U(1) \to U(1)$

We find an invariant Hermitian inner product on $\mathbb{C}$ by Weyl's unitariant trick. (This works here because $U(1)$ is compact) So we know there is a $\langle , \rangle$

(Hermitian) on $\mathbb{C}$ s.t. $\langle R(e^{i\phi})v, R(e^{i\phi})w \rangle =$
$= \langle v, w \rangle \quad \forall \phi, v, w$

On $\mathbb{C}$ there is only one Hermitian inner product up to scale.

$\langle av, w \rangle = \bar{a} \langle v, w \rangle$
$\langle v, bw \rangle = b \langle v, w \rangle$

On $\mathbb{C}$ $\langle a, b \rangle = \bar{a} b \langle 1, 1 \rangle$

$\Rightarrow$ Any Hermitian inner product is some multiple of $\bar{z} w = \langle z, w \rangle$

If $\langle R(e^{i\phi})v, R(e^{i\phi})w \rangle = \langle v, w \rangle$

$$\overline{R(e^{i\phi})} Re^{i\phi} \langle v, w \rangle = \langle v, w \rangle$$

$$\Rightarrow \overline{R(e^{i\phi})} Re^{i\phi} = 1 \quad \forall \phi$$

so $Re^{i\phi} \in U(1) \ \forall \phi$

$\Rightarrow R(U(1)) \subseteq U(1) \Rightarrow R = F_n$ for some n by our earlier classification. ∎

n is called the weight of the rep.

Any finite dim rep of $U(1)$ splits as a direct sum of irreducible reps, because $U(1)$ is compact. So any f.d. rep of $U(1)$ is isomorphic to $F_{n_1}^{m_1} \oplus F_{n_2}^{m_2} \oplus \dots \oplus F_{n_k}^{m_k}$

Lie Groups and Lie Algebras 11^{th} Feb

Alternatively, any f.d. rep V splits as $V = \bigoplus_{n \in \mathbb{Z}} V_n$, where V_n is the span of all subreps isomorphic to F_n i.e. $V_n = \{ v \in V : R(e^{i\phi}) v = e^{in\phi} v \}$. This is called the weight space decomp.

Example

$$R(e^{i\phi}) = \begin{pmatrix} e^{i\phi} & 0 & 0 & 0 \\ 0 & e^{-i\phi} & 0 & 0 \\ 0 & 0 & e^{i\phi} & 0 \\ 0 & 0 & 0 & e^{2i\phi} \end{pmatrix}$$

The weights -1, 1 and 2 occur in this 4-dim rep.

$$\left.\begin{aligned} V_{-1} &= \mathbb{C}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \\ V_1 &= \mathbb{C}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \\ V_2 &= \mathbb{C}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \end{aligned}\right\}$$ This is the weight space decomp of $R(e^{i\phi})$ as above

Pictorially, I will write

$$\begin{array}{ccccccc} -3 & -2 & -1 & 0 & 1 & 2 & 3 \\ \cdot & \cdot & \circ & \circ & \circ & \circ & \circ \\ & & 1 & & 2 & 1 & \end{array}$$

$\mathbb{C}$-dim of V_n

$V_n = \{ v \in V : R(e^{i\phi}) v = e^{in\phi} v \ , \forall \phi \}$

Remember V is also Lie algebra rep.

$R_* : u(1) \longrightarrow gl(V)$
$R(\exp X) = \exp(R_* X) v$

So $R(e^{i\phi}) v = e^{in\phi} v$
$\Leftrightarrow e^{R_*(i\phi)} v = e^{in\phi} v$

i.e we can also write
$V_n = \{ v \in V : R_*(i\phi)(v) = in\phi v \}$

$\Leftrightarrow R_*(i) v = inv$

9.2 Representations of T^n for $n > 1$

For any n-tuples of integers $k_1, \dots, k_n$ there is a rep, 1-dim $\mathbb{C}$ irred. rep for $T^n = \underbrace{U(1) \times \dots \times U(1)}_{n \text{ times}}$

$$F_{k_1, \dots, k_n}(e^{i\phi_1}, \dots, e^{i\phi_n}) = e^{i(k_1\phi_1 + k_2\phi_2 + \dots k_n\phi_n)}$$

Lemma: Any irreducible rep over $\mathbb{C}$ of T^n is isomorphic to $F_{k_1, \dots, k_n}$ for some $(k_1, \dots, k_n) \in \mathbb{Z}^n$.

Proof: Let V be such an irreducible rep, $R : U(1)^n \longrightarrow GL(V)$ then we can restrict it

Lie Groups and Lie Algebras 11^{th} Feb

to $U(1) \times \{(1, \dots, 1)\} \subseteq U(1)^n$ and we get a rep of $U(1)$ on V. So we get a weight space decomp of $V = \bigoplus_{n \in \mathbb{Z}} V_n$

of V as a rep of $U(1)_1$

Now each V_n is a subrep of $R: U(1)^n \to GL(V)$ because

$V_{k_1} = \{ v \in V : R(e^{i\phi_1}, 1, \dots, 1) v = e^{ik\phi_1} v \}$ & if $v \in V_{k_1}$ then

$$R(e^{i\phi}, 1, \dots, 1) R(e^{i\phi_1}, \dots, e^{i\phi_n}) v =$$

$$= R(e^{i\phi_1}, \dots, e^{i\phi_n}) R(e^{i\phi}, 1, \dots, 1) v$$

$$= e^{ik_1\phi} R(e^{i\phi_1}, \dots, e^{i\phi_n}) v$$

$$\Rightarrow R(e^{i\phi_1}, \dots, e^{i\phi_n}) v \in V_{k_1}$$

$\Rightarrow V_{k_1}$ is invariant subspace of the representation for $U(1)^n$ i.e. it is a subrepresentation

$\Rightarrow V = V_{k_1}$ for some n, as V is irreducible so has no proper subrep.

Now we've seen that $R(e^{i\phi}, 1, \dots, 1)$ acts as $e^{ik_1\phi}$.

We now apply the same argument for each factor and we deduce that

$$R(1, \dots, 1, \underbrace{e^{i\phi}}_{m^{th}\text{ place}}, 1, \dots, 1) v = e^{ik_m} v \quad \text{for some } k_m \in \mathbb{Z}$$

$$\Rightarrow R(e^{i\phi_1}, \dots, e^{i\phi_n}) v = R(e^{i\phi_1}, 1, \dots, 1) R(1, e^{i\phi_2}, 1, \dots, 1) \cdot$$

$$\ldots R(1, \ldots, 1, e^{i\phi_n})v =$$

$$= e^{ik_1\phi_1} \cdot e^{ik_2\phi_2} \cdots e^{ik_n\phi_n} v =$$

$$= e^{i(k_1\phi_1 + \ldots k_n\phi_n)} v$$

we see this is 1-dim as it's irreducible

Corollary: We can decompose any f-d. $\mathbb{C}$-rep of T^n as "weight space decomp"

$$V = \bigoplus_{(k_1, \ldots, k_n) \in \mathbb{Z}^n} V_{k_1, \ldots, k_n}$$

where

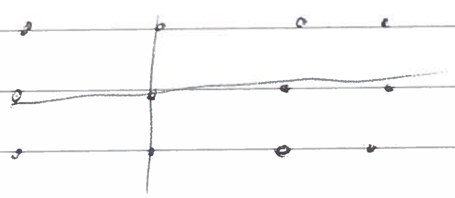
$$V_{k_1, \ldots, k_n} = \{ v : R(e^{i\phi_1}, \ldots, e^{i\phi_n})v = e^{i(k_1\phi_1 + \ldots + k_n\phi_n)} v \}$$

"weight space"

$$= \{ v : R_*(i\phi_1, \ldots, i\phi_n)v = ik_1\phi_1 + \ldots + ik_n\phi_n v \}$$

e.g. if $n = 2$ we can represent this pictorially as a lattice

"lattice of weights"



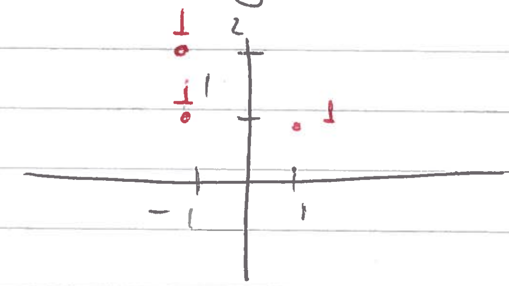
Each lattice pt (k_1, k_2) is labelled with $\dim_{\mathbb{C}} V_{k_1, k_2}$

$$\text{e.g. } R(e^{i\phi_1}, e^{i\phi_2}) = \begin{pmatrix} e^{i(\phi_1+\phi_2)} & 0 & 0 \\ 0 & e^{-i\phi_1+i\phi_2} & 0 \\ 0 & 0 & e^{-i\phi_1+2i\phi_2} \end{pmatrix}$$

Lie Groups and Lie Algebras 11th Feb

The weights are $(1,1)$
$(-1,1)$
$(-1,2)$

Pictorially



3 weight spaces

$V_{1,1} = \mathbb{C}\begin{pmatrix}1\\0\\0\end{pmatrix}$, $V_{-1,1} = \mathbb{C}\begin{pmatrix}0\\1\\0\end{pmatrix}$

$V_{-1,2} = \mathbb{C}\begin{pmatrix}0\\0\\1\end{pmatrix}$

Recall 22nd Feb

If $R : U(1)^n \longrightarrow GL(V)$ a finite dim $\mathbb{C}$-rep then $V = \bigoplus_{(k_1,\dots,k_n)\in\mathbb{Z}^n} V_{k_1,\dots,k_n}$

where $V_{k_1,\dots,k_n} = \{ v \in V : R(e^{i\theta_1}, \dots, e^{i\theta_n})v = e^{i(k_1\theta_1 + \dots + k_n\theta_n)} \}$

$= \{ v \in V : R_*\underbrace{(i\theta_1,\dots,i\theta_n)}_{\in \mathrm{Lie}\, U(1)^n} v = i(k_1\theta_1 + \dots + k_n\theta_n)\}$

9.3. Lattice of weights

Let $\mathfrak{t}$ denotes $[\mathrm{Lie}\, U(1)^n] \otimes \mathbb{C}$. An element $\lambda \in \mathfrak{t}^*$ is (by def) a linear map sending lie alg. elements to $\mathbb{C}$ numbers.

Claim : $V = \bigoplus_{\lambda \in \mathfrak{t}^*} V_\lambda$ where to each

$(k_1, \dots, k_n) \in \mathbb{Z}^n$, we have associated $\lambda \in \mathfrak{t}^*$ by $\lambda(i\theta_1, \dots, i\theta_n) = i(k_1\theta_1 + \dots + k_n\theta_n)$

Now $V_\lambda = \{v \in V : R(\exp X)v = \exp(\lambda(X))v \ , \forall X \in \mathrm{Lie}(U(1)^n)$

$X = (i\theta_1, \dots, i\theta_n)$

$\exp X = (e^{i\theta_1}, \dots, e^{i\theta_n})$

$= \{v \in V : R_*(X)v = \lambda(X)v \ , \forall X \in \mathrm{Lie}(U(1)^n)$

Remark : In the sum $\bigoplus_{\lambda \in \mathfrak{t}^*} V_\lambda$ it looks like we're summing over a continuum of λ_s.

Lemma : Let $\mathrm{Ker}\exp = \{X \in \mathrm{Lie}\, U(1)^n \text{ s.t. } \exp(X) = 1\}$
$= \{(i2\pi m_1, \dots, i2\pi m_n) : (m_1, \dots, m_n) \in \mathbb{Z}^n\}$
Then if $\lambda \in \mathfrak{t}^*$ has the form $\lambda(i\theta_1, \dots, i\theta_n) = \sum i a_\ell \theta_\ell$ then

$a_\ell \in \mathbb{Z} \ \forall \ell \iff \lambda(X) \in 2\pi i \mathbb{Z} \ \forall X \in \mathrm{ker}\exp$

We call $\mathfrak{t}^*_{\mathbb{Z}} = \{\lambda \in \mathfrak{t}^* \mid \lambda(X) \in 2\pi i \mathbb{Z} \ \forall X \in \mathrm{Ker}\exp\}$

$\subseteq \mathfrak{t}^*$

I know the only weights that occur in

Lie Groups and Lie Algebras 22nd Feb

$\bigoplus_{\lambda \in \mathfrak{t}^*} V_\lambda$ are those corresponding to $(k_1, \dots, k_n) \in \mathbb{Z}^n$

so I can restrict to $\bigoplus_{\lambda \in \mathfrak{t}^*_{\mathbb{Z}}} V_\lambda$.

Proof of lemma

$$\sum i a_\ell 2\pi m_\ell \in 2\pi i \mathbb{Z} \quad \forall m_\ell \in \mathbb{Z} \iff a_\ell \in \mathbb{Z} \ \forall \ell$$

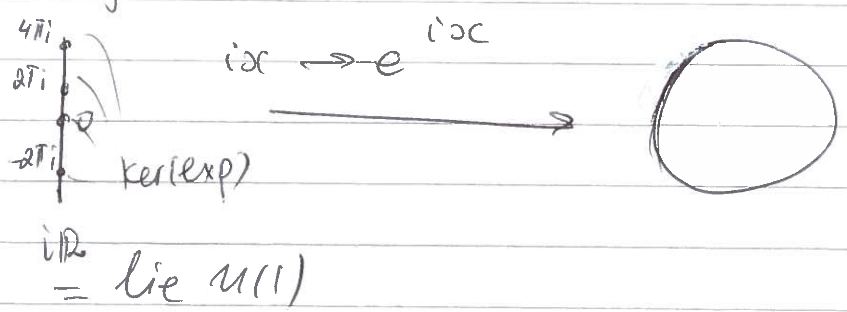
$\Leftarrow$ easy

$\Rightarrow$ take $m = (1, 0, \dots, 0) \Rightarrow 2\pi i a_1 \in 2\pi i \mathbb{Z} \Rightarrow a_1 \in \mathbb{Z}$
$m = (0, 1, \dots, 0) \Rightarrow 2\pi i a_2 \in 2\pi i \mathbb{Z} \Rightarrow a_2 \in \mathbb{Z}$
etc

∎

Summary: Inside Lie $U(1)^n$ we have an integral lattice ker exp and $\mathfrak{t}^*_{\mathbb{Z}}$ the lattice of possible weights for representation is the dual lattice to ker exp.

e.g: if $n = 1$



9.4 Tensor products

Lemma:

To find the weights in the tensor product $V \otimes W$: $V = \bigoplus_{\alpha \in A} V_\alpha$ for some $A \subseteq t^*_{\mathbb{Z}}$, $W = \bigoplus_{\beta \in B} W_\beta$, for some $B \subseteq t^*_{\mathbb{Z}}$

then $V \otimes W = \bigoplus_{\gamma \in C} (V \otimes W)_\gamma$, where

$$(V \otimes W)_\gamma = \bigoplus_{\substack{\alpha+\beta=\gamma \\ \alpha \in A \\ \beta \in B}} (V_\alpha \otimes W_\beta)$$ and $C = \{\alpha + \beta \mid \alpha \in A \text{ \& } \beta \in B\}$

Proof: $$V \otimes W = \left[\bigoplus_{\alpha \in A} V_\alpha\right] \otimes \left[\bigoplus_{\beta \in B} W_\beta\right] =$$

$$= \bigoplus_{\substack{\alpha \in A \\ \beta \in B}} V_\alpha \otimes W_\beta$$

Need to show that if $v \in V_\alpha$, $w \in W_\beta$ then $V \otimes W$ has weight $\alpha + \beta$. The lemma will then follow if we group these pieces by total weight.

α is like an eigenvalue that depends linearly on X

$v \in V_\alpha$ then $R_1(\exp X) v = \exp(\alpha(X)) v$

$w \in W_\beta$ then $R_2(\exp X) w = \exp(\beta(X)) w$

$(R_1 \otimes R_2)(\exp X)(v \otimes w) =$

$= [R_1(\exp X) v] \otimes [R_2(\exp X) w] =$

$$= \left(e^{\alpha(x)} v\right) \otimes \left(e^{\beta(x)} w\right) =$$

$$= e^{\alpha(x)} e^{\beta(x)} v \otimes w =$$
$$= e^{(\alpha+\beta)(x)} v \otimes w$$

Alternatively,

$$v \in V_\alpha \Rightarrow (R_1)_*(X) v = \alpha(X) v$$

$$w \in W_\beta \Rightarrow (R_2)_*(X) w = \beta(X) w$$

$$(R_1 \otimes R_2)_*(X)(v \otimes w) =$$

$$= [R_{1*}(X) v] \otimes w + v \otimes [(R_2)_*(X) w]$$

$$= \alpha(X) v \otimes w + v \otimes \beta(X) w =$$

$$= [\alpha(x) + \beta(x)] v \otimes w$$ $\blacksquare$

10. Representations of $su(2)$

10.1 Relation between $su(2)$ and $sl(2, \mathbb{C})$

$su(2)$ is 2×2 skew Hermitian, Trace free
$sl(2, \mathbb{C})$ 2×2 complex, trace free.

$$su(2) \otimes \mathbb{C} \cong sl(2, \mathbb{C})$$
$$\frac{1}{2}(A + A^\dagger) + \frac{1}{2}(A - A^\dagger) = A$$

The following is a basis for $su(2)$.

$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \sigma_1$, $\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$

$[\sigma_i, \sigma_j] = 2\sigma_k$, i, j, k cyclic perm of $1, 2, 3$

The following is a basis for $sl(2, \mathbb{C})$

$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$[H, X] = 2X$
$[H, Y] = -2Y$
$[X, Y] = H$

$H = -i\sigma_1$, $X = \frac{1}{2}(\sigma_2 - i\sigma_3)$, $Y = \frac{-1}{2}(\sigma_2 + i\sigma_3)$

These equations hold in $su(2) \otimes \mathbb{C} \cong sl(2, \mathbb{C})$

10.2 Diagonalising σ_1 and H

Suppose $\rho : su(2) \longrightarrow gl(V)$ is a $\mathbb{C}$-rep of $su(2)$ $\rightsquigarrow$ a $\mathbb{C}$ rep of $sl(2, \mathbb{C}) = su(2) \otimes \mathbb{C}$

Lemma : V decomposes as a direct sum [NOT OF SUBREPS!] $V = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda$,

where $V_\lambda = \{ v \in V \mid \rho(\sigma_1) v = \lambda v \}$

The eigenvalues λ in the sum are in $i\mathbb{Z}$ are called the weights of V.

Proof: Since $SU(2)$ is a simply connected Lie group with Lie algebra $su(2)$ Lie's thm $\Rightarrow$ $\exists\ R : SU(2) \rightarrow GL(V)$ s.t.

Lie Groups and Lie Algebras 22^{nd} Feb

$R(\exp X) = \exp(\rho(X)) \quad \forall \; X \in su(2)$

Now inside $SU(2)$ the subgroup $\{\exp(t\sigma_1) : t \in \mathbb{R}\} \subseteq SU(2)$ is a torus i.e. just a copy of $U(1)$

Now take the weight space decomposition of V considered as a rep of $U(1) \subseteq SU(2)$

$\Rightarrow \quad V = \bigoplus_\lambda V_\lambda$

Each weight space $V_\lambda = \{v \in V : R(\exp_\rho(t\sigma_1))v = e^{\lambda t} v\}$

$= \{v \in V : \rho(\sigma_1) v = \lambda v\}$

The weights are imaginary integers

A $\mathbb{C}$-rep of $su(2)$ gives us a $\mathbb{C}$-rep of $sl(2, \mathbb{C}) = su(2) \otimes \mathbb{C}$, $\underset{su(2,\mathbb{C})}{\underset{\cap}{H}} = -i\sigma_1$.

So corollary: If V is a $\mathbb{C}$-linear $\mathbb{C}$-rep of $sl(2, \mathbb{C})$ then $V = \bigoplus_\lambda V_\lambda$

where $V_\lambda = \{v \in V : \rho(H) v = -i\lambda v\}$

where $\lambda \in i\mathbb{Z}$

$\Rightarrow V_\lambda = \{v \in V : \rho(H) v = \lambda v\}$ for $\lambda \in \mathbb{Z}$

Note we could not have proved this the same way as

$$\left\{ \exp(tH) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \right\} \cong \mathbb{R} \text{ not } U(1)$$

Example : 1) The adjoint rep of $sl(2,\mathbb{C})$

$ad_\xi \eta = [\xi, \eta]$, $ad_\xi \in gl(g)$

For $sl(2,\mathbb{C})$ take basis H, X, Y

$$ad_H X = [H, X] = 2X$$

$$ad_H H = [H, H] = 0$$

$$ad_H Y = [H, Y] = -2Y$$

If $V = sl(2,\mathbb{C})$ is the vector space for adjoint rep then we see that

$$V = \bigoplus_\lambda V_\lambda = \underset{\overset{\parallel}{\mathbb{C}Y}}{V_{-2}} \oplus \underset{\overset{\parallel}{\mathbb{C}H}}{V_0} \oplus \underset{\overset{\parallel}{\mathbb{C}X}}{V_2}$$

$V_\lambda = \{ v \in H : ad_H v = \lambda v \}$

2) Let x, y be a basis for standard rep $\mathbb{C}^2$ of $sl(2,\mathbb{C})$ and let x^2, xy, y^2 be basis for $Sym^2\mathbb{C}^2$

$$(Sym^2 H) \begin{cases} x^2 & = 2x^2 \\ xy & = 0 \\ y^2 & = -2y^2 \end{cases}$$

$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $Hx = x$, $Hy = -y$

Liebnitz Rule:

$$(Sym^2 H)(x \otimes x) = 2x \otimes x = 2x^2$$

Lie Groups and Lie Algebras 22^{nd} Feb

So weight decomposition of $\mathrm{Sym}^2\mathbb{C}^2$ is

$$V_{-2} \oplus V_0 \oplus V_2$$
$$\begin{array}{ccc} \| & \| & \| \\ \mathbb{C}y^2 & \mathbb{C}xy & \mathbb{C}x^2 \end{array}$$

This is the same as the weight decomp of adjoint rep. We will see that this means $\mathrm{Sym}^2\mathbb{C}^2 \cong \mathrm{ad}$

We now understand the action of H completely: it can be diagonalised i.e. V splits into subspaces where H acts as scalar multiplication by integer. We still need to understand how X & Y act in any given rep

Lemma: If $\rho: \mathfrak{sl}(2,\mathbb{C}) \longrightarrow \mathfrak{gl}(V)$ is a $\mathbb{C}$-linear $\mathbb{C}$ rep of $\mathfrak{sl}(2,\mathbb{C})$ and $V = \bigoplus_{\lambda\in\mathbb{Z}} V_\lambda$ is the weight decomp.

Then $v \in V_\lambda \Rightarrow \rho(X)v \in V_{\lambda+2}$

$\rho(Y)v \in V_{\lambda-2}$

[And $\rho(H)v \in V_\lambda$]

Proof: Omit ρ and just write Xv for $\rho(X)v$

$v \in V_\lambda \Rightarrow Hv = \lambda v$

$$\rho(H)\rho(X)v = (\rho(H)\rho(X) - \rho(X)\rho(H))\, v + \rho(X)\rho(H)v$$

$= [\rho(H), \rho(X)] v + \rho(X)\rho(H) v$

$= \rho([H, X]) v + \rho(X) \lambda v$ using ρ is a rep, $\rho(H)v = \lambda v$

$= \rho(2X) v + \rho(X) \lambda v$

$= (\lambda + 2) \rho(X) v = (\lambda + 2) X v$

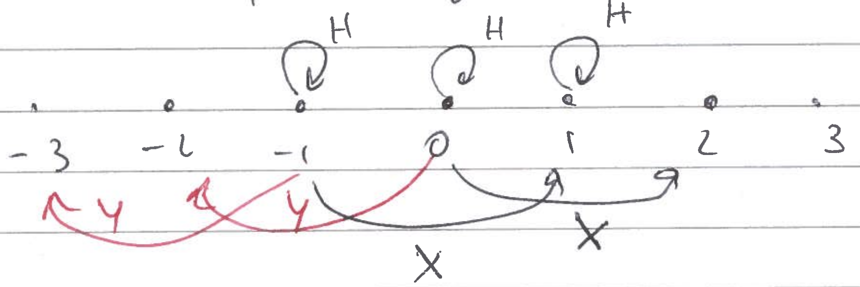
$\Rightarrow Xv \in V_{\lambda+2}$

Similarly for Y

$$H Y v = [H, Y] v + Y H v =$$
$$= -2 Y v + Y \lambda v =$$
$$= (-2 + \lambda) Y v$$

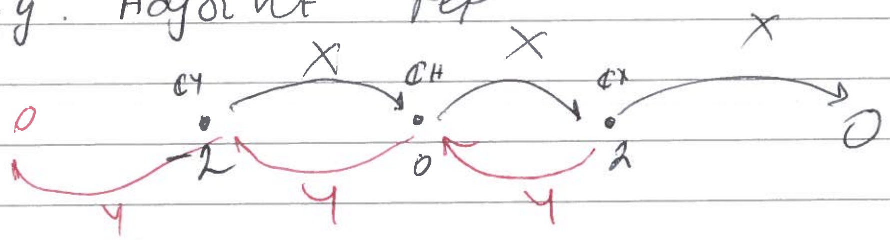
$$\Rightarrow Yv \in V_{\lambda - 2}$$

We can visualise this by drawing the lattice of weights



$\subseteq \mathbb{Z}$

e.g. Adjoint rep



$ad_X Y = H = [X, Y]$, $ad_X X = [X, X] = 0$

$ad_X H = [X, H] = -2X$,

Lie Groups and Lie Algebras 22nd Feb

$ad_Y X = [Y, X] = -H$

$ad_Y H = [Y, H] = 2Y$

$ad_Y Y = 0$

Theorem (Classification Thm)

Suppose that $\rho: sl(2,\mathbb{C}) \to gl(V)$ is a $\mathbb{C}$-linear $\mathbb{C}$-rep of $sl(2,\mathbb{C})$ which is finite-dim. and irreducible. Then the weight spaces V_λ are 1-dim and the weights form an uninterrupted chain $-m, -m+2, \dots, m-4, m-2, m$ for some $m \in \mathbb{N}$. m is called the highest weight.

The proof also gives formulae for the action of $\rho(X)$ and $\rho(Y)$ so this is a complete classification there is one irrep for each $m \in \mathbb{N}$ (up to iso)

Proof: Let m be the biggest integer s.t. $V_m \neq 0$. This exists because V is f.d.

Pick $v \in V_m$, $\rho(X)v = 0$ because $\rho(X)v \in V_{m+2}$ and $m+2 > m \Rightarrow V_{m+2} = 0$

We call such a vector a ~~highest~~ weight vector.

Consider the sequence starting with v $v, \rho(Y)v, \rho(Y)^2 v, \dots, \rho(Y)^K v$, where K is maximal s.t. $\rho(Y)^K v \neq 0$. This exists because V is f.d.

Want to show that this sequence is a basis for V and $k = m$

Henceforth drop ρ!

We need to show that span of $v, Yv, Y^2v, \ldots, Y^k v$ is a subrep. Irreducibility then implies that $W = V$, i.e. this sequence is a basis.

Need to show $w \in W$ then $Xw \in W$, $Hw \in W$, $Yw \in W$

As $v, Yv, \ldots, Y^k v$ are eigenvectors of H,
$HY^\ell v = (m-2\ell)Y^\ell v$ as $Y^\ell v \in V_{m-2\ell}$

$Y\,Y^\ell v = Y^{\ell+1} v \in W \quad \forall \ell$

So I only need to check

$XY^\ell v$ is a multiple of $Y^{\ell-1} v$

WTS
$XY^\ell v = (m-\ell+1)\ell\, Y^{\ell-1} v$

Before we prove this we will first note that a) $Y^\ell v \neq 0$ if $\ell \leq m$
b) $Y^{m+1} v = 0$

a) $v \neq 0 \quad \ell = 0$, Assume $Y^n v \neq 0$
if $Y^{n+1} v = 0$
then $XY^{n+1} v = 0$
$\Rightarrow (m-n)\underbrace{(n+1)}_{\neq 0}\underbrace{Y^n v}_{\neq 0} = 0$
$\Rightarrow m = n$

Lie Groups and Lie Algebras 25th Feb

$\Rightarrow Y^n v \neq 0 \quad \Leftarrow n \leq m$

b)

Let k be minimal number s.t. $Y^n v = 0$

we will prove that $k = m+1$

suppose $k \neq m+1$ $\quad Y^{k-1} v \neq 0$

$$X Y^k v = \underbrace{(m-k+1)k}_{\neq 0} \underbrace{Y^{k-1} v}_{\neq 0} \neq 0$$

$\Rightarrow k = m+1$

We will now prove: $X Y^l v = (m-l+1) l \, Y^{l-1} v$ $(*)$

if $l = 0$ $\quad X v = 0$ by def. of v being the highest weight vector

$$X Y v = [X, Y] v + Y X v$$

$$= H v + 0$$

$$= m v$$

Assume $(*)$ holds for l

We want to compute $X Y^{l+1} v$ $=$; since $[X, Y] = H$

$= X Y Y^l v = Y X Y^l v + H Y^l v =$

$= Y (m-l+1) l \, Y^{l-1} v + (m-2l) Y^l v =$

$= [(m-l+1)l + m - 2l] Y^l v =$

$= [(m-(l+1)+1)(l+1)] Y^l v$ $\blacksquare$

This proof gives formulae for the action of $\rho(H)$, $\rho(Y)$ and $\rho(X)$ in the rep w.r.t. given basis $\Rightarrow$ it determines the rep up to isomorphism.

An alternative proof that $\exists!$ irreducible rep with highest weight m.

Proof:

Suppose V and W are irreps with highest weight m. Consider $V \oplus W$. if $v \in V_e$ and $w \in W_e$ then $u = v \oplus w \in (V \oplus W)_e$

Now the Theorem $\Rightarrow \exists$ a subrep $U \leq V \oplus W$
$U = \mathrm{span}(u, \rho(Y)u, \rho(Y)^2 u, \dots)$

Now U is a subrep of $V \oplus W$. project onto V & W i.e. $pr_V: U \to V$ and $pr_W: U \to W$ } morphisms of irreducible reps

By Schur's lemma $pr_V: U \to V$ & $pr_W: U \to W$ are isomorphisms, because they are not 0 as $pr_V(v \oplus w) = v \neq 0$ & $pr_W(v \oplus w) = w \neq 0$

Therefore $pr_W \circ pr_V^{-1}: V \to W$ is an iso
Therefore $V \cong W$

What we actually proved:
Theorem if $\rho: sl(2, \mathbb{C}) \to gl(V)$ is a rep of $sl(2, \mathbb{C})$ not necessarily irreducible. Then for any highest weight vector $v \in V$ $\exists!$ irred. subrep. $U \leq V$ containing v and U satisfies the conclusions of the previous thm.

Remark: U is irred. because by construction, $U = U_{-m} \oplus U_{-m-2} \oplus \cdots \oplus U_m$ and each U_k is 1-dim weight space.

If $U' \subseteq U$ were a proper subrep and so is $(U')^\perp$ so the weight decomp. of U' and $(U')^\perp$ divide $\{-m, -m+2, \ldots, m-2, m\}$ into two disjoint subsets.

Now suppose $U_m \subseteq U'$ then by the thm U' contains a subrep U'' containing U_m and (by thm) it also contains $U_{m-2}, U_{m-4}, \ldots, U_{-m}$

$\Rightarrow (U')^\perp = 0 \Rightarrow U = U'$ ∎

We saw that the adjoint rep has weight decomp.

$$\begin{array}{ccc} \bullet & \bullet & \bullet \\ -2 & 0 & 2 \end{array}$$

and $\mathrm{Sym}^2 \mathbb{C}^2$ has this also. Therefore by the thm, they are isomorphic

Example! $\mathbb{C}^2$ standard rep of $\mathfrak{sl}(2, \mathbb{C})$. What is the weight decomp.

The weights are the eigenvalues of $\rho(H) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Similarly $\rho(X) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ & $\rho(Y) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

The eigenvalues of $\rho(H)$ are 1 and -1

Hence the weight decomp. is

$$\begin{array}{cc} \bullet & \bullet \\ -1 & 1 \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ y & x \end{array}$$

$\mathrm{Sym}^n \mathbb{C}^2$? If x and y are basis of $\mathbb{C}^2$ then $x^n, x^{n-1}y, \ldots, y^n$ is a basis of $\mathrm{Sym}^n \mathbb{C}^2$

$$(\mathrm{Sym}^n H)(x^n) = x \cdot x^{n-1} + x \cdot x \cdot x^{n-2} + \cdots = n x^n$$

$\Rightarrow x^n$ has weight n

$$(\mathrm{Sym}^n H)(x^{n-k} y^k) = ((n-k) - k)\, x^{n-k} y^k = (n-2k)\, x^{n-k} y^k$$

So weight decomp of $\mathrm{Sym}^n \mathbb{C}^2$ is

$$\begin{array}{ccccccc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ y^n & & \cdots & & & x^{n-1}y & x^n \\ -n & & & & & n-2 & n \end{array}$$

$\Rightarrow \mathrm{Sym}^n \mathbb{C}^2$ is the unique irreducible rep with highest weight n

Lie Groups and Lie Algebras 3rd March

So far we know:

Theorem. All irreducible f.d. $\mathbb{C}$-reps of $sl(2, \mathbb{C})$ are $Sym^k \mathbb{C}^2$, $k \in \{0, 1, 2, \ldots\}$

Theorem: If V is a f.d. $\mathbb{C}$-rep of $sl(2, \mathbb{C})$ and $v \in V$ is a highest weight vector (i.e $\rho(X)v = 0$ and $\rho(H)v = \lambda v$ for some λ) then v is contained in an irreducible subrep

$$v, \rho(Y)v, \rho^2(Y)v, \ldots, \rho^\lambda(Y)v$$

Decomposing tensor products of reps

Given V, W reps $V \otimes W$ is a new rep.

So eg. given $Sym^k \mathbb{C}^2$, $Sym^l \mathbb{C}^2$ what is $Sym^k \mathbb{C}^2 \otimes Sym^l \mathbb{C}^2$?
[Clebsch - Gordon decomposition]

Example: $Sym^2 \mathbb{C}^2 \otimes Sym^3 \mathbb{C}^2$

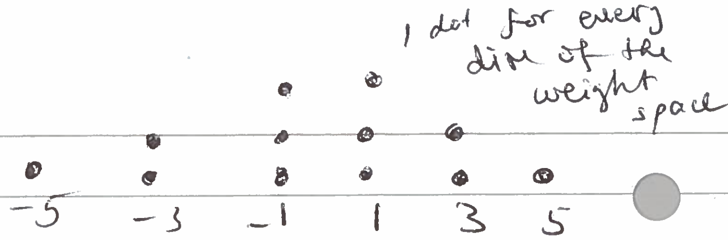
1. Write down the weight decomp of $Sym^2 \mathbb{C}^2 \otimes Sym^3 \mathbb{C}^2$?

2. Then decompose into irreducibles

$$Sym^2 \mathbb{C}^2 = \begin{array}{ccc} x & y & z \\ \circ & \circ & \circ \\ -2 & 0 & 2 \end{array}$$

$$Sym^3 \mathbb{C}^2 = \begin{array}{cccc} a & b & c & d \\ \circ & \circ & \circ & \circ \\ -3 & -1 & 1 & 3 \end{array}$$

weight diagrams

$\mathrm{Sym}^2\mathbb{C}^2 \otimes \mathrm{Sym}^3\mathbb{C}^2 =$ 

If x, y, z is a basis of $\mathrm{Sym}^2\mathbb{C}^2$ and a, b, c, d is a basis of $\mathrm{Sym}^3\mathbb{C}^2$

$x \otimes a$	$x \otimes b$, $y \otimes a$	$x \otimes c$, $y \otimes b$, $z \otimes a$	$x \otimes d$, $y \otimes c$, $z \otimes b$	$y \otimes d$, $z \otimes c$	$z \otimes d$
-5	-3	-1	1	3	5

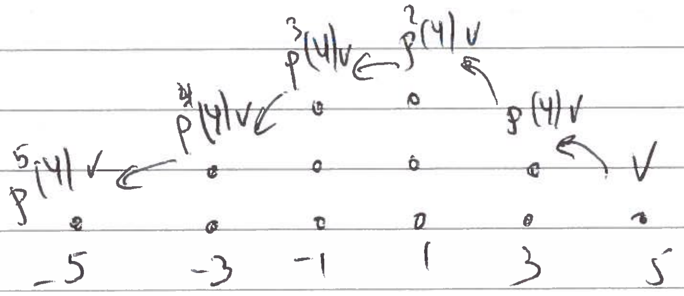
The above is a basis for $\mathrm{Sym}^2\mathbb{C}^2 \otimes \mathrm{Sym}^3\mathbb{C}^2$ listed by weight

<u>Recall</u>: x has weight 2 means

$$\mathrm{Sym}^2(H)x = 2x$$
$$\mathrm{Sym}^3(H)a = 3a$$
$$\rho = \mathrm{Sym}^2(H) \otimes \mathrm{Sym}^3(H)$$
$$\Rightarrow \rho(H)(x \otimes a) = 2x \otimes a + 3x \otimes a = 5x \otimes a$$

So we add the weights!

2. Pick $v \in V_5$ which is a highest weight vector. Look at the irrep containing v



Then I can just take the orthogonal complement of this sub rep with respect

Lie Groups and Lie Algebras 3^{rd} Mar

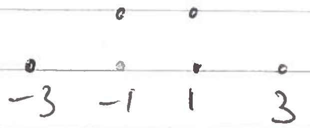
to some invariant Hermitian inner product.

$$V \cong W \oplus W^{\perp}$$

irreducible

$W =$  $\cong Sym^5 \mathbb{C}^2$

Now, what is the weight decomp of $W^{\perp}$?

$W^{\perp} =$ 

It's this because $W^{\perp}_3 =$ orth. complement of $\rho(Y)v$ in V_3

Since $\rho(Y)^k v$ spans 1-dim subspace of V_{5-2k}

$$\dim_{\mathbb{C}} W^{\perp}_{5-2k} = \dim_{\mathbb{C}} V_{5-2k} - 1$$

~~To get the decomp of $W^{\perp}$ we just repeat the previous argument~~

$$W^{\perp} \cong Sym^3 \mathbb{C}^2 \oplus \underbrace{Sym^1 \mathbb{C}^2}_{\mathbb{C}^2}$$

Overall $V \cong Sym^5 \mathbb{C}^2 \oplus Sym^3 \mathbb{C}^2 \oplus \mathbb{C}^2$

$\parallel$

$Sym^2 \mathbb{C}^2 \otimes Sym^3 \mathbb{C}^2$

Recap $V =$ 

V contains W, whose decomp. is . . . . . .

$\Rightarrow W^{\perp}$ has decomp the same as ~~that~~ for V except with one fewer dot in each column because in each weight space I am ~~thinking~~ taking the orthogonal complement of a 1-dim subspace, namely $\mathbb{C}(\rho(Y)^k v)$

! Each dot is a 1-dim v. space.

~~Theorem~~

Theorem (Clebsch - Gordon)

$$\mathrm{Sym}^{|m-n|}\mathbb{C}^2 \oplus \mathrm{Sym}^{|m-n|+2}\mathbb{C}^2 \oplus \dots \oplus \mathrm{Sym}^{m+n-2}\mathbb{C}^2 \oplus \mathrm{Sym}^{m+n}\mathbb{C}^2$$

Binary quadratic Forms

$$\underbrace{(x \ \ y)}_{v^T} \underbrace{\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}}_{M} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{v} = ax^2 + bxy + cy^2$$

Action of $SL(2,\mathbb{C})$ on binary quadratic forms $g \in SL(2,\mathbb{C})$. $M \longrightarrow (g^T)^{-1} M (g)^{-1}$

$$(gv)^T (g^T)^{-1} M g^{-1} (gv) = v^T M v.$$

The diagonal matrix $g = \begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix}$

acts as $\begin{pmatrix} e^{-i\theta} & \\ & e^{i\theta} \end{pmatrix} \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} =$

$$= \begin{pmatrix} a e^{-2ei\theta} & b/2 \\ b/2 & c e^{2i\theta} \end{pmatrix}$$

$\Rightarrow$ if $V = \{$ space of binary quadratic forms $\}$ considered as an $SL(2,\mathbb{C})$-rep then

$$V = \underset{\substack{\uparrow \\ \mathbb{C}\cdot a}}{V_{-2}} \oplus \underset{\substack{\uparrow \\ \mathbb{C}\cdot b}}{V_0} \oplus \underset{\substack{\uparrow \\ \mathbb{C}c}}{V_2} \cong \mathrm{Sym}^2\mathbb{C}^2 \text{ ~~or~~ adjoint rep.}$$

Lie groups and Lie algebras 3^{rd} Mar.

What qualities associated to quadratic forms are invariant under the action of cordinate changes by $g \in SL(2,\mathbb{C})$?
i.e. Is there a (polynomial) function of a, b, c which is invariant under $SL(2,\mathbb{C})$ action.

a, b, c are components of $\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$ i.e.

they are linear maps $V \longrightarrow \mathbb{C}$

$\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \begin{array}{l} \mapsto a \\ \mapsto b \\ \mapsto c \end{array}$

$\Rightarrow a, b, c \in V^*$

$\Rightarrow$ ~~*~~ homogeneous polynomials of degree k in a, b, c live in $Sym^k(V^*)$

Take $k=2$ $Sym^2(V^*)$ i.e. ~~Sym^2~~
$Sym^2((Sym^2\mathbb{C}^2)^*)$

Take a basis a, b, c for $V^* = (Sym^2\mathbb{C}^2)^*$
and write polys of degree 2.
$a^2 \quad ab \quad ac \quad b^2 \quad bc \quad c^2$

In V a, b, c had weights $-2, 0, 2$
But in V^* a, b, c have weights $2, 0, -2$
$\rho(X)v$, $(\rho^*(X)f)(v) = -f(\rho(X)v)$
so the weights switch sign

So $\begin{array}{ccccc} a^2 & ab & \overset{b^2}{ac} & bc & c^2 \\ 4 & 2 & 0 & -2 & -4 \end{array}$

This tells us that

$$\mathrm{Sym}^2 V^* = \cdot \quad \cdot \quad \dot{\cdot} \quad \cdot \quad \cdot$$

$$\mathrm{Sym}^2 V^* \cong \mathrm{Sym}^4 \mathbb{C}^2 \oplus \underbrace{\mathrm{Sym}^0 \mathbb{C}^2}_{\text{trivial rep}}$$

A vector in this trivial subrep is a polynomial in a, b, c s.t. it is fixed by the action of $SL(2,\mathbb{C})$ i.e. it is $\overset{\text{poly}}{\wedge}$ invariant of the coordinate changes.

In fact this 1-dim trivial subrep is spanned by the polynomial $b^2 - 4ac$ and you can check it as follows:

- Pick an invariant Hermitian inner product on $\mathrm{Sym}^2 V^*$
- Pick a highest weight vector c^2
- Apply $\rho(Y)^2 c^2$ to get something in the 0-weight space $\mathbb{C}.ac \oplus \mathbb{C}.b^2$
- Take its orthogonal complement.

  and you get $b^2 - 4ac$

All we will take away from this is that $\exists!$ invariant up to scale which is quadratic in a, b, c since the trivial part of $\mathrm{Sym}^2(V^2)$ is 1-dim

Lie Groups and Lie Algebras 3rd Mar

Let $k = 4$

$$\begin{array}{ccccccccc} & & & & ab^2c & & & & \\ \mathrm{Sym}^4(V^*) & c^3a & abc^2 & & a^2c^2 & a^2bc & a^3c & & \\ c^4 & c^3b & c^2b^2 & cb^3 & b^4 & ab^3 & a^2b^2 & a^3b & a^4 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -8 & -6 & -4 & -2 & 0 & 2 & 4 & 6 & 8 \end{array}$$

$$\mathrm{Sym}^4(V^*) = \mathrm{Sym}^8\mathbb{C}^2 \oplus \mathrm{Sym}^4\mathbb{C}^2 \oplus \underset{\uparrow \text{trivial}}{\mathbb{C}}$$

We have a trivial 1 dim subrep thus there is a quartic invariant $(b^2-4ac)^2$ Has to be up to scale because this is an ~~quartic~~, invariant of deg 4 And ~~invariant of d~~ this subrep is 1-dim so any other invariant of degree 4 is a multiple of this one.

Representations of SO(3)

We've seen that complex reps of $SU(2)$ $\overset{\text{Lie Thm}}{=}$ are the same as $\mathbb{C}$-reps of $su(2)$ $\overset{\text{complex.}}{\text{the same}}$ as $\mathbb{C}$ reps of $sl(2,\mathbb{C})$ $\overset{\text{Lie Thm}}{\text{the}}$ same as $\mathbb{C}$-reps of $SL(2,\mathbb{C})$

And we know ~~$so(3) \cong su(2)$ as Lie algebra~~
$so(3) \otimes \mathbb{C} \cong sl(2,\mathbb{C})$

$$\begin{array}{ll} K_x \to \sigma_1 & H = i\sigma_1 \\ K_y \to \sigma_2 & X = \frac{1}{2}(\sigma_2 + i\sigma_3) \\ K_z \to \sigma_3 & Y = \frac{1}{2}(\sigma_2 - i\sigma_3) \\ \quad \in su(2) & \end{array}$$

But, reps of $SO(3)$ are not necessarily the same as the reps of $SU(2)$ because $SO(3)$ is not simply connected $\pi_1(SO(3)) = \mathbb{Z}/2$

We have a double cover $SU(2) \xrightarrow{\pi} SO(3)$ because elements of $SU(2)$ acts as rotations and ± 1 act as the identity.
So given a rep ~~of~~ $R : SO(3) \longrightarrow GL(V)$ I get a rep $SU(2) \xrightarrow{\pi} SO(3) \xrightarrow{R} GL(V)$

Not every rep of $SU(2)$ arises this way

Lemma: If $R \circ \pi$ is a rep of $SU(2)$ then $(R \circ \pi)(-1) = 1$, the converse is also true

Proof: $\Rightarrow$ easy $(R \circ \pi)(-1) = R(\pi(-1)) = R(1) = 1$

$\Leftarrow$ Conversely, if $R' : SU(2) \to GL(V)$ is a rep s.t. $R'(-1) = 1$ then $\exists\, R : SO(3) \to GL(V)$ s.t. $R' = R \circ \pi$.

For every $A \in SO(3)$ pick $\tilde{A} \in SU(2)$ s.t. $\pi(\tilde{A}) = A$ and define
$R(A) = R'(\tilde{A})$ This gives a well defined map & then s.t. by construction $R \circ \pi = R'$. But it is not clear it is a rep.
Need to check
$$R'(\tilde{A}) \cdot R'(\tilde{B}) = R'(\widetilde{AB})$$

$R'(\tilde{A}) = R(A)$ $\quad$ $\tilde{A}$ has two possible values $\pi \tilde{A} = A$. They are $\pm \tilde{A}$

Lie Groups and Lie Algebras 3rd Mar

If I change my choice $\tilde{A}$ by "$-$" sign then $R'(-\tilde{A}) = R'(-1)R'(\tilde{A}) = R'(\tilde{A}) =: R(A)$

$$\cancel{\text{so } R'(\tilde{A})R'(\tilde{B}) = R(\tilde{A})R(\tilde{B}) = R(\tilde{A}\tilde{B})} =$$

~~Is~~ $\tilde{A}\tilde{B}$ $\quad R(A)R(B) = R'(\tilde{A})R'(\tilde{B}) =$
$= R'(\tilde{A}\tilde{B})$

Is $\tilde{A}\tilde{B} = \widetilde{AB}$?

~~Maybe not~~ $R'(\tilde{A}\tilde{B}) = R'(\widetilde{AB})$ because $R'(-1) = 1$

$\Rightarrow R(A)R(B) = R(AB)$.

Which reps R' of $su(2)$ satisfy $R'(-1) = 1$

$$-1 = \exp\begin{pmatrix} +i\pi & 0 \\ 0 & -i\pi \end{pmatrix}$$

$$R'\left(\exp\begin{pmatrix} i\pi & 0 \\ 0 & -i\pi \end{pmatrix}\right)v = e^{\pi i m} v = v$$

$v \in V_m$ $\Rightarrow m$ is even

~~The~~ So we need all weights to be even

$Sym^0 \mathbb{C}^2$, $Sym^2 \mathbb{C}^2$, $Sym^4 \mathbb{C}^2$, ...

1 dim $\quad$ 3 dim

$\Rightarrow$ precisely half of the irreducible reps of $su(2)$ come from reps of $SO(3)$ the ones with even weight.

Define "spin" of a rep to be $\frac{1}{2}$ (highest weight) so there are reps of integral spin.

Non-exam

Hydrogen atom

Energy in quantum mechanics arises as an eigenvalue of an "energy operator"

$$(\Delta - V)\,\Psi = H\Psi$$

Δ: Laplac. $\quad V$: potential

Separate variables like in Methods 3

$$\Psi = R(r)\,Y(u) \qquad u \in S^2$$

$$\Delta - V = \underbrace{\Delta_r}_{\text{radial}} + \frac{1}{r^2}\Delta_{S^2}$$

spherical Laplacian

$$\Delta_u = J_x^2 + J_y^2 + J_z^2$$

$$J_x = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \ \ldots$$

$$\Delta_r(RY) + \frac{1}{r^2}\Delta_u(RY) = \lambda RY$$

$$\cancel{r}^2\frac{\Delta_r R}{R} + \lambda r^2 = -\frac{\Delta_u Y}{Y} = -K$$

$$\Delta_u Y = KY$$

Spherical Harmonics are Y s.t.

Thm: Eigenval's K are $m(m+1)$, $m \in \mathbb{N}$

J_x, J_y, J_z are basis of $so(3)$. These are diff operators so when they act on functions they obey the Leibnitz Rule. $J_x^2 + J_y^2 + J_z^2 = -\frac{1}{4}(H)^2 + \frac{1}{2}(XY + YX)$

$= m(m+1)$

Lie Groups and Lie Algebras 10th Mar.

Representation theory of SU(3)

Strategy for SU(2)

Given a rep $R : SU(2) \to GL(V)$

1. Consider the subgroup

$$T = \left\{ \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{i\varphi} \end{pmatrix} : \varphi \in [0, 2\pi) \right\} \subseteq SU(2)$$

This is a 1-dim torus

2. Restrict $R|_T : T \longrightarrow GL(V)$

This gave us a weight decomp. $V = \bigoplus_m V_m$ where

$$V_m = \left\{ v \in V : R\begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{i\varphi} \end{pmatrix} v = e^{im\varphi} v \right\} =$$

$$= \left\{ v \in V : R_* \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} v = imv \right\}$$

$$= \{ v \in V : \rho(H) v = mv \}$$

where ρ is the complexification of R_*, a rep of $\mathfrak{sl}(2, \mathbb{C})$ and $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Then

3. Describe the action of $\rho(X)$ and $\rho(Y)$ on V
   - $\rho(X) V_m \subseteq V_{m+2}$
   - $\rho(Y) V_m \subseteq V_{m-2}$

4. Using $[X, Y] = H$ we showed that a highest weight vector v generates a subrepresentation $v, \rho(Y)v, \rho^2(Y)v, \dots$

For $SU(3)$ we will do the same steps.

1. Pick the corresponding torus

$$T = \left\{ \begin{pmatrix} e^{i\varphi_1} & 0 & 0 \\ 0 & e^{i\varphi_2} & 0 \\ 0 & 0 & e^{-i(\varphi_1+\varphi_2)} \end{pmatrix}, \varphi_1, \varphi_2 \in [0, 2\pi) \right\} \subseteq SU(3)$$

$$T \cong U(1)^2$$

Let t denote the Lie algebra T, so $t \cong \mathbb{R}^2$

2. Restrict given representation $R: SU(3) \longrightarrow GL(V)$ to T

This representation $R|_T : T \longrightarrow GL(V)$ has a weight decomposition $V = \bigoplus_{\alpha \in t^*_{\mathbb{Z}}} V_\alpha$

where the weights α now belong to the weight lattice $t^*_{\mathbb{Z}}$.

$$V_\alpha = \left\{ v \in V : R \begin{pmatrix} e^{i\varphi_1} & 0 & 0 \\ 0 & e^{i\varphi_2} & 0 \\ 0 & 0 & e^{-i(\varphi_1+\varphi_2)} \end{pmatrix} v = e^{i(m_1\varphi_1 + m_2\varphi_2)} v \right\}$$

($\alpha = m_1 L_1 + m_2 L_2$, m_1, m_2 are fixed by α)

$$= \{ v \in V, \rho(H) = \alpha(H) v \quad \forall H \in t \}$$

Diagonal matrices in the lie algebra $su(3)$

$$H_{12} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in t, \quad H_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in t$$

$$m_1 - m_2 = \alpha H_{12} \quad \text{and} \quad m_2 = \alpha(H_{23})$$

Recall $t^*_{\mathbb{Z}} = \{ \alpha_{\text{linear}} : t \to \mathbb{C} \mid \alpha(X) \in 2\pi i \mathbb{Z} \text{ if } \exp X = 1 \}$

Lie Groups and Lie algebras 10th Mar

α assigns a number to each $H \in t$

So what is $\{H \in t : \exp H = 1\}$ for $SU(3)$?

$$t = \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} : a_1 + a_2 + a_3 = 0 \right\}$$

$$\exp \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} = \begin{pmatrix} e^{a_1} & 0 & 0 \\ 0 & e^{a_2} & 0 \\ 0 & 0 & e^{a_3} \end{pmatrix} = 1$$

$\Leftrightarrow a_1, a_2, a_3 \in 2\pi i \mathbb{Z}$

$\Leftrightarrow a_1$ and $a_2 \in 2\pi i \mathbb{Z}$, because $a_3 = -a_1 - a_2$

So $t^* = \{$ linear maps $t \to \mathbb{C}\}$

It is spanned by the linear maps

$$L_1 \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} = a_1, \quad L_2 \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} = a_2$$

$$L_3 \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} = a_3 \qquad \text{where } L_1 + L_2 + L_3 = 0$$

And inside $t^*_{\mathbb{Z}} = \{\alpha = m_1 L_1 + m_2 L_2 : \alpha \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} \in 2\pi i \mathbb{Z}$

$\Leftrightarrow a_1, a_2 \in 2\pi i \mathbb{Z} \}$

$\Leftrightarrow t^*_{\mathbb{Z}} = \{ m_1 L_1 + m_2 L_2 : m_1, m_2 \in \mathbb{Z} \}$

$$(m_1 L_1 + m_2 L_2)\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} = m_1 a_1 + m_2 a_2$$

$\in 2\pi i \mathbb{Z} \quad \forall\, a_1, a_2 \in 2\pi i \mathbb{Z}$ iff $m_1, m_2 \in \mathbb{Z}$

3. We need to find suitable X, Y analogues in $su(3) \otimes \mathbb{C} \cong sl(3, \mathbb{C})$.

Here is a basis for $sl(3, \mathbb{C}) = su(3) \otimes \mathbb{C}$

$$H_{12} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad H_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$sl(3, \mathbb{C})$ is 8-dim $\mathbb{C}$

$$E_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

E_{ij} = 1 in the ij^{th} entry, 0's everywhere else

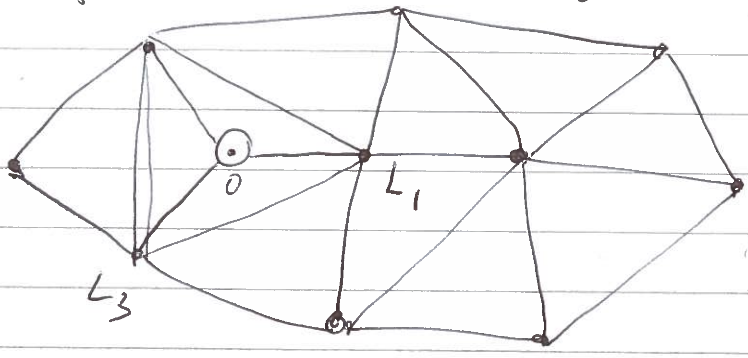
$E_{13}, E_{23}, E_{21}, E_{31}, E_{32}$ form a basis for $sl(3,\mathbb{C})$

We have already seen that $t^*_{\mathbb{Z}} = \{m_1 L_1 + m_2 L_2 : m_1, m_2 \in \mathbb{Z}\}$

where $L_i \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} = a_i$

We will draw $t^*_{\mathbb{Z}}$ as a triangular lattice because there is also $L_3 = -L_2 - L_1$ with $L_1 + L_2 + L_3 = 0$

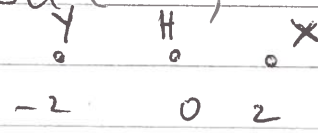
Lie groups and Lie algebras 10th Mar.



Picture of $\mathfrak{t}^*_{\mathbb{Z}}$

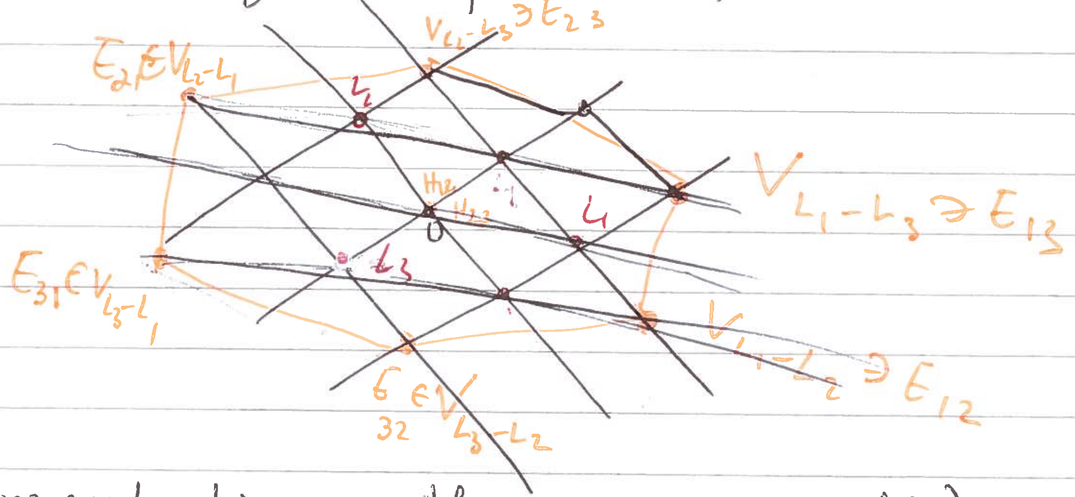
The fact that $L_1 \cdot L_2 = \cos \frac{2\pi}{3}$ really makes sense because $\mathfrak{t}^*_{\mathbb{Z}}$ comes equipped with a natural dot product called the Killing form

For $su(2)$, the adjoint rep looks like



So X, Y are determined up to scale by the choice of torus spanned by H

The analagous picture of $su(3)$ is as follows,



Representations theory of $su(3)$

$$\mathrm{ad}\begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} E_{ij} = \left[\begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix}, E_{ij}\right] =$$

$$\begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} \cdot E_{ij} - E_{ij} \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}$$

$$= a_i \cdot E_{ij} - a_j E_{ij} = (a_i - a_j) E_{ij}$$

$$(L_i - L_j)\begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} = a_i - a_j$$

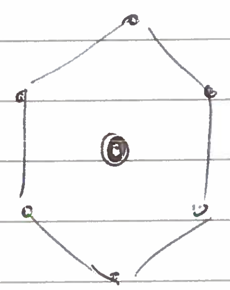
$\Rightarrow E_{ij} \in V_{L_i - L_j}$, where V is the adjoint rep. $= sl(3, \mathbb{C})$

Recall $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$

$$X \to (\underbrace{Y \to [X, Y]}_{ad_X})$$

The remaining elements H_{12} and H_{23} satisfy $ad \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} H_{ij} = 0$

$\Rightarrow H_{12}, H_{23}$ span 2-dim weight space V_0.



Definition: The weights / weight spaces of ad are called roots / root spaces

Lie Groups and Lie algebras 10th Mar.

Definition: We write $R = \{L_i - L_j\}_{i \neq j}$ $\subset \mathfrak{t}^*_{\mathbb{C}}$ for the set of roots of $sl(3, \mathbb{C})$

If $\alpha \in R$ then we write $sl(3, \mathbb{C})_\alpha$ for the corresponding root space.

The vector space $V = sl(3, \mathbb{C})$

Lemma Let V be the rep of $sl(3, \mathbb{C})$ and $X \in sl(3, \mathbb{C})_\alpha$ [so $X \in \{E_{12}, E_{21}, E_{13}, E_{31}, E_{23}, E_{32}\}$]
depending on α

If $v \in V_\beta$ then $\rho(X) v \in V_{\alpha+\beta}$, $\alpha, \beta \in \mathfrak{t}^*_{\mathbb{C}}$

Remark
In the $sl(2, \mathbb{C})$ case $X \in sl(2, \mathbb{C})_2$ so $\rho(X) v \in V_{\beta+2}$ and $Y \in sl(2, \mathbb{C})_{-2}$ so $\rho(Y) v \in V_{\beta-2}$.

Proof: Want to check $\rho(X) v \in V_{\alpha+\beta}$

$V_{\alpha+\beta} = \{ w \in V \text{ s.t. } \rho(H) w = \alpha(H) w, \forall H \in \mathfrak{t}\}$

We need to find $\rho(H)\rho(X)v =$

$\rho(H)\rho(X)v = \rho(H, \cancel{X}) v + \rho(X)\rho(H) v =$

$= \rho(ad_H X) v + \rho(X) \beta(H) v$ as $v \in V_\beta$

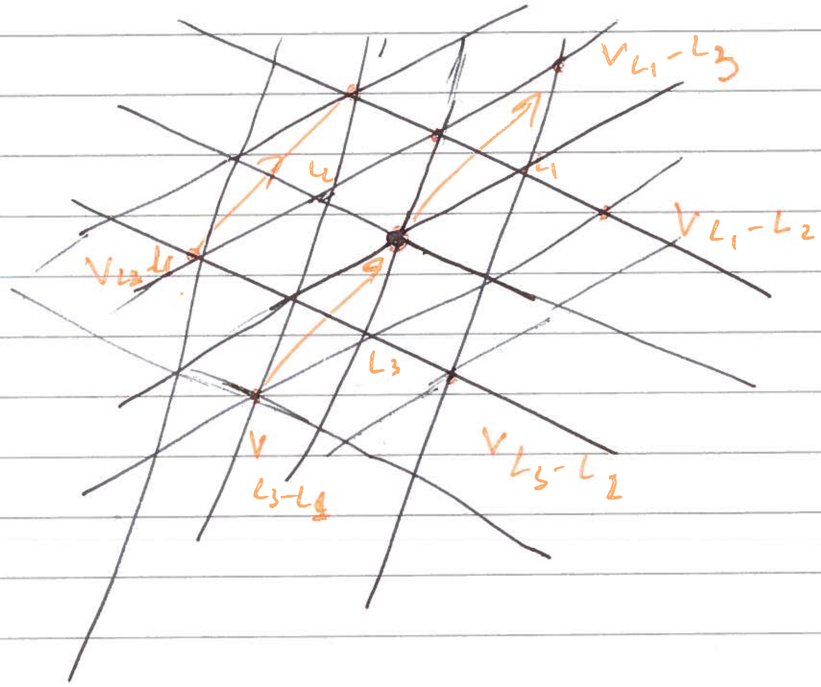
$= \rho(\alpha(H) X) v + \rho(X) \beta(H) v$
as $X \in sl(3, \mathbb{C})_\alpha$

$= (\alpha(H) + \beta(H)) \rho(X) v \Rightarrow \rho(X)v \in V_{\alpha+\beta}$ ∎

<u>Corollary</u>: If $X \in sl(3,\mathbb{C})_\alpha$ and $Y \in sl(3,\mathbb{C})_\beta$

$$[X, Y] = ad_X(Y) \in sl(3,\mathbb{C})_{\alpha+\beta}$$

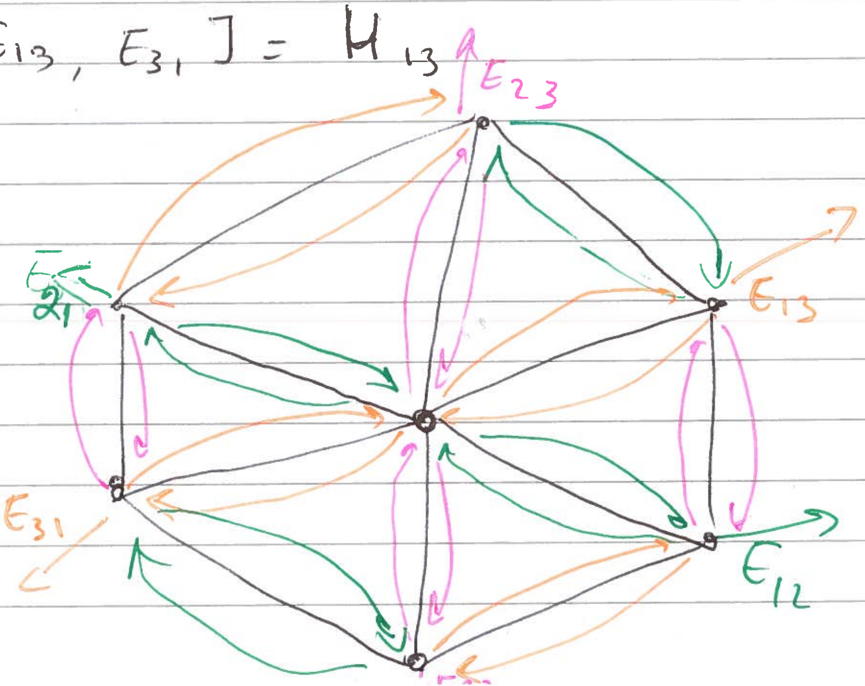
apply lemma with $\rho = ad_X$ and $v = Y$ ∎



$$\Rightarrow [E_{13}, E_{21}] = \begin{pmatrix} 0 & 0 & 1 \\ & 0 & \end{pmatrix}\begin{pmatrix} 0 & & \\ 1 & 0 & \\ 0 & & \end{pmatrix} = 0$$

$$- \begin{pmatrix} 0 & & \\ 1 & 0 & \\ 0 & & \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ & 0 & \end{pmatrix} = -E_{23}$$

$$[E_{13}, E_{31}] = H_{13}$$



Lie Groups and Lie Algebras 10th Mar.

In fact $\mathbb{C}E_{12} \oplus \mathbb{C}H_{12} \oplus \mathbb{C}E_{21}$,
$\mathbb{C}E_{13} \oplus \mathbb{C}H_{13} \oplus \mathbb{C}E_{31}$,
$\mathbb{C}E_{23} \oplus \mathbb{C}H_{23} \oplus \mathbb{C}E_{32}$

are 3 Lie subgroups of $sl(3, \mathbb{C})$ each is isomorphic to $sl(2, \mathbb{C})$

Proof: Just do 1-2 case

We've seen $[E_{12}, E_{21}] = H_{12}$

$ad_{H_{12}} E_{12} = [H_{12}, E_{12}] = 2E_{12}$

$ad_{H_{12}} E_{21} = [H_{12}, E_{21}] = -2E_{21}$

Since $E_{12} \in sl(3, \mathbb{C})_{L_1 - L_2}$ so $ad_{H_{12}} E_{12} = (L_1 - L_2)(H_{12})$ $\binom{1}{-1}$ E_{12}

$\cancel{E_{12}} = ad_{H_{12}} E_{12} = (1-(-1))E_{12} = 2E_{12}$

and $E_{21} \in sl(3, \mathbb{C})_{L_2 - L_1}$

Take $X = E_{12}$, $Y = E_{21}$, $H = H_{12}$

this gives us an isomorphism with $sl(2, \mathbb{C})$

$\boxtimes$

We call these three $sl(2, \mathbb{C})$-subalgebras and write them as

$S_{L_1 - L_2}$
$S_{L_1 - L_3}$
$S_{L_2 - L_3}$

} more generally as S_α for $\alpha \in R$.

Example: Standard representation of $sl(3, \mathbb{C})$

$\mathbb{C}^3 = V$

e_1, e_2, e_3 basis for V.

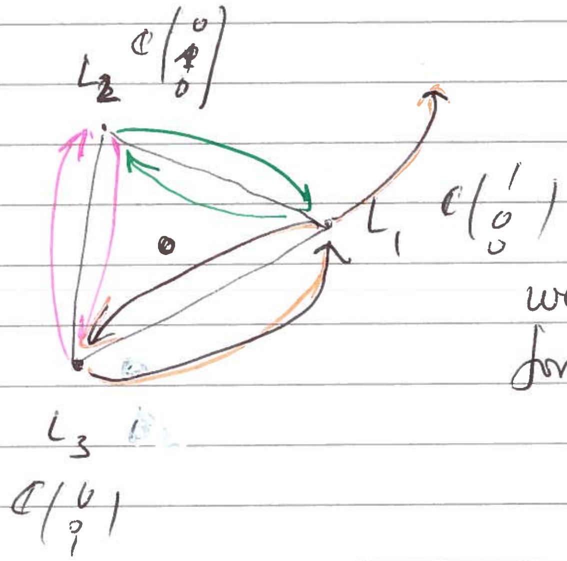
What are the weight spaces and what are

the weights?

$$\begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} e_i = a_i e_i$$

so weights are L_1, L_2, L_3 and the weight spaces are spanned by

$$\mathbb{C}\begin{pmatrix}1\\0\\0\end{pmatrix}, \mathbb{C}\begin{pmatrix}0\\1\\0\end{pmatrix}, \mathbb{C}\begin{pmatrix}0\\0\\1\end{pmatrix}$$



weight diagram for standard rep.

If we act using E_{13} : $E_{13}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}$

$$E_{13}\begin{pmatrix}1\\0\\0\end{pmatrix} = 0$$

$$E_{13}\begin{pmatrix}0\\1\\0\end{pmatrix} = 0 \qquad E_{31}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

Example : Take V^* dual to standard rep.

$$\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$$

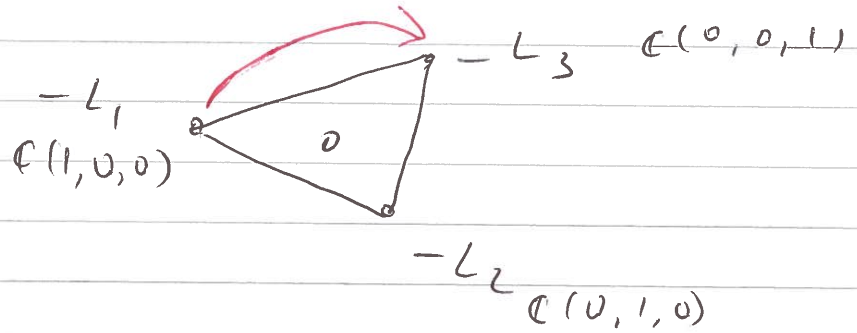
$$\rho : \mathfrak{g} \to \mathfrak{gl}(V^*)$$

$$(\rho^*(x)f)(v) = -f(\rho(x)v)$$

Lie Groups and Lie algebras 10^{th} Mar

So if α is a weight of V then $-\alpha$ is a weight of V^*

So the weight diagram for $(\mathbb{C}^3)^*$ is



Recall that row vectors are dual to column vectors.

$$(1,0,0)\underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{E_{13}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Remark: From these weight decomps, we see that $\mathbb{C}^3 \not\cong (\mathbb{C}^3)^*$ as reps of $\mathfrak{sl}(3,\mathbb{C})$

Recap: In $SU(3)$, we looked at the torus $T = \left\{ \begin{pmatrix} e^{i\varphi_1} & & \\ & e^{i\varphi_2} & \\ & & e^{-i(\varphi_1+\varphi_2)} \end{pmatrix} : ~~and~~ \varphi_1, \varphi_2 \in [0, 2\pi) \right\}$

For each rep $R: SU(3) \longrightarrow GL(V)$ we took the weight decomp of V as a rep of T, i.e. $R|_T$ $V = \bigoplus_{\substack{\alpha \in A \\ A \subseteq \mathfrak{t}^*_{\mathbb{R}}}} V_\alpha$

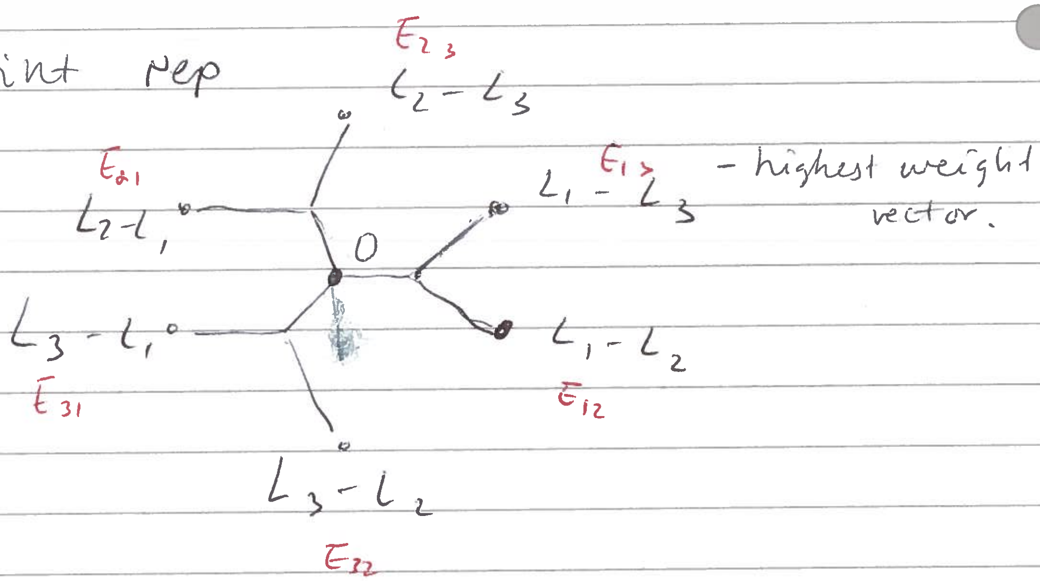
$V_\alpha = \{ v \in V : R(\exp(iH))v = e^{i\alpha(H)} v \quad \forall H \in t \}$

$$t = \left\{ \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} : a_1 + a_2 + a_3 = 0 \right\}$$

α is an integer linear combination of L_1, L_2, L_3 which are the weights

$L_i \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} = a_i \quad , \quad L_1 + L_2 + L_3 = 0$

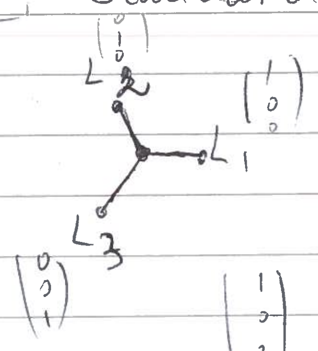
e.g. adjoint rep



$E_{12} \in L_1 - L_2$ means $ad_H E_{12} = (a_1 - a_2) E_{12}$ if

$H = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}$ etc. (i.e. E_{12} is an eigenvector for H with eigenval $(a_1 - a_2)$)

Example : standard rep $\mathbb{C}^3$



$$\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ This is an eigenvector for H and the eigenval L_1

Finally, we saw that there are three $sl(2,\mathbb{C})$ subalgebras

$\langle E_{12}, E_{21}, H_{12}\rangle$, $\langle E_{13}, E_{31}, H_{13}\rangle$ and

$\langle E_{23}, E_{32}, H_{23}\rangle$ we call them

$S_{L_1-L_2}$, $S_{L_1-L_3}$, $S_{L_2-L_3}$ since

$E_{12} \in sl(3,\mathbb{C})_{L_1-L_2}$, $E_{13} \in sl(3,\mathbb{C})_{L_1-L_3}$, $E_{23} \in sl(3,\mathbb{C})_{L_2-L_3}$

Lemma: If $X \in s_\alpha$ and $v \in V_\beta$ then $\rho(x)v \in V_{\alpha+\beta}$

Classifying irreps of $sl(3,\mathbb{C})$

We need a notion of highest weight vector.

Def. A linear function π on a v.space is said to be irrational w.r.t. a given lattice Λ if $\pi(\alpha) = \pi(\beta) \Rightarrow \alpha = \beta$

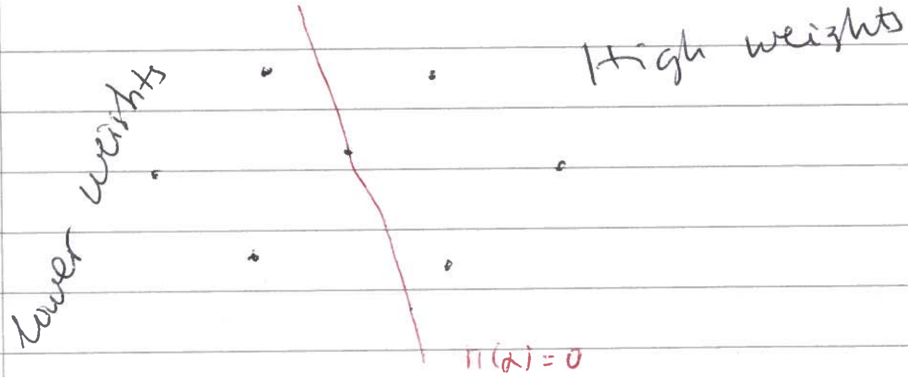
e.g. Take $e_1, \dots, e_k$ an integral basis for Λ
Take $\mu_1, \dots, \mu_k \in \mathbb{R}$ which are l.i. over $\mathbb{Q}$
Then $\pi = \mu_1 e_1^* + \dots + \mu_k e_k^*$ (e_i^* is dual basis)
this is irrational w.r.t. Λ.

Why? if $\alpha, \beta \in \Lambda$ apply π to α
$\alpha = a_i e_i$ and $\beta = b_i e_i$

$\Rightarrow \pi(\alpha) = \sum a_i \mu_i$, $\pi(\beta) = \sum b_i \mu_i$
if $\pi(\alpha) = \pi(\beta)$
$\Rightarrow \sum (a_i - b_i)\mu_i = 0$ but μ_i are l.i. over $\mathbb{Q}$ =
this linear dependence over $\mathbb{Q}$

$\Rightarrow \quad a_i = b_i \quad \forall i.$

In our example Λ = triangular lattice and $\pi(\alpha) = 0$ in $\mathbb{R}^2$ we just need this to be a line not intersecting any Λ-points ($\Leftrightarrow \mathrm{Ker}(\pi) = 0$)



Define A highest weight vector v is one s.t. it's weight α is maximal i.e. $\pi(\alpha)$
$$\pi(\alpha) = \max_{\beta \in A} \pi(\beta) \quad , \quad V = \bigoplus_{\alpha \in A} V_\alpha$$

Definition: Our roots (weights of ad rep) split into two types: positive roots and negative roots: R_+ & R_-

$$R_+ = \{ \alpha \in R : \pi(\alpha) > 0 \}$$
$$R_- = \{ \alpha \in R : \pi(\alpha) < 0 \}$$

In our example $R_+ = \{ L_1 - L_2, L_2 - L_3, L_1 - L_3 \}$
$R_- = \{ L_2 - L_1, L_3 - L_2, L_3 - L_1 \}$

Corollary: $v \in V$ is a highest weight vector when $\rho(X)v = 0 \quad \forall X \in R_+$

Lie Groups and Lie algebras 14.03

Proof: $v \in V_\alpha$ is highest weight vector
$\Leftrightarrow \pi(\alpha) = \max_{\beta \in A} \pi(\beta)$

$\rho(X) v \in V_{\alpha+\gamma}$ if $X \in sl(3, \mathbb{C})_\gamma$

$\pi(\gamma) > 0$ as $\gamma \in R_+$

$\Rightarrow \pi(\alpha+\gamma) = \pi(\alpha) + \pi(\gamma) > \pi(\alpha)$
$\Rightarrow \alpha+\gamma$ is not in A

$\Rightarrow V_{\alpha+\gamma} = 0 \Rightarrow \rho(X) v = 0$ ■

Theorem (Classification)

Let V be a f.d. rep of $sl(3, \mathbb{C})$ and $v \in V_\alpha$ be a highest weight. Then the following elements generate a irred. subrep containing v:

$\rho(Y_1) \cdots \rho(Y_k) v$

$Y_i \in \{E_{21}, E_{32}, E_{31}\}$

Proof: Let W denote the subspace spanned by $\{\rho(Y_1) \cdots \rho(Y_k) v : Y_1, \dots, Y_k \in \{E_{21}, E_{32}, E_{31}\}\}$

Need to show that this is a subrep.

Need to show that $\rho(X) w \in W$ $\forall w \in W, X \in sl(3, \mathbb{C})$

This is easy for $X \in \{E_{21}, E_{31}, E_{32}\}$ as it will just increase the length of the string of Y's.

Let's check it for $H = \begin{pmatrix} a_1 & & 0 \\ & a_2 & \\ 0 & & a_3 \end{pmatrix}$

Assume

$\rho(H)\rho(Y_1) \dots \rho(Y_n) v \in W$ for all strings of length at most n.

$$\rho(H)\rho(Y_1)\dots\rho(Y_{n+1}) =$$

$$= \rho([H, Y_1])\rho(Y_2), \dots \rho(Y_{n+1}) +$$

$$+ \rho(Y_1)\rho(H)\rho(Y_2), \dots \rho(Y_{n+1}) =$$

$$= \rho(ad_H Y_1)\rho(Y_2)\dots\rho(Y_{n+1}) v +$$

$$+ \rho(Y_1) w' \quad , w' \in W \text{ by inductive assumption}$$

$Y_1 \in sl(3, \mathbb{C})_{\alpha_1}$, for some $\alpha_1 \in R_-$

$\Rightarrow ad_H Y_1 = \alpha_1(H) Y_1$

$\Rightarrow \quad = \alpha_1(H)\rho(Y_1)\dots\rho(Y_{n+1})v + \rho(Y_1)w'$

$\in W \Rightarrow$ if $w \in W$ $\rho(H) w \in W$

On sheet 7 we check E_{12}, E_{13}, E_{23} ∎

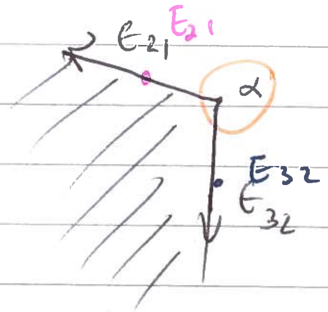
Lie Groups and Lie Algebras 14th Mar.

Structure of weight diagram

Pick $\rho \in \mathfrak{sl}(3, \mathbb{C}) \to \mathfrak{gl}(V)$ with highest weight $\alpha \in t_{\mathbb{Z}}^*$.

Observations

- the weights of V are contained in the shaded region (because starting with $v \in V_\alpha$, I just apply $\rho(Y_i)$ with Y_i a negative root vector)



- The weight spaces V_α, $V_{\alpha+k(L_3-L_2)}$, $V_{\alpha+\ell(L_2-L_1)}$, $k, \ell \in \mathbb{N}$ are at most 1-dim.

To see this look at the basis

v, $\rho(E_{32})v$, $\rho(E_{32})^2 v, \dots$, $\rho(E_{21})v$, $\rho(E_{21})^2 v, \dots$, $\rho(E_{21})\rho(E_{32})v$, etc.

v is the only basis vector with weight α

$\rho(E_{21})v$ is the only basis vector with weight $\alpha + (L_2 - L_1)$

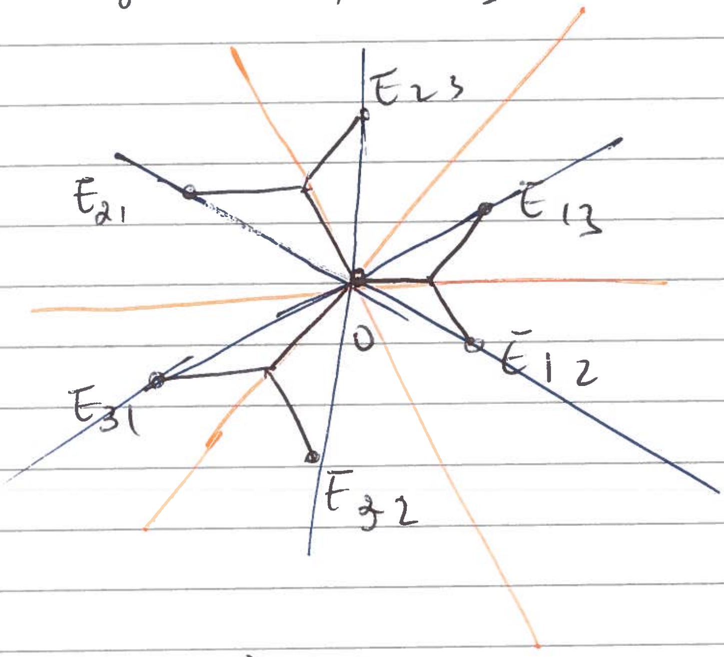
similarly $\rho(E_{21})^k v$ is the only — " — $\alpha + k(L_2 - L_1)$

$\rho(E_{32})^\ell v$ — " — weight $\alpha + \ell(L_3 - L_2)$

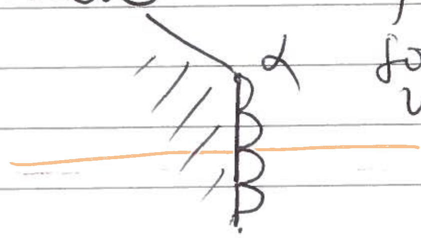
Thus they are all 1 dim.

Theorem: The weight diagram of a irreducible $\mathfrak{sl}(3, \mathbb{C})$ rep is:

a) symmetric under the action of the Weyl group: $W(\mathfrak{sl}(3,\mathbb{C}))$, which is a group of reflections in the lines through L_1, L_2, L_3, the orange lines in the picture



b) In particular, α is the highest weight then $\rho(E_{32})^{e+1}\alpha = 0$ for the first time when $l = \alpha(H_{32})$



c) As a consequence of part a) the weight diagram is either a hexagon (like the adjoint rep) or a triangle (like the standard rep) and there is an algorithm which tells you the dimensions of the weight spaces at each weight. (To be given later). This gives the value 1 to the vertices along the edges.

Lie Groups and Lie Algebras 14th Mar

P : First prove (B)

$S_{L_3-L_2} = \langle E_{32}, E_{23}, H_{23} \rangle$

E_{32}, E_{23} and H_{23} span an $sl(2,\mathbb{C})$ subalg. so we can actually think of V as a rep of $sl(2,\mathbb{C})$. The direct sum of weight spaces $\bigoplus_{l} V_{\alpha+l(L_3-L_2)}$ is

preserved by E_{32}, E_{23}, H_{23} so it gives a subrep of $S_{L_3-L_2}$ and $\dim_{\mathbb{C}} V_{\alpha+l(L_3-L_2)} \leq 1$

So this is an irred. $sl(2,\mathbb{C})$ rep. going from weight $\alpha H_{23} = m$ to $[\alpha+l(L_3-L_1)](H_{23}) = -m$

$\Rightarrow \ell(E_{32})^{m+1} v = 0$ (B)

We also know that weight diagrams of $sl(2,\mathbb{C})$ reps are symmetric around the origin. $\Rightarrow$ this line of dots is also symmetric around the point where the line iL_1 intersects this edge.

Every vertical line of dots is preserved by $S_{L_3-L_2}$ therefore symmetric around the red axis.

If we run exactly the same argument with a different choice of irrational linear function. This gives the same conclusions for the other axis

(A)

Take $\alpha = L_1 - 2L_3$. We will plot the weight diagram for the unique irreducible rep of $sl(3, \mathbb{C})$ with this highest weight.

<u>Step 1</u>: Plot the orbit of α under the action of Weyl group

<u>Step 2</u>: The weight diagram is contained in the convex hull of these six points

<u>Step 3</u>: Which weights occur?

By applying $\rho(E_{31})$ to v we get a weight vector with weight $L_1 - 2L_3 + L_3 - L_1 = -L_3$

Applying again I get a vector in weight space with weight $-L_3 + L_3 - L_1 = -L_1$

Applying $\rho(E_{31})$ to $\rho(E_{32})v$ I get to $\rho(E_{31})\rho(E_{32})v$ ~~of~~ weight $-L_2$

These are the only weights I can get to by applying E_{32}, E_{31}, E_{21} in some order

<u>Step 4</u>: What dimension are the weight spaces? The weights β on the edges have $\dim_{\mathbb{C}} V_\beta = 1$

<u>Algorithm</u>:

Weight diagram splits into nested hexagonal / triangular shells.

$X_0 \supseteq X_1 \supseteq X_2 \supseteq \dots$

0

Lie Groups and Lie Algebras 14th Mar.

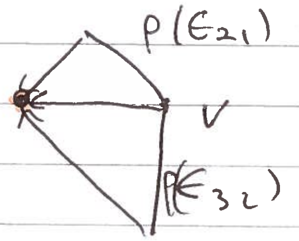
The dimensions of the weight spaces are constant on each shell, and the multiplicity for weights in $X_k \setminus X_{k-1} = m_k$ then $m_{k+1} = \begin{cases} m_k & \text{if } X_k \text{ is a } \triangle \\ m_k + 1 & \text{if } X_k \text{ is a hexagon} \end{cases}$

In our examples

X_0 then $m_0 = 1$

X_1 then $m_1 = m_0 + 1 = 2$ since X_0 is hexagon.

<u>Remark</u>: There are 3 ways to get to ~~the~~ orange



i.e. $\rho(E_{32})\rho(E_{21})v$

$\rho(E_{21})\rho(E_{32})v$

$\rho(E_{31})v$

$\Rightarrow$ orange weight space has $\dim_{\mathbb{C}} \leq 3$

In fact, since $[E_{32}, E_{21}] = E_{31}$

$\rho(E_{32})\rho(E_{21})v - \rho(E_{21})\rho(E_{32})v = \rho(E_{31})v$

$\Rightarrow$ we have a linear dependence, so this weight space has $\dim_{\mathbb{C}} \leq 2$.

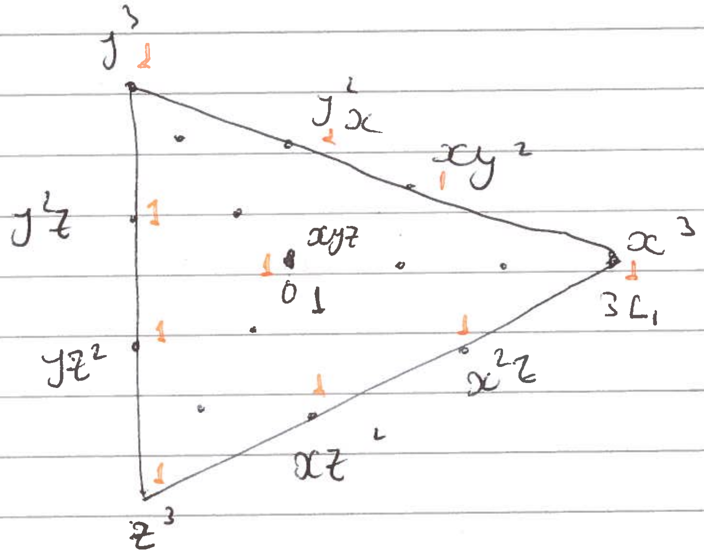
Example: $Sym^3 \mathbb{C}^3$.

Picks a basis x, y, z for $\mathbb{C}^3$

x^3 x^2y, xy^2, y^3, x^2z, xz^2, z^3,

y^2z, yz^2, xyz

This is a basis of $Sym^3(\mathbb{C}^3)$ so the weight diagram is



y^3
y^2x
xy^2
y^2z
xyz
x^3
0 1
$3L_1$
yz^2
x^2z
xz^2
z^3

Lie Groups and Lie Algebras

21st Mar

Compact semisimple Lie groups

1. Find a Torus $T \subseteq G$
2. Given a rep $R: G \to GL(V)$ weight decomposition of $R|_T$

$$V = \bigoplus_{\alpha \in t_{\mathbb{Z}}^*} V_\alpha$$

3. Study the adjoint rep of the group G: weight spaces are called root spaces (E_{ij}) Study how root vectors act on the weight spaces.

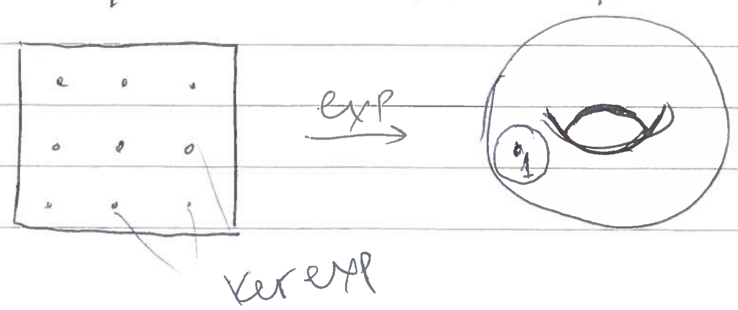
Compact Abelian Lie Groups

Proposition: A compact abelian Lie group G is a torus $\cong U(1)^n$ for some n.

Proof: Look at $\exp: \mathfrak{g} \to G$. As G is abelian the lie bracket $[\,,\,]$ is zero on $\mathfrak{g}$. Thus $\exp(A+B) = \exp A \cdot \exp B$ this follows from Baker-Campbell-Hausdorff formula.

$\Rightarrow$ exp gives a homomorphism from $(\mathfrak{g}, +) \to (G, \cdot)$. So by the 1st Isomorphism Thm, $G = \mathfrak{g}/\ker(\exp)$

$\ker(\exp) \subseteq \mathfrak{g}$ is a discrete lattice i.e. for every $p \in \ker(\exp)$ $\exists$ ball $B_p \ni p \subseteq \mathfrak{g}$ s.t. $B_p \cap \ker(\exp) = \{p\}$



We showed that exp is a local diffeo i.e. there exists a ball $B \subseteq \mathfrak{g}$ and ball $C \subseteq G$ s.t. $\exp|_B : B \to C$ is a diffeo i.e. it is the bijective, so $0 \in B$ is the only point of B in $\ker(\exp)$.

Either use the same argument with an exponential chart at each point $p \in \ker(\exp)$ or just use the ball $B_p = p + B$. This shows us that $\ker\exp \subseteq \mathfrak{g}$ is a discrete lattice and any lattice in $\mathbb{R}^n$ is of the form $\mathbb{Z}^m \subseteq \mathbb{R}^n$ i.e. $\exists$ a integral basis for the lattice with $m \leq n$ elements and the quotient $\mathfrak{g}/\ker\exp$ is then $\cong \mathbb{R}^n/\mathbb{Z}^m \cong (\mathbb{R}/\mathbb{Z})^m \oplus \mathbb{R}^{n-m}$ (by extending this basis. But our group $G \cong \mathfrak{g}/\ker(\exp) \cong (\mathbb{R}/\mathbb{Z})^m \oplus \mathbb{R}^{n-m}$ is compact if only if $n = m$. Thus $G \cong (\mathbb{R}/\mathbb{Z})^m$ and $\mathbb{R}/\mathbb{Z} \cong U(1)$ via $\theta \to e^{i2\pi\theta}$ ∎

Remark: This actually proves that $\exp : \mathfrak{g} \to G$ is a covering map.

Proposition: If G is a compact group then G contains a torus T. Moreover it contains a maximal torus i.e. it is not contained in any strictly bigger torus

Proof: Take $X \in \mathfrak{g}$ and take $T' = \{\exp(sX) | s \in \mathbb{R}\}$

T' is a one parameter subgroup as $\exp(sX)\exp(tX) = \exp((s+t)X) \in T'$

Lie Groups and Lie Algebras 21st Mar.

Take $\overline{T'} \leq G$ (topological closure) i.e. add all possible limit points to T' so that I get a closed set.
We proved that this is an abelian group of G

[Reminder proof $g = \lim_{i \to \infty} g_i$, $h = \lim_{i \to \infty} h_i$, then

$gh = \lim(g_i \cdot h_i)$ because multiplication is continuous i.e. $(\lim g_i)\cdot(\lim h_i) = \lim(g_i h_i)$
$= \lim(h_i \cdot g_i)$
since T' is abelian $= hg$. ■]

Closed subset of a compact set is compact.
so $T := \overline{T'}$ is a compact abelian group
Therefore a Torus.

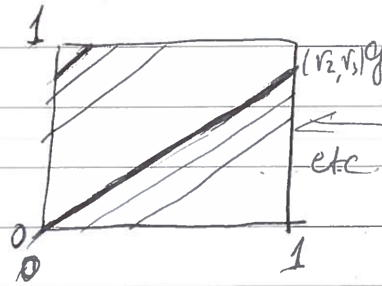
For maximality consider the partially ordered set of abelian subalgebras in the Lie algebra of the group $\mathfrak{g}$.
For each Torus its Lie algebra is an abelian subalgebra of $\mathfrak{g}$ and each subalgebra $\mathfrak{t}$ defines a torus $\exp \mathfrak{t}$. The p.o. set has maximal element (with respect to inclusion) as $\mathfrak{g}$ is finite dim. v.s. so increasing sequences of nested subspaces terminate. Take the corresponding torus and it will be maximal ■

Definition: An element X in the Lie alg. of a torus T is called a topological generator if $T = \overline{\{\exp(sX) : s \in \mathbb{R}\}}$ i.e. X generates a dense one parameter subgroup of T.

Lemma: Any Torus has a topological generator.

Proof:
The idea is if $X = (i\theta_1, \ldots, i\theta_n) \in u(1)^n$.
WTS that the set $(t\theta_1, \ldots, t\theta_n) \bmod \mathbb{Z}^n$ is dense in $\mathbb{R}^n/\mathbb{Z}^n$.
Theorem (Kronecker) This is true if $\theta_1, \ldots, \theta_n$ are s.t. $k_1\theta_1 + \ldots k_n\theta_n \notin \mathbb{Z}$ $\forall (k_1, \ldots, k_n) \in \mathbb{Z}^n$

e.g. $(\sqrt{2}, \sqrt{3}) = (\theta_1, \theta_2)$



~~gives a dence subset of the square.~~
this is the line of slope $\sqrt{\frac{3}{2}}$ repeated mod $\mathbb{Z}^2$.

Theorem: Let G be a connected compact group, T max torus
a) Any element of the group G is conjugate to $x^{-1}gx \in T$
b) T' is another maximal torus then $\exists x \in G$ s.t. $T = xT'x^{-1}$

c) $\exp: \mathfrak{g} \to G$ is surjective.

~~Non exam!~~
Proof: Use Lefschetz fixed point Theorem.
Recall if $f: X \to X$ is a map homotopic to the identity then $L(f)$ which can be computed in terms of fixed points
$L(f) = \chi(X)$ in particular if $\chi(X) \neq 0$
then $\exists$ a fixed point.

Lie Groups and Lie Algebras 21st Mar.

$x^{-1}gx \in T \Leftrightarrow gx \in xT \Leftrightarrow g(xT) = xT$ (*)

Consider the space G/T of cosets of G
"flag manifold". Condition (*) means
that the map $f_g : G/T \longrightarrow G/T$
$xT \mapsto g(xT)$
has a fixed point.

$f_g \simeq f_1 = id$. because $\exists$ g_t s.t. $g_0 = 1$ and $g_1 = g$ giving a homotopy between f_g and $f_1 = id$. Thus if $\chi(G/T) \neq 0$

then by the Lefschetz fixed point Thm
$\exists$ a fixed point of f_g which proves a)

What is G/T when $G = SU(2)^2 \cong S^3$
And $T \cong U(1)$. ~~SU(2)~~

$U(1) \longrightarrow SU(2)$ fibre bundle
$\downarrow$
$SU(2)/U(1) = G/T$
$\|$
S^2 Hopf fibration.

$\chi(S^2) = 2 \neq 0$ so $\exists$ a fixed point.

G/T has a cell-structure with only even-dimensional cells and its homology is concentrated in even degrees $\Rightarrow \chi(G/T) > 0$.
$\Rightarrow \exists$ fixed point of $f_g \Rightarrow$ a) ∎

b) Suppose $t \in T'$ is a topological generator i.e. $\overline{\langle\{\exp(st) \mid s \in \mathbb{R}\}} = T'\}$ by a) $\exists\, x \in G$ s.t. $x^{-1} t x \in T \Rightarrow x^{-1} T' x \subseteq T$. But these are maximal ~~tori~~ so $\Leftrightarrow T' \subseteq x T x^{-1}$

$$T' = x T x^{-1} \text{ by maximality}$$

$$\Rightarrow x^{-1} T' x = T.$$

$\blacksquare$

c) Note that $\exp : \mathfrak{u}(1)^n \longrightarrow U(1)^n$ is surjective (for any torus). Now conjugate arbitrary $g \in G$ so that $x g x^{-1} \in T \Rightarrow$ $x g x^{-1} = \exp X$ as T is a torus so $g = \exp(x^{-1} X x)$ $\blacksquare$

Killing Form

Lemma: There is a natural symmetric bilinear form $K : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{R}$ invariant in the sense that $K(X, [Y, Z]) = K(Y, [X, Z])$

Proof: K is $K(X, Y) = \mathrm{Tr}(ad_X \cdot ad_Y)$

Invariance follows from the Jacobi identity $\blacksquare$

Example: $\mathfrak{su}(2)$

$$\left.\begin{aligned} \sigma_1 &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \\ \sigma_2 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ \sigma_3 &= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \end{aligned}\right\} \text{ in } \mathfrak{su}(2)$$

Lie Groups and Lie Algebras

$Ad_{\sigma_i} \sigma_i = 0$

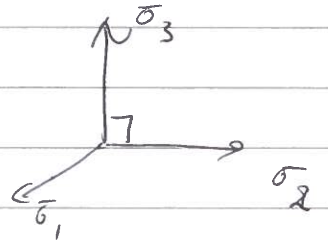
$ad_{\sigma_1} \sigma_2 = [\sigma_1, \sigma_2] = 2\sigma_3$
$ad_{\sigma_1} \sigma_3 = -2\sigma_2$

$$ad_{\sigma_1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix}$$

$$K(\sigma_1, \sigma_1) = Tr(ad_{\sigma_1} ad_{\sigma_1}) = Tr\begin{pmatrix} 0 & & \\ & -4 & \\ & & -4 \end{pmatrix} = -8$$

$Tr(ad_{\sigma_1} ad_{\sigma_2}) = 0$

So Lie alg $su(2)$ are



orthogonal w.r.t. Killing form

The Killing form in this example was the standard dot product $\mathbb{R}^3 \cong su(2)$ (up to scale)

Definition: G is called semisimple if K is nondegenerate symm. bilinear form on $\mathfrak{g}$.

E.g. if $K = 0$ then G is abelian.
e.g. $su(2)$ is semisimple.

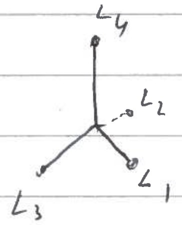
The weight lattice lives in t^* = dual of lie algebra of max torus. If K is nondegenerate then it gives an iso.

between $\mathfrak{g} \to \mathfrak{g}^*$ hence a dot product on the dual $\mathfrak{g}^*$

$su(3)$ is semisimple. (Just compute all $Tr(ad_{E_{ij}} \, ad_{E_{kl}})$) And so t^* has an inner product and one can compute that e.g. $L_i \cdot L_i = +2$ and $L_i \cdot L_j = -1$ up to scale!

For $Su(4)$: $L_i \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_4 \end{pmatrix} = a_i \qquad a_1 + \dots + a_4 = 0$

$\in \mathbb{R}^3 = Lie\, T$



$$T = \left\{ \begin{pmatrix} e^{ia_1} & & & \\ & e^{ia_2} & & \\ & & e^{ia_3} & \\ & & & e^{-i(a_1+a_2+a_3)} \end{pmatrix} \right\}$$

$$E_{12} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ etc.} \qquad ad_{(a_1 \dots a_4)} E_{ij} = (a_i - a_j) E_{ij}$$

E_{ij} lives in a root space with root $L_i - L_j$

The twelve external vertices correspond to off diagonal E_{ij} root spaces. The central vertex at 0 corresponds to diagonal entries : 3 dim. weight space

$$\left\{ \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & a_3 & \\ & & & a_4 \end{pmatrix} : a_1 + a_2 + a_3 + a_4 = 0 \right\}$$

These root systems are very symmetric, because the following group acts on them.

$N(T)/T$ where $N(T)$ is the normaliser of T.

" Weyl group

Lie Groups and Lie Algebras 21st March

	Weyl group	
$SU(2)$	$\mathbb{Z}/2\mathbb{Z}$	reflection
$SU(3)$	S_3	symmetries of Δ
$SU(4)$	S_4	symmetries of tetrahedron
$su(n)$	S_n	

Geometry of Root Systems

Fix a compact semisimple Lie group and a maximal torus T. Then $\mathfrak{g}_{\mathbb{C}} = \underbrace{\bigoplus_{\alpha \neq 0} \mathfrak{g}_\alpha}_{\text{def}} \oplus \underbrace{t_{\mathbb{C}}}_{0\text{ root space}}$

root space decomposition of ad. rep.

We know that $ad_H X = \alpha(H) X \quad \forall X \in \mathfrak{g}_\alpha, H \in t_{\mathbb{C}}$

How does ad_X act on $\bigoplus \mathfrak{g}_\alpha \oplus t_{\mathbb{C}}$ when $X \in \mathfrak{g}_\alpha$?

Lemma: If $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_\beta$ then

$$ad_X Y = [X, Y] \in \mathfrak{g}_{\alpha+\beta}$$

Proof: $X \in \mathfrak{g}_\alpha \Leftrightarrow ad_H X = \alpha(H) X$
$Y \in \mathfrak{g}_\beta \qquad ad_H Y = \beta(H) Y$

Jacobi identity

$$\begin{aligned} ad_H [X, Y] &= [H, [X, Y]] = -[X, [Y, H]] - \\ &\quad - [Y, [H, X]] = \\ &= [X, [H, Y]] - [Y, [H, X]] \\ &= ad_X \beta(H) Y - ad_Y \alpha(H) X \\ &= \beta(H)[X, Y] - \alpha(H)[Y, X] \\ &= (\alpha(H) + \beta(H))[X, Y] \end{aligned}$$

$\Leftrightarrow [X, Y] \in \mathfrak{g}_{\alpha+\beta}$ ∎

Corollary. If $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_\beta$ and $\alpha+\beta \neq 0$
Then $K(X, Y) = 0$

Proof: Consider $(ad_X ad_Y)^N$, for $Z \in \mathfrak{g}_\gamma$

$$(ad_X ad_Y)^N Z \in \mathfrak{g}_{\gamma + N(\alpha+\beta)}$$

If $\alpha+\beta \neq 0$ then for large N, this weight space is 0. $\Rightarrow (ad_X ad_Y)^N = 0$

$\Rightarrow Tr(ad_X ad_Y) = 0 = K(X, Y)$ ∎

As $\mathfrak{g}$ is semisimple, $K(X, Y)$ cannot vanish $\forall Y$ since it is a nondegenerate form, but then by the corollary if $X \in \mathfrak{g}_\alpha$ then $Y \in \mathfrak{g}_{-\alpha}$ to get the killing product nonzero. Thus if $\mathfrak{g}$ is semisimple then $\mathfrak{g}_\alpha \neq 0 \iff \mathfrak{g}_{-\alpha} \neq 0$

Lie Groups and Lie Algebras 24th Mar

Introduced the Killing form
$\underline{K}(X, Y) = Tr(ad_X, ad_Y)$

<u>Assume</u>: K is nondegenerate

$$X \xrightarrow{b} K(X, -) = X^b$$
$$\alpha^{\#} \quad \mathfrak{g} \xleftarrow{\#} \mathfrak{g}^* \ni \alpha$$

i.e. is an isomorphism.

In fact For $U(n)$, $K(X, X) \leq 0$ with equality iff $X = 0$

$u(n) = \{$ skew Hermitian Matrices $\}$

$$Tr(ad_X \cdot ad_X) = -Tr(ad_X \, ad_X^{\dagger}) = -\sum |x_{ij}|^2$$

where x_{ij} are the entries of ad_X

If G is a compact group then any representation is unitary i.e. the adjoint rep ~~is~~ $ad : G \to \mathfrak{gl}(\mathfrak{g})$ is unitary (i.e. $\exists$ invariant inner product).

The same argument shows that

$$K(X, X) \leq 0$$

It could be that X is non zero but $ad_X = 0$. However this violates non degeneracy $\Rightarrow$ Compact semi simple group (i.e. Killing form non degenerate) has negative definite Killing form.

We saw that $X \in \mathfrak{g}_\alpha = \{ v \in \mathfrak{g} : ad_H v = \alpha(H) v$ $\forall H \in \mathfrak{t}^*\}$, $Y \in \mathfrak{g}_\beta$ then

$$[X, Y] \in \mathfrak{g}_{\alpha+\beta}$$

$$K(X, Y) = 0 \quad \text{if} \quad \alpha+\beta \neq 0$$

Lemma:

If $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_{-\alpha}$ then

$$[X, Y] = K(X, Y)\alpha^\# \qquad , \alpha^\# \in \mathfrak{g} \text{ s.t. } K(\alpha^\#, Z) = \alpha(Z)$$

Note that $K(X, Y) \neq 0$ by non degeneracy

Proof: $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_{-\alpha} \Rightarrow [X, Y] \in \mathfrak{g}_0 = \mathfrak{t}$

We only need to check the lemma when applied to $H \in \mathfrak{t}$.

$$K(H, [X, Y]) \overset{\text{To show}}{=} K(X, Y)\, \underbrace{K(\alpha^\#, H)}_{\alpha(H)} \quad \forall H \in \mathfrak{t}$$

This follows from invariance of K.

$$K(H, [X, Y]) = K([H, X], Y) = \underset{X \in \mathfrak{g}_\alpha}{\alpha(H)} K(X, Y)$$

■

Define $H_\alpha = \dfrac{2\alpha^\#}{K(\alpha^\#, \alpha^\#)}$ so that $\cancel{\alpha(H_\alpha) = 2\alpha(\alpha^\#) = 2K(\alpha^\#, \alpha^\#)}$

$$\alpha(H_\alpha) = \frac{2\alpha(\alpha^\#)}{K(\alpha^\#, \alpha^\#)} = \frac{2K(\alpha^\#, \alpha^\#)}{K(\alpha^\#, \alpha^\#)} = 2$$

Lie Groups and Lie Algebras 24th Mar.

In Lie alg. $sl(2, \mathbb{C})$ we have weights $-2, 0, 2$ for adjoint rep.

Corollary: Let $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_{-\alpha}$, $H = H_\alpha$.

Then $\mathbb{C}X \oplus \mathbb{C}H_\alpha \oplus \mathbb{C}Y = S_\alpha$ is isomorphic to $sl(2, \mathbb{C})$

Proof: $[X, Y] =$ multiple of H (from previous lemma). Rescale X we can get $[X, Y] = H$.

We also know $[H, X] = 2X$, because $X \in \mathfrak{g}_\alpha$ so $ad_H X = \alpha(H) X = 2X$

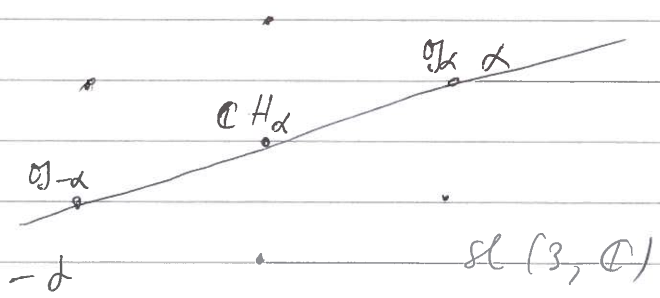
And $[H, Y] = -2Y$. Thus $S_\alpha \cong sl(2, \mathbb{C})$ ■

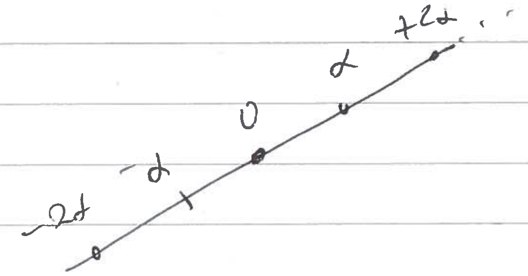
This gives for every opposite pair of nonzero roots a subalgebra $S_\alpha \cong sl(2, \mathbb{C})$

Lemma: For any root α, the subspace

$$\dots \oplus \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus \mathbb{C}H_\alpha \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha} \oplus \dots$$

is an irreducible $sl(2, \mathbb{C})$ rep of S_α





Proof: a) it is a subrep
b) it is irred.

a) show that ad_X, ad_Y, ad_H preserve this V

$$ad_X \mathfrak{g}_{k\alpha} \subseteq \mathfrak{g}_{(k+1)\alpha} \qquad X \in \mathfrak{g}_\alpha$$

$$ad_Y \mathfrak{g}_{k\alpha} \subseteq \mathfrak{g}_{(k-1)\alpha}$$

So all we need is to see that

$(*)$ $ad_X \mathfrak{g}_{-\alpha} \subseteq \mathbb{C}H_\alpha \subseteq \mathfrak{g}_0$

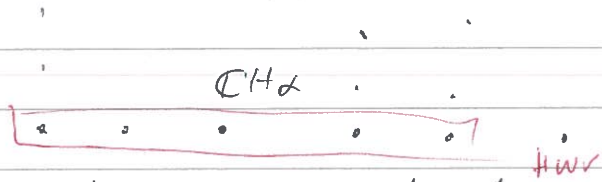
We have seen that if $X \in \mathfrak{g}_\alpha$, $Z \in \mathfrak{g}_{-\alpha}$ then

$$[X, Z] = ad_X Z = K(X, Z)\alpha^\# \in \mathbb{C}H_\alpha$$

$\Rightarrow (*)$. Similarly for $ad_Y \mathfrak{g}_\alpha \subseteq \mathbb{C}H_\alpha$

which proves a)

For b) The rep V looks like



Let $v \in V$ be a highest weight vector (HWV)
Get an irrep containing, say V'.
$(V')^+ \subseteq V$ is a subrep whose zero weight

space ~~is zero~~ $\Rightarrow (V')^{\perp} = 0 \Rightarrow V$ is irred.
$\Rightarrow \mathfrak{g}_{k\alpha}$ is 1-dim $\forall$ k ∎

Corollary $\mathfrak{g}_{k\alpha} = 0 \quad \forall \; k \neq -1, 0, 1$

Proof: We know that in an irrep of $sl(2, \mathbb{C})$ if $v \in V_\alpha$ then $\rho(X) v = 0 \iff \alpha$ is highest weight

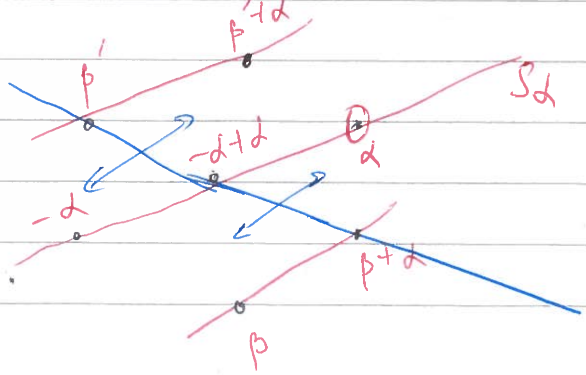
$$\begin{array}{ccccc} & & & X & \\ \cdot & \mathfrak{g}_{-\alpha} & \mathbb{C}H & \mathfrak{g}_\alpha & \mathfrak{g}_{2\alpha} \end{array}$$

$$\mathrm{ad}_X X = [X, X] = 0$$

$\Rightarrow \alpha$ is highest weight ∎

$\Rightarrow$ Root diagram for a compact semi simple group only has roots $-\alpha, 0, \alpha$ on any given line. ∎

Using the $sl(2, \mathbb{C})$ subalgebras S_α as in $sl(3, \mathbb{C})$ case we deduce that the root system is symmetric under the Weyl group of reflections in the planes orthogonal to the roots.



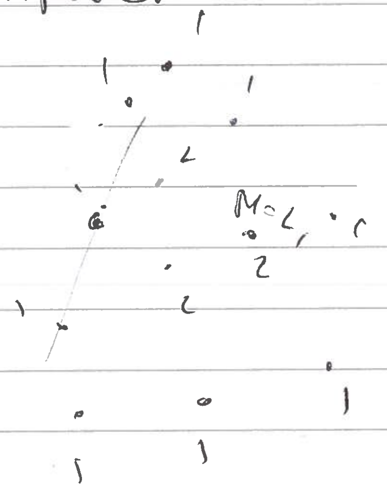
Each subspace $\bigoplus \mathfrak{g}_{\beta + k\alpha}$ is an irreducible S_α-rep.

$\Rightarrow$ weights are symmetric around the axis/plane orthogonal to α.

This leads to a complete classification of compact semi-simple Lie groups.

And rep theory goes just as for $sl(3, \mathbb{C})$

Finally: How do we figure out the multiplicities on weight diagrams?



$\Gamma_{2,1}$

Freudental multiplicity formula:

G compact semisimple group
$G \to GL(V)$ with highest weight λ

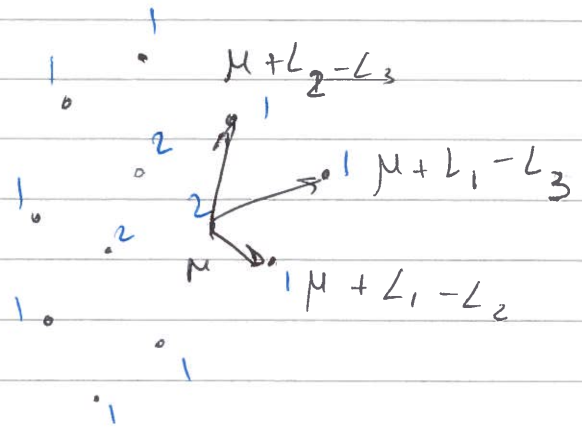
$$\dim V_\mu = \frac{2 \sum_{\alpha \in R^+} \sum_{j \geq 1} \langle \mu + j\alpha, \alpha \rangle \dim V_{\mu + j\alpha}}{\| \lambda + \rho \|^2 - \| \mu + \rho \|^2}$$

$$\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$$

In our example $\lambda = 2L_1 - L_3$
$R^+ = \{L_1 - L_2, L_1 - L_3, L_2 - L_3\}$ $\rho = L_1 - L_3$

Lie Groups and Lie Algebras

Take $\mu = L_1$ sum over $j=1$



$$\langle \mu + L_2 - L_3 , L_2 - L_3 \rangle \underbrace{\dim V_{\mu + L_2 - L_3}}_{1}$$

$$\langle L_i , L_j \rangle = \begin{cases} 2 & i=j \\ -1 & i \neq j \end{cases}$$

$$\Rightarrow \langle 2L_1 + L_2 - L_3 , L_2 - L_3 \rangle =$$

$$= -1 + 1 + 2 + 1 + 1 + 3 = 6$$

$$\langle L_1 + L_1 - L_3 , L_1 - L_3 \rangle =$$

$$= 9$$

$$\langle L_1 + L_1 - L_2 , L_1 - L_2 \rangle =$$

$$= 9$$

$$\Rightarrow \dim V_{L_1} = \frac{2.(6+9+9)}{\|\lambda + \rho\|^2 - \|\mu + \rho\|^2} = \frac{2 \times 24}{}$$

$$\|\lambda + \rho\|^2 = \langle 2L_1 - L_2 + L_1 - L_3 , 2L_1 - L_2 + L_1 - L_3 \rangle$$
$$= 38$$

$\|\mu+\rho\|^2 = \langle L_1 + L_1 - L_3, \; 2L_1 - L_3 \rangle =$

$= 14$

$\Rightarrow \dim V_{L_1} = 2 \times \dfrac{24}{38-14} = 2$

Fulton & Harris have proof
"Rep Theory"