MATH0075 Lie Groups and Lie Algebras Notes

Based on the 2018 autumn lectures by Dr P C L Humphries

The Author(s) has made every effort to copy down all the content on the board during lectures. The Author(s) accepts no responsibility for mistakes on the notes nor changes to the syllabus for the current year. The Author(s) highly recommends that the reader attends all lectures, making their own notes and to use this document as a reference only.

MATH0075

02-10-18

Lie Groups & Lie Algebras - Peter Humphries (room 602)

8 Hw assignments - due Thursdays 5pm
Lecture notes on Moodle
(Different to last year, rather similar to Jonny Evans' course in 2016/17).
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What is a representation?

A rep. ρ of a group G on an n-dim. vector space over a field K is an assignment of a K-linear map $\rho(g): V \to V$ to each $g \in G$ st.

- $\rho(gh) = \rho(g)\rho(h)$
- $\rho(1_G) = 1_V$

Equivalently,

1). ... an assignment of an $n \times n$ matrix $\rho(g)$ to each $g \in G$ st. $\rho(gh) = \rho(g)\rho(h)$.

2). ... a homomorphism $\rho: G \to GL(V)$,
$GL(V)$ = space of invertible linear transformations of V
homomorphism $\Rightarrow \rho(gh) = \rho(g)\rho(h)$, $\rho(1_G) = 1_V$.

3). ... a group action of G on V by linear maps
$\Leftrightarrow \tilde{\rho}: G \times V \to V$, $\tilde{\rho}(g, v) = \rho(g)(v)$.
group action $\Leftrightarrow \tilde{\rho}(gh, v) = \tilde{\rho}(g, \tilde{\rho}(h, v)) \Leftrightarrow \rho(gh) = \rho(g)\rho(h)$

"Let V be a rep of G."
"Consider the action ρ of G on V".

Example (Invariant theory for binary quadratic forms).
$K = \mathbb{C}$.
A binary quadratic form is an expression of the form $ax^2 + bxy + cy^2$ in the variables x, y.
These form a 3-dim vector space V.

Equivalently, $ax^2 + bxy + cy^2 = \underline{x}^T M \underline{x} = (x \ y)\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

$\Leftrightarrow V \cong$ space of 2×2 symmetric matrices.

What happens when we change coordinates?
$\Leftrightarrow$ act by conjugation by a matrix $S \in SL_2(\mathbb{C})$
$SL_2(\mathbb{C})$ = space of (invertible) 2×2 matrices (over $\mathbb{C}$) with det 1.

$$\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} = M \longmapsto M' := S^T M S$$

This is a 3-dim rep of $SL_2(\mathbb{C})$ on V
Note that
$$\det M' = \det S^T \det M \det S = (\det S)^2 \det M = \det M = ac - \frac{b^2}{4} = -\frac{1}{4}(b^2 - 4ac) =: \Delta$$

Δ is an invariant of this binary quadratic form.
M can be diagonalised: M is equivalent to $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$
Action by $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ swaps λ_1 & λ_2.

Use $S = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ $(\lambda \neq 0)$ to get $M = \begin{pmatrix} \lambda^2\lambda_1 & 0 \\ 0 & \lambda^2\lambda_2 \end{pmatrix}$

$\Delta = \lambda_1\lambda_2$ is an invariant, so M is equivalent to either $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} \Delta & 0 \\ 0 & 1 \end{pmatrix}$
so Δ essentially characterises M up to equivalence.

MATH0075

02-10-18

Consider quadratic polynomials in the entries of M:

$Aa^2 + Bb^2 + Cc^2 + Dab + Eac + Fbc$

This is a 6-dim vector space (in A, B, C, D, E, F).

Action of S on M acts on the coefficients of such a polynomial,

i.e. this is a 6-dim rep $R : SL_2(\mathbb{C}) \to Q$

The vector $\Delta = -\frac{1}{4}(b^2 - 4ac)$ in Q

$(A = C = D = F = 0, \quad B = -\frac{1}{4}, \quad E = \frac{1}{4})$

is fixed by $R(g)$ $\forall g \in SL_2(\mathbb{C})$

$\Rightarrow$ this gives a 1-dim subspace of Q invariant under R.

Question: When can we find invariant subspaces of a rep? (not just 1-dim subspaces).

Goal: Break down a rep into irreducible components.

(Further goal: classify irreducibles).

Smoothness

Example

$G = \mathbb{R}$ (under addition)

$\mathbb{R}$ is a vector space over $\mathbb{Q}$ assuming the axiom of choice.

Each $r \in \mathbb{R}$ can be written as

$\sum_{a \in A} c_a a$, $c_a \in \mathbb{Q}$ (all but finitely many $= 0$)

A = basis for $\mathbb{R}$ over $\mathbb{Q}$.

Let $\lambda : A \to \mathbb{R}$ be any function

$\sum_{a \in A} c_a a \longmapsto \sum_{a \in A} c_a \lambda(a) a$ is a homomorphism $\mathbb{R} \to \mathbb{R}$

(ugly!)

However, if we demand smoothness, then these go away.

$\mathbb{R}$ is a Lie group: notion of differentiability on $\mathbb{R}$.

Let $F : \mathbb{R} \to \mathbb{R}$ be a smooth homomorphism
$F(s+t) = F(s) + F(t)$
$\frac{d}{dt}\big|_{t=0}$ both sides : $F'(s) = 0 + F'(0)$

so $F(s)$ is linear $\Rightarrow F(s) = \lambda s$, $s \in \mathbb{R}$ fixed

Matrix exponential

Example
Consider the matrix group $U(1) = \{ z \in \mathbb{C} : z\bar{z} = 1 \}$
$z \in U(1) \iff z = e^{i\theta}, \ \theta \in \mathbb{R}.$
$\Rightarrow$ every element of $U(1)$ is the exponential of an element of $i\mathbb{R}$

Def
The exponential of an $n \times n$ matrix A is
$$\exp(A) := 1 + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \dots = \sum_{k=0}^{\infty} \frac{1}{k!} A^k .$$

Example
$$A = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} \Rightarrow A^2 = -\theta^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\exp(A) = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

MATH0075

02-10-18

Convergence:

Convergence in operator norm

$\|A\|^2 := \inf\{c \in \mathbb{R} : |Av| \leq c|v|, \forall v \in \mathbb{R}^n\}$

So $A_i \to A$ iff $\|A_i - A\| \to 0$.

Lemma

The power series in $\exp(A)$ converges absolutely.

Proof

Note $\|AB\| \leq \|A\|\|B\|$ since $|ABv| \leq \|A\||Bv| \leq \|A\|\|B\||v|$ $\forall v$.

$$\sum_{n=0}^{N} \frac{1}{n!}\|A^n\| \leq \sum_{n=0}^{N} \frac{1}{n!}\|A\|^n \leq e^{\|A\|}.$$

$\square$

Lemma

This series converges.

WTS: $\sum_{n=0} \frac{1}{n!} A^n$ is Cauchy.

$$\Leftrightarrow \left\|\sum_{n=M+1}^{N} \frac{1}{n!}A^n\right\| \to 0$$

$$\leq \sum_{n=M+1}^{N} \frac{1}{n!}\|A^n\| \leq \sum_{n=M+1}^{N} \frac{1}{n!}\|A\|^n$$

This is Cauchy since $\sum_{n=0}^{N} \frac{1}{n!}\|A\|^n \to e^{\|A\|}$.

Corollary

The function $t \mapsto \exp(tA)$ satisfies $\frac{d}{dt}\exp(tA) = A\exp(tA)$.

Proof

Differentiate $\sum_{k=0}^{\infty} \frac{1}{k!}(tA)^k$ term by term to get

$$\sum_{k=0}^{\infty} \frac{1}{k!} k t^{k-1} A^k = \sum_{k=1}^{\infty} \frac{t^{k-1}}{(k-1)!} A^k = A\sum_{k=0}^{\infty} \frac{1}{k!} t^k A^k.$$

$\square$

Corollary

$\exp(A)\exp(B) = \sum_{k=0}^{\infty} \sum_{i+j=k} \frac{1}{i!j!} A^i B^j$

Corollary

$\exp(A)\exp(-A) = 1$ (identity matrix)

Proof

Take $B = -A$ above. $\sum_{k=0}^{\infty} \frac{1}{k!}(A-A)^k$

$\square$

So the inverse of $\exp(A)$ is $\exp(-A)$.
In particular, $\exp(A)$ is invertible (even if A isn't).

Corollary

If $AB = BA$, $\exp(A)\exp(B) = \exp(B)\exp(A)$.

Proof

Use the above, then reindex $(i,j) \leftrightarrow (j,i)$.

$U(n)$

Unitary matrices $U(n) \subset GL_n(\mathbb{C})$.
n-dim generalisation of the complex numbers.

A^+ = conjugate transpose of A
$U(n) = \{$ $n \times n$ matrices, complex entries, $A^+A = 1$ (id. matrix)$\}$
$GL_n(\mathbb{C}) = \{$ $n \times n$ matrices, complex entries, $\det \neq 0\}$
(general linear group)

Lemma

A matrix B is skew-Hermitian iff $\exp(tB) \in U(n)$ $\forall t \in \mathbb{R}$.

skew-Hermitian means $B^+ = -B$.

MATH0075

02-10-18

Proof

[$\Rightarrow$] Note $(\exp(tB))^{+} = \exp(tB^{+})$.

So if $B^{+} = -B$, then

$(\exp(tB))^{+} = \exp(tB^{+}) = \exp(-tB) = (\exp(tB))^{-1}$

so $\exp(tB)$ is unitary.

[$\Leftarrow$] $\exp(tB^{+}) = \exp(-tB)$, differentiate both sides and set $t=0$:

we get $B^{+} = -B$.

$\square$

04-10-18 Lie algebras and the local logarithm

$\exp : gl_n(\mathbb{C}) \longrightarrow GL_n(\mathbb{C})$

$\uparrow$ (space of $n \times n$ matrices with complex entries)

(same space but invertible)

$$\exp(A) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

Def^n

If $G \subset GL_n(\mathbb{C})$ is a subgroup, its Lie algebra is

$g = \{B \in gl_n(\mathbb{C}) : \exp(tB) \in G \quad \forall t \in \mathbb{R}\}$

Example

$U(n) = \{A \in GL_n(\mathbb{C}) : A^{+}A = 1\}$

Lie algebra is $u(n) = \{B \in gl_n(\mathbb{C}) : B^{+} = -B\}$

Example

$SU(2) = \{A \in GL_2(\mathbb{C}) : A^{+}A = 1, \det A = 1\}$ "special unitary"

Lie algebra is $su(2) = \{B \in gl_2(\mathbb{C}) : B^{+} = -B, \operatorname{tr} B = 0\}$

such a B can be written as

$$M_u := \begin{pmatrix} ix & y+iz \\ -y+iz & -ix \end{pmatrix}, \quad u = (x,y,z) \in \mathbb{R}^3$$

In particular, $su(2)$ is a 3-dim vector space over $\mathbb{R}$.

Lemma

An element of $SU(2)$ has the form $\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$, $a, b \in \mathbb{C}$, $|a|^2 + |b|^2 = 1$.

Proof

Write it as $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $A^{+} = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}$

$$A^{+}A = 1 \Rightarrow A^{+} = A^{-1}, \ \det A = 1$$

$$\begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \qquad \Rightarrow d = \bar{a}, \ c = -b$$

$$\det A = |a|^2 + |b|^2 = 1 \qquad \square$$

Can write $a = x_1 + ix_2$, $b = x_3 + ix_4$

$|a|^2 + |b|^2 = 1 \iff x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$

So we can identify $SU(2)$ with $\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$

$$\cong S^3$$

Assignment Q2: if $u = (x, y, z) \in \mathbb{R}^3$ is a unit vector, then

$$\exp(\theta M_u) = \cos\theta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sin\theta \, M_u$$

$$= \begin{pmatrix} \cos\theta + ix\sin\theta & y\sin\theta + iz\sin\theta \\ -y\sin\theta + iz\sin\theta & \cos\theta - ix\sin\theta \end{pmatrix}$$

This had better be in $SU(2)$!

Lemma

Any matrix in $SU(2)$ can be written in this form.

Proof

We know that every element is of the form

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} = \begin{pmatrix} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - ix_2 \end{pmatrix}$$

$x_1 = \cos\theta$, $x_2 = x\sin\theta$

$x_3 = \sin\theta$, $x_4 = z\sin\theta$

$\square$

MATH0075

04-10-18

$\exp : su(2) \to SU(2)$ is surjective but not injective.

Goal: prove that in a nbhd of $0 \in U' \subset gl_n(\mathbb{R})$ and $1 \in V' \subset GL_n(\mathbb{R})$, $\exp : gl_n(\mathbb{R}) \to GL_n(\mathbb{R})$ admits an inverse $\log : V' \to U'$.

Calculus of several variables

Def^n

$U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^n$ are open, $(x_1, \dots, x_m)$ variables, $F : U \to V$ a map. $F = (F_1, \dots, F_n)$ is smooth if $\frac{\partial^k F_i}{\partial x_{i_1} \dots \partial x_{i_k}}$ exist, continuous $\forall k$, $\forall i \in \{1, \dots, m\}$.

Def^n

$d_p F$ is the matrix of partial derivatives at $p \in U$

$$d_p F = \begin{pmatrix} \frac{\partial F_1}{\partial x_1}(p) & \cdots & \frac{\partial F_n}{\partial x_1}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial F_1}{\partial x_m}(p) & \cdots & \frac{\partial F_n}{\partial x_m}(p) \end{pmatrix}$$

linear map $\mathbb{R}^n \to \mathbb{R}^m$ "best linear approx to F at p".

$F(p+v) = F(p) + d_p F(v) + o(|v|)$ ← goes to zero faster than $|v|$.

or $d_p F(v) = \frac{d}{dt}\Big|_{t=0} F(p+tv)$

Example

$F(x) = x^2$

$F(x+t) = (x+t)^2 = x^2 + 2xt + o(|t|)$

$d_x F(t) = 2xt$

Example

$H = \{A \in gl_n(\mathbb{C}) : A^+ = A\}$

$F : gl_n(\mathbb{C}) \longrightarrow H \quad , \quad F(A) = A^+A$

Want to find $d_A F(B)$

$F(A+B) = (A+B)^+(A+B)$

$= A^+A + (B^+A + A^+B) + o(B)$

So $d_A F(B) = B^+A + A^+B$

or

$$d_A F(B) = \frac{d}{dt}\Big|_{t=0} F(A+tB)$$

$$= \frac{d}{dt}\Big|_{t=0} \left(A^+A + (B^+A + A^+B)t + B^+Bt^2\right)$$

$$= B^+A + A^+B.$$

Chain rule:

If $U_i \in \mathbb{R}^{n_i}$, $i \in \{1,2,3\}$ are open sets, and $U_1 \xrightarrow{F_1} U_2 \xrightarrow{F_2} U_3$ is a sequence of maps with composite $F_3 = F_2 \circ F_1$, then $d_x F_3(v) = d_{F_1(x)} F_2 (d_x F_1(v))$

i.e. $d_x F_3 = d_{F_1(x)} F_2 \circ d_x F_1$ where $\circ$ denotes matrix product.

Inverse function thm

(Smooth inverse function exists if derivative is invertible).
Let U, V be open subsets of $\mathbb{R}^n$, $F : U \rightarrow V$ smooth.
If $d_p F$ is invertible, $\exists$ nbhd $p \in U' \subset U$, $f(p) \in V' \subset V$ st. $F|_{U'} : U' \rightarrow V'$ is a bijection with smooth inverse.

A diffeomorphism is a bijective smooth function $F : U \rightarrow V$, $U, V \subset \mathbb{R}^n$, with a smooth inverse.

MATH0075

04-10-18

Thm

In a nbhd of $0 \in U' \subset gl_n(\mathbb{R})$ and $1 \in V' \subset GL_n(\mathbb{R})$, $\exp : gl_n(\mathbb{R}) \to GL_n(\mathbb{R})$ admits a smooth inverse $\log : V' \to U'$ (i.e. exp is a diffeomorphism $U' \to V'$).

Proof

$\exp(A) = 1 + A + o(\|A\|^2)$

so $d_0 \exp(A) = A$

i.e. $d_0 \exp = Id : gl_n(\mathbb{R}) \to gl_n(\mathbb{R})$, obviously invertible.

Now apply the inverse function thm. $\square$

The Baker - Campbell - Hausdorff formula

Lemma

log has a power series expansion about $1 \in GL_n(\mathbb{R})$ with radius of convergence 1 (in the operator norm) given by

$$\log(1+X) = X - \tfrac{1}{2}X^2 + \tfrac{1}{3}X^3 - \dots$$

Proof

We know $\log(1+X)$ exists for $\|X\|$ sufficiently small

$\log(1+X) = X + b_2 X^2 + b_3 X^3 + \dots$

$\exp(X) = 1 + X + \frac{1}{2!}X^2 + \dots$

$X = \log(1 + X + \frac{1}{2!}X^2 + \dots) = X - \frac{1}{2}X^2 + \frac{1}{3}X^3 - \dots$

$\square$

$$\begin{aligned}\exp(A)\exp(B) &= (1 + A + \tfrac{1}{2!}A^2 + \dots)(1 + B + \tfrac{1}{2!}B^2 + \dots) \\ &= 1 + A + B + AB + \tfrac{1}{2}(A^2+B^2) + \dots\end{aligned}$$

$$\begin{aligned}\text{So } \log(\exp(A)\exp(B)) &= A + B + AB + \tfrac{1}{2}(A^2+B^2) + \dots \\ &\quad - \tfrac{1}{2}(A + B + AB + \tfrac{1}{2}(A^2+B^2) + \dots)^2 + \tfrac{1}{3}(\dots) + \dots \\ &= A + B + \tfrac{1}{2}(AB - BA) + \dots\end{aligned}$$

- to first order approximations, we get the usual law of logarithms $\log(e^x e^y) = x + y$

- the second order term is a correction term involving the commutator $[A, B] := AB - BA$

Next order term is
$\frac{1}{12}\left([A,[A,B]] - [B,[A,B]]\right)$ (Assignment 2).

Theorem

All higher order terms can be expressed in terms of $[\cdot,\cdot]$.

⚠

$\log(\exp(A)\exp(B))$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sum_{\substack{r_1, s_1 = 0 \\ r_1 + s_1 > 0}}^{\infty} \cdots \sum_{\substack{r_n, s_n = 0 \\ r_n + s_n > 0}}^{\infty} \frac{\left(\sum_{i=1}^{n}(r_i + s_i)\right)^{-1}}{\prod_{i=1}^{n} r_i!\, s_i!} \, ad_A^{r_1} ad_B^{s_1} \cdots ad_A^{r_{n-1}} ad_B^{s_{n-1}} K_{r_n, s_n}$$

where for $X \in gl_n(\mathbb{R})$, $ad_x : gl_n(\mathbb{R}) \to gl_n(\mathbb{R})$,

$ad_x Y = [X, Y] = XY - YX$,

$$K_{r_i s_i} = \begin{cases} ad_A^{r_i} B & \text{if } s_i = 1 \\ A & \text{if } r_i = 1,\ s_i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Takeaway from this:

$[\cdot,\cdot]$ determines the group law on $GL_n(\mathbb{R})$.

$[\cdot,\cdot]$ is also called a Lie bracket.

Def^n

Let V be a K-vector space, let $[\cdot,\cdot] : V \times V \to V$ be a bilinear bracket satisfying $[X, X] = 0$, and
$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$ (Jacobi identity)
$(V, [\cdot,\cdot])$ is a Lie algebra.

Note: if char $K \neq 2$, then $[X,X] = 0 \Leftrightarrow [X,Y] = -[Y,X]$.

MATH0075

04-10-18

If we write $ad_X Y = [X,Y]$ then the Jacobi identity is $ad_{[X,Y]} = ad_X ad_Y - ad_Y ad_X$.

Matrix groups

$GL_n(\mathbb{R})$ is a metric space via the operator norm, so we can talk about convergence of sequences of matrices.

Defn
$G \subset GL_n(\mathbb{R})$ is closed if for every sequence $A_i \in G$ converging to $A \in GL_n(\mathbb{R})$, $A \in G$.

Defn
A matrix group G is a closed subgroup of $GL_n(\mathbb{R})$.

Lemma
The closure $\overline{G}$ of a subgroup G of $GL_n(\mathbb{R})$ is a subgroup.

Proof
Suppose that $\lim_{i\to\infty} \overset{\in G}{g_i} \to g \in \overline{G}$, $\lim_{i\to\infty} \overset{\in G}{h_i} \to h \in \overline{G}$.
Multiplication depends continuously on g_i, h_i,
so $g_ih_i \to gh \in \overline{G}$.
g_i^{-1} is a rational function of the entries of g_i,
so it is continuous. So g_i^{-1} is a convergent sequence, converging to $c \in GL_n(\mathbb{R})$.
$g_i^{-1}g_i = 1 \Rightarrow cg = 1 \Rightarrow c = g^{-1}$. So $g \in \overline{G} \Rightarrow g^{-1} \in \overline{G}$. □

Examples : Stabilisers of quadratic forms

Q a matrix, real entries.

$G = \{A \in GL_n(\mathbb{R}) : A^T Q A = Q\}$

Lemma

G is closed.

Proof

Suppose $A_i \in G$ converges to $A \in GL_n(\mathbb{R})$.

$A \longmapsto A^T Q A$ is a continuous map w.r.t. $\|\cdot\|$:

$$\|(A+B)^T Q(A+B) - A^T Q A\| = \|B^T Q A + A^T Q B + B^T Q B\|$$
$$\leq \|B\| (2\|A\| + \|B\|) \|Q\| \qquad ?$$
$$\to 0 \text{ as } \|B\| \to 0$$

$A_i^T Q A_i = Q \Rightarrow A^T Q A = Q \Rightarrow A \in G.$ $\square$

Example

- $Q = 1$, so that $A^T A = 1$
  $G = O(n)$, group of orthogonal matrices.

- $Q = \text{diag}(-1, 1, 1, \dots, 1)$, $G = O(1, n-1)$, Lorentz group
- replace n with $2n$, $Q = \text{blockdiag}\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right) =: J$
  $G = Sp_{2n}(\mathbb{R})$, symplectic group.

Take $G = O(1,1)$. We claim $o(1,1)$ is
$B \in gl_2(\mathbb{R})$ s.t. $B^T \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} B$.
$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow a, d = 0, \; b = c \Rightarrow B = \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}$
We can also show that $\exp(B) = \begin{pmatrix} \cosh b & \sinh b \\ \sinh b & \cosh b \end{pmatrix}$

Claim

$GL_n(\mathbb{C})$ is a closed subgroup of $GL_{2n}(\mathbb{R})$.

04-10-18

This is because $GL_n(\mathbb{C})$ is isomorphic to
$G := \{A \in GL_{2n}(\mathbb{R}) : AJ = JA\}$
This isomorphism replaces a complex entry $a+ib$ of an element of $GL_n(\mathbb{C})$ with the 2×2 real matrix $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$

09-10-18 <u>The Lie Algebra</u>

$\{A \in gl_n(\mathbb{R}) : \exp(tA) \in G \ \ \forall t \in \mathbb{R}\} = g$

<u>Lemma</u>
H a matrix group
$h_n \in gl_n(\mathbb{R})$ s.t. $\exp(h_i) \in H$, $h_i \to 0$, $\frac{h_i}{|h_i|} \to v$,
then $\exp(tv) \in H \ \ \forall t \in \mathbb{R}$.

Recall $gl_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$, so $|h_i|$ makes sense.

<u>Proof</u>
First fix $t \in \mathbb{R}$. Let $m_n \in \mathbb{Z}$ be the largest integer less than $t/|h_n|$ s.t. $m_n \to \infty$ as $h_n \to 0$.
(n here not necessarily n from $gl_n(\mathbb{R})$)

So $\frac{t}{|h_n|} - 1 \leq m_n \leq \frac{t}{|h_n|} + 1$ and so

we have $t - |h_n| \leq m_n|h_n| \leq t + |h_n|$
$\Rightarrow m_n|h_n| \to t$
$\Rightarrow \exp(m_n h_n) \to \exp(tv)$.

Now $\exp(m_n h_n) = (\exp(h_n))^{m_n} \in H$.
Since H is topologically closed, $\longleftrightarrow$ since it is a matrix group
$\exp(tv) = \lim_{n\to\infty} \exp(m_n h_n) \in H$.

$\square$

Thm

Let $G \subset GL_n(\mathbb{R})$ be a matrix group.
$g := \{v \in gl_n(\mathbb{R}) : \exp(tv) \in G \ \forall t \in \mathbb{R}\}$
is a vector subspace of $gl_n(\mathbb{R})$.

Proof

NTS: g is closed under scalar multiplication.
True by definition. NTS: $Id \in g$. True trivially.
NTS: g is closed under addition: $\omega_1, \omega_2 \in g \Rightarrow \omega_1 + \omega_2 \in g$
i.e. if $\exp(t\omega_1), \exp(t\omega_2) \in G \ \forall t \in \mathbb{R} \Rightarrow \exp(t(\omega_1+\omega_2)) \in G$
$\forall t \in \mathbb{R}$.

Note that $\gamma(t) := \exp(t\omega_1)\exp(t\omega_2) \in G$, and for t sufficiently small, $\gamma(t)$ is contained in the image of exp.
$\Rightarrow \gamma(t) = \exp(f(t))$, $f : (-\varepsilon, \varepsilon) \to g$ s.t. $f(t) \to 0$ as $t \to 0$.

Since $\exp(t\omega_i) = 1 + t\omega_i + o(t^2)$ it follows that
$\exp(f(t)) = 1 + t\dot{f}(0) + o(t^2) = 1 + t(\omega_1 + \omega_2) + o(t^2)$.
So $\dot{f}(0) = \lim_{t\to 0} \dfrac{f(t)}{t} = \omega_1 + \omega_2$

So $\lim_{t\to 0} \dfrac{f(t)}{|f(t)|} = \lim_{t \to 0} \dfrac{f(t)}{t} \dfrac{t}{|f(t)|} = \dfrac{\omega_1 + \omega_2}{|\omega_1 + \omega_2|} =: v$.
Now set $h_n = f(\frac{1}{n})$ in the previous lemma,
so that $\exp(tv) \in G \ \forall t \in \mathbb{R}$
$\Rightarrow \exp(t(\omega_1 + \omega_2)) \in G \quad \forall t \in \mathbb{R}$.

$\square$

Def^n

Given a matrix group $G \subset GL_n(\mathbb{R})$,
$g := \{v \in gl_n(\mathbb{R}) : \exp(tv) \in G \ \forall t \in \mathbb{R}\}$
is the Lie algebra of G.

MATH 0075

09-10-18

Examples

$B^+ := \overline{B^T}$

- $G = U(n) : g = u(n) = \{B \in gl_n(\mathbb{C}) : B^+ = -B\}$
- $G = SL_n(\mathbb{R}) : g = sl_n(\mathbb{R}) = \{B \in gl_n(\mathbb{R}) : tr\, B = 0\}$
- $G = SU(n) : g = su(n) = \{B \in gl_n(\mathbb{C}) : B^+ = -B,\ Tr\, B = 0\}$

skew symmetric matrices

- $G = O(n) : g = so(n) = \{B \in gl_n(\mathbb{R}) : B^T = -B\}$
  (note that this implies $Tr\, B = 0$)
- $G = SO(n) : g = so(n)$ again.

$O(n) = SO(n) \sqcup \tau SO(n)$ where $\tau \in O(n)$ is some fixed reflection matrix with $\det = -1$.
exp can never map to $\tau SO(n)$ [hence $O(n)$ and $SO(n)$ have the same Lie algebra]

Corollary

$\exp : g \rightarrow G$ need not be globally surjective.

Exponential charts on matrix groups

We often need to work in local coordinates.

Def^n

G a matrix group with Lie algebra g.
$0 \in U'' \subset g$, $1 \in V'' \subset G$ s.t. $\exp : U'' \rightarrow V''$ is a bijection.
We will call $\exp : U'' \rightarrow V''$ an exponential chart for G near 1.

We have previously seen exponential charts $U' \to V'$ from $\mathfrak{gl}_n(\mathbb{R}) \to GL_n(\mathbb{R})$.
We want to use this for $\mathfrak{g} \to G$.

Thm
$\exists\ 0 \in U' \subset \mathfrak{gl}_n(\mathbb{R})$, $I \in V' \subset GL_n(\mathbb{R})$
s.t. $\exp|_{U' \cap \mathfrak{g}} : U' \cap \mathfrak{g} \to V' \cap G$ is an exp. chart.

Proof
We know that there is an exp chart $U' \to V'$.
Want to use this to define $U' \cap \mathfrak{g} \to V' \cap G$.
$\exp : U' \to V'$ is injective, but $U' \cap \mathfrak{g} \to V' \cap G$ might not be. Suppose that it isn't.
Then $\exists\ U', V'$ s.t. $\exists\ \overset{\text{element}}{g} \in V' \cap G$ not contained in $\exp(\overset{\text{Lie algebra}}{\mathfrak{g}})$.
By shrinking V' we can ensure $\exists\ g_i \in G \setminus \exp(\mathfrak{g})$
s.t. $g_i \to I$.

Lemma
Suppose $\mathfrak{gl}_n(\mathbb{R}) = W_1 \oplus W_2$, W_1, W_2 complementary subspaces of $\mathfrak{gl}_n(\mathbb{R})$.
$\exists$ neighbourhoods U' of $0 \in W_1 \oplus W_2$, V' of $I \in GL_n(\mathbb{R})$
s.t. $F : W_1 \oplus W_2 \to GL_n(\mathbb{R})$, $F(w_1, w_2) := \exp(w_1)\exp(w_2)$
is a diffeomorphism $F|_{U'} : U' \to V'$.

Proof
Assignment 2 (similar to earlier thm). □

Returning to proof of thm:
Take $W_1 = \mathfrak{g}$, $W =$ complementary subspace.
g_i eventually lies in the image of F, so that
$g_i = \exp(w_{i,1})\exp(w_{i,2})$.

MATH0075

09-10-18

Note g_i and $\exp(-\omega_{i,1})$ both are in $G \subset GL_n(\mathbb{R})$, so multiply both sides on the left by $\exp(-\omega_{i,1})$

$\Rightarrow \exp(\omega_{i,2}) = \exp(-\omega_{i,1}) g_i \in G$.

Define $\widetilde{\omega_i} = \dfrac{\omega_{i,2}}{|\omega_{i,2}|}$.

This has a convergent subsequence with limit $\omega \in W_2$, $|\omega| = 1$.

Lemma at beginning of lecture $\Rightarrow \exp(t\omega) \in G \; \forall t \in \mathbb{R}$.

$\Rightarrow \omega \in \mathfrak{g}$, a contradiction.

□

Def^n

$\gamma = (\gamma_1, \dots, \gamma_n) : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}^n$ a cont. diff. path.

Tangent vector $\dot{\gamma}(t)$ to γ is

$\dot{\gamma}(t) = \left(\dfrac{d\gamma_1(t)}{dt}, \dots, \dfrac{\partial \gamma_n(t)}{dt}\right)$

By def^n, $\dot{\gamma}(0) = \lim\limits_{\varepsilon \to 0} \left(\dfrac{\gamma(\varepsilon) - \gamma(0)}{\varepsilon}\right)$.

Def^n

$x \in X \subset \mathbb{R}^n$. $\dot{\gamma}(0)$ is a tangent vector to X at x if $\gamma : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}^n$ is a path with $\gamma((-\varepsilon, \varepsilon)) \subset X$, $\gamma(0) = x$.

Tangent cone is the space of all tangent vectors to X at x. If this is a subspace we call it the tangent space.

Prop^n

$\overset{\text{Lie algebra}}{\swarrow}$

The vector space $\mathfrak{g}$ is the tangent space of G at 1. More generally, $g\mathfrak{g}$ is the tangent space of G at $g \in G$.

Proof

Suppose $v \in \mathfrak{g}$. Then $g\exp(tv) \in G$ $\forall t \in \mathbb{R}$.
So $\gamma(t) := g\exp(tv)$ is a path in G, $\gamma(0) = g$,
$\dot{\gamma}(t) = gv\exp(tv)$, $\dot{\gamma}(0) = gv$
$\Rightarrow gv$ is a tangent vector at g.

Conversely suppose gv is a tangent vector to G at g. $\exists \gamma : (-\varepsilon, \varepsilon) \longrightarrow G$ s.t. $\gamma(0) = g$, $\dot{\gamma}(0) = gv$.
Then $g^{-1}\gamma$ is a path satisfying $g^{-1}\gamma(0) = 1$, $g^{-1}\dot{\gamma}(0) = v$.
$\exists f(t) \in \mathfrak{g}$, $f(0) = 0$, s.t. for sufficiently small t,
$g^{-1}\gamma(t) = \exp(f(t))$.

Define $h_n = f(\frac{1}{n})$, so

$$h_n = \log(g^{-1}\gamma(\tfrac{1}{n})) = g^{-1}\gamma(\tfrac{1}{n}) + o(\tfrac{1}{n^2}) = \frac{g^{-1}\dot{\gamma}(0)}{n} + o(\tfrac{1}{n^2})$$

and so $|h_n| = \dfrac{|g^{-1}\dot{\gamma}(0)|}{n} + o(\frac{1}{n^2})$

$$\text{so} \quad \lim_{n \to \infty} \frac{\exp(h_n) - 1}{|h_n|} = \lim_{n \to \infty} \frac{1}{|g^{-1}\dot{\gamma}(0)|} \cdot \frac{g^{-1}\gamma(\frac{1}{n}) - g^{-1}\gamma(0)}{1/n}$$

$$= \frac{g^{-1}\dot{\gamma}(0)}{|g^{-1}\dot{\gamma}(0)|}.$$

Let $v := \dfrac{\dot{\gamma}(0)}{|\dot{\gamma}(0)|}$. Then $\displaystyle\lim_{n \to \infty} \frac{\exp(h_n) - 1}{|h_n|} = v$.

By previous lemma, this implies that $\exp(tv) \in G$ $\forall t$.
So $v \in \mathfrak{g} \Rightarrow \dot{\gamma}(0) \in \mathfrak{g}$. $\square$

MATH0075

09-10-18

Quick way to find tangent spaces:

Lemma

$F = (F_1, \dots, F_n) : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ a smooth map,

$F(p) = q$, $F(p+v) = q + d_p F(v) + \dots$

$$d_p F = \begin{pmatrix} \frac{\partial F_1}{\partial x_1}(p) & \dots & \frac{\partial F_1}{\partial x_m}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1}(p) & \dots & \frac{\partial F_n}{\partial x_m}(p) \end{pmatrix}$$

If $d_p F$ is surjective, then the tangent space to $F^{-1}(q) \in \mathbb{R}^m$ at p is $\mathrm{Ker}\, d_p F$.

Corollary

The tangent space of $G = U(n)$ at $\mathbb{1}$ is $u(n) = \{B \in gl_n(\mathbb{C}) : B^+ = -B\}$

Proof

Let $H = \{B \in gl_n(\mathbb{C}) : B^+ = -B\}$ be the space of Hermitian matrices. Take $F : gl_n(\mathbb{C}) \to H$,

$F(A) = A^+ A$. $U(n) = F^{-1}(\mathbb{1})$.

$F(1+tB) = (1+tB)^+(1+tB) = 1 + t(B+B^+) + o(t^2)$

$d_1 F(B) = B + B^+$.

If $C \in H$, then $C = \frac{1}{2}C + (\frac{1}{2}C)^+ = d_1 F(\frac{C}{2})$

so $d_1 F$ is surjective.

$\Rightarrow$ tangent space of $U(n)$ at 1 is

$\mathrm{Ker}\, d_1 F = \{B \in gl_n(\mathbb{C}) : B + B^+ = 0\}$

Lie Bracket

Lemma

Given a matrix group G, the subspace $\mathfrak{g}$ is preserved by the Lie bracket.

Proof

Given $X, Y \in \mathfrak{g}$, define

$C_{s,t} := \exp(sX)\exp(tY)\exp(-sX)\exp(-tY) \in G$

By Baker - Campbell - Hausdorff

$$\exp(sX)\exp(tY) = \exp(sX + tY + st\tfrac{1}{2}[X,Y] + \tfrac{1}{12}[X,[X,Y]] - \tfrac{1}{12}[Y,[X,Y]] + o(s^2) + o(t^2))$$

So $C_{s,t} = \exp(st[X,Y] + st(o(s) + o(t)))$

Set $\gamma(u) = C_{\sqrt{u}\sqrt{u}}$, so $\gamma(u) = \exp(u[X,X] + O(u^{3/2}))$
This defines a path in G whose tangent vector at $\gamma(0) = 1$ is $[X,Y]$. So $[X,Y] \in \mathfrak{g}$. $\square$

Upshot:

Thm

Given a matrix group G,
$\mathfrak{g} := \{v \in gln(\mathbb{R}) : \exp(tv) \in G \ \forall t\}$
is a vector space that is the tangent space of G at 1.
It is closed under the Lie bracket $[\cdot,\cdot]$, so has the structure of an abstract Lie algebra.
exp carries $\mathfrak{g}$ to G and is a local diffeomorphism from $0 \in U'' \subset \mathfrak{g}$ to $1 \in V'' \subset G$.

MATH0075

09-10-18

Smoothness in exp charts

Suppose $\phi : G \to H$ is a homomorphism between matrix groups.
ϕ is said to be smooth if it is smooth when written in exp charts:

Def^n
G_1, G_2 matrix groups with Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2$.
Let $F : G_1 \to G_2$ be a homo.
Let $\exp : B_i \to C_i$ be exponential charts,
$B_i \subset \mathfrak{g}_i$, $C_i \subset G_i$, $F(C_1) = C_2$.
F is smooth if $f := \exp^{-1} \circ F \circ \exp$ is smooth.
$$B \longmapsto C_1 \longmapsto C_2 \longmapsto B_2$$

By def^n, $\exp(f(X)) = F(\exp(X))$ for X in the domain of an exp. chart.

Example
$R : SU(2) \longrightarrow SO(3)$
$\exp(\theta M_u) \mapsto$ rotation through 2θ about $u = (x, y, z)$.
$$K_u = \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix} \in so(3) \Rightarrow \exp(2\theta K_u) \in SO(3)$$
rotates about u by 2θ.

So in an exponential neighbourhood,
$$\exp^{-1} \circ R \circ \exp \begin{pmatrix} ix & y+iz \\ -y+iz & -ix \end{pmatrix} = 2\begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}$$

Upshot: smooth homomorphisms of matrix groups are linear in an exp. chart.

One parameter subgroups

Example

$\mathbb{R}$ is a closed subgroup of $GL_2(\mathbb{R})$

$$x \longmapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x+y \\ 0 & 1 \end{pmatrix}$$

Def^n

A smooth homomorphism $\phi : \mathbb{R} \to G$ is called a one-parameter subgroup.

Example

$X \in \mathfrak{g} \Rightarrow \phi(t) = \exp(tX)$ is a one-parameter subgroup since $\exp((s+t)X) = \exp(sX)\exp(tX)$.

11-10-18

Lie Algebra homomorphisms & Lie's Thm

One parameter subgroups

homo $\phi : \mathbb{R} \longrightarrow G$, $\phi(t) = \exp(tX)$, $X \in \mathfrak{g}$ is a one parameter subgroup.

Note $\dot{\phi}(t) = X\phi(t)$

$\Psi(s) := \phi(s+t)$ solves $\dfrac{d\Psi}{ds} = X\Psi(s)$, $\Psi(0) = \phi(t)$

$\Theta(s) := \phi(s)\phi(t)$ solves $\dfrac{d\Theta}{ds} = X\Theta(s)$, $\Theta(s) = \phi(t)$

ODEs have unique solns with same initial condition.

So $\Theta = \Psi \Rightarrow \phi$ is a homomorphism.

$\Leftrightarrow \exp((s+t)X) = \exp(sX)\exp(tX)$.

MATH0075

11-10-18

Prop^n

Suppose $\phi : \mathbb{R} \to G$ is a one-parameter subgroup.
Then $\exists X \in \mathfrak{g}$ s.t. $\phi(t) = \exp(tX)$.

Proof

$\phi(s+t) = \phi(s)\phi(t)$, $\frac{d}{ds}\Big|_{s=0}$ both sides.

$\Rightarrow \dot\phi(t) = \dot\phi(0)\phi(t)$
Define $X = \dot\phi(0)$
$\Rightarrow \psi(t) = \exp(tX)$ also satisfies this
So $\phi = \psi$. $\square$

Linearity in exp charts

Thm

Suppose $F : G_1 \to G_2$ is a homo, smooth.
Let $\exp : B_1 \to C_1$, $\exp : B_2 \to C_2$ be exp charts
of open neighbourhoods of $\mathfrak{g}_1 \to G$, $\mathfrak{g}_2 \to G_2$.
Let $f : B_1 \to B_2$ be the map F viewed in exp coordinates.
Then f is a linear map.

Proof

For $X \in \mathfrak{g}_1$, $\exp(tX)$ is a one-parameter subgroup
of G_1. Since F is smooth, $F(\exp(tX))$ is a
one-parameter subgroup of G_2.
So $F(\exp(tX)) = \exp(tY)$ for some $Y \in \mathfrak{g}_2$.
Since $\exp(f(X)) = F(\exp(X))$ for $X \in B$ and so
$\exp(f(tX)) = F(\exp(tX)) = \exp(tY)$ for t
sufficiently close to 0.
Take logarithms, then $\frac{d}{dt}\Big|_{t=0}$

We get $d_0 f(X) = Y$.
We know $f(tX) = tY$ (by taking logs in previous step)
So $d_0 f(tX) = f(tX)$
So $f = d_0 f \Rightarrow f$ is linear.

$\square$

<u>Def</u>

Def^n

Given a smooth homo $F : G_1 \to G_2$, define
$F_* : \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$ by $F_* = d_0 f$.

"Linearisation of F at 1"
"Induced map on Lie algebras"
"Differential of F at the identity"

By def^n, $\exp(f(X)) = F(\exp(X))$ $\forall X \in B$,
$= \exp(F_* X)$

Lemma (proof omitted)

This is true for all $X \in \mathfrak{g}$.

Example

$G_2 = GL_1(\mathbb{R}) = \mathbb{R}^\times$
$F = \det$. Then $F_* = Tr$
$\Rightarrow \det(\exp(X)) = \exp(Tr(X))$.

Any smooth homo $F : G_1 \to G_2$ has the form
$F(\exp(X)) = \exp(F_* X)$ for some linear map
$F_* : \mathfrak{g}_1 \to \mathfrak{g}_2$.

Which linear maps arise?

MATH0075

11-10-18

Def^n

A homomorphism of Lie algebras: $f: \mathfrak{g} \to \mathfrak{h}$ that is linear and satisfies
$[f(X), f(Y)] = f([X,Y])$ $\forall X, Y \in \mathfrak{g}$.

Thm

If $F: G_1 \to G_2$ is a smooth homomorphism, then F_* is a homomorphism.

Proof

We have already shown that

$$\exp(tX)\exp(tY)\exp(-tX)\exp(-tY) = \exp(t^2[X,Y] + o(t^2)).$$

Apply the homo. F:

$$F(\exp(tX))F(\exp(tY))F(\exp(-tX))F(\exp(-tY)) = F(\exp(t^2[X,Y]+o(t^2)))$$

So $\exp(tF_*X)\exp(tF_*Y)\exp(-tF_*X)\exp(-tF_*Y) = \exp(t^2F_*[X,Y] + o(t^2))$

Now do the same from the beginning with $X, Y \in \mathfrak{g}_1$ replaced by $F_*X, F_*Y \in \mathfrak{g}_2$.

We get

$$\to\ = \exp(t^2[F_*X, F_*Y] + o(t^2))$$

$\Rightarrow [F_*X, F_*Y] = F_*[X,Y]$ (Taylor series, the differentiate$_{t=0}$)

$\square$

Example

$F: U(1) \to U(1)$

Lie algebra is $i\mathbb{R} = \{ z \in GL_1(\mathbb{C}) = \mathbb{C}^\times : \overline{z} = -z \}$

$F_* : i\mathbb{R} \to i\mathbb{R}$ linear $\Rightarrow F_*(ix) = i\lambda x$ for some $\lambda \in \mathbb{R}$

$[ix, iy] = xy[i,i] = 0 \Rightarrow$ automatically a Lie algebra homo.

If F_* is the linearisation of F, then

$F(\exp(ix)) = \exp(F_*(ix)) = \exp(i\lambda x)$

But $e^{i2\pi k} = 1$, $F(1) = 1$, so $\exp(i2\pi k\lambda) = 1 \Rightarrow \lambda \in \mathbb{Z}$

Prop^n

Given a smooth homo $F: U(1) \to U(1)$
$\exists \lambda \in \mathbb{Z}$ s.t. $F(z) = z^{\lambda}$.

Simply connectedness

For $U(1) \to U(1)$, there were $|\mathbb{R}|$ possible Lie algebra homos, but only $|\mathbb{Z}|$ that were linearisations of matrix group homos.
The problem is due to the topology of $U(1)$.
Loops in $U(1)$ have a winding number.

Def^n
$X \subset \mathbb{R}^n$ is simply-connected if for any loop
$\gamma: [0,1] \to X$, $\gamma(0) = \gamma(1) = x \in X$ $\exists$ cont. map
$H: [0,1] \times [0,1] \to X$ s.t.
$H(0,t) = H(1,t) = x \quad \forall t$
$H(s,0) = x$, $H(s,1) = \gamma(s)$.
H is a null homotopy of γ.

Think of H contracting $\gamma(s)$ to a point through a family of cont. loops $\gamma_s(t) = H(s,t)$.

Fails for $X = U(1)$: the loop $\gamma(t) = e^{i2\pi t}$ is not contractible.

This is the <u>only</u> obstruction to exponentiating Lie algebra homos.

MATH0075

11-10-18

Thm (Lie's Thm on homomorphisms)
G_1, G_2 path-connected matrix groups with Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2$, with G_1 simply connected.
If $f: \mathfrak{g}_1 \to \mathfrak{g}_2$ is a Lie algebra homo, $\exists$ a matrix group homo. $F: G_1 \to G_2$ st. $F_* = f$.

Example
S^3 is simply connected:
A loop $\gamma([0,1])$ cannot fill out S^3 (it has to have measure zero).
Take $p \in S^3 \setminus \gamma([0,1])$, stereographically project γ to $\mathbb{R}^3$. Any loop in $\mathbb{R}^3$ is contractible to a point. Project this nullhomotopy back to S^3.
Recall $SU(2)$ is topologically homeomorphic to S^3
$\Rightarrow$ $SU(2)$ is simply connected.

In fact, one can show that $SU(n)/SU(n-1) \cong S^{2n-1}$.
The same proof works to show that S^{2n-1} $(n \geq 2)$ is simply connected, and one can then use techniques from algebraic topology (& induction) to show that $SU(n)$ is simply connected (consider the long exact sequence $1 \to SU(n) \to SU(n-1) \to S^{2n-1} \to 1$).

Example
Given a matrix group G, $\exists$ universal cover $\tilde{G}$ that is simply connected: locally isomorphic to G but not necessarily a matrix group.
e.g. universal cover of $SL_2(\mathbb{R})$ does not embed in $GL_n(\mathbb{R})$ for any n.
BUT $\tilde{G}$ has a well-defined notion of exp map and Lie algebra.

$\tilde{G}$ is a Lie group: a smooth manifold that has a group structure with the multiplication and inverse maps being smooth.

Matrix groups are Lie groups, but not vice versa.

Thm (Lie correspondence) vector space with Lie bracket ?

For any abstract Lie algebra $\mathfrak{g}$, $\exists$ a path-connected, simply-connected Lie group G with Lie algebra $\mathfrak{g}$. (Not true for matrix groups)

Thm (Lie again)

Suppose that $\mathfrak{g}$ is the Lie algebra of G. Given a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ $\exists$ a Lie subgroup $H \subset G$ with Lie algebra $\mathfrak{h}$.

Thm (Ado - Hochschild)

Any finite-dimensional Lie algebra occurs as a subalgebra of $\mathfrak{gl}_n(\mathbb{R})$ for some n.

Representations of Lie groups

Let K be a field. A K-representation of a group G consists of a K-vector space V and a homo. $\rho : G \to GL(V)$.

$GL(V)$ = group of invertible linear transformations of V.
e.g. $V = \mathbb{R}^n$, $GL(V) = GL_n(\mathbb{R})$.

Examples

- The zero rep $\rho = 0 : G \to GL(0)$ on the 0-dim vector space.

MATH0075

11-10-18

- The trivial rep $\rho : G \to GL(V)$, $\rho(g) = 1 \ \forall g \in G$ (the identity transformation).

- $K = \mathbb{C}$, $V = \mathbb{C}^n$, $G = U(n)$
  ρ the inclusion $U(n) \hookrightarrow GL_n(\mathbb{C})$
  $K = \mathbb{R}$, $V = \mathbb{R}^n$, $G = SO(n)$
  ρ the inclusion $SO(n) \longrightarrow GL_n(\mathbb{R})$
  } both called the standard rep^n.

- G a matrix group, $\mathfrak{g}$ its Lie algebra ($\mathfrak{g} = V$)
  $\rho = Ad : G \to GL(\mathfrak{g})$, $g \mapsto Ad_g$
  $Ad_g(v) = gvg^{-1}$ for $v \in \mathfrak{g}$ "adjoint rep^n"

Adjoint rep^n of $SU(2)$, 2-to-1 homo
$SU(2) \longrightarrow SO(3) \subset GL_3(\mathbb{R})$
$\exp(\theta M_u) \longmapsto \exp(2\theta K_u)$
where $M_u = \begin{pmatrix} ix & y+iz \\ -y+iz & -ix \end{pmatrix}$, $K_u = \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}$

Want to say when two rep^{n}s are isomorphic.

Def^n
$\rho_1 : G \to GL(V)$, $\rho_2 : G \to GL(W)$ rep^{n}s.
A morphism of rep^{n}s is a linear map $L : V \to W$ s.t. $L(\rho_1(g)v) = \rho_2(g)L(v) \ \forall g \in G, v \in V$,
also called an equivarient map / intertwiner / intertwining map.
L is an isomorphism if it is an isomorphism of vector spaces & an intertwiner.

Example
V the standard rep^n of $SO(3) \Rightarrow \exp(\theta K_u)$ acts via rotation by θ about u.

W the adjoint rep^n of $SO(3)$ on $so(3)$, the Lie algebra of skew-symmetric matrices K_v.
Let $L: V \to W$ be the map $v \mapsto K_v$.
We claim that this is an isomorphism.

Need to check that $Ad_{\exp(\theta K_u)} K_v = K_{\exp(\theta K_u) v}$ if $|u|^2 = 1$

LHS is $\exp(\theta K_u) K_v \exp(-\theta K_u)$.
One can check that this is equal to
$K_v \cos\theta + K_{u \times v} \sin\theta + (1-\cos\theta)(u \cdot v) K_u$.
By Rodrigues' formula, this is $K_{\exp(\theta K_u) K_v}$.

There are much shorter proofs that we will see later (in more generality).

16-10-18

Lie Group Representations

Subrep^{n}s:
A subrepresentation W of V is a subspace s.t. $\rho(g) w \in W \quad \forall g \in G, w \in W$.

This defines a homomorphism $Res_W \rho : G \to GL(W)$
$Res_W \rho(g) := \rho(g)|_W$

Examples
$U(n)$ acts on $gl_n(\mathbb{C})$ by conjugation: $g \mapsto g^{-1} A g$.
This is a rep^n.
Conjugate of a skew-hermitian matrix by g is skew-hermitian $\Rightarrow$ this map preserves $u(n)$.
(This is the adjoint rep^n of $U(n)$).

MATH0075

16-10-18

Conjugate of a trace free skew-hermitian matrix is trace free, skew-hermitian
$\Rightarrow$ this rep^n preserves $su(n)$.

This gives a subrep^n of $u(n)$ which in turn is a subrep^n of $gl_n(\mathbb{C})$.

Examples

If $\rho : G \to GL(V)$ is a rep^n,
- ρ is a subrep^n of ρ
- the zero-dim rep^n $G \to GL(0)$ is a subrep^n of ρ.

Def^n

- A subrep^n of ρ is *proper* if it is not ρ itself or the 0-dim rep^n.
- A rep^n is *irreducible* if it has no proper subrep^{n}s.

New rep^{n}s from old:

Direct sums

Given $\rho_1 : G \to GL(V)$, $\rho_2 : G \to GL(W)$, we can construct $\rho_3 : G \to GL(V \oplus W)$ by $(\rho_1 \oplus \rho_2)(g) := \begin{pmatrix} \rho_1(g) & 0 \\ 0 & \rho_2(g) \end{pmatrix}$

ie. for $v \in V$, $w \in W$ $(\rho_1 \oplus \rho_2)(g)(v, w) := (\rho_1(g)v, \rho_2(g)w)$

If $\rho_1 \neq 0$, $\rho_2 \neq 0$, then $\rho_1 \oplus \rho_2$ contains ρ_1 and ρ_2 as proper subrep^{n}s. $\Rightarrow$ not irreducible.

Duals

V a finite-dim K-vector space.
Its K-dual is the vector space V^* of K-linear functionals $f : V \to K$. (Dual of irred. rep^n is irred.)

Given a rep^n $\rho : G \to GL(V)$, there exists a dual rep^n $\rho^* : G \to GL(V^*)$, $\rho^*(g)f$ is the linear functional whose value on $v \in V$ is $(\rho^*(g)f)(v) = f(\rho(g^{-1})v)$.

Pick a basis of V, so V is identified with column vectors $\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in K^n$, its dual can be viewed as the space of row vectors $(x_1 \ \dots \ x_n) : V \to K$,
$$(x_1 \ \dots \ x_n)\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \sum x_i y_i$$

We can view $\rho(g)$ as an $n \times n$ invertible matrix, acting on the left of column vectors: $\rho(g)\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$
On row vectors, act on the right by $(x_1 \dots x_n)\rho(g^{-1})$.
(g^{-1} instead of g to ensure this is a homo)

Tensor products

Given vector spaces V, W with bases $\{e_i\}$, $\{f_i\}$, form the vector space $V \otimes W$ with basis $\{e_i \otimes f_j\}_{i,j}$.

If V has dim n, W has dim m, then $V \otimes W$ has dim $n \times m$. ($V \oplus W$ has dim $n+m$).

Lemma (Universal property of tensor products)
The bilinear map $\psi : V \times W \to V \otimes W$ defined by
$$\psi\left(\sum_i v_i e_i , \sum_j w_j f_j\right) = \sum_{i,j} (v_i w_j)(e_i \otimes f_j)$$
has the following universal property:
any bilinear map $h : V \times W \to X$, X any vector space, factors uniquely as $h' \circ \psi$ for a linear map $h' : V \otimes W \to X$.

$$\begin{tikzcd} V \times W \arrow[r, "\psi"] \arrow[dr, "h"'] & V \otimes W \arrow[d, dashed, "h'"] \\ & X \end{tikzcd}$$

MATH0075

16-10-18

Proof

Take a basis $\{g_k\}$ for X.

A linear map $h : (\sum_i v_i e_i, \sum_j w_j f_j) \mapsto \sum_k A_k(v,w) g_k$, is bilinear if $A_k(v,w) = \sum_{i,j} A_{ijk} v_i w_j$.

Such a bilinear map factors as $h' \circ \psi$ where $h'(e_i \otimes f_j) = \sum_k A_{ijk} g_k$.

So h' exists.

It is unique: if we instead had $h''(e_i \otimes f_j) = \sum_k B_{ijk} g_k$, then we can work backwards to show $B_{ijk} = A_{ijk}$.

□

Remark

Not every element of $V \otimes W$ is of the form $v \otimes w$ for some $v \in V$, $w \in W$.

Such an element is a pure tensor, but

e.g. $e_1 \otimes f_1 + e_2 \otimes f_2$ is in $V \otimes W$ but is not a pure tensor.

$e_1 \otimes f_1 + e_2 \otimes f_1 = (e_1 + e_2) \otimes f_1$ on the other hand.

Pure tensors form a subvariety.

Any map on $V \otimes W$ can be defined on pure tensors, then defined linearly for everything else.

Def^n

$\rho_1 : G \to GL(V)$, $\rho_2 : G \to GL(W)$ rep^{n}s.

Define $\rho_1 \otimes \rho_2 : G \to GL(V \otimes W)$, by

$(\rho_1 \otimes \rho_2)(g)(v \otimes w) = \rho_1(g) v \otimes \rho_2(g) w$ on pure tensors $v \otimes w \in V \otimes W$, then extend linearly to all tensors.

This may or may not be irreducible.

e.g. $\underset{\text{span}\{e_1, e_2\}}{\mathbb{R}^2} \otimes \underset{\text{span}\{f_1, f_2, f_3\}.}{\mathbb{R}^3} \cong \mathbb{R}^6 = \text{span}\{e_1 \otimes f_1, e_2 \otimes f_1, e_1 \otimes f_2, e_2 \otimes f_2, e_1 \otimes f_3, e_2 \otimes f_3\}$

<u>Hom. spaces</u>

Let $\text{Hom}(V, W)$ denote the vector space of linear maps from V to W.

Given $\rho_1 : G \to GL(V)$, $\rho_2 : G \to GL(W)$, define $\tau : G \to GL(\text{Hom}(V, W))$ by $(\tau(g)F)(v) := \rho_2(g) F(\rho_1(g^{-1})v)$ for $F \in \text{Hom}(V, W)$, $v \in V$.

It can be shown that this is isomorphic to $V^* \otimes W$ (i.e. $\rho_1^* \otimes \rho_2$).

<u>Symmetric powers</u>

K a field with characteristic zero.

<u>Def^n</u>

Consider the action of the symmetric group S_n on $V^{\otimes n} := \underbrace{V \otimes V \otimes \dots \otimes V}_{n \text{ times}}$ defined on pure tensors by $\sigma(v_1 \otimes \dots \otimes v_n) = v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(n)}$, $\sigma \in S_n$.

The n-th symmetric power of V is the subspace $\text{Sym}^n(V) \subset V^{\otimes n}$ consisting of vectors fixed by every $\sigma \in S_n$.

e.g. $V = K$-span of e_1, e_2.

$e_1 \otimes e_2 + e_2 \otimes e_1 \in \underset{\subset V \otimes V}{\text{sym}^2(V)}$, but $e_1 \otimes e_2 \notin \text{Sym}^2(V)$

<u>Def^n</u>

The averaging map is $Av : V^{\otimes n} \longrightarrow V^{\otimes n}$,

$Av(v) = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma(v)$ for $v \in V^{\otimes n}$

Av projects onto $\text{Sym}^n(V)$.

MATH0075

16-10-18

Example

$A_V(e_1 \otimes e_2) = \frac{1}{2}(e_1 \otimes e_2 + e_2 \otimes e_1)$.

Prop^n

Given $\rho : G \to GL(V)$, $Sym^n(V)$ is a proper subrep^n of $V^{\otimes n}$ (or $sym^n(\rho)$ is a subrep^n of $\rho^{\otimes n}$).

Corollary

$\rho^{\otimes n}$ is never irreducible for $n \geq 2$ because $sym^n(\rho)$ is a proper subrep^n of it.

Proof (of prop^n)

We only prove that A_V is an intertwiner from $V^{\otimes n}$ to $V^{\otimes n}$. Later we will show that the image of a subrep^n.

WTS: $A_V(\rho^{\otimes n}(g)v) = \rho^{\otimes n}(g)A_V(v)$.

Suffices to check on pure tensors $v = v_1 \otimes \ldots \otimes v_n$.

$$LHS = \frac{1}{n!}\sum_{\sigma \in S_n} \sigma(\rho(g)v_1 \otimes \ldots \otimes \rho(g)v_n)$$

$$= \frac{1}{n!}\sum_{\sigma \in S_n} \rho(g)v_{\sigma(1)} \otimes \ldots \otimes \rho(g)v_{\sigma(n)}$$

$$= \rho^{\otimes n}(g)\left(\frac{1}{n!}\sum_{\sigma \in S_n} v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)}\right) = RHS$$

$\square$

Sometimes we will write $v_1 \cdots v_n := A_V(v_1 \otimes \ldots \otimes v_n)$

e.g. $xy = \frac{1}{2}(x \otimes y + y \otimes x)$.

$\Leftrightarrow$ think of $Sym^n(V)$ as homogeneous polys in the elements of V.

Exterior powers

Another action of S_n on $V^{\otimes n}$ is
$\sigma(v_1 \otimes \dots \otimes v_n) = \operatorname{sgn}(\sigma)\, v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(n)}$.
Define the vector subspace $\Lambda^n(V) \subset V^{\otimes n}$ as the subspace of tensors fixed under this action.

Example

$e_1 \otimes e_2 - e_2 \otimes e_1 \in \Lambda^2(V)$, but $e_1 \otimes e_2 \notin \Lambda^2(V)$

Def^n

The alternating map $Alt : V^{\otimes n} \longrightarrow V^{\otimes n}$ is defined by
$$Alt(v) := \frac{1}{n!} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, \sigma(v) \quad , \; v \in V^{\otimes n}.$$

This projects from $V^{\otimes n}$ to $\Lambda^n(V)$.

Prop^n

Given $\rho : G \to GL(V)$, $\Lambda^n(V)$ is a subrep^n of $V^{\otimes n}$
(i.e. $\Lambda^n(\rho)$ is a subrep^n of $\rho^{\otimes n}$).
Λ^n is called the n-th exterior power.

We write $v_1 \wedge \dots \wedge v_n := Alt(v_1 \otimes \dots \otimes v_n)$,
projection of the pure tensor $v_1 \otimes \dots \otimes v_n$ onto $\Lambda^n(V)$.

e.g. $x \wedge y = \frac{1}{2}(x \otimes y - y \otimes x)$.

Lemma

If $\dim V = m$, then $\dim \Lambda^n(V) = \binom{m}{n}$.
In particular, $\Lambda^n(V)$ is 0-dim if $n > m$.

Proof

If $e_1, \dots, e_m$ is a basis of V, then a basis of $\Lambda^n(V)$ is $e_{i_1} \wedge \dots \wedge e_{i_n}$, $i_1 < \dots < i_n$.
There are $\binom{m}{n}$ such vectors. $\square$

MATH 0075

16-10-18

Rep^{n}s that aren't irreducible

Direct sum of two nonzero rep^{n}s is not irreducible.

Def^n

A rep^n $\rho : G \to GL(V)$ is completely reducible if $\exists$ irred. subrep^{n}s $V_1, \dots, V_k$ s.t. $V = \bigoplus_{i=1}^{k} V_k$

(equiv $\rho_1, \dots, \rho_k$ s.t. $\rho = \bigoplus_{i=1}^{k} \rho_k$).

ρ is irreducible if $k=1$.

$\exists$ rep^{n}s that are not completely reducible.

$G = \mathbb{C}$, viewed as a group under addition.

This admits a rep^n of $V = \mathbb{C}^2$ given by $z \mapsto \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$

$\exists$ a 1-dim subrep^n : $A = \left\{ \begin{pmatrix} a \\ 0 \end{pmatrix} \in \mathbb{C}^2 : a \in \mathbb{C} \right\}$

(1-dim vector subspace of V).

$G = \mathbb{C}$ acts trivially on A.

Suppose that B is a complementary subrep^n, spanned by $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ s.t. $b_2 \neq 0$.

Then $\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} b_1 + zb_2 \\ b_2 \end{pmatrix}$.

For B to be a subrep^n, for each $z \in \mathbb{C}$, $\exists \lambda(z)$ s.t. $\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \lambda(z) \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} b_1 + zb_2 \\ b_2 \end{pmatrix}$

$\Rightarrow \lambda(z) = 1$ since $b_2 \neq 0$

however $b_1 = b_1 + zb_2$ cannot be satisfied for all z.

Deep Thm

If G is a compact Lie group, then every rep^n is completely reducible.

Unitarity

A Hermitian inner product on a complex vector space is $\langle \cdot , \cdot \rangle : V \times V \to \mathbb{C}$ s.t.

- $\langle \lambda u + \mu v , w \rangle = \lambda \langle u, w \rangle + \mu \langle v, w \rangle$
- $\langle u, w \rangle = \overline{\langle w, u \rangle}$ $\quad [\forall u,v,w \in V,\ \lambda, \mu \in \mathbb{C}]$
- $\langle u, u \rangle \geqslant 0$, $\langle u, u \rangle = 0 \Leftrightarrow u = 0$.

18-10-18

Unitary reps

A unitary rep of a Lie group G is a homo $G \to U(n)$ for some n.

Equivalently, a homo $G \to GL_n(\mathbb{C})$ (identifying V with $\mathbb{C}^n$ for some n) for which there exists an inner product $\langle \cdot , \cdot \rangle$ st. $\rho(g)$ preserves $\langle \cdot , \cdot \rangle$ $\forall g \in G$.
This is called a Hermitian invarient inner product.
["preserves" means $\langle \rho(g) v , \rho(g) w \rangle = \langle v, w \rangle$ $\forall g \in G,\ v, w \in \mathbb{C}^n$]

Lemma

For any rep ρ of a finite group G $\exists$ a Hermitian invarient inner product.
(Known as the Weyl unitary trick).

Proof

Let $\langle \cdot , \cdot \rangle'$ be any Hermitian inner product on $\mathbb{C}^n$, which may not be invarient.
Define $\langle v, w \rangle = \sum_{g \in G} \langle \rho(g) v , \rho(g) w \rangle'$ $\forall v, w \in \mathbb{C}^n$.
Easy to check that this is a Hermitian inner product.
It is invarient: given $h \in G$,
$\langle \rho(h) v , \rho(h) w \rangle = \sum_{g \in G} \langle \rho(gh) v , \rho(gh) w \rangle'$
Relabel $g \mapsto gh^{-1}$, we still run over all elements of G so $\sum_{g \in G} \langle \rho(g) v , \rho(g) w \rangle' = \langle v, w \rangle$. □

Trick: averaging over the group.

MATH0075

18-10-18

Note: We cannot do this trick for arbitrary Lie groups with infinitely many elements.
We can do this for compact Lie groups

Def^n

A manifold X is compact if every open cover $\{U_i\}$ of X has a finite subcover.

Examples

- $\mathbb{R}$ is not compact.
- A matrix group is compact iff it is a bounded subset of $GL_n(\mathbb{R})$.
- $SL_n(\mathbb{R})$ is not compact for $n \geq 2$
- $SU(n)$, $U(n)$, $SO(n)$, $O(n)$ are all compact

Lemma

For any rep. ρ of a compact Lie group, $\exists$ a Hermitian invarient inner product.
(Weyl unitary trick).

Proof

Take any Hermitian inner product $\langle \cdot , \cdot \rangle'$.
Define
$$\langle v, w \rangle = \frac{\int_G \langle \rho(g)v, \rho(g)w \rangle' \, dg}{\int_G dg}$$

Here dg is the Haar measure:
(i) We need $\int_G dg$ to be finite
(ii) We need to be able to make the change of variables $g \mapsto gh^{-1}$ without changing the measure,
i.e. $\int_G f(gh) \, dg = \int_G f(g) \, dg \ \forall h \in G$.

Key thm (Haar)

Such a measure exists (and is unique up to scalar mult

Using this, the proof follows as for the case of finite groups. □

More on the Haar measure:

For noncompact Lie groups, Haar measures exist but don't give G finite volume.

Examples

- Lebesgue measure on $\mathbb{R}^n$
- Finite groups with counting measure.
- $G = U(1) = \{e^{i\theta} : \theta \in [0, 2\pi)\}$

  The Haar integral is $\int_0^{2\pi} f(e^{i\theta})\, d\theta$

  Action of $e^{i\phi} \in U(1)$ sends $e^{i\theta}$ to $e^{i(\theta+\phi)}$ and $d(\theta+\phi)$ to $d\theta$.

Complete reducibility

Recall $\rho : G \to GL(V)$ is completely reducible if it can be written as the direct sum of irreducible repⁿs.

Propⁿ

Let G be a compact Lie group.

Any finite-dim rep is completely reducible.

Proof

Induction on dim of rep.

$n=1$: one-dim repⁿ ρ can only have subrepⁿs of dim 0 or 1, i.e. 0-repⁿ or ρ itself, so ρ is irreducible.

Suppose $\rho : G \to GL(V)$ is an n-dim rep, and assume induction hypothesis for all m-dim reps s.t. $m \leq n-1$.

MATH0075

18-10-18

$\exists$ Hermitian invariant inner product on V, $\langle \cdot , \cdot \rangle$.
If ρ is irreducible, we are done.
If not, $\exists$ proper subrep^n $W \subset V$.
Take the orthogonal complement $W^\perp$ w.r.t. $\langle \cdot , \cdot \rangle$.
ρ preserves the inner product, so it preserves $W^\perp$.
So this is a complementary subrep^n.
$V = W \oplus W^\perp$, ρ = direct sum of the corresponding two subrep^{n}s.
Result follows from induction hypothesis. □

Upshot:
The study of finite-dim reps of compact Lie groups reduces to the study of irred. rep^{n}s.

Goal:
Classify irred. finite dim rep^{n}s of compact Lie groups.
This is completely solved!
[For noncompact groups & infinite-dim rep^{n}s, ongoing (connections to the Langlands program).
(Especially working over other fields other than $\mathbb{R}$ or $\mathbb{C}$).]

Rep^{n}s of Lie algebras

V a K vector space.
A K-rep^n of a Lie group is a homo $\rho: G \to GL(V)$.
A K-rep^n of a Lie algebra is a homo $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$.

Recall: $\mathfrak{gl}(V)$ = vector space of linear transformations of V (not necessarily invertible).
and a Lie algebra homo is a linear map between Lie algebras that preserves Lie brackets:

$$\rho([X,Y]) = [\rho(X), \rho(Y)] = \rho(X)\rho(Y) - \rho(Y)\rho(X).$$

Example

$\mathfrak{g}$ a subalgebra of $\mathfrak{gl}(V)$ such as $su(n)$ in $\mathfrak{gl}_n(\mathbb{C})$ or $so(n)$ in $\mathfrak{gl}_n(\mathbb{R})$,
$\rho: \mathfrak{g} \to \mathfrak{gl}(V)$ is the inclusion map.
This is the standard repⁿ.

Example

Adjoint repⁿ of a Lie algebra on itself,
$X \mapsto ad_X$, $ad_X Y = [X, Y]$.
We will later see that $ad_{[X,Y]} = ad_X ad_Y - ad_Y ad_X$

Example

$\rho: G \to GL(V)$ a repⁿ of a matrix group.
Its linearisation ρ_* is a rep $\mathfrak{g} \to \mathfrak{gl}(V)$
Recall that the linearisation F_* of a map F satisfies $F(\exp(X)) = \exp(F_* X)$.

So $F_* X = \frac{d}{dt}\Big|_{t=0} F(\exp(X))$

Example

Take ρ to be the adjoint repⁿ Ad of G

$$\frac{d}{dt}\Big|_{t=0} Ad_{\exp(tX)} Y = \frac{d}{dt}\Big|_{t=0} \exp(tX) Y \exp(-tX)$$

By the product rule, this is $XY - YX = [X, Y] = ad_X Y$, so Ad_* is ad.

Rep^{n}s of Lie algebras

New rep^{n}s from old

For Lie groups we showed how to create new rep^{n}s via direct sums / duals / tensor products / symmetric powers / exterior powers.
If $\rho : G \to GL(V)$ and $\rho_* : \mathfrak{g} \to \mathfrak{gl}(V)$ are related by $\rho(\exp(X)) = \exp(\rho_* X)$ and V is one of the list above, then we get corresponding direct sums / duals / ... etc of rep^{n}s of the Lie algebra $\mathfrak{g}$.

Direct sum

Suppose $\rho = \rho_1 \oplus \rho_2 : G \to GL(V \oplus W)$ is a direct sum of rep^{n}s $\rho_1 : G \to GL(V)$, $\rho_2 : G \to GL(W)$.
Then $\rho(\exp(tX))(v \oplus w) = \exp(t\rho_* X)(v \oplus w)$
Differentiate w.r.t. t at $t=0$.
$\rho_*(X)(v \oplus w) = \rho_{*1}(X)v \oplus \rho_{*2}(X)w$

Def^n

If $\rho_1 : \mathfrak{g} \to \mathfrak{gl}(V)$, $\rho_2 : \mathfrak{g} \to \mathfrak{gl}(W)$ are rep^{n}s, then we define the rep^n $\rho_1 \oplus \rho_2 : \mathfrak{g} \to \mathfrak{gl}(V \oplus W)$ by $(\rho_1 \oplus \rho_2)(X)(v \oplus w) := \rho_1(X)v \oplus \rho_2(X)w$.

Dual

Recall the dual $\rho^* : G \to GL(V^*)$ of a rep^n $\rho : G \to GL(V)$
Then $(\rho^*(\exp(tX))f)(v) = (\exp(t\rho^*{}_* X)f)(v)$.
Differentiate at $t=0$: $(\rho^*{}_* X)(f)(v) = -f((\rho_* X)v)$.

Def^n

Given a rep^n $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, the dual rep^n $\rho^* : \mathfrak{g} \to \mathfrak{gl}(V^*)$ is defined by $(\rho^*(X)f)v := -f(\rho(X)v)$.

Tensor products

ρ_1, ρ_2 rep^{n}s of G with tensor product

$\rho_1 \otimes \rho_2 = \rho.$

$\rho(\exp(tX))(v \otimes w) = \rho_1(\exp(tX))v \otimes \rho_2(\exp(tX))w$

Differentiate at $t=0$ (using Liebnitz' rule)

$\rho_*(X)(v\otimes w) = (\rho_{1*}(X)v)\otimes w + v \otimes(\rho_{2*}(X)w)$

Def^n

The tensor product of two rep^{n}s $\rho_1 : \mathfrak{g} \to gl(V)$, $\rho_2 : \mathfrak{g} \to gl(W)$ is $\rho_1 \otimes \rho_2 : \mathfrak{g} \to gl(V \otimes W)$,

$((\rho_1 \otimes \rho_2)(X))(v \otimes w) = (\rho_1(X)v)\otimes w + v \otimes (\rho_2(X)w)$

Symmetric and exterior powers

same idea.

Example

Standard rep ρ of $\mathfrak{g} = sl_2(\mathbb{C})$ on $V = \mathbb{C}^2$,

(since $sl_2(\mathbb{C}) \subset gl_2(\mathbb{C}) = gl(V)$).

$sl_2(\mathbb{C})$ is 3-dim : generated by H, X, Y

$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is the usual basis of $V = \mathbb{C}^2$

$He_1 = e_1$, $He_2 = -e_2$

$Xe_1 = 0$, $Xe_2 = e_1$

$Ye_1 = e_2$, $Ye_2 = 0$

We will describe the symmetric square rep^n $Sym^2\rho$

A basis for $Sym^2(\mathbb{C}^2)$ is

$e_1 \otimes e_1$, $\frac{1}{2}(e_1 \otimes e_2 + e_2 \otimes e_1)$, $e_2 \otimes e_2$

Action of H on this basis is

$Sym^2(H)(e_1 \otimes e_1) = (He_1)\otimes e_1 + e_1 \otimes (He_1) = 2(e_1 \otimes e_1)$

$Sym^2(H)(e_2 \otimes e_2) = -2(e_2 \otimes e_2)$

$Sym^2(H)(\frac{1}{2}(e_1 \otimes e_2 + e_2 \otimes e_1)) = 0$

MATH0075

18-10-18

For X:

$Sym^2(X)(e_1 \otimes e_1) = 0$, $Sym^2(X)(\frac{1}{2}(e_1 \otimes e_2 + e_2 \otimes e_1)) = e_1 \otimes e_1$

$Sym^2(X)(e_2 \otimes e_2) = 2\frac{1}{2}(e_1 \otimes e_2 + e_2 \otimes e_1)$

For Y:

$Sym^2(Y)(e_1 \otimes e_1) = 2\ \frac{1}{2}(e_1 \otimes e_2 + e_2 \otimes e_1)$

$Sym^2(Y)(\frac{1}{2}(e_1 \otimes e_2 + e_2 \otimes e_1)) = e_2 \otimes e_2$, $Sym^2(Y)(e_2 \otimes e_2) = 0$.

Example

Exterior square of the standard repⁿ of $sl_2(\mathbb{C})$, $\Lambda^2(\mathbb{C}^2)$ is spamed by $e_1 \wedge e_2 = \frac{1}{2}(e_1 \otimes e_2 + e_2 \otimes e_1)$

$(\Lambda^2 H)(e_1 \wedge e_2) = (He_1) \wedge e_2 + e_1 \wedge (He_2) = 0$ $[e_1 \wedge e_2 = -e_2 \wedge e_1]$

Similarly $(\Lambda^2 X)(e_1 \wedge e_2) = 0$, $(\Lambda^2 Y)(e_1 \wedge e_2) = 0$

So Λ^2 of the standard rep is the zero rep.

Complexification

Suppose that $\mathfrak{g}$ is a Lie algebra over $\mathbb{R}$ (i.e. a real vector space with a Lie bracket).

We can complexify it. $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$.

Here we consider this as the vector space $\mathfrak{g} \oplus \mathfrak{g}$ extended over $\mathbb{C}$, so that we view $v \oplus w \in \mathfrak{g} \oplus \mathfrak{g}$ as the element $v + iw \in \mathfrak{g}_{\mathbb{C}}$.

Then $\mathbb{C}$ acts on $v+iw$ in the obvious way, so this is a complex vector space.

The Lie bracket also extends in the obvious way:

$$\underbrace{[v_1 + iw_1, v_2 + iw_2]}_{\text{on } \mathfrak{g}_{\mathbb{C}}} := ([v_1, v_2] - [w_1, w_2]) + i\underbrace{([v_1, w_2] + [w_1, v_2])}_{\text{on } \mathfrak{g}.}$$

Example

$u(n)_{\mathbb{C}} = gl_n(\mathbb{C})$, $su(n)_{\mathbb{C}} = sl_n(\mathbb{C})$

because i times a skew-Hermitian matrix is Hermitian and every $A \in gl_n(\mathbb{C})$ can be written as the sum of skew-Hermitian and Hermitian matrices,

$A = \frac{1}{2}(A + A^+) + \frac{1}{2}(A - A^+)$

Example

$gl_n(\mathbb{R})_{\mathbb{C}} = gl_n(\mathbb{C})$, $sl_n(\mathbb{R})_{\mathbb{C}} = sl_n(\mathbb{C})$.

In particular, different Lie algebras may have the same complexification.

23-10-18

Representations of Tori

coming soon ↓

Complexifications

Suppose $\mathfrak{g}$ is a real Lie algebra.

- A complex repⁿ of $\mathfrak{g}$ is a real linear Lie algebra homo $\mathfrak{g} \to gl_n(\mathbb{C})$
- A complex-linear complex repⁿ of $\mathfrak{g}_{\mathbb{C}}$ is a complex-linear homo $\mathfrak{g}_{\mathbb{C}} \to gl_n(\mathbb{C})$.

Lemma

There is a 1-1 correspondence between complex repⁿs of $\mathfrak{g}$ and complex-linear complex repⁿs of $\mathfrak{g}_{\mathbb{C}}$.

Proof

Given $\rho : \mathfrak{g} \to gl_n(\mathbb{C})$ a real-linear Lie algebra homo, define $\rho_{\mathbb{C}} : \mathfrak{g}_{\mathbb{C}} \to gl_n(\mathbb{C})$ by $\rho_{\mathbb{C}}(X+iY) = \rho(X) + i\rho(Y)$ for $X+iY \in \mathfrak{g}_{\mathbb{C}} \Leftrightarrow X, Y \in \mathfrak{g}$.

Easy to check that this is a complex-linear complex repⁿ.

MATH0075

23-10-18

Conversely, if $\sigma : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{gl}_n(\mathbb{C})$ is a complex linear map, then $\sigma(X+iY) = \sigma(X) + i\sigma(Y)$ so define $\rho = \sigma|_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{gl}_n(\mathbb{C})$
Easy to check that ρ is a complex rep^n and $\rho_{\mathbb{C}} = \sigma$.

$\square$

Rep^{n}s of $U(1)$.

Define the homo $F_n : U(1) \to U(1)$ by $F_n(e^{i\theta}) = e^{in\theta}$, $n \in \mathbb{Z}$. We showed previously that the only smooth homomorphisms $U(1) \to U(1)$ are F_n.
We call n the weight of F_n.

Lemma (Schur's - baby case!)
If $\rho : U(1) \to GL(V)$ is an irred. finite dim. rep^n of $U(1)$, and $L \in GL(V)$ is an irreducible linear transformation that commutes with every element in $\rho(U(1)) \subset GL(V)$, then $L = \lambda\, id$ for some $\lambda \in \mathbb{C}$.

Proof
Let λ be an eigenvalue of L with eigenspace $E_\lambda = \{ v \in V : Lv = \lambda v \} \neq \{0\}$.
Then E_λ is a subrep^n of ρ:

$L(\rho(g)v) = \rho(g)Lv$ since L commutes with $\rho(g)$
$= \rho(g)\lambda v$ since $v \in E_\lambda$
$= \lambda\rho(g)v$ since ρ is linear

So $\rho(g)v \in E_\lambda$ $\forall g \in G$, $v \in E_\lambda$.
But ρ was assumed to be irreducible, so $E_\lambda = V$
$\Rightarrow Lv = \lambda v$ $\forall v \in V \Rightarrow L = \lambda id$.

Corollary

If $\rho : U(1) \to GL(V)$ is an irreducible complex rep^n of $U(1)$, then V is one-dim.

Proof

As $U(1)$ is abelian and ρ is a homo, $\rho(e^{i\theta})$ and $\rho(e^{i\phi})$ commute $\forall\, \theta, \phi \in \mathbb{R}$.

$\Rightarrow$ Schur's Lemma tells us that $\rho(e^{i\theta})v = \lambda v$ for some $\lambda \in \mathbb{C}^\times$.

If $v \in V \setminus \{0\}$, then $\mathbb{C}v$ is a 1-dim subrep^n: since ρ is irreducible, it must be V.

$\square$

Remark: True for any compact abelian group (including finite abelian groups).

Lemma

Let $\rho : U(1) \to GL_1(\mathbb{C})$ be a 1-dim complex rep^n. Then $\rho(e^{i\theta}) = \rho(e^{in\theta})$ for some $n \in \mathbb{Z}$.

So the set of isomorphism classes of irred. rep^{n}s of $U(1)$ is in bijection with $\mathbb{Z}$:

$$\cdots \bullet \quad \bullet \quad \bullet \quad \circ \quad \bullet \quad \bullet \quad \bullet \cdots$$

each dot represents an integer ($\circ$ represents 0), which corresponds to a rep^n.

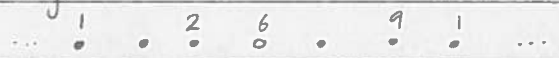
$U(1)$ is compact, so every rep^n is completely reducible $\Rightarrow$ every rep^n can be written as a direct sum of irred. rep^{n}s.

In particular, given a complex rep^n $\rho : U(1) \to GL(V)$ we may write $\rho = \bigoplus_{n \in \mathbb{Z}} F_n^{m_n}$, $m_n \in \mathbb{N}_0$ (≥ 0 integer) i.e. F_n occurs m_n times in ρ.

23-10-18

We call m_n the *multiplicity* of F_n in ρ.

E.g.



corresponds to $\rho = F_{-3} \oplus F_{-1}^{2} \oplus F_0^{6} \oplus F_2^{9} \oplus F_3^{1}$.

The vector space V breaks up into subrep^{n}s

$V = \bigoplus_{n \in \mathbb{Z}} V_n$, $V_n = \{ v \in V : \rho(e^{i\theta})v = e^{in\theta}v \}$

$\dim V_n = m_n$

We call the V_n the *weightspaces* of ρ.
One can think of them as the $e^{in\theta}$-eigenspaces of $\rho(e^{i\theta})$.

The n-torus $T = T_n$ is $U(1)^n$.
This is abelian and compact.

For any n-tuple of integers $(k_1, \dots, k_n) \in \mathbb{Z}^n$,
consider the rep^n $F_{k_1,\dots,k_n} : U(1)^n \to GL_1(\mathbb{C})$
given by $F_{k_1,\dots,k_n}(e^{i\theta_1}, \dots, e^{i\theta_n}) = e^{i(k_1\theta_1 + \dots + k_n\theta_n)}$

Lemma

Any irred. rep^n. of $U(1)^n$ is isomorphic to some $F_{k_1,\dots,k_n}$.

Given a rep^n V of $U(1)^n$, we can restrict to
$U(1)_l := \{ (1, \dots, 1, \underset{\nwarrow\, l \text{ th place}}{e^{i\theta_l}}, 1, \dots, 1) \in U(1)^n \}$, a subgroup ≅ to $U(1)$.

This restricted rep^n has a weight space decomp. $V = \bigoplus_{j \in \mathbb{Z}} V_j$.
Since $U(1)^n$ is abelian, the elements of the subgroups $U(1)_1$ and $U(1)_l$ commute $\forall\ 2 \le l \le n$.
So if $v \in V_j = \{ v \in V : \rho(e^{i\theta}, 1, \dots, 1)v = e^{ij\theta}v$

$h = (e^{i\theta}, 1, \dots, 1) \in U(1)_1$, $g \in U(1)_i$,

then $\rho(h)\rho(g)v = \rho(g)\rho(h)v$ since h, g commute

$= \rho(g) e^{ij\theta} v$ since $v \in V_j$

$= e^{ij\theta} \rho(g) v$ since ρ is linear

so $\rho(g)v \in V_j$.

So V_j is a subrep^n of $V \Rightarrow V_j = V$ since ρ is irred.

So there is only one nonempty weight space, say $j = k_1$,

so $V = V_{k_1}$.

Now decompse $V = V_{k_1}$ into weightspaces of ρ restricted to $U(1)_2$. By the same argument, $V = V_{k_2}$ for some $k_2 \in \mathbb{Z}$, so that

$\rho(1, e^{i\theta}, 1, \dots, 1)v = e^{ik_2\theta} v \quad \forall v \in V$

By induction, we find that V is simultaneously an eigenspace for $(e^{i\theta}, 1, \dots, 1), (1, e^{i\theta}, 1, \dots, 1), \dots, (1, \dots, 1, e^{i\theta})$ with eigenvalues $k_1, k_2, \dots, k_n$.

So $\rho(e^{i\theta_1}, \dots, e^{i\theta_n})v = e^{i(k_1\theta_1 + \dots + k_n\theta_n)} v \quad \forall v \in V.$ $\square$

We just showed that the set of isomorphism classes of irreducible rep^{n}s of $U(1)^n$ are in bijection with the lattice $\mathbb{Z}^n$

$$F_{k_1, \dots, k_n} \longleftrightarrow (k_1, \dots, k_n)$$

$U(1)^n$ compact, so any finite-dim rep^n is completely reducible. So any finite-dim complex rep^n $\rho : U(1)^n \to GL(V)$ decomposes as $\rho = \bigoplus_{(k_1,\dots,k_n) \in \mathbb{Z}^n} F_{k_1,\dots,k_n}^{m_1,\dots,m_n}$

where $m_1, \dots, m_n$ is the multiplicity of $F_{k_1,\dots,k_n}$.

Correspondingly, $V = \bigoplus_{(k_1,\dots,k_n) \in \mathbb{Z}^n} V_{k_1,\dots,k_n}$

where $V_{k_1,\dots,k_n} = \{ v \in V : \rho(e^{i\theta_1}, \dots, e^{i\theta_n})v = e^{i(\theta_1 k_1, \dots, \theta_n k_n)} v \}$

We call $V_{k_1,\dots,k_n}$ the weightspace associated to the weight

$(k_1,\dots,k_n)$; think of the weight space as the simultaneous eigenspaces with eigenvalue $e^{ik_j\theta_j}$ of $\rho|_{U(1)_j}$

Lattice of Weights

This notation is very cumbersome.
Instead let $\mathfrak{t}$ be the complexified Lie algebra of $T = U(1)^n$, $\mathfrak{t}^*$ its dual space;
an element $\lambda \in \mathfrak{t}^*$ is a linear map sending Lie algebra elements to complex numbers.

Consider the element $u_k = (0,\dots,0,i,\overset{\leftarrow k\text{-th place}}{0,\dots,0}) \in \mathfrak{t}$
Then $\exp(tu_k) = (1,\dots,1,\underset{\nwarrow k^{th} \text{ place}}{e^{it}},1,\dots,1) \in U(1)^n$ for $t \in \mathbb{R}$

Given a complex finite-dim rep^n $\rho : U(1)^n \to GL(V)$, the weight spaces are the subspaces of V of the form $V_{k_1,\dots,k_n} = \{v \in V : \rho(\exp(t_1u_1 + \dots + t_nu_n))v = \exp(it_1k_1 + \dots + it_nk_n)v\}$

Equivalently, if ρ_* is the corresponding Lie algebra rep^n, i.e. $\rho(\exp(X)) = \exp(\rho_*(X))$, or $\rho_*X = \frac{d}{dt}\Big|_{t=0} \rho(\exp(tX))$,

then $\rho_*(t_1u_1 + \dots + t_nu_n) = it_1k_1 + \dots + it_nk_n$ when restricted to $V_{k_1,\dots,k_n}$.

Recall that the k_i are integers.
The weights $(k_1,\dots,k_n) \in \mathbb{Z}^n$ can be encoded as an element $\lambda \in \mathfrak{t}^*$: $\lambda(t_1u_1 + \dots + t_nu_n) = it_1k_1 + \dots + it_nk_n$, $t_j \in 2\pi\mathbb{Z}$
so $t_1u_1 + \dots + t_nu_n \in \mathfrak{t}$ are mapped to the identity under the exponential map.
Under λ they are mapped to elements of $2\pi i\mathbb{Z}$

Write $\operatorname{Ker}\exp = \{X \in \mathfrak{t} : \exp X = id\}$ and define the <u>weight lattice</u> $\mathfrak{t}^*_{\mathbb{Z}} := \{\lambda \in \mathfrak{t}^* : \lambda(X) \in 2\pi i\mathbb{Z} \ \forall\, X \in \operatorname{Ker}\exp\}$.

The weight space decomposition can instead be written as $V = \bigoplus_{\lambda \in t^*_{\mathbb{Z}}} V_\lambda$,

$$V_\lambda = \{v \in V : \rho(\exp(X))v = \exp(\lambda(X))v \quad \forall X \in t\}$$
$$= \{v \in V : \rho_*(X)v = \lambda(X)v \quad \forall X \in t\}.$$

We will usually just write $V = \bigoplus_{\alpha \in A} V_\alpha$ where $A \subset t^*_{\mathbb{Z}}$ is the subset of the weight lattice for which $\dim V_\alpha \neq 0$.

Tensor products

Lemma

Let T be a torus with Lie algebra t.
If $\rho_1 : T \to GL(V)$, $\rho_2 : T \to GL(W)$ are rep^{n}s with weight space decompositions
$V = \bigoplus_{\alpha \in A} V_\alpha$, $W = \bigoplus_{\beta \in B} W_\beta$,
then $\rho_1 \otimes \rho_2 : T \to GL(V \otimes W)$ has the weight space decomposition $\bigoplus_\gamma (V \otimes W)_\gamma$ where
$$(V \otimes W)_\gamma = \bigoplus_{\substack{\alpha \in A \\ \beta \in B \\ \alpha+\beta=\gamma}} V_\alpha \otimes W_\beta .$$

Proof

$$V \otimes W = \left(\bigoplus_{\alpha \in A} V_\alpha\right) \otimes \left(\bigoplus_{\beta \in B} W_\beta\right) = \bigoplus_{(\alpha,\beta) \in A \times B} V_\alpha \otimes W_\beta$$

If $v \in V_\alpha$, then $\rho_1(\exp(X))v = \exp(\alpha(X))v \quad \forall X \in t$
$w \in W_\beta$, then $\rho_2(\exp(X))w = \exp(\beta(X))w \quad \forall X \in t$
so
$$\begin{aligned}(\rho_1 \otimes \rho_2)(\exp(X))(v \otimes w) &= \rho_1(\exp(X))v \otimes \rho_2(\exp(X))w \\ &= \exp(\alpha(X))v \otimes \exp(\beta(X))w \\ &= \exp(\alpha(X))\exp(\beta(X))(v \otimes w) \\ &= \exp((\alpha+\beta)(X))(v \otimes w)\end{aligned}$$

$\Rightarrow V_\alpha \otimes W_\beta \subset (V \otimes W)_{\alpha+\beta}$

□

MATH0075

23-10-18

Rep^{n}s of SU(2)

The Lie algebra su(2) consists of trace free skew-Hermitian complex 2×2 matrices

This is a 3-dim real vector space with basis

$\sigma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$

If we think of $\sigma_1, \sigma_2, \sigma_3$ as an oriented basis of $\mathbb{R}^3$, then the Lie bracket is just twice the cross product.

The complexification $su(2)_{\mathbb{C}}$ of $su(2)$ is $sl_2(\mathbb{C})$, the Lie algebra of trace free complex 2×2 matrices

Complex rep^{n}s of su(2) $\longleftrightarrow$ complex linear complex rep^{n}s of $sl_2(\mathbb{C})$.

Basis of $sl_2(\mathbb{C})$:

$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ (as a complex vector space)

It is easily checked that

$[H, X] = 2X$, $[H, Y] = -2Y$, $[X, Y] = H$

Note that $H = -i\sigma_1$, $X = \frac{1}{2}(\sigma_2 - i\sigma_3)$, $Y = -\frac{i}{2}(\sigma_2 + i\sigma_3)$

Lemma

Suppose that $\rho_* : su(2) \to gl(V)$ is a finite-dim. complex rep^n of su(2). Then V decomposes as $V = \bigoplus_\lambda V_\lambda$, where $V_\lambda = \{v \in V : \rho_*(\sigma_1)v = \lambda v\}$.

The collection of such $\lambda \in \mathbb{C}$ are called the weights of V, and V_λ is called the weight space.

Moreove $\lambda \in i\mathbb{Z}$.

Proof

Let $\rho : su(2) \to GL(V)$ be the corresponding rep^n of Lie groups.

$H = \{ exp(t\sigma_1) : t \in [0, 2\pi) \}$ is a subgroup of $SU(2)$ isomorphic to $U(1)$

So $\rho|_H$ is a rep^n of $U(1)$

$\Rightarrow$ V decomposes as a direct sum of $U(1)$ rep^{n}s

$V = \bigoplus_{m \in \mathbb{Z}} V_m$, where $exp(t\sigma_1)$ acts by scalar multiplication by e^{imt} on V_m.

Descending to the Lie algebra, this means that $exp(t\rho_*(\sigma_1))$ acts by e^{imt}, so $\rho_*(\sigma_1)$ acts by im on V_m. (recall $\rho(exp(t\sigma_1)) = exp(t\rho_*(\sigma_1))$).

□

25-10-18

Rep^{n}s of $sl_2(\mathbb{C})$, $su(2)$ & $SU(2)$

Recall $su(2)$ has a basis (over $\mathbb{R}$)

$\sigma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$

$sl_2(\mathbb{C})$ has a basis (over $\mathbb{C}$)

$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

So $H = -i\sigma_1$, $X = \frac{1}{2}(\sigma_2 - i\sigma_3)$, $Y = -\frac{1}{2}(\sigma_2 + i\sigma_3)$

$[H, X] = 2X$, $[H, Y] = 2Y$, $[X, Y] = H$

We showed that a finite dim complex rep^n $\rho : su(2) \to gl(V)$, then $V = \bigoplus_{\lambda} V_\lambda$ where $V_\lambda = \{ v \in V : \rho(\sigma_1)v = \lambda v \}$ and $\lambda \in i\mathbb{Z}$ (because $\{ exp(t\sigma_1) : t \in \mathbb{R} \} \subset SU(2)$ is isomorphic to the torus $U(1)$).

A complex rep^n of $su(2)$ extends to a complex linear complex rep^n ρ of $sl_2(\mathbb{C}) = su(2)_{\mathbb{C}}$

MATH0075

25-10-18

Corollary

If $\rho : sl_2(\mathbb{C}) \to gl(V)$ is a complex linear complex finite dim rep, then $V = \bigoplus_\lambda V_\lambda$, $V_\lambda = \{v \in V : \rho(H)v = \lambda v\}$ and $\lambda \in \mathbb{Z}$.

Note that $\{\exp(tH) : t \in \mathbb{R}\} \subset SL_2(\mathbb{C})$ is not isomorphic to $U(1)$, but rather $\mathbb{R}$.

Now we consider the actions of X and Y. We already have decomposed V into eigenspaces V_λ of $\rho(H)$. What about $\rho(X)$ and $\rho(Y)$? We will use the fact that

$[H, X] = 2X$, $[H, Y] = 2Y$, $[X, Y] = H$.

Lemma

If $\rho : sl_2(\mathbb{C}) \to gl(V)$ is a finite-dim complex linear complex repⁿ with weight space decomposition $V = \bigoplus_\lambda V_\lambda$, then

- $v \in V_\lambda \Rightarrow \rho(X)v \in V_{\lambda+2}$
- $v \in V_\lambda \Rightarrow \rho(Y)v \in V_{\lambda-2}$

[i.e. $\rho(X)$ raises the weight by 2
$\rho(Y)$ lowers the weight by 2.]

Proof

If $v \in V_\lambda$, then $\rho(H)v \in V_\lambda$, so

$$\begin{aligned}\rho(H)\rho(X)v &= \rho(HX - XH + XH)v \\ &= \rho([H, X])v + \rho(X)\rho(H)v \quad \text{by linearity and def}^n \text{ of Lie bracket} \\ &= 2\rho(X)v + \lambda\rho(X)v \\ &= (\lambda+2)\rho(X)v \quad \text{so } \rho(X)v \in V_{\lambda+2}\end{aligned}$$

$$\begin{aligned}\rho(H)\rho(Y)v &= \rho([H, Y])v + \rho(Y)\rho(H)v \\ &= -2\rho(Y)v + \lambda\rho(Y)v = (\lambda-2)\rho(Y)v\end{aligned}$$

so $\rho(Y)v \in V_{\lambda-2}$

□

We can visualise the rep^n $\rho : sl_2(\mathbb{C}) \to gl(V)$ as a chain of weight spaces

$$\begin{tikzcd}
\cdots & V_{\lambda-2} \arrow[loop above, "H"] \arrow[r, bend left, "X"] & V_{\lambda} \arrow[loop above, "H"] \arrow[r, bend left, "X"] \arrow[l, bend left, "Y"] & V_{\lambda+2} \arrow[loop above, "H"] \arrow[r, bend left] \arrow[l, bend left, "Y"] & \cdots \arrow[l, bend left]
\end{tikzcd}$$

where $\rho(H)$ sends V_λ to itself,
$\rho(X)$ sends V_λ to $V_{\lambda+2}$,
$\rho(Y)$ sends V_λ to $V_{\lambda-2}$.

Here each $\lambda \in \mathbb{Z}$, and only finitely many weight spaces V_λ can be nontrivial since V is finite dim.

Classification of irred. rep^n's of $sl_2(\mathbb{C})$

Thm

Suppose that $\rho : sl_2(\mathbb{C}) \to gl(V)$ is an irred. finite dim complex linear complex rep^n.
Then the associated weight spaces V_λ are one-dim, and the weights live in an uninterupted chain $-m, -m+2, \ldots, m-2, m$ from $-m$ to m for some nonnegative integer m.
(so ρ has dim $m+1$).

Proof

V is finite dim, so there exists a maximal integer m for which $\dim V_m > 0$.
Since $\rho(X)v \in V_{m+2}$ for $v \in V_m$, we know that $\rho(X)v = 0 \;\; \forall\, v \in V_m$. Such a $v \in V_m$ is called a heighest-weight vector.
Pick such a vector $v \neq 0$.
Theorem will follow if we can show that a basis for V is $\{V, \rho(Y)v, \rho(Y^2)v, \ldots, \rho(Y^m)v\}$.

MATH0075

25-10-18

Consider the sequence $v, \rho(Y)v, \dots, \rho(Y^k)v$, where k is the smallest integer for which $\rho(Y^k)v \neq 0$ but $\rho(Y^{k+1})v = 0$.

Consider the subspace W of V spanned by these vectors. This is invarient under the action of $\rho(Y)$ (clearly), and also by $\rho(H)$, because $\rho(Y^n)v$ is an eigenvector of $\rho(H)$ with eigenvalue $m-2n$.
It is also invarient under the action of $\rho(X)$.
I claim that $\rho(X)\rho(Y^n)v = B_n\rho(Y^{n-1})v$ for some $B_n \in \mathbb{R}$:

$$\begin{aligned}\rho(X)\rho(Y)v &= \rho([X,Y])v + \rho(Y)\rho(X)v \\ &= \rho(H)v + \rho(Y)0 \\ &= mv\end{aligned}$$

This solves $n=1$. More generally, since
$XY^n = [X,Y]Y^{n-1} + YX^{n-1}, \leftarrow YXY^{n-1}?$

$\underline{\rho(X)\rho(Y^n)v} = \rho[X,Y]\rho(Y^{n-1})v + \rho(Y)\rho(X)\rho(Y^{n-1})v$

by induction hypothesis, $\rho(X)\rho(Y^{n-1})v = B_{n-1}\rho(Y^{n-2})v$

$$\begin{aligned}&= \rho(H)\rho(Y^{n-1})v + B_{n-1}\rho(Y)\rho(Y^{n-2})v \\ &= (m-2n+2)\rho(Y^{n-1})v + B_{n-1}\rho(Y^{n-1})v \\ &= (m-2n+2+B_{n-1})\rho(Y^{n-1})v\end{aligned}$$

Using this we find that
$\rho(X)\rho(Y^n)v = (m-n+1)n\rho(Y^{n-1})v$

More is true:

- $\rho(Y^n)v \neq 0 \quad \forall n \in \{0,\dots,m\}$ by induction
  clearly true for $n=0$
  $\rho(X)\rho(Y^{n+1})v = (m-n)(n+1)\rho(Y^n)v \neq 0$ by hypothesis
  so $\rho(Y^{n+1})v \neq 0$ so long as $n < m$
- $\rho(Y^{m+1})v = 0$. Indeed, let n be the smallest integer such that $\rho(Y^n)v = 0$.
  Then $0 = \rho(X)\rho(Y^n)v = (m-n+1)n\rho(Y^{n-1})v \neq 0$ unless $m = n+1$.

- This implies that the weights of W occur in an uninterrupted chain
  $-m, -m+2, \dots, m-2, m$
  Each space is one-dim, spanned by $\rho(Y^n)v$, where v is a heighest-weight vector.

Finally I claim that $V = W$.
This is because W is preserved by $\rho(H)$, $\rho(X)$ and $\rho(Y)$, and H, X, Y span the Lie algebra, so W is an invariant subspace of V.
Since V is irreducible, $V = W$. $\square$

Thm
The finite-dim complex linear complex irred. rep^{n}s of $sl_2(\mathbb{C})$ are in bijection with the non-negative integers. The bijection sends a rep^n to its highest weight.

We have shown uniqueness but not yet existence.

Half the weight (in $\frac{1}{2}\mathbb{Z}$) is called the spin of the rep^n (will come up later in rep^{n}s of $so(3)$).

This theorem also holds for complex rep^{n}s of $su(2)$ and $SU(2)$ because $sl_2(\mathbb{C}) = su(2)_{\mathbb{C}}$,
$H = -i\sigma_1$, $X = \frac{1}{2}(\sigma_2 - i\sigma_3)$, $Y = -\frac{1}{2}(\sigma_2 + i\sigma_3)$.
So rep^{n}s of $sl_2(\mathbb{C})$ restrict to $su(2)$ and rep^{n}s of $su(2)$ complexify to rep^{n}s of $sl_2(\mathbb{C})$.

To go from rep^{n}s ρ_* of $su(2)$ to rep^{n}s ρ of $SU(2)$ we define $\rho(\exp(X)) = \exp(\rho_*(X))$, and note that $\exp: su(2) \longrightarrow SU(2)$ is surjective.

MATH0075

25-10-18

It remains to construct rep^{n}s of heighest weight m for each non-negative integer m.
We will start with a couple of examples, and then do everything at once via symmetric powers.

Example : the adjoint rep^n. $[\rho = ad, \rho(H) = ad_H]$
Adjoint rep^n ad of $sl_2(\mathbb{C})$, $V = sl_2(\mathbb{C})$
V is spanned by H, X, Y.
$ad_H X = [H, X] = 2X$, $ad_H Y = -2Y$, $ad_H H = 0$.
Weight spaces: $V_{-2} = \mathbb{C}Y$, $V_0 = \mathbb{C}H$, $V_2 = \mathbb{C}X$.
X is a heighest weight vector with weight 2.
$ad_X H = -2X$, $ad_X X = 0$, $ad_X Y = H$
so ad_X sends V_{-2} to V_0, V_0 to V_2, and V_2 to $V_4 = \{0\}$.
$ad_Y H = 2Y$, $ad_Y X = -H$, $ad_Y Y = 0$
so ad_Y sends V_{-2} to $V_{-4} = \{0\}$, V_0 to V_{-2}, V_2 to V_0.

Example : the standard rep^n
$V = \mathbb{C}^2$, $gl(V) = gl_2(\mathbb{C})$
$\rho : sl_2(\mathbb{C}) \to gl(V) = gl_2(\mathbb{C})$ the inclusion map.
$\rho(H), \rho(X), \rho(Y)$ act on elements $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{C}^2$ by
$\rho(H)v = H\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, $\rho(X)v = Xv$, $\rho(Y)v = Yv$
The weight spaces are spanned by $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
with weights $1, -1$.
Standard rep^n has dim 2 $(V = \mathbb{C}^2)$, adjoint rep^n has dim 3 $(V = sl_2(\mathbb{C}))$.

Prop^n
The n-th symmetric power $\mathrm{Sym}^n \rho$ of the standard rep^n ρ of $sl_2(\mathbb{C})$ is irreducible with highest weight n, $\dim = n+1$.

This will complete the proof of the earlier theorem by showing existence.
Note also that by uniqueness, the adjoint rep^n of $\mathfrak{sl}_2(\mathbb{C})$ is isomorphic to the symmetric square of the standard rep^n.

Proof
Let $e_1 = \binom{1}{0}$, $e_2 = \binom{0}{1} \in \mathbb{C}^2$ be eigenvectors of $\rho(H)$ $(= H)$ in the ± 1 eigenspaces of $\mathbb{C}^2$.
The vector $e_1^{\otimes n} = \underbrace{e_1 \otimes \dots \otimes e_n}_{n \text{ times}} \in \text{Sym}^n(\mathbb{C}^2) \subset (\mathbb{C}^2)^{\otimes n}$ has weight n, and so is contained in an irred. subrep^n $W \subset \text{Sym}^n(\mathbb{C}^2)$ of highest weight at least n.
The dim of this subrep^n is at least $n+1$ by the proof of the classification thm of irred. rep^ns of $\mathfrak{sl}_2(\mathbb{C})$, as it contains the non zero vectors
$e_1^{\otimes n}, Ye_1^{\otimes n}, \dots, Y^n e_1^{\otimes n}$.

Recall that we may identify $\text{Sym}^n(\mathbb{C}^2)$ with homogeneous polynomials of degree n in the elements of $\mathbb{C}^2$.
In particular, $\dim(\text{Sym}^n(\mathbb{C}^2))$ is the dim of the space of degree n homogeneous polynomials in 2 variables.
This is spanned by the polynomials
$x^n, x^{n-1}y, x^{n-2}y^2, \dots, xy^{n-1}, y^n$
so $\dim = n+1$.
So $n+1 \le \dim W \le \dim \text{Sym}^n(\mathbb{C}^2) = n+1$
So $W = \text{Sym}^n(\mathbb{C}^2) \Rightarrow \text{Sym}^n(\mathbb{C}^2)$ is irred. with highest weight n.
$\square$

MATH0075

25-10-18

Decomposing Tensor products

$\mathrm{Sym}^n \rho$ is a subrep^n of $\rho^{\otimes n}$.
For $sl_2(\mathbb{C})$, ρ the standard rep^n, $\mathrm{Sym}^n \rho$ is irred.

In general, how can we decompose tensor powers of rep^{n}s into irreducibles?
If V, W decompose as
$V = V_1 \oplus \dots \oplus V_m$, $W = W_1 \oplus \dots \oplus W_n$,
then $V \otimes W = \bigoplus_{j=1}^{m} \bigoplus_{l=1}^{n} V_j \otimes W_l$.
So the hard part is understanding the decomposition of $V \otimes W$ when V & W are both irreducible.

For $sl_2(\mathbb{C})$, every irreducible rep^n is of the form $\mathrm{Sym}^n(\mathbb{C}^2)$.
Question: For any two nonnegative integers n, m, what is the decomposition of $\mathrm{Sym}^n(\mathbb{C}^2) \otimes \mathrm{Sym}^m(\mathbb{C}^2)$?

The answer must be of the form
$\bigoplus_{j=0}^{\infty} \mathrm{Sym}^j(\mathbb{C}^2)^{m_j}$ where $m_j \geq 0$ is the multiplicity of $\mathrm{Sym}^j(\mathbb{C}^2)$ in $\mathrm{Sym}^n(\mathbb{C}^2) \otimes \mathrm{Sym}^m(\mathbb{C}^2)$ and only finitely many j with $m_j > 0$.

30-10-18

Decompose $\mathrm{Sym}^m \mathbb{C}^2 \otimes \mathrm{Sym}^n \mathbb{C}^2$ into irreducibles.

Example

Claim: $\mathrm{Sym}^2 \mathbb{C}^2 \otimes \mathrm{Sym}^3 \mathbb{C}^2 = \mathrm{Sym}^5 \mathbb{C}^2 \oplus \mathrm{Sym}^3 \mathbb{C}^2 \oplus \mathrm{Sym}^1 \mathbb{C}^2$
Of course $\mathrm{Sym}^1 \mathbb{C}^2 = \mathbb{C}^2$
Sketch of proof: Let v & w be highest weight vectors of $V = \mathrm{Sym}^2 \mathbb{C}^2$ and $W = \mathrm{Sym}^3 \mathbb{C}^2$
So $Hv = 2v$, $Hw = 3w$.

$Sym^2 \mathbb{C}^2$ is spanned by v, Yv, Y^2v.
$Sym^3 \mathbb{C}^2$ is spanned by w, Yw, Y^2w, Y^3w.

The tensor product is spanned by $Y^k v \otimes Y^l w$ where $0 \leq k \leq 2$, $0 \leq l \leq 3$, which have weight
$2+3-2k-2l = 5-2(k+l)$
i.e. $H(Y^k v \otimes Y^l w) = (5-2(k+l))\, Y^k v \otimes Y^l w$.
So the weight space decomposition of the tensor product is $Z_{-5} \oplus Z_{-3} \oplus Z_{-1} \oplus Z_1 \oplus Z_3 \oplus Z_5$
where
$$Z_m = \bigoplus_{\substack{k=0 \\ 5-2(k+l)=m}}^{2} \bigoplus_{l=0}^{3} \mathbb{C}\langle Y^k v \otimes Y^l w\rangle .$$

So $\dim Z_{\pm 5} = 1$, $\dim Z_{\pm 3} = 2$, $\dim Z_{\pm 1} = 3$.

The vector $v \otimes w$ (i.e. $k = l = 0$) generates an irreducible subrep^n $\xi_5 := \{u = A(v\otimes w) : A \in sl_2(\mathbb{C})\}$
$\subset Sym^2 \mathbb{C}^2 \otimes Sym^3 \mathbb{C}^2$
of highest weight 5.

Since ξ_5 is irred, it has the weight space decomposition
$\xi_5 = \xi_{5,-5} \oplus \xi_{5,-3} \oplus \xi_{5,-1} \oplus \xi_{5,1} \oplus \xi_{5,3} \oplus \xi_{5,5}$
where each space is one-dim.

Now consider the complement of ξ_5 in $Sym^2 \mathbb{C}^2 \otimes Sym^3 \mathbb{C}^2$.
This has weight spaces with weights $-3, -1, 1, 3$
and dims $1, 2, 2, 1$.
Call this complement ξ. Then a vector in ξ of highest weight 3 generates an irred. subrep^n.
Call this subrep^n ξ_3.
The complement of ξ_3 in ξ has weight spaces with weights $-1, 1$ and dims $1, 1$.

A vector in this complement of heighest weight 1 generates

MATH0075

30-10-18

an irred. subrep^n that is all of this complement (which we will call ξ_1).

So $\mathrm{Sym}^2\mathbb{C}^2 \otimes \mathrm{Sym}^3\mathbb{C}^2 = \xi_5 \oplus \xi_3 \oplus \xi_1$
$= \mathrm{Sym}^5\mathbb{C}^2 \oplus \mathrm{Sym}^3\mathbb{C}^2 \oplus \mathrm{Sym}^1\mathbb{C}^2$.

The general version of this theorem is the following.

Thm (Clebsch - Gordan thm)
The tensor product $\mathrm{Sym}^m\mathbb{C}^2 \otimes \mathrm{Sym}^n\mathbb{C}^2$ of irred. rep^{n}s of $\mathfrak{sl}_2(\mathbb{C})$ (or of $\mathfrak{su}(2)$, or of $SU(2)$) decomposes into subrep^{n}s:

$$\mathrm{Sym}^m\mathbb{C}^2 \otimes \mathrm{Sym}^n\mathbb{C}^2 = \bigoplus_{\substack{k=|m-n| \\ k \equiv m+n \ (\mathrm{mod}\ 2)}}^{m+n} \mathrm{Sym}^k\mathbb{C}^2$$

Proof
In homework (eventually!)

Note: every rep^n on the RHS occurs with multiplicity one. This is not the case for more complicated groups.

Symmetric Powers & Homogeneous Polynomials

Let V_m denote the $(m+1)$-dim complex vector space of homogeneous polynomials in two complex variables with total degree $m \geqslant 0$.
An element f of V_m is of the form
$f(z_1, z_2) = \sum_{j=0}^{m} a_j z_1^{m-j} z_2^{j}$ for $a_j \in \mathbb{C}$, $(z_1, z_2) \in \mathbb{C}^2$.

A basis is given by the monomials $z_1^k z_2^{m-k}$, $k \in \{0, \dots, m\}$

For $(z_1, z_2) \in \mathbb{C}^2$ viewed as a column vector $\binom{z_1}{z_2}$, define the finite-dim complex rep^n ρ_m of $SU(2) \ni g$ by

$\rho_m(g)f(z) = f(g^{-1}z)$.

There is a corresponding Lie algebra rep^n ρ_{m*} of $su(2) = \{X \in gl_2(\mathbb{C}) : \mathrm{Tr}\, X = 0,\ X^{+} = -X\}$ given by

$$\rho_{m*}(X)f(z) = \frac{d}{dt}\Big|_{t=0} f(\exp(-tX)z)$$

we parameterise $z(t) = \exp(-tX)z$ and use the chain rule: $\rho_{m*}(X)f(z) = \frac{\partial f}{\partial z_1}\frac{dz_1}{dt}\Big|_{t=0} + \frac{\partial f}{\partial z_2}\frac{dz_2}{dt}\Big|_{t=0}$

$$= -\frac{\partial f}{\partial z_1}(X_{11}z_1 + X_{12}z_2) - \frac{\partial f}{\partial z_2}(X_{21}z_1 + X_{22}z_2)$$

for $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \in su(2)$

The rep^n ρ_{m*} of $su(2)$ extends to a complex linear complex rep^n of $sl_2(\mathbb{C})$,

$su(2)_{\mathbb{C}} = sl_2(\mathbb{C}) = \{X \in gl_2(\mathbb{C}) : \mathrm{Tr}(X) = 0\}$

$\rho_{\mathbb{C}}(X + iY) = \rho_{m*}(X) + i\rho_{m*}(Y)$.

$sl_2(\mathbb{C})$ is generated by $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

so $\rho_{\mathbb{C}}(H) = -z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2}$

$\rho_{\mathbb{C}}(X) = -z_2\frac{\partial}{\partial z_1}$

$\rho_{\mathbb{C}}(Y) = -z_1\frac{\partial}{\partial z_2}$

We apply these to the basis $z_1^k z_2^{m-k}$ of V_m

$\rho_{\mathbb{C}}(H) z^k z^{m-k} = (m-2k) z_1^k z_2^{m-k}$

$\rho_{\mathbb{C}}(X) z^k z^{m-k} = -k z_1^{k-1} z_2^{m-k+1}$

$\rho_{\mathbb{C}}(Y) z^k z^{m-k} = (k-m) z_1^{k+1} z_2^{m-k-1}$.

In particular, $z_1^k z_2^{m-k}$ is an eigenfunction of $\rho_{\mathbb{C}}(H)$ with eigenvalue $(m-2k)$.

30-10-18

Let W be an irred. subrep^n of V_m containing a nonzero element $w = a_0 z_1^m + a_1 z_1^{m-1} z_2 + \dots + a_m z_2^m$.
Let $k \in \{0, \dots, m\}$ be the largest integer for which $a_k \neq 0$.

$\rho_{\mathbb{C}}(X^k) w$ is a non zero multiple of z_2^m (easy to check).
So W contains z_2^m, and hence contains $\rho_{\mathbb{C}}(X^j) z_2^m$ $\forall j \in \{0, \dots, m\}$.
Each of these is a non zero multiple of $z_1^j z_2^{m-j}$ so W contains $z_1^j z_2^{m-j}$ $\forall j \in \{0, \dots, m\}$ and hence $W = V_m$. So $\rho_{\mathbb{C}}$ is irreducible and has dim $m+1$.

In particular, by the classification of irred. rep^{n}s of $\mathfrak{sl}_2(\mathbb{C})$ this is isomorphic to $\mathrm{Sym}^n \mathbb{C}^2$.

<u>Remark</u>

One can, in this way, construct irred. rep^{n}s of $SU(n)$ (and $\mathfrak{su}(n)$ and $\mathfrak{sl}_n(\mathbb{C})$) by the analogous action of this on the vector space of homogeneous polynomials in n variables of total degree $m \geq 0$.

(More accurately, the space of homogeneous harmonic polynomials).
Similarly for $SO(n)$ with real homogeneous polys.
However, you don't get all irred. subrep^{n}s in this way.

Recall that a binary quadratic form is an expression $ax^2 + bxy + cy^2$ in two variables x, y.
These form a 3-dim vector space.
Alternatively we can view binary quadratic forms as symmetric 2×2 matrices, since

$$ax^2 + bxy + cy^2 = (x\ y)\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} =: x^T M x$$

That is, binary quad. forms can be viewed as the 3-dim space $\left\{ M = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \right\} = V$.

A matrix $g \in SL_2(\mathbb{C})$ acts on V via $M \mapsto g^T M g$ (coordinate change of binary quad. forms).

This is a 3-dim repⁿ of $SL_2(\mathbb{C})$.
The diagonal matrix $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ acts by

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} = \begin{pmatrix} ae^{2i\theta} & b/2 \\ b/2 & ce^{-2i\theta} \end{pmatrix}$$

So the weight space decomposition of this repⁿ is $V_{-2} \oplus V_0 \oplus V_2$, where each summand is 1-dim corresponding to $\begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}$, $\begin{pmatrix} 0 & b/2 \\ b/2 & 0 \end{pmatrix}$, $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$.

By the classification of repⁿs of $sl_2(\mathbb{C})$, this is (iso to) the adjoint repⁿ (or $Sym^2(\mathbb{C})$).

Let's find the polynomials in a, b, c that are invarient under this action.
We consider the matrix entries a, b, c as linear coordinate functions on V
$\Leftrightarrow$ elements of the dual repⁿ V^*.

Then we can identify homogeneous polynomials of degree d in a, b, c as elements of $Sym^d(V^*)$.

An invarient poly is one that is fixed by the action of $SL_2(\mathbb{C})$. That is, it spans a one-dim trivial subrepⁿ of $sym^d(V^*)$.
So we can find such invarient polynomials by decomposing $Sym^d(V^*)$ into irreducible subrepⁿs and looking for one-dim trivial subrepⁿs.

30-10-18

Example

$d = 2$

$V^* = V^*_{-2} \oplus V^*_0 \oplus V_2^*$ spanned by c, b, a resp.

The weight spaces for $Sym^2(V^*)$ are:

$Sym^2(V^*)_{-4} = \mathbb{C}\langle c^2 \rangle$

$Sym^2(V^*)_{-2} = \mathbb{C}\langle bc \rangle$

$Sym^2(V^*)_0 = \mathbb{C}\langle b^2, ac \rangle$

$Sym^2(V^*)_2 = \mathbb{C}\langle ab \rangle$

$Sym^2(V^*)_4 = \mathbb{C}\langle a^2 \rangle$

Here we are viewing $c^2, bc, b^2, \dots$ as homogeneous polynomials $\Leftrightarrow$ elements of $Sym^2(V^*)$.

In particular, a^2 is a highest weight vector and generates an irred subrep^n isomorphic to $Sym^4(\mathbb{C}^2)$. The orthogonal complement of this is a trivial one-dim subrep^n.

So some linear combination of b^2 and ac must be invarient.

1). Let $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{C})$. 2) consider a^2, Ya^2, Y^2a^2, Y^3a^2, Y^4a^2 – this is a basis for the highest weight space containing a^2.

3) Pick an invarient inner product, and let $\Delta \in V^*$ be a vector orthogonal to Y^2a^2.

Then Δ will be a scalar multiple of $b^2 - 4ac$.

Example

$d=4$. If we decompose $Sym^4(V^*)$ into weight spaces, we find that

$Sym^4(V^*)_{-8} = \mathbb{C}\langle c^4 \rangle$

$Sym^4(V^*)_{-6} = \mathbb{C}\langle bc^3 \rangle$

$Sym^4(V^*)_{-4} = \mathbb{C}\langle b^2c^2, ac^3 \rangle$

$Sym^4(V^*)_{-2} = \mathbb{C}\langle b^3c, abc^2 \rangle$

$Sym^4(V^*)_0 = \mathbb{C}\langle b^4, ab^2c, a^2c^2 \rangle$

$Sym^4(V^*)_2 = \mathbb{C}\langle ab^3, a^2bc \rangle$

$Sym^4(V^*)_4 = \mathbb{C}\langle a^2b^2, a^3c \rangle$

$Sym^4(V^*)_6 = \mathbb{C}\langle a^3b \rangle$

$Sym^4(V^*)_8 = \mathbb{C}\langle a^4 \rangle$

Decomposition into irreducible subrep^{n}s is
$Sym^4(V^*) = Sym^8(\mathbb{C}^2) \oplus Sym^4(\mathbb{C}^2) \oplus \mathbb{C}$.
So there is some linear combination of b^4, ab^2c, a^2c^2 that is invarient under the action of $SL_2(\mathbb{C})$. One can show that this is Δ^2 (or a scalar multiple thereof).

Rep^{n}s of $SO(3)$

Recall $SO(3) = \{A \in GL_3(\mathbb{R}) : \det A = 1,\ A^TA = 1\}$
From the homework$^{(1)}$, there exists a Lie group homomorphism $SU(2) \to SO(3)$ that is 2-to-1.

More precisely, the adjoint rep^n of $SU(2)$ is
$Ad: SU(2) \to GL(su(2))$, $Ad_g X = gXg^{-1}$
for $g \in SU(2)$, $X \in su(2)$.

One can show that this is an orthogonal linear transformation of the vector space $su(2)$ w.r.t. the invarient inner product $\langle X, Y\rangle := 2\,Tr(XY)$.
So we can identify Ad_g with an element of $O(3)$.

Ad is a continuous homo: it maps connected sets (such as $SU(2)$) to connected sets, it maps subgroups to subgroups, it maps the identity to the identity.

So the image of $SU(2)$ under Ad can be identified with a connected subgroup of $O(3)$ (in particular, it must contain the identity).

Every such subgroup is contained in in $O(3)$, and one can show that $Ad(SU(2)) \cong SO(3)$.
Moreover, this is a 2-to-1 map.

MATH0075

30-10-18

The 2-to-1 homo $Ad: SU(2) \to SO(3)$ comes from a Lie algebra homo $ad: su(2) \to so(3)$.

Explicitly, ad sends σ_i to $2K_i$ where tK_i is the matrix that exponentiates to a rotation by angle t about the x_i-axis.
Recall that $\sigma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$

$$K_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad K_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

↖ sign error?

Since $su(2)$ is spanned by $\sigma_1, \sigma_2, \sigma_3$
$so(3)$ is spanned by K_1, K_2, K_3,
$ad: su(2) \to so(3)$ is an isomorphism of Lie algebras
$Ad: SU(2) \to SO(3)$ is <u>not</u> an isomorphism since it is 2-to-1. In particular, $SU(2)$ is simply connected but $SO(3)$ is not [$SU(2)$ is a double cover of $SO(3)$].

<u>Lemma</u>
The finite-dim repⁿs of $SU(2)$ that arise as lifts of repⁿs of $SO(3)$ are precisely those with even highest weight (or integer spin).

We say that a repⁿ $\tilde{\rho}: SU(2) \to GL(V)$ is a <u>lift</u> of a repⁿ $\rho: SO(3) \to GL(V)$ if it factors through $Ad: SU(2) \to SO(3)$, i.e. $\tilde{\rho} = \rho \circ Ad$.

01-11-18

Rep^{n}s of $SU(3)$

Weights and weight spaces of the adjoint rep^{n} of $sl_2(\mathbb{C})$ are special: they are called roots and rootspaces.
Special for the following reason:
Suppose that we didn't pick H, X, Y to begin with.
The choice of H was such that iH exponentiated to give a subgroup $U(1) \subset SU(2)$ isomorphic to the 1-torus - this was the key property of H.
Having picked this H, we got weight spaces indexed by $\lambda \in \mathbb{Z}$. For the adjoint rep^{n} we got $-2, 0, 2$. (These are the roots).

For a different choice of H, these might be scaled differently (e.g. replace H with $2H$), but we can always rescale H to ensure that 2 is a root.

Let X be a generator of the 1-dim root space associated to the root 2.
Similarly let Y be a generator of the 1-dim root space for the root -2.

By the def^{n} of the root space,
$$\mathrm{ad}_H X = [H, X] = 2X \quad , \quad \mathrm{ad}_H Y = [H, Y] = -2Y.$$

Then via the Jacobi identity,
$$\mathrm{ad}_H [X, Y] = [H, [X, Y]] = [X, [H, Y]] - [Y, [H, X]]$$
$$= -2[X, Y] - 2[Y, X] = 0.$$

So $[X; Y]$ is in the root space with root 0, spanned by H.

01-11-18

So $[X, Y] = \mu H$ for some $\mu \in \mathbb{C} \setminus \{0\}$
Now we rescale X (or Y) so that $\mu = 1$.
So we have recovered H, X, Y just by starting with the roots and root spaces.

ie. the root space decomposition of the adjoint repⁿ of $sl_2(\mathbb{C})$ is exactly what is needed to read off the Lie bracket relations
$[H, X] = 2X$, $[H, Y] = -2Y$, $[X, Y] = H$,
which were the crucial properties needed to classify the irred. repⁿs of $sl_2(\mathbb{C})$.

Strategy
Given a complex irred repⁿ $\rho: SU(2) \to GL(V)$ we did the following:

1). We took the torus subgroup $T = U(1) \subset SU(2)$ of diagonal matrices $\begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}$ and considered the restricted repⁿ
$\rho|_T : U(1) \to SU(2) \to GL(V)$.

2). We decomposed V into weight spaces $V = \bigoplus_n V_n$,
$V_n = \{v \in V : \rho\begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} v = e^{in\phi} v\}$

3). We considered the corresponding complexified Lie algebra repⁿ $\rho_{\mathbb{C}}$ of $sl_2(\mathbb{C}) = su(2)_{\mathbb{C}}$.
This meant we could consider the action of $H \in sl_2(\mathbb{C})$, so that $V_n = \{v \in V : \rho_{\mathbb{C}}(H)v = nv\}$

4). We took a (complex) basis H, X, Y of $sl_2(\mathbb{C})$ and analysed the action of $\rho_{\mathbb{C}}(X)$ and $\rho_{\mathbb{C}}(Y)$ on weight spaces: $\rho_{\mathbb{C}}(X) V_i \subset V_{i+2}$, $\rho_{\mathbb{C}}(Y) V_i \subset V_{i-2}$
This relied crucially on the fact that
$[H, X] = 2X$, $[H, Y] = -2Y$ (that is, X and Y are bases of root spaces).

5). By these relations and $[X, Y] = H$, we showed that a highest weight vector $v \in V_m$, m maximal, generates an irred. subrep^n of ρ with one-dim weight spaces $V_m, V_{m-2}, \ldots, V_{-m+2}, V_{-m}$.

Given a complex rep^n $\rho : SU(3) \to GL(V)$, our strategy is the following:

1). We will take the subgroup of diagonal matrices of the form $\begin{pmatrix} e^{i\phi_1} & 0 & 0 \\ 0 & e^{i\phi_2} & 0 \\ 0 & 0 & e^{-i(\phi_1+\phi_2)} \end{pmatrix}$, which is isomorphic to a 2-torus $T = U(1)^2 \subset SU(3)$.
Let $\mathfrak{t}$ denote the Lie algebra of T (viewed as a subalgebra of $su(3)$), and consider the restricted rep^n $\rho|_T : T \to SU(3) \to GL(V)$.
Crucially, T is a torus, and we understand completely the irred. rep^{n}s of tori.

2). We decompose V into weight spaces $V = \bigoplus_\lambda V_\lambda$
$V_\lambda = \{ v \in V : \rho(\exp(t))v = e^{i\lambda(t)} v \quad \forall t \in \mathfrak{t} \}$.

3). We take the complexified rep^n $\rho_{\mathbb{C}} : sl_3(\mathbb{C}) \to gl(V)$ and decompose into weight spaces
$V_\lambda = \{ v \in V : \rho_{\mathbb{C}}(H)v = \lambda(H)v \quad \forall H \in \mathfrak{t}_{\mathbb{C}} \}$.

4). We take a (complex) basis of $sl_3(\mathbb{C})$
E_{jk}, $1 \leq j \neq k \leq 3$, and analyse how $\rho_{\mathbb{C}}(E_{jk})$ acts on weight spaces.
We will pick E_{jk} to be weight vectors for the adjoint rep^n (i.e. basis elements of root spaces).

5). We will consider the Lie bracket relations between the elements E_{jk} and also a basis of $\mathfrak{t}_{\mathbb{C}}$ (two

01-11-18

elements; H_{12} and H_{23}). Using this, we will show the existence of a "highest weight vector" v generating an irred subrep".

The Lie algebra $su(3)$

Recall that $su(3) \subset gl_3(\mathbb{C})$ is the (real) vector space of complex 3×3 matrices that are trace free and satisfy $A^+ = -A$.

Its complexification $sl_3(\mathbb{C}) \cong su(3)_{\mathbb{C}}$ is the (complex) vector space of 3×3 complex trace-free matrices.

A basis is as follows:

$$H_{12} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$E_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$E_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{31} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad E_{32} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$sl_3(\mathbb{C})$ is an 8-dim complex vector space.

Write $\mathfrak{h} = \mathfrak{t}_{\mathbb{C}} = \mathfrak{t} \oplus i\mathfrak{t}$ for the abelian Lie subalgebra of complex trace free diagonal 3×3 matrices i.e. $\mathfrak{h} = \mathbb{C}\langle H_{12}, H_{23}\rangle$,

note that it consists of diagonal matrices in $sl_3(\mathbb{R})$ so we will write this as $\mathfrak{h}_{\mathbb{R}}$

(i.e. $i\mathfrak{t} = \mathfrak{h}_{\mathbb{R}} = \mathbb{R}\langle H_{12}, H_{23}\rangle$)

Note that $\mathfrak{h} = \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} : a_i \in \mathbb{C}, \ a_1 + a_2 + a_3 = 0 \right\}$

Recall that $\exp(\mathfrak{t})$ is the subgroup of $SU(3)$ isomorphic to the 2-torus.

The weight lattice

We want to think of weights as elements of $\mathfrak{h}^*$.
The weight lattice is the set of all $\lambda \in \mathfrak{h}^*$ such that $\lambda(X) \in 2\pi i \mathbb{Z} \quad \forall X \in \operatorname{Ker} \exp$.

What is $\operatorname{Ker}\exp$ in this case?

$$\exp\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} = \begin{pmatrix} e^{a_1} & 0 & 0 \\ 0 & e^{a_2} & 0 \\ 0 & 0 & e^{a_3} \end{pmatrix}$$

which is the identity iff $a_1, a_2, a_3 \in 2\pi i \mathbb{Z}$
For $k \in \{1, 2, 3\}$, let $L_k : \mathfrak{h} \to \mathbb{C}$ denote the coordinate function $L_k\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} = a_k$.

So $\begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} \in \operatorname{Ker}\exp \iff L_k\begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} \in 2\pi i \mathbb{Z} \quad \forall k \in \{1,2,3\}$

So L_1, L_2, L_3 span the weight lattice and satisfy
$L_1 + L_2 + L_3 = 0$ (since $a_1 + a_2 + a_3 = 0$ for $\begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} \in \mathfrak{h}$).

How should we view the weight lattice?
It should be a lattice in $\mathbb{R}^2$ (thinking of $\mathfrak{h}^*_{\mathbb{R}} \cong \mathbb{R}^2$), and the elements L_1, L_2, L_3 should have centre of mass at the origin and sit in a symmetric way
e.g. $L_1 = (1, 0)$, $L_2 = (\cos\frac{2\pi}{3}, \sin\frac{2\pi}{3})$, $L_3 = (\cos\frac{4\pi}{3}, \sin\frac{4\pi}{3})$.

We haven't justified why we can view the weight lattice in this way, we will do so (much) later.

Returning to the rep^n $\rho : SU(3) \longrightarrow GL(V)$, restrict to the diagonal 2-torus T and decompose into weight spaces: $V = \bigoplus_{\lambda \in \Lambda \subset \ell^*_{\mathbb{Z}}} V_\lambda$

$V_\lambda = \{ v \in V : \rho_*(H)v = \lambda(H)v \quad \forall H = \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} \}$

If $\lambda = A_1 L_1 + A_2 L_2 + A_3 L_3$, then $\lambda(H) = A_1 a_1 + A_2 a_2 + A_3 a_3$

MATH0075

01-11-18

i.e. $V_\lambda = \{ v \in V : \rho(\exp(iH))v = \exp(i\lambda(H))v \quad \forall H = \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix}$

$a_1, a_2, a_3 \in \mathbb{R} \quad a_1 + a_2 + a_3 = 0 \}$

$= i\mathfrak{t} = \mathfrak{h}_{\mathbb{R}}$

Root spaces

We want to find elements in $\mathfrak{sl}_3(\mathbb{C})$ analogous to $X, Y \in \mathfrak{sl}_2(\mathbb{C})$.

We will find these via the weight space decomposition of the adjoint repⁿ.

We call these weight spaces root spaces and the of the adjoint repⁿ are roots.

If $H = \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix}$, then $\mathrm{ad}_H E_{jk} = (a_j - a_k) E_{jk}$

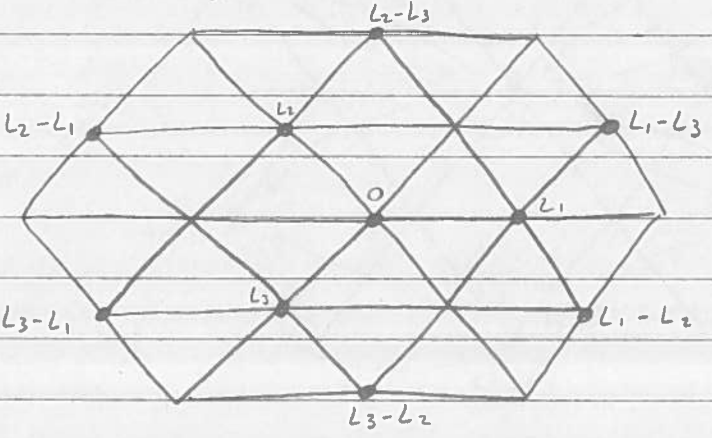
Lemma

The roots of $\mathfrak{sl}_3(\mathbb{C})$ are 0 and $L_j - L_k$ for each $j, k \in \{1, 2, 3\}$, $j \neq k$.

The corresponding root spaces are $\mathfrak{h}$ and $\mathbb{C}\langle E_{jk} \rangle$.

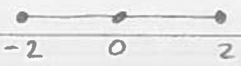
Here $L_j \in \mathfrak{h}^*$ is the linear functional that sends $\begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} \in \mathfrak{h}$ to $a_j \in \mathbb{C}$.

We decided to represent L_1, L_2, L_3 as the vertices of an equilateral triangle, so we can visualise the roots as follows:



These are the roots (weights of the adjoint repⁿ) for $\mathfrak{sl}_3(\mathbb{C})$

For $\mathfrak{sl}_2(\mathbb{C})$:



We write $\Phi := \{ L_j - L_k \}_{j,k \in \{1,2,3\},\ j \neq k}$
and call Φ the *root system* of $sl_3(\mathbb{C})$.

The *root diagram* is the picture of the roots as vectors in $\mathfrak{h}_{\mathbb{R}}^* \cong \mathbb{R}^2$.
For each $\alpha \in \Phi$, we write $sl_3(\mathbb{C})_\alpha$ as the corresponding *root space*

$$sl_3(\mathbb{C})_\alpha := \{ v \in V : ad_H v = \alpha(H) v \ \forall H \in \mathfrak{h} \}.$$

Lemma

If $v \in sl_3(\mathbb{C})_\alpha$, $w \in sl_3(\mathbb{C})_\beta$, then $[v,w] \in sl_3(\mathbb{C})_{\alpha+\beta}$.

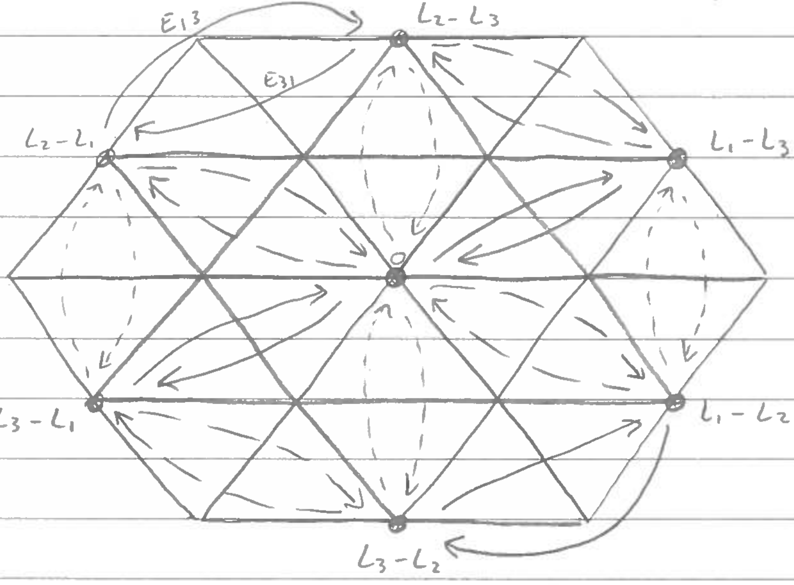
Proof

Given $H \in \mathfrak{h}$,

$$\begin{aligned} ad_H [v,w] &= [H, [v,w]] = [v, [H,w]] - [w, [H,v]] \\ &= (\beta(H) + \alpha(H))[v,w] \end{aligned}$$

by the Jacobi identity. So $[v,w] \in sl_3(\mathbb{C})_{\alpha+\beta}$. $\square$

We can visualise this lemma as follows:



MATH0075

01-11-18

Lemma

For each root $\alpha \in \Phi$, the subspace
$$S_\alpha = sl_3(\mathbb{C})_{-\alpha} \oplus [sl_3(\mathbb{C})_\alpha, sl_3(\mathbb{C})_{-\alpha}] \oplus sl_3(\mathbb{C})_\alpha$$
of $sl_3(\mathbb{C})$ is a Lie subalgebra isomorphic to $sl_2(\mathbb{C})$

Proof

Consider the root $\alpha = L_j - L_k$.
The root spaces with $\alpha \neq 0$ are one-dimensional.
Pick generators $E_{jk} := X_\alpha$, $E_{kj} := Y_\alpha$
One can check that $[X_\alpha, Y_\alpha] = E_{jj} - E_{kk} =: H_\alpha \overset{\neq 0}{\in} \mathfrak{h}$
This generates $[sl_3(\mathbb{C})_\alpha, sl_3(\mathbb{C})_{-\alpha}]$.
Moreover $[H_\alpha, X_\alpha] = \alpha(H) X_\alpha$
$[H_\alpha, Y_\alpha] = -\alpha(H) Y_\alpha$
So this subspace is isomorphic to $sl_2(\mathbb{C})$ by identifying $H_\alpha, X_\alpha, Y_\alpha$ with H, X, Y.
We can insure that $\alpha(H_\alpha) = 2$ by rescaling.
□

Remark

Hiding in this proof is the key fact that $\alpha[X_\alpha, Y_\alpha] \neq 0$.

Remark

There are three distinguished subalgebras of $sl_3(\mathbb{C})$ corresponding to α that are isomorphic to $sl_2(\mathbb{C})$.

13-11-18

Representations of SU(3) (cont.)

Recap of strategy: Given a repⁿ $\rho : SU(3) \to GL(V)$

1). We took the subgroup $T \cong U(1)^2$ of diagonal matrices $\begin{pmatrix} e^{i\phi_1} & & \\ & e^{i\phi_2} & \\ & & e^{-i(\phi_1+\phi_2)} \end{pmatrix}$ and considered the restricted repⁿ $\rho|_T : T \to GL(V)$.

2). We decomposed V into weight spaces
$V_\lambda = \{ v \in V : \rho(\exp(t))v = e^{i\lambda(t)} v \quad \forall t \in \ell \}$,
where ℓ is the Lie algebra of L.

3). We took the complexified Lie algebra repⁿ $\rho_{\mathbb{C}} : sl_3(\mathbb{C}) \to gl(V)$, which allows us to make sense of $\rho_{\mathbb{C}}(H_{12})$, $\rho_{\mathbb{C}}(H_{23})$ where
$H_{12} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $H_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ etc.

Then $V_\lambda = \{ v \in V : \rho_{\mathbb{C}}(H)v = \lambda(H)v \quad \forall H \in \ell_{\mathbb{C}} \}$

4). We considered basis elements H_{12}, H_{23}, and E_{jk} of $sl_3(\mathbb{C})$ where $j, k \in \{1, 2, 3\}$, $j \neq k$, and we analysed how the elements $\rho_{\mathbb{C}}(E_{jk})$ acted on the weight spaces. This worked because the E_{jk} are weight vectors for the adjoint repⁿ.

5). By considering the commutation relations between H_{12}, H_{23}, and E_{jk} we will show that a "heighest weight vector" $v \in V_\lambda$ generates an irred. subrepⁿ (in a suitable sense).

Highest weight vectors

Suppose that $\rho_{\mathbb{C}} : sl_3(\mathbb{C}) \to gl(V)$ is an irred. repⁿ with weight space decomposition $V = \bigoplus_{\alpha \in \Lambda} V_\alpha$

Recall $s_\alpha = sl_3(\mathbb{C})_{-\alpha} \oplus [sl_3(\mathbb{C})_\alpha, sl_3(\mathbb{C})_{-\alpha}] \oplus sl_3(\mathbb{C})_\alpha$

$sl_3(\mathbb{C})_\alpha = \{ X \in sl_3(\mathbb{C}) : ad_H X = \alpha(H) X \quad \forall H \in \mathfrak{h} \}$

MATH0075

13-11-18

Lemma

If $X \in sl_3(\mathbb{C})_\alpha$, $v \in V_\beta$, then $\rho_{\mathbb{C}}(X)v \in V_{\alpha+\beta}$.

Proof

If $H \in \mathfrak{h}$, then

$$\begin{aligned}\rho_{\mathbb{C}}(H)(\rho_{\mathbb{C}}(X)v) &= \rho_{\mathbb{C}}([H,X])v + \rho_{\mathbb{C}}(X)(\rho_{\mathbb{C}}(H)v)\\ &= \alpha(H)\rho_{\mathbb{C}}(X)v + \beta(H)\rho_{\mathbb{C}}(X)v\\ &= (\alpha+\beta)(H)\rho_{\mathbb{C}}(X)v\end{aligned}$$

So $\rho_{\mathbb{C}}(X)v \in V_{\alpha+\beta}$ $\square$

We need an analogue of the highest weight vector. This is not straightforward: while weights for $sl_2(\mathbb{C})$ lay on the real line, weights for $sl_3(\mathbb{C})$ lie in $\mathbb{R}^2$ which doesn't have a natural ordering.

Recall that $\mathfrak{h}_{\mathbb{R}} = sl_3(\mathbb{R}) = \{X \in gl_3(\mathbb{R}) : Tr(X) = 0\}$, $\mathfrak{h}_{\mathbb{R}}^* \cong \mathbb{R}^2$.

A lattice L in a real k-dim vector space $V \cong \mathbb{R}^k$ is a subgroup isomorphic to $\mathbb{Z}^k$ that spans V.

Def^n

A linear functional $\pi : \mathfrak{h}_{\mathbb{R}}^* \to \mathbb{R}$ is irrational w.r.t. a lattice L if for any $\alpha, \beta \in L$, $\pi(\alpha) = \pi(\beta)$ iff $\alpha = \beta$.

Example

$e_1, \dots, e_k$ an integral basis for L, $\mu_1, \dots, \mu_k \in \mathbb{R}$ linearly independent over $\mathbb{Q}$, then $\pi(\sum_{j=1}^{k} \alpha_j e_j) = \sum_{j=1}^{k} \alpha_j \mu_j$ is irrational w.r.t. L.

If $\alpha = \sum \alpha_j e_j$, $\beta = \sum \beta_j e_j$, then
$\pi(\alpha) - \pi(\beta) = \sum (\alpha_j - \beta_j)\mu_j$ and if this is non-zero then $\mu_1, \ldots, \mu_n$ aren't linearly independent over $\mathbb{Q}$.

Def^n (Highest weight vector)
Given a linear functional $\pi : \mathfrak{h}_{\mathbb{R}}^* \to \mathbb{R}$ that is irrational w.r.t. $\mathfrak{h}_{\mathbb{Z}}^*$, the weight α s.t. $\pi(\alpha) = \max_{\beta} \pi(\beta)$ is called the highest weight (w.r.t. π).
Any $v \in V_\alpha$ is called a highest weight vector.

There are only finitely many weights, so the highest weight exists.
The irrationality of π means that the highest weight is unique.

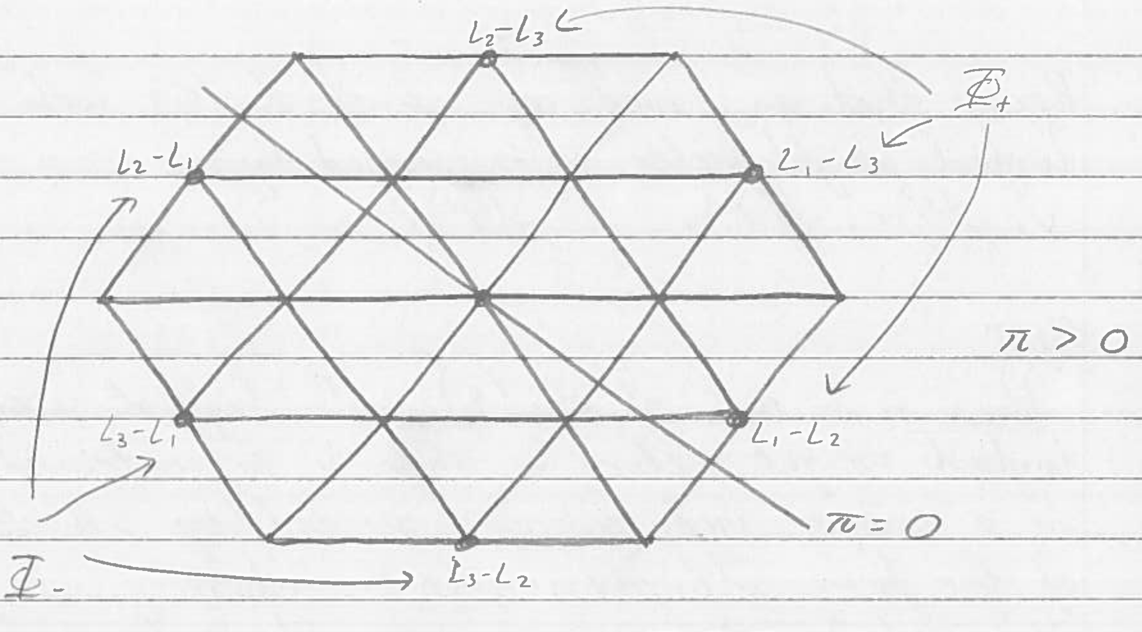
Since π is irrational w.r.t. $\mathfrak{h}_{\mathbb{Z}}^*$, which contains the roots, none of the nonzero roots $\gamma \in \Phi$ has $\pi(\gamma) = 0$.
If $\gamma \in \Phi$, then $-\gamma \in \Phi$. So if $\pi(\gamma) > 0$ then $\pi(-\gamma) < 0$.

Def^n (Positive and negative roots)
We write $\Phi = \Phi_+ \sqcup \Phi_-$, and call the elements of Φ_+ the positive roots (w.r.t. π), and elements of Φ_- the negative roots.

Now we pick π s.t.

$$\Phi_+ = \{L_2 - L_3, L_1 - L_3, L_1 - L_2\}$$
$$\Phi_- = \{L_2 - L_1, L_3 - L_1, L_3 - L_2\}$$

13-11-18



π was basically arbitrary.
There are 6 ways to divide Φ into Φ_+ & Φ_-.

Defⁿ
If u lies in some weight space V_δ for some repⁿ ρ, we write $w_\rho(u) = \delta$ for the weight of u.
[If the repⁿ ρ is clear from context, just write $w(u)$.]
If ρ is the adjoint repⁿ, we write $r(u) = \alpha$, which is a root.

Corollary
If $v \in V_\alpha$ is a highest weight vector for a repⁿ ρ, and $X \in sl_3(\mathbb{C})$, and $r(X) \in \Phi_+$, then $\rho_{\mathbb{C}}(X)v = 0$.

Proof
Previous lemma implies that $\rho_{\mathbb{C}}(X)v \in V_{\alpha + r(X)}$
but $\pi(\alpha + r(X)) = \pi(\alpha) + \pi(r(X)) > \pi(\alpha)$, yet $\pi(\alpha)$ is maximal. So $\rho_{\mathbb{C}}(X)v = 0$.

□

Recall that an irred. rep^n ρ of $sl_2(\mathbb{C})$ with highest weight vector v is spanned by $v, \rho(Y)v, \rho(Y^2)v, \dots, \rho(Y^n)v$.

Prop^n

If ρ is a rep^n of $sl_3(\mathbb{C})$ with highest weight α, highest weight vector v, then v is contained in a unique irred. subrep^n spanned by all elements of the form $\rho(X_1)\rho(X_2)\cdots\rho(X_n)v$ where $X_i \in sl_3(\mathbb{C})_{\alpha_i}$ for some sequence of negative roots $\alpha_j \in \Phi_-$.

Choosing π as previously, we may assume that $X_j \in \{E_{21}, E_{31}, E_{32}\}$. In particular, if ρ is irreducible it is equal to this subrep^n, and so is spanned by $\rho(X_1)\cdots\rho(X_n)v$ taken over all negative root vectors X_j.

Proof

It suffices to show that the subspace W spanned by these elements is preserved by $sl_3(\mathbb{C})$.

By construction, it is preserved by the action of E_{21}, E_{31}, E_{32}.

Next, if $H \in \mathfrak{h}$, then $\rho(H)W \subset W$, as

$$\rho(H)\rho(X_1)\cdots\rho(X_n)v = \rho([H,X_1])\rho(X_2)\cdots\rho(X_n)v + \rho(X_1)\rho(H)\rho(X_2)\cdots\rho(X_n)v$$

$$= \alpha(H)\rho(X_1)\cdots\rho(X_n)v + \sum_{j=1}^{n} \alpha_j(H)\rho(X_1)\cdots\rho(X_n)v$$

by iterating this process

$$= \left(\alpha(H) + \sum_{j=1}^{n}\alpha_j(H)\right)\rho(X_1)\cdots\rho(X_n)v$$

$\Rightarrow \rho(H)W \subset W$, since $[H, X_j] = \alpha_j(H)X_j$, $Hv = \alpha(H)v$.

Lastly, we need to check that E_{12}, E_{13}, E_{23} preserve W. This is a relatively easy inductive exercise. (Similar idea to the proof for $sl_2(\mathbb{C})$). $\square$

MATH0075

13-11-18

Applying $\mathfrak{s}_\alpha$

We know that

- the highest weight space is 1-dim, spanned by v : all other vectors $u = \rho(X_1)\cdots\rho(X_n)v$, $\omega(X_j) \in \Phi_-$, have $\pi(\omega(u)) < \pi(\alpha)$.
- the weight spaces $V_{\alpha+k(L_2-L_1)}$, $V_{\alpha+k(L_3-L_2)}$ are at most 1-dim, spanned by $\rho(E_{21}^k)v$, $\rho(E_{32}^k)v$ resp.
- all the weights that occur in V are contained in the following shaded subspace of $\mathfrak{h}_{\mathbb{R}}^* \cong \mathbb{R}^2$:

To get more information, we apply the subalgebras $\mathfrak{s}_\beta \cong \mathfrak{sl}_2(\mathbb{C})$ to the highest weigh vector v for $\beta = L_2-L_1, L_3-L_2 \in \Phi_-$

The upper edge

Weights on the upper edge are of the form $\alpha + k(L_2-L_1)$.

$V_{\alpha+k(L_2-L_1)}$ gives a rep^n of $\mathfrak{s}_{L_2-L_1} \cong \mathfrak{sl}_2(\mathbb{C})$.

This is an irreducible rep^n of $\mathfrak{s}_{L_2-L_1}$ generated by v.

The weight of $V_{\alpha+k(L_2-L_1)}$ as a $\mathfrak{s}_{L_2-L_1}$ rep^n are $\alpha(H_{12}) - 2k$, recalling that the diagonal element in $\mathfrak{s}_{L_2-L_1}$ is H_{12} (which we view as $H \in \mathfrak{sl}_2(\mathbb{C})$), $(L_2-L_1)(H_{12}) = 2$.

So from our knowledge of $\mathfrak{sl}_2(\mathbb{C})$ reps, each weight space along the upper edge is at most 1-dim, and after a certain point, they are all 0-dim.

In fact, terminates when k reaches $\alpha(H_{12})$.

Geometrically, the edge is parallel to the line through O and L_2-L_1, and so orthogonal to $l_{12} = \{\beta \in \mathfrak{h}_{\mathbb{R}}^* : \beta(H_{12}) = 0\}$ which is parallel to L_3.

The weights of the $\mathfrak{s}_{L_2-L_1}$-rep^n along the edge are the values $\beta(H_{12})$ which are symmetric about zero.

The right edge

Similarly, we can analyse the action of $\mathfrak{s}_{L_3-L_2}$ on the weight spaces $V_{\alpha+k(L_3-L_2)}$, which correspond to weights on the right edge.
Again these come in an unbroken sequence of 1-dim weight spaces that terminate at $k=\alpha(H_{23})$.

Rotating π

Recall that we chose π such that
$\Phi_+ = \{L_1-L_2, L_1-L_3, L_2-L_3\}$, $\Phi_- = \{L_2-L_1, L_3-L_1, L_3-L_2\}$.

Weight space decomposition was independent of π: only highest weight depended on π.
In particular, we could instead have $\alpha+\alpha(H_{12})(L_2-L_1)$ be the highest weight by instead choosing π s.t. $\Phi_+ = \{L_2-L_3, L_1-L_3, L_2-L_1\}$
There are six ways to divide Φ into $\Phi_+ \sqcup \Phi_-$, which gives us six possible choices of highest weight, depending on our choice of π.

By doing the same analysis as before on the edges, we find that the weights lie inside a hexagon whose vertices are the possible highest weights.

By analysing the action of the $\mathfrak{sl}_2(\mathbb{C})$ subalgebras we see that

- the weight spaces along the edges of the hexagon are 1-dim.
- the hexagon is symmetric under reflections along the lines $l_{jk} = \{\beta \in \mathfrak{h}_{\mathbb{R}}^* : \beta(H_{jk})=0\}$ (though sides may have different lengths).
- points β contained in the intersection of the

MATH0075

13-11-18

weight lattice with the interior of the hexagon occur as weights of the repⁿ, each weight space on an edge generates an irred. repⁿ of one of the distinguished $sl_2(\mathbb{C})$ subalgebras $\mathfrak{s}_\alpha$.

Defⁿ

The group of reflections in the lines l_{jk} is called the Weyl group of $sl_3(\mathbb{C})$, and is isomorphic to S_3.

Remark

If $\alpha(H_{12})$ or $\alpha(H_{23}) = 0$, where α is the highest weight, then the hexagon is actually a triangle (or a "degenerate hexagon").

Uniqueness

WTS: $\exists$ unique irred. repⁿ with given highest weight vector.

Lemma (Schur's Lemma).

If V, W are irred. repⁿs of a Lie algebra $\mathfrak{g}$, a homo. $f: V \to W$ is either 0 or an isomorphism.

Proof

Ker f is a subrepⁿ of V so is either 0 or V

Im f is a subrepⁿ of W so is either 0 or W.

□

Lemma (Uniqueness)

$\exists$ unique (up to iso) irred. repⁿ of $sl_3(\mathbb{C})$ with a given highest weight.

Proof Exercise using previous lemma.

□

15-11-18

More Rep^{n}s of $SU(3)$

Uniqueness

Thm (Classification of irred rep^{n}s of $SU(3)/su(3)/sl_3(\mathbb{C})$).
Irred. rep^{n}s are as follows:
Take a weight $\alpha = aL_1 - bL_3$, a, b nonnegative integers, consider its reflections under the Weyl group.
This gives us six points in the weight lattice.
The convex hull of these points is a hexagon X that is possibly degenerate.
There is a unique irred. rep^n $\Gamma_{a,b}$ whose weights that occur are translates of α by roots.

Write $W(\Lambda) \subset \mathfrak{h}^*_{\mathbb{Z}}$ the $\mathbb{Z}$-lattice spanned by the weights Λ, and call $X \cap W(\Lambda)$ the hexagon of weights.
The hexagon of weights is layered by concentric hexagons:
$X \cap W(\Lambda) = X_0 \cap W(\Lambda) \supset X_1 \cap W(\Lambda) \supset \dots \supset X_m \cap W(\Lambda)$.
The dim of a weight space on each layer is constant and given as follows:

- For a weight on the boundary of the hexagon the dimension is one.
- On each subsequent layer, the dim increases by one if the hexagon is non-degenerate, and otherwise remains the same.

"Proof" by example - we will construct some rep^{n}s

The part of the thm on dims of weight spaces follows from a thm in a much more general setting (arbitrary compact Lie groups): the Freudenthal multiplicity thm

MATH0075

15-11-18

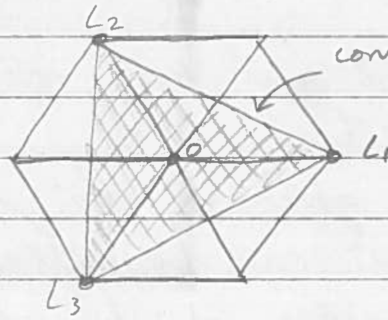
1). The standard repⁿ

Simplest nontrivial repⁿ of $SU(3)$ is the standard repⁿ on $\mathbb{C}^3$.
Then $e^{isH_{12}}$, $e^{itH_{23}}$ act by

$$\begin{pmatrix} e^{is} & 0 & 0 \\ 0 & e^{-is} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{it} & 0 \\ 0 & 0 & e^{-it} \end{pmatrix}$$

Take a standard basis e_1, e_2, e_3 of $\mathbb{C}^3$ so that the simultaneous eigenspaces are $\mathbb{C}e_1$, $\mathbb{C}e_2$, $\mathbb{C}e_3$ with corresponding weights $\alpha_1 = L_1$, $\alpha_2 = L_2$, $\alpha_3 = L_3$.

i.e. $e^{itH} e_k = e^{itL_k(H)} e_k$, recalling that
$L_k \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} = a_k$, $k \in \{1, 2, 3\}$.



Weight space diagram of standard repⁿ.

Each of $L_1, L_2, L_3, 0$ are weights with one-dim weight space. This repⁿ is $\Gamma_{1,0}$.

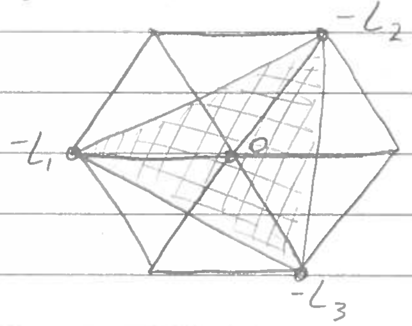
2) The dual standard repⁿ

Now we consider $(\mathbb{C}^3)^*$, the dual repⁿ:
view these as row vector instead of column vectors, with group action $\rho(A)(a\ b\ c) = (a\ b\ c)A^{-1}$, $A \in SU(3)$.
(A^{-1} to ensure that this is a homo).
Take a dual basis e_1^*, e_2^*, e_3^* of $(\mathbb{C}^3)^*$, so that $e_j^*(e_k) = \delta_{jk}$.
Then the action of $e^{isH_{12}}$, $e^{itH_{23}}$ is

$$\begin{pmatrix} e^{-is} & 0 & 0 \\ 0 & e^{is} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-it} & 0 \\ 0 & 0 & e^{it} \end{pmatrix}$$ with simultaneous eigenspaces $\mathbb{C}e_1^*$, $\mathbb{C}e_2^*$, $\mathbb{C}e_3^*$.

Corresponding weights are $\alpha_1 = -L_1$, $\alpha_2 = -L_2$, $\alpha_3 = -L_3$

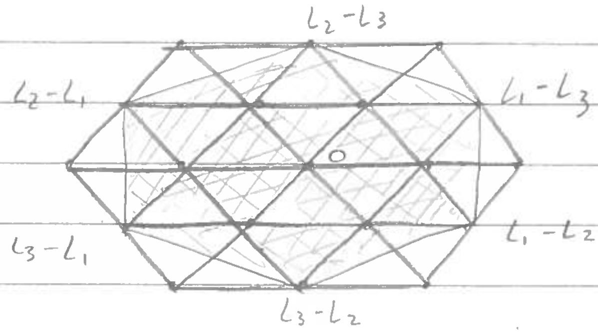
Weight diagram is



Again the convex hull is degenerate: each weight $-L_1$, $-L_2$, $-L_3$, 0 has a 1-dim weight space. This rep^n is $\Gamma_{0,1}$.

In particular, taking the dual rep^n reflected the hexagon along the vertical axis.
This is always true: given a rep^n ρ, the weight diagram of ρ^* can be obtained by applying the transformation $-1 : \mathfrak{h}^*_{\mathbb{R}} \to \mathfrak{h}^*_{\mathbb{R}}$ to the weight diagram of ρ.

3) The adjoint rep^n
Weights are roots.



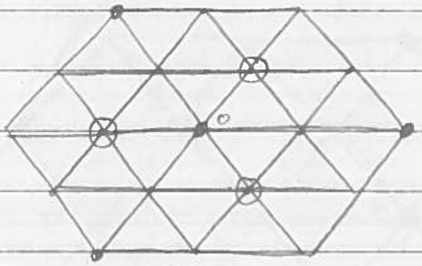
Each weight L_1-L_3, L_1-L_2, L_3-L_2, L_3-L_1, L_2-L_3 has an associated 1-dim weight space.
The weight 0 has a 2-dim weight space.
This rep^n is $\Gamma_{1,1}$.

MATH0075

15-11-18

4) The tensor square $(\mathbb{C}^3)^{\otimes 2}$

We take the tensor product of the standard rep^n with itself. The new weight spaces are $\mathbb{C}(e_i \otimes e_j) \subset (\mathbb{C}^3)^{\otimes 2}$, with corresponding weight $L_i + L_j$. The weight diagram is the following



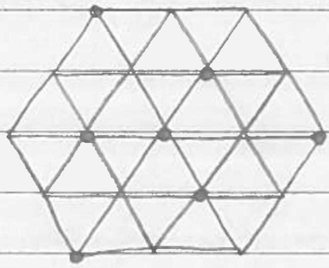
where each $\odot$ has a 2-dim weight space.

The weight spaces of weights on the boundary of the convex hull are not all 1-dim.

So this is not irreducible.

More generally, the tensor square of a non trivial rep^n is never irred: it always contains the symmetric square and exterior square rep^{n}s.

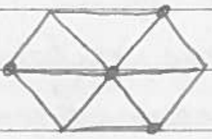
We can factorise this rep^n into irreducibles by inspecting the weight diagram: it contains



where each weight space is 2-dim (since this is a degenerate hexagon.

This is the rep^n $\Gamma_{2,0}$.

It also contains



$\Gamma_{0,1}$ the dual standard rep^n.

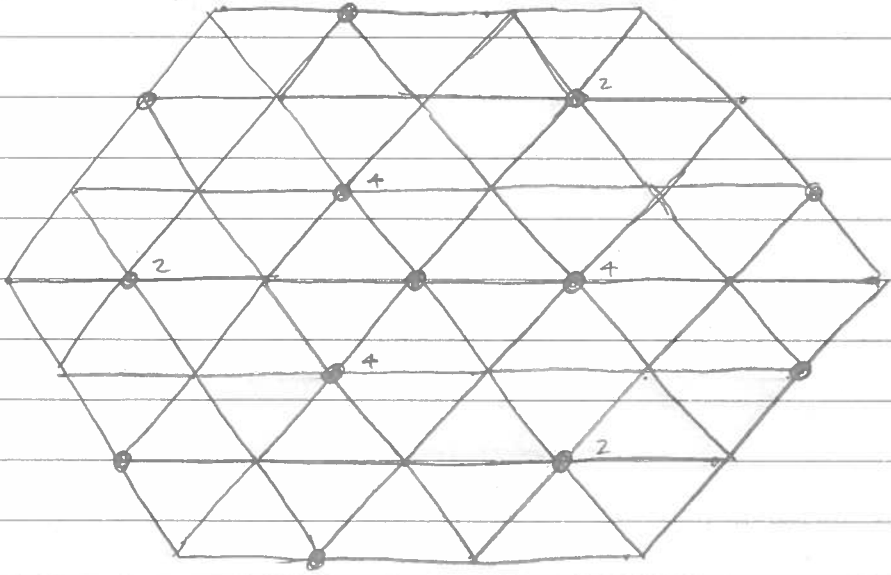
So we conclude that $\mathbb{C}^3 \otimes \mathbb{C}^3 = \Gamma_{2,0} \oplus \Gamma_{0,1}$

In fact, $\mathrm{Sym}^2 \mathbb{C}^3 = \Gamma_{2,0}$

$\Lambda^2 \mathbb{C}^3 = \Gamma_{0,1} = (\mathbb{C}^3)^*$

More generally, $\mathrm{Sym}^n \mathbb{C}^3 = \mathrm{Sym}^n \Gamma_{1,0} = \Gamma_{n,0}$
$\Lambda^n (\mathbb{C}^3)^* = \mathrm{Sym}^n \Gamma_{0,1} = \Gamma_{0,n}$.

5). The tensor product $\Gamma_{1,0} \otimes \Gamma_{1,1}$
(standard and adjoint rep^{n}s)
Weight diagram is



This is reducible : it contains a copy of $\Gamma_{2,1}$ generated by $e_1 \otimes E_{13}$ (recall the basis e_1, e_2, e_3 of $\Gamma_{1,0} = \mathbb{C}^3$, E_{13}, E_{23}, etc of $\Gamma_{1,1} = sl_3(\mathbb{C})$).

Three ways of applying E_{21}, E_{31}, E_{32} to $e_1 \otimes E_{13}$ to obtain a vector with weight L_1 :
$\rho(E_{21})\rho(E_{32})(e_1 \otimes E_{13})$
$\rho(E_{23})\rho(E_{21})(e_1 \otimes E_{13})$
$\rho(E_{31})\ (e_1 \otimes E_{13})$ where ρ is the tensor rep^n.
$\Rightarrow$ weight space of $\Gamma_{2,1}$ with weight L_1 is spanned by these three vectors.
There is a linear dependence since
$\rho(E_{21})\rho(E_{32}) - \rho(E_{32})\rho(E_{21}) = \rho([E_{21}, E_{32}]) = -\rho(E_{31})$

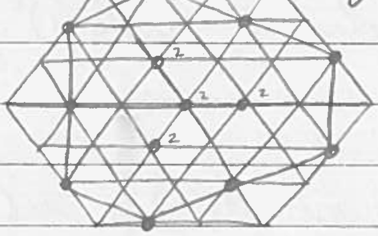
So this weight space is at most 2-dim.
It is at least 1-dim since it is one layer lower

15-11-18

than the heighest weight.
We have that
$E_{32}e_1 = 0$, $ad_{E_{32}} E_{13} = -E_{12}$, $E_{21}e_1 = e_2$, $ad_{E_{21}} E_{12} = -H_{12}$
so $\rho(E_{21})\rho(E_{32})(e_1 \otimes E_3) = e_2 \otimes E_{12} - e_1 \otimes H_{12}$

Similarly, $\rho(E_{32})\rho(E_{21})(e_1 \otimes E_{13}) = e_3 \otimes E_{13} - e_2 \otimes E_{12} - e_1 \otimes H_{23}$.
These are clearly L.I. so L_1 is a weight with 2-dim weight space.

Similarly, L_2 & L_3 are weights with 2-dim weight spaces.



< Weight diagram of $\Gamma_{2,1}$

We can decompose $\Gamma_{1,0} \otimes \Gamma_{1,1}$ further to get $\Gamma_{2,1} \oplus \Gamma_{0,2} \oplus \Gamma_{1,0}$.

<u>Strategy for classifying rep^ns of arbitrary Lie groups</u>
(In fact, for compact semisimple Lie groups or equiv. semisimple Lie algebras)

Recall that we have solved this problem for tori.

Prop^n
Every compact abelian Lie group G is a torus.

Proof
Since G is abelian, the Lie bracket on its Lie algebra vanishes. This means that the exponential map is a homo from $\mathfrak{g} \cong \mathbb{R}^n$ to G
(i.e. usual log law holds).
Ker exp is a lattice in $\mathfrak{g}$: it is a subgroup

s.t. $\forall p \in \mathrm{Ker}\exp$, $\exists$ ball $B_p \subset \mathfrak{g}$ containing p s.t. no other elements of Ker exp lie in this ball.
Lattice is finitely generated, $\cong \mathbb{Z}^n$ by classification of finitely generated abelian groups (torsion free since $\mathfrak{g} \cong \mathbb{R}^n$ is torsion free).
Take generators $a_1, \dots, a_n$ of Ker exp : these form an $\mathbb{R}$-basis for $\mathfrak{g} \cong \mathbb{R}^n$.
For $t \in \mathfrak{g}$, write $t = \sum t_j a_j$, then
$\exp(t) = (\exp(t_1), \dots, \exp(t_n))$
so every element in the Lie group can be identified with an element in $U(1)^n$.
$\square$

Maximal tori

Key to the structure of $SU(2)$, $SU(3)$ and their rep^{n}s was the existence of a torus of diagonal elements H, isomorphic to $U(1)$ (for $SU(2)$) and $U(1)^2$ (for $SU(3)$).

Def^n

A torus in a Lie group G is a subgroup of G that is isomorphic to $U(1)^n$ for some $n \geq 1$.
A maximal torus is a torus not contained in any other torus.

Prop^n

A compact Lie group G has a maximal torus.

Proof

Let $X \in \mathfrak{g}$, take the one parameter subgroup $\exp(tX)$.
The image of this in G is abelian, and its topological closure in G is also abelian and is compact since it is closed and G is compact. So by the previous prop^n it is a torus. Its Lie algebra contains X :

15-11-18

if $X \neq 0$, this torus must be at least 1-dim.
Consider the partially ordered set of abelian subalgebras of $\mathfrak{g}$. Since subalgebras are vector subspaces there are maximal elements (in terms of dim).
We claim that the exponential image of a maximal abelian subalgebra $\mathfrak{t}$ is a maximal torus.
Consider $T = \exp(\mathfrak{t})$: this is a torus with abelian Lie algebra containing $\mathfrak{t}$, so equal to $\mathfrak{t}$ by maximality. If $T \subset T'$ for some other torus T', then its Lie algebra $\mathfrak{t}'$ is abelian and contains $\mathfrak{t}$, contradicting maximality. So T is maximal. □

27-11-18

Maximal Tori and Killing Forms

Recall that a torus in a Lie group G is a subgroup of G isomorphic to $U(1)^n$ for some n.

A maximal torus is a torus not contained in any other torus, e.g. diagonal matrices in $SU(2)$, $SU(3)$.

We showed that a compact Lie group has a maximal torus.
We will choose a maximal torus. This won't affect the classification of of rep^{n}s due to the following.

Thm
Any two maximal tori T_1, T_2 in a connected compact Lie group are conjugate to each other: given T_1, T_2, $\exists\, g \in G$ s.t. $T_1 = g T_2 g^{-1}$.

Moreover, every element in G lies in some maximal torus.

This is very important: in classifying rep^{n}s of G, we want to abelianise the problem.

The proof is quite long and heavily uses the corresponding Lie algebras $\mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{g}$ of T_1, T_2, G.
A key aspect of the proof is that there exists a "nice" inner product on $\mathfrak{g}$.
This is related to the Killing form (named after mathematician called Killing!)

For $SU(3)$, we drew a picture of the roots (a hexagon in $\mathbb{R}^2$) and were able to do geometry in $\mathfrak{h}_{\mathbb{R}}^*$, where $\mathfrak{h}$ was the maximal abelian subalgebra of $\mathfrak{g}_{\mathbb{C}}$, i.e. $\mathfrak{h} = \mathfrak{t}_{\mathbb{C}}$, $\mathfrak{h}_{\mathbb{R}} := i\mathfrak{t}$, where $\mathfrak{t}$ was the maximal abelian subalgebra of $\mathfrak{g}$.

A the time, this involved an arbitrary choice of metric by declaring $\mathfrak{h}_{\mathbb{R}}^*$ to be a quotient of the inner product space

$\mathbb{R}L_1 \oplus \mathbb{R}L_2 \oplus \mathbb{R}L_3$ with basis $\{L_1, L_2, L_3\}$

For more general Lie algebras, there is a more canonical description.

Lemma

There is a natural bilinear form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ that is symmetric and invariant:

$$B([X,Y],Z) = B(Y,[X,Z]).$$

This is called the Killing form.

MATH 0075

27-11-18

Proof

We define

$B(X,Y) = Tr(ad_X \circ ad_Y)$

$ad_X \circ ad_Y$ is a linear transformation $\mathfrak{g} \to \mathfrak{g}$, whose trace is defined by picking any basis of $\mathfrak{g}$, writing $ad_X \circ ad_Y$ as a matrix w.r.t. that basis, and taking the trace; this is independent of the choice of basis.

The symmetry of $B(X,Y)$ follows by considering ad_X, ad_Y as matrices w.r.t. some basis; the trace map is invariant under cyclic permutations, so $Tr(ad_X \circ ad_Y) = Tr(ad_Y \circ ad_X)$.

Finally, invariance follows from the Jacobi identity. □

We will assume that $\mathfrak{g}$ is such that the Killing form is negative definite, i.e. $B(X,X) \leq 0$, $B(X,X)=0 \Leftrightarrow X=0$. This is not true for arbitrary Lie algebras, but only special ones.

Def^n

A complex Lie algebra is called semisimple if its Killing form is nondegenerate (i.e. $B(X,X)=0 \Leftrightarrow X=0$).

A Lie group is semisimple if its complexified Lie algebra is semisimple.

Remark

A connected compact Lie group is in fact semisimple iff its centre is finite. (We won't show this)

Eg $SU(n)$.

Lemma

The Killing form of a compact semisimple Lie group is negative definite.

Proof

For a compact group, any finite dim repⁿ is unitary for some choice of Hermitian inner product (via the Weyl unitary trick).
Take this repⁿ to be the adjoint repⁿ Ad, and differentiate: this allows us to view the adjoint repⁿ ad of $\mathfrak{g}$ as a map $ad: \mathfrak{g} \to u(n)$, where $n = \dim(G)$.
$u(n)$ = skew Hermitian matrices, so for all $X \in \mathfrak{g}$, ad_X can be viewed as a skew Hermitian matrix.

$$\text{So } B(X,X) = Tr(ad_X \circ ad_X) = -Tr(ad_X \circ ad_X^\dagger)$$
$$= -\sum_{i,j} |x_{ij}|^2 \quad (\text{writing } ad_X = (x_{ij}))$$
$$\leq 0.$$

Since the Lie algebra is semisimple, $B(X,Y) = 0$ $\forall Y \in \mathfrak{g}$ iff $X = 0$.
In particular, $B(X,X) = 0 \Leftrightarrow X = 0$. $\square$

Example

Killing form on $su(2)$. We have a basis $\sigma_1, \sigma_2, \sigma_3$, and these satisfy
$ad_{\sigma_1}\sigma_1 = 0$, $ad_{\sigma_1}\sigma_2 = 2\sigma_3$, $ad_{\sigma_1}\sigma_3 = -2\sigma_2$
$$\Rightarrow ad_{\sigma_1} \circ ad_{\sigma_1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{pmatrix}$$
So $B(\sigma_1, \sigma_1) = Tr(ad_{\sigma_1} \circ ad_{\sigma_1}) = -8$
Similarly $B(\sigma_2, \sigma_2) = B(\sigma_3, \sigma_3) = -8$
Moreover $B(\sigma_1, \sigma_2) = B(\sigma_1, \sigma_3) = B(\sigma_2, \sigma_3) = 0$

MATH 0075

27-11-18

An inner product on $\mathfrak{g}$ gives a vector space isomorphism $\flat : \mathfrak{g} \to \mathfrak{g}^*$ with inverse $\# : \mathfrak{g}^* \to \mathfrak{g}$ given by $\flat(v)(w) = \langle v, w \rangle$ i.e. $\flat(v)$ is a continuous linear functional $\mathfrak{g} \to \mathbb{C}$. So we can also think of this inner product as being defined on the dual space $\mathfrak{g}^*$.

Now we check the Killing form for $su(3)$ gives the inner product we worked with earlier.

Example

We consider the Killing form on $su(3)$. We have a basis consisting of E_{ij} and $H_{ij} = E_{ii} - E_{jj}$. We can compute the matrix for $ad_{H_{12}}$ w.r.t. this basis: it is the block diagonal matrix with blocks $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Its square is $\mathrm{diag}(0, 0, -4, -4, -1, -1, -1, -1)$ with trace $B(H_{12}, H_{12}) = -12$

Similarly, $B(H_{23}, H_{23}) = -12$, $B(H_{12}, H_{23}) = 6$?

We can define an inner product

$$\langle X, Y \rangle = -\frac{1}{12} B(X, Y) \text{ on } su(3)$$

The two elements $I = H_{12}$, $J = \frac{2}{\sqrt{3}}(H_{23} + \frac{1}{2} H_{12})$ are orthonormal w.r.t. this inner product $\Rightarrow I^\flat, J^\flat$ form a basis of the dual space, we find that

$$L_1 = I^\flat + \frac{1}{\sqrt{3}} J^\flat \quad , \quad L_2 = -I^\flat + \frac{1}{\sqrt{3}} J^\flat$$

After rescaling, we find that

$$\langle L_1, L_1 \rangle = \langle L_2, L_2 \rangle = \underset{?}{2} \quad , \quad \langle L_1, L_2 \rangle = -1$$

Recall Thm

Any two maximal tori T_1, T_2 in a connected compact Lie group G are conjugate: $\exists g \in G$ st. $T_1 = g T_2 g^{-1}$. Moreover, every element in G lives in some maximal torus.

Def^n

A maximal abelian subalgebra of a Lie algebra is called a Cartan subalgebra.

Lemma 1

Given a Cartan subalgebra $\mathfrak{t}$ of a Lie algebra $\mathfrak{g}$, $\exists X \in \mathfrak{t}$ st. $\mathfrak{t} = \zeta_{\mathfrak{g}}(X)$, where $\zeta_{\mathfrak{g}}(X) := \{Y \in \mathfrak{g} : [Y, X] = 0\}$ is the centraliser of X in $\mathfrak{g}$.

Lemma 2

Let $\mathfrak{t}$ be a Cartan subalgebra of a Lie algebra $\mathfrak{g}$ of a Lie group G, and suppose that the Killing form on $\mathfrak{g}$ is non degenerate. Then $\forall Y \in \mathfrak{g}$, $\exists g \in G$ st. $Ad_g(Y) \in \mathfrak{t}$.

Proof of Thm

We claim that any maximal torus T is of the form $T = \exp(\mathfrak{t})$ for some Cartan subalgebra $\mathfrak{t}$.
Follows from two things:
$\exp: \mathfrak{t} \to T$ is surjective, and $\mathfrak{t}$ is maximal abelian.

The first fact follows more generally for any compact connected group, not just for tori; compact and connected means there is a path connecting the identity to any element $g \in G$ of the form $\exp(tX)$ for some $X \in \mathfrak{g}$.

27-11-18

The second fact is easy: clearly abelian, since T is, and maximal since T is.

Now suppose $\mathfrak{t}_1, \mathfrak{t}_2$ are Cartan subalgebras of $\mathfrak{g}$.
By Lemma 1, $\exists\, X_1 \in \mathfrak{t}_1, X_2 \in \mathfrak{t}_2$ s.t. $\mathfrak{t}_1 = \zeta_{\mathfrak{g}}(X_1)$, $\mathfrak{t}_2 = \zeta_{\mathfrak{g}}(X_2)$.
By Lemma 2, $\exists\, g \in G$ s.t. $Ad_g(X_1) \in \mathfrak{t}_2 = \zeta_{\mathfrak{g}}(X_2)$.
So $Ad_g(\mathfrak{t}_1) = \{Ad_g(Y) : Y \in \mathfrak{g}, [Y, X_1] = 0\}$
(as $\mathfrak{t}_1 = \zeta_{\mathfrak{g}}(X_1) = \{Y \in \mathfrak{g}$ s.t. $[Y, X_1] = 0\}$)

$$\Rightarrow Ad_g(\mathfrak{t}_1) = \{Z \in \mathfrak{g} : [Ad_{g^{-1}} Z, X_1] = 0\}$$

(writing $Z = Ad_g Y$).

$$= \{Z \in \mathfrak{g} : [Z, Ad_g X_1] = 0\}$$
$$= \zeta_{\mathfrak{g}}(Ad_g(X_1))$$

Now $Ad_g(X_1) \in \mathfrak{t}_2$ which is abelian,
so $Ad_g(\mathfrak{t}_1) = \zeta_{\mathfrak{g}}(Ad_g(X_1)) \supset \mathfrak{t}_2$.
But $\mathfrak{t}_2$ is maximal, so $Ad_g(\mathfrak{t}_1) = \mathfrak{t}_2$.

So if T_1, T_2 are maximal tori with Cartan subalgebras $\mathfrak{t}_1, \mathfrak{t}_2$, $\exists\, g \in G$ s.t. $Ad_g(\mathfrak{t}_1) = \mathfrak{t}_2$, and so
$gT_1g^{-1} = g\exp(\mathfrak{t}_1)g^{-1} = \exp(Ad_g(\mathfrak{t}_1)) = \exp(\mathfrak{t}_2) = T_2$.
So T_1, T_2 are conjugate.

Furthermore,

$$\bigcup_{g \in G} gT_1g^{-1} = \bigcup_{g \in G} g\exp(\mathfrak{t}_1)g^{-1} = \bigcup_{g \in G} \exp(Ad_g\, \mathfrak{t}_1) = \exp(\mathfrak{g}) = G$$

since every $X \in \mathfrak{g}$ is such that $Ad_g(X) = Y$ for some $g \in G$, $Y \in \mathfrak{t}$, and hence $Ad_{g^{-1}}(Y) = X \in \mathfrak{g}$.
Since gT_1g^{-1} is a maximal torus, every element in G lies in some maximal torus. $\square$

Note: Needed compact & connected to ensure surjectivity of exp map.

Proof of Lemma 1

WTS: $\exists X. \in \mathfrak{t}$ s.t. $\mathfrak{t} = \zeta_{\mathfrak{g}}(X) = \{Y \in \mathfrak{g} : [Y, X] = 0\}$.

Choose a basis for $\mathfrak{t}$ (finite dim vector space) and write $\mathfrak{t} = \bigcap_{j=1}^{n} \operatorname{Ker} ad_{X_j}$ because $\mathfrak{t}$ is abelian, so $\underset{= ad_X Y}{[X, Y]} = 0 \quad \forall X, Y \in \mathfrak{t}$.

We will show the existence of $t \in \mathbb{R}$ s.t. $\operatorname{Ker} ad_{X_1 + tX_2} = \operatorname{Ker} ad_{X_1} \cap \operatorname{Ker} ad_{X_2}$.
The result will follow by induction.

Endow $\mathfrak{g}$ with an inner product that makes ad skew-symmetric (i.e. use the Weyl unitary trick on the rep^n Ad of G, then differentiate)
i.e. $\langle [X, Y_1], Y_2 \rangle = - \langle Y_1, [X, Y_2] \rangle \quad \forall X, Y_1, Y_2 \in \mathfrak{g}$.

W.r.t. this inner product, we have the orthogonal decomposition $\mathfrak{g} = \operatorname{Ker} ad_{X_1} \oplus (\operatorname{Ker} ad_{X_1})^{\perp}$.
Both spaces are ad_{X_1}-invariant.
Similarly, if $Y \in \mathfrak{t}$ then $[Y, X_1] = 0$, so $0 = ad_{[X_1, Y]} = [ad_{X_1}, ad_Y]$. So ad_Y preserves both spaces.

This allows us to write

$$\mathfrak{g} = (\operatorname{Ker} ad_{X_1} \cap \operatorname{Ker} ad_{X_2}) \oplus (\operatorname{Ker} ad_{X_1} \cap (\operatorname{Ker} ad_{X_2})^{\perp}) \oplus ((\operatorname{Ker} ad_{X_1})^{\perp} \cap \operatorname{Ker} ad_{X_2}) \oplus ((\operatorname{Ker} ad_{X_1})^{\perp} \cap (\operatorname{Ker} ad_{X_2})^{\perp})$$

If $(\operatorname{Ker} ad_{X_1})^{\perp} \cap (\operatorname{Ker} ad_{X_2})^{\perp} = \{0\}$, then $\operatorname{Ker} ad_{X_1 + X_2} = \operatorname{Ker} ad_{X_1} \cap \operatorname{Ker} ad_{X_2}$, so we have proved the desired result with $t = 1$.

MATH 0075

27-11-18

Otherwise, consider the nontrivial map
$ad_{X_1+tX_2} : (\text{Ker}\, ad_{X_1})^\perp \cap (\text{Ker}\, ad_{X_2})^\perp \to \mathfrak{g}$

Take the det of this map: this gives a polynomial in $t \in \mathbb{R}$, nonvanishing at $t=0$, so not the zero polynomial.
So $\exists t_0 \neq 0$ st. $ad_{X_1+t_0X_2}$ is invertible on $(\text{Ker}\, ad_{X_1})^\perp \cap (\text{Ker}\, ad_{X_2})^\perp$ and hence
$\text{Ker}\, ad_{X_1+t_0X_2} = \text{Ker}\, ad_{X_1} \cap \text{Ker}\, ad_{X_2}$.
$\square$

<u>Proof of Lemma 2</u>
WTS: $\forall Y \in \mathfrak{g}$, $\exists g \in G$ st. $Ad_g(Y) \in \mathfrak{t}$.

We fix $Y \in \mathfrak{g}$, and consider the function
$f: G \to \mathbb{C}$, $f(g) := \langle X, Ad_g(X) \rangle$
(same inner product as before)
where X is such that $\mathfrak{t} = \mathfrak{z}_{\mathfrak{g}}(X)$ (by Lemma 1).

G is compact, f is continuous, so f has a maximum at some point $g_0 \in G$ (or rather $|f|$ has a max).
So for each $Z \in \mathfrak{g}$, the function $\tilde{f}_Z : \mathbb{R} \to \mathbb{C}$ given by $\tilde{f}_Z(t) := \langle X, Ad_{\exp(tZ)} \circ Ad_{g_0}(Y) \rangle$
has a max at t_0 (or rather $|\tilde{f}_Z|$ has a max).

Differentiate and evaluate at $t=0$, using the fact that the inner product is skew-symmetric w.r.t. ad:
$0 = \langle X, [Z, Ad_{g_0}(Y)] \rangle = -\langle X, [Ad_{g_0}(Y), Z] \rangle$
$= \langle [Ad_{g_0}(Y), X], Z \rangle$.
This is true $\forall Z \in \mathfrak{g}$. $\langle \cdot, \cdot \rangle$ is nondegenerate, so this can only occur if $[Ad_{g_0}(Y), X] = 0$
ie. $Ad_{g_0}(Y) \in \mathfrak{z}_{\mathfrak{g}}(X) = \mathfrak{t}$ by Lemma 1
(Note: we used nondegeneracy of inner product at the end).

29-11-18

The general strategy

Given a compact group G and complex repⁿ $\rho: G \to GL(V)$

- Pick a maximal torus T and restrict the repⁿ to T to obtain a repⁿ $\rho|_T$ of T.
- Decompose $\rho|_T : T \to GL(V)$ into weight spaces $V = \bigoplus_{\lambda \in \mathfrak{t}^*_{\mathbb{Z}}} V_\lambda$
- Let $\rho_* : \mathfrak{g} \to \mathfrak{gl}(V)$ be the associated Lie algebra repⁿ. We now know that the elements $X \in \mathfrak{t}$ act by $\rho_*(X)v = \lambda(X)v$ for $v \in V_\lambda$.

So it remains to figure out how the other elements of $\mathfrak{g}$ (not in $\mathfrak{t}$) act on V_λ.
This will involve careful analysis of the adjoint repⁿ.

Key ideas / tricks:

- We know how to classify repⁿs of abelian groups (in particular tori).
- We will be able to understand the weight space decomposition of the adjoint repⁿ in some detail (roots & root spaces).
- We will reduce down to copies of $\mathfrak{sl}_2(\mathbb{C})$, which we understand.

Geometry of roots

Given a compact semisimple Lie group G with Lie algebra $\mathfrak{g}$, we have

(a) a maximal torus T with Lie algebra $\mathfrak{t}$ (a maximal abelian subalgebra of $\mathfrak{g}$), and

corresponding Cartan subalgebra $\mathfrak{h} := \mathfrak{t}_{\mathbb{C}} \subset \mathfrak{g}_{\mathbb{C}}$ which is the zero weight space of the adjoint repn.
We then have the root decomposition

$\mathfrak{g}_{\mathbb{C}} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$ where Δ is the set of roots $\{\alpha\}$, where $\mathfrak{g}_\alpha := \{X \in \mathfrak{g}_{\mathbb{C}} : ad_H X = \alpha(H) X \ \forall H \in \mathfrak{h}\}$
(This is the weight space decomposition of the adjoint repn).

(b) We have the definite Killing form on $\mathfrak{h}_{\mathbb{R}}^*$, where $\mathfrak{h}_{\mathbb{R}} = i\mathfrak{t}$.

Given this we can start proving things about root systems of compact semisimple Lie groups.

Lemma

If $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_\beta$, then $ad_X Y = [X, Y] \in \mathfrak{g}_{\alpha+\beta}$.

Proof

Same as proof for $\mathfrak{g}_{\mathbb{C}} = sl_3(\mathbb{C})$ $\square$

Corollary

If $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_\beta$, and $\alpha + \beta \neq 0$, then $B(X, Y) = 0$.

Proof

By the previous lemma, the matrix $(ad_X \circ ad_Y)^n$ applied to a vector $Z \in \mathfrak{g}_\gamma$ gives
$(ad_X \circ ad_Y)^n Z \in \mathfrak{g}_{\gamma + n(\alpha+\beta)}$.
Since $\alpha+\beta \neq 0$, and since $\mathfrak{g}_{\mathbb{C}}$ is finite dim this must always be zero once n is sufficiently large
This means that $ad_X ad_Y$ is a nilpotent matrix.

$A^k = 0$ for some k.

The trace of a $\overbrace{\text{nilpotent matrix}}$ is zero: we can write it in Jordon normal form, where nilpotence mean
diagonal entries are all zero.
So $B(X,Y) = Tr(ad_X \circ ad_Y) = 0$

□

Take away from this: the only way for two root vectors to pair nontrivially w.r.t. the Killing form is if they live in opposite root spaces $\mathfrak{g}_\alpha$, $\mathfrak{g}_{-\alpha}$

We are assuming that G is semisimple, so that the Killing form is nondegenerate
$B(X,Y) = 0 \ \forall X, Y \Rightarrow X = 0.$

Corollary

If $\alpha \in \Delta$, then $-\alpha \in \Delta$
(If $\mathfrak{g}_\alpha$ is non empty then $\mathfrak{g}_{-\alpha}$ is nonempty).

What is $B(X,Y)$ if $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_{-\alpha}$?

Lemma

If $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_{-\alpha}$, then $[X,Y] = B(X,Y)\alpha^\#$
where $\alpha \mapsto \alpha^\#$ is the natural isomorphism.

Proof

We just need to show that
$B(H, [X,Y]) = \alpha(H) B(X,Y)$ (equiv to $[X,Y] = B(X,Y)\alpha^\#$)
$\forall H \in \mathfrak{h}_{\mathbb{R}}$ = ie.
Since the Killing form is invarient,
$B(H, [X,Y]) = B([H,X], Y) = \alpha(H) B(X,Y)$.

□

MATH0075

29-11-18

$\mathfrak{sl}_2(\mathbb{C})$-subalgebras

Pick $X \in \mathfrak{g}_\alpha$ $(X \neq 0)$. Then there exists $Y_\alpha \in \mathfrak{g}_{-\alpha}$ st. $B(X_\alpha, Y_\alpha) \neq 0$.
Define $H_\alpha := [X_\alpha, Y_\alpha]$.
This is equivalent to $B(X_\alpha, Y_\alpha)\alpha^\#$ by the previous lemma. We may rescale X_α, Y_α, to ensure that $B(X_\alpha, Y_\alpha) = \frac{2}{\|\alpha\|^2}$, where $\|\alpha\|^2 = \langle\alpha,\alpha\rangle$

(Note that $\|\alpha\|^2 \neq 0$ since α is a nonzero root and the Killing form is definite).
Then $\alpha(H_\alpha) = 2$

(We choose this normalisation to ensure that $H_\alpha, X_\alpha, Y_\alpha$ give an $\mathfrak{sl}_2(\mathbb{C})$-subalgebra).

Lemma

The subspaces $\mathfrak{s}_\alpha := \mathbb{C}H_\alpha \oplus \mathbb{C}X_\alpha \oplus \mathbb{C}Y_\alpha$ of $\mathfrak{g}_\mathbb{C}$ is a subalgebra isomorphic to $\mathfrak{sl}_2(\mathbb{C})$.

This is an amazing feature of semisimple Lie algebras: we can break them up into root spaces with corresponding copies of $\mathfrak{sl}_2(\mathbb{C})$, which is the smallest nontrivial semisimple Lie algebra.

We now show that $\mathfrak{g}_\alpha$ is one-dim, so that there really is no choice (or flexibility) in constructing this $\mathfrak{sl}_2(\mathbb{C})$-subalgebra $\mathfrak{s}_\alpha$.

Lemma

The subspace $V := \mathbb{C}H_\alpha \oplus \bigoplus_{k\in\mathbb{Z}\setminus\{0\}} \mathfrak{g}_{k\alpha}$ of $\mathfrak{g}_\mathbb{C}$ is an irred. rep^n of $\mathfrak{s}_\alpha$. In particular, the root spaces $\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}$ are one-dim.

Proof

We need to check that this is a repⁿ.
It is preserved by the adjoint action of $\mathfrak{s}_\alpha$:
Clearly $ad_{X_\alpha} \mathfrak{g}_{k\alpha} \subset \mathfrak{g}_{(k+1)\alpha}$, $ad_{Y_\alpha} \mathfrak{g}_{-k(\alpha)} \subset \mathfrak{g}_{(-k-1)\alpha}$
for $k \neq -1$.
$ad_{X_\alpha} \mathfrak{g}_{-\alpha} \subset \mathbb{C} H_\alpha$, $ad_{Y_\alpha} \mathfrak{g}_\alpha \subset \mathbb{C} H_\alpha$.

Also $v \in \mathfrak{g}_{k\alpha} \Rightarrow ad_{H_\alpha} v = k\alpha(H_\alpha) v = 2kv$
So V is preserved by the generators $X_\alpha, Y_\alpha, H_\alpha$ of $\mathfrak{s}_\alpha$.
Moreover, the weight space decomposition of this repⁿ has $\mathfrak{g}_{k\alpha}$ as the $2k$ weight space, and $\mathbb{C}H_\alpha$ as the 0 weight space.

V decomposes into irred. subrepⁿs, each with even weight and a weight zero subspace.
The direct sum of these weight zero subspaces is $\mathbb{C}H_\alpha$, which is 1-dim.
So there is only 1 irred subrepⁿ $\Rightarrow$ V is irred.

Then from the classification of $\mathfrak{sl}_2(\mathbb{C})$ irred. repⁿs, the weight spaces $\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}$ of V must be one-dim.

□

Corollary

On the line connecting $-\alpha$ to α, there are no roots other than $-\alpha, 0, \alpha$.

Proof

$\mathfrak{s}_\alpha$ preserves the direct sum of the root spaces along this line, since X_α & Y_α translate in either direction. If $\beta = h\alpha$ is a root on this line, then the weight of H_α acting on $\mathfrak{g}_\beta$ is $\frac{2B(\alpha,\beta)}{B(\alpha,\alpha)} = 2b$.

$2b$ must be an integer.
Reversing the roles of α, β we see that $2/b$ must also be an integer and so $b \in \{\pm 1, \pm 2, \pm 1/2\}$.

W.l.o.g. we may assume that $b \in \{1, 2\}$ by swapping α and β and changing β to $-\beta$ as necessary.

We want to rule out the possibility that $b = 2$.
By the previous lemma, $V = \mathbb{C}H_\alpha \oplus \bigoplus_{k \in \mathbb{Z} \setminus \{0\}} \mathfrak{g}_{k\alpha}$ is an irred. rep^n. of $\mathfrak{s}_\alpha$. If $b = 2$ then this would be

$$\mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus \mathbb{C}H_\alpha \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}.$$

We know that X_α generates $\mathfrak{g}_\alpha$, so that $ad_{X_\alpha} \mathfrak{g}_\alpha = 0$
This means that $\mathfrak{g}_{2\alpha} = ad_{X_\alpha} \mathfrak{g}_\alpha = 0$. This implies the result.

□

Lemma

Suppose that β is a root linearly independent from α. Then $\bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_{\beta + k\alpha}$ is an irred. rep^n of $\mathfrak{s}_\alpha$.

Proof

This is preserved by $\mathfrak{s}_\alpha$. It decomposes into weight spaces $\mathfrak{g}_{\beta+k\alpha}$ with weight $\beta(H_\alpha) + 2k$.
Each weight space is one-dim, because they are nonzero root spaces (since $\beta + k\alpha \neq 0$).
1-dim weight spaces $\Rightarrow$ irred.

□

Two more important facts about $\mathfrak{sl}_2(\mathbb{C})$ irred. rep^{n}s that we can exploit:

- the weights are integers
- the weights are distributed symmetrically about 0.

Since H_α acts with weight $\beta(H_\alpha) = \dfrac{2B(\alpha, \beta)}{B(\alpha, \alpha)}$ on $\mathfrak{g}_\beta$, the first fact means the following.

Corollary

For any nonzero roots α, β, $\frac{2B(\alpha,\beta)}{B(\alpha,\alpha)} \in \mathbb{Z}$.

The second fact means that:

Corollary

The reflection operator $s_\alpha(\beta) := \beta - \frac{2B(\alpha,\beta)}{B(\alpha,\alpha)}\alpha$,

which reflects in the hyperplane $\Omega_\alpha := \{\beta : B(\alpha,\beta) = 0\}$ preserves the set of roots in $\mathfrak{h}^*_{\mathbb{R}}$.

The group generated by these reflection operators is called the Weyl group.

Summary

Thm

If G is a compact semisimple Lie group with Lie algebra $\mathfrak{g}$, then

- the Killing form $B(X,Y) = Tr(ad_x \circ ad_x)$ on $\mathfrak{g}$ is negative definite
- there exists a maximal torus $T \subset G$, unique up to conjugation, with Lie algebra $\mathfrak{t} \subset \mathfrak{g}$, and corresponding Cartan subalgebra $\mathfrak{h} = \mathfrak{t}_{\mathbb{C}} \subset \mathfrak{g}_{\mathbb{C}}$
- under the adjoint action of the maximal torus, the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$ decomposes as $\mathfrak{g}_{\mathbb{C}} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$, where Δ is the set of roots and $\mathfrak{g}_\alpha$ is the root space corresponding to α.
- if $\alpha \in \Delta$ then $-\alpha \in \Delta$. Moreover, the only roots on the line between $-\alpha$ and α are $-\alpha, 0, \alpha$.
- each root space $\mathfrak{g}_\alpha$ with nonzero weight α is one-dim.
- for each pair of roots $\alpha, \beta \in \Delta$, $\frac{2B(\alpha,\beta)}{B(\alpha,\alpha)} \in \mathbb{Z}$.
- the reflection operator $s_\alpha(\beta) = \beta - \frac{2B(\alpha,\beta)\alpha}{B(\alpha,\alpha)}$, which reflects in the hyperplane Ω_α, preserves the roots.

MATH0075

29-11-18

- these reflections generate a finite group called the Weyl group of G.
- if β is linearly independent of α, then $V_{\beta,\alpha} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_{\beta + k\alpha}$ is an irred. rep^n of $\mathfrak{s}_\alpha$.
- As a rep^n of $\mathfrak{s}_\alpha$, $\mathfrak{g}_{\mathbb{C}}$ decomposes into irred. rep^{n}s
  $\mathfrak{s}_\alpha \oplus \bigoplus_{\substack{\beta \in \Delta \\ \text{L.I. of } \alpha}} V_{\beta,\alpha} \oplus \left(\mathfrak{h} / \mathbb{C}H_\alpha\right)$.

Irred. Rep^{n}s

We can analyse irred rep^{n}s of arbitrary compact semisimple Lie groups, just as we did for SU(2) and SU(3).
We take a weight space decomp. w.r.t. a maximal torus, and we get a collection of vertices in the weight lattice $\mathfrak{t}^*_{\mathbb{Z}} := \{ f \in i\mathfrak{t}^* : f(v) \in 2\pi\mathbb{Z}, \forall v \in \ker \exp \}$.

A highest weight vector (w.r.t. an irrational linear function for the weight lattice) generates an irred. subrep^n by acting using negative roots.
For each weight α, $\exists$ a unique irred. rep^n containing a highest weight vector with weight α, and the weight diagram for this rep^n is:

- symmetric under the action of the Weyl group
- obtained by
  - reflecting the highest weight under all elements of the Weyl group
  - taking the convex hull of these points
  - looking at all the lattice points in this convex hull that can be obtained from the highest weight by translating along a root.

04-12-18

More on the general strategy

We previously discussed irred. repⁿs of arbitrary compact semisimple Lie groups.
We left some aspects unresolved:

1). how exactly to index the irred. repⁿs in terms of highest weights.
2). how to prove the existence of a repⁿ with given highest weights.
3). what multiplicities to put on the weights.

We first clarify 1).
For 2), the standard approach of constructing repⁿs is via Verma modules, we won't discuss this.
For 3), there are several approaches:
Weyl's character formula, the Littelmann path model, Freudenthal's multiplicity formula. We will just state the latter.

More on the Killing form & inner products

Recall that the Killing form is the symmetric bilinear form $B: \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ given by
$B(X,Y) = \mathrm{Tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y)$.
This extends naturally to a bilinear form
$B: \mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} \to \mathbb{C}$.

We define the Cartan involution on $\mathfrak{g}_{\mathbb{C}}$ as follows:
$\theta: \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}, \quad \theta(X \otimes z) = X \otimes \bar{z}$
recalling that $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$.

We can also write $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$, so that for
$X + iY \in \mathfrak{g} \oplus i\mathfrak{g}$, $\theta(X+iY) = X - iY$.

MATH0075

04-12-18

Now we define an inner product (assuming $\mathfrak{g}_\mathbb{C}$ is semisimple) on $\mathfrak{g}_\mathbb{C}$: $\langle X, Y\rangle := -B(X, \Theta Y)$ for $X, Y \in \mathfrak{g}_\mathbb{C}$.

Note that for $X, Y \in i\mathfrak{g}$, then $\langle X, Y\rangle = B(X, Y)$. In particular, this is the case for $X, Y \in \mathfrak{h}_\mathbb{R} = i\mathfrak{t} \subset i\mathfrak{g}$.

We can also define the Killing form and inner product on $\mathfrak{g}_\mathbb{C}^*$. Every linear functional $\lambda \in \mathfrak{g}_\mathbb{C}^*$ can be uniquely identified with an element $\lambda^\# \in \mathfrak{g}_\mathbb{C}$ by $\lambda(Y) = \langle \lambda^\#, Y\rangle \; \forall Y \in \mathfrak{g}_\mathbb{C}$.

The inverse map is such that given $X \in \mathfrak{g}_\mathbb{C}$, we define $X^\flat \in \mathfrak{g}_\mathbb{C}^*$ by $X^\flat(Y) = \langle X, Y\rangle$.
This is the Riesz representation thm.

This allows us to define the Killing form on $\mathfrak{g}_\mathbb{C}^*$ by $B(\alpha, \beta) := B(\alpha^\#, \beta^\#)$ for $\alpha, \beta \in \mathfrak{g}_\mathbb{C}^*$, and corresponding $\alpha^\#, \beta^\# \in \mathfrak{g}_\mathbb{C}$.

Similarly we define $\langle \alpha, \beta\rangle := \langle \alpha^\#, \beta^\#\rangle$.
Once again, the Killing form is the inner product for $\alpha, \beta \in i\mathfrak{g}^*$ - in particular for $\alpha, \beta \in \mathfrak{h}_\mathbb{R}^* = i\mathfrak{t}^*$.

More on the Weyl group

Associated to each root $\alpha \in \Delta$ (the set of roots) we define the reflection operator $s_\alpha : \mathfrak{h}_\mathbb{R}^* \longrightarrow \mathfrak{h}_\mathbb{R}^*$ by $s_\alpha(\beta) := \beta - \dfrac{2\langle \alpha, \beta\rangle \alpha}{\langle \alpha, \alpha\rangle}$.

This reflects in the hyperplane $\Omega_\alpha := \{\beta \in \mathfrak{h}_\mathbb{R}^* : \langle \alpha, \beta\rangle = 0\}$

This hyperplane divides into 3 parts:
- the hyperplane itself
- the part $\{\beta : \langle\alpha,\beta\rangle > 0\}$
- the part $\{\beta : \langle\alpha,\beta\rangle < 0\}$

<u>Def^n</u>

The reflection operators break up the set $\mathfrak{h}_{\mathbb{R}}^* \setminus \bigcup_{\alpha\in\Delta} \Omega_\alpha$ into finitely many connected components. These are called the open Weyl chambers of $\mathfrak{h}_{\mathbb{R}}^*$.

The Weyl group of $\mathfrak{g}_{\mathbb{C}}$ is the subgroup of $GL(\mathfrak{h}_{\mathbb{R}}^*)$ generated by the reflection operators s_α; this group acts simply transitively on the open Weyl chambers.

<u>Def^n</u>

Let C be an open Weyl chamber.
We say that a root α is C-positive if $\langle\alpha,\beta\rangle > 0 \ \forall \beta\in C$.

We say that a C-positive root is <u>indecomposible</u> if it cannot be written as the non-trivial sum of two other C-positive roots.
We let <u>$\Pi(C)$</u> denote the set of indecomposible C-positive roots, and we call these <u>simple roots</u> w.r.t. C.

<u>Lemma</u>

A system of simple roots has the property that they form a basis of $\mathfrak{h}_{\mathbb{R}}^*$, and every $\beta\in\Delta$ can be written as $\beta = \sum_{\alpha\in\Pi(C)} k_\alpha \alpha$ for integers k_α all of the same sign.

MATH0079

04-12-18

In fact, we could have used this as the def^n of the simple roots, then showed that there exists a corresponding Weyl chamber.

Def^n

Given a system of simple roots $\Pi(C)$ we define the set of positive roots w.r.t. Π (or C) to be

$$\Delta^+ = \Delta^+(\Pi) = \{\beta \in \Delta : \beta = \sum_{\alpha\in\Pi} k_\alpha \alpha \ , \ k_\alpha \geq 0\}$$

and similarly the negative roots Δ^-.

In particular, $\Delta^+ \cap \Delta^- = \phi$,
$\alpha \in \Delta^+$ iff $-\alpha \in \Delta^-$, $\Delta^+ \cup \Delta^- = \Delta$

So each Weyl chamber gives a division of Δ into positive and negative roots.

In particular the choice of a Weyl chamber is the same as the choice of a linear functional on $\mathfrak{h}_{\mathbb{R}}^*$ that is irrational w.r.t. $\mathfrak{h}_{\mathbb{Z}}^*$, just as we did for $sl_3(\mathbb{C})$.

For $sl_3(\mathbb{C})$ we chose C s.t.

$$\sum_{\alpha\in\Delta^+} \mathfrak{g}_\alpha = \bigoplus_{1\leq j<k\leq 3} \mathbb{C}E_{jk} \ , \ \sum_{\alpha\in\Delta^-} \mathfrak{g}_\alpha = \bigoplus_{1\leq j<k\leq 3} \mathbb{C}E_{kj}$$

Indexing irred. rep^{n}s

Given a semisimple Lie algebra $\mathfrak{g}_{\mathbb{C}}$ with Cartan subalgebra $\mathfrak{h}$ and a corresponding set of roots $\Delta \subset \mathfrak{h}_{\mathbb{R}}^*$ and root space decomp
$\mathfrak{g}_{\mathbb{C}} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$, we have the following thm.

Thm

Choose an open Weyl chamber C and simple roots $\Pi(C)$, and positive / negative roots $\Delta^{\pm}$.

- the weights of an irred. rep^n of $\mathfrak{g}_{\mathbb{C}}$ lie in the lattice $\{\beta \in \mathfrak{h}_{i\mathbb{R}}^* : 2\frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}\}$

- $\exists!$ highest weight β_0 (w.r.t. C) with weightspace V_{β_0}, and a unique (up to scalar) highest weight vector $v_0 \in V_{\beta_0}$ s.t. v_0 satisfies $X v_0 = 0 \ \forall X \in \mathfrak{g}_\alpha, \alpha \in \Delta^+$

- the highest weight β_0 lives in the closed Weyl chamber $\overline{C}$, so that $\langle \beta_0, \alpha \rangle \geq 0 \ \forall \alpha \in \Pi$.

- the highest weight satisfies $\beta_0(H) \in 2\pi i \mathbb{Z}$ $\forall H \in \mathfrak{t} \cap \operatorname{Ker} \exp$.

- every weight β is of the form $\beta_0 - \sum_{\alpha \in \Pi} k_\alpha \alpha$, $k_\alpha \geq 0$, and β lies in the convex hull of reflections of β_0 under the action of the Weyl group

- every weight β satisfies $\langle \beta, \beta \rangle \leq \langle \beta_0, \beta_0 \rangle$, with equality iff β is the image of β_0 under the action of some element of the Weyl group.

- up to isomorphism, $\exists$ a unique irred. rep^n. of $\mathfrak{g}_{\mathbb{C}}$ (or $\mathfrak{g}$ or G) with highest weight β_0

- Conversely, given an element β_0 of the weight lattice lying in the closed Weyl chamber $\overline{C}$ and satisfying $\beta_0(H) \in 2\pi i \mathbb{Z}$ $\forall H \in \mathfrak{t} \cap \operatorname{Ker} \exp$, $\exists$ a unique (up to isomorphism) irred. rep^n of $\mathfrak{g}_{\mathbb{C}}$ with highest weight β_0.

MATH0075

04-12-18

Multiplicities of weight spaces

Thm (Freudental's multiplicity formula)
Suppose that G is a compact semisimple Lie group and $\rho: G \to GL(V)$ is an irred. rep^n of highest weight λ. Then the mult. of a weight μ is

$$\dim V_\mu = \frac{2 \sum_{\alpha \in \Delta^+} \sum_{j=1}^{\infty} \langle \mu + j\alpha, \alpha \rangle \dim V_{\mu + j\alpha}}{\|\lambda + \delta\|^2 - \|\mu + \delta\|^2}$$

where $\delta = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$, $\|X\|^2 := \langle X, X \rangle$.

(assuming $\mu \neq s_\alpha \lambda$ for some s_α in the Weyl group).

- This is a recursive formula: you start on the edge of the convex hull, work out $\dim V_\mu$, then work your way inwards, using the information you just learned.

- The sum over j is finite since $\dim V_{\mu+j\alpha} = 0$ for j sufficiently large.

- The proof involves the action of a Casimir operator on $\mathfrak{g}$, namely an element C s.t. $\forall$ rep^n ρ, $\rho(C)$ commutes with $\rho(X)$ $\forall X \in \mathfrak{g}$.

Classifying Lie algebras

Def^n
A Lie algebra $\mathfrak{g}$ is said to be simple if it is non abelian (i.e. $\exists X, Y \in \mathfrak{g}$ s.t. $[X, Y] \neq 0$) and its ideals (subspaces $\mathfrak{g}'$ s.t. $[\mathfrak{g}, \mathfrak{g}'] \subseteq \mathfrak{g}'$) are either 0 or itself.

Thm

A Lie algebra is semisimple iff it is the direct sum of finitely many simple Lie algebras.

This is usually taken to be the def^n of a semisimple Lie algebra.

In our setting, sum of simple Lie algebras $\Leftrightarrow$ nondegenerate Killing form.

We will completely classify all simple Lie algebras (over $\mathbb{C}$), just as one can completely classify finite abelian groups (or f.g. abelian groups) or even finite simple groups.
We will do this via Dynkin diagrams.

Dynkin diagrams

Lemma

Let $\mathfrak{g}_{\mathbb{C}}$ be a complex semisimple Lie algebra and let $\alpha, \beta \in \Delta$ be roots. Then $\frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} \in \{0, \pm 1, \pm 2, \pm 3\}$

Proof

$$\frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} = \beta(H_\alpha) \quad \text{for} \quad H_\alpha := [X_\alpha, Y_\alpha] = B(X_\alpha, Y_\alpha)\alpha^{\#}.$$

with $X_\alpha \in \mathfrak{g}_\alpha$, $Y_\alpha \in \mathfrak{g}_{-\alpha}$, normalised s.t. $B(X_\alpha, Y_\alpha) = 2/\langle\alpha,\alpha\rangle$.
Moreover, for $X_\beta \in \mathfrak{g}_\beta$, we have that $\mathrm{ad}_{H_\alpha} X_\beta = \beta(H_\alpha) X_\beta$, so $\beta(H_\alpha)$ is a weight for a rep^n of $\mathfrak{sl}_2(\mathbb{C})$, and hence is an integer.
Next let θ be the angle between α and β
Then $\alpha(H_\beta)\beta(H_\alpha) = \frac{4\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle}\frac{\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle} = 4\cos^2\theta$.

MATH0075

04-11-18

This must be a nonnegative integer $\Rightarrow \in \{0,1,2,3,4\}$
So $\beta(H_\alpha)$ must be an integer between -4 and 4.
It remains to note that ± 4 is impossible, because then $\theta \in \{0, \pi\}$, so α is a mult. of β, but in this case $\alpha(H_\beta) = \beta(H_\alpha) \in \{\pm 2\}$. $\square$

Def^n
The Dynkin diagram of a semisimple Lie algebra is defined as follows.
Given a system of simple roots $\pi = \{\alpha_1, \dots, \alpha_n\}$ the Dynkin diagram is an oriented graph such that:

- there is a vertex associated to each simple root
- there are no edges joining the vertices accociated to colinear roots α_j, α_k
- there are edges joining the vertices associated to noncollinear roots α_j, α_k with multiplicity
$$\alpha_j(H_{\alpha_k})\alpha_k(H_{\alpha_j}) = 4\frac{\langle \alpha_j, \alpha_k\rangle}{\langle \alpha_k, \alpha_k\rangle}\frac{\langle \alpha_k, \alpha_j\rangle}{\langle \alpha_j, \alpha_j\rangle} = 4\cos^2\theta_{jk}$$
where θ_{jk} is the angle between α_j and α_k.
- the edges are non oriented if the mult is at most 1.
- the edges are oriented if they occur with mult. > 1 (i.e. 2 or 3), with an arrow pointing towards the vertex whose associated simple root is of shorter length.

One can check that the Dynkin diagram is independent of the choice of simple roots (i.e. choice of open Weyl chamber).

Lemma

Suppose that $\mathfrak{g}$ is a semisimple Lie algebra so that $\mathfrak{g} = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_m$ with each $\mathfrak{g}_i$ simple. Then the Dynkin diagram has m connected components, each one being the Dynkin diagram of some $\mathfrak{g}_i$.

Dynkin diagrams are useful for classifying simple Lie algebras for the following reason:

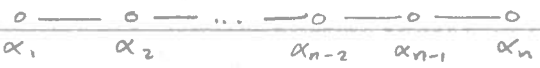
Prop^n

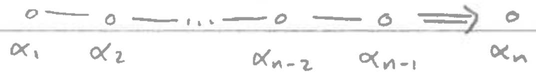
Two simple Lie algebras with identical Dynkin diagrams are isomorphic.

Next time: Will draw all possible Dynkin diagrams
- 4 infinite families
- 5 exceptional simple Lie algebras.

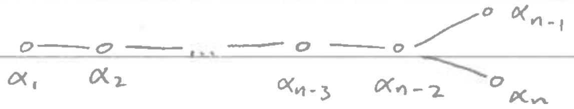
06-12-18 Thm

There are 4 infinite families of simple Lie groups with the following Dynkin diagrams:

- For $n \geq 1$, the family A_n : 

- For $n \geq 2$, the family B_n : 

- For $n \geq 3$, the family C_n : 

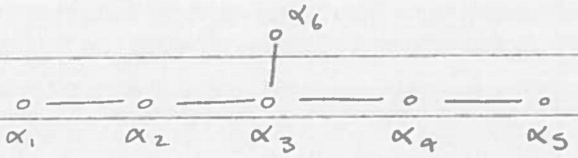
- For $n \geq 4$, the family D_n : 

There are 5 exceptional Dynkin diagrams:

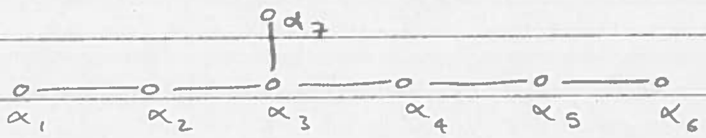
MATH 0075

06-12-18

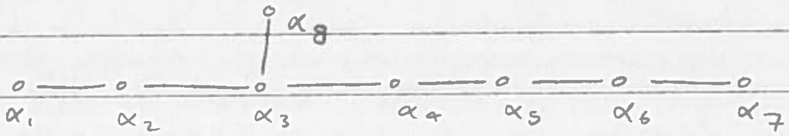
- E_6 :



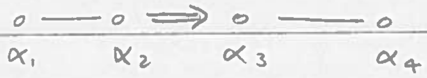
- E_7 :



- E_8 :



- F_4 :



- G_2 :

See online notes! Rest of course nonexaminable.