MATH0016 Mathematical Methods 3 Notes (Part 1 of 2)

Based on the 2019 autumn lectures by Prof N R McDonald

METHODS 3

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$$\int f(x)\,dx = \lim_{x \to 0} \sum f(x) \cdot \Delta x$$

MATHEMATICAL METHODS 3

Robb Mc Donald → room 413 (not on Wednesdays or Thursdays) → office hour: Tues 12-1

email → n.r.mcdonald@ucl.ac.uk.

10% → HW.

90%.

First deadline → 11 o'clock Friday 11th.

CHAPTER SUMMARY:

I. Vector calculus.

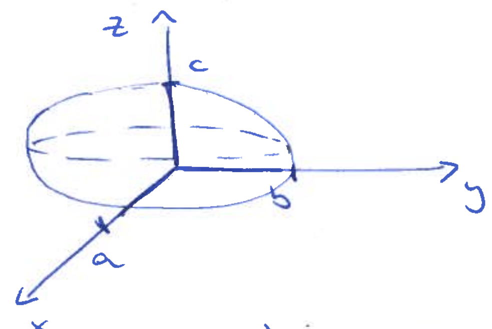
II. Fourier Series

III. Calculus of variations

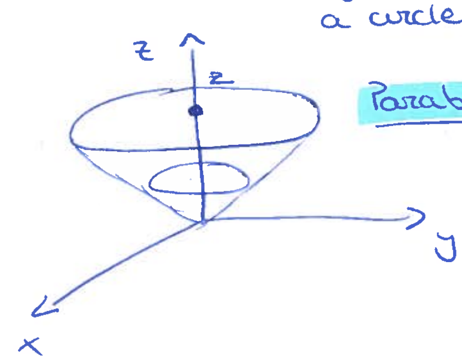
IV. PDEs.

REVIEW

- $x^2+y^2+z^2=d^2$ → sphere of radius d, centred in (x,y,z).
- $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ → ellipsoid



- $x^2+y^2=1$ → cilinder (since we are in 3D) if we are in 2D it would be a circle
- $z=x^2+y^2 \quad 0 \le z \le 1$ → Paraboloid bowl



CHAPTER 1:

Unit normal. Let $z = f(x,y)$ be a surface.

Recall, the unit normal to the surface is

$$\boxed{\underline{n} = \frac{-\frac{\partial f}{\partial x}\underline{i} - \frac{\partial f}{\partial y}\underline{j} + \underline{k}}{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}}}$$

surface $z = f(x,y)$

$\underline{r}(u,v) = (u, v, f(u,v))$

El vector normal a una surface

$\underline{N} = \underline{r}'_u \times \underline{r}'_v$

$$\begin{vmatrix} i & j & k \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = -f_x\underline{i} - f_y\underline{j} + \underline{k}$$

$$\underline{n} = \frac{\underline{N}}{|\underline{N}|} = \frac{-f_x\underline{i} - f_y\underline{j} + \underline{k}}{\sqrt{(f_x)^2 + (f_y)^2 + 1}}$$

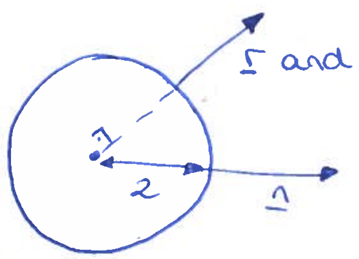
Example. $x^2 + y^2 + z^2 = 4$.

$z = \sqrt{4 - x^2 - y^2} = f(x,y)$.

$$\frac{\partial f}{\partial x} = \frac{-x}{\sqrt{4-x^2-y^2}} = \frac{-x}{z} \qquad \frac{\partial f}{\partial y} = \frac{-y}{\sqrt{4-x^2-y^2}} = \frac{-y}{z}.$$

$$\underline{n} = \frac{\frac{x}{z}\underline{i} + \frac{y}{z}\underline{j} + \underline{k}}{\sqrt{\frac{x^2}{z^2} + \frac{y^2}{z^2} + 1}} = \frac{\frac{1}{z}\cdot(x\underline{i} + y\underline{j} + z\underline{k})}{\frac{1}{z}\sqrt{x^2+y^2+z^2}} = \frac{x\underline{i} + y\underline{j} + z\underline{k}}{2} = \frac{\underline{r}}{2}$$

where $(\underline{r} = x\underline{i} + y\underline{j} + z\underline{k})$.



Surface integrals

$z = f(x,y)$ defines a surface S in $\mathbb{R}^3$.

Let $g: \mathbb{R}^3 \to \mathbb{R}^3$ be a function in $\mathbb{R}^3$, so that $g(x,y,z) = g(x,y,f(x,y))$ is the value of g on S.

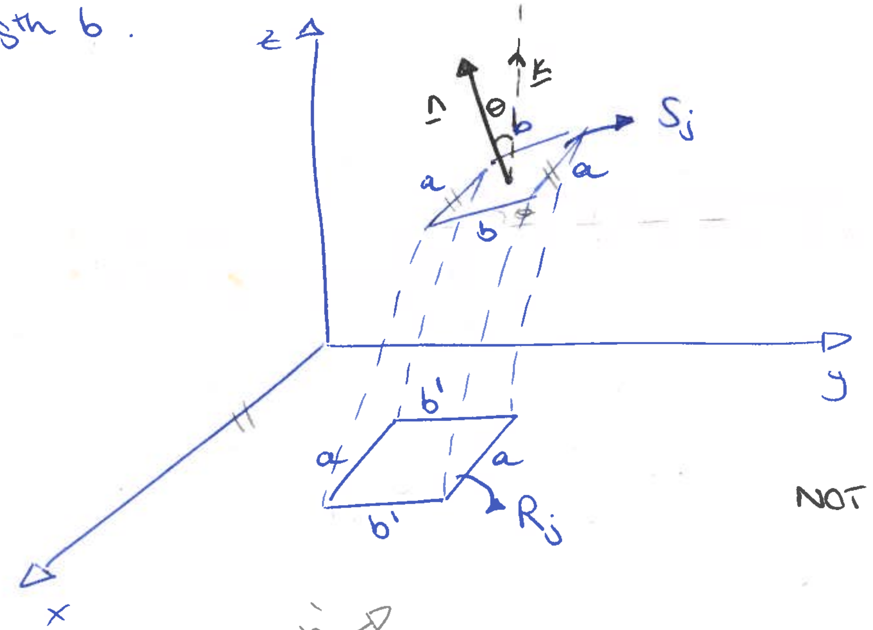
Consider the usual limiting process.

i.e. divide S into many smaller regions S_j having areas ΔS_j.

$$\boxed{\iint_S g(x,y,z)\, dS = \lim_{\Delta S_j \to 0} \sum_{j=1}^{N} g(x_j, y_j, z_j)\, \Delta S_j}$$

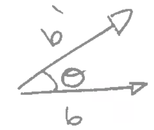
Project S_j onto the xy-plane to obtain R_j.

For simplicity, suppose S_j is a rectangle with two sides parallels to the xy-plane and of length a. The other 2 sides have length b.



NOT FOR EXAM (NFE).

$b' = b \cdot \cos\theta \to$



ΔR_j = area R_j $= ab'$

$\Delta S_j = ab$

$$\Delta S_j = \frac{\Delta R_j}{\cos\theta} = \frac{\Delta R_j}{\underline{n}_j \cdot \underline{k}}$$

$\to$ since $|\underline{n}_j| = 1 \Rightarrow \underline{n}_j \cdot \underline{k} = |\underline{n}_j| \cdot |\underline{k}| \cos\theta = \cos\theta$

Also θ is angle the surface makes with the xy-plane

Thus,
$$\iint_S g(x,y,z)\, dS = \lim_{\Delta R_j \to 0} \sum_{j=1}^{N} g(x_j, y_j, z_j) \frac{\Delta R_j}{\underline{n}_j \cdot \underline{k}}$$

But
$$\frac{g(x_j, y_j, z_j)}{\underline{n}_j \cdot \underline{k}} \sim \frac{g(x_j, y_j, z)}{\underline{n}(x_j, y_j, z_j) \cdot \underline{k}}$$

$$\boxed{\iint_S g(x,y,z)\, dS = \iint_R \frac{g(x,y,z)}{\underline{n}(x,y,z)\cdot\underline{k}}\, dx\, dy} = \iint_R \frac{g(x,y,f(x,y))}{\underline{n}(x,y,f(x,y))\cdot\underline{k}}\, dx\, dy$$

$\to z = f(x,y)$.

Also,
$$\underline{n}\cdot\underline{k} = \frac{1}{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}}$$

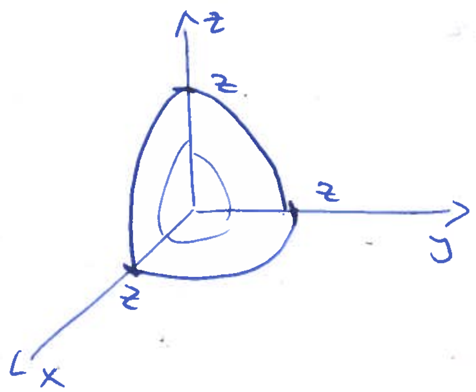
$$\left(\frac{-\frac{\partial f}{\partial x}\underline{i} - \frac{\partial f}{\partial y}\underline{j} + \underline{k}}{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}} \cdot (0\underline{i} + 0\underline{j} + \underline{k})\right)$$

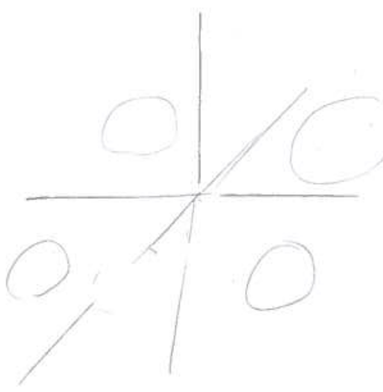
$$\iint_S g(x,y,z)\,dS = \iint_R g(x,y,z)\sqrt{\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+1}\;dx\,dy$$

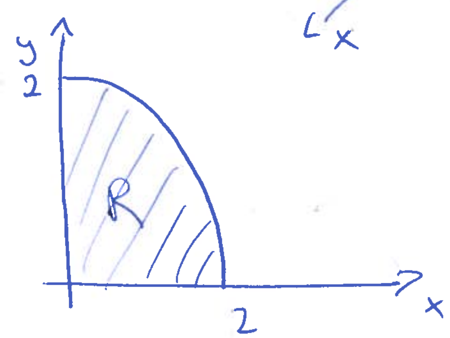
$y = h(x,z)$

$$\iint_S g(x,y,z)\,dS = \iint_R g(x,h(x,z),z)\sqrt{\left(\frac{\partial f}{\partial x}\right)^2+1+\left(\frac{\partial f}{\partial z}\right)^2}\;dx\,dz$$

Ex: $\iint_S z^2\,dS$ where S is an octant of the sphere of radius 2, centre $(0,0,0)$







since it's a sphere of radius 2

$z=\sqrt{4-x^2-y^2} = f(x,y)$. $x^2+y^2+z^2=2^2$

Recall, $\sqrt{\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+1} = \frac{2}{\sqrt{4-x^2-y^2}}$

So, $\iint_S z^2\,dS = \iint_R (4-x^2-y^2)\frac{2}{\sqrt{4-x^2-y^2}}\,dx\,dy \overset{\text{rationalize}}{=} 2\iint_R \sqrt{4-x^2-y^2}\,dx\,dy =$

$r^2=x^2+y^2$ → plane polar coordinates

$= 2\iint_R \sqrt{4-r^2}\;r\,dr\,d\theta = 2\int_0^{\pi/2}\int_0^2 \sqrt{4-r^2}\;r\,dr\,d\theta =$ → one octant

$= 2\int_0^{\pi/2}\left[-\frac{1}{3}(4-r^2)^{3/2}\right]_0^2 d\theta = 2\cdot\frac{8}{3}\cdot\frac{\pi}{2} = \boxed{\frac{8\pi}{3}}$

Neat trick

área esphere

$\iint_S x^2\,dS = \iint_S y^2\,dS = \iint_S z^2\,dS$. $\frac{4}{3}\iint 1\,dS = \frac{4}{3}\boxed{4\pi 2^2}\frac{1}{8} =$

$\iint z^2\,dS = \frac{1}{3}\iint_S (x^2+y^2+z^2)\,dS = \frac{4}{3}\iint 1\,dS = \frac{4}{3}$ (sphere are of octant) $=$

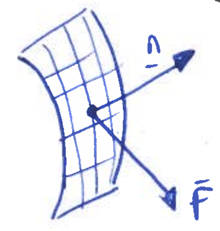
$4=2^2$ since $x^2+y^2+z^2$

$= \frac{4}{3}\cdot\frac{1}{8}(4\pi\cdot 2^2) = \frac{8\pi}{3}$

Flux integrals

$\iint_S (F\cdot n)\,dS$. → unit normal vector field to S . $F = F_1\,i + F_2\,j + F_3\,k$

→ scalar function



The amount of "stuff" passing across S. (Notes).

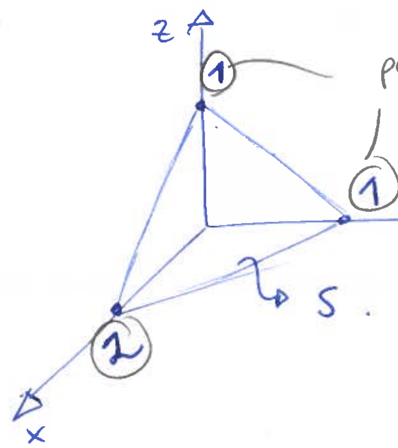
$S:\ z=f(x,y)$ then

$$\iint_S F\cdot n\,dS = \iint_R F\cdot n\sqrt{\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+1}\;dx\,dy =$$

$$=\iint_R (F_1 i + F_2 j + F_3 k)\cdot\frac{\left(-\frac{\partial f}{\partial x}i - \frac{\partial f}{\partial y}j + k\right)}{\cancel{\sqrt{\quad}}}\cancel{\sqrt{\quad}}\;dx\,dy =$$

$$=\iint_R\left(-F_1\frac{\partial f}{\partial x} - F_2\frac{\partial f}{\partial y} + F_3\right)dx\,dy$$

Ex: $\underline{F} = z\,\underline{i} - y\,\underline{j} + x\,\underline{k}$. Compute flux of $\underline{F}$ over the $x+2y+2z=2$, $x,y,z \geq 0$ plane.



para encontrarlos voy sustituyendo en el plano x, y, z por 0.

$F_1 = z = -\frac{1}{2}x - y + 1$

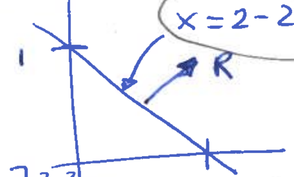
$F_2 = -y$

$F_3 = x$

viene de ahí

$z = -\frac{1}{2}x - y + 1 = f(x,y)$.

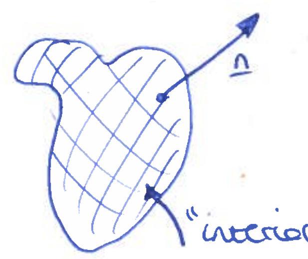
$$\iint_S \underline{F}\cdot\underline{n}\, dS = \iint_R \left[\left(\tfrac{1}{2}x + y - 1\right)\cdot\left(-\tfrac{1}{2}\right) + y(-1) + x\right] dx\,dy =$$



$x = 2 - 2y$

$$= \int_0^1\int_0^{2-2y}\left[\frac{3}{4}x - \frac{3}{2}y + \frac{1}{2}\right]dx\,dy = \int_0^1\left[\frac{3}{8}x^2 - \frac{3}{2}yx + \frac{x}{2}\right]_0^{2-2y} dy = \frac{1}{2}$$

Close surface: divides space into two pieces s.t it is impossible to go from one piece to another without passing through the surface.



"interior" (bounded)

exterior

Flux over closed surfaces
- $>0 \Rightarrow$ net flow out
- $<0 \Rightarrow$ net flow in
- $=0 \Rightarrow$ no net flow

Divergence: Let $\underline{F}: \mathbb{R}^3 \to \mathbb{R}^3$. Choose a point (x_0, y_0, z_0) and consider a family of spheres S_ε centred at (x_0, y_0, z_0) and of radius ε.

Definition: the

$$\boxed{\operatorname{div}\underline{F} = \lim_{\varepsilon\to 0}\frac{1}{\underbrace{\Delta S_\varepsilon}_{\text{volume of sphere}}}\iint_{S_\varepsilon}\underline{F}\cdot\underline{n}\, dS}$$

THIS IS A SCALAR

Remarks:

1. I don't need to consider spheres.
2. $\operatorname{div}\underline{F}$ is a scalar.
3. Gives information on how $\underline{F}$ is 'flowing near a given point'.
4. This definition is not practical.

Ex: 'First principles' calculation of $\operatorname{div}\underline{r}$ at $(0,0,0)$:

$\underline{r} = x\,\underline{i} + y\,\underline{j} + z\,\underline{k}$ (position vector for the radius) $\to |r| = \varepsilon$.

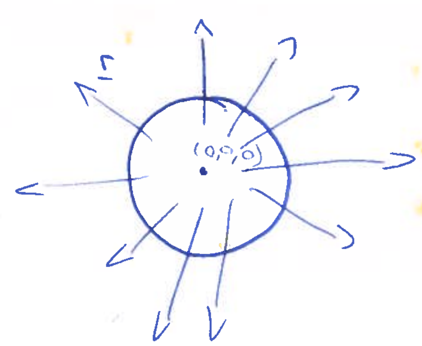
(since S is a sphere, they can be the same)

$\underline{n} = \frac{\underline{r}}{\varepsilon}$ $\quad \underline{r}\cdot\underline{n} = \underline{r}\cdot\left(\frac{\underline{r}}{\varepsilon}\right) = \frac{|\underline{r}|^2}{\varepsilon} = \frac{\varepsilon^2}{\varepsilon} = \varepsilon$

$$\iint_{S_\varepsilon}\underline{r}\cdot\underline{n}\, dS = \iint_{S_\varepsilon}\varepsilon\, dS = \varepsilon\iint_{S_\varepsilon} 1\, dS = \varepsilon\left(4\pi\varepsilon^2\right) = 4\pi\varepsilon^3$$

área sphere

$$\operatorname{div}\underline{r}\Big|_{(0,0,0)} = \frac{4\pi\varepsilon^3}{\frac{4\pi}{3}\varepsilon^3} = 3.$$



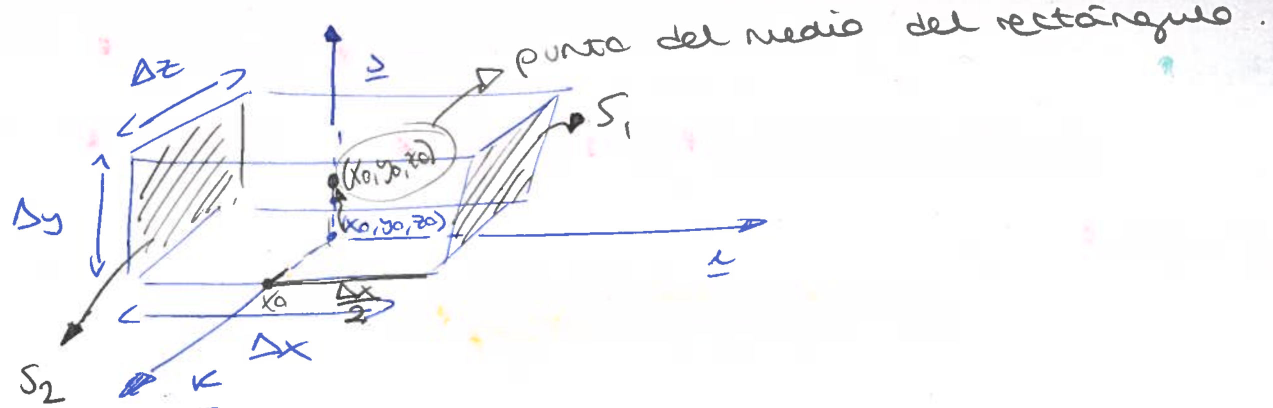
net 'flow' out.

Differential form of divergence NFE.

Let $\underbrace{SRP}_{\text{small rectangular parallelepiped}}$ be a box with edges $\Delta x, \Delta y, \Delta z$ parallel to the coordinate axes and with centre (x_0, y_0, z_0)

Let $\underline{F} = F_1\underline{i} + F_2\underline{j} + F_3\underline{k}$.

$$\operatorname{div}\underline{F}(x_0, y_0, z_0) = \lim_{\Delta x, \Delta y, \Delta z \to \infty}\frac{1}{\Delta x\Delta y\Delta z}\iint_{SRP}\underline{F}\cdot\underline{n}\, dS$$



① $\iint_{S_1} \underline{F}\cdot\underline{i}\, dS = \iint_{S_1} F_1(x,y,z)\, dS \simeq F_1\left(x_0+\frac{\Delta x}{2}, y_0, z_0\right)\Delta y\Delta z$.

② S_2 be the opposite face, which has normal $(-i)$

$$\iint_{S_2} \underline{F}\cdot\underline{n}\, dS = -\iint_{S_2} \underline{F}\cdot\underline{i}\, dS = -\iint_{S_2} F_1\, dS = -F_1\left(x_0-\frac{\Delta x_0}{2}, y_0, z_0\right)\Delta y\Delta z .$$

$$\iint_{S_1+S_2} \underline{F}\cdot\underline{n} = \left[\frac{F_1\left(x_0+\frac{\Delta x}{2}, y_0, z_0\right) - F_1\left(x_0-\frac{\Delta x}{2}, y_0, z_0\right)}{\Delta x}\right]\Delta y\,\Delta z$$

$$\lim_{\Delta x,\Delta y,\Delta z\to 0} \frac{1}{\Delta x\Delta y\Delta z}\iint_{S_1+S_2} \underline{F}\cdot\underline{n}\, dS = \lim_{\Delta x\Delta y\Delta z\to 0} \frac{F_1\left(x_0+\frac{\Delta x}{2}, y_0, z_0\right) - F_1\left(x_0-\frac{\Delta x}{2}, y_0, z_0\right)}{\Delta x} =$$

$$= \left.\frac{\partial f_1}{\partial x}\right|_{(x_0,y_0,z_0)} .$$

Similarly,

$$\lim_{\Delta x,\Delta y,\Delta z\to 0} \iint_{S_3+S_4} \underline{F}\cdot\underline{n}\, dS = \frac{\partial f_2}{\partial y}(x_0,y_0,z_0)$$

$$\lim_{\Delta x,\Delta y,\Delta z\to 0} \iint_{S_5+S_6} \underline{F}\cdot\underline{n}\, dS = \frac{\partial f_3}{\partial z}(x_0,y_0,z_0)$$

$$\Rightarrow \boxed{\operatorname{div} \underline{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}}$$

Remarks:

1. Revisit $\boxed{\operatorname{div} \underline{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} =}$ $\quad \underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$

2. $\underline{F} = 3\underline{i} + 4\underline{j} + (-12)\underline{k} \quad \operatorname{div} \underline{F} = 0$.

3. $\underline{F} = 3x\underline{i} - 2y\underline{j} - z\underline{k} \quad \operatorname{div} \underline{F} = 3-2-1$

4. $\underline{F} = xy\,\underline{i} + 4xz^2\underline{j} + \cos x\, z^5\underline{k}$

$\operatorname{div} \underline{F} = y + 0 + 5z^4\cdot\cos x$.

Definition: $\underline{F}$ is divergence free if $\operatorname{div} \underline{F} = 0$. or Solenoidal

$(\operatorname{div} B = 0) \to$ example.
magnetic field

Del operator

$$\underline{\nabla} = \frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}\underline{j} + \frac{\partial}{\partial z}\underline{k}$$

$$\operatorname{div} \underline{F} = \underline{\nabla}\cdot\underline{F} = \left(\frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}\underline{j} + \frac{\partial}{\partial z}\underline{k}\right)\left(F_1\underline{i} + F_2\underline{j} + F_3\underline{k}\right) =$$

$$= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} .$$

The divergence theorem

For a closed surface S and a vector field $\underline{F}$ defined in a region of $\mathbb{R}^3$ containing S and its interior V, then:

$$\underbrace{\iint_S \underline{F}\cdot\underline{n}\, dS}_{\text{flux integral}} = \iiint_V \underline{\nabla}\cdot\underline{F}\, dV$$

Proof: (see notes) NFE

Slice S into 2 slices  S_1, S_2.

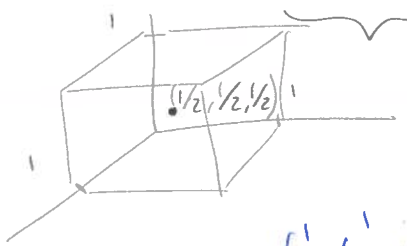
Can show $\iint_S \underline{F}\cdot\underline{n}\, dS = \iint_{S_1} \underline{F}\cdot\underline{n}\, dS + \iint_{S_2} \underline{F}\cdot\underline{n}\, dS$.

** slice again .....

$$\Rightarrow \iint \underline{F}\cdot\underline{n}\, dS = \sum_{j=1}^{N} \iint_{S_j} \underline{F}\cdot\underline{n}_j\, dS = \sum_{j=1}^{N}\left(\frac{1}{\Delta V_j}\iint \underline{F}\cdot\underline{n}_j\, dS\right)\Delta V_j =$$

$$= \sum_{j=1}^{N} \operatorname{div}\underline{F}\ \Delta V_j = \iiint_V \operatorname{div}\underline{F}\, dV \qquad \blacksquare$$

Ex: Calculate flux of $\underline{F} = (x^2+y^2)\underline{i} + (y^2+z^2)\underline{j} + (x^2+z^2)\underline{k}$ over the surface of a unit cube centred at $\underbrace{(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2})}$, and edges parallel to coordinate axes.



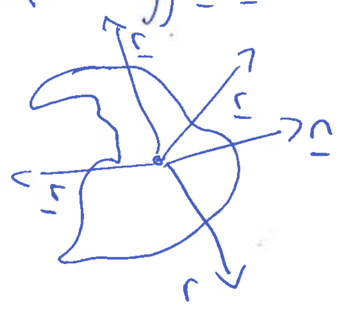
$$\underline{\nabla}\cdot\underline{F} = 2x+2y+2z$$

$$\iint \underline{F}\cdot d\underline{s} = \iiint_V \underline{\nabla}\cdot\underline{F}\ dV = 2\iiint_V (x+y+z)\,dV = 2\int_0^1\int_0^1\int_0^1 (x+y+z)\,dx\,dy\,dz =$$

$$= 2\int_0^1\int_0^1 \left[\frac{x^2}{2}+yx+zx\right]_0^1 dy\,dz = 2\int_0^1\int_0^1 \left(\frac{1}{2}+y+z\right)dy\,dz =$$

$$= \dots = 3$$

Ex: Compute $\iint \underline{r}\cdot\underline{n}\,ds$ for any closed surface S



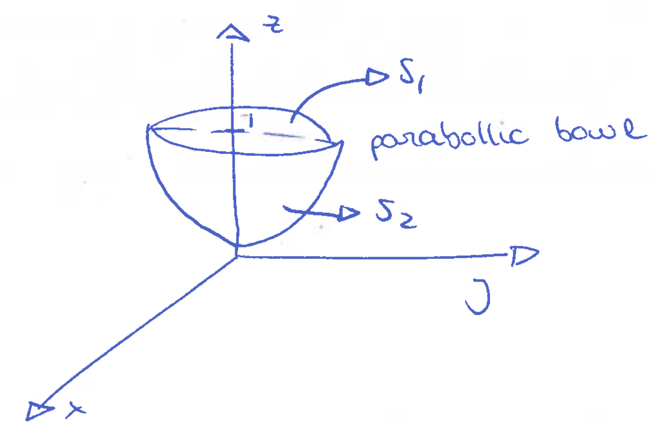
$$\iint \underline{r}\cdot\underline{n}\,ds = \iiint \operatorname{div}\underline{r}\cdot dV = 3\iiint_V dV = 3\cdot(\text{volume of interior of } S)$$

October 11th 2019

Extra example: (not in the printed notes).

$S: z = x^2+y^2,\ 0 \le z \le 1 \quad z=1$ upper surface.

S:

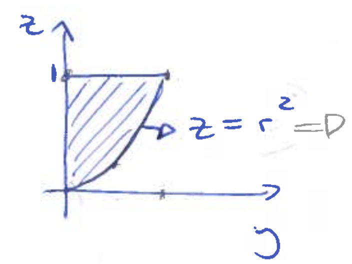


Verify the Divergence Theorem with $\underline{F} = y\underline{i} + x\underline{j} + z^2\underline{k}$.

$$\iint_S \underline{F}\cdot\underline{n}\,ds = \iiint_V \operatorname{div}\underline{F}\,dV$$

$\underline{\nabla}\cdot\underline{F}$

RHS $\iiint_V \operatorname{div}\underline{F}\,dV = 2\iiint_V z\,dV = 2\iiint_V z\cdot r\,dr\,d\theta\,dz =$



$r = \sqrt{z}$

$0 \le r \le \sqrt{z}$

$$= 2\int_0^{2\pi}\int_0^1\int_{r^2}^1 z\cdot r\,dz\,dr\,d\theta = 2\int_0^{2\pi}\int_0^1 \left[r\cdot\frac{z^2}{2}\right]_{r^2}^1 dr\,d\theta =$$

$$= 2\int_0^{2\pi}\int_0^1 \frac{1}{2}(r - r^5) = \boxed{\frac{2\pi}{3}}$$

LHS $= \iint_S \underline{F}\cdot\underline{n}\,ds = \iint_{S_1} \underline{F}\cdot\underline{n}_1 ds + \iint_{S_2} \underline{F}\cdot\underline{n}_2\,ds = \iint_{\text{"Lid"}} \underline{F}\cdot\underline{k}\,ds + \iint_{S_2} \underline{F}\cdot\underline{n}_2\,ds$

$$\iint_{\text{"Lid"}} \underline{F}\cdot\underline{k}\,ds = \iint_{\substack{\text{unit}\\ \text{disk}\\ (z=1)}} z^2\,ds = \iint_{\substack{\text{unit}\\ \text{disk}}} 1\,ds = \pi r^2 = \pi\cdot 1^2 = \pi.$$

$$\iint_{S_2} \underline{F}\cdot\underline{n}_2\, dS \qquad \underline{n}_2 = \frac{2x\underline{i}+2y\underline{j}-\underline{k}}{\sqrt{4x^2+4y^2+1}} \qquad z = x^2+y^2 = f(x,y)$$

$$\iint_{S_2} \underline{F}\cdot\underline{n}_2 = \iint_{\substack{\text{unit}\\\text{disk}}} \frac{(4xy - z^2)}{\cancel{\sqrt{\quad}}} \cdot \cancel{\sqrt{\quad}}\, dA =$$

$$= \iint_{\text{unit disk}} (4xy - r^4)\, dA \underset{\substack{\uparrow\\ z=f(x,y)=r^2}}{=} \iint_{\text{unit disk}} (\cancel{4xy}^{\,0} - r^4)\, dA =$$

$$\left[\begin{array}{l} \iint xy\, dA = 0 \\ \quad \hookrightarrow xy = \sin(\theta)\cdot\cos(\theta)\, r^2 \quad (\text{formula } \sin(2\theta) = 2\cos(\theta)\sin(\theta)) \\ \iint \frac{r^2}{2}\sin(2\theta)\, dr\, d\theta = \int_0^{2\pi} \sin 2\theta\, d\theta = 0. \end{array}\right]$$

$$= -\iint r^4 \cdot r\, dr\, d\theta = -\int_0^{2\pi}\int_0^1 r^5\, dr\, d\theta = \boxed{-\pi/3}$$

$$\iint_S \underline{F}\cdot\underline{n}\, dS = \pi - \pi/3 = \frac{2\pi}{3} = \boxed{\text{RHS}}$$

PART 1: VECTOR CALCULUS

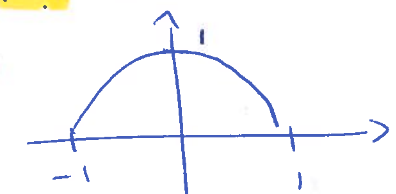
CHAPTER 2: line integrals, the curl and Stokes' theorem

Line integrals / Path integral: A line integral of a vector field F over a curve C with parametrisation r is defined by: $\int_C \underline{F}(\underline{r}(t))\cdot d\underline{r} = \int_a^b F(r(t))\cdot r'(t)\, dt$.

Let $\underline{r}(t) = x(t)\underline{i} + y(t)\underline{j} + z(t)\underline{k}$ be a curve in 3D. $a \le t \le b$.

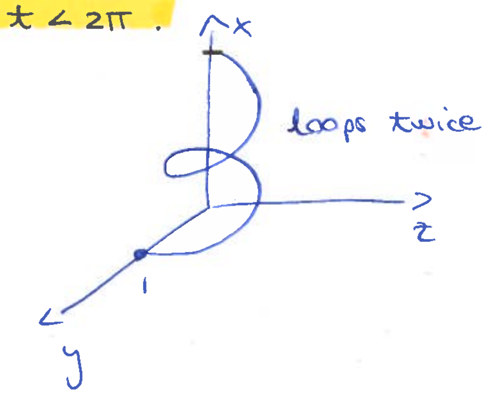
eg: (1): $r(t) = \cos(t)\,\underline{i} + \sin t\,(\underline{j})$, $0 \le t \le \pi$.

is a 2D example $\Rightarrow$ a semi-circle of radius (1)



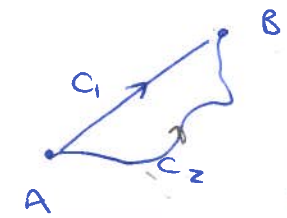
(2) $\underline{r}(t) = t^2\underline{i} + \cos t\,\underline{j} + \sin t\,\underline{k}$ $\quad 0 \le t < 2\pi$.

is a spiral around the x-axis.



Recall, if $\int_C \underline{F}\cdot d\underline{r}$ is path independent, then the line integral around any closed path is 0.

i.e. $\oint \underline{F}\cdot d\underline{r} = 0$



i.e. $\int_{C_1} \underline{F}\cdot d\underline{r} = \int_{C_2} \underline{F}\cdot d\underline{r}$

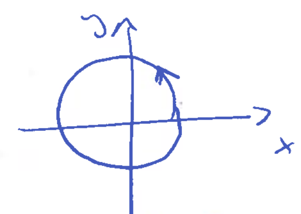
$\oint \underline{F}\cdot d\underline{r} = 0$: PATH INDEPEND.

Example:

(1) C circle in the (xy)-plane, centre $(0,0)$ radius 2 with anticlockwise.

Choose $\underline{F} = \underline{i}$

$\underline{r}(t) = 2\cos t\,\underline{i} + 2\sin t\,\underline{j} + 0\underline{k}$ $\quad 0 \le t < 2\pi$



$$\oint \underline{F}\cdot d\underline{r} = \int_0^{2\pi} \underline{F}\cdot\frac{d\underline{r}}{dt}\, dt = \int_0^{2\pi} (\underline{i}\cdot(-2\sin t\,\underline{i} + 2\cos t\,\underline{j} + 0\underline{k}))\, dt =$$

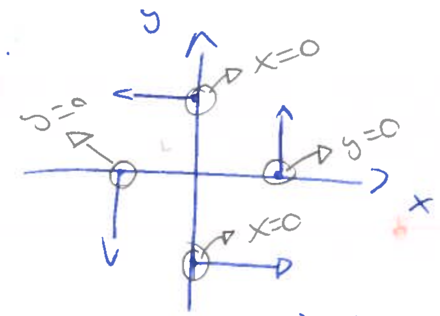
$$= -2\int_0^{2\pi} \sin t\, dt = 0.$$

Definition: Let C be a closed path. The line integral $\oint \underline{F} \cdot d\underline{r}$ is called the circulation of $\underline{F}$ around C.

$\Rightarrow$ ex(1) has 0 circulation.

(2) Same path as (1), $\underline{F} = -y\underline{i} + x\underline{j}$

$\underline{r}(t)$ from example (1) ; $y = 2\sin(t)$, $x = 2\cos(t)$



$$\oint \underline{F} \cdot d\underline{r} = \int_0^{2\pi} (-2\sin t\, \underline{i} + 2\cos(t)\, \underline{j})(-2\sin t\, \underline{i} + 2\cos t\, \underline{j})\, dt =$$

$$= 4\int_0^{2\pi} 1\, dt = 8\pi.$$

(3) C is the circle in the (yz-plane), centre origin with radius 2 and anti-clockwise direction $\underline{F} = -y\underline{i} + x\underline{j}$.

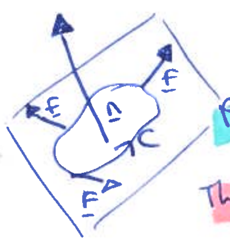
Here $\underline{r}(t) = 2\cos t\, \underline{j} + 2\sin t\, \underline{k}$ $\quad 0 \le t < 2\pi$.

$\underline{F} = -2\cos t\, \underline{i} + 0\underline{j}$

$\underline{F} \cdot \frac{d\underline{r}}{dt} = 0 \Rightarrow$ zero circulation

$\Downarrow$

$(-2\cos t\, \underline{i} + 0\underline{j} + 0\underline{k}) \cdot (0\underline{i} - 2\sin t\, \underline{j} + 2\cos t\, \underline{k}) = 0.$



Planar curve C

The circulation $\oint_C \underline{F} \cdot d\underline{r}$ measures the extent to which the vector $\underline{F}$ rotates around $\underline{n}$.

NB: The choice $\underline{n}$ and direction around C are consistent with RH rule.

What happens as C shrinks enclosing a point (x, y, z)?

Consider $\lim_{\ell \to \infty} \frac{1}{\Delta S_\ell} \oint_{C_\ell} \underline{F} \cdot d\underline{l}$

where C_ℓ is a sequence of closed curves in the plane with decreasing area ΔS_ℓ.

As $\ell \to \infty$ curves shrink down onto (x, y, z).

$\lim_{\ell \to \infty} \frac{1}{\Delta S_\ell} \oint_{C_\ell} \underline{F} \cdot d\underline{l}$ is called the curl of $\underline{F}$ in direction $\underline{n}$.

Example:

Point $(0,0,0)$ in yz-plane $\Rightarrow \underline{n} = \underline{i}$

Consider C_ℓ to circles of radius a, centre $(0,0,0)$.

$\underline{r}(t) = a\cos t\, \underline{j} + a\sin t\, \underline{k}$.

$\frac{d\underline{r}}{dt} = -a\sin t\, \underline{j} + a\cos t\, \underline{k}$.

Choose: $\underline{F} = z^2\underline{i} + z\underline{j} - y\underline{k} = a^2\sin^2 t\, \underline{i} + a\sin t\, \underline{j} - a\cos t\, \underline{k}$.

$\underline{F} \cdot \frac{d\underline{r}}{dt} = -a^2$.

$$\oint_{C_\ell} \underline{F} \cdot \frac{d\underline{r}}{dt} = -\int_0^{2\pi} a^2\, dt = -2\pi a^2$$

$$\lim_{a \to 0} \frac{1}{\pi a^2} \oint_{C_\ell} \underline{F} \cdot d\underline{r} = -2.$$

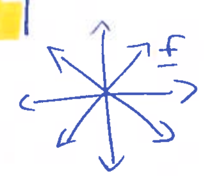
Thus -2 is curl of $\underline{F}$ in direction $\underline{i}$.

Differential formula for the curl

Can show (similar to the "div" approach)

$$\text{curl } \underline{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\underline{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\underline{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\underline{k} =$$

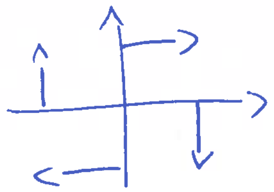
$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \nabla \times \underline{F}.$$



Examples (a) $\underline{F} = x\underline{i} + y\underline{j}$

$$\text{curl } \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & 0 \end{vmatrix} = \underline{i}\,0 - \underline{j} \cdot 0 + \underline{k} = \underline{0}$$

(b) $\underline{F} = y\underline{i} - x\underline{j}$

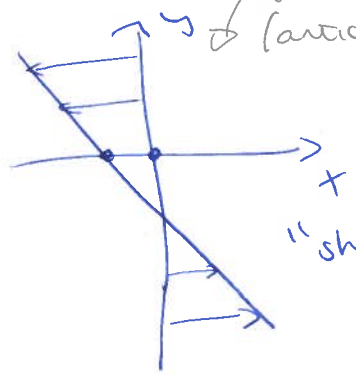


$$\text{curl}\,\underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x & 0 \end{vmatrix} = 0\underline{i} - 0\underline{j} - 2\underline{k} = -2\underline{k}.$$

(c) $\underline{F} = -(y+1)\underline{i}$

$$\text{curl}\,\underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -(y+1) & 0 & 0 \end{vmatrix} = 0\underline{j} + 1\underline{k} = +1\underline{k}.$$

from picture (anticlockwise)



Definition A vector field is called irrotational if $\nabla \times \underline{F} = 0$ everywhere that $\underline{F}$ is defined.

Suppose $\underline{F}$ is conservative ie. $\int_C \underline{F}\cdot d\underline{r}$ is path independent

$$\Rightarrow \oint \underline{F}\cdot d\underline{r} = 0.$$

So for fixed direction $\hat{n}$, consider closed curve C enclosing area ΔS.

curl $\underline{F}$ $\cdot \hat{n}$ = curl $\underline{F}$

$$\text{curl}\,\underline{F}\cdot\hat{n} = \lim_{\Delta S \to 0} \frac{1}{\Delta S}\oint \underline{F}\cdot d\underline{r} = 0 \quad \text{i.e. } \underline{F} \text{ is irrotational}$$

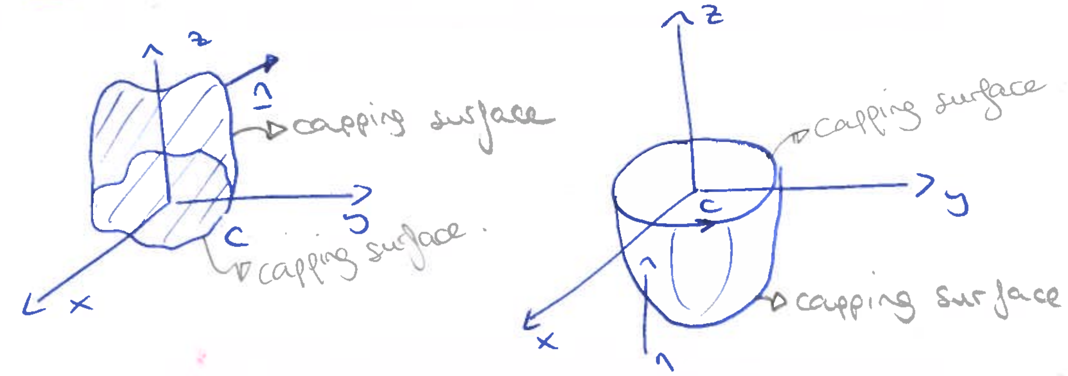
Summary: $\underline{F}$ path independent $\Rightarrow$ $\underline{F}$ is irrotational $(\nabla \times \underline{F}) = 0$

So if $\nabla \times \underline{F} \neq 0 \Rightarrow \underline{F}$ is not path independent.

Stoke's Theorem

Definition Given a closed curve C, a capping surface for C is any smooth surface with C as its boundary.

The orientation of the normal of the capping surface is taken so that it is consistent with direction around C according to RH rule.



[NFE]

Step 1 Take a closed curve C with capping surface S.

Consider vector field $\underline{F}$ and 2 distinct points on C called a and b.

Introduce a new path A from a->b on the capping surface S.

C_1: From a->b via A and then traverse along C in the position direction to get to a.

C_2: From a->b along C and then along -A to back to a.

Adding

$$\oint_C \underline{F}\cdot d\underline{r} = \oint_{C_1} \underline{F}\cdot d\underline{r} + \oint_{C_2} \underline{F}\cdot d\underline{r}$$

Further subdividing S into N small pieces S_p, $p = 1, \dots, N$ with boundaries represented by closed curves C_p leads it

$$\oint_C \underline{F}\cdot d\underline{r} = \sum_{p=1}^{N} \oint \underline{F}\cdot d\underline{r}.$$

Step 2: Let ΔS_p be the area of S_p.

$$\oint \underline{F}\cdot d\underline{r} = \sum_{p=1}^{N} \frac{1}{\Delta S_p} \oint \underline{F}\, d\underline{r}\ \Delta S_p$$

Now, recall $\frac{1}{\Delta S_p}\oint \underline{F}\cdot d\underline{r}$ is very similar to the quantity

we took the limit of to get curl $\underline{F}$ in a particular direction.

As $\Delta S_p \to 0$ the normal to S_p approaches the normal $\underline{n}_p$ at point (x_p, y_p, z_p) laying in S_p.

So, $$\frac{1}{\Delta S_p}\oint_{C_p} \underline{F}\cdot d\underline{r} = \nabla\times\underline{F}\Big|_{(x_p,y_p,z_p)} \cdot \underline{n}_p$$

So $$\oint_C \underline{F}\cdot d\underline{r} = \lim_{\substack{N\to\infty \\ \Delta S_p\to 0}} \sum_{p=1}^{N} \left(\nabla\times\underline{F}(x_p,y_p,z_p)\cdot \underline{n}_p\right)\Delta S_p$$

$$\boxed{\oint_C \underline{F}\cdot d\underline{r} = \iint_S (\nabla\times\underline{F})\cdot\underline{n}\, dS}$$

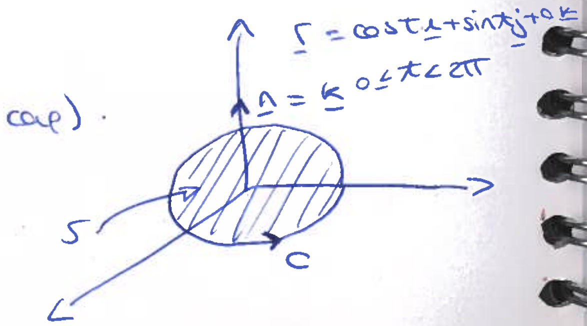
Theorem (Stokes)

Given a curve C and capping surface S, if $\underline{F}$ is smooth vector defined on C and S $\Rightarrow$ $\oint_C \underline{F}\cdot d\underline{r} = \iint_S (\text{curl}\,\underline{F})\cdot\underline{n}\, dS$.

Example: $\underline{F} = z\underline{i} + x\underline{j} - x\underline{k}$

C is the unit circle in the (xy)-plane centred at $(0,0,0)$ in the anti-clockwise direction

S: unit disk in the (xy)-plane ("flat" cap).



Verify Stokes

$$LHS = \oint_C \underline{F}\cdot d\underline{r} = \oint_C (0\underline{i} + \cos t\,\underline{j} - \cos t\,\underline{k})\cdot(-\sin t\,\underline{i} + \cos t\,\underline{j} + 0\underline{k})\, dt =$$

$$= \int_0^{2\pi} \cos^2 t\, dt = \pi$$

$$RHS = \iint_S (\nabla\times\underline{F})\cdot\underline{k}\, dS$$

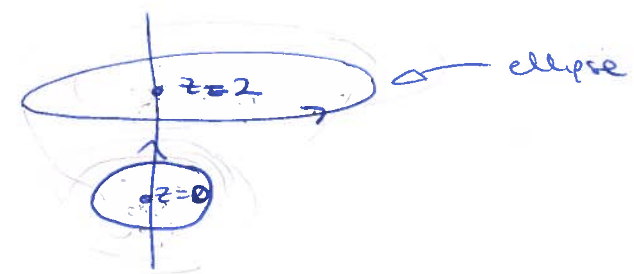
$$\nabla\times\underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & -x \end{vmatrix} = 0\underline{i} + 2\underline{j} + \underline{k}$$

$$RHS = \iint 1\, dS = \iint_{\substack{\text{unit}\\ \text{circle}}} 1\, dA = \boxed{\pi}$$

Ex: Current I flows along z-axis in the $+\underline{k}$ direction. This gives a magnetic field: $\underline{B}(x,y,z) = \frac{2I}{c}\left(\frac{-y\underline{i} + x\underline{j}}{x^2+y^2}\right)$

Maxwell's Equation: curl $\underline{B} = 0$ whenever $\underline{B}$ is defined

Compute circulation of $\underline{B}$ around ellipse $x^2 + 9y^2 = 9$ in the plane $z = 2$.



Compute: $\oint_C \underline{B}\cdot d\underline{r}$

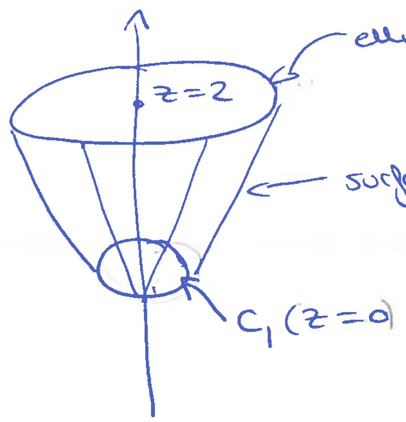
Let's use Stokes' Theorem

Step 1: Compute circulation of B around unit circle centred at origin.

C_1: $\underline{r} = \cos t\,\underline{i} + \sin t\,\underline{j}$ $0 \le t < 2\pi$

$\frac{d\underline{r}}{dt} = -\sin t\,\underline{i} + \cos t\,\underline{j}$ $\qquad \underline{B} = \frac{2I}{c}\left(\frac{-\sin t\,\underline{i} + \cos t\,\underline{j}}{1}\right)$

$$\oint_{C_1} \underline{B} \cdot d\underline{r} = \frac{2I}{c}\int_0^{2\pi} 1\, dt = \boxed{\frac{4\pi I}{c}}$$



Keeping C_1 anticlockwise forces $\underline{n}$ outwards

$\Rightarrow$ clockwise orientation on C.

$\therefore$ There are 2 boundary pieces to this surface.

$$\oint_{C_1} \underline{B} \cdot d\underline{r} + \oint_{-C} \underline{B} \cdot d\underline{r} = \iint_S \text{curl}\, \underline{B} \cdot \underline{n}\, dS .$$

($-C$: clockwise)

But $\text{curl}\,\underline{B} = 0$ (Maxwell!).

$$\Rightarrow \oint_{C_1} \underline{B} \cdot d\underline{r} = -\oint_{-C} \underline{B} \cdot d\underline{r} = \oint_{C_1} \underline{B} \cdot d\underline{r} = \frac{4\pi I}{c} .$$

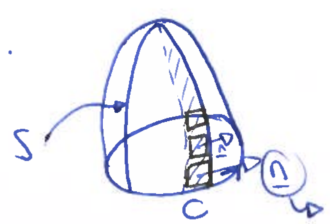
October 18th 2019

Remark: choice of $\underline{n}$ in Stoke's theorem.

Recall Stoke's theorem: $\iint_S \text{curl}\, \underline{F} \cdot \underline{n}\, dS = \oint_C \underline{F} \cdot d\underline{r}$

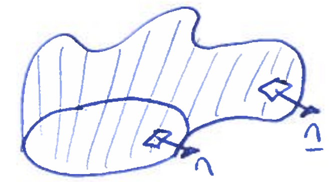
where S is a capping surface to the closed curve C.

In deriving the above we split the capping surface into many small pieces.

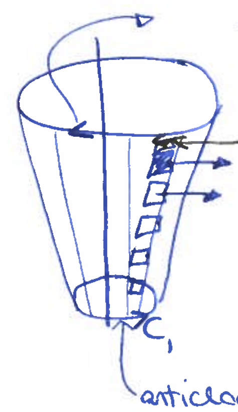


this are the small pieces and by RH rule we define each normal to be $\Rightarrow$ normal outward to the surface.

Apply RH rule to every piece $\Rightarrow$ normal is outward in this example. This is true even if the surface "overhangs".



In our example involving the $\oint \underline{B} \cdot d\underline{r}$ owing to an electric current:



they must be in $\neq$ directions so that those pieces go anticlockwise and when we get to the top we can see that the above arrow goes clockwise

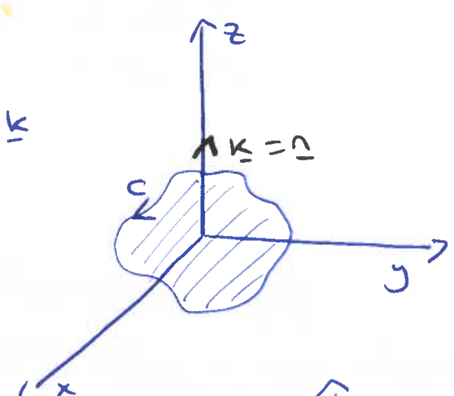
Connection with Green's Theorem

Let's restrict ourselves to the xy-plane.

Consider a closed curve C in this plane with anticlockwise orientation, and $\underline{F} = F_1 \underline{i} + F_2 \underline{j}$

Choose the surface capping to be flat.

i.e. it lies in the (xy)-plane $\Rightarrow \hat{n} = \underline{k}$



$$(\text{curl}\,\underline{F}) \cdot \underline{k} = (\text{curl}\,\underline{F}) = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} .$$

$$\Rightarrow \oint_{C} \underline{F} \cdot d\underline{r} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\, dy$$

$\Rightarrow$ me sirve solo con una suma de todos los trozos de la región

GREEN'S THM.

where R is the region in the (xy)-plane enclosed by C.

(Green's Thm is just a specialization of Stoke's thm)

Stoke's Theorem, path independence and the gradient

Recall: $\int_C \underline{F} \cdot d\underline{r}$ is path independent $\equiv \oint_C \underline{F} \cdot d\underline{r} = 0$, and

this implies $\text{curl}\,\underline{F} = 0$. (see definition of curl $\underline{F}$).

But what about reverse implication? (*)

$$\text{curl}\,\underline{F} = 0 \overset{?}{\Longrightarrow} \oint_C \underline{F} \cdot d\underline{r} = 0 .$$

Now, Stoke's theorem may help:

$$\oint \underline{F} \cdot d\underline{r} = \iint_S (\text{curl}\,\underline{F}) \cdot \underline{n}\, dS = 0 .$$

So (*) is true only if we can use Stoke's theorem:

In fact, we can apply Stoke's theorem if S is simply-connected

(see notes: NFE).

Definition: A region is simply-connected if every curve lying in the region can be shrunk to a point in the region.

Ex:

SC

can shrink to a point

If I take this curve it cannot be shrunk to a point

not SC (it has a hole)

Hence,

$$\boxed{\begin{array}{c}\oint \underline{F} \cdot d\underline{r} = 0 \\ \text{path independent}\end{array}} \;\begin{array}{c}\xleftarrow{\text{only SC}} \\ \longrightarrow\end{array}\; \boxed{\text{curl}\,\underline{F} = 0}$$

Recall, the gradient of a scalar function $f(x,y,z)$

→ In cartesian coordinates

$$\underline{F} = \nabla f = \frac{\partial f}{\partial x}\underline{i} + \frac{\partial f}{\partial y}\underline{j} + \frac{\partial f}{\partial z}\underline{k}$$

If $\underline{F}$ is the gradient of $f(x,y,z)$ then note

r_b $t=b$

r_a $t=a$

$$\underline{F} = \frac{\partial f}{\partial x}\underline{i} + \frac{\partial f}{\partial y}\underline{j} + \frac{\partial f}{\partial z}\underline{k} .$$

$$\int_C \underline{F} \cdot d\underline{r} = \int_a^b \underline{F} \cdot (\dot{x}\underline{i} + \dot{y}\underline{j} + \dot{z}\underline{k})\, dt =$$

$$\frac{d\underline{r}}{dt} = \frac{dx(t)}{dt}\underline{i} + \frac{d(y(t))}{dt}\underline{j} + \frac{dz(t)}{dt}\underline{k}$$

$$= \int_a^b \frac{d}{dt} f(x(t),y(t),z(t))\, dt = f\Big|_{t=b} - f\Big|_{t=a} =$$

$$\underbrace{x'(t)}_{\frac{dx}{dt}} \frac{\partial f}{\partial x} + \underbrace{y'(t)}_{\frac{dy}{dt}} \frac{\partial f}{\partial y} + z'(t)\frac{\partial f}{\partial z} = \frac{d}{dt} f(x(t),y(t),z(t))$$

$$= f\Big|_{r_b} - f\Big|_{r_a} = \text{i.e path independent} .$$

And if $r_b = r_a$ (closed curve) $\Rightarrow \int_C \underline{F} \cdot d\underline{r} = 0$.

So, if $\underline{F} = \nabla f \Rightarrow \oint_C \underline{F} \cdot d\underline{r} = 0$.

Also, (not shown) if $\oint_C \underline{F} \cdot d\underline{r} = 0 \Rightarrow \underline{F} = \nabla f$

Also, can show (see later exercise): $\text{curl}\,\nabla f = 0$.

Summary

$$\boxed{\oint_C \underline{F} \cdot d\underline{r} = 0 \text{ i.e. path independent}}$$

$$\boxed{\underline{F} = \nabla f} \;\begin{array}{c}\xleftarrow{\text{SC only}} \\ \longrightarrow\end{array}\; \boxed{\text{curl}\,\underline{F} = 0}$$

$\boxed{\oint_C \underline{F} \cdot d\underline{r} = 0} \leftrightarrow \boxed{\underline{F} = \nabla f}$; $\boxed{\text{curl}\,\underline{F} = 0} \xrightarrow{\text{SC only}} \boxed{\oint_C \underline{F} \cdot d\underline{r} = 0}$

$\underline{F}$ = "conservative"

Summation / index notation

"Sum over repeated indices" i, j, k, l taking as values 1, 2, 3.

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \qquad \text{eg: } \delta_{23} = 0 .$$

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{even permutation of } 1,2,3 \Rightarrow \varepsilon_{123} = \varepsilon_{312} = \varepsilon_{231} = 1 \\ -1 & \text{odd permutation of } 1,2,3 \Rightarrow \varepsilon_{132} = \varepsilon_{321} = \varepsilon_{213} = -1 \\ 0 & \text{otherwise} \end{cases}$$

$$\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3 = \delta_{\ell\ell} \quad \text{etc.}$$

$$(\underline{a} \cdot \underline{b}) = a_k b_k = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$\underline{a} = a_1 \underline{e}_1 + a_2 \underline{e}_2 + a_3 \underline{e}_3$$

$$\underline{b} = b_1 \underline{e}_1 + b_2 \underline{e}_2 + b_3 \underline{e}_3$$

$$(\underline{a} \times \underline{b})_i = \varepsilon_{ijk} a_j b_k$$

$$\delta_{ij} T_j = \delta_{i1} T_1 + \delta_{i2} T_2 + \delta_{i3} T_3 = T_i$$ "substitution property".

$i = j$

eg: $\delta_{ij} T_{ik\ell m} = T_{jk\ell m}$

$i = j$

del operator

$$(\underline{\nabla})_i = \frac{\partial}{\partial x_i}$$

$$\text{div}\,\underline{a} = \underline{\nabla} \cdot \underline{a} = \frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3} = \frac{\partial a_i}{\partial x_i} = \partial_i a_i = a_{i,i}$$

$$(\text{curl}\,\underline{a})_i = (\underline{\nabla} \times \underline{a})_i = \varepsilon_{ijk} \frac{\partial a_k}{\partial x_j}$$

$f = f(x, y, z)$.

$$(\underline{\nabla} f)_i = \frac{\partial f}{\partial x_i}$$

Let's prove $\text{div}(\text{curl}\,\underline{a}) = 0$.

$$\text{div}(\text{curl}\,\underline{a}) = \frac{\partial}{\partial x_i} (\text{curl}\,\underline{a})_i = \frac{\partial}{\partial x_i} \varepsilon_{ijk} \frac{\partial a_k}{\partial x_j} = \varepsilon_{ijk} \frac{\partial^2 a_k}{\partial x_i \partial x_j} =$$

$$= \underbrace{\varepsilon_{jik} \frac{\partial^2 a_k}{\partial x_j \partial x_i}}_{i \to j,\ j \to i} = \underbrace{\varepsilon_{jik} \frac{\partial^2 a_k}{\partial x_i \partial x_j}}_{\text{swap order of diff}} = \underbrace{-\varepsilon_{ijk} \frac{\partial^2 a_k}{\partial x_i \partial x_j}}_{\text{odd} \to \text{even},\ \text{even} \to \text{odd}} = 0$$

HW $\text{curl}(\text{grad}\,\phi) = 0$ similarly.

PART II : FOURIER THEORY

Theorem: Any sufficiently nice function $F : [-L, L] \to \mathbb{R}$ can be written as a Fourier series

$$F(x) = C + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right)$$

eg: $L = \pi$, any function on $[-\pi, \pi]$ can be written as

$$F(x) = C + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

The a_n, b_n, c are called the fourier coefficients of F.

"Sufficiently nice" could mean "differentiable except at a finite collection of points".

"can be written as" means $\Rightarrow$ the series converges to the original function

No proof.

How to calculate the fourier coefficients?

Lemma.

- If $n \geq 0$ is an integer $\Rightarrow$ $$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) dx = 0.$$

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) dx = 2L\,\delta_{n0}$$

- If $m, n > 0$ are integers $\Rightarrow$ $$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = L\,\delta_{mn}$$

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) \cdot \cos\left(\frac{m\pi x}{L}\right) dx = L\,\delta_{mn}$$

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = 0.$$

eg. Recall

$$2\sin A \cdot \sin B = \cos(A-B) - \cos(A+B)$$

So,

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \cdot \sin\left(\frac{m\pi x}{L}\right) dx =$$

$$= \frac{1}{2}\int_{-L}^{L}\left[\cos\left(\frac{(n-m)\pi}{L}x\right) - \cos\left(\frac{(n+m)\pi x}{L}\right)\right]dx$$

Now, $n+m \neq 0$ so $\int_{-L}^{L} \cos\left(\frac{(m+n)\pi x}{L}\right) dx = 0$.

and $\frac{1}{2}\int_{-L}^{L} \cos\left(\frac{(n-m)}{L}\pi x\right) dx = L\,\delta_{mn}$.

Theorem: If $F(x) = C + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$

$$\Rightarrow C = \frac{1}{2L}\int_{-L}^{L} F(x)\,dx$$

$$a_m = \frac{1}{L}\int_{-L}^{L} F(x)\cos\left(\frac{m\pi x}{L}\right)dx$$

$$b_m = \frac{1}{L}\int_{-L}^{L} F(x)\sin\left(\frac{m\pi x}{L}\right)dx$$

eg: Proof:

$$\int_{-L}^{L} f(x)\,dx = \int_{-L}^{L} C\,dx + \sum_{n=1}^{\infty}\int_{-L}^{L} a_n \cos\left(\frac{n\pi x}{L}\right)dx + \sum_{n=1}^{\infty}\int_{-L}^{L} b_n \sin\left(\frac{n\pi x}{L}\right)dx =$$

$$= 2LC + 0 + 0$$

$$C = \frac{1}{2L}\int_{-L}^{L} F(x)\,dx$$

and

$$\int_{-L}^{L} F(x)\cos\left(\frac{m\pi x}{L}\right)dx = \int_{-L}^{L} C \cdot \cos\left(\frac{m\pi x}{L}\right)dx +$$

$$+ \sum_{n=1}^{\infty} a_n \int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) \cdot \cos\left(\frac{m\pi x}{L}\right) dx +$$

$$+ \sum_{n=1}^{\infty} b_n \int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \cdot \cos\left(\frac{m\pi x}{L}\right) dx = L a_n \delta_{mn} = L a_m$$

$$\boxed{a_m = \frac{1}{L}\int_{-L}^{L} F(x)\cos\left(\frac{m\pi x}{L}\right)dx}$$

The proof that

$$\boxed{b_m = \frac{1}{L}\int_{-L}^{L} F(x)\sin\left(\frac{m\pi x}{L}\right)dx}$$

is for you!

Example:

Consider $F(x) = x$ on $[-\pi, \pi]$ (i.e. $L = \pi$) $\Rightarrow C = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,dx = 0 =$

$$= \frac{1}{2\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{\pi} = 0.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cdot \cos(nx)\,dx = 0 \quad \text{(odd or by parts)}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\,dx = \frac{1}{\pi}\left[-x\frac{\cos nx}{n}\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{\cos nx}{n}\,dx =$$

$$= \frac{1}{\pi}\left[-\pi\frac{\cos n\pi}{n} - \frac{\pi\cos(n\pi)}{n}\right] + \frac{1}{n^2\pi}\left[\sin(nx)\right]_{-\pi}^{\pi} =$$

$$= -\frac{2}{n}\cos(n\pi) = -\frac{2}{n}(-1)^n = \frac{2}{n}(-1)^{n+1}$$

$$\boxed{\cos n\pi = (-1)^n}$$

$$x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin(nx)$$

that is,

$$x = 2\cdot\left(\frac{\sin x}{1} - \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3} - \dots\right)$$

LOOK AT NOTES! (page 7)

Note we used a shortcut to computing Fourier series when the function has certain symmetries.

odd $F(-x) = -F(x)$ → sin

even $F(-x) = F(x)$ → cos

Lemma: Suppose F has Fourier series

$$F(x) = C + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right)$$

If F is even

- $b_n = 0$
- $a_n = \frac{2}{L}\int_0^L F(x)\cos\left(\frac{n\pi x}{2}\right)dx$
- $C = \frac{1}{L}\int_0^L F(x)\,dx$

If F is odd

- $b_n = \frac{2}{L}\int_0^L F(x)\sin\left(\frac{n\pi x}{L}\right)dx$
- $a_n = 0$
- $C = 0$

Proof: Let's show $F(x)$ is even implies $b_n = 0$

We have

$$b_n = \frac{1}{L}\int_{-L}^{L} F(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\left(\int_{-L}^{0} F(x)\sin\left(\frac{n\pi x}{L}\right)dx + \int_0^L F(x)\sin\left(\frac{n\pi x}{L}\right)dx\right)$$

$$\begin{bmatrix} u = -x \\ F(-u) = F(u) \\ \sin\left(\frac{n\pi(-u)}{L}\right) = -\sin\left(\frac{n\pi u}{L}\right) \end{bmatrix}$$

$$+\int_L^0 F(u)\sin\left(\frac{n\pi u}{L}\right)du = -\int_0^L F(u)\sin\left(\frac{n\pi u}{L}\right)du =$$

$$= -\int_0^L F(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$\Rightarrow b_n = 0$ //

Exercise: $L = 1$

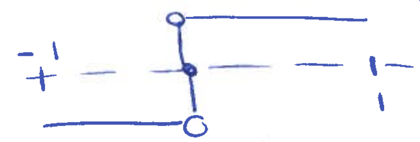
$$F(x) = \begin{cases} 0 & x \in [-1, 0) \\ 1/2 & x = 0 \\ 1 & x \in (0, 1] \end{cases}$$

F is neither even nor odd, but if we substract 1/2, then we get an odd function:

$$G(x) = \begin{cases} -1/2 & x \in [-1, 0) \\ 0 & x = 0 \\ 1/2 & x \in (0, 1] \end{cases}$$

odd

STEP FUNCTION



$$b_n = 2\int_0^1 G(x)\sin(n\pi x)\,dx = \int_0^1 \sin(n\pi x)\,dx = \frac{1}{n\pi}\Big[-\cos(n\pi x)\Big]_0^1 = \frac{1}{n\pi}\left[1 - (-1)^n\right]$$

$b_n = 2/n\pi$ n odd ; $b_n = 0$ n even.

$$G(x) = \frac{2}{\pi}\sin(\pi x) + \frac{2}{3\pi}\sin(3\pi x) + \dots$$ and

$$F(x) = \frac{1}{2} + \frac{2}{\pi}\sin(\pi x) + \frac{2}{3\pi}\sin(3\pi x) + \dots$$

Ex: $F(x) = x^2$ on $[-\pi, \pi]$ → f is even.

$$C = \frac{1}{\pi}\int_0^{\pi} x^2\,dx = \frac{\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2\cos(nx)\,dx = \frac{2}{\pi}\left(\left[x\frac{\sin nx}{n}\right]_0^{\pi} - \frac{2}{n}\int_0^{\pi} x\sin(nx)\,dx\right) =$$

$$= -\frac{4}{n\pi}\left(\left[-x\cdot\frac{\cos nx}{n}\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos(nx)\,dx\right) =$$

$$= \frac{4}{n^2}(-1)^n - \frac{4}{n^2\pi}\left[\cancel{\;}\right]^0 = \frac{4(-1)^n}{n^2}$$

$$x^2 = \frac{\pi^2}{3} - 4\left(\cos x - \frac{\cos 2x}{4} + \frac{\cos^3 x}{9} - \dots\right)$$

October 24^{th} 2019

$F: [-L, L] \to \mathbb{R}$

$$F(x) = C + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right)$$

$$C = \frac{1}{2L}\int_{-L}^{L} F(x)\,dx .$$

$$a_m = \frac{1}{L}\int_{-L}^{L} F(x)\cos\left(\frac{m\pi x}{L}\right)dx .$$

$$b_m = \frac{1}{L}\int_{-L}^{L} F(x)\sin\left(\frac{m\pi x}{L}\right)dx \qquad m = 1, 2, \dots$$

$F(x)$ even $\Rightarrow$ $b_n = 0$

$F(x)$ odd $\Rightarrow$ $a_n = 0$, $C = 0$.

REVIEW LAST WEEK.

Half-range Fourier Series

Definition: Suppose $F(x)$ is a function $[0, L] \to \mathbb{R}$

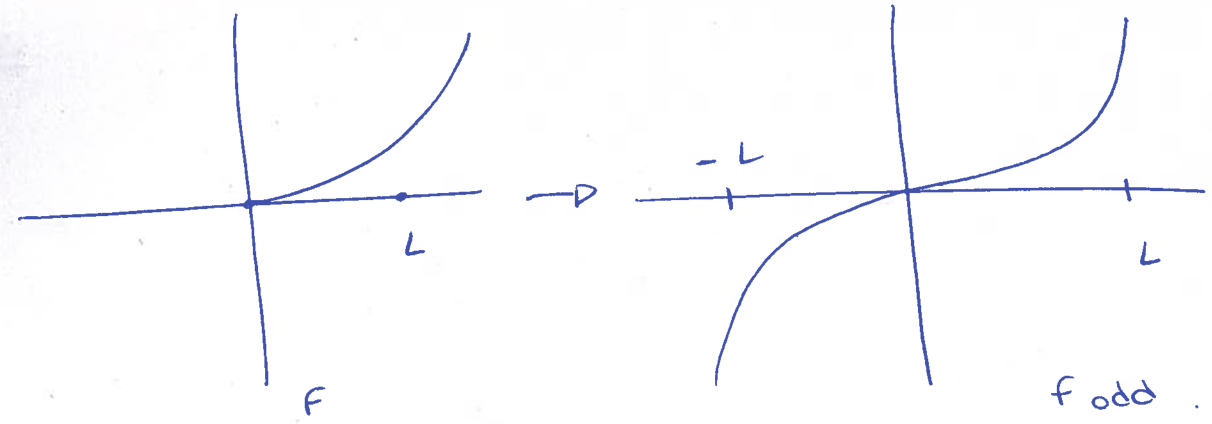
Define its odd extension to be.

$$F_{odd}(x) = \begin{cases} F(x) & \text{if } x > 0 \\ -F(-x) & \text{if } x < 0 \end{cases}$$

The half-range sine series of F is then defined to be the fourier series of F_{odd} i.e.

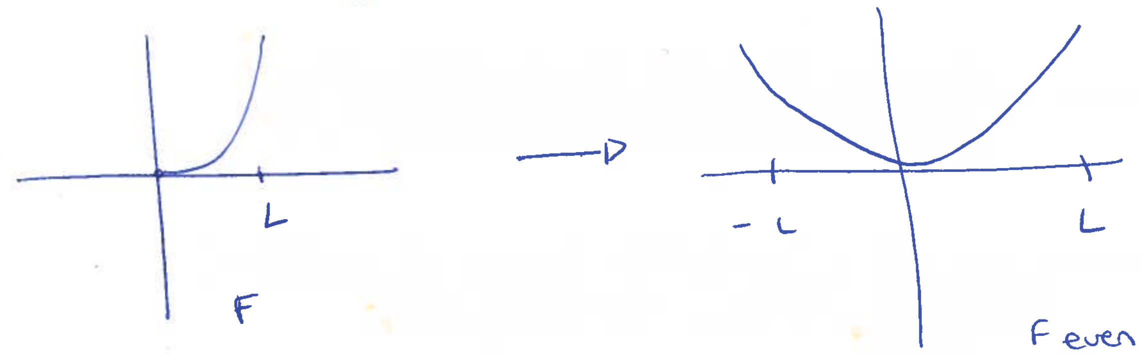
$$F(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

on $[0, L]$ where $b_n = \frac{2}{L}\int_0^L F(x)\sin\left(\frac{n\pi x}{L}\right)dx$.



Analogously, we define half-range cosine series by taking the Fourier series of the even extension.

$$f_{even}(x) = \begin{cases} f(x) & x > 0 \\ f(-x) & x < 0 \end{cases}$$



Ex: $f(x) = x(\pi - x)$ on $[0, \pi]$

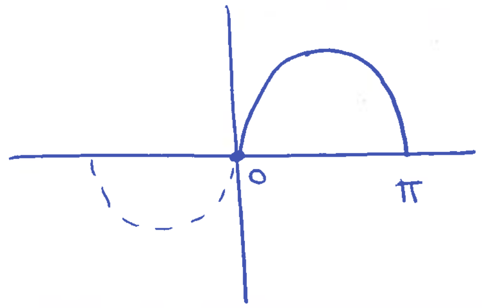
Odd extension $\Rightarrow$ half-range fourier series

$$b_n = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin(nx)\,dx$$

$\vdots$ $\leftarrow$ by yourself

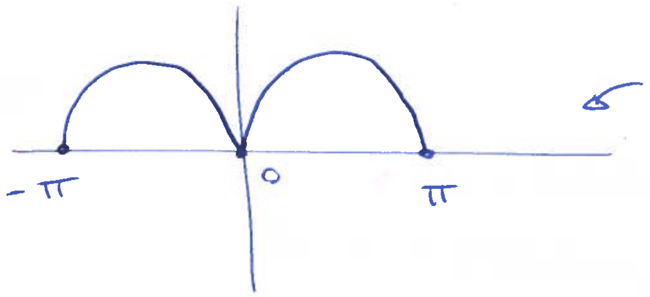
$$= \begin{cases} 8/n^3\pi & n \text{ odd} \\ 0 & n \text{ even.} \end{cases}$$

$$x(\pi - x) = 8 \cdot \left(\frac{\sin x}{\pi} + \frac{\sin(3x)}{27\pi^3} + \dots \right) \quad \text{on } [0, \pi].$$



odd extension

or



by yourself
i.e. compute cosine
half-range expansion

Parseval's Theorem

If $f(x) = c + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right)$

then,
$$\frac{1}{L} \int_{-L}^{L} f(x)^2 \, dx = 2L^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

Proof:

$$\frac{1}{L} \int_{-L}^{L} f(x)^2 \, dx = \frac{1}{L} \int_{-L}^{L} f(x) \left[c + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right) \right] dx =$$

$$= \frac{c}{L} \int_{-L}^{L} f(x) \, dx + \sum_{n=1}^{\infty} \frac{a_n}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx +$$

$$+ \sum_{n=1}^{\infty} \frac{b_n}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx =$$

$$= 2c^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

Ex: $\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$

Why? Take $f(x) = x$ on $[-\pi, \pi]$.

Recall, $f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nx)$.

i.e. $c = 0$, $a_n = 0$, $b_n = \frac{2(-1)^{n+1}}{n}$

Parseval says: $\frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \, dx = \sum_{n=1}^{\infty} b_n^2 = \sum_{n=1}^{\infty} \frac{4}{n^2}$

but LHS $= \frac{2\pi^2}{3}$.

Ex: $\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$

Ex: $\frac{\pi^4}{90} = \sum_{n=1}^{\infty} \frac{1}{n^4}$

Why? Recall, $f(x) = x^2$ on $[-\pi, \pi]$.

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(nx).$$

Parseval $\Rightarrow$ $\left(c = \frac{\pi^2}{3},\ a_n = \frac{4(-1)^n}{n^2},\ b_n = 0 \right)$

$$\frac{1}{\pi} \int_{-\pi}^{\pi} x^4 \, dx = \frac{2\pi^4}{9} + \sum_{n=1}^{\infty} \frac{16}{n^4}$$

$$\frac{2\pi^4}{5} = \frac{2\pi^4}{9} + \sum_{n=1}^{\infty} \frac{16}{n^4}$$

$$\boxed{\frac{\pi^4}{90} = \sum_{n=1}^{\infty} \frac{1}{n^4}}$$

Note: $\sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s)$ Riemann zeta function

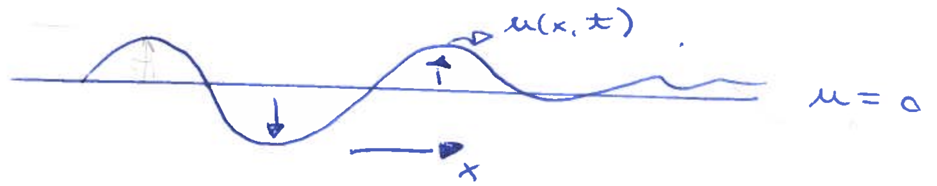
Ex: Separation of variables : use of Fourier series

al estudiar la he considerado ej.

Consider the 1D wave equation.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad u = u(x,t).$$

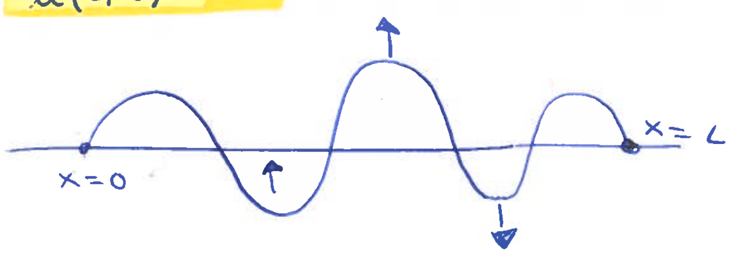
i.e. u = displacement / amplitude of wave.



This is a partial differential equation - PDE. Much more about PDE's later in this course.

Suppose a string vibrates according to the wave equation. It is fixed at ends $x=0$ and $x=L$.

i.e. $\left.\begin{array}{l} u(0,t) = 0 \\ u(L,t) = 0 \end{array}\right\} \forall t$ Boundary condition



Suppose the initial string deflection $(t=0)$ is $f(x)$ and has initial velocity $g(x)$.

Initial condition $\left\{ u(x,0) = f(x) \text{ and } \frac{\partial u}{\partial t}(x,0) = g(x) \quad 0 \le x \le L \right.$.

Task: solve $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ to find $u(x,t)$:

Step 1 We use the method of separation of variables to reduce the PDE into ODE's.

Let $u(x,t) = F(x)\,G(t)$ — $G(t)$: solo depende de t. — $F(x)$: solo depende de x

$$\frac{\partial^2 u}{\partial x^2} = F''G \qquad \left(F'' = \frac{d^2F}{dx^2}\right)$$

$$\frac{\partial^2 u}{\partial t^2} = F\ddot{G} \qquad \left(\ddot{G} = \frac{d^2G}{dt^2}\right)$$

Substituting into the wave equation:

$$F\ddot{G} = c^2 F'' G.$$

Dividing by $c^2 F G$ gives:

$$\frac{\ddot{G}}{c^2 G} = \frac{F''}{F}$$

$\frac{F''}{F}$: function of x only.

$\frac{\ddot{G}}{c^2 G}$: function of t only

$$\Rightarrow \frac{\ddot{G}}{c^2 G} = k = \frac{F''}{F}$$

and so $\left.\begin{array}{l} F'' - kF = 0 \\ \ddot{G} - c^2 k G = 0 \end{array}\right\}$ a pair of ODEs.

k is called the separation constant.

Step 2 Now we employ the BCs: (BCs → Boundary conditions)

Recall $u(0,t) = F(0)G(t) = 0$.

$u(L,t) = F(L)G(t) = 0$.

Now $G(t) = 0 \; \forall t$, not interesting; ~~X~~ — busco una función que dependa del tiempo.

So, we must have:

$$F(0) = F(L) = 0.$$

Then, what is the value of k?

(i) $k = 0 \Rightarrow F = ax + b$

But, $F(0) = F(L) = 0 \Rightarrow a = b = 0$ — $F(0) = b = 0$, $F(L) = aL + b$

and, $F \equiv 0$. Not interesting

(ii) $K>0$ $\Rightarrow K = \mu^2$ = al decir que $K=\mu^2$ aseguras que $K>0$

$\Rightarrow F'' - \mu^2 F = 0$ and so

$F = a e^{\mu x} + b e^{-\mu x}$

$a + b = 0 \Rightarrow a = b$

$a e^{\mu L} + a e^{-\mu L} = 0$

but again $F(0) = F(L) = 0$ $\Rightarrow a = b = 0$ reject.

$\Rightarrow a\,(e^{\mu L} + e^{-\mu L}) = 0$

$\Rightarrow a = 0$

$\Rightarrow e^{\mu L} + e^{-\mu L} = 0$

(iii) $K = -\mu^2$ $\Rightarrow F'' + \mu^2 F = 0$

so $F = a\cos\mu x + b\sin\mu x$

$F(0) = 0 \Rightarrow a = 0$; $F(L) = 0$ $\quad b\sin\mu L = 0$.

Demand $b \neq 0$ (otherwise $F \equiv 0$ again!)

$\Rightarrow \sin\mu L = 0 \Rightarrow \boxed{\mu L = n\pi}$ $\quad n = 1, 2, 3, \ldots$

$\mu = \dfrac{n\pi}{L}$

depende de n

Hence: $F = F_n = b\sin\left(\dfrac{n\pi x}{L}\right)$

Now, let's consider the equation for G.

Recall: $\ddot{G} - c^2 K G = 0$ ya vale $K = -\mu^2$

$\Rightarrow \ddot{G} + c^2\mu^2 G = 0$. $\left(\mu = \dfrac{n\pi}{L}\right)$

So $G = \displaystyle\sum_{n=1}^{\infty} G_n$ where $\ddot{G}_n + \dfrac{c^2 n^2 \pi^2}{L^2} G_n = 0$.

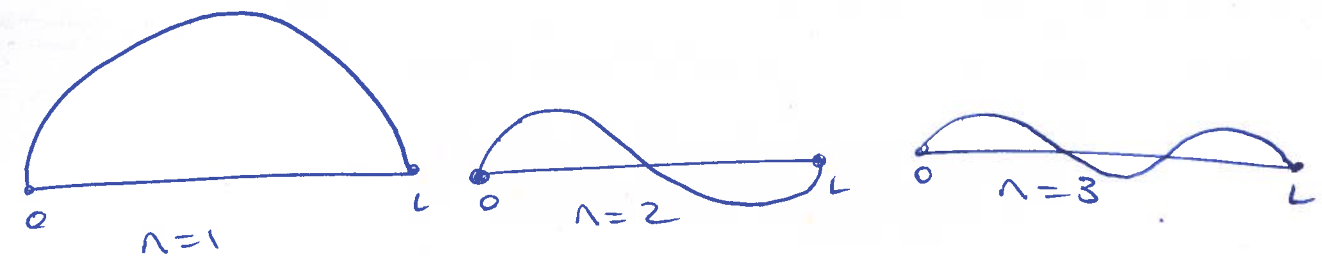
Thus, $G_n = C_n\cos\left(\dfrac{cn\pi}{L}t\right) + d_n\sin\left(\dfrac{cn\pi}{L}t\right)$.

So, $u_n(x,t) = G_n F_n$

$= \left(C_n^*\cos\left(\dfrac{cn\pi t}{L}\right) + d_n^*\sin\left(\dfrac{cn\pi t}{L}\right)\right)\cdot\sin\left(\dfrac{n\pi x}{L}\right)$

where $C_n^* = bc_n$ and $d_n^* = bd_n$ are arbitrary.

Remark: $\sin\left(\dfrac{n\pi x}{L}\right) = 0$ at $x = \dfrac{L}{n}, \dfrac{2L}{n}, \ldots, \dfrac{n-1}{n}L$



Normal modes.

STEP 3

$u(x,t) = \displaystyle\sum_{n=1}^{\infty}\left(C_n\cos\lambda_n t + d_n\sin\lambda_n t\right)\sin\left(\dfrac{n\pi x}{L}\right)$, where we

have dropped the $*$s and $\lambda_n = \dfrac{cn\pi}{L}$.

Now, we satisfy the initial conditions

From above, $u(x,0) = \displaystyle\sum_{n=1}^{\infty} C_n\sin\left(\dfrac{n\pi x}{L}\right) = f(x)$

Initial condition

The C_n's are s.t. $u(x,0)$ is the Fourier sine series of the (odd extension) $f(x)$. (porque me fijo y es la misma fórmula)

por def.

$C_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\left(\dfrac{n\pi x}{L}\right)dx$.

Now, we use the Initial Conditions for the velocity of the string.

$\dfrac{\partial u}{\partial t}(x,0) = g(x)$.

First, differentiate our series solutions for $u(x,t)$ w.r.t t

to $\dfrac{\partial u}{\partial t}$ and then put $t = 0$.

$\dfrac{\partial u}{\partial t} = \displaystyle\sum_{n=1}^{\infty}\left(-C_n\lambda_n\sin\lambda_n t + d_n\lambda_n\cos\lambda_n t\right)\sin\left(\dfrac{n\pi x}{L}\right)$

So $\left.\dfrac{\partial u}{\partial t}\right|_{t=0} = \displaystyle\sum_{n=1}^{\infty}\lambda_n d_n\sin\left(\dfrac{n\pi x}{L}\right) = g(x)$.

IC

which is a Fourier sine series for $g(x)$.

Hence, $\lambda_n d_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$ $\quad n = 1, 2, 3, \dots$

$$d_n = \frac{2}{cn\pi}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

The final solution is determined:

$$u(x,t) = \sum_{n=1}^{\infty}\left(c_n\cos\lambda_n + d_n\sin\lambda_n t\right)\cdot\sin\frac{\pi x}{L},$$

where $\lambda_n = \frac{cn\pi}{L}$

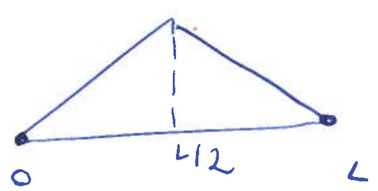
$$c_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$d_n = \frac{2}{cn\pi}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

Example:

Suppose the initial string displacement is triangular

$$f(x) = \begin{cases} x/L & 0 < x < L/2 \\ \frac{1}{L}(L-x) & \frac{L}{2} < x < L \end{cases}$$



$g(x) = 0$ i.e. string initially stationary.

$$\Rightarrow d_n = 0, \quad c_n = \dots \begin{cases} \frac{4}{\pi^2}\frac{(-1)^n}{n} & n \text{ odd.} \\ 0 & n \text{ even.} \end{cases}$$

$$u(x,t) = \frac{4}{\pi^2}\left[\sin\frac{\pi x}{L}\cos\frac{\pi ct}{L} - \frac{1}{3^2}\sin\frac{3\pi x}{L}\cos\frac{3\pi c}{L}t + \dots\right]$$

Remark $\quad 2\cdot\cos A\cdot\sin B = \sin(A-B) + \sin(A+B)$

$$\cos\frac{cn\pi}{L}t\,\sin\frac{n\pi x}{L} = \frac{1}{2}\sin\left[\frac{n\pi}{L}(x-ct)\right] + \frac{1}{2}\sin\left[\frac{n\pi}{L}(x+ct)\right]$$

$$\text{so, } u(x,t) = \underbrace{\frac{1}{2}\sum_{n=1}^{\infty} b_n\sin\left[\frac{n\pi}{L}(x-ct)\right]}_{\text{Right-travelling wave}} + \underbrace{\frac{1}{2}\sum_{n=1}^{\infty} b_n\sin\left[\frac{n\pi}{L}(x+ct)\right]}_{\text{left-travelling wave.}}$$

PART III: CALCULUS OF VARIATIONS

CHAPTER 4: STRAIGHT LINES ARE THE SHORTEST PATHS.

Max / min problems over infinite-dimensional spaces

e.g.
- minimising length of a path between 2 points.
- minimising surface tension of a soap film.
- minimising the length of a loop enclosing a given area.

Theorem: A straight line is the shortest path between 2 points in the plane.

Recall, length of a path $\gamma: [0,1] \to \mathbb{R}^2$ is $\int_0^1 |\dot{\gamma}(t)|\,dt$ (derivada)

$$\left\{ \gamma := x(t)\,\underline{i} + y(t)\,\underline{j} = x_1(t)\,\underline{i} + x_2(t)\,\underline{j} \right.$$

i.e. $\int_0^1 \sqrt{\dot{x}_1^2 + \dot{x}_2^2}$

Proof: 1. Let A be the action of a path

$$A(\gamma) = \int_0^1 |\dot{\gamma}(t)|^2\,dt$$

Now, by Cauchy-Schwarz inequality

$$\left|\int_0^1 |\dot{\gamma}|\,dt\right|^2 \leq \int_0^1 |\dot{\gamma}(t)|^2\,dt$$

[Equality if $|\dot{\gamma}| = \text{constant}$]

We can always parametrise so that $|\dot{\gamma}| = \text{constant}$

$\Rightarrow$ Suffice to show a straight line minimises action.

2. Proposition: a straight line minimises the action integral

$$\int_0^1 (\dot{x}_1^2 + \dot{x}_2^2)\, dt$$

among all smooth paths connecting 2 points.

Proof: Let $\gamma : [0,1] \to \mathbb{R}^2$ be the straight line connecting 2 points with constant speed

write $\gamma(t) = (\gamma_1(t), \gamma_2(t))$

Suppose $\delta(t)$ is another path joining the 2 points

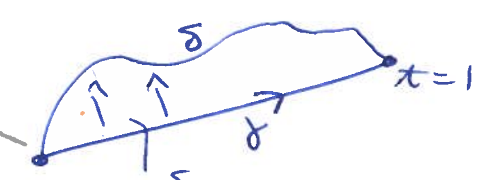
$$\delta(t) = (\delta_1(t), \delta_2(t))$$

MIRA DIBUJO

Definition $\varepsilon_i(t) = \delta_i(t) - \gamma_i(t) \quad i = 1, 2,$

so, $\delta = \gamma + \varepsilon$. Also, insist

$\varepsilon(0) = \varepsilon(1) = 0$.



ε = the difference

Now, $A(\delta) = \int_0^1 |\dot{\delta}|^2 dt = \int_0^1 (\dot{\varepsilon} + \dot{\gamma}) \cdot (\dot{\varepsilon} + \dot{\gamma})\, dt =$

$$= \int_0^1 \left(|\dot{\gamma}|^2 + |\dot{\varepsilon}|^2 + 2\dot{\varepsilon}\dot{\gamma} \right) dt$$

$$= A(\gamma) + A(\varepsilon) + 2\int_0^1 \dot{\varepsilon} \cdot \dot{\gamma}\, dt$$

Now, $\dot{\gamma} = \text{constant}$ (it is tangent to a straight line) so the integral term is

$$2\int_0^1 \dot{\gamma} \cdot \dot{\varepsilon}\, dt = 2\dot{\gamma} \cdot \int_0^1 \dot{\varepsilon}\, dt$$

$$= 2\dot{\gamma} \cdot (\varepsilon(1) - \varepsilon(0))$$

$$= 0.$$

$\Rightarrow A(\delta) = A(\gamma) + A(\varepsilon)$.

Equality occurs only if $A(\varepsilon) = 0$

or $\dot{\varepsilon} = 0 \Rightarrow$ or $\varepsilon(t) = \varepsilon(0) = 0$.

$\Rightarrow$ straight line.

November 1st 2019

Recall we showed a straight line (among other paths in an infinite dimensional vector space of paths) minimised the action

$$\int_0^1 (\dot{x}_1^2 + \dot{x}_2^2)\, dt$$

This was equivalent to minimising the path length.

How to generalise this approach?

Definition Let $A : V \to \mathbb{R}$ be a functional on a vector space (possibly infinite-dimensional). The Gâteaux derivative of A in the ε-direction at γ is

$$dA(\gamma; \varepsilon) = \frac{d}{dt}\bigg|_{t=0} A(\gamma + t\varepsilon)$$

Definition: γ is a critical point of A if $dA(\gamma;\varepsilon)=0 \ \forall \varepsilon \in V$.

What if we don't know a straight line is a minimum of the Action?

Proposition: Let V be the space of paths $\gamma:[0,1]\to\mathbb{R}$ connecting 2 given points in the plane and $A: V\to\mathbb{R}$ is the action functional. Then the critical points of A is precisely the straight line.

Proof: $A(\gamma+t\varepsilon)=\int_0^1 |\dot\gamma+t\dot\varepsilon|^2 dt =$

$$=\int_0^1|\dot\gamma|^2dt+2t\int_0^1\dot\gamma\cdot\dot\varepsilon\,dt+t^2\int_0^1|\dot\varepsilon|^2dt=$$

$$=A(\gamma)+2t\int_0^1\dot\gamma\cdot\dot\varepsilon\,dt+t^2A(\varepsilon)$$

So $dA(\gamma;\varepsilon)=\frac{d}{dt}\left\{ \text{above} \right\}\Big|_{t=0}=$

$$=2\int_0^1\dot\gamma\cdot\dot\varepsilon\,dt$$

This vanishes at critical points (this means $dA(\gamma,\varepsilon)=2\int_0^1\dot\gamma\dot\varepsilon\,dt=0$)

Integrating by parts gives: $dA(\gamma;\varepsilon)=-2\int_0^1\ddot\gamma\cdot\varepsilon\,dt$ since $\begin{cases}\varepsilon(0)=0\\ \varepsilon(1)=0\end{cases}$.

$u=\dot\gamma \quad dv=\dot\varepsilon\,dt$
$du=\ddot\gamma\,dt \quad v=\varepsilon$
$\Rightarrow 2\left(\dot\gamma\,\varepsilon\Big]_0^1-\int_0^1\varepsilon\cdot\ddot\gamma\,dt\right)=-2\int_0^1\varepsilon\,\ddot\gamma\,dt$

We will see by the next theorem that if this vanishes $\forall\varepsilon(t)\Rightarrow\ddot\gamma(t)=0\Rightarrow\gamma(t)$ is linear in t

i.e. $\left.\begin{aligned}x_1(t)&=at+b\\ x_2(t)&=ct+d\end{aligned}\right\}$ straight line

Theorem: Fundamental theorem of the calculus of Variations. NFE

Suppose that $y:[0,1]\to\mathbb{R}^n$ is a vector-valued function

If $\int_0^1 y(t)\,\varepsilon(t)\,dt=0 \quad \forall$ smooth functions $\varepsilon:[0,1]\to\mathbb{R}^n$

$\Rightarrow y(t)=0 \quad \forall t\in[0,1]$.

Proof: Let $y(t)=(y_1(t),y_2(t),y_3(t))$.

Suppose for contradiction $\exists\, t_0\in[0,1]$ s.t. $g(t_0)\neq 0$. Therefore one of the components $y_i(t_0)\neq0$ and wlog, assume that $y_1(t_0)>0$. Then $y_1(t)>0$ in some interval $t\in(t_0-\delta,t_0+\delta)$

Define a "bump" function $F:[0,1]\to\mathbb{R}$

$$F(t)=\begin{cases}\exp\left(\frac{1}{(1-t_0)^2-\delta^2}\right) & t\in(t_0-\delta,t_0+\delta)\\ 0 & \text{otherwise}\end{cases}.$$

Thus, consider $\varepsilon(t)=(F(t),0,\dots,0)$ and so $\int_0^1 y(t)\cdot\varepsilon(t)\,dt=$

$$=\int_{t_0-\delta}^{t_0+\delta}F(t)\,y_1(t)\,dt>0$$

which is a contradiction.

CHAPTER 5: THE EULER-LAGRANGE EQUATION I

Let V be the spaces of functions s.t. $y:[a,b]\to\mathbb{R}$ satisfying the BCs $y(a)=y_a$ and $y(b)=y_b$ for given numbers $y_a,y_b\in\mathbb{R}$.

If $y\in V\Rightarrow$ any other function in this space can be written as $y+\varepsilon$ for some $\varepsilon(x)$ satisfying $\varepsilon(a)=\varepsilon(b)=0$.

A function L of 3-variables $(L(p,q,r))$ is called a Lagrangian.
It defines a functional $A: V\to\mathbb{R}$.

$$A(y) = \int_a^b L(x, y(x), y'(x))\, dx \qquad \left(y'(x) = \frac{dy}{dx} \right).$$

(We will derive an equation satisfied by the critical points of A - The Euler-Lagrange Equation).

Recall the Gâteaux derivative

$$dA(y;\varepsilon) = \frac{d}{dt}\Big|_{t=0} A(y+\varepsilon t)$$

Theorem: If A is a functional of the form

$$\int_a^b L(x, y(x), y'(x))\, dx$$

defined for function $y(x)$ s.t. $y(a) = y_a$ and $y(b) = y_b$

then $dA(y;\varepsilon) = \int_a^b \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)\right] \varepsilon(x)\, dx$.

The function y is a critical point of A iff the Euler-Lagrange eq. holds. i.e.

$$\boxed{\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = 0}$$

Proof: $dA(y;\varepsilon) = \frac{d}{dt}\Big|_{t=0} \int_a^b L(x, y+t\varepsilon, y'+t\varepsilon')\, dx =$

$$= \int_a^b \left(\frac{\partial L}{\partial y}\varepsilon + \frac{\partial L}{\partial y'}\varepsilon' \right) dx \quad \Big\} \text{ chain rule.}$$

$$\left(L(x, y+t\varepsilon, y'+t\varepsilon') = L(x,y,y') + \varepsilon t \frac{\partial L}{\partial y} + \varepsilon' t \frac{\partial L}{\partial y'} + o(t^2) \right)$$

But $\int_a^b \frac{\partial L}{\partial y'} \varepsilon'\, dx = \left[\cancel{\frac{\partial L}{\partial y'}\varepsilon} \right]_a^b - \int_a^b \varepsilon \cdot \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) dx =$

$$= -\int_a^b \varepsilon \cdot \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) dx$$

$$\Rightarrow dA(y;\varepsilon) = \int_a^b \left(\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)\right) \varepsilon(x)\, dx$$

So, by the Fundamental theorem of C of V.

$$\boxed{\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = 0}$$

Examples

① $L = \sqrt{1+(y')^2}$. Then $A(y) = \int_a^b \sqrt{1+(y')^2}\, dx$

i.e. this functional measures distance between (a, y_a) and (b, y_b).

$$\frac{\partial L}{\partial y} = 0, \quad \frac{\partial L}{\partial y'} = \frac{y'}{\sqrt{1+(y')^2}}$$

Euler-Lagrange equation

E-L gives $\cancel{\frac{\partial L}{\partial y}} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = 0 \Rightarrow \frac{d}{dx}\left(\frac{y'}{\sqrt{1+(y')^2}}\right) = 0$

Integrating, $\frac{y'}{\sqrt{1+(y')^2}} = c$

$$(y')^2 = c^2(1+(y')^2) \Rightarrow (y')^2 - c^2(y')^2 = c^2, \; (y')^2(1-c^2) = c^2, \; (y')^2 = \frac{c^2}{1-c^2}$$

$$\Rightarrow y' = \frac{c}{\sqrt{1-c^2}} = k$$

$$y' = \frac{dy}{dx} = k \Rightarrow y = kx + \ell$$

Using $y(a) = y_a$ and $y(b) = y_b$.

$$y = \frac{y_b - y_a}{b-a}(x-a) + y_a \quad (= y')$$

i.e. straight line. This means that the minimum distance between y_a and y_b is given by a straight line

(2) $L = \frac{1}{2}(my'^2 - ky^2)$

$$\frac{\partial L}{\partial y} = -ky \qquad \frac{\partial L}{\partial y'} = my'$$

E-L $\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = 0$

$$\Rightarrow -ky - \frac{d}{dx}(my') = 0 \; ; \; -ky - my'' = 0$$

$$\Rightarrow y'' = \frac{-k}{m} y$$

$$y = A \cdot \cos\sqrt{\frac{k}{m}}\,x + B\sin\sqrt{\frac{k}{m}}\,x$$

You can find A and B in terms of a, b, y_a, y_b etc.

Beltrami's Identity

Theorem: If $L(x, y, y')$ is independent of x and y is a solution of the E-L equation, then:

there is no x.

$$\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = 0$$

$$L(x, y, y') - y'\frac{\partial L}{\partial y'} = c \text{ (constant)}.$$

Proof:

Diff $L - y'\frac{\partial L}{\partial y'}$ w.r.t. x:

differenciate

$$\Rightarrow \frac{d}{dx}\left(L - y'\frac{\partial L}{\partial y}\right) = \frac{\partial L}{\partial x} + \frac{\partial L}{\partial y}\frac{dy}{dx} + \frac{\partial L}{\partial y'}\frac{dy'}{dx} - \frac{d}{dx}\left(y'\frac{\partial L}{\partial y'}\right) =$$

by assumption

$$y''\cdot\left(\frac{\partial L}{\partial y'} - \frac{\partial L}{\partial y'}\right) = y''\cdot 0$$

$$= \cancel{\frac{\partial L}{\partial x}}^{\,0} + \frac{\partial L}{\partial y}y' + \cancel{\frac{\partial L}{\partial y'}y''} - \cancel{y''\frac{\partial L}{\partial y'}} - y'\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) =$$

$$= y'\underbrace{\left[\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)\right]}_{E-L} = 0 \Rightarrow \boxed{L - y'\frac{\partial L}{\partial y'} = c}$$

If the derivative $= 0 \Rightarrow$ the function = constant.

Ex: Recall $L = \sqrt{1 + (y')^2}$

Here, $\frac{\partial L}{\partial x} = 0$.

$$\Rightarrow L - y'\frac{\partial L}{\partial y'} = c$$

$$\sqrt{1+(y')^2} - y'\cdot\frac{y'}{\sqrt{1+(y')^2}} = c \; ; \; \frac{1 + \cancel{(y')^2} - \cancel{(y')^2}}{\sqrt{1+(y')^2}} = c$$

$$\frac{1}{\sqrt{1+(y')^2}} = c.$$

$$\Rightarrow y' = k \quad \text{etc.}$$

Ex: Recall $L = \frac{1}{2}(m(y')^2 - ky^2)$

Here $\frac{\partial L}{\partial x} = 0$.

$$\Rightarrow L - y'\frac{\partial L}{\partial y'} = c.$$

$$\frac{1}{2}(my'^2 - ky^2) - my'^2 = C$$

$$\frac{1}{2}my'^2 + \frac{1}{2}ky^2 = -C \; ; \text{ multiply by: } \frac{2}{k}.$$

$$\frac{m}{k}y'^2 = -\frac{2c}{k} - y^2$$

$$\frac{y'}{\sqrt{\frac{-2c}{k} - y^2}} = \sqrt{\frac{k}{m}}$$

Put $y = \sqrt{\frac{-2c}{k}}\sin\theta \Rightarrow \frac{y'}{\sqrt{\frac{-2c}{k} + \frac{2c}{k}\sin^2\theta}} = \sqrt{\frac{k}{m}}$; Solve differential equation

Integrate $\theta = x\sqrt{\frac{K}{m}} + d$

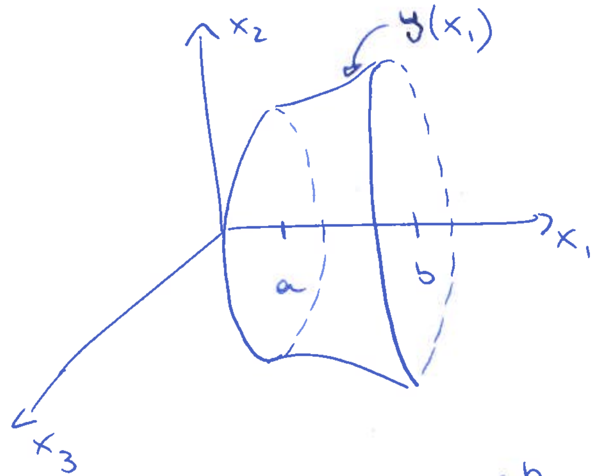
So $y = e \sin\left(\sqrt{\frac{K}{m}} + d\right)$.

Which is another way of writing $y = A\cos\sqrt{\frac{K}{m}}\,x + B\sin\sqrt{\frac{K}{m}}\,x$

Ex: Catenoid

Consider the surface of revolution $(x_1, x_2, x_3) \in \mathbb{R}^3$; $x_1 \in [a,b]$

and $\sqrt{x_2^2 + x_3^2} = y(x_1) > 0$ for $x_1 \in [a,b]$



Surface area: $A(y) = 2\pi \int_a^b y\sqrt{1+(y')^2}\,dx$

What function y minimises the surface for given y_a and y_b?

$L = y\sqrt{1+(y')^2}\,dx$.

Note $\frac{\partial L}{\partial x} = 0 \Rightarrow$ Beltrami identity $L - y'\frac{\partial L}{\partial y'} = C$

$$y\sqrt{1+(y')^2} - \frac{y'yy'}{\sqrt{1+(y')^2}} = C.$$

$$y\cdot\sqrt{1+(y')^2} - \frac{y'yy'}{\sqrt{1+(y')^2}} = C \Rightarrow \frac{y}{\sqrt{1+(y')^2}} = C.$$

$$y' = \sqrt{\frac{y^2}{C^2} - 1}$$

$$\int \frac{dy}{\sqrt{\frac{y^2}{C^2} - 1}} = x + d.$$

$y = C\cdot\cosh\theta$

$\vdots$

$\theta = \frac{x+d}{c}$

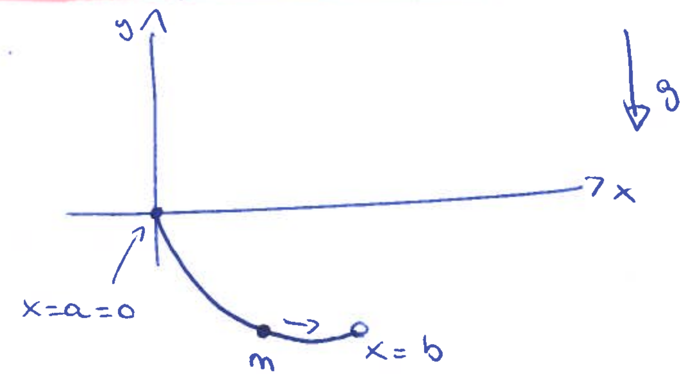
$y(x) = c\cdot\cosh\left(\frac{x+d}{c}\right)$

(finding c, d is messy)

Ex. Brachistochrone

Let a bead of mass m slide on a wire underneath the x-axis.
The shape of the wire is $y = y(x)$ and let w.l.o.g $y(0) = 0$

i.e.



The speed of the bead is $v(x)$ and it starts from rest at $x = y = 0$

ie. $v = 0$ at $y = 0$. $\left(v = \frac{dt}{ds}\right)$

Time taken to go from $a \to b$ is

$$A(y) = \int_a^b \frac{ds}{v(x)}$$ where $ds = \sqrt{1+(y')^2}\,dx$

arclength

velocity

But conservation of energy says $\frac{1}{2}mv^2 + gmy = 0$

$\Rightarrow v = \sqrt{-2gy}$

$$A(y) = \int_a^b \frac{\sqrt{1+(y')^2}}{\sqrt{-2gy}}\,dx$$

$$L = \frac{\sqrt{1+(y')^2}}{\sqrt{-2gy}}$$ note $\frac{\partial L}{\partial x} = 0 \Rightarrow$ Beltrami

$$\frac{\sqrt{1+(y')^2}}{\sqrt{-2gy}} - \frac{y'y'}{\sqrt{-2gy}\cdot\sqrt{1+(y')^2}} = c$$

or

$$\frac{1}{\sqrt{-2gy}\sqrt{1+(y')^2}} = c \ , \ \frac{1}{-2gy(1+(y')^2)} = c^2 \ , \ \frac{1}{-2gy - 2gy(y')^2} = c^2$$

$$\Rightarrow 1+(y')^2 = \frac{1}{-2c^2gy} \ \bigstar$$

$$\Rightarrow y' = \sqrt{\frac{1}{-2c^2gy} - 1}$$ saco "factor" común $\sqrt{\frac{-1}{y}}$

$$\frac{dy}{dx} = \sqrt{\frac{1}{-y}}\cdot\sqrt{\frac{1}{2c^2g}+y} = \frac{1}{\sqrt{-y}}\sqrt{\frac{1}{2c^2g}+y}$$

$$\int \frac{\sqrt{-y}\,dy}{\sqrt{\frac{1}{2c^2g}+y}} = x + d$$

Let $y = \frac{-\sin^2\theta}{2gc^2}$

⋮

$$\frac{x+d}{1} = \frac{-1}{2gc^2}\left[\sin^{-1}\sqrt{-2gc^2y} - \sqrt{2gc^2}\sqrt{-y}\sqrt{1+2gc^2y}\right]$$

reading week please check.

$x = y = 0 \Rightarrow d = 0$.

or Alternative parametric solution: from $\bigstar$

$$(1+(y')^2)(-y) = \frac{1}{2gc^2} = k^2 \quad (*)$$ because it's positive

has solution

$$\left.\begin{aligned} x &= \tfrac{1}{2}k^2(\theta - \sin\theta) \\ y &= \tfrac{1}{2}k^2(\cos\theta - 1)\end{aligned}\right\}$$ cycloid — curve traced out by a point on the rim of a rolling circle — borde aro

Check: $\frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta} = \frac{-\sin\theta}{1-\cos\theta} = -\cot\frac{\theta}{2}$

$$\text{LHS of } (*) = \left(1+\cot^2\frac{\theta}{2}\right)\frac{1}{2}k^2(1-\cos\theta)$$

$$= \frac{1}{\sin^2\frac{\theta}{2}}\cdot\frac{1}{\cancel{2}}k^2\left(\cancel{2}\sin^2\frac{\theta}{2}\right)$$

$$= k^2 = \text{RHS}$$

Extra example:

$$L = \frac{\sqrt{1+(y')^2}}{x} \quad \left(\frac{\text{dist}}{\text{speed}}\right)$$

Note that $\frac{\partial L}{\partial y} = 0$. (E-L)

$$\Rightarrow \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = 0$$

$$\frac{d}{dx}\frac{y'}{x\sqrt{1+(y')^2}} = 0 \ , \ \int \frac{d}{dx}\frac{y'}{x\sqrt{1+(y')^2}} = \int 0 \to 0$$

$$\Rightarrow \frac{y'}{x\sqrt{1+(y')^2}} = c$$

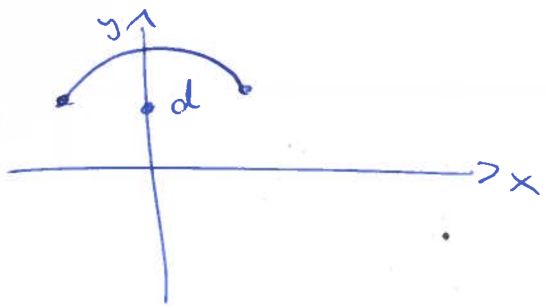
$$(y')^2 = c^2x^2(1+(y')^2) \ ; \ (y')^2 = c^2x^2 + c^2x^2(y')^2 \ ; \ (1-c^2x^2)(y')^2 = c^2x^2$$

$$\frac{dy}{dx} = \frac{cx}{\sqrt{1-c^2x^2}}$$

$$y - d = \frac{-\sqrt{1 - c^2x^2}}{c}$$

$$\Rightarrow c^2(y-d)^2 = 1 - c^2x^2$$

$$x^2 + (y-d)^2 = 1/c^2 \quad \Rightarrow \text{circle of radius } \tfrac{1}{c} \text{ lying on the } y\text{-axis}.$$



CHAPTER 6: THE EULER-LAGRANGE EQUATION II: CONSTRAINTS

$\underbrace{\text{Imposing constraints}}_{\text{restriction}}$ (i.e. "maximise ... subject to ...")

We only consider constraints of the form

$$G(y) = \int_a^b M(x, y, y')\,dx = 0$$

We consider modified functional $\tilde{A}(y, \lambda)$ of the form:

$$\tilde{A}(y, \lambda) = \int_a^b [L(x, y, y') - \lambda M(x, y, y')]\,dx$$

constraint

and find the extremal functions y for

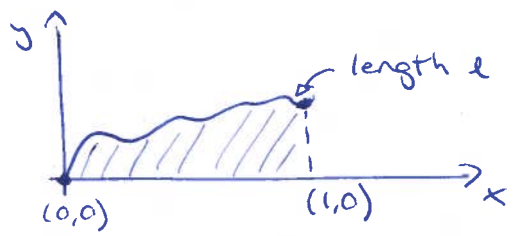
$$L(x, y, y') - \lambda M(x, y, y')$$

and finally, find λ by satisfying the constraint $G(y) = 0$.

(λ = constant, "Lagrange multipliers")

Why this works? - see moodle supplement notes (NFE).

Ex: Maximise the area under the curve $y(x)$, $y > 0$, going through $(0,0)$ and $(1,0)$ with given curve length ℓ.



Maximise $A(y) = \int_0^1 y\,dx$.

Subject to $G(y) = \int_0^1 \sqrt{1 + (y')^2}\,dx = \ell$

i.e. $$G^*(y) = \int_0^1 \sqrt{1 + (y')^2}\,dx - \int_0^1 \ell\,dx = $$

$$= \int_0^1 \Big[\underbrace{\sqrt{(1 + (y')^2)} - 1}_{M(x, y, y')}\Big]\,dx = 0$$

i.e. $$\tilde{A}(y, \lambda) = \int_0^1 \left[y - \lambda\sqrt{1 + (y')^2} + \lambda \ell\right] dx$$

So, $L = y - \lambda\sqrt{1 + (y')^2} + \lambda \ell$.

$$E\text{-}L \Rightarrow \frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = 0$$

$$1 - \frac{d}{dx}\left(\frac{-\lambda y'}{\sqrt{1 + (y')^2}}\right) = 0.$$

$$1 + \lambda \frac{d}{dx} \frac{y'}{\sqrt{1 + (y')^2}} = 0;$$

$$; \quad \frac{d}{dx}\left(\frac{y'}{\sqrt{1 + (y')^2}}\right) = -\frac{1}{\lambda}.$$

$$\frac{y'}{\sqrt{1 + (y')^2}} = -\frac{x}{\lambda} + d.$$

$$\vdots$$

$$\boxed{(y + E)^2 + (x + D)^2 = \lambda^2} \Rightarrow \text{circle}.$$

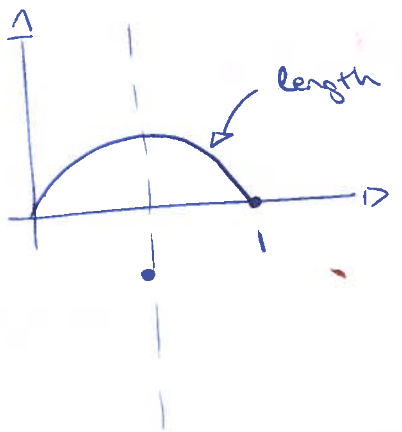
Now, find out the arbitrary constants:

$(0,0) \Rightarrow E^2 + D^2 = \lambda^2$

$(1,0) \Rightarrow E^2 + (D+1)^2 = \lambda^2$

$$\Rightarrow \boxed{D = -1/2, \quad E = \sqrt{\lambda^2 - \tfrac{1}{4}}}$$

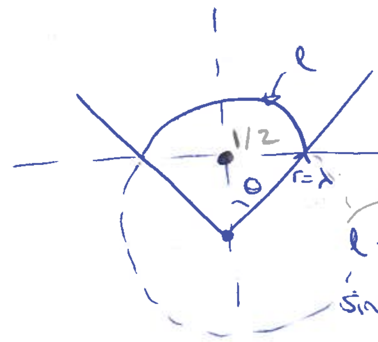
$$\left(y + \sqrt{\lambda^2 - \tfrac{1}{4}}\right)^2 + (x - 1/2)^2 = \lambda^2$$



Now, find λ using the constraint:

Could use $\underbrace{\int_0^1 \sqrt{1+(y')^2}\, dx = \ell}$.

Así encuentro ℓ y después lo meto



Perímetro $= 2\pi r$.

$\ell = 2 \cdot \lambda\theta$

$\sin\theta = 1/2\lambda$

$\Rightarrow \frac{\ell}{2\lambda} = \theta = \sin^{-1}\left(\frac{1}{2\lambda}\right)$

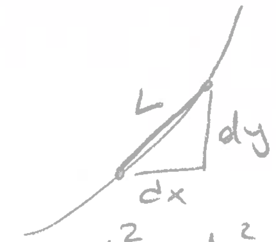
$$\Rightarrow \boxed{2\lambda \sin\left(\frac{\ell}{2\lambda}\right) = 1}$$

Ex: (Catenary) Consider a chain hanging above the x-axis with endpoints (a, y_a) and (b, y_b). It hangs s.t its potential energy minimised. If the chain has uniform density ρ (kgm^{-1}) then the segment lying over an infinitesimal segment dx has mass $\rho\sqrt{1+(y')^2}\,dx$. The potential energy is ("mgh")

$\rho g y \sqrt{1+(y')^2}\, dx$

So we minimise

$$A(y) = \int_a^b \rho g y \sqrt{1+(y')^2}\, dx$$



$L^2 = dx^2 + dy^2$

$L = \sqrt{dx^2 + dy^2}$

$L = dx\sqrt{1 + \frac{dy^2}{dx^2}} = dx\sqrt{1+(y')^2}$

The chain has fixed length ℓ. so $\longrightarrow$

$$G(y) = \int_a^b \sqrt{1+(y')^2}\, dx = \ell \longleftarrow \text{can ignore this constant in the E-L eq.}$$

$\Rightarrow G(y) = \int dx \sqrt{1+(y')^2}$ → length of all circle

$$\tilde{A}(y,\lambda) = \int_a^b \underbrace{\left[\rho g y\sqrt{1+(y')^2} - \lambda\sqrt{1+(y')^2}\right]}_{L} dx$$

Beltrami: $\frac{\partial L}{\partial x} = 0$.

$\Rightarrow L - y'\frac{\partial L}{\partial y'} = c$.

$$\rho g y\sqrt{1+(y')^2} - \lambda\sqrt{1+(y')^2} - y'\left\{\frac{y'}{\sqrt{1+(y')^2}}(\rho g y - \lambda)\right\} = c$$

$\vdots$

$$\Rightarrow \frac{(\rho g y - \lambda)^2}{c^2} - 1 = (y')^2$$

$$\frac{dy}{dx} = \sqrt{\frac{(\rho g y - \lambda)^2}{c^2} - 1}$$

$$\cosh\theta = \frac{\rho g y - \lambda}{c^2} \Rightarrow x = \frac{c}{\rho g}\cosh\left[\frac{\rho g y - \lambda}{c}\right] - E$$

so
$$\boxed{y = \frac{c}{\rho g}\cosh\left(\frac{\rho g}{c}(x+E)\right) + \frac{\lambda}{\rho g}}$$

Catenary curve.

To find λ it is messy.

CHAPTER 7: THE EULER-LAGRANGE EQUATION III: MORE VARIABLES

Theorem 7.1

Let V be the space of functions $\underset{\sim}{y} : [a,b] \to \mathbb{R}^n$ satisfying the BC's $\underset{\sim}{y}(a) = \underset{\sim}{y}_a$ and $\underset{\sim}{y}(b) = \underset{\sim}{y}_b$.

We will write $\underset{\sim}{y}(x) = (y_1(x), y_2(x), \dots, y_n(x))$. Let A be a functional defined by a Lagrangian

$$A(\underset{\sim}{y}) = \int_a^b L(x, y_1(x), \dots, y_n(x), y_1'(x), \dots, y_n'(x))\, dx$$

Let $\underset{\sim}{\varepsilon}(x)$ be a function s.t. $\underset{\sim}{\varepsilon}(a) = \underset{\sim}{\varepsilon}(b) = 0$.

$\Rightarrow$ The Gâteaux derivative of A at y in the $\underset{\sim}{\varepsilon}$ direction

$$dA(\underset{\sim}{y}; \underset{\sim}{\varepsilon}) = \sum_{i=1}^{n} \int_a^b \left[\frac{\partial L}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial L}{\partial y_i'}\right) \right] \varepsilon_i(x)\, dx$$

which vanishes $\forall\, \underset{\sim}{\varepsilon}$ iff the n E-L equation holds.

$$\frac{\partial L}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial L}{\partial y_i'}\right) = 0 \qquad i = 1, \dots, n$$

Proof: as before

Ex: Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be a curve with coordinates $\gamma(t) = (x(t), y(t))$.

We assume γ is a closed curve

$$\gamma(t + 2\pi) = \gamma(t)$$

We assume it has action

$$\int_0^{2\pi} (\dot{x}^2 + \dot{y}^2)\, dt = K,$$

and we want to maximise the area it bounds.

By Green's theorem the area U is

$$\int_U dx\,dy = \oint_{\partial U} x\, dy \quad \text{or} \quad \int_0^{2\pi} x(t)\dot{y}(t)\, dt$$

∂U — boundary of U

Therefore, we find critical points of the constrained problem:

$$\int_0^{2\pi} \left[x\dot{y} - \lambda(\dot{x}^2 + \dot{y}^2 - K) \right] dt$$

$\Rightarrow L(t, x, y, \dot{x}, \dot{y})$

Two E-L equations:

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0 \Rightarrow \dot{y} - \frac{d}{dt}(-\lambda 2\dot{x}) = 0$$

$$\frac{\partial L}{\partial y} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = 0 \Rightarrow -\frac{d}{dt}(x - 2\lambda\dot{y}) = 0$$

$\Rightarrow \ddot{x} = -\dot{y}/2\lambda$

$\ddot{y} = \dfrac{\dot{x}}{2\lambda}$

$\Rightarrow \dddot{x} = -\dot{x}/4\lambda^2 \quad \left(\dfrac{d\ddot{x}}{dx} = \dfrac{d(-\dot{y}/2\lambda)}{dx} \right) \Rightarrow \dddot{x} = -\dfrac{1}{2\lambda}\cdot\ddot{y} = -\dfrac{\dot{x}}{4\lambda}$

$\dddot{y} = -\dfrac{\dot{y}}{4\lambda^2}$

$$\ddot{x} + \frac{x}{4\lambda^2} = c$$

$$x - c = C\cos\frac{t}{2\lambda} + D\sin\frac{t}{2\lambda}$$

$$y - d = A\cos\frac{t}{2\lambda} + B\sin\frac{t}{2\lambda}$$

But $\ddot{x} = -\dfrac{\dot{y}}{2\lambda} \Rightarrow C = B,\ D = -A$.

Also $\ddot{y} = \dfrac{\dot{x}}{2\lambda} \Rightarrow -A = D,\ B = C$

$$y - d = A\cdot\cos\frac{t}{2\lambda} + B\cdot\sin\frac{t}{2\lambda}$$

$$x - c = B\cdot\cos\frac{t}{2\lambda} - A\sin\frac{t}{2\lambda}.$$

$$\Rightarrow (x-c)^2 + (y-d)^2 = A^2\cos^2\frac{t}{2\lambda} + 2AB\cos\frac{t}{2\lambda}\sin\frac{t}{2\lambda} + B^2\sin^2\frac{t}{2\lambda}$$
$$+ B^2\cos^2\frac{t}{2\lambda} - 2AB\cos\frac{t}{2\lambda}\sin\frac{t}{2\lambda} + A^2\cos^2\frac{t}{2\lambda} =$$
$$= A^2 + B^2 \Rightarrow \boxed{\text{CIRCLE}}$$

Recall: $L = L(x, \underline{y}, \underline{y}')$

$$\frac{\partial L}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial L}{\partial y_i'}\right) = 0 \qquad i = 1, \ldots, n.$$

Note: if $\frac{\partial L}{\partial y_i} = 0 \Rightarrow \frac{d}{dx}\left(\frac{\partial L}{\partial y_i'}\right) = 0 \Rightarrow \frac{\partial L}{\partial y_i'} = \text{constant}$.

In mechanics if $\frac{\partial L}{\partial y_i} = 0 \Rightarrow y_i$ is called "ignorable". The presence of an ignorable coordinate immediately implies the existence of a conservation law $\frac{\partial L}{\partial y_i'} = \text{constant}$ $\left(\frac{\partial L}{\partial y_i'} = \text{generalised notation}\right)$

Symmetry $\Leftrightarrow$ conservation

Ex: $L = L(t, \theta, r, \dot\theta, \dot r) = \underbrace{\frac{m}{2}(\dot r^2 + r^2\dot\theta^2)}_{\text{Kinetic energy}} - \underbrace{V(r)}_{\text{potential}}$

Note: $\frac{\partial L}{\partial \theta} = 0$ θ is ignorable

$\frac{\partial L}{\partial \dot\theta} = \text{constant} \Rightarrow mr^2\dot\theta^2 = \text{constant}$

i.e. angular momentum conservation

Beltrami's identity for vector-valued functions.

$$\frac{d}{dx}\left(L - y'\frac{\partial L}{\partial y'}\right) = \underbrace{\cancelto{0}{\frac{\partial L}{\partial x}} + \frac{\partial L}{\partial y_i}\frac{\partial y_i}{\partial x} + \frac{\partial L}{\partial y_i'}\cdot\frac{\partial y_i'}{\partial x}}_{\frac{dL}{dx}} - \frac{d}{dx}\left(y_i'\frac{\partial L}{\partial y_i'}\right)$$

by assumption (i.e. $\frac{\partial L}{\partial x}$ for Beltrami).

Summation over i is implied since from $i = 1, \ldots, n$.

$$= \frac{\partial L}{\partial y_i}y_i' + \frac{\partial L}{\partial y_i'}y_i'' - y_i''\frac{\partial L}{\partial y_i'} - y_i'\frac{d}{dx}\left(\frac{\partial L}{\partial y_i'}\right) =$$

$$= y_i'\underbrace{\left[\frac{\partial L}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial L}{\partial y_i'}\right)\right]}_{E-L \Rightarrow = 0} = 0$$

i.e. $L - y_i'\frac{\partial L}{\partial y_i'} = \text{const}$ (sum over i).

Ex: $L = \frac{m}{2}(\dot r^2 + r^2\dot\theta^2) - V(r)$

Note: here $L = L(t, r, \theta, \dot r, \dot\theta)$

Also note $\frac{\partial L}{\partial t} = 0 \Rightarrow$ I can apply Beltrami's identity

$$\underbrace{L - \dot r\frac{\partial L}{\partial \dot r} - \dot\theta\frac{\partial L}{\partial \dot\theta}}_{-y_i'\frac{\partial L}{\partial y_i'}} = \text{const} \qquad \begin{cases} y_1 = r \\ y_2 = \theta \end{cases}$$

$$\frac{m}{2}(\dot r^2 + r^2\dot\theta^2) - V - \dot r(m\dot r) - \dot\theta(mr^2\dot\theta) = \text{const}.$$

$$\frac{1}{2}m(\dot r^2 + r^2\dot\theta^2) + V = \text{const}.$$

$\frac{\partial L}{\partial t} = 0 \Rightarrow$ conservation of energy.

Functions of several variables: (proof NFE)

Let ϕ be a m-variable function defined $U \subset \mathbb{R}^m$ where U has smooth boundary ∂U; $\phi: U \to \mathbb{R}$.

Suppose $\phi(\underline{x}) = \phi_0(\underline{x})$ for $\underline{x} \in \partial U$.

Perturbation: $\varepsilon(\underline{x})$ satisfy $\varepsilon(\underline{x}) = 0$ for $\underline{x} \in \partial U$.

Our Lagrangian will depend on $\underline{x} = (x_1, \ldots, x_m)$, $\phi(\underline{x})$, and $\frac{\partial \phi}{\partial x_i}$, $i = 1, \ldots, m$

$$A(y) = \int_U L(\underline{x}, \phi, \nabla\phi)\, dx_1 dx_2 \cdots dx_m$$

many integrals $\int\ldots\int$

$\nabla\phi = \left(\frac{\partial \phi}{\partial x_1}, \ldots, \frac{\partial \phi}{\partial x_m}\right)$

Theorem:

$$dA(\phi;\varepsilon) = \int_U \left(\frac{\partial L}{\partial \phi} - \sum_{i=1}^{m} \frac{\partial}{\partial x_i} \frac{\partial L}{\partial\left(\frac{\partial \phi}{\partial x_i}\right)} \right) \varepsilon(\underline{x})\, dx_1 \cdots dx_m$$

which vanishes $\forall\, \varepsilon(\underline{x})$ iff $\displaystyle \frac{\partial L}{\partial \phi} - \sum_{i=1}^{m} \frac{\partial}{\partial x_i} \frac{\partial L}{\partial\left(\frac{\partial \phi}{\partial x_i}\right)} = 0$.

Proof: NFE.

$$\frac{d}{dt}\bigg|_{t=0} L(\phi + t\varepsilon) = \frac{\partial L}{\partial \phi}\varepsilon + \sum_{i=1}^{m} \frac{\partial \varepsilon}{\partial x_i} \frac{\partial L}{\partial\left(\frac{\partial \phi}{\partial x_i}\right)}$$

Integrating over U gives the Gâteaux derivative of the action:

$$d'A(y;\varepsilon) = \int_U \left(\varepsilon \frac{\partial L}{\partial \phi} + \underbrace{\sum_{i=1}^{m} \frac{\partial \varepsilon}{\partial x_i} \frac{\partial L}{\partial\left(\frac{\partial \phi}{\partial x_i}\right)}}_{\nabla \varepsilon \cdot \frac{\partial L}{\partial(\nabla \phi)}} \right) dx_1 \cdots dx_m$$

Now, $\displaystyle \underline{\nabla} \cdot \left(\varepsilon \frac{\partial L}{\partial(\underline{\nabla}\phi)} \right) = \underline{\nabla}\varepsilon \cdot \frac{\partial L}{\partial(\underline{\nabla}\phi)} + \varepsilon\, \underline{\nabla} \cdot \frac{\partial L}{\partial(\underline{\nabla}\phi)}$ | $\underline{\nabla} \cdot (f \underline{a}) = \underline{\nabla}(f) \cdot \underline{a} + f\, \underline{\nabla} \cdot \underline{a}$

But, $\displaystyle \int_U \underline{\nabla} \cdot \left(\varepsilon \frac{\partial L}{\partial(\underline{\nabla}\phi)} \right) dx_1 \cdots dx_m = \int_{\partial U} \varepsilon \cdot \frac{\partial L}{\partial(\underline{\nabla}\phi)}\, \hat{n}\, dS$ $\Rightarrow$ c.f. Divergence theorem.

$= 0$.

$$\Rightarrow \int_U \underline{\nabla}\varepsilon \cdot \frac{\partial L}{\partial(\underline{\nabla}\phi)} dx_1 \cdots dx_m = -\int_U \varepsilon \cdot \underline{\nabla} \cdot \frac{\partial L}{\partial(\underline{\nabla}\phi)} dx_1 \cdots dx_m$$

So
$$dA(\phi;t) = \int_U \varepsilon \left[\frac{\partial L}{\partial \phi} - \underline{\nabla} \cdot \frac{\partial L}{\partial(\underline{\nabla}\phi)} \right] dx_1 \cdots dx_m$$

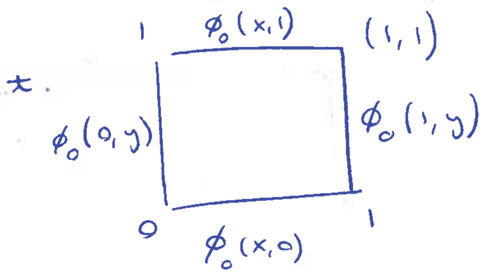
$$= \int_U \varepsilon \left[\frac{\partial L}{\partial \phi} - \sum_{i=1}^{m} \frac{\partial}{\partial x_i} \frac{\partial L}{\partial\left(\frac{\partial \phi}{\partial x_i}\right)} \right] dx_1 \cdots dx_m$$

$= 0$ (for an extremum).

By fundamental theorem of calculus of variations we have:

$$\frac{\partial L}{\partial \phi} - \sum_{i=1}^{m} \frac{\partial}{\partial x_i} \frac{\partial L}{\partial\left(\frac{\partial \phi}{\partial x_i}\right)} = 0.$$

Ex: $U = [0,1]^2$ (square) and s.t.



$m = 2$, $x_1 = x$, $x_2 = y$.

E-L equation is

$$\boxed{\frac{\partial L}{\partial \phi} - \frac{\partial}{\partial x}\frac{\partial L}{\partial(\phi_x)} - \frac{\partial}{\partial y}\frac{\partial L}{\partial(\phi_y)} = 0} \qquad \begin{cases} \phi_x = \dfrac{\partial \phi}{\partial x} \\ \phi_y = \dfrac{\partial \phi}{\partial y} \end{cases}$$

$$A(\phi) = \int_0^1\!\!\int_0^1 \left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 \right] dx\,dy \leftarrow \text{real flows so minimise this quantity.}$$

$L = \phi_x^2 + \phi_y^2$, so $\dfrac{\partial L}{\partial \phi} = 0$, $\dfrac{\partial L}{\partial \phi_x} = 2\phi_x$, $\dfrac{\partial L}{\partial \phi_y} = 2\phi_y$.

So E-L gives

$$0 - \frac{\partial}{\partial x}(2\phi_x) - \frac{\partial}{\partial y}(2\phi_y) = 0$$

or
$$\left.\begin{aligned} &\phi_{xx} + \phi_{yy} = 0 \\ &\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \\ &\nabla^2 \phi = 0 \\ \phi_{xx} + \phi_{yy} = \; &\Delta \phi = 0 \end{aligned}\right\} \text{LAPLACE'S EQUATION}$$

Ex: $A(\phi) = \iint_{\mathcal{U}} \sqrt{1+\phi_x^2+\phi_y^2}\, dx\,dy$

$L = \sqrt{1+\phi_x^2+\phi_y^2} = \sqrt{1+|\nabla\phi|^2}$

E-L says: $\frac{\partial L}{\partial \phi} - \frac{\partial}{\partial x}\frac{\partial L}{\partial \phi_x} - \frac{\partial}{\partial y}\frac{\partial L}{\partial \phi_y} = 0$

$$\frac{\partial}{\partial x}\frac{\phi_x}{\sqrt{1+|\nabla\phi|^2}} + \frac{\partial}{\partial y}\frac{\phi_y}{\sqrt{1+|\nabla\phi|^2}} = 0$$

$\vdots$

$$\frac{1}{(1+|\nabla\phi|^2)^{3/2}}\left[\phi_{xx}(1+\phi_y^2) + \phi_{yy}(1+\phi_x^2) - 2\phi_x\phi_y\phi_{xy}\right] = 0$$

So the equation for a minimal surface is

$$\phi_{xx}(1+\phi_y^2) + (1+\phi_x^2)\phi_{yy} - 2\phi_x\phi_y\phi_{xy} = 0.$$

$\phi = ax+by+c \Rightarrow$ trivial solution

$\phi = \tan^{-1}(y/x)$ helicoid

$\phi = \frac{1}{a}\cosh^{-1}(a\sqrt{x^2+y^2})$ catenoid $(a = \text{constant})$

$\phi = \frac{1}{a}\log\frac{\cos(ay)}{\cos(ax)}$ Scherk surface

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22nd November 2019

PART IV Partial differential equations

Chapter 8: Method of characteristics I: linear case

Linear change of coordinates:

Ex: $\phi(x,y)$: $\frac{\partial\phi}{\partial x} = 0 \Rightarrow \phi = c(y)$ ← arbitrary function

Ex: $\phi(x,y)$:

$$\frac{\partial\phi}{\partial x} - \frac{\partial\phi}{\partial y} = 0. \quad (*)$$

Consider the LHS and note:

$$\frac{\partial\phi}{\partial u} = \frac{\partial x}{\partial u}\cdot\frac{\partial\phi}{\partial x} + \frac{\partial y}{\partial u}\cdot\frac{\partial\phi}{\partial y} \quad \text{(chain rule)}$$

Now, if $\frac{\partial x}{\partial u} = 1$ and $\frac{\partial y}{\partial u} = -1 \Rightarrow$ LHS of $(*)$ $= \frac{\partial\phi}{\partial u}$.

So let's change to a new (linear) coordinates (u,v) satisfying $\frac{\partial x}{\partial u} = 1$ and $\frac{\partial y}{\partial u} = -1$.

For example,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \Rightarrow \begin{matrix} x = u \\ y = -u+v \end{matrix} \quad \left\{ \begin{matrix} \frac{\partial x}{\partial u} = 1 \\ \frac{\partial y}{\partial u} = -1 \end{matrix} \right.$$

you choose this column

or

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 7 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \Rightarrow \begin{matrix} x = u+7v \\ y = -u \end{matrix}.$$

The only conditions that need to be met:

- First column of the matrix is (1) and (−1) i.e. the coefficients of $\frac{\partial\phi}{\partial x}$ and $\frac{\partial\phi}{\partial y}$ in $(*)$.
- The matrix is invertible.

Let's use $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$

so inverting,

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \begin{cases} u = x \\ v = x + y \end{cases}.$$

$$\frac{\partial \varphi}{\partial u} = \frac{\partial x}{\partial u} \cdot \frac{\partial \varphi}{\partial x} + \frac{\partial y}{\partial u} \cdot \frac{\partial \varphi}{\partial y} = \frac{\partial \varphi}{\partial x} - \frac{\partial \varphi}{\partial y}.$$

$(*) \Rightarrow \frac{\partial \varphi}{\partial u} = 0 \Rightarrow \varphi = c(v)$.

But $v = x+y \Rightarrow \boxed{\varphi = c(x+y)}$ eg: $\varphi = \sin(x+y)$.

What if: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 7 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 0 & 7 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow v = \frac{1}{7}(x+y)$$

So $\frac{\partial \varphi}{\partial u} = 0 = \frac{\partial x}{\partial u} \cdot \frac{\partial \varphi}{\partial x} + \frac{\partial y}{\partial u} \cdot \frac{\partial \varphi}{\partial y} =$

$$= \frac{\partial \varphi}{\partial x} - \frac{\partial \varphi}{\partial y}.$$

$$\Rightarrow \boxed{\varphi = c(v) = c\left(\frac{x+y}{7}\right)} = D(x+y)$$

D is another arbitrary function.

Example: $\frac{\partial \varphi}{\partial x} - \frac{\partial \varphi}{\partial y} = x$

Use the transformation as before:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}; \quad \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{cases} u = x \\ v = x+y \end{cases}.$$

We get $\frac{\partial \varphi}{\partial u} = x$,

$$\frac{\partial \varphi}{\partial u} = u.$$

$$\varphi = \frac{1}{2} u^2 + C(v)$$

$$\boxed{\varphi = \frac{1}{2} x^2 + C(x+y)}.$$

In general, the approach works for any PDE $\varphi(x_1, x_2, \dots, x_n)$ of the

$$\sum_{i=1}^{n} A_i \frac{\partial \varphi}{\partial x_i} = 0 \quad (A_i = \text{const}).$$

In new coordinates $(u_1, \dots, u_n)$

$$\frac{\partial \varphi}{\partial u_1} = \sum_{i=1}^{n} \frac{\partial x_i}{\partial u_i} \cdot \frac{\partial \varphi}{\partial x_i},$$ so choose $\frac{\partial x_i}{\partial u_i} = A_i$. Thus is

a suitable change of coordinates

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} A_1 & \cdots & * \\ \vdots & & \\ A_n & \cdots & * \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$$

where $*$ can be anything so long as the matrix is invertible.

So the PDE:

$$\sum_{i=1}^{n} A_i \frac{\partial \varphi}{\partial x_i} = \frac{\partial \varphi}{\partial u_i} = 0,$$ so

$\varphi = C(u_1, \dots, u_n)$ for C arbitrary function

Ex: $\frac{\partial\varphi}{\partial x} + 2\frac{\partial\varphi}{\partial y} = \sin(y)$

Use $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \Rightarrow y = 2u+v$.

you choose.

$\Rightarrow \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{cases} u = x \\ v = -2x+y \end{cases}$

$\Rightarrow \frac{\partial\varphi}{\partial u} = \sin y$

$= \sin(2u+v)$

$\varphi = -\frac{1}{2}\cos(2u+v) + C(v)$.

$$\boxed{\varphi = -\frac{1}{2}(\cos(y)) + C(-2x+y)}$$

Boundary conditions are needed to find C.

Ex: Solve $\frac{\partial\varphi}{\partial x} + 2\frac{\partial\varphi}{\partial y} = \sin(y)$ subject to $\varphi(s,0) = s^2$.

General solution:

$\varphi(x,y) = -\frac{1}{2}\cos(y) + C(-2x+y)$.

so $\varphi(s,0) = s^2$.

$\Rightarrow s^2 = -\frac{1}{2}\cdot 1 + C(-2s+0)$

$= -\frac{1}{2} + C(-2s)$.

let $w = -2s$; $s = w/-2$.

$\Rightarrow C(w) = \frac{w^2}{4} + \frac{1}{2}$.

$\Rightarrow \varphi(x,y) = -\frac{1}{2}\cos(y) + \frac{1}{4}(-2x+y)^2 + \frac{1}{2}$.

Nonlinear change of coordinates

Ex: Plane polar $\longleftrightarrow$ cartesian

$x = r\cdot\cos\theta$
$y = r\cdot\sin\theta$ and $\theta = \tan^{-1}(y/x)$.

$$\frac{\partial\varphi}{\partial r} = \frac{\partial\varphi}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial\varphi}{\partial y}\cdot\frac{\partial y}{\partial r} = \cos\theta\cdot\frac{\partial\varphi}{\partial x} + \sin\theta\,\frac{\partial\varphi}{\partial y} =$$

$$= \frac{x}{r}\frac{\partial\varphi}{\partial x} + \frac{y}{r}\frac{\partial\varphi}{\partial y}$$

So, for example, the PDE $\frac{\partial\varphi}{\partial r} = 0$.

$x\frac{\partial\varphi}{\partial x} + y\frac{\partial\varphi}{\partial y} = 0$.

has the solution $\varphi = C(\theta)$.

But $\theta = \tan^{-1}(y/x)$, so $\varphi = F(y/x) \Rightarrow F$ arbitrary function

Characteristic vector field

Lemma: Given an expression of the form: $\sum_{i=1}^{n} A_i(x_1,\dots,x_n)\frac{\partial\varphi}{\partial x_i}$,

(non-constant coefficients)

Suppose $\exists\,(u_1,\dots,u_n)$ s.t. $\frac{\partial x_i}{\partial u_1} = A_i(x_1,\dots,x_n)$,

then $\frac{\partial\varphi}{\partial u_1} = \sum_{i=1}^{n} A_i(\underline{x})\frac{\partial\varphi}{\partial x_i}$

Proof chain rule:

$$\frac{\partial\varphi}{\partial u_1} = \sum_{i=1}^{n}\frac{\partial\varphi}{\partial x_i}\frac{\partial x_i}{\partial u_1} = \sum_{i=1}^{n} A_i(\underline{x})\frac{\partial\varphi}{\partial x_i}$$

So, how do we find $(u_1,\dots,u_n)$ s.t.

$\frac{\partial x_i}{\partial u_1} = A_i(x)$?

Ex: Consider

$$x \frac{\partial \varphi}{\partial x} + y \frac{\partial \varphi}{\partial y} = 0 .$$

$x = A_1$, $y = A_2$

we want to solve

$$\begin{pmatrix} \partial x / \partial u \\ \partial y / \partial u \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \frac{\partial x}{\partial u} = x \Rightarrow \int \frac{1}{x} \partial x = \int \partial u, \quad \ln x = u + c, \quad x = e^u \underbrace{e^c}_{A}$$

A solution is: $x = Ae^u$, $y = B \cdot e^u$

$[A = A(v) ; B = B(v)]$.

Now, let $A = 1$ and $B = v$ to get the coordinate transform

$x = e^u$, $y = ve^u$

$$\Rightarrow u = \log x, \quad v = \frac{y}{e^{\log x}} = y/x .$$

So, by the lemma,

$$\frac{\partial \phi}{\partial u} = 0 \Rightarrow \phi = c(v) = c(y/x) .$$

Definition: Consider a PDE of the form $A(x,y)\frac{\partial \varphi}{\partial x} + B(x,y)\frac{\partial \varphi}{\partial y} + C(x,y)\varphi + D(x,y) = 0$. (inhomogeneous linear).

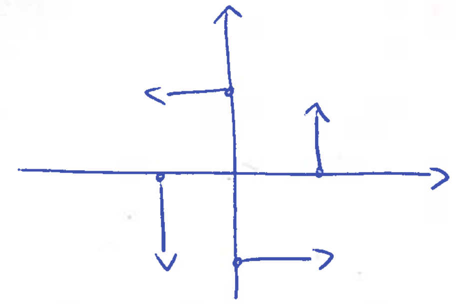
The differential equation (Characteristic vector field)

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} A(x,y) \\ B(x,y) \end{pmatrix} \qquad \text{"}\cdot\text{"} = \frac{\partial}{\partial u} .$$

are called the characteristic equations, and a curve $(x(u), y(u))$ satisfying these equations is called characteristic curve.

The method for solving PDEs of this form is called method of characteristics.

Ex: $-y \frac{\partial \varphi}{\partial x} + x \frac{\partial \varphi}{\partial y} = 0$ has characteristic vector field $\begin{pmatrix} -y \\ x \end{pmatrix}$



Characteristic equations: $\dot{x} = \frac{\partial x}{\partial u} = -y$

$\dot{y} = \frac{\partial y}{\partial u} = x$.

$$\Rightarrow \ddot{x} = -\dot{y} = -x \Rightarrow \ddot{x} + x = 0 \left[\Rightarrow \lambda = \dot{x} \Rightarrow \lambda^2 + 1 = 0, \quad \lambda = \begin{cases} \lambda_1 = -i \\ \lambda_2 = i \end{cases} \right]$$

$\Rightarrow x = C\cos(u) + D\sin(u)$

$y = C\sin(u) - D\cos(u)$

Choose $D = 0$, $C = v$

$\Rightarrow (x,y) = (v\cos u, v\sin u)$.

$\Rightarrow v = r = \sqrt{x^2 + y^2}$

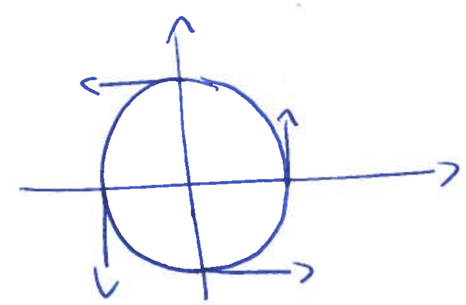
$u = \tan^{-1}(y/x)$

$x = v\cos(u)$, $y = v\sin(u)$ (r, θ) $\Rightarrow x/v = \cos(u)$, $y/v = \sin(u)$ $\Rightarrow tg(u) = \frac{y}{x} \Downarrow u = \tan^{-1}(y/x)$

So our PDE becomes (by lemma)

$$\frac{\partial \varphi}{\partial u} = 0 \Rightarrow \varphi = F(v) = F\left(\sqrt{x^2 + y^2}\right) .$$

Remark:



characteristic curve: $x(u) = C \cdot \cos(u)$, $y(u) = C\sin(u)$ is a circle given by C.

Ex: $x \frac{\partial \phi}{\partial x} - \frac{\partial \phi}{\partial y} = 0$.

Characteristic vector field $(x, -1)$

$\Rightarrow \dot{x} = x$, $\dot{y} = -1$.

ODE's.

$\Rightarrow x = Ce^{u}$, $y = D - u$.

Choose $c = v$, $D = 0$.

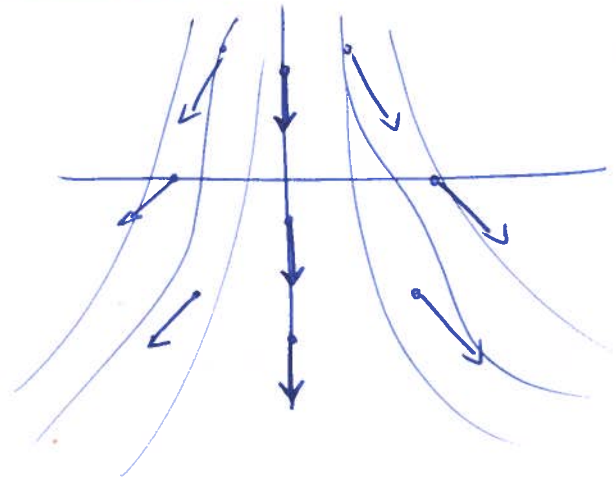
$\Rightarrow x = ve^{u}$, $y = -u$.

$\Rightarrow v = xe^{-u} = xe^{y}$.

So, by the Lemma,

$$\frac{\partial \varphi}{\partial u} = \left(x \frac{\partial \phi}{\partial x} - \frac{\partial \varphi}{\partial y} \right) = 0$$

$\varphi = C(v) = C(xe^{y})$



Ex: Recall

$$x \frac{\partial \phi}{\partial x} - \frac{\partial \phi}{\partial y} = 0.$$

characteristic vector field $(x, -1)$.

$\begin{cases} \dot{x} = x \\ \dot{y} = -1 \end{cases} \Rightarrow x = Ce^{u}$, $y = D - u$.

$\boxed{1^{st}}$ Choose $D = 0$, $c = v$

$\Rightarrow x = v.e^{u}$, $y = -u$.

$v = xe^{y}$

$\phi = C(v) = c(xe^{y})$.

$\boxed{2^{nd}}$ Different choice repeating the above example,

$C = 1$, $D = v$.

$\Rightarrow x = e^{u} \rightarrow x > 0$.

$y = v - u$.

$v = y + u = y + \log x$.

$\frac{\partial \phi}{\partial u} = 0 \Rightarrow \varphi = C(v) = C(y + \log x)$.

$= C(\log(e^{y}x))$

$= D(e^{y}x)$

= Solution

Ex: $\frac{1}{x} \frac{\partial \varphi}{\partial x} - y \frac{\partial \varphi}{\partial y} = 0$.

$\begin{cases} \dot{x} = \frac{1}{x} \\ \dot{y} = -y. \end{cases} \Rightarrow \begin{cases} \frac{\partial x}{\partial u} = \frac{1}{x} \Rightarrow \int x \, \partial x = \int \partial u, \; \frac{x^2}{2} = u + C \\ \frac{\partial y}{\partial u} = -y \Rightarrow \int -\frac{1}{y} \partial y = \int \partial u, \; -\ln|y| = u + D; \end{cases}$

$\begin{cases} x^2 = (u + C)2, \; x = \sqrt{2u + A} \\ \ln y = -u - D \Rightarrow y = e^{(-u-D)} = Be^{-u} \end{cases}$

Choose $A=0 \quad B=v$.

$\Rightarrow \quad u = x^2/2$

$v = y\, e^{x^2/2}$

$$\frac{\partial \phi}{\partial u} = 0 \Rightarrow \varphi = C(v).$$

$$\boxed{\varphi = C\left(y e^{x^2/2}\right)}$$

Ex: $\frac{1}{x}\frac{\partial \phi}{\partial x} - y \frac{\partial \phi}{\partial y} = y$.

Skip first steps since they are as above,

$$\Rightarrow \frac{\partial \phi}{\partial u} = y = v e^{-u}$$

$\Rightarrow$ Integrating $\varphi = -v e^{-u} + C(v)$.

$$\varphi(x,y) = -y e^{x^2/2} e^{-x^2/2} + C\left(y e^{x^2/2}\right) = -y + C\left(y e^{x^2/2}\right).$$

Ex: $\frac{\partial \phi}{\partial x} + 2x \frac{\partial \phi}{\partial y} = 1$.

$$\begin{cases} \dot{x} = 1 \Rightarrow \frac{\partial x}{\partial u} = 1 \ ; \ x = u + A \\ \dot{y} = 2x \Rightarrow \frac{\partial y}{\partial u} = 2x = 2(u+A) \end{cases}$$

$$\Rightarrow \partial y = (2u+2A)\partial u,$$

$$; \ y = (u+A)^2 + B.$$

Choose $A=0$, $B=v$. $\Rightarrow$ $y = u^2 + v$. $x = u$

$$\begin{cases} u = x \\ v = y - x^2. \end{cases}$$

Now, $\frac{\partial \phi}{\partial u} = 1$

$$\Rightarrow \boxed{\varphi = u + C(v) = x + C(y - x^2)}$$

$\Downarrow$ see that It works.

Ex: $y \frac{\partial \phi}{\partial x} + x \frac{\partial \phi}{\partial y} = 0$.

$$\begin{cases} \dot{x} = y \\ \dot{y} = x \end{cases} \Rightarrow \text{WAY TO APPROACH IT = diff. again}$$

$\ddot{x} = \dot{y} = x$ ODE's

$$\ddot{x} = x \Rightarrow x = A e^{u} + B e^{-u}$$

$$\dot{x} = y = A e^{u} - B e^{-u}$$

Note, $x^2 - y^2 = 4AB$. So if A or $B = 0 \Rightarrow \underbrace{x^2 - y^2 = 0}_{\text{two lines}}$.

[And I need curves, not 2 lines].

$\Rightarrow$ let's try something more involved, $B=1, A=v$.

$$\Rightarrow \begin{aligned} x &= v e^{u} + e^{-u} \\ y &= v e^{u} - e^{-u} \end{aligned}$$

$$\Rightarrow x - y = 2e^{-u} > 0.$$

This choice holds only for $x > y$.

$$\begin{cases} u = -\log\left(\frac{x-y}{2}\right) \\ v = ? \end{cases}$$

$$e^{u} v = x - e^{-u} = x - \left(\frac{x-y}{2}\right) = \frac{x+y}{2}$$

$$v = \left(\frac{x+y}{2}\right) e^{-u} = \left(\frac{x+y}{2}\right)\cdot\left(\frac{x-y}{2}\right) = \frac{x^2-y^2}{4}$$

$$\frac{\partial \phi}{\partial u} = 0 \Rightarrow \boxed{\varphi = C(v) = C\left(\frac{x^2-y^2}{4}\right)}$$

Question: If we try $B=-1$ and $A=v$?

We get = answers

December 7^{th} 2019

Ex: Solve $y\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y} + xy\,\phi = 0$

Subject to $\phi(s,1) = \sin(s)$

$$\begin{cases}\dot{x} = y \\ \dot{y} = x\end{cases} \quad \left(\text{"}\cdot\text{"} = \frac{d}{du}\right) \implies \begin{aligned} x &= Ae^{u} + Be^{-u} \\ y &= Ae^{u} - Be^{-u}\end{aligned}$$

Choose $B=1$, $A=v$

$u = -\log\left(\frac{1}{2}(x-y)\right)$

$v = \frac{x^2-y^2}{4}$

$$\implies \frac{\partial\phi}{\partial u} = -xy\phi = -\left(v^2e^{2u} - e^{-2u}\right)\phi$$

$$\log\phi = -\frac{v^2}{2}e^{2u} - \frac{1}{2}e^{-2u} + C(v)$$

$$= -\frac{1}{2}\left[\left(\frac{x^2-y^2}{4}\right)^2\frac{4}{(x^2-y^2)^2} + \frac{(x-y)^2}{4}\right] + C\left(\frac{x^2-y^2}{4}\right)$$

$$= -\frac{1}{4}(x^2+y^2) + C\left(\frac{x^2-y^2}{4}\right)$$

$$\phi = e^{-(x^2+y^2)/4}\cdot k\left(\frac{x^2-y^2}{4}\right) \qquad k = e^{c\left(\frac{x^2-y^2}{4}\right)}$$

Now, $\phi(s,1) = \sin(s)$

$$\implies e^{-(s^2-1)/4}\,k\left(\frac{s^2-1}{4}\right) = \sin(s)$$

$$\implies k\left(\frac{s^2-1}{4}\right) = \sin s\; e^{(s^2+1)/4}$$

$$w = \frac{s^2-1}{4} \implies s = \sqrt{1+4w}\;;\quad \frac{s^2+1}{4} = w + \frac{1}{2}$$

$$k(w) = \sin\sqrt{1+4w}\;e^{w+1/2}$$

$$\phi(x,y) = e^{-(x^2+y^2)/4}\sin\sqrt{1+x^2-y^2}\;e^{\left(\frac{x^2-y^2}{4}+\frac{1}{2}\right)}$$

Linear $A\frac{\partial\phi}{\partial x} + B\frac{\partial\phi}{\partial y} + C\phi = 0$.

If there's $\left(\frac{\partial\phi}{\partial y}\right)^2$ is non-linear or $B=\phi$ neither

CHAPTER 9: METHOD OF CHARACTERISTICS II

Quasilinear case

Consider $A(x,y,\phi)\frac{\partial\phi}{\partial x} + B(x,y,\phi)\frac{\partial\phi}{\partial y} + C(x,y,\phi) = 0$.

⚠ $e^x\frac{\partial\phi}{\partial x} + \phi\left(\frac{\partial\phi}{\partial y}\right)^2 + 2\phi = 0$ is $\boxed{\text{not}}$ quasilinear

(term B will depend on (x,y,ϕ,ϕ^2))

but

$e^x\frac{\partial\phi}{\partial x} + \phi\left(\frac{\partial\phi}{\partial y}\right) + 2\phi = 0$ $\boxed{\text{is}}$

i.e. coefficients A, B and C are "smooth" and can now depend on ϕ as well x and y.

$\left(\frac{\partial\phi}{\partial x}\right)^1 \implies$ first order.

The equation is linear in $\frac{\partial\phi}{\partial x}$ and $\frac{\partial\phi}{\partial y}$ but not necessarily in ϕ: this is "quasilinear".

Definition If ϕ is a solution for some $(x,y)\in\mathcal{U}\subset\mathbb{R}^2$ then the graph of ϕ is the set of points

$$\{(x,y,\phi(x,y)) : (x,y)\in\mathcal{U}\}$$

i.e. the surface in $\mathbb{R}^3 = \{(x,y,z) ; x,y,z\in\mathbb{R}\}$ cut out by $z = \phi(x,y)$.

eg. $z = 1-x^2-y^2$, $x,y\in\mathcal{U}\in(-1,1)\times(-1,1)$.

Definition The characteristic vector field of

$$A(x,y,\phi)\frac{\partial\phi}{\partial x} + B(x,y,\phi)\frac{\partial\phi}{\partial y} + C(x,y,\phi) = 0.$$

is $(A,B,-C)$

A characteristic curve is a solution $(x(t), y(t), z(t))$ to

$$\frac{dx}{dt} = A$$

$$\frac{dy}{dt} = B$$

$$\frac{dz}{dt} = -C.$$

Definition A 1-parameter family of characteristic curves (it is a surface) is a smooth map $\mathbb{R}^2 \supset U \ni (s,t) \longrightarrow (x(s,t), y(s,t), z(s,t)) \in \mathbb{R}^3$ where, for fixed s, each curve $(x(s,t), y(s,t), z(s,t))$ is a characteristic curve.

The image of this map $z = \phi(x,y)$ is a surface — the solution surface.

Definition: A surface in $\mathbb{R}^3$ is a graph if it is of the form $z = \phi(x,y)$.

Note:
- eg: $\{y = 0\} \in \mathbb{R}^3$ is not a graph.
- $\{z^2 = 1\}$ is not a graph. {union of 2 graphs $z = +1$, $z = -1$}

Theorem: Let $(s,t) \longrightarrow (x_s(t), y_s(t), z_s(t))$ be a solution surface which is the graph of a function ϕ. Then ϕ is a solution of $A\frac{\partial \phi}{\partial x} + B\frac{\partial \phi}{\partial y} + C\phi = 0$

Proof Since the surface is a graph, we have

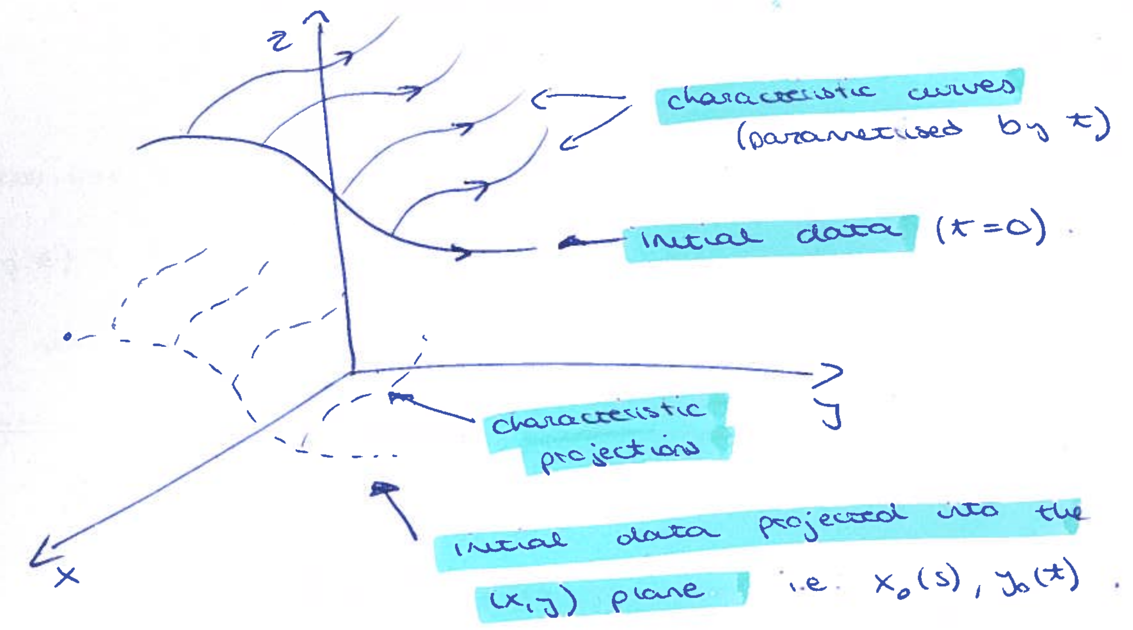
$$z_s(t) = \phi(x_s(t), y_s(t))$$

fix s and differentiate w.r.t. t:

$$\frac{dz}{dt} = \frac{\partial \phi}{\partial x} \cdot \frac{dx_s}{dt} + \frac{\partial \phi}{\partial y}\frac{dy}{dt}$$

and since the solution surface is a 1-parameter family of characteristic curve $\frac{dx_s}{dt} = A$, $\frac{dy_s}{dt} = B$, $\frac{dz_s}{dt} = -C$

$$\Longrightarrow -C = A\frac{\partial \phi}{\partial x} + B\frac{\partial \phi}{\partial y}$$ as required $\square$



Ex: $\frac{\partial \phi}{\partial x} = 0$ with I.C. $\phi(0,s) = s$.

$\dot{x} = 1 \Longrightarrow$ (because of the coefficient of $\frac{\partial \phi}{\partial x}$)

$\dot{y} = 0 \Longrightarrow$ (because of coefficient of $\frac{\partial \phi}{\partial y}$) $\Longrightarrow$ $x = t + a$, $y = b$, $z = c$.

$\dot{z} = 0$

Fixing a, b, c and letting t vary, gives the characteristic curves.

i.e. The characteristic curves are a "bunch" of straightness through (a,b,c) parallel to the x-axis.

$\phi(x,y) \Longrightarrow x = 0,\ y = s,\ z = \phi(x,y)$

Now, $\phi(0,s) = s$ at $t = 0$.

$\Longrightarrow x = 0 + a = 0$

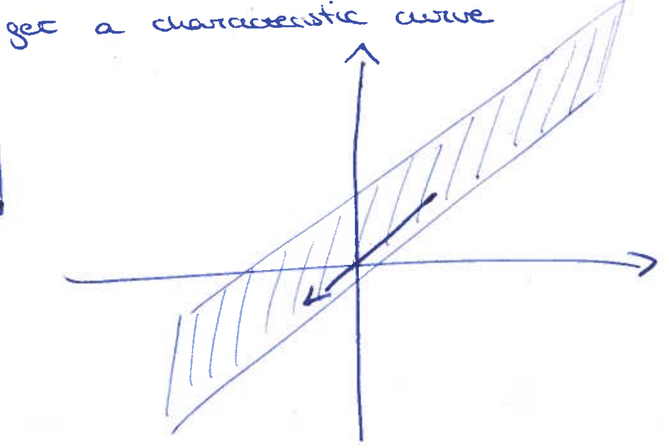
$y = b = s$

$z = c = s$.

For each value of s, we get a characteristic curve

$(x,y,z) = (t, s, s)$.

$\Longrightarrow$ surface $\boxed{y = z}$



Ex: Burger's equation

$$\frac{\partial \phi}{\partial x} + \phi \frac{\partial \phi}{\partial y} = 0 , \quad \phi(0,s) = s .$$

$A = 1$, $B = \phi$, $c = 0$.

$$\begin{cases} \dot{x} = 1 \\ \dot{y} = \phi = z \\ \dot{z} = 0 \end{cases} \Rightarrow \left.\begin{aligned} x &= t + c \\ y &= at + b \\ z &= a \end{aligned}\right\} \quad z = \phi(x,y)$$

How do we get the solution surface

i.e. choose a, b, c? Use $\phi(0,s) = s$

$x \quad y \quad z$

when $t = 0$ $\Rightarrow$ $z = s$, $y = s$, $x = 0$.

$\Rightarrow$ $a = s$, $b = s$, $c = 0$.

Solution surface: $(x, y, z) = (t, st + s, s)$

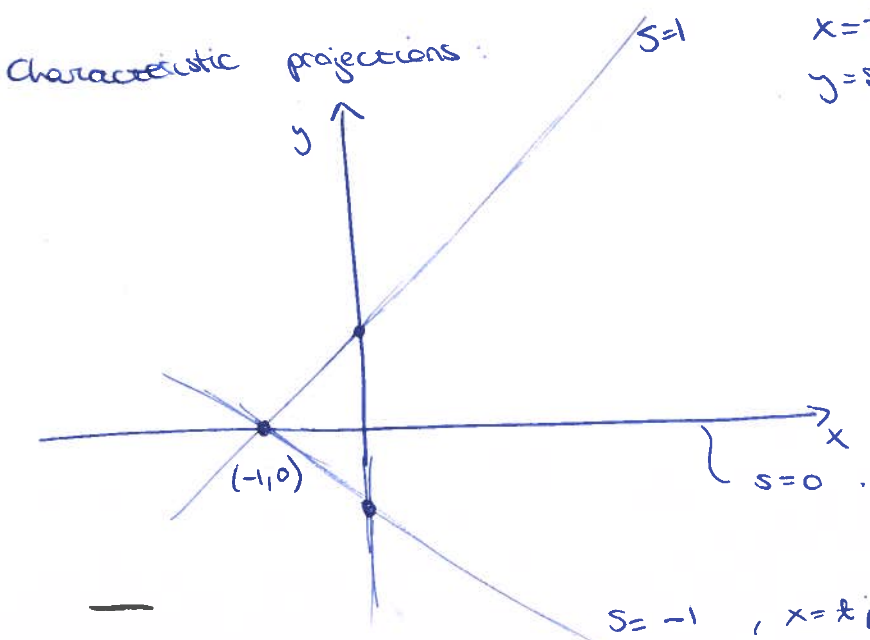
Express $z = \phi(x,y)$.

$$z = s = \frac{y}{t+1} = \boxed{\frac{y}{x+1} = z}$$

$y = st + s$.

Characteristic projections:

$x = t$
$y = st + s$.



$S = -1$, $x = t$ y $s = t - 1$

Ex: Solve $\dfrac{\partial \phi}{\partial x} + \phi \dfrac{\partial \phi}{\partial y} = 0$ with $\phi(0,s) = s^2$.

Characteristic curves:

$(x, y, z) = (t + c, at + b, a)$.

but now, $\phi(0,s) = s^2$

$$\Rightarrow \left.\begin{aligned} t + c &= 0 \\ at + b &= s \\ a &= s^2 \end{aligned}\right\} \text{ put } t = 0 .$$

$\Rightarrow$ $c = 0$, $b = s$, $a = s^2$

Surface $(x, y, z) = (t, s^2 t + s, s^2)$

Express z in terms of (x,y):

$y = s^2 t + s$, $x = t$.

$\Rightarrow$ $xs^2 + s - y = 0$.

$\Leftrightarrow$ $s = \dfrac{-1 \pm \sqrt{1 + 4xy}}{2x}$

$$z = s^2 = \left(\frac{-1 \pm \sqrt{1+4xy}}{2x} \right)^2$$

Note, this fails to be well-defined wherever

(i) $x = 0$ (ii) $1 + 4xy < 0$ ← near here the characteristic projections begin to cross each other.

Extra example: Solve $\dfrac{\partial \phi}{\partial x} + y\phi \dfrac{\partial \phi}{\partial y} = \phi$

for $\phi(x,y)$ in $x > 0$ subject to $\phi = y$ when $x = 0$, $0 < y < 1$.

$$\Rightarrow \frac{dx}{dt} = 1$$

$$\frac{dy}{dt} = y\phi$$

$$\frac{dz}{dt} = -(\phi) = \phi .$$

recall $z = \phi(x,t) \Rightarrow \dfrac{d\phi}{dt} = \phi \Rightarrow \phi = ce^t$

$$\begin{cases} x = t + a \\ \dfrac{dy}{dt} = yce^t \Rightarrow \log y = ce^t + k \\ \qquad y = \exp(ce^t + k) \end{cases}$$

Now, find c, a, k using our IC, $t = 0$ initial condition

i.e. $\begin{cases} y = s \quad (0 < s < 1) \\ x = 0 \\ \phi = s \end{cases}$

$$\Rightarrow \begin{cases} a = 0 \\ s = \exp(c + k) \\ s = c \end{cases}$$

from $s = e^{(c+k)} = e^c \cdot e^k = s$

Thus, $e^k = s/e^s$

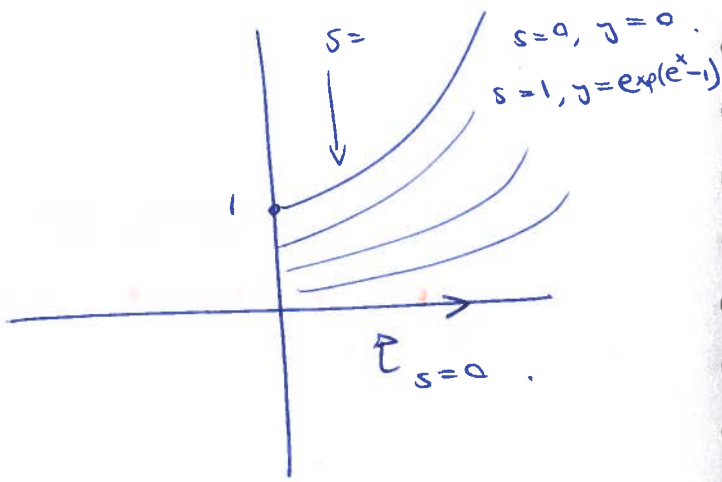
$x = t \quad \phi = ce^t$

$\phi = se^t = s \cdot e^t$

$y = \exp(se^t)\exp(k)$

$= \exp(s \cdot e^t)\dfrac{s}{e^s}$

$= s \cdot \exp(s(e^t - 1))$



$\phi \dfrac{\partial \phi}{\partial x} + y \dfrac{\partial \phi}{\partial y} = \sin \phi$ quasilinear

$\dfrac{\partial \phi}{\partial x} + y \dfrac{\partial \phi}{\partial y} = \sin \phi$ quasilinear

no hay función de ϕ, es ϕ directamente

$\dfrac{\partial \phi}{\partial x} + y \dfrac{\partial \phi}{\partial y} = \phi \Rightarrow$ linear

$\dfrac{\partial \phi}{\partial x} + y \left(\dfrac{\partial \phi}{\partial y}\right)^2 = \phi \Rightarrow$ non linear

Caustics

Let $(s,t) \to (x(s,t), y(s,t), z(s,t))$ be a surface in $\mathbb{R}^3$.

The surface has vertical tangency if some linear combination of the vectors:

$$\begin{pmatrix} \partial x/\partial s \\ \partial y/\partial s \\ \partial z/\partial s \end{pmatrix} \text{ and } \begin{pmatrix} \partial x/\partial t \\ \partial y/\partial t \\ \partial z/\partial t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Projecting the set of points the surface has vertical tangency to the (x,y)-plane gives the caustic of the surface.

The surface has projection $\Pi(s,t) = (x(s,t), y(s,t))$ in the (x,y)-plane. (s,t) is a critical point of Π if

$$\det \begin{pmatrix} \partial x/\partial s & \partial y/\partial s \\ \partial x/\partial t & \partial y/\partial t \end{pmatrix} = 0.$$

Lemma: The set of critical points contains the caustic of the surface

Typically, the characteristic projections start to cross each other near a caustic.

often $|\ \ | = 0$ gives a curve and corresponds to when ϕ folds over itself. $\to$ ceases to be single valued.

This is to be avoided in physical application, eg: ϕ = pressure, multi-valued pressure is unphysical.

Ex: Consider the surface $(t, st+s, s)$

(cf earlier example: $z = y/_{x+1}$)

$\pi(s,t) = (t, st+s)$

$$\begin{vmatrix} \partial x/\partial s & \partial y/\partial s \\ \partial x/\partial t & \partial y/\partial t \end{vmatrix} = \begin{vmatrix} 0 & t+1 \\ 1 & s \end{vmatrix} = -(t+1)$$

vanishes when $t=-1 \Rightarrow x=-1$

Ex: Consider the surface (t, s^2t+s, s^2)

$\pi(s,t) = (t, s^2t+s)$

$$\begin{vmatrix} \partial x/\partial s & \partial y/\partial s \\ \partial x/\partial t & \partial y/\partial t \end{vmatrix} = \begin{vmatrix} 0 & 2st+1 \\ 1 & s^2 \end{vmatrix} = -2st-1$$

$\Rightarrow$ vanishes when $s = -1/2t$, $y = \frac{1}{4t} - \frac{1}{2t} = -\frac{1}{4t} = \frac{-1}{4x}$

$y = -1/4x$

$\Rightarrow$ $4xy+1=0$ gives the caustic, and is the "bad" locus found earlier.

Ex: $-\sin\phi \dfrac{\partial\phi}{\partial x} + \cos\phi \dfrac{\partial\phi}{\partial y} = 1 \qquad \phi(s,0)=0.$

This has characteristic field $\begin{pmatrix} -\sin\phi \\ \cos\phi \\ 1 \end{pmatrix}$ $(\phi = z)$

and so characteristic curves are found from

$$\begin{cases} \dfrac{dx}{dt} = -\sin z \\ \dfrac{dy}{dt} = \cos z \\ \dfrac{dz}{dt} = 1 \Rightarrow z = t + C. \end{cases} \Rightarrow \begin{cases} x = \cos(t+c)+a \\ y = \sin(t+c)+b. \end{cases}$$

$\Rightarrow$ these are helices spiralling up as $t\uparrow$.

Now $\phi(s,0) = 0 \Rightarrow z=0$ when $x=s$, $y=0$

Choosing $t=0 \Rightarrow \begin{cases} z = c = 0 \Rightarrow \boxed{c=0} \\ s = x = \cos(0)+a = 1+a \,; \boxed{a = s-1} \\ 0 = y = \sin(0)+b \,; \boxed{b=0} \end{cases}$

$(s,t) \longrightarrow (\cos t + s - 1, \underbrace{\sin t}_{y}, \underbrace{t}_{z})$ $\Rightarrow$ comes from def of caustics

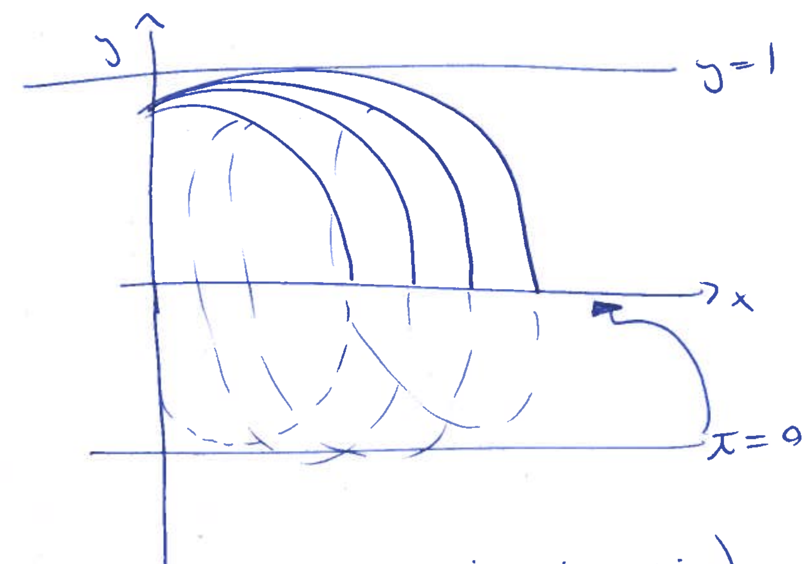
Note, $z = \sin^{-1} y$ $y=1$, $z = \frac{\pi}{2}, \frac{5\pi}{2}, \dots$

$t = \sin^{-1} y$.

Characteristic projections: $\underbrace{(\cos t + s - 1, \sin t)}$

i.e. $(x+(1-s))^2 + y^2 = 1$. $\cos^2 t + \sin^2 t = 1$, $\sin t = y$, $\cos t = (x-(1-s))$

circles centre $(1-s, 0)$ radius 1.



Critical values of the projection (caustics)

$$\begin{vmatrix} x_s & y_s \\ x_t & y_t \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -\sin t & \cos t \end{vmatrix} = \cos t = 0$$

when $t = \pi/2 + n\pi$

$y = \pm 1$ caustic.

CHAPTER 10 : Method of characteristics III NFE

December 13^{th} 2019

Ex : sheet 8 exercise 2a) : NON-EXAMINABLE .

$\phi_y + \phi_x^2 = 0$. Non quasilinear or linear : It is a 1^{st} order partial differential equation .

BC's : $\phi(s,0) = s$.

In the form $G(x,y,\phi,\phi_x,\phi_y)=0$ → Let $u=\phi$, $p=\phi_x$, $q=\phi_y$.

↳ $G=0$; $G=p^2+q=0$.

The solution $u=\phi(x,y)$ to the PDE $G=0$ is equivalent to :

$$\frac{dx}{d\tau}=\frac{\partial G}{\partial p} \qquad \dot p = \frac{\partial G}{\partial x} - p\frac{\partial G}{\partial u}$$

$$\frac{dy}{d\tau}=\frac{\partial G}{\partial q} \qquad \dot q = -\frac{\partial G}{\partial y} - q\frac{\partial G}{\partial u}$$

$$\dot u = \frac{\partial G}{\partial \phi}\dot p + \frac{\partial G}{\partial q}\cdot \dot q$$

Usual to have initial data , $x(s), y(s)$ and $u(s)$ on $t=0$.

⇒ Need IC's for p and q. These IC's satisfy

(i)

(ii) $\underline{du}$

$$\frac{dx}{dt}=2p \qquad \frac{dp}{dt}=0$$

$$\frac{dy}{dt}=1 \qquad \frac{dq}{dt}=0 .$$

$$\frac{du}{dt}=2p^2+q .$$

Initial data :

$x(s)=s$

$y(s)=0$

$u(s)=s$.

ψ_1 and ψ_2 ?

$$\left.\begin{array}{ll}(i) & \psi_1^2+\psi_2^2=0 \\ (ii) & 1=\psi_1+\psi_2\cdot 0\end{array}\right\} \begin{array}{l}\psi_1=1 \\ \psi_2=-1\end{array}$$

Substituting and simplifying

$$\frac{dp}{dt}=0 \Rightarrow p=cte=\psi_1=1$$

$$\frac{dy}{dt}=0 \Rightarrow q=cte=\psi_2=-1$$

$$\frac{dx}{dt}=2p=2\psi_1=2 \Rightarrow x=2t+a .$$

$$\frac{dy}{dt}=1 \Rightarrow y=t+b$$

$$\frac{du}{d\tau}=2p^2+q=2\psi_1^2+\psi_2=1 \Rightarrow u=t+c$$

At $t=0$.

$$\left.\begin{array}{l}x=s=a\\ y=b=0\\ u=c=s\end{array}\right\} \Rightarrow \begin{array}{l}x=2t+s\\ y=t\\ u=t+s .\end{array}$$

$$u=\phi(x,y)=t+s .$$
$$=y+s$$
$$=y+x-2y .$$

$$\Rightarrow \boxed{\phi = x-y}$$

CHAPTER 11 : D'ALEMBERT'S METHOD Examinable .

Recall The wave equation

$$\frac{1}{c^2}\cdot\frac{\partial^2\phi}{\partial t^2}=\frac{\partial^2\phi}{\partial x^2}$$

c = wave speed .

Let $X_+ = x+ct$ and $X_- = x-ct$.

$$x=\frac{X_+ + X_-}{2} \quad \text{and} \quad t=\frac{1}{2c}(X_+ - X_-)$$

Note :
$$\frac{\partial}{\partial X_+}=\frac{\partial x}{\partial X_+}\frac{\partial}{\partial x}+\frac{\partial t}{\partial X_+}\frac{\partial}{\partial t}=\frac{1}{2}\left(\frac{\partial}{\partial x}+\frac{1}{c}\frac{\partial}{\partial t}\right) \to b$$

$$\frac{\partial}{\partial X_-}=\frac{1}{2}\left(\frac{\partial}{\partial x}-\frac{1}{c}\frac{\partial}{\partial t}\right) \to a$$

So,
$$\underbrace{\frac{\partial^2}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}}_{\text{multiplying } a\cdot b}=4\frac{\partial^2}{\partial X_+\partial X_-}$$

The wave equation becomes

$$\frac{\partial^2\phi}{\partial x_+\partial x_-}=0.$$

Integrating wrt x_- : $\int \frac{\partial^2\phi}{\partial x_+\cancel{\partial x_-}}\cancel{\partial x_-}$

$$\frac{\partial\phi}{\partial x_+}=C(x_+)$$

Integrating wrt x_+

$$\int \frac{\partial^2\phi}{\partial x_+\partial x_-}\cancel{\partial x_+} \Rightarrow \phi = C_+(x_+)+C_-(x_-)$$

$$= C_+(x+ct)+C_-(x-ct).$$

Ex: Solve the wave equation with IC's

$\phi(x,0)=e^{-x^2}$, $\frac{\partial\phi}{\partial t}(x,0)=0$.

General solution $\phi = C_-(x-ct)+C_+(x+ct)$

$\Rightarrow e^{-x^2}=C_-(x)+C_+(x)$. (1)

$\frac{\partial\phi}{\partial t} \Rightarrow -cC_-'(x)+cC_+'(x)=0$. (2)

Integrate (2) $\Rightarrow$ $-cC_-(x)+cC_+(x)=k\cdot c$

constant.

or $C_+=k+C_-$

(1) $\Rightarrow$ $C_-+C_+=2C_-+k=e^{-x^2}$

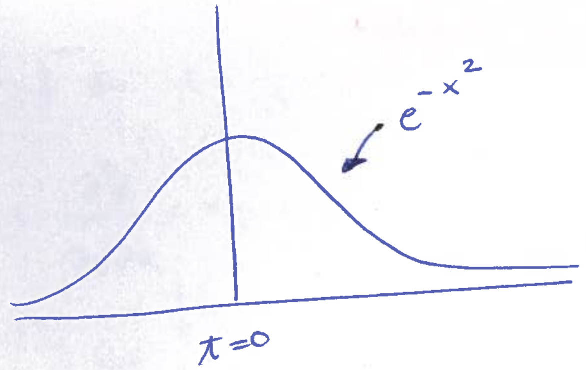
$\Rightarrow C_-(x)=\frac{1}{2}\left(e^{-x^2}-k\right)$

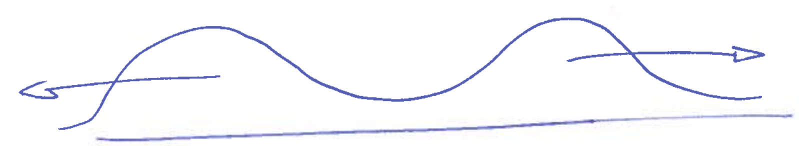
$C_+(x)=\frac{1}{2}\left(e^{-x^2}+k\right)$

$\Rightarrow \phi=\frac{1}{2}\left(e^{-(x+ct)^2}+k\right)+\frac{1}{2}\left(e^{-(x-ct)^2}-k\right)$

Cancel

$=\frac{1}{2}e^{-(x+ct)^2}+\frac{1}{2}e^{-(x-ct)^2}$





Note: k cancels out: this is always the case.

$\Rightarrow$ No need to include k in our calculation.

Hyperbolic equations

The wave equation belongs to a class of hyperbolic, 2nd-order linear PDE's.

They have general form

$$A\frac{\partial^2\phi}{\partial x^2}+B\frac{\partial^2\phi}{\partial x\partial y}+C\frac{\partial^2\phi}{\partial y^2}=D(x,y).$$

where A, B and C are constants.

We will find coordinates (s,t) s.t. the PDE becomes

$$A\frac{\partial^2\phi}{\partial s\,\partial t}=0 \quad (A\neq 0).$$

ie. $A\frac{\partial^2\phi}{\partial s\partial t}=A\frac{\partial^2\phi}{\partial x^2}+B\frac{\partial^2\phi}{\partial x\partial y}+C\frac{\partial^2\phi}{\partial y^2}$

Lemma: Suppose $x=s+t$, $y=-\beta s-\alpha t$

$$\Rightarrow \frac{\partial^2}{\partial s\partial t}=\frac{\partial^2}{\partial x^2}-(\alpha+\beta)\frac{\partial^2}{\partial x\partial y}+\alpha\beta\frac{\partial^2}{\partial y^2}.$$

Proof: chain rule

$$\frac{\partial}{\partial s}=\frac{\partial x}{\partial s}\frac{\partial}{\partial x}+\frac{\partial y}{\partial s}\frac{\partial}{\partial y}=\frac{\partial}{\partial x}-\beta\frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial t}=\frac{\partial}{\partial x}-\alpha\frac{\partial}{\partial y}.$$

So, $\frac{\partial^2}{\partial s\partial t}=\left(\frac{\partial}{\partial x}-\beta\frac{\partial}{\partial y}\right)\cdot\left(\frac{\partial}{\partial x}-\alpha\frac{\partial}{\partial y}\right)=\frac{\partial^2}{\partial x^2}-(\alpha+\beta)\frac{\partial^2}{\partial x\partial y}+\alpha\beta\frac{\partial^2}{\partial y^2}$

Choose α,β s.t. $\frac{B}{A}=-\alpha-\beta$, $\frac{C}{A}=\alpha\beta$.

Lemma: If α and β are roots of the quadratic

$$AT^2 + BT + C = 0$$

$$\Rightarrow \frac{B}{A} = -\alpha - \beta \quad \text{and} \quad \frac{C}{A} = \alpha\beta$$

Proof: $AT^2 + BT + C = A(T-\alpha)\cdot(T-\beta)$
$= A(T^2 - (\alpha-\beta)T + \alpha\beta)$

$$\Rightarrow \frac{C}{A} = \alpha\beta \quad \text{and} \quad \frac{B}{A} = -\alpha-\beta.$$

Definition

A PDE: $A\phi_{xx} + B\phi_{xy} + C\phi_{yy} = D$.

is: (i) hyperbolic: $B^2 - 4AC > 0$.
(ii) parabolic: $B^2 - 4AC = 0$
(iii) elliptic: $B^2 - 4AC < 0$.

Idea: If $A\phi_{xx} + B\phi_{xy} + C\phi_{yy} = D(x,y)$ is a hyperbolic PDE and α, β are roots of $AT^2 + BT + C = 0$, then, under a change of coordinates

$$x = s + t \qquad y = -\beta s - \alpha t$$

then PDE simplifies to: $A\dfrac{\partial^2\phi}{\partial s\,\partial t} = D$.

Example $\dfrac{\partial\phi^2}{\partial x^2} + 5\dfrac{\partial\phi}{\partial x\partial y} + 4\dfrac{\partial\phi^2}{\partial y^2} = xy$.

$T^2 + 5T + 4 = 0$.
$(T+4)(T+1) = 0$.
$\alpha = -4 \quad \beta = -1$

Change coordinates $\begin{cases} x = s+t \\ y = -\beta s - \alpha t \\ \;\; = s + 4t. \end{cases}$

[so $t = \frac{1}{3}(y-x) \qquad s = \frac{1}{3}(4x-y)$]

$$\frac{\partial^2\phi}{\partial s\partial t} = xy = s^2 + 5st + 4t^2.$$

By integrating wrt. t

$$\frac{\partial\phi}{\partial s} = s^2 + \frac{5}{2}st^2 + \frac{4}{3}t^3 + g(s)$$

By integrating w.r.t. s — constants

$$\phi = \frac{1}{3}s^3t + \frac{5}{4}s^2t^2 + \frac{4}{3}st^3 + G(s) + H(t).$$

$$\phi(x,y) = \frac{1}{81}\left[\frac{1}{3}(4x-y)^3(y-x) + \frac{5}{4}(4x-y)^2(y-x)^2 + \frac{4}{3}(4x-y)(y-x)^3\right] + G\left(\frac{4x-y}{3}\right) + H\left(\frac{y-x}{3}\right)$$

Ex: $\dfrac{\partial^2\phi}{\partial x^2} + 5\dfrac{\partial^2\phi}{\partial x\partial y} + 4\dfrac{\partial^2\phi}{\partial y^2} = 0$.

subject to $\phi(x,0) = x$ and $\dfrac{\partial\phi}{\partial y}(x,0) = x^2$

From previous example, the general solution is:

$$\phi = G\left(\frac{4x-y}{3}\right) + H\left(\frac{y-x}{3}\right).$$

Now, $\phi(x,0) = x \Rightarrow G\left(\frac{4x}{3}\right) + H\left(\frac{-x}{3}\right) = x$.

$$\Rightarrow -\frac{1}{3}G'\left(\frac{4}{3}x\right) + \frac{1}{3}H'\left(\frac{-x}{3}\right) = x^2.$$

Integrating $\boxed{-\frac{3}{4}G\left(\frac{4}{3}x\right) - 3H\left(\frac{-x}{3}\right) = x^3}$.

$$\left(3 - \frac{3}{4}\right)G\left(\frac{4}{3}x\right) = 3x + x^3$$

$$\left(\frac{3}{4} - 3\right)H\left(-\frac{x}{3}\right) = \frac{3x}{4} + x^3.$$

So, $G\left(\frac{4x}{3}\right) = \frac{4}{9}\left(x^3 + 3x\right)$

$H\left(\frac{-x}{3}\right) = -\frac{4}{9}\left(\frac{3x}{4} + x^3\right)$

$\Rightarrow G(u) = \frac{3}{16}u^3 + u \qquad \left(u = \frac{4}{3}x\right).$

$H(v) = v + 12v^3 \qquad \left(v = -x/3\right).$

$$\phi = G\left(\frac{4x-y}{3}\right) + H\left(\frac{y-x}{3}\right)$$

$$= \frac{3}{16}\left(\frac{4x-y}{3}\right)^3 + \frac{1}{3}(4x-y) + \frac{y-x}{3} + 12\left(\frac{1}{3}(y-x)\right)^3$$

[see sheet #9 for examples/practice].